Differential Equations Report on Assignment 1 (Numerical Methods)

github link is here: https://github.com/dariamikl/Numerical-methods-project

Exact solution of the Initial Value problem

$$y' = \frac{y}{x} + \frac{x}{y}$$

$$y' - \frac{1}{x}y = xy^{-1}$$

First-order Bernoulli differential equation

Using substitution:

$$v = y^{1-n}$$

We get:

$$\frac{v'(x)}{2} - \frac{v(x)}{x} = x$$

After integration:

$$v(x) = 2\ln(x)x^2 + c_1x^2$$

Substituting back

$$v(x) = y^2$$

We get:

$$y^2 = 2\ln(x)x^2 + c_1x^2$$

We get:

$$y = x\left(-\sqrt{c_1 + 2\ln(x)}\right),\ y = x\sqrt{c_1 + 2\ln(x)}$$

In [106]:
```python
import pandas as pd
```

In [120]:
```python
l = {'y0':[1],'x0':[1], 'xb':[2.3]}
init = pd.DataFrame(data = l)
```

In [124]:
```python
init.head()
```

Out[124]:

	y0	x0	xb
0	1	1	2.3

Finding c1

Since X lies in the positive x-axis, we take into consideration one of the solutions, since the root can not be negative.

Substituting initial value:

$$1 = \sqrt{c_1 + 2\ln(1)} = \sqrt{c_1}$$

We get, that c1=1, then for this IVP:

$$y = x\sqrt{1 + 2\ln(x)}$$

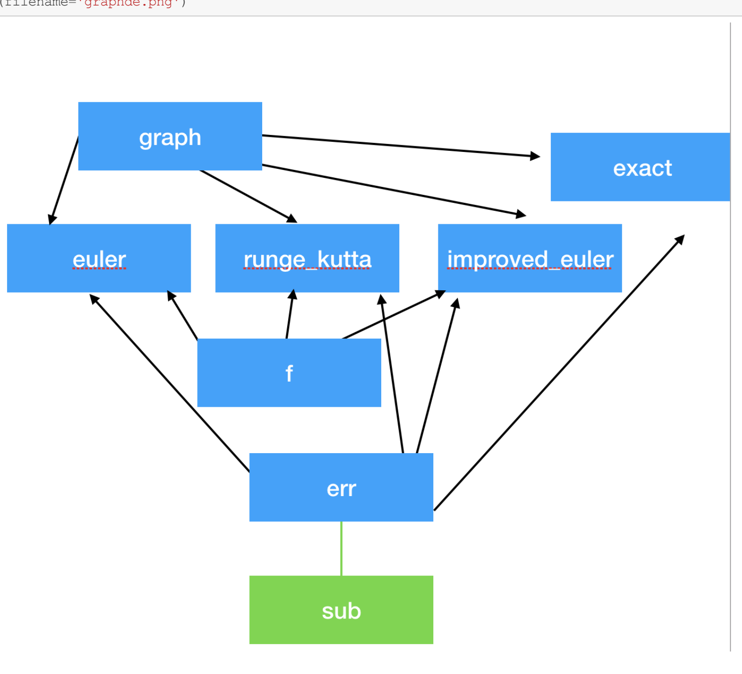
Structure of the program

The program has the following methods:

- **euler** (Euler method implementation),
- **improved_euler** (Improved Euler method implementation)
- **f** (Helper function for calculating f(x,y))
- **runge_kutta** (Runge-Kutta method implementation)
- **exact** (Calculation of exact values of the function for estimating the error)
- **graph** (For graphing the forementioned four functions with the ability of chnaging initial values and the grid step)
- **sub** (Inner submethod for array substraction)
- **err** (Method for outputting the error)

In [129]:
```python
from IPython.display import Image
Image(filename='graphde.png')
```

Out[129]:



Procedures diagram

Description of the methods

Calculating the step: $h = \frac{xb - x0}{n}$

Calculating x: $xi = x0 + i \cdot h$

Euler method: $y_{i+1} = y_i + hf\left(x_i, y_i\right)$ which is derived simply from the equation of the tangent line the assumption that tangent line to the integral curve approximates the intergal curve

In [145]:
```python
def euler(n, y0, x0, xb):
    h = (xb-x0)/float(n) #the step
    x = np.linspace(x0, xb, n) #array of x's with equally spaces intervals (same as h)
    y = np.zeros([n]) #fill initial vals with 0
    y[0] = y0
    for i in range (1,n):
        y[i] = y[i-1] + h*f(x[i-1], y[i-1]) #here is the main formula but starting from i, not i+1
    return [x, y]
```

Improved Euler method uses the concept of the average of the slopes of the tangents to the integral curve:

$y_{i+1} = y_i + \frac{h}{2}\left(f\left(x_i, y_i\right) + y_p\right)$

where $y_p = f\left(x_{i+1}, y_i + hf\left(x_i, y_i\right)\right)$

In [146]:
```python
def improved_euler(n, y0, x0, xb):
    h = (xb-x0)/float(n) #the step
    x = np.linspace(x0, xb, n) #array of x's with equally spaces intervals (same as h)
    y = np.zeros([n]) #fill initial vals with 0
    y_p = np.zeros([n]) #same for y_p
    y_p[0]=-1
    y[0] = y0
    for i in range (1,n):
        y_p[i]=y[i-1] + h*f(x[i-1], y[i-1]) #y_p
        y[i] = y[i-1] + (h/2)*(f(x[i-1], y[i-1]) + f(x[i], y_p[i])) #here is the main formula but starting from i, not i+1
    return [x,y]
```

Runge Kutta method with k=4, the most widely used method:

$k_{1i} = f\left(x_i, y_i\right)$

$k_{2i} = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}\right)$

$k_{3i} = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{2i}\right)$

$k_{4i} = f\left(x_i + h, y_i + hk_{3i}\right)$

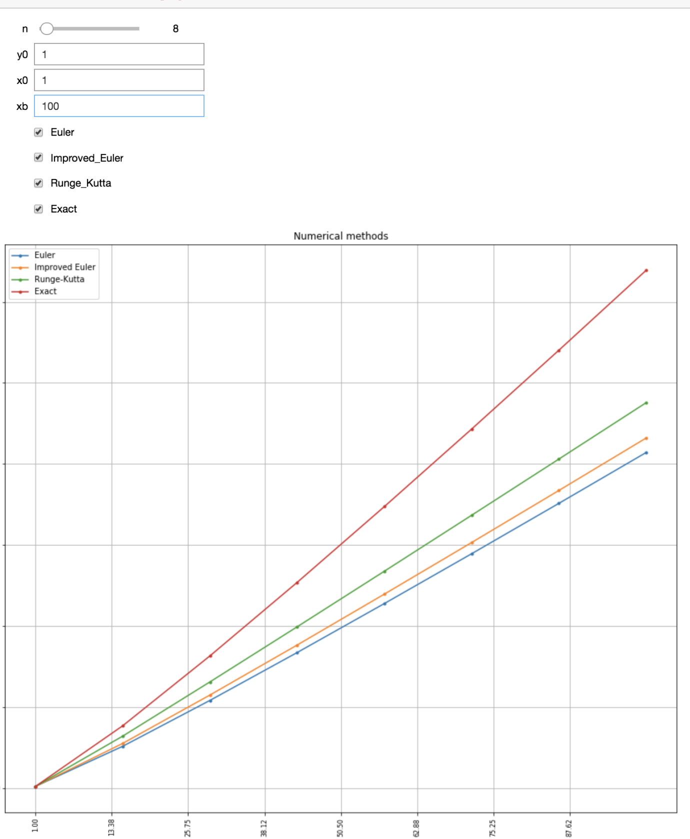
$y_{i+1} = y_i + \frac{h}{6}\left(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i}\right)$

In [ ]:
```python
def runge_kutta(n, y0, x0, xb):
    h = (xb-x0)/float(n) #the step
    x = np.linspace(x0, xb, n) #filling out x's
    y = np.zeros([n]) #y's initialization
    y[0] = y0
    for i in range (1,n):
        k1 = h*f(x[i-1], y[i-1]) #multiplied by h here
        k2 = h*f(x[i-1] + (h*0.5), y[i-1] + (k1*0.5))
        k3 = h*f(x[i-1] + (h*0.5), y[i-1] + (k2*0.5))
        k4 = h*f(x[i-1] + h, y[i-1] + k3)
        delta_y = (k1+2*k2+2*k3+k4)/6 #since all k's are already multiplied by h, no need to do it here
        y[i] = y[i-1] + delta_y #main formula
    return [x,y]
```

Display of the graphs

In [150]:
```python
Image(filename='numerical.png')
```

Out[150]:

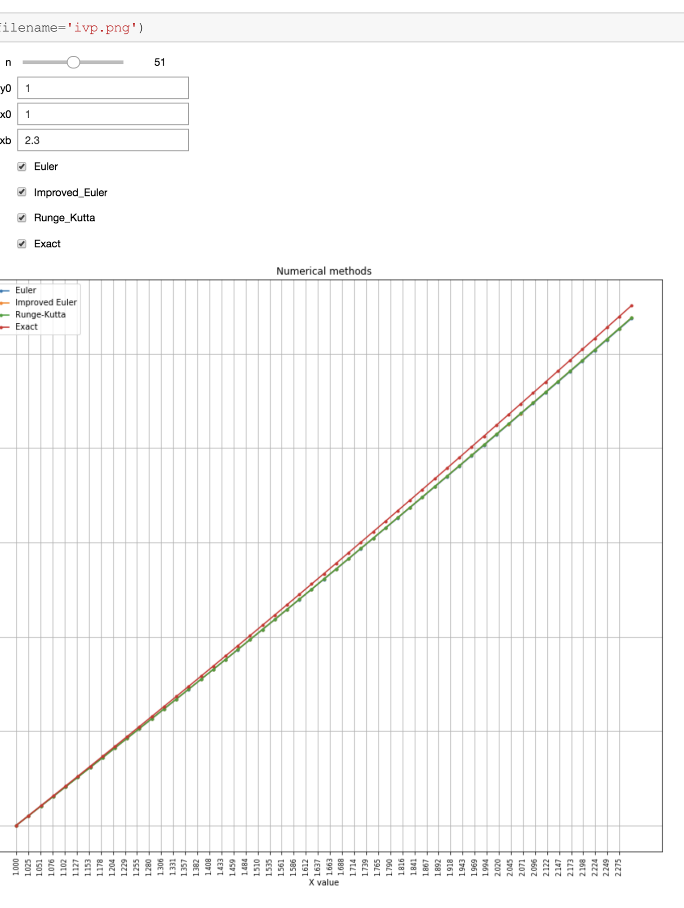


Having large enough interval we can see that Runge-Kutta method remains closer to the exact functions than the rest (initial values and grid size are adjustible), obviously, the further - the greater the error

And this is the result with the default initial values (still very close)

In [152]:
```python
Image(filename='ivp.png')
```

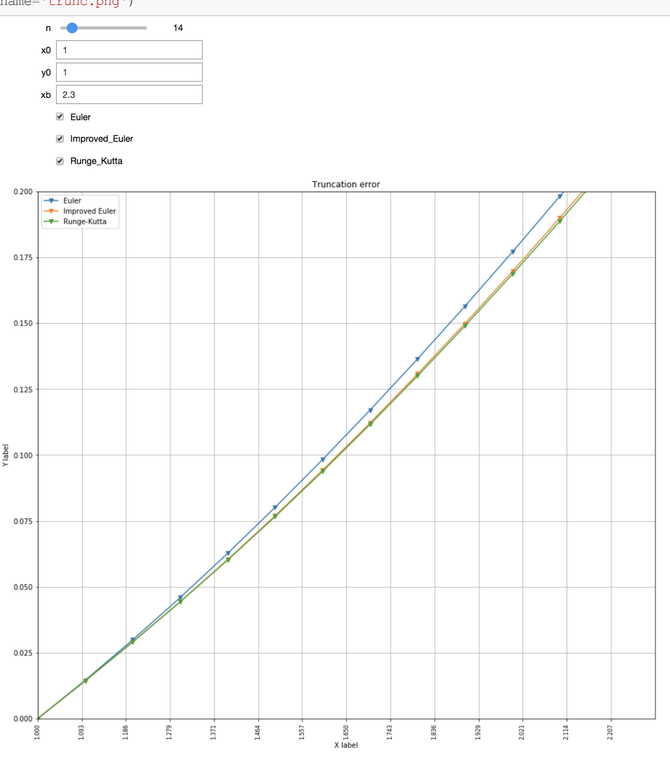
Out[152]:



Here we can compare the global trucantion error for the values given. As we see, Runge-Kutta method performs better.

In [154]:
```python
Image(filename='trunc.png')
```

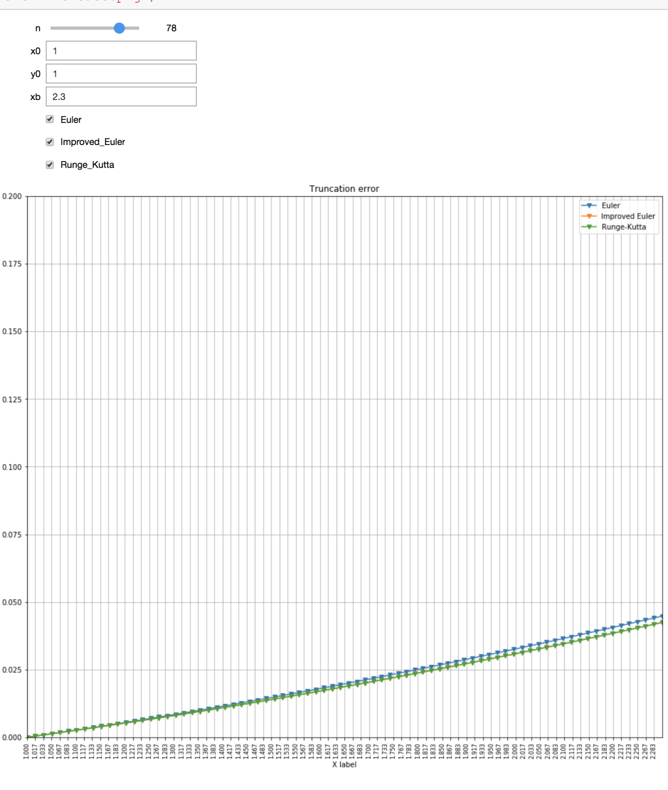
Out[154]:



And as the number of step increses - the value descreases, like here:

In [155]:
```python
Image(filename='increase.png')
```

Out[155]:



Result:Again, Runge-Kutte method has shown better performance

Source code

In [156]:
```python
import numpy as np
import matplotlib.pyplot as plt
import math
from __future__ import print_function
from ipywidgets import interact, interactive, fixed, interact_manual
import ipywidgets as widgets

def f(x,y):
        return y/x + x/y

def euler(n, y0, x0, xb):
    h = (xb-x0)/float(n)
    x = np.linspace(x0, xb, n)
    y = np.zeros([n])
    y[0] = y0
    for i in range (1,n):
        y[i] = y[i-1] + h*f(x[i-1], y[i-1])

    return [x, y]

def runge_kutta(n, y0, x0, xb):
    h = (xb-x0)/float(n)
    x = np.linspace(x0, xb, n)
    y = np.zeros([n])
    y[0] = y0
    for i in range (1,n):
        k1 = h*f(x[i-1], y[i-1])
        k2 = h*f(x[i-1] + (h*0.5), y[i-1] + (k1*0.5))
        k3 = h*f(x[i-1] + (h*0.5), y[i-1] + (k2*0.5))
        k4 = h*f(x[i-1] + h, y[i-1] + k3)
        delta_y = (k1+2*k2+2*k3+k4)/6
        y[i] = y[i-1] + delta_y
    return [x,y]

def improved_euler(n, y0, x0, xb):
    h = (xb-x0)/float(n)
    x = np.linspace(x0, xb, n)
    y = np.zeros([n])
    y_p = np.zeros([n])
    y_p[0]=-1
    y[0] = y0
    for i in range (1,n):
        y_p[i]=y[i-1] + h*f(x[i-1], y[i-1])
        y[i] = y[i-1] + (h/2)*(f(x[i-1], y[i-1]) + f(x[i], y_p[i]))

    return [x,y]
def exact(n, y0, x0, xb):
    c1=(y0/x0)**2 - 2*math.log(x0)

    #h = 0.1
    h = (xb-x0)/float(n)
    x = np.linspace(x0, xb, n)
    y = np.zeros([n])
    y[0] = y0
    for i in range (0, n):
        y[i] = x[i]*math.sqrt(c1+math.log(x[i]**2))
    return [x,y]
def graph(n, y0, x0, xb, Euler, Improved_Euler, Runge_Kutta, Exact)  :

    x0 = float(x0)
    y0 = float(y0)
    xb = float(xb)
    n = int(n)
    h = (xb-x0)/float(n)
    %matplotlib inline
    [x_e, y_e] = euler(n, y0, x0, xb)
    [x_ie, y_ie] = improved_euler(n, y0, x0, xb)
    [x_rk, y_rk] = runge_kutta(n, y0, x0, xb)
    [x, y] = exact(n, y0, x0, xb)
    plt.rcParams["figure.figsize"] = (15,13)
    if Euler:
        plt.plot(x_e,y_e,marker='.', label = "Euler")
    if Improved_Euler:
        plt.plot(x_ie, y_ie, marker ='.', label="Improved Euler")
    if Runge_Kutta:
        plt.plot(x_rk, y_rk, marker='.', label = "Runge-Kutta")
    if Exact:
        plt.plot(x, y,marker='.', label="Exact")
    plt.xticks(np.arange(x0, xb, step=h),rotation='vertical', fontsize=8)
    plt.grid( linestyle='-')
    plt.legend()
    plt.xlabel('X value')
    plt.ylabel('Y label')
    plt.title("Numerical methods")
    plt.show()
interactive_plot = interactive(graph, n=(2,100), y0='1', x0='1', xb='2.3', Euler = True, Improved_Euler=True, Runge_Kutta=True, Exact=True)
output = interactive_plot.children[-1]
output.layout.height = '1000px'
interactive_plot
```

In [157]:
```python
def err(n, x0, y0, xb, Euler, Improved_Euler, Runge_Kutta): #in here we just calculate the difference between yi and y(xi)
    def sub(a,b):
        e = np.zeros([len(a)])
        for i in range(len(a)):
            e[i] = abs(a[i] -b[i])
        return e
    x0 = float(x0)
    y0=float(y0)
    xb=float(xb)
    h = (xb-x0)/float(n)

    %matplotlib inline
    [x_e, y_e] = [euler(n, y0, x0, xb)[0], sub(euler(n, y0, x0, xb)[1], exact(n, y0, x0, xb)[1])]
    [x_ie, y_ie] = [improved_euler(n, y0, x0, xb)[0], sub(improved_euler(n, y0, x0, xb)[1], exact(n, y0, x0, xb)[1])]
    [x_rk, y_rk] = [runge_kutta(n, y0, x0, xb)[0], sub(runge_kutta(n, y0, x0, xb)[1], exact(n, y0, x0, xb)[1])]
    #[x, y] = exact(h)
    plt.rcParams["figure.figsize"] = (16,14)
    if Euler:
        plt.plot(x_e,y_e,marker='v', label = "Euler")
    if Improved_Euler:
        plt.plot(x_ie, y_ie, marker ='v', label="Improved Euler")
    if Runge_Kutta:
        plt.plot(x_rk, y_rk, marker='v', label = "Runge-Kutta")

    plt.legend()
    #plt.plot(x, y, 'r')
    plt.ylim(0,0.2)
    plt.xticks(np.arange(x0, xb, step=h),rotation='vertical', fontsize=8)
    plt.xlim(x0, xb)
    plt.grid( linestyle='-')
    plt.xlabel('X label')
    plt.ylabel('Y label')
    plt.title("Truncation error")

    plt.show()
interactive_plot = interactive(err, n=(2, 100), x0='1', y0='1', xb='2.3', Euler = True, Improved_Euler=True, Runge_Kutta=True)
output = interactive_plot.children[-1]
output.layout.height = '1000px'
interactive_plot
```