Knowledge Representation for the Semantic Web
Lecture 3: Description Logics II

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slides based on Reasoning Web 2011 tutorial "*Foundations of Description Logics and OWL*" by S. Rudolph

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WS 2017/18

Unit Outline

Semantics of Description Logics

DL Nomenclature

Equivalences

Semantics of Description Logics

"Now! *That* should clear up a few things around here!"

Interpretations

Semantics for DLs is defined in a **model theoretic** way, i.e., based on "abstract possible worlds", called interpretations.

Def.: An **interpretation** $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ **consists of:**

- a nonempty set $\Delta^{\mathcal{I}}$, called the interpretation domain (of $\mathcal{I}$)
- an interpretation function $\cdot^{\mathcal{I}}$, which maps
  - each atomic concept A to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$
  - each atomic role r to a subset $r^{\mathcal{I}}$ of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

individual names N_I	class names N_C	role names N_R
$\ldots\ a\ \ldots$	$\ldots\ C\ \ldots$	$\ldots\ r\ \ldots$

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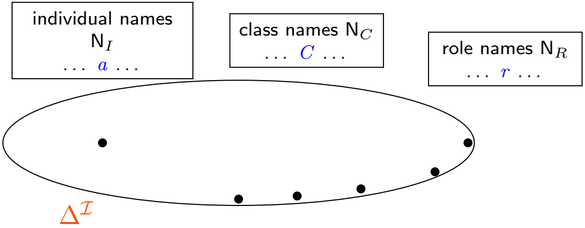

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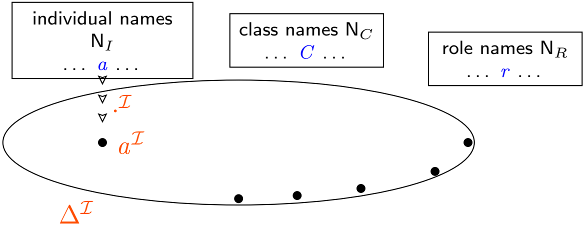

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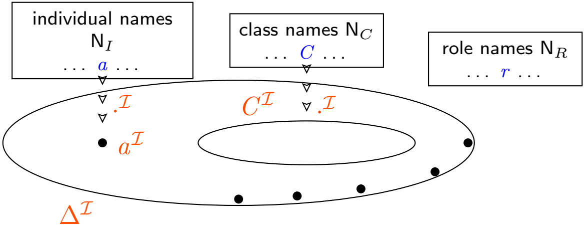

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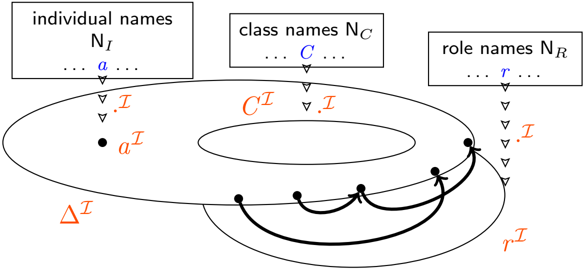

Interpretations: an Example

$\mathsf{N}_I = \{\texttt{sun}, \texttt{morning\_star}, \texttt{evening\_star}, \texttt{moon}, \texttt{home}\}$.
$\mathsf{N}_C = \{\texttt{Planet}, \texttt{Star}\}$.
$\mathsf{N}_R = \{\texttt{orbitsAround}, \texttt{shinesOn}\}$.

$$\Delta^{\mathcal{I}} = \{\text{☉}, \text{☿}, \text{♀}, \text{♁}, \text{☾}, \text{♂}, \text{♃}, \text{♄}, \text{⛢}, \text{♆}, \text{♇}\}$$

$$\begin{aligned}
\texttt{sun}^{\mathcal{I}} &= \text{☉} \\
\texttt{morning_star}^{\mathcal{I}} &= \text{♀} \\
\texttt{evening_star}^{\mathcal{I}} &= \text{♀} \\
\texttt{moon}^{\mathcal{I}} &= \text{☾} \\
\texttt{home}^{\mathcal{I}} &= \text{♁}
\end{aligned}$$

$$\begin{aligned}
\texttt{Planet}^{\mathcal{I}} &= \{\text{☿}, \text{♀}, \text{♁}, \text{♂}, \text{♃}, \text{♄}, \text{⛢}, \text{♆}\} \\
\texttt{Star}^{\mathcal{I}} &= \{\text{☉}\} \\
\texttt{orbitsAround}^{\mathcal{I}} &= \{\langle \text{☿}, \text{☉} \rangle, \langle \text{♀}, \text{☉} \rangle, \langle \text{♁}, \text{☉} \rangle, \langle \text{♂}, \text{☉} \rangle, \langle \text{♃}, \text{☉} \rangle, \\
&\quad \langle \text{♄}, \text{☉} \rangle, \langle \text{⛢}, \text{☉} \rangle, \langle \text{♆}, \text{☉} \rangle, \langle \text{♇}, \text{☉} \rangle, \langle \text{☾}, \text{♁} \rangle\} \\
\texttt{shinesOn}^{\mathcal{I}} &= \{\langle \text{☉}, \text{☿} \rangle, \langle \text{☉}, \text{♀} \rangle, \langle \text{☉}, \text{♁} \rangle, \langle \text{☉}, \text{☾} \rangle, \langle \text{☉}, \text{♂} \rangle, \\
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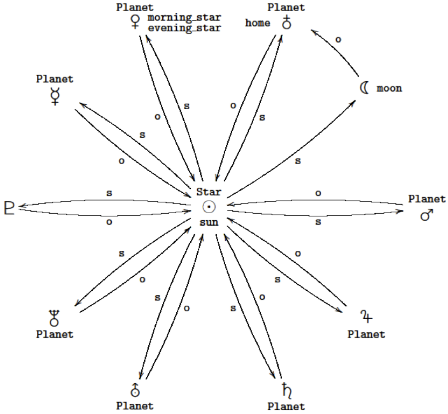
\end{aligned}$$



Interpretation of Individuals

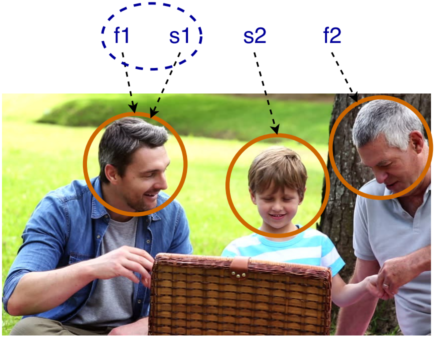
Unique Name Assumption (UNA)

If c_1 and c_2 are two individuals such that $c_1 \neq c_2$, then $c_1^{\mathcal{I}} \neq c_2^{\mathcal{I}}$

Note: When the UNA holds, equality and distincntness assertions are meaningless. In DLs one can drop UNA.

Example: absence of UNA

Two fathers $(f1, f2)$ and two sons $(s1, s2)$ went to a pizzeria and bought three pizzas for picnic lunch. When they started their lunch, everyone had a whole pizza. How could this happen?



Interpretation of Individuals

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If c_1 and c_2 are two individuals such that $c_1 \neq c_2$, then $c_1^{\mathcal{I}} \neq c_2^{\mathcal{I}}$

Note: When the UNA holds, equality and distincntness assertions are meaningless. In DLs one can drop UNA.

Standard Name Assumption (SNA)

The UNA holds, and moreover individuals are interpreted in the same way in all interpretations. Hence, we can assume that $\Delta^{\mathcal{I}}$ contains the set of individuals, and that for each interpretation $\mathcal{I}$, we have that $c^{\mathcal{I}} = c$ (then c is called standard name)

Interpretation of Concept Expressions

Construct	Syntax	Example	Semantics
atomic concept	A	$Doctor$	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
atomic role	r	$hasChild$	$r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
atomic negation	$\neg A$	$\neg Doctor$	$\Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}$
conjunction	$C \sqcap D$	$Human \sqcap Male$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
unqual. exist. res.[1]	$\exists r$	$\exists hasChild$	$\{o \mid \exists o'.(o, o') \in r^{\mathcal{I}}\}$
value res.	$\forall r.C$	$\forall hasChild.Male$	$\{o \mid \forall o'.(o, o') \in r^{\mathcal{I}} \rightarrow o' \in C^{\mathcal{I}}\}$
bottom	$\bot$		$\emptyset$

C, D denote arbitrary concepts and r denotes an arbitrary role.

The above constructs form the basic language $\mathcal{AL}$

[1] Unqualified existential restriction

Interpretation of Concept Expressions, cont'd

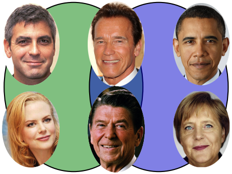
Construct	$\mathcal{AL}$	Syntax	Semantics	
disjunction	$\mathcal{U}$	$C \sqcup D$	$Singer \sqcup Dancer$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
qual. exist. res.[2]	$\mathcal{E}$	$\exists R.C$	$\exists hasChild.Male$	$\{o \mid \exists o'.(o,o') \in r^{\mathcal{I}} \wedge o' \in C^{\mathcal{I}}\}$
(full) negation	$\mathcal{C}$	$\neg C$	$\neg(\exists hasSibling.Female)$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
num. res.	$\mathcal{N}$	$(\geq k\ r)$	$\geq 2\ hasSister$	$\{o \mid \#\{o' \mid (o,o') \in r^{\mathcal{I}}\} \geq k\}$
		$(\leq k\ r)$	$\leq 3\ hasBrother$	$\{o \mid \#\{o' \mid (o,o') \in r^{\mathcal{I}}\} \leq k\}$
qual. num. res.	$\mathcal{Q}$	$(\geq k\ r.C)$	$\geq 2\ hasSibling.Female$	$\{o \mid \#\{o' \mid (o,o') \in r^{\mathcal{I}} \wedge o' \in C^{\mathcal{I}}\} \geq k$
		$(\geq k\ r.C)$	$\leq 3\ hasSibling.Male$	$\{o \mid \#\{o' \mid (o,o') \in r^{\mathcal{I}} \wedge o' \in C^{\mathcal{I}}\} \leq k$
top		$\top$		$\Delta^{\mathcal{I}}$

Many different DL constructs and their combinations have been investigated.

Combining various constructs we obtain a concrete DL fragment, i.e., language (see slide 26 for further details).

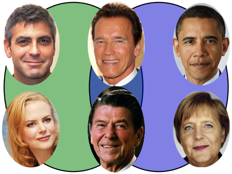
[2] Qualified existential restriction

Boolean Concept Expressions



Boolean Concept Expressions

$\neg$ *Politician*



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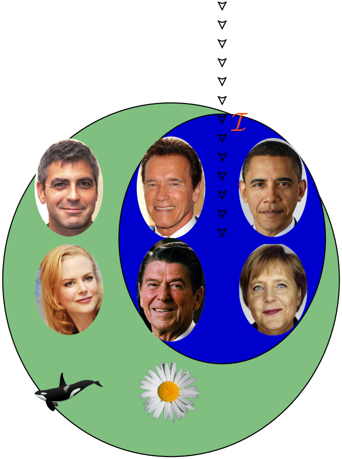

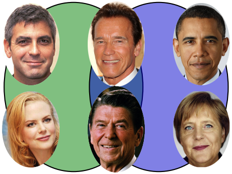

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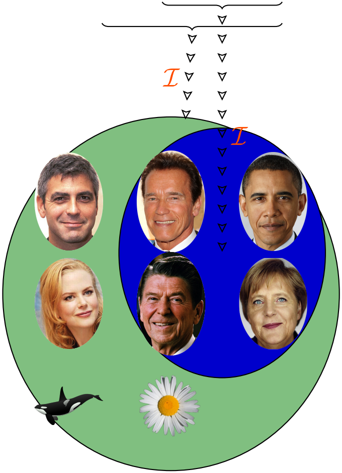

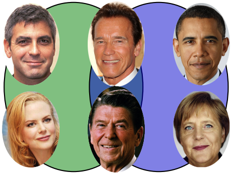

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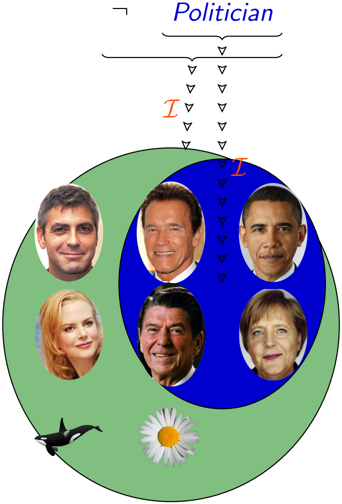

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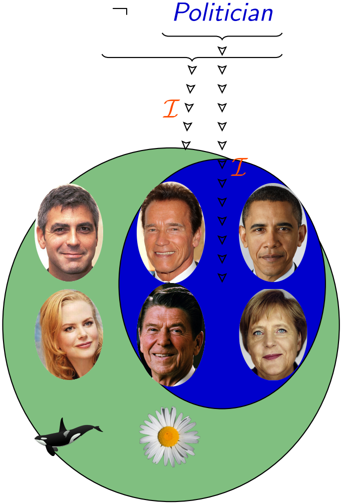

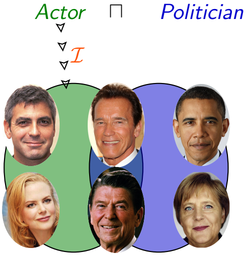

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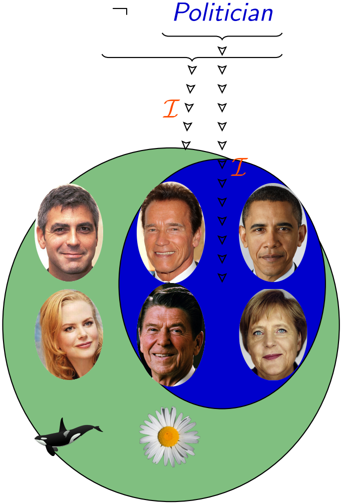

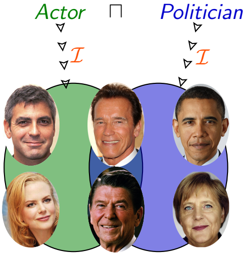

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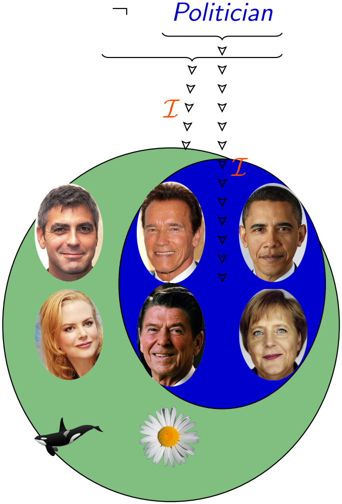

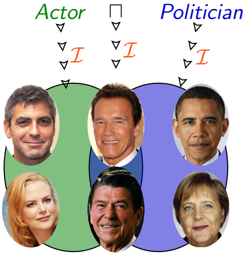

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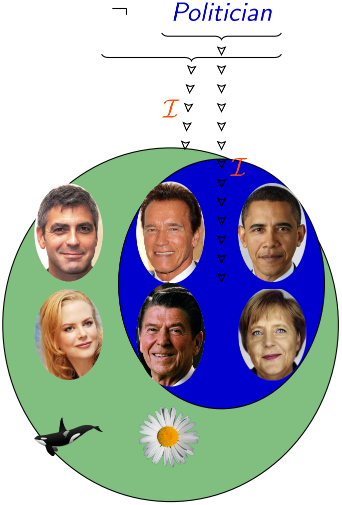

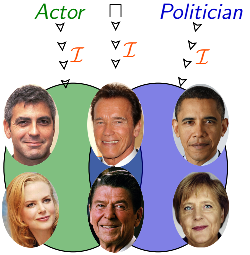

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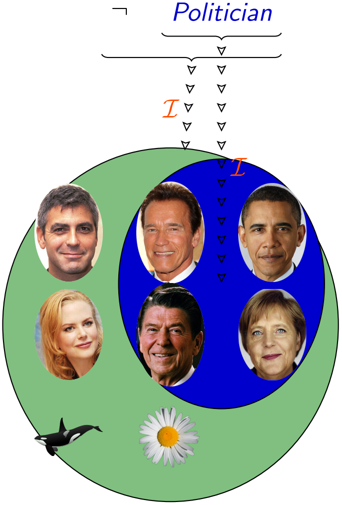

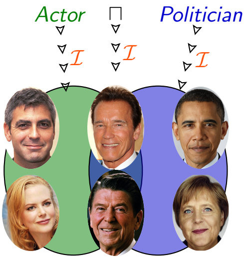

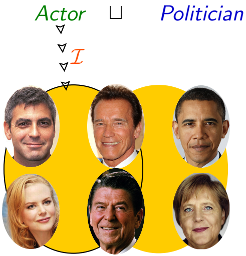

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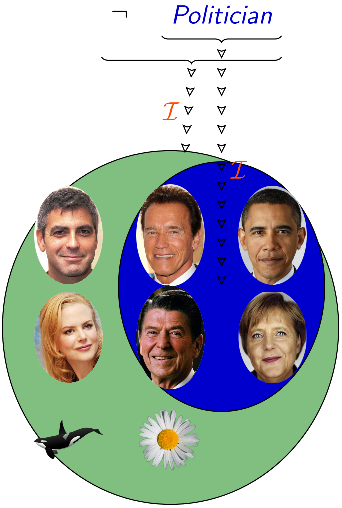

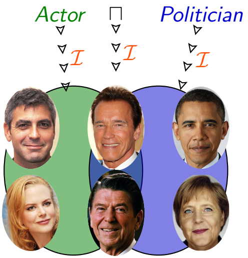

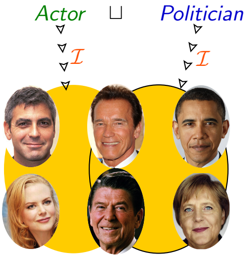

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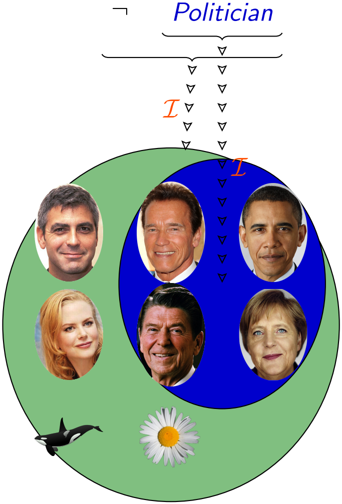
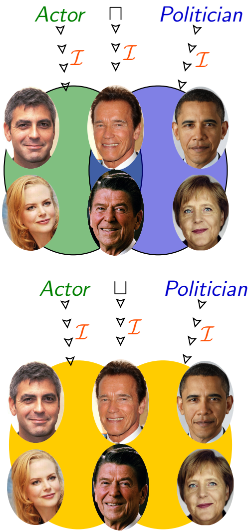
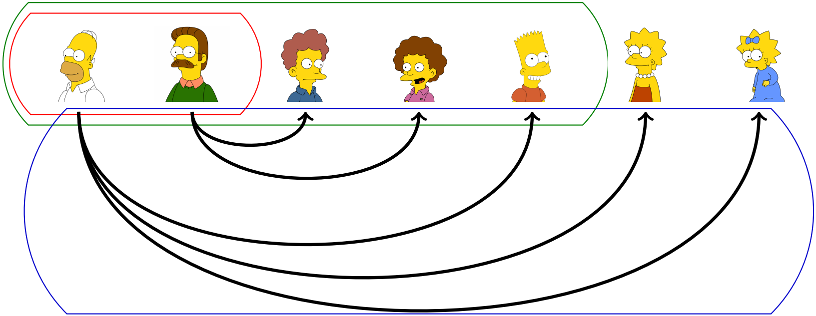




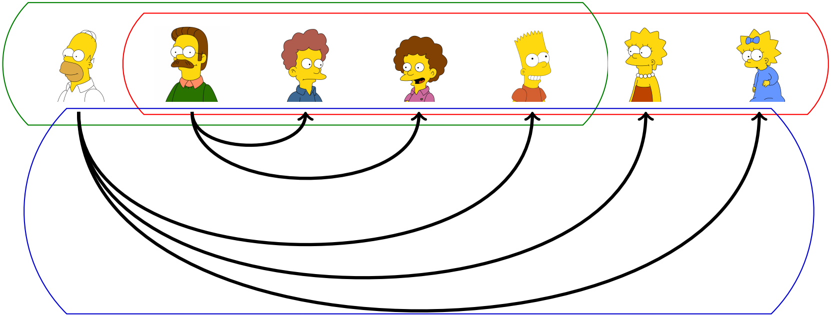
Existential Role Restrictions

$$\underbrace{\exists\ \underbrace{parentOf}\ .\ \underbrace{Male}}$$



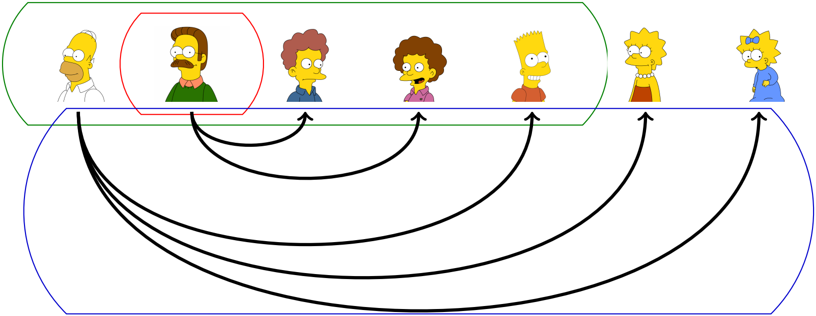
Universal Role Restrictions

$$\underbrace{\forall\ \underbrace{parentOf}.\ \underbrace{Male}}$$



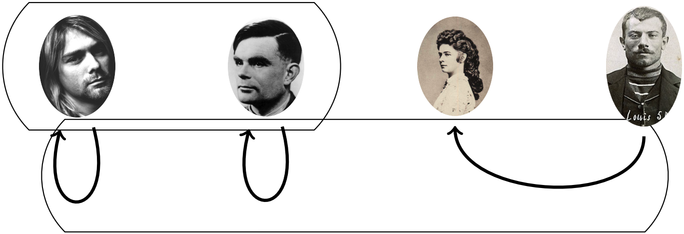
Qualified Number Restrictions

$$\underbrace{\geq 2\ \underbrace{parentOf.}_{}\ \underbrace{Male}_{}}_{}$$



Self-Restrictions

$$\underbrace{\exists \quad \underbrace{killed.}_{} \quad \underbrace{\mathsf{Self}}_{}}_{}$$

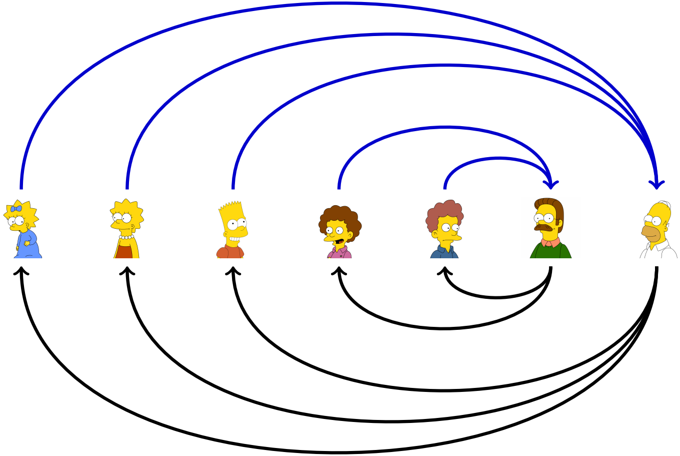


Interpretation of Role Expressions

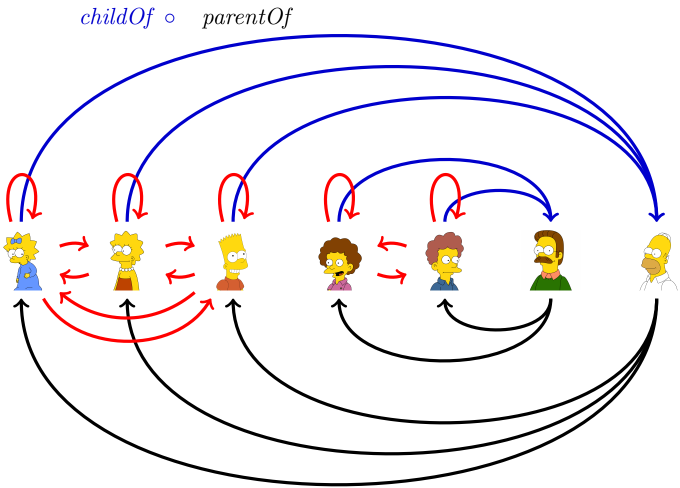
Construct	Syntax	Example	Semantics
atomic role	r	$hasChild$	$r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
role negation	$\neg r$	$\neg hasSister$	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \backslash \{(o, o') \in r^{\mathcal{I}}\}$
inverse role	r^{-}	$hasParent^{-}$	$\{(o, o') \mid (o', o) \in r^{\mathcal{I}}\}$
transitivity	$r \circ r'$	$hasChild \circ hasParent$	$\{(o, o') \mid (o, o'') \in r^{\mathcal{I}}, (o'', o') \in r'^{\mathcal{I}}\}$

Inverse Role

$childOf^{-} \quad = \quad parentOf$



Role Chain

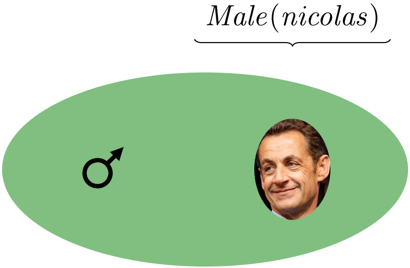


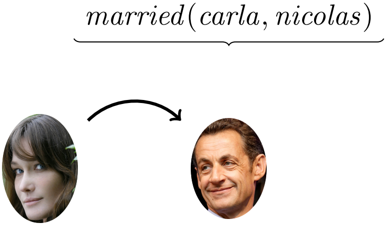
Semantics of Axioms

Given a way to determine a semantic counterpart for all expressions, we now define the criteria for checking whether an interpretation $\mathcal{I}$ satisfies an axiom alpha α (written: $\mathcal{I} \models \alpha$).

- $\mathcal{I} \models r_1 \circ \ldots \circ r_n \sqsubseteq r$ if $(r_1 \circ \ldots \circ r_n)^\mathcal{I} \subseteq r^\mathcal{I}$
- $\mathcal{I} \models Dis(s_1, s_2)$ if $s_1^\mathcal{I} \cap s_2^\mathcal{I} = \{\}$
- $\mathcal{I} \models C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$
- $\mathcal{I} \models C(a)$ if $a^\mathcal{I} \in D^\mathcal{I}$
- $\mathcal{I} \models r(a, b)$ if $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$
- $\mathcal{I} \models \neg r(a, b)$ if $(a^\mathcal{I}, b^\mathcal{I}) \notin r^\mathcal{I}$
- $\mathcal{I} \models a \approx b$ if $a^\mathcal{I} = b^\mathcal{I}$
- $\mathcal{I} \models a \not\approx b$ if $a^\mathcal{I} \neq b^\mathcal{I}$

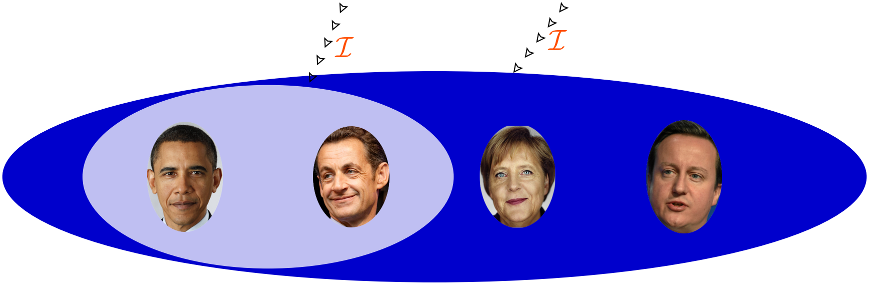
Concept and Role Membership





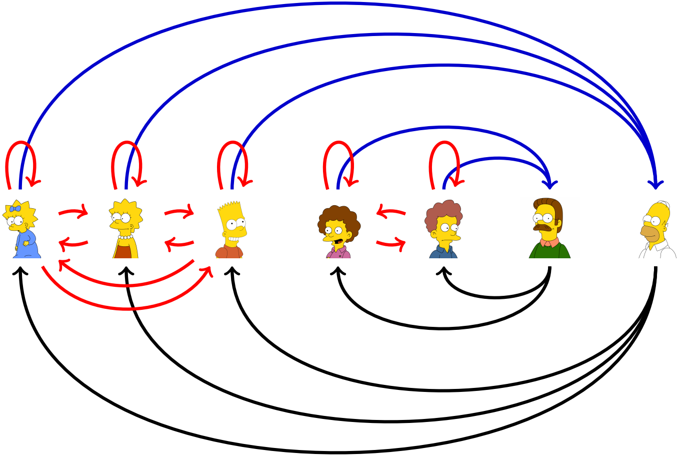
General Inclusion Axioms

$$ExPresident \sqsubseteq Politician$$



Role Inclusion Axioms

$$\textit{childOf} \circ \textit{parentOf} \sqsubseteq \textit{siblingOf}$$



DL to First Order Logic

Syntax	FOL formalization
$A_1 \sqsubseteq A_2$	$\forall x(A_1(x) \rightarrow A_2(x))$
$R_1 \sqsubseteq R_2$	$\forall x, y(R_1(x,y) \rightarrow R_2(x,y))$
$A_1 \sqsubseteq \neg A_2$	$\forall x(A_1(x) \rightarrow \neg A_2(X))$
$R_1 \sqsubseteq \neg R_2$	$\forall x, y(R_1(x,y) \rightarrow \neg R_2(x,y))$
$\exists R \sqsubseteq A$	$\forall x(\exists y(R(x,y)) \rightarrow A(x))$
$\exists R^- \sqsubseteq A$	$\forall x(\exists y(R(y,x)) \rightarrow A(x))$
$A \sqsubseteq \exists R$	$\forall x(A(x) \rightarrow \exists y(R(x,y)))$
$funct(R)$	$\forall x, y, y'(R(x,y) \wedge R(x,y') \rightarrow y = y')$
$A_1 \sqcap A_2 \sqsubseteq A_3$	$\forall x A_1(x) \wedge A_2(x) \rightarrow A_3(x)$
$\exists R.A_1 \sqsubseteq A_2$	$\forall x(\exists y(R(x,y) \wedge A_1(y)) \rightarrow A_2(x)$
$A_1 \sqsubseteq \exists R.A_2$	$\forall x(A(x) \rightarrow \exists y(R(x,y) \wedge A_2(y)))$
$\ldots$	$\ldots$

Semantics via Translation into FOL

As (common) DLs can be seen as fragments of FOL, one can also define the semantics by providing a translation of DL axioms into FOL formulas.

- $\tau_{\mathbf{R}}(r, x, y)$: produce for $r(x, y)$ a formula with free variables x, y
- $\tau_{\mathbf{C}}(C, x)$: produce for $C(x)$ a formula with free variable x
- define transformations recursively

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Bottom rewriting:

$$\tau_{\mathbf{R}}(u, x_i, x_j) = \mathbf{true}$$
$$\tau_{\mathbf{R}}(r, x_i, x_j) = r(x_i, x_j)$$
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$$\tau_{\mathbf{C}}(A, x_i) = A(x_i)$$

$$\tau_{\mathbf{C}}(\top, x_i) = \mathbf{true}$$

$$\tau_{\mathbf{C}}(\bot, x_i) = \mathbf{false}$$

$$\tau_{\mathbf{C}}(\{a_1, \ldots, a_n\}, x_i) = \bigvee_{j=1}^{n} x_i = a_j$$

Semantics via Translation into FOL (ctd.)

Complex concepts:

$$
\begin{aligned}
\tau_{\mathbf{C}}(C \sqcap D, x_i) &= \tau_{\mathbf{C}}(C, x_i) \land \tau_{\mathbf{C}}(D, x_i)\\
\tau_{\mathbf{C}}(C \sqcup D, x_i) &= \tau_{\mathbf{C}}(C, x_i) \lor \tau_{\mathbf{C}}(D, x_i)\\
\tau_{\mathbf{C}}(\neg C, x_i) &= \neg\tau_{\mathbf{C}}(C, x_i)\\
\tau_{\mathbf{C}}(\exists r.C, x_i) &= \exists x_{i+1}.(\tau_{\mathbf{R}}(r, x_i, x_{i+1}) \land \tau_{\mathbf{C}}(C, x_{i+1}))\\
\tau_{\mathbf{C}}(\forall r.C, x_i) &= \forall x_{i+1}.(\tau_{\mathbf{R}}(r, x_i, x_{i+1}) \to \tau_{\mathbf{C}}(C, x_{i+1}))\\
\tau_{\mathbf{C}}(\exists r.\mathsf{Self}, x_i) &= \tau_{\mathbf{R}}(r, x_i, x_i)\\
\tau_{\mathbf{C}}(\geq nr.C, x_i) &= \exists x_{i+1} \ldots x_{i+n}.(\bigwedge_{j=i+1}^{i+n} \bigwedge_{k=j+1}^{i+n} (x_j \neq x_k)\\
&\qquad \land \bigwedge_{j=i+1}^{i+n} \bigwedge_{k=j+1}^{i+n} (\tau_{\mathbf{R}}(r, x_i, x_j) \land \tau_{\mathbf{C}}(C, x_j))\\
\tau_{\mathbf{C}}(\leq nr.C, x_i) &= \neg\tau_{\mathbf{C}}(\geq (n+1)r.C, x_i)
\end{aligned}
$$

Semantics via Translation into FOL (ctd.)

Axioms:

$$
\begin{aligned}
\tau(C \sqsubseteq D) &= \forall x_0(\tau_{\mathbf{C}}(C, x_0) \rightarrow \tau_{\mathbf{C}}(D, x_0)) \\
\tau(r_1 \circ \ldots \circ r_n \sqsubseteq r) &= \forall x_0 \ldots x_n(\textstyle\bigwedge_{i=1}^{n} \tau_{\mathbf{R}}(r_i, x_{i-1}, x_i)) \rightarrow \tau_{\mathbf{R}}(r, x_0, x_n) \\
\tau(Dis(r, r')) &= \forall x_0, x_1(\tau_{\mathbf{R}}(r, x_0, x_1) \rightarrow \neg\tau_{\mathbf{R}}(r', x_0, x_1) \\
\tau(Ref(r, r')) &= \forall x\, \tau_{\mathbf{R}}(r, x, x) \\
\tau(Asym(r)) &= \forall x_0, x_1.(\tau_{\mathbf{R}}(r, x_0, x_1) \rightarrow \neg\tau_{\mathbf{R}}(r, x_1, x_0))
\end{aligned}
$$

Semantics via Translation into FOL (ctd.)

Axioms:

$$
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\tau(C \sqsubseteq D) &= \forall x_0(\tau_{\mathbf{C}}(C, x_0) \to \tau_{\mathbf{C}}(D, x_0)) \\
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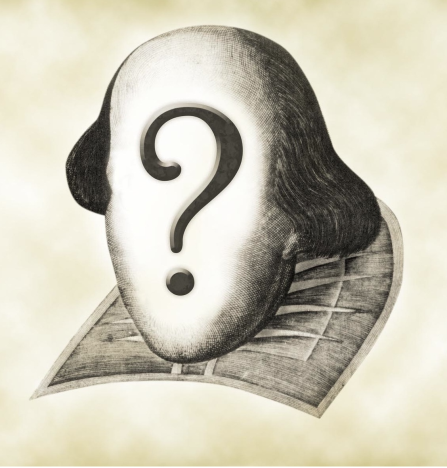
\tau(Ref(r, r')) &= \forall x\, \tau_{\mathbf{R}}(r, x, x) \\
\tau(Asym(r)) &= \forall x_0, x_1.(\tau_{\mathbf{R}}(r, x_0, x_1) \to \neg\tau_{\mathbf{R}}(r, x_1, x_0))
\end{aligned}
$$

Assertions:

$$
\begin{aligned}
\tau(C(a)) &= \tau_{\mathbf{C}}(C, x_0)[x_0/a] \\
\tau(r(a, b)) &= \tau_{\mathbf{R}}(r, x_0, x_1)[x_0/a][x_1/b] \\
\tau(\neg r(a, b)) &= \neg\tau(r(a, b)) \\
\tau(a \approx b) &= a = b \\
\tau(a \not\approx b) &= \neg(a = b)
\end{aligned}
$$

Description Logics Nomenclature



Naming Scheme for Expressive DLs

$$((\mathcal{ALC} \mid \mathcal{S})[\mathcal{H}] \mid \mathcal{SR})[\mathcal{O}][\mathcal{I}][\mathcal{F} \mid \mathcal{N} \mid \mathcal{Q}]$$

- $\mathcal{S}$ stands for $\mathcal{ALC}$ + role transitivity
- $\mathcal{H}$ stands for role hierarchies
- $\mathcal{O}$ stands for nominals, i.e., closed classes $\{o\}$ such as $\{john, mary, tom\}$
- $\mathcal{I}$ stands for inverse roles (seen soon)
- $\mathcal{F}$ stands for role functionality ($\top \sqsubseteq \leq 1.r$)
- $\mathcal{N}$ ($\mathcal{Q}$) stands for arbitray (qualified) cardinality restrictions
- $\mathcal{R}$ stands for role box with all kinds of role axioms plus self concepts

Note:

- $\mathcal{S}$ subsumes $\mathcal{ALC}$, $\mathcal{SR}$ subsumes $(\mathcal{ALC} \mid \mathcal{S})[\mathcal{H}]$ $\mathcal{ALCH}$
- $\mathcal{SROIQ}$ subsumes all the other description logics in this scheme.
- $\mathcal{N}$ makes $\mathcal{F}$ obsolete
- $\mathcal{Q}$ makes $\mathcal{N}$ (and $\mathcal{F}$) obsolete

DL Syntax - Overview

Concepts		
$\mathcal{ALC}$	Atomic	A, B
	Not	$\neg C$
	And	$C \sqcap D$
	Or	$C \sqcup D$
	Exists	$\exists r.C$
	For all	$\forall r.C$
$\mathcal{Q}$ ($\mathcal{N}$)	At least	$\geq n\ r.C$ $(\geq n\ r)$
	At most	$\leq n\ r.C$ $(\leq n\ r)$
$\mathcal{O}$	Closed class	$\{i_1, \ldots, i_n\}$
$\mathcal{R}$	Self	$\exists r.\mathsf{Self}$

Roles		
$\mathcal{I}$	Atomic	r
	Inverse	r^-

DL Syntax - Overview ctd.

Ontology (=Knowledge Base)

Concept Axioms *TBox*	
Subclass	$C \sqsubseteq D$
Equivalent	$C \equiv D$

Assertional Axioms *ABox*	
Instance	$C(a)$
Role	$r(a, b)$
Same	$a \approx b$
Different	$a \not\approx b$

Role Axioms *RBox*		
$\mathcal{H}$	Subrole	$r \sqsubseteq s$
$\mathcal{S}$	Transitivity	$Trans(r)$
$\mathcal{SR}$	Role Chain	$r \circ r' \sqsubseteq s$
	Role Disjointness	$Disj(s, r)$

- Transitivity and Disjointness are *role characteristics*.
- Further characteristics in $\mathcal{SROIQ}$ are asymmetry, $Asym(r)$, and reflexivity, $Ref(r)$.

Concept Equivalences

$C \equiv D$

Two concept expressions C and D are called equivalent (written: $C \equiv D$), if for every interpretation $\mathcal{I}$ holds $C^{\mathcal{I}} = D^{\mathcal{I}}$.

- Commutativity, Associativity, Idempotence:

$$\begin{aligned} C \sqcap D &\equiv D \sqcap C \\ (C \sqcap D) \sqcap E &\equiv C \sqcap (D \sqcap E) \\ C \sqcap C &\equiv C \end{aligned} \qquad \begin{aligned} C \sqcup D &\equiv D \sqcup C \\ (C \sqcup D) \sqcup E &\equiv C \sqcup (D \sqcup E) \\ C \sqcup C &\equiv C \end{aligned}$$

- Double Negation:

$$\neg\neg C \equiv C$$

- Complement, de Morgan laws:

$$\begin{aligned} \neg\top &\equiv \bot \\ C \sqcap \neg C &\equiv \bot \\ \neg(C \sqcap D) &\equiv \neg D \sqcup \neg C \end{aligned} \qquad \begin{aligned} \neg\bot &\equiv \top \\ C \sqcup \neg C &\equiv \top \\ \neg(C \sqcup D) &\equiv \neg D \sqcap \neg C \end{aligned}$$

Concept Equivalences ctd.

- Distributivity, Absorption:

$$\begin{aligned}(C \sqcup D) \sqcap E &\equiv (C \sqcap E) \sqcup (D \sqcap E)\\ (C \sqcap D) \sqcup E &\equiv (C \sqcup E) \sqcap (D \sqcup E)\\ C \sqcup (C \sqcap D) &\equiv C\end{aligned}$$

$$\begin{aligned}(C \sqcup D) \sqcap C &\equiv C\\ (C \sqcap D) \sqcup C &\equiv C\\ C \sqcup (C \sqcap D) &\equiv C\end{aligned}$$

- Quantifiers and Counting:

$$\begin{aligned}\neg\exists r.C &\equiv \forall r.\neg C\\ \neg\forall r.C &\equiv \exists r.\neg C\\ \neg \leq nr.C &\equiv \geq (n+1)r.C\\ \neg \geq (n+1)r.C &\equiv \leq nr.C\end{aligned}$$

$$\begin{aligned}\geq 0r.C &\equiv \top\\ \geq 1r.C &\equiv \exists r.C\\ \leq 0r.C &\equiv \forall r.\neg C\end{aligned}$$

Axiom and KB Equivalences

- Lloyd-Topor equivalences

$$\begin{aligned}\{A \sqcup B \sqsubseteq C\} &\iff \{A \sqsubseteq C, B \sqsubseteq C\}\\ \{A \sqsubseteq B \sqcap C\} &\iff \{A \sqsubseteq B, A \sqsubseteq C\}\end{aligned}$$

- Turning GCIs into universally valid concept descriptions

$$C \sqsubseteq D \iff \top \sqsubseteq \neg C \sqcup D$$

- Internalisation of ABox into TBox

$$\begin{aligned}C(a) &\iff \{a\} \sqsubseteq C\\ r(a,b) &\iff \{a\} \sqsubseteq \exists r.\{b\}\\ \neg r(a,b) &\iff \{a\} \sqsubseteq \neg\exists r.\{b\}\\ a \approx b &\iff \{a\} \sqsubseteq \{b\}\\ a \not\approx b &\iff \{a\} \sqsubseteq \neg\{b\}\end{aligned}$$

(Non-)Concept Equivalences

Exercise: Show that the following equivalences are not valid.

$$\exists r.(C \sqcap D) \equiv \exists r.C \sqcap \exists r.D \tag{1}$$

$$C \sqcap (D \sqcup E) \equiv (C \sqcap D) \sqcup E \tag{2}$$

$$\exists r.\{a\} \sqcap \exists r.\{b\} \equiv \geq 2\, r.\{a, b\} \tag{3}$$

$$\exists r.\top \sqcap \exists s.\top \equiv \exists r.\exists r^{-}.\exists s.\top \tag{4}$$

Exercise: Show that the following equivalences are valid.

$$\exists r^{-}.C \sqsubseteq D \equiv C \sqsubseteq \forall r.D \tag{5}$$

$$C \sqsubseteq \forall r^{-}.D \equiv \exists r.C \sqsubseteq D \tag{6}$$

Concept Subsumption

$C \sqsubseteq D$

A concept expression C is subsumed by a concept expression D (written: $C \subseteq D$), if for every interpretation $\mathcal{I}$ holds $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

Some elementary properties:

- $C \sqsubseteq D \iff C \equiv C \sqcup D$
- $C \equiv D \iff C \sqsubseteq D$ and $D \sqsubseteq C$
- $C \sqsubseteq D$ and $D \sqsubseteq E$ implies $C \sqsubseteq E$ (transitivity)
- $C \sqsubseteq D \iff \neg D \sqsubseteq \neg C$
- $C \sqsubseteq D$ implies $C \sqcap E \sqsubseteq D$
- $C \equiv D$ implies $C \sqcap E \equiv D \sqcap E$

Summary

1. Semantics of DLs
    - Interpretation of
        - individuals
        - concept expressions
        - role expressions
    - Semantics of axioms
    - DLs to FOL
2. DL nomenclature
    - Naming schema
    - DL syntax overview
3. Concept equivalences

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