Cable Project

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0.1 Stability of the Traveling Front Solutions

Now that we have found the traveling front solutions, it is important to understand their stability using linear stability analysis. Consider the solution $v(x,t) = V(\xi) + \epsilon\psi(\xi,t)$, where $0 < \epsilon << 1$ and $\psi(\xi,t)$ represents a small perturbation to the traveling wave solution, $V(\xi)$. The goal of the linear stability analysis is to understand how $\psi(\xi,t)$ behaves over time. Recall the governing partial differential equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - v + H(v - \theta) \tag{1}$$

The chain rule allows us to obtain the following partial derivatives of $v(x,t) = V(\xi) + \epsilon\psi(\xi,t)$.

$$\frac{\partial v}{\partial t} = \frac{\partial \xi}{\partial t}\frac{dV}{d\xi} + \epsilon(\frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \psi}{\partial t}) = -c\frac{dV}{d\xi} + \epsilon(-c\frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial t})$$

$$\frac{\partial v}{\partial x} = \frac{\partial \xi}{\partial x}\frac{dV}{d\xi} + \epsilon\frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial \xi} = \frac{dV}{d\xi} + \epsilon\frac{\partial \psi}{\partial \xi}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial \xi}{\partial x}\frac{d^2 V}{d\xi^2} + \epsilon\frac{\partial^2 \psi}{\partial \xi^2}\frac{\partial \xi}{\partial x} = \frac{d^2 V}{d\xi^2} + \epsilon\frac{\partial^2 \psi}{\partial \xi^2}$$

Equation (1) is rewritten as

$$-c\frac{dV(\xi)}{d\xi} + \epsilon(-c\frac{\partial \psi(\xi,t)}{\partial \xi} + \frac{\partial \psi(\xi,t)}{\partial t}) = \frac{d^2 V(\xi)}{d\xi^2} + \epsilon\frac{\partial^2 \psi(\xi,t)}{\partial \xi^2} + f(V(\xi) + \epsilon\psi(\xi,t)) \tag{2}$$

Analysis of equation (2) can be simplified by using the Taylor expansion of $f\big(V(\xi) + \epsilon\psi(\xi,t)\big)$ with respect to $V(\xi)$, and discounting $O(\epsilon^2)$ terms.

$$f\big(V + \epsilon\psi\big) \approx f(V) + \frac{\partial f(V)}{\partial V}\epsilon\psi = -V + \mathcal{H}(V - \theta) - \epsilon\psi + \delta(V - \theta)\epsilon\psi$$

Equation 2 becomes

$$-c\frac{dV}{d\xi} + \epsilon(-c\frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial t}) = \frac{d^2 V}{d\xi^2} + \epsilon\frac{\partial^2 \psi}{\partial \xi^2} - V + H(V - \theta) + \epsilon\psi\big(-1 + \delta(V - \theta)\big)$$

Look at the terms involving ψ and ϵ to analyze the long term behavior of the disturbance. The partial differential equation that models the behavior of the perturbation is

$$\frac{\partial \psi}{\partial t} = c\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial \psi}{\partial \xi} - (1 - \delta(V - \theta))\psi \tag{3}$$

0.2 Separation of Variables

This equation can be solved using separation of variables, where solutions have the form of $\psi(\xi, t) = S(\xi)T(t) \neq 0$.

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{S}\frac{\partial^2 S}{\partial \xi^2} + \frac{c}{S}\frac{\partial S}{\partial \xi} - 1 + \delta(V - \theta) = \lambda$$

Before proceeding with the analysis, we need to define the boundary conditions. Recall that the boundary condition $\lim_{x\to\pm\infty} \frac{\partial v(x,t)}{\partial x} = 0$ is necessary to ensure a physically realistic solution. Because we previously imposed the boundary condition that $\lim_{\xi\to\pm\infty} \frac{dV}{d\xi} = 0$, it must also be true that

$$\lim_{x\to\pm\infty} \frac{\partial\big(V(\xi) + \epsilon\psi(\xi, t)\big)}{\partial x} = \lim_{\xi\to\pm\infty} \frac{dV}{d\xi} + \lim_{\xi\to\pm\infty} \epsilon\frac{\partial \psi}{\partial \xi} = 0 \implies \lim_{\xi\to\pm\infty} \frac{d\psi}{d\xi} = 0$$

The time-dependent equation is a homogeneous ODE with the following solutions, where $a \in \mathbb{R}$.

$$\frac{1}{T}\frac{dT}{dt} = \lambda \implies \int \frac{dT}{T} = \int \lambda dt \implies T(t) = ae^{\lambda t} \tag{4}$$

For the spatial dependence, we split the domain of the equation into three regions, depending on the value of $\delta(V(\xi) - \theta)$, with the boundary condition that $\lim_{\xi\to\pm\infty} \psi = 0$:

Region 1: $\xi < 0$ and $\delta(V(\xi) - \theta) = 0$
Region 2: $\xi > 0$ and $\delta(V(\xi) - \theta) = 0$
Region 3: $\xi = 0$ and $\delta(V(\xi) - \theta) = \infty$

0.2.1 Region 1 and 2

In Region 1 and 2, we solve the equation:

$$\frac{d^2 S}{d\xi^2} + c\frac{dS}{d\xi} - (\lambda + 1)S = 0 \tag{5}$$

The boundary conditions are in Region 1: $\lim_{\xi\to-\infty} S_1 = 0$; and in Region 2: $\lim_{\xi\to\infty} S_2 = 0$. The solutions have the form:

$$\begin{aligned} S_1(\xi) &= c_1 e^{\frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}\xi} \\ S_2(\xi) &= c_2 e^{\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2}\xi} \end{aligned} \tag{6}$$

0.2.2 Region 3

At Region 3, we integrate the equation at a small interval around the point of discontinuity to examine the effect $\delta(V(\xi)-\theta)$ has on the solution. The spatial equation in this region is

$$\frac{d^2S}{d\xi^2}+c\frac{dS}{d\xi}-\big[\lambda+1+\delta(V(\xi)-\theta)\big]S=0 \tag{7}$$

Since there is a discontinuity due to the dirac delta function here at $V(\xi)=\theta$, $\xi=0$, we integrate a small interval $(+\epsilon,-\epsilon)$ around the discontinuity. Then, we take the limit $\epsilon\to 0$ gives us information as to how the solutions in Region 1 and 2, (6), connect in Region 3.

$$\int_{-\epsilon}^{+\epsilon}\frac{d^2S}{d\xi^2}d\xi+c\int_{-\epsilon}^{+\epsilon}\frac{dS}{d\xi}d\xi-\int_{-\epsilon}^{+\epsilon}(\lambda+1)Sd\xi-\int_{-\epsilon}^{+\epsilon}\delta(V(\xi)-\theta)Sd\xi=0$$

For the first two terms, the fundamental theorem of calculus gives us the first derivative evaluated at the endpoints and the function evaluated at the endpoints.

$$\frac{dS}{d\xi}(+\epsilon)-\frac{dS}{d\xi}(-\epsilon)+c[S(+\epsilon)-S(-\epsilon)]-\int_{-\epsilon}^{+\epsilon}(\lambda+1)Sd\xi-\int_{-\epsilon}^{+\epsilon}\delta(V(\xi)-\theta)Sd\xi=0$$

Now we take the limit:

$$\lim_{\xi\to 0}\left[\frac{dS}{d\xi}(+\epsilon)-\frac{dS}{d\xi}(-\epsilon)+c[S(+\epsilon)-S(-\epsilon)]-\int_{-\epsilon}^{+\epsilon}(\lambda+1)Sd\xi-\int_{-\epsilon}^{+\epsilon}\delta(V(\xi)-\theta)Sd\xi\right]=0$$

Of the remaining two integrals, the first one goes to zero as the bounds shrink to zero since the integral is continuous, even if S is not. The second integral involving the dirac delta function evaluates to $S(0)$ by the *sampling property*, since we are specifically looking at the small region centered around where $V(\xi)=\theta$. Therefore we get:

$$\lim_{\xi\to 0}\left[\frac{dS}{d\xi}(+\epsilon)-\frac{dS}{d\xi}(-\epsilon)+c[S(+\epsilon)-S(-\epsilon)]\right]=S(0)$$

S must be continuous if we are assuming that the derivative $\frac{dS}{d\xi}$ exists so $S(0^+)=S(0^-)$ and the constants in front of the homogeneous solutions (6) are equal, $c_1=c_2=K$ (using K to avoid confusion with the speed c), revealing a relationship between the value of the spatial term and the discontinuity between its derivative at $\xi=0$.

$$S(0)=\lim_{\xi\to 0}\left[K\Big(\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2}e^{\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2}\xi}-\frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}e^{\frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}\xi}\Big)\right]$$

Taking the limit, we find the following, where $S(0)\in\mathbb{R}$ and $K\in\mathbb{R}$ since this is a physical system and we assume the solutions are real-valued.

$$S(0)=K\left(\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2}-\frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}\right) \tag{8}$$

S is the spatial part of some arbitrary perturbation function that satisfies $\lim_{\xi\to\pm\infty} S(\xi) = 0$, and its value at zero could be either non-zero, or it could be 0. Since the boundary conditions provided do not fix $S(0)$ or K, we will examine both cases. In the latter case, if $S(0) = 0$ then $K = c_1 = 0$ and we find the trivial solution of $S = 0$ which gives us $\psi(\xi, t) = 0$. In the following section, we explore the former case.

0.3 Eigenvalues

0.3.1 $\lambda = 0$ Case

Now to examine the family of solutions that depend on λ, we start with the $\lambda = 0$ case. Assuming we can rescale $S(0)$ to unity, eq (8) simplifies to:

$$1 = K\left(\frac{-c - \sqrt{c^2 + 4)}}{2} - \frac{-c + \sqrt{c^2 + 4)}}{2}\right) = -K\sqrt{c^2 + 4}$$

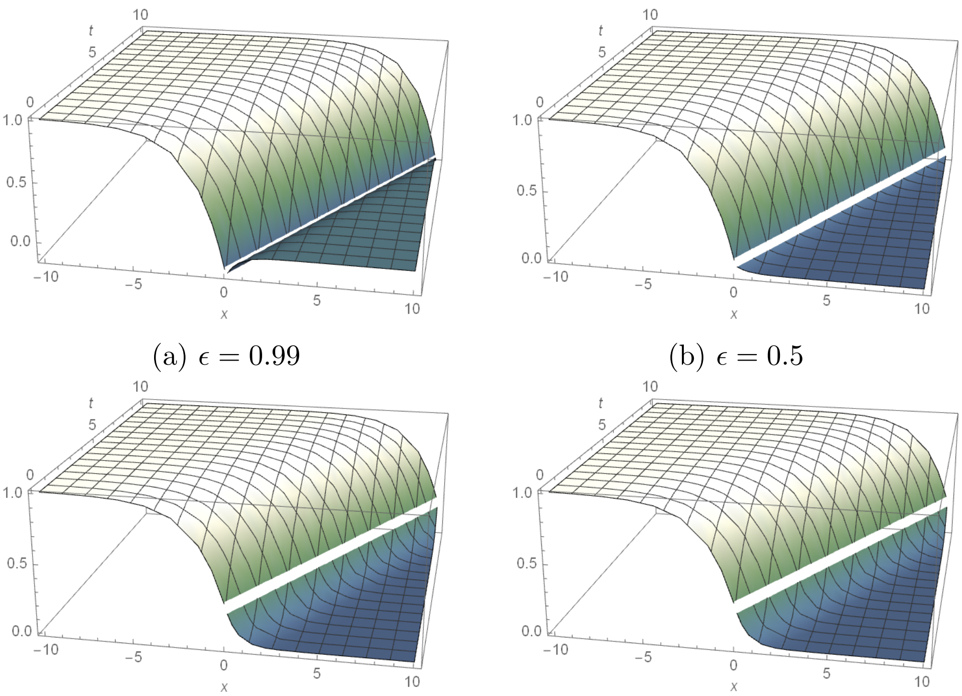
or

$$K = -\frac{1}{\sqrt{c^2 + 4}}$$

Plugging this constant back into our solutions for Region 1 and 2 (6) and multiplying it by the time-dependent solution (4), we find a solution for $\psi(\xi, t)$:

$$\psi_{\lambda=0}(\xi, t) = \begin{cases} \frac{-1}{\sqrt{c^2+4}} e^{\frac{-c+\sqrt{c^2+4}}{2}\xi}, & \xi < 0 \\ \frac{-1}{\sqrt{c^2+4}} e^{\frac{-c-\sqrt{c^2+4}}{2}\xi}, & \xi > 0 \end{cases} \tag{9}$$

Recalling our traveling wave solution $V(\xi)$ (**??**), we note that $\psi(\xi, t) = V'(\xi)$. This solution corresponds to $v(x, t) = V(\xi) + \epsilon V'(\xi)$, a Taylor expansion linearization around the traveling wave solution. This perturbation is independent of time. We plot $v(x, t)$ with various values of ϵ to show its effect on $V(\xi)$.



(a) $\epsilon = 0.99$ (b) $\epsilon = 0.5$

(c) $\epsilon = 0.01$ (d) $\epsilon = 0$

Figure 1: $v(x,t) = V(\xi) + \epsilon V'(\xi)$ with various ϵ values

While this perturbation is independent of time, $v(x,t)$ actually solves the original ODE for the traveling wave solution (**??**). Therefore, it is another traveling wave solution.

0.3.2 $\lambda \neq 0$ Case

Looking at this case, we revert back to using $S(0)$ as a parameter and combine it with the time-dependent solutions in eq(4),we find solutions of the form.

$$\psi_{\lambda>0}(\xi,t) = \begin{cases} \frac{-S(0)}{\sqrt{c^2+4(\lambda+1)}} e^{\frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}\xi+\lambda t}, & \xi < 0 \\ \frac{-S(0)}{\sqrt{c^2+4(\lambda+1)}} e^{\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2}\xi+\lambda t}, & \xi > 0 \end{cases} \tag{10}$$

The derivatives are given by:

$$\partial_\xi \psi_{\lambda>0}(\xi,t) = \begin{cases} \frac{-S(0)}{2} e^{\frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}\xi+\lambda t}, & \xi < 0 \\ \frac{S(0)}{2} e^{\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2}\xi+\lambda t}, & \xi > 0 \end{cases} \tag{11}$$

For $\lambda < 0$, we pull out the negative to find solutions of the form:

$$\psi_{\lambda<0}(\xi,t) = \begin{cases} \frac{-S(0)}{\sqrt{c^2-4(\lambda-1)}} e^{\frac{-c+\sqrt{c^2-4(\lambda-1)}}{2}\xi-\lambda t}, & \xi < 0 \\ \frac{-S(0)}{\sqrt{c^2-4(\lambda-1)}} e^{\frac{-c-\sqrt{c^2-4(\lambda-1)}}{2}\xi-\lambda t}, & \xi > 0 \end{cases} \tag{12}$$

$$\partial_\xi \psi_{\lambda<0}(\xi,t) = \begin{cases} \frac{-S(0)}{2} e^{\frac{-c+\sqrt{c^2-4(\lambda-1)}}{2}\xi-\lambda t}, & \xi < 0 \\ \frac{S(0)}{2} e^{\frac{-c-\sqrt{c^2-4(\lambda-1)}}{2}\xi-\lambda t}, & \xi > 0 \end{cases} \tag{13}$$

We note that if $c^2 - 4(\lambda - 1) < 0$ then the solutions will be complex. Without further boundary conditions, it is difficult to see what the difference between these two families of solutions are and why solutions with $\lambda > 0$ are not valid.

References

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