Please checkmark exercises that you solved before **11.01.2018**. The details of the checkmarking process will be available on the course website from **11.12.2017** onwards. Be sure to tick only those exercises which you can solve and explain on the blackboard. Do not leave the exercise work for the very last moment. Start preparing solutions as early as possible!

Description Logic Syntax

Problem 1. Suppose that C and D are concept names and s, r are role names. State for every expression from below whether it is

- a concept
- a concept equivalence
- a concept inclusion
- a role inclusion

(1) $C \sqcap D$

(2) $C \sqcap D \sqcup C$

(3) $\neg C$

(4) $C \sqsubseteq D$

(5) $\forall r.(C \sqcup D)$

(6) $C \equiv \exists r^{-}.(C \sqcup D)$

(7) $\exists s.(\exists r.C) \sqsubseteq D$

(8) $C \sqcap D \sqsubseteq C$

(9) $C \equiv C \sqcap D$

(10) $r \sqsubseteq s$

Solution:

(1) $C \sqcap D$: Concept

(2) $C \sqcap D \sqcup C$: Concept

(3) $\neg C$: Concept

(4) $C \sqsubseteq D$: Concept inclusion

(5) $\forall r.(C \sqcup D)$: Concept

(6) $C \equiv \exists r^{-}.(C \sqcup D)$: Concept equivalence

(7) $\exists s.(\exists r.C) \sqsubseteq D$: Concept inclusion

(8) $C \sqcap D \sqsubseteq C$: Concept inclusion

(9) $C \equiv C \sqcap D$: Concept equivalence

(10) $r \sqsubseteq s$: role inclusion

Problem 2. Let *Author*, *Book* and *Article* be concept names, and let *writes* be a role name.

- Express the following statements in natural language:
  (1) $\exists writes.Book \sqsubseteq Author$
  (2) $\forall writes.Book$
  (3) $\exists writes.Book$
  (4) $\exists writes^{-}.\top \sqsubseteq Book \sqcup Article$
  (5) $Author \sqsubseteq \exists writes.\top$
  (6) $Author \sqsubseteq \exists writes.\bot$
  (7) $\geq 11\ writes.\top \sqsubseteq Author$
  (8) $\geq 9\ writes.Book \sqsubseteq Author$
  (9) $\forall writes.\top \sqsubseteq \exists writes.Book$
  (10) $\exists writes.\top \sqsubseteq (\geq 5 writes.\top)$
  (11) $\geq 5\ writes.\top \sqsubseteq \exists writes.\top$
  (12) $\leq 1\ writes.\top \sqsubseteq \neg Author$
- For every expression from above state whether it is (a) a concept, (b) a concept inclusion, (c) a role inclusion, (d) none of the above.

Solution:

(1) $\exists writes.Book \sqsubseteq Author$: Concept inclusion
Anyone who writes a book is an author.

(2) $\forall writes.Book$: Concept
This concept describes a set of individuals, who wrote only books, and possibly nothing.

(3) $\exists writes.Book$: Concept
This concept describes a set of individuals who wrote a book

(4) $\exists writes^{-}.\top \sqsubseteq Book \sqcup Article$: Concept inclusion
Whatever written is either a book or an article.

(5) $Author \sqsubseteq \exists writes.\top$: Concept inclusion
Every author has written something.

(6) $Author \sqsubseteq \exists writes.\bot$: Concept inclusion
Every author has written nothing.

(7) $\geq 11\ writes.\top \sqsubseteq Author$: Concept inclusion
Everyone, who wrote more than 11 artifacts, is an author.

(8) $\geq 9\ writes.Book \sqsubseteq Author$: Concept inclusion
Everyone, who wrote more than 9 books, is an author.

(9) $\forall writes.\top \sqsubseteq \exists writes.Book$: Concept inclusion
Everything writes a book.
Explanation: $\forall writes.\top$ contains individuals, who either did not write anything at all or wrote something (no matter what it is), i.e., the left hand-side of the given concept inclusion describes the whole domain, since $\forall writes.\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$ for any $\mathcal{I}$.

(10) $\exists writes.\top \sqsubseteq \geq 5 writes.\top$: Concept inclusion
Everyone, who wrote something, wrote at least 5 artifacts.

(11) $\geq 5\ writes.\top \sqsubseteq \exists writes.\top$: Concept inclusion
Those who wrote at least 5 things, wrote something.

(12) $\leq 1\ writes.\top \sqsubseteq \neg Author$: Concept inclusion
Those who wrote at most 1 thing are not authors.

Problem 3. Write the following statements in description logic $\mathcal{SHOIQ}$. Explicitly mention, which of the used symbols are concept names, role names and nominals.

Solution:

(1) Every student at Saarland university is a person;
$\exists studentAt.\{saarland\_uni\} \sqsubseteq Person$

(2) MPI for Informatics has at least 500 students;
$\{mpi\} \sqsubseteq \geq 500\, studentAt^{-}.\top$

(3) Every citizen of Germany is a European;
$\exists citizenOf.\{germany\} \sqsubseteq European$

(4) There are at least 150.000 people in Saarland;
$\{Saarland\} \sqsubseteq \geq 150000\, citizenOf^{-}$

(5) The domain of the relation “lives in” comprises of people;
$\exists livesIn.\top \sqsubseteq Person$

(6) The range of the relation “has nationality” comprises of countries;
$\exists hasNationality^{-}.\top \sqsubseteq Country$

(7) Bob lived in at least 3 countries;
$\geq 3\, livesIn.Country(bob)$

(8) Everybody who has a happy friend is also happy;
$\exists hasFriend.Happy \sqsubseteq Happy$

(9) Brad and Angelina played in at least 2 movies together;
$\{brad\} \sqsubseteq \geq 2\, actedIn.(Film \sqcap \exists actedIn^{-}.\{angelina\})$

(10) Brad and Charlie did not work together.
$\exists workedAt^{-}.\{brad\} \sqsubseteq \neg\exists workedAt^{-}.\{charlie\}$, or
$\exists workedAt^{-}.\{brad\} \sqcap \exists workedAt^{-}.\{charlie\} \sqsubseteq \bot$ or
$\neg workedWith(brad, charlie)$

Description Logic Semantics

Problem 4. Consider the description logic statements from **Problem 2**.

- For every concept inclusion $\mathcal{E}$, check whether $\mathcal{E}$ follows from the empty TBox (i.e., check whether $\emptyset \models \mathcal{E}$), and if it is not the case, construct an interpretation $\mathcal{I}$, such that $\mathcal{I} \not\models \mathcal{E}$.
- For every concept $\mathcal{E}$, test whether $\mathcal{E}$ is satisfiable. If this is the case, define an interpretation $\mathcal{I}$ such that $\mathcal{E}^{\mathcal{I}} \neq \emptyset$.

Solution:

- For every concept inclusion $\mathcal{E}$, check whether $\mathcal{E}$ follows from the empty TBox (i.e., check whether $\emptyset \models \mathcal{E}$), and if it is not the case, construct an interpretation $\mathcal{I}$, such that $\mathcal{I} \not\models \mathcal{E}$.

  (1) $\exists writes.Book \sqsubseteq Author$: Concept inclusion
  Let us construct an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$ as follows:
  * $\Delta^{\mathcal{I}} = \{a, b, a2\}$
  * $Author^{\mathcal{I}} = \{a2\}$
  * $Book^{\mathcal{I}} = \{b\}$
  * $writes^{\mathcal{I}} = \{(a, b)\}$

  We have that $a \in (\exists writes.Book)^{\mathcal{I}}$, but $a \notin Author^{\mathcal{I}}$.
  Hence, for the constructed interpretation $\mathcal{I}$ it holds that $\mathcal{I} \not\models \exists writes.Book \sqsubseteq Author$, meaning that $\emptyset \not\models \exists writes.Book \sqsubseteq Author$.

  (4) $\exists writes^{-}.\top \sqsubseteq Book \sqcup Article$: Concept inclusion
  Consider the following interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$:
  * $\Delta^{\mathcal{I}} = \{a, b, c, d, e\}$
  * $Book^{\mathcal{I}} = \{b\}$
  * $Article^{\mathcal{I}} = \{d\}$
  * $writes^{\mathcal{I}} = \{(a, b), (c, d), (a, e)\}$

  We have that $e \in (\exists writes^{-}.\top)^{\mathcal{I}}$; however, $e \notin (Book \sqcup Article)^{\mathcal{I}}$, therefore $\mathcal{I} \not\models \exists writes^{-}.Book \sqsubseteq Author$, and thus the given concept inclusion does not follow from an empty TBox.

  (5) $Author \sqsubseteq \exists writes.\top$: Concept inclusion
  Let us consider the following interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$:
  * $\Delta^{\mathcal{I}} = \{a, a_2, b, d\}$

- $*$ $Author^{\mathcal{I}} = \{a\}$
- $*$ $writes^{\mathcal{I}} = \{(a_2, b), (a_2, d)\}$

We have that $a \in Author^{\mathcal{I}}$, but $a \notin (\exists writes.\top)^{\mathcal{I}}$, hence $\mathcal{I} \not\models \mathcal{E}$ and $\emptyset \not\models \mathcal{E}$.

(6) $Author \sqsubseteq \exists writes.\bot$: Concept inclusion
Consider $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$:

- $*$ $\Delta^{\mathcal{I}} = \{a, b\}$;
- $*$ $Author^{\mathcal{I}} = \{a\}$;
- $*$ $writes^{\mathcal{I}} = \{(a, b)\}$.

It holds that $a \in Author^{\mathcal{I}}$; however, $a \notin (\exists writes.\bot)^{\mathcal{I}}$. Note that $\exists writes.\bot$ is unsatisfiable, i.e., it is always empty, and hence no interpretation $\mathcal{I}$, such that $Author^{\mathcal{I}} \neq \emptyset$ can be a model of $Author \sqsubseteq \exists writes.\bot$.

(7) $\geq 11\ writes.\top \sqsubseteq Author$: Concept inclusion
Let us define $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$ as follows:

- $*$ $\Delta^{\mathcal{I}} = \{a, a_2, b_1, b_2, ..., b_{12}, c\}$
- $*$ $Author^{\mathcal{I}} = \{a_2\}$
- $*$ $writes^{\mathcal{I}} = \{(a, b_1), (a, b_2), ..., (a, b_{12}), (a_2, c)\}$

We have that $a \in (\geq 11\ writes.\top)^{\mathcal{I}}$, but $a \notin Author^{\mathcal{I}}$, and thus $\mathcal{I} \not\models \geq 11\ writes.\top \sqsubseteq Author$, i.e., the inclusion does not follow from the empty TBox.

(8) $\geq 9\ writes.Book \sqsubseteq Author$: Concept inclusion
Assume that $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$ is given as follows:

- $*$ $\Delta^{\mathcal{I}} = \{a, a_2, b_1, b_2, ..., b_{10}\}$
- $*$ $Author^{\mathcal{I}} = \{a_2\}$
- $*$ $Book^{\mathcal{I}} = \{b_1, ..., b_{10}\}$
- $*$ $writes^{\mathcal{I}} = \{(a, b_1), (a, b_2), ..., (a, b_9), (a_2, b_{10})\}$

It holds that $a \in (\geq 9 writes.Book)^{\mathcal{I}}$, but $a \notin Author^{\mathcal{I}}$, thus $\mathcal{I} \not\models \mathcal{E}$.

(9) $\forall writes.\top \sqsubseteq \exists writes.Book$: Concept inclusion
Consider $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$ defined as follows:

- $*$ $\Delta^{\mathcal{I}} = \{a\}$
- $*$ $writes^{\mathcal{I}} = \emptyset$
- $*$ $Book^{\mathcal{I}} = \emptyset$

As earlier mentioned $\forall writes.\top$ is a set of individuals who either did not write anything or wrote something, i.e., that it is the set $\{a\}$ in our interpretation, i.e., $a \in (\forall writes.\top)^{\mathcal{I}}$; however, $a \notin (\exists writes.Book)^{\mathcal{I}}$, i.e., $\mathcal{I} \not\models \mathcal{E}$.

(10) $\exists writes.\top \sqsubseteq (\geq 5\ writes.\top)$: Concept inclusion
Consider $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$ defined as follows:

- $*$ $\Delta^{\mathcal{I}} = \{a, b\}$
- $*$ $writes^{\mathcal{I}} = \{(a, b)\}$

We have $a \in (\exists writes)^{\mathcal{I}}$; however, $a \notin (\geq 5\ writes.\top)^{\mathcal{I}}$, since a wrote only b. Thus it holds that $\mathcal{I} \not\models \mathcal{E}$.

(11) $\geq 5 writes.\top \sqsubseteq \exists writes.\top$: Concept inclusion.
This concept inclusion follows from the empty TBox, i.e., any interpretation $\mathcal{I}$ satisfies it. Indeed, whenever someone wrote at least 5 things, he must have written something.

(12) $\leq 1\ writes.\top \sqsubseteq \neg Author$: Concept inclusion
For the below interpretation $\mathcal{I}$, it holds that $\mathcal{I} \not\models \mathcal{E}$:

* $\Delta^{\mathcal{I}} = \{a, b_1\}$
* $Author^{\mathcal{I}} = \{a\}$
* $writes^{\mathcal{I}} = \emptyset$

We have $a \in (\leq 1 writes.\top)^{\mathcal{I}}$; however, $a \notin (\neg Author)^{\mathcal{I}}$, since $a \in Author^{\mathcal{I}}$

- For every concept $\mathcal{E}$, test whether $\mathcal{E}$ is satisfiable. If this is the case, define an interpretation $\mathcal{I}$ such that $\mathcal{E}^{\mathcal{I}} \neq \emptyset$.

(2) $\forall writes.Book$: Concept
The following interpretation $\mathcal{I}$ satisfies this concept:

- $\Delta^{\mathcal{I}} = \{a, a_2, b_1\}$
- $Book^{\mathcal{I}} = \{b_1\}$
- $writes^{\mathcal{I}} = \{(a, b_1)\}$

We have $(\forall writes.Book)^{\mathcal{I}} = \{a, a_2, b_1\}$

(3) $\exists writes.Book$: Concept
This concept is satisfied by the following interpretation $\mathcal{I}$:

- $\Delta^{\mathcal{I}} = \{a, b\}$
- $Book^{\mathcal{I}} = \{b\}$
- $writes^{\mathcal{I}} = \{(a, b)\}$

It holds that $(\exists writes.Book)^{\mathcal{I}} = \{a\}$.

Problem 5. Consider the following interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$ given as

- $\Delta^{\mathcal{I}} = \{a, b, c\}$
- $C^{\mathcal{I}} = \{a\}$
- $D^{\mathcal{I}} = \{b, c\}$
- $s^{\mathcal{I}} = \{(c, a), (a, a), (a, b)\}$

Compute the extension of the following concepts under $\mathcal{I}$:
Solution:

(1) $(C \sqcup D)^{\mathcal{I}} = \{a, b, c\}$;

(2) $(C \sqcap D)^{\mathcal{I}} = \emptyset$;

(3) $(\top \sqcap \neg(C \sqcup \neg D))^{\mathcal{I}} = (\top \sqcap (\neg C \sqcap D))^{\mathcal{I}} = (\top \sqcap (\{b, c\} \sqcap \{b, c\}))^{\mathcal{I}} = \{b, c\}$;

(4) $(\forall s.(C \sqcup D))^{\mathcal{I}} = \{a, b, c\}$;

(5) $(\forall s.(C \sqcap D))^{\mathcal{I}} = (\forall s.(\{\})^{\mathcal{I}} = \{b\}$;

(6) $(\forall s.C \sqcap \exists s.D)^{\mathcal{I}} = (\{c, b\} \sqcap \{a\})^{\mathcal{I}} = \emptyset$;

(7) $(\neg(\neg C \sqcup \neg D))^{\mathcal{I}} = (C \sqcap D)^{\mathcal{I}} = \emptyset$;

(8) $(\exists s.(\exists s.(\exists s.(C \sqcup D))))^{\mathcal{I}} = (\exists s.(\exists s.(\exists s.(\{a, b, c\}))))^{\mathcal{I}} = (\exists s.(\exists s.(\{a, c\})))^{\mathcal{I}} =$
$= (\exists s.(\{a, c\}))^{\mathcal{I}} = \{a, c\}$;

(9) $(\exists s^{-}.D)^{\mathcal{I}} = \{a\}$;

(10) $(\forall s^{-}.C)^{\mathcal{I}} = \{b, c\}$.

Problem 6. Consider the following TBox $\mathcal{T} = \{Author \sqsubseteq \exists writes.Book, Novelist \sqsubseteq Author\}$. Formally prove that $\mathcal{T} \not\sqsubseteq Author \sqsubseteq Novelist$ by constructing an interpretation $\mathcal{I}$, such that $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \not\models Author \sqsubseteq Novelist$.

Solution: We construct the following interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$:

- $\Delta = \{tom, bob, alice\}$
- $Author^{\mathcal{I}} = \{tom, bob, alice\}$
- $Novelist^{\mathcal{I}} = \{tom, bob\}$
- $Book^{\mathcal{I}} = \{book1, book2, book3\}$
- $writes^{\mathcal{I}} = \{(tom, book1), (bob, book2), (alice, book3)\}$.

It holds that $alice \in (Author \sqcap \neg Novelist)^{\mathcal{I}}$, hence we have that $\mathcal{I} \models \mathcal{T}$, but $\mathcal{I} \not\models Author \sqsubseteq Novelist$.

Problem 7. Suppose you are given an ABox consisting of the following axioms:

- $takesCourse(olly, databases)$;
- $takesCourse(olly, data\_structures)$;
- $takesCourse(olly, data\_modeling)$;
- $takesCourse(olly, kr)$;
- $\leq 2\ takesCourse(olly)$.

Is this ABox satisfiable? If the answer is yes, then construct an interpretation that satisfies it. Under which assumption it is not satisfiable?

Solution: Yes, the ABox is satisfiable by the following interpretation:

- $\Delta^{\mathcal{I}} = \{db101, kr102\}$

- $databases^{\mathcal{I}} = db101$
- $data\_structures^{\mathcal{I}} = db101$
- $kr^{\mathcal{I}} = kr102$
- $data\_modeling^{\mathcal{I}} = kr102$

This ABox will not be satisfiable under the Unique Name Assumption.

Problem 8. Consider the following ontology $\mathcal{O} = \langle \mathcal{T}, \mathcal{A} \rangle$, where the TBox $\mathcal{T}$ and the ABox $\mathcal{A}$ are given as:

- $\mathcal{T} = \{E \sqsubseteq F, B \sqsubseteq \neg(\neg C \sqcap \neg E), A \sqsubseteq \neg\forall s.\neg B\}$
- $\mathcal{A} = \{\neg\exists s.F(j_1), A(j_1)\}$

Is the given knowledge base satisfiable? Formally prove your answer

- using semantics and
- using the $\mathcal{ALC}$ tableu algorithm.

Solution:
Semantics. Using the equivalence axioms the given TBox can be rewritten to

$$\mathcal{T} = \{E \sqsubseteq F, B \sqsubseteq C \sqcup E, A \sqsubseteq \exists s.B\},$$

its translation to the Negation Normal Form is as follows:

$$NNF(\mathcal{T}) = \{\neg E \sqcup F, \neg B \sqcup C \sqcup E, \neg A \sqcup \exists s.B\}.$$

Moreover, the ABox can be simplified to

$$\mathcal{A} = \{\forall s.\neg F(j_1), A(j_1)\}.$$

We have that for any $\mathcal{I}$, such that $\mathcal{I} \models \mathcal{T}$, it holds that $E^{\mathcal{I}} \subseteq F^{\mathcal{I}}$, $B^{\mathcal{I}} \subseteq C^{\mathcal{I}} \cup E^{\mathcal{I}}$. Since we have that $A(j_1) \in \mathcal{A}$ by the last axiom we must have some b, such that $(a, b) \in s^{\mathcal{I}}$ and $b \in B^{\mathcal{I}}$, and, thus also $b \in (C \sqcup E)^{\mathcal{I}}$. Two possibilities exist: either $b \in C^{\mathcal{I}}$ or $b \in E^{\mathcal{I}}$. In the latter case we have that $b \in F^{\mathcal{I}}$. However, this leads to a contradiction, since $j_1 \in \forall s.\neg F$. Hence, $b \in (\neg F)^{\mathcal{I}}$ must hold.
We have constructed the following interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$ that satisfies $\mathcal{O} = \langle \mathcal{T}, \mathcal{A} \rangle$:

- $\Delta^{\mathcal{I}} = \{j_1, b\}$
- $E^{\mathcal{I}} = \emptyset$
- $F^{\mathcal{I}} = \emptyset$
- $B^{\mathcal{I}} = \{b\}$
- $C^{\mathcal{I}} = \{b\}$
- $A^{\mathcal{I}} = \{j_1\}$

- $s^{\mathcal{I}} = \{(j_1, b)\}$

Therefore, the given ontology is satisfiable.

Tableau.

$$\begin{tikzcd} j_1 \; \mathcal{L}(j_1) = \{A, \forall s.\neg F, \exists s.B\} \arrow[d, "s"] \\ b \; \mathcal{L}(b) = \{B, \neg F, \neg E, C\} \end{tikzcd}$$

Reasoning

Problem 9. Suppose that r is a role name and i, j are individual names. Prove formally that the following expressions are not valid by constructing appropriate countermodels:

(1) $\exists r.\{i\} \sqcap \exists r.\{j\} \equiv \geq 2r.\{i, j\}$;

(2) $\exists r.\{i\} \sqcap \exists r.\{j\} \equiv \exists r.\{i, j\}$.

Solution:

(1) Let $\mathcal{I} = (\Delta, \circ^{\mathcal{I}})$ be an interpretation, where $\Delta^{\mathcal{I}} = \{a, b\}$, $i^{\mathcal{I}} = b$, $j^{\mathcal{I}} = b$, and $r^{\mathcal{I}} = \{(a, b)\}$. Then $(\exists r.\{i\} \sqcap \exists r.\{j\})^{\mathcal{I}} = \{a\}$, but $(\geq 2r.\{i, j\})^{\mathcal{I}} = \emptyset$.

(2) Let $\mathcal{I} = (\Delta, \circ^{\mathcal{I}})$ be an interpretation, where $\Delta^{\mathcal{I}} = \{a, b\}$, $i^{\mathcal{I}} = a$, $j^{\mathcal{I}} = b$, and $r^{\mathcal{I}} = \{(a, b)\}$. Then $(\exists r.\{i, j\})^{\mathcal{I}} = \{a\}$, but $(\exists r.\{i\} \sqcap \exists r.\{j\})^{\mathcal{I}} = \emptyset$.

Problem 10. Consider the following two concept definitions:

(1) $\mathit{BinaryTree} \equiv \leq 2\ \mathit{hasBranch} \sqcup \forall \mathit{hasBranch}.\mathit{BinaryTree}$;

(2) $\mathit{List} \equiv \leq 1\ \mathit{hasBranch} \sqcup \forall \mathit{hasBranch}.\mathit{List}$

Show formally that for any interpretation $\mathcal{I}$, we have $\mathcal{I} \models \mathit{List} \sqsubseteq \mathit{BinaryTree}$.

Solution:
By the theorem on reducing reasoning problems to KB satisfiability, we have that
$\mathcal{T} \models \mathit{List} \sqsubseteq \mathit{BinaryTree}$ iff $\mathcal{T} \cup \mathit{List}(a) \cup \neg\mathit{BinaryTree}(a)$ is unsatisfiable for a fresh a.
Towards a contradiction, assume that $\mathcal{T} \cup \mathit{List}(a) \cup \neg\mathit{BinaryTree}(a)$ is satisfiable. Let us assume that $\mathcal{I} = (\Delta^{\mathcal{I}}, \circ^{\mathcal{I}})$ is an interpretation that satisfies it. We have that $a \in \mathit{List}^{\mathcal{I}}$ and $a \in \neg\mathit{BinaryTree}^{\mathcal{I}}$. Since $a \in \mathit{List}^{\mathcal{I}}$, it must hold that either

(i) $a \notin (\exists \mathit{hasBranch})^{\mathcal{I}}$ or

(ii) $hasBranch^{\mathcal{I}} = \{(a, b)\}$ for some fresh b or

(iii) $(a, b_1) \in hasBranch^{\mathcal{I}} \ldots (a, b_n) \in hasBranch^{\mathcal{I}}$ for some $n > 1$, and for all b_i it holds that $b_i \in List^{\mathcal{I}}$.

If either (i) or (ii) hold then obviously $a \in (\leq 2 hasBranch)^{\mathcal{I}}$, and hence by (1) $a \in BinaryTree^{\mathcal{I}}$, which contradicts our assumption that $a \in \neg BinaryTree^{\mathcal{I}}$.

Assume now that (iii) holds. We have that $b_1, \ldots, b_n \in List^{\mathcal{I}}$, hence for every b_i again one of the following cases must apply:

(i) $b_i \notin \exists hasBranch^{\mathcal{I}}$ or

(ii) $hasBranch^{\mathcal{I}} = \{(b_i, c)\}$ for some fresh c or

(iii) $hasBranch^{\mathcal{I}} = \{(b_i, c_1), \ldots, (b, c_m)\}$ for some $m > 1$, and for all c_i it holds that $c_i \in List^{\mathcal{I}}$.

If either (i) or (ii) holds then, $c \in BinaryTree^{\mathcal{I}}$, and hence due to (1) $b_i, a \in BinaryTree^{\mathcal{I}}$. Otherwise (iii) must hold, i.e., more than 1 outgoing *hasBranch* link exists from b_i. Applying the above argument further, we obtain that the interpretation that we are trying to construct at some point will contradict our assumption. Thus, we have that $\mathcal{T} \models List \sqsubseteq BinaryTree$ holds.[1]

[1] Another possibility to prove the statement is to use the Tableau algorithm with blocking, which for the considered TBox, however, requires additional expansion rules, i.e., $(\leq k\, r.C)$-rule and $(\geq k\, r.C)$-rule. As these have not been discussed in the lecture, only proof using semantics is presented here.