Multivariable Calculus

1) a) $f(x,y,z) = 4xy^3 - e^2 y^4 + y\ln(x) + x\ln(y)$

$$\frac{df}{dx}(x,y,z) = \frac{df}{dx}\left(4xy^3 - e^2 y^4 + y\ln(x) + x\ln(y)\right) =$$
$$= 4y^3 - 0 + y \cdot \frac{1}{x} + \ln(y) =$$
$$= 4y^3 + \frac{y}{x} + \ln(y)$$

$$\frac{df}{dy}(x,y,z) = \frac{df}{dx}\left(4xy^3 - e^2 y^4 + y\ln(x) + x\ln(y)\right) =$$
$$= 4x \cdot 3y^2 - e^2 4y^3 + \ln(x) + \frac{x}{y} =$$
$$= 12xy^2 - 4y^3 e^2 + \ln(x) + \frac{x}{y}$$

$$\frac{df}{dz}(x,y,z) = \frac{df}{dx}\left(4xy^3 - e^2 y^4 + y\ln(x) + x\ln(y)\right) =$$
$$= 0 - y^4 e^2 + 0 + 0 =$$
$$= -y^4 e^2$$

e) $f(x,y,z) = \sin(x + 2y^2) - e^{4x-y+2z}$

$$\frac{df}{dx}(x,y,z) = \cos(x + 2y^2) \cdot 1 - e^{4x-y+2z} \cdot 4 =$$
$$= \cos(x+2y^2) - 4e^{4x-y+2z} =$$

$$\frac{df}{dz}(x,y,z) = \sin(x+2y^2) - e^{4x-y+2z} =$$
$$= \cos(x+2y^2) \cdot 2 \cdot 2y^1 - e^{4x-y+2z} \cdot (-1) =$$
$$= 4y(\cos(x+2y^2)) + e^{4x-y+2z}$$

$$\frac{df}{dy}(x,y,z) = \sin(x+2y^2) - e^{4x-y+2z} =$$
$$= 0 - e^{4x-y+2z} \cdot 2 =$$
$$= -2e^{4x-y+2z}$$

c) $f(x,y,z) = y\sqrt{x^2+yz} = y(x^2+yz)^{\frac{1}{2}}$

$\frac{df}{dx}(x,y,z) = y(x^2+yz)^{\frac{1}{2}} = y\cdot(x^2+yz)^{\frac{1}{2}} =$

$= 0\cdot(x^2+yz)^{\frac{1}{2}} + y\cdot\frac{1}{2}(x^2+yz)^{-\frac{1}{2}}\cdot 2x =$

$= 0 + y\cdot\frac{1}{2\sqrt{x^2+yz}}\cdot 2x =$

$= \frac{xy}{\sqrt{x^2+yz}}$

$\frac{df}{dy}(x,y,z) = y(x^2+yz)^{\frac{1}{2}} = \frac{df}{dx}y\cdot(x^2+yz)^{\frac{1}{2}} + \frac{df}{dx}(x^2+yz)^{\frac{1}{2}} =$

$= 1\cdot(x^2+yz)^{\frac{1}{2}} + y\cdot\frac{1}{2}(x^2+yz)^{-\frac{1}{2}}\cdot z =$

$= (x^2+yz)^{\frac{1}{2}} + y\cdot\frac{1}{2\sqrt{x^2+yz}}\cdot z =$

$= (x^2+yz)^{\frac{1}{2}} + \frac{yz}{2\sqrt{x^2+yz}} =$

$= \frac{\sqrt{x^2+yz}\cdot 2\sqrt{x^2+yz}}{2\sqrt{x^2+yz}} + \frac{yz}{2\sqrt{x^2+yz}} =$

$= \frac{2(x^2+yz)+yz}{2\sqrt{x^2+yz}}$

$\frac{df}{dz}(x,y,z) = (y(x^2+yz)^{\frac{1}{2}}) \frac{d}{dx}y\cdot(x^2+yz)^{\frac{1}{2}} + \frac{d}{dx}y\cdot(x^2+yz)^{\frac{1}{2}} =$

$= 0\cdot(x^2+yz)^{\frac{1}{2}} + y\cdot\frac{1}{2}(x^2+yz)^{-\frac{1}{2}}\cdot y =$

$= 0 + y\cdot\frac{1}{2\sqrt{x^2+yz}}\cdot y =$

$= \frac{y^2}{2\sqrt{x^2+yz}}$

d) $f(x,y,z) = \frac{x^2}{y^2+1} - \frac{y^2}{x^3+z^2} = x^2(y^2+1)^{-1} - y^2(x^3+z^2)^{-1}$

$\frac{df}{dx}(x,y,z) = x^2(y^2+1)^{-1} - y^2(x^2+z^2)^{-1} =$

$= x^2(y^2+1)^{-1} + x^2(y^2+1)^{-1} =$

$= y^2(x^3+z^2)^{-1} + y^2(x^3+z^2)^{-1} =$

$= 2x(y^2+1)^{-1} + x^2\cdot(-1)(y^2+1)^{-2}\cdot 0 -$

$- 0\cdot(x^3+z^2)^{-1} \cdot y^2\cdot(-1)(x^3+z^2)^{-2}\cdot 3x^2 =$

$= \frac{2x}{y^2+1} - 0 - 0 - \left(-\frac{3x^2y^2}{(x^3+z^2)^2}\right) =$

$= \frac{2x}{y^2+1} + \frac{3x^2y^2}{(x^3+z^2)^2}$

$$\frac{df}{dy}(x,y,z) = x^2(y^2+1)^{-1} - y^2(x^3+z^2)^{-1} =$$

$$= x^2(y^2+1)^{-1} + x^2\cdot(y^2+1)^{-1} - y^2(x^3+z^2)^{-1} + y(x$$

$$= 0\cdot(y^2+1)^{-1} + x^2\cdot(-1)(y^2+1)^{-2}\cdot 2y - (2y(x^3+z^2)$$

$$+y^2\cdot 0) =$$

$$= 0 + \frac{(-x^2)\cdot 2y}{(y^2+1)^2} - \frac{2y}{x^3+z^2} - 0 =$$

$$= \frac{-2x^2y}{(y^2+1)^2} - \frac{2y}{x^3+z^2}$$

$$\frac{df}{dz}(x,y,z) = x^2(y^2+1)^{-1} - y^2(x^3+z^2)^{-1} =$$

$$= x^2(y^2+1)^{-1} + x^2(y^2+1)^{-1} - y^2(x^3+z^2)^{-1} + y^2(x^2+z^2$$

$$= 0\cdot(y^2+1)^{-1} + x^2\cdot 0 - 0\cdot(x^3+z^2)^{-1}\cdot y^2\cdot(-1)(x^3+z^2$$

$$= 0 + 0 - 0 - \left(\frac{-2y^2z}{(x^3+z^2)^2}\right) =$$

$$= \frac{2y^2z}{(x^3+z^2)^2}$$

2) a) $f(x,y,z) = \ln\left(\frac{y}{x}\right) + \ln\left(\frac{1}{x+y}\right) - \ln\left(\frac{x}{6}\right) =$

$$= \ln(y\cdot x^{-1}) + \ln(x+y)^{-1} + \ln\left(x\cdot\frac{1}{6}\right)$$

$$\frac{df}{dx}(x,y,z) = \ln(y\cdot x^{-1}) + \ln(x+y)^{-1} - \ln\left(\frac{x}{6}\right) =$$

$$= \frac{1}{y\cdot x^{-1}}\cdot y(-1)(x^{-2}) + \frac{1}{(x+y)^{-1}}\cdot(-1)(x+y)^{-2}(1)$$

$$- \frac{1}{\frac{x}{6}}\cdot\frac{1}{6} =$$

$$= \frac{x}{y}\cdot(-y)(x^{-2}) + (x+y)\cdot\frac{-1}{(x+y)^2} - \frac{1}{x} =$$

$$= \frac{-\cancel{x}y}{yx^{\cancel{2}}} - \frac{(x+y)}{(x+y)^2} - \frac{1}{x} =$$

$$= -\frac{2}{x} - \frac{1}{x+y}$$

$$\frac{df}{dy}(x,y,z) = \ln(y \cdot x^{-1}) + \ln(x+y)^{-1} - \ln\left(\frac{x}{6}\right) =$$

$$= \frac{1}{y \cdot x^{-1}} \cdot x^{-1} + \frac{1}{(x+y)^{-1}} \cdot (-1)(x+y)^{-2} \cdot 1 - 0 =$$

$$= \frac{x}{y} \cdot \frac{1}{x} + (x+y) \cdot \frac{-1}{(x+y)^2} - 0 =$$

$$= \frac{1}{y} - \frac{1}{x+y}$$

$$\frac{df}{dz}(x,y,z) = \ln\left(\frac{y}{x}\right) + \ln\left(\frac{1}{x+y}\right) - \ln\left(\frac{x}{6}\right) = 0$$

$$f(x,y,z) = \begin{bmatrix} \frac{df}{dx}(x,y,z) \\ \frac{df}{dx}(x,y,z) \\ \frac{df}{dx}(x,y,z) \end{bmatrix} = \begin{bmatrix} -\frac{2}{x} - \frac{1}{x+y} \\ \frac{1}{y} - \frac{1}{x+y} \\ 0 \end{bmatrix}$$

h) $f(x,y,z) = \sin(z(z^2+x)^{-1}) - (6x^2+y)(y^2-z^2)^{-1} =$

$$= \cos(z(z^2+x)^{-1}) \cdot z\,((-1)(z^2+x)^{-2}) -$$

$$- 12x\,(y^2-z^2)^{-1} + (6x^2+y) \cdot 0 =$$

$$= -\frac{\cos\left(\frac{z}{z^2+x}\right) \cdot z}{(z^2+x)^2} - \frac{12x}{y^2-z^2}$$

$$\frac{df}{dx}(x,y,z) = \frac{df}{dx}(\sin(z(z^2+x)^{-1}) - (6x^2+y)(y^2-z^2)^{-1} =$$

$$= 0 - 1 \cdot (y^2-z^2)^{-1} + (6x^2+y) \cdot (-1)(y^2-z^2)^{-2} \cdot 2y =$$

$$= \frac{-1}{y^2-z^2} + \frac{(12x^2y + 2y^2)(y^2-z^2)^{-1}}{y^2-z^2} =$$

$$= \frac{-1 + (12x^2y + 2y^2)(y^2-z^2)^{-1}}{y^2-z^2}$$

$$\vec{v}_{\nabla f} = \vec{v} \cdot \nabla f = \begin{pmatrix} \frac{3}{5}, \cancel{\frac{4}{5}} \\ \frac{4}{5} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{12}\pi \\ \frac{1}{2} \end{pmatrix} = \frac{3}{5} \cdot \frac{-13\pi}{12} + \frac{4}{5} \cdot \frac{1}{2} =$$

$$= \frac{-3\pi\sqrt{3}}{60} + \frac{4}{10} = \frac{-\sqrt{3}\pi}{20} + \frac{8}{20} = \frac{8-\sqrt{3}\pi}{20}$$