ДЗ 7

7.2. Найти математические ожидания и дисперсии декартовых координат точки в задаче 5.7.

$$p_x(t) = \begin{cases} 0, & t<0 \\ 1-\frac{1}{2}t, & t\in[0,2] \\ 0, & t>2 \end{cases}$$

$$p_y(t) = \begin{cases} 0, & t<0 \\ 2-2t, & t\in[0,1] \\ 0, & t>1 \end{cases}$$

- $\mathbb{E}X = \int_0^2 \left(t-\frac{1}{2}t^2\right)dt = \frac{t^2}{2}\Big|_0^2 - \frac{t^3}{6}\Big|_0^2 = 2-\frac{8}{6} = \frac{2}{3}$
- $\mathbb{E}X^2 = \int_0^2 t^2\left(1-\frac{1}{2}t\right)dt = \frac{t^3}{3}\Big|_0^2 - \frac{t^4}{8}\Big|_0^2 = \frac{8}{3} - 2 = \frac{2}{3}$
- $\mathbb{D}X = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$

miro

- $\mathbb{E}Y = \int_0^1 t(2-2t)\,dt = \int_0^1 (2t-2t^2)\,dt = \frac{2t^2}{2}\Big|_0^1 - \frac{2t^3}{3}\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$
- $\mathbb{E}Y^2 = \int_0^1 t^2(2-2t)\,dt = \frac{2t^3}{3}\Big|_0^1 - \frac{t^4}{2}\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$
- $\mathbb{D}Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \frac{1}{6} - \frac{1}{9} = \frac{3-2}{18} = \frac{1}{18}$

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7.4. Найти математическое ожидание и дисперсию случайной личины Y в задаче 6.1.

$$p_Y = \begin{cases} 0, & t<0 \\ 2\dfrac{1}{\pi\sqrt{1-t^2}}, & t\in[0,1] \\ 0, & t>1 \end{cases}$$

$$\bullet\ \mathbb{E}Y = 2\int_0^1 \frac{t}{\pi\sqrt{1-t^2}}\,dt = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{\underbrace{1-z}_{u}}}\,dz = \frac{1}{\pi}\int_{-1}^{0} u^{-\frac{1}{2}}du = -\frac{1}{\pi}2\sqrt{u}\Big|_{-1}^{0} = -\frac{1}{\pi}2\sqrt{1-t^2}\Big|_0^1 = \frac{2}{\pi}$$

$du = dz$

$z = t^2$

$dz = 2t\,dt, \quad dt = \dfrac{dz}{2t}$

$$\bullet\ \mathbb{E}Y^2 = 2\int_0^1 \frac{t^2}{\pi\sqrt{1-t^2}}\,dt = \frac{2}{\pi}\int_0^1 \frac{t^2}{\sqrt{1-t^2}}\,dt = \frac{2}{\pi}\int_0^{\pi/2} \frac{\sin^2 u\cdot\cos u}{\underbrace{\sqrt{1-\sin^2 u}}_{\cos u}}\,du =$$

$t = \sin u \quad dt = \cos u\,du$

$$= \frac{2}{\pi}\int_0^{\pi/2}\sin^2 u\,du = \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{2}\cos 2u\right)du = \frac{1}{\pi}u\Big|_0^{\pi/2} - \frac{1}{\pi}\int_0^{\pi/2}\cos 2u\,du =$$

Формулы половинного аргумента

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$
$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$

$$\sin^2 u = \frac{1-\cos 2u}{2}$$

$z = 2u$

$dz = 2\,du$

$$= \frac{1}{2} - \frac{1}{2\pi}\int_0^{\pi/2}\cos z\,dz = \frac{1}{2} - \frac{1}{2\pi}\cdot\sin 2u\Big|_0^{\pi/2} = \frac{1}{2} - \frac{\sin u\cdot\cos u}{\pi}\Big|_0^{\pi/2} = \frac{1}{2}$$

$$\bullet\ \mathbb{D}X = \frac{1}{2} - \frac{4}{\pi^2} = \frac{\pi^2-8}{2\pi^2}$$

$$\mathbb{D}X = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

7.7. Случайные величины X и Y независимы, X имеет стандартное нормальное распределение, Y имеет распределение Бернулли с параметром 1/3. Найти математические ожидания и дисперсии случайных величин: а) $2X+3Y$; б) $X-9Y-1$.

б) $\mathbb{E}X = 0 \qquad \mathbb{E}Y = \frac{1}{3}$

$\mathbb{D}X = 1 \qquad \mathbb{D}Y = \frac{2}{9}$

$\mathbb{E}Y^2 = \frac{1}{3}$

$$\mathbb{E}(X-9Y-1) = \mathbb{E}X - 9\cdot\mathbb{E}Y - 1 = 0 - 9\cdot\frac{1}{3} - 1 = -4$$

$$\mathbb{D}(X-9Y-1) = \mathbb{D}X + 9^2\cdot\mathbb{D}Y - 0 = 1 + 9^2\cdot\frac{2}{9} = 19$$

7.5. Найти математическое ожидание и дисперсию случайной величины $X-Y$ в задаче 6.4.

$$f_{X-Y} = \begin{cases} t+1, & t \in [-1;0] \\ 1-t, & t \in (0,1] \\ 0, & \text{иначе} \end{cases}$$

$$\mathbb{E}(X-Y) = \int_0^1 (t-t^2)\,dt + \int_{-1}^0 (t^2+t)\,dt = \frac{t^2}{2}\Big|_0^1 - \frac{t^3}{3}\Big|_0^1 + \frac{t^3}{3}\Big|_{-1}^0 + \frac{t^2}{2}\Big|_{-1}^0 = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{2} = 0$$

$$\mathbb{E}(X-Y)^2 = \int_0^1 (t^2-t^3)\,dt + \int_{-1}^0 (t^3+t^2)\,dt = \frac{t^3}{3}\Big|_0^1 - \frac{t^4}{4}\Big|_0^1 + \frac{t^4}{4}\Big|_{-1}^0 + \frac{t^3}{3}\Big|_{-1}^0 = \frac{1}{3} - \frac{1}{4} - \frac{1}{4} + \frac{1}{3} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$\mathbb{D}(X-Y) = \frac{1}{6} - 0 = \frac{1}{6}$$

7.8

а) $$\mathbb{E}X = \sum_{k=0}^{\infty} k\cdot\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\cdot\sum_{k=0}^{\infty} k\cdot\frac{\lambda^k}{k!} = e^{-\lambda}\cdot\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!} =$$

$$= \lambda\, e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda\cdot e^{-\lambda}\cdot e^{\lambda} = \lambda$$

$$\mathbb{E}X^2 = \sum_{k=0}^{\infty} k^2\,\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{k\cdot\lambda^{k-1}}{(k-1)!} = e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{(k-1)\lambda^{k-1}}{(k-1)!} + e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} =$$

$$= e^{-\lambda}\lambda\left[\lambda\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!} + \sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}\right] = e^{-\lambda}\lambda\left[e^{\lambda}\cdot\lambda + e^{\lambda}\right] = \lambda^2+\lambda$$

$$\mathbb{D}X = \lambda^2+\lambda-\lambda^2 = \lambda$$

в) $\mathbb{E}X = \int\limits_a^b t \cdot \frac{1}{b-a}\,dt = \frac{1}{b-a}\cdot\frac{t^2}{2}\Big|_a^b = \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] = \frac{b+a}{2}$

$$\mathbb{E}X^2 = \int\limits_a^b t^2 \cdot \frac{1}{b-a}\,dt = \frac{1}{b-a}\cdot\frac{t^3}{3}\Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ba+a^2)}{3\cdot(b-a)} = \frac{b^2+ba+a^2}{3}$$

$$\mathbb{D}X = \frac{a^2+ba+b^2}{3} - \frac{b^2+2ba+a^2}{4} = \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12} = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

g) $\gamma_{\alpha,\beta} = \begin{cases} \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha t} t^{\beta-1}, & t>0 \\ 0, & t\le 0 \end{cases}$ $\qquad \Gamma(\beta) = \int\limits_0^\infty e^{-t} t^{\beta-1}\,dt$

$\alpha t = u \quad du = \alpha dt$

$t = \frac{u}{\alpha} \quad dt = \frac{du}{\alpha}$

$$\mathbb{E}X = \int\limits_0^{+\infty} \frac{t^\beta \alpha^\beta e^{-\alpha t}}{\Gamma(\beta)}\,dt = \frac{\alpha^\beta}{\Gamma(\beta)}\int\limits_0^{+\infty}\frac{u^\beta}{\alpha^\beta}\cdot\frac{e^{-u}}{\alpha}\,du = \frac{1}{\alpha\cdot\Gamma(\beta)}\int\limits_0^{+\infty} u^\beta \cdot e^{-u}\,du =$$

$\Gamma(z+1) = z\Gamma(z)$

$$= \frac{\Gamma(\beta+1)}{\alpha\,\Gamma(\beta)} = \frac{\beta\cdot\Gamma(\beta)}{\alpha\,\Gamma(\beta)} = \frac{\beta}{\alpha}$$

$$\mathbb{E}X^2 = \int\limits_0^{+\infty}\frac{t^{\beta+1}\cdot\alpha^\beta e^{-\alpha t}}{\Gamma(\beta)}\,dt = \frac{\alpha^\beta}{\Gamma(\beta)}\int\limits_0^{+\infty} t^{\beta+1} e^{-\alpha t}\,dt =$$

$\alpha t = u \quad du = \alpha dt$

$t = \frac{u}{\alpha} \quad dt = \frac{du}{\alpha}$

$$= \frac{\alpha^\beta}{\Gamma(\beta)}\cdot\int\limits_0^{+\infty}\frac{u^{\beta+1}}{\alpha^{\beta+1}}\cdot\frac{e^{-u}}{\alpha}\,du = \frac{\Gamma(\beta+2)}{\alpha^2\Gamma(\beta)} = \frac{(\beta+1)\beta\,\Gamma(\beta)}{\alpha^2\,\Gamma(\beta)} = \frac{\beta(\beta+1)}{\alpha^2}$$

$$\mathbb{D}X = \frac{(\beta+1)\beta}{\alpha^2} - \frac{\beta^2}{\alpha^2} = \frac{\beta}{\alpha^2}$$

7.9. Найти математические ожидания и дисперсии случайных величин Y_1 и Y_n, введенных в задаче 5.12.

$$f_{Y_1} = \begin{cases} 0, & t\le 0 \\ 1 + \frac{n\left(1-\frac{t}{a}\right)^{n-1}}{a} & t\in[0,a] \\ 0, & t>a \end{cases}$$

$$\left(\left(1-\frac{t}{a}\right)^n\right)' = n\left(1-\frac{t}{a}\right)^{n-1}\left(-\frac{1}{a}\right) = -\frac{n}{a}\left(1-\frac{t}{a}\right)^{n-1}$$

$$\mathbb{E}Y_1 = \int\limits_0^a \frac{nt\left(1-\frac{t}{a}\right)^{n-1}}{a}\,dt = -\int\limits_0^a t\,d\left(\left(1-\frac{t}{a}\right)^n\right) = \left[\int u\,dv = uv - \int v\,du\right]$$

$$= -t\cdot\left(1-\frac{t}{a}\right)^n\Big|_0^a + \int\limits_0^a\left(1-\frac{t}{a}\right)^n dt = -t\left(1-\frac{t}{a}\right)^n\Big|_0^a + \frac{a}{n+1} = \frac{a}{n+1}$$

$$z = 1 - \frac{t}{a}$$

$$dz = -\frac{dt}{a} \quad dt = -a \cdot dz$$

$$= -\int z^n \cdot a\,dz = -a\int z^n dz = -a\frac{z^{n+1}}{n+1} = -a\frac{\left(1-\frac{t}{a}\right)^{n+1}}{n+1}$$

$$dt^2 = 2t\,dt$$

$$EY_1^2 = \int_0^a \left[\frac{nt^2\left(1-\frac{t}{a}\right)^{n-1}}{a}\right] dt = -\int_0^a t^2\, d\left(\left(1-\frac{t}{a}\right)^n\right) = -t^2\left(1-\frac{t}{a}\right)^n\Big|_0^a + \int_0^a \left(1-\frac{t}{a}\right)^n dt^2 =$$

$$= -t^2\left(1-\frac{t}{a}\right)^n\Big|_0^a + 2\int_0^a t\left(1-\frac{t}{a}\right)^n dt = 0 + 2\cdot\frac{a}{n+1}$$

$$DY_1 = \frac{2a}{n+1} - \frac{a^2}{(n+1)^2} = \frac{2a(n+1)-a^2}{(n+1)^2} = \frac{2an+2a-a^2}{(n+1)^2}$$

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7.6

7.6. Доказать, что $\mathbf{E}X = \sum_{k=1}^{\infty} \mathbf{P}\,(X \geq k)$, если известно, что $\sum_{k=1}^{\infty} \mathbf{P}\,(X = k) = 1$.

$$EX \overset{\text{дискр}}{=} \sum_k k\,P(x=k)$$

$$EX = 1P(x=1) + 2P(x=2) + \ldots + kP(x=k) + \ldots =$$

$$= P(x=1) +$$
$$+ P(x=2) + P(x=2) +$$
$$+ P(x=3) + P(x=3) + P(x=3) +$$
$$+ P(x=4) \ldots + P(x=4) + P(x=4) + \ldots$$

$$\sum_{k=1}^{\infty} P(x=k) \qquad \sum_{k=1}^{\infty} P(x=k) - P(x=1) \qquad \sum_{k=1}^{\infty} P(x=k) - P(x=1) - P(x=2)$$

$$= \underset{1-P(x=0)}{1} + \underbrace{1 - P(x=1)}_{P(x\geq 2)} + \underbrace{1 - P(x=1) - P(x=2)}_{P(x\geq 3)} + \ldots =$$

$$= \sum_{k=1}^{\infty} P(x \geq k)$$

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