Propagation of Voltage in a Neuron

The Cable Equation

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Overview

1. Motivation
2. Neuronal Cable Equation
3. Passive Membrane
4. Bi-stable Ion Channels

How Do Neurons Communicate?

Within one cell

- Electrochemical signals
- Membrane Potential: $\Delta V_m = V_i - V_e$
- Ions: charge-carriers
- Ion Channels in Membrane

Between cells

- Neurotransmitters

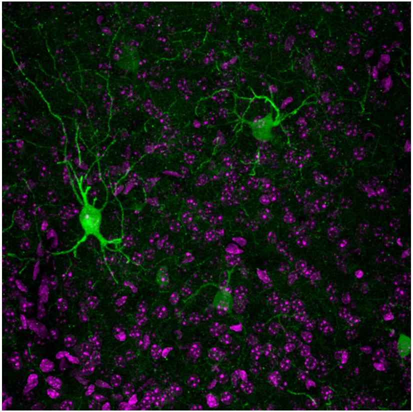


Figure: Mouse neurons, 40X. Bosch Institute Advanced Microscopy Facility, The University of Sydney

Action Potentials

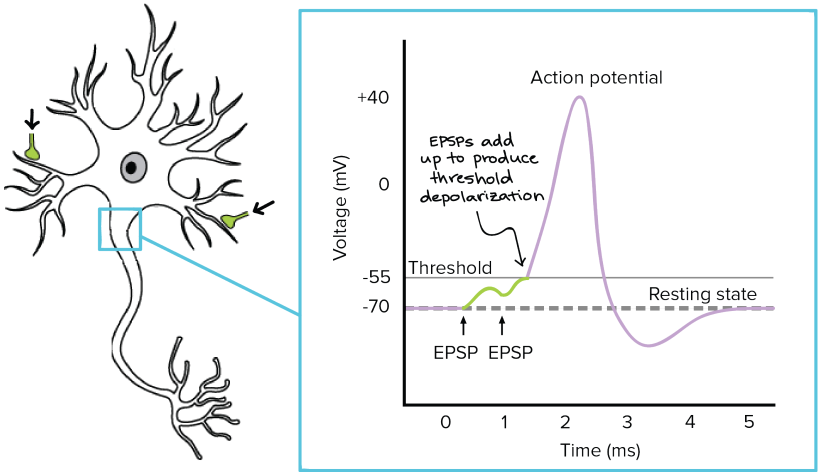


Figure: Changes in axonal membrane voltage due to an action potential. Image from Khan Academy

Hodgkin-Huxley's Neuronal Cable Model (1952)

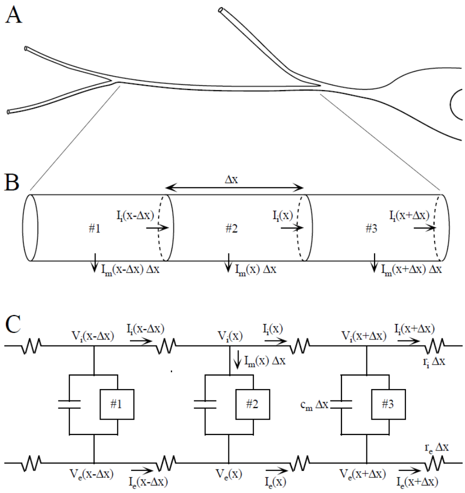


Figure: Differential membrane patches as circuit. Image from jh.edu/motn

- 1-D & Ohmic assumption
- Intracellular current
- Extracellular current
- Membrane current
- Membrane as capacitor
- Ion channels as conductances
- Length Constant:
  $\lambda = \sqrt{\frac{r_m}{r_i + r_e}}$
- Time Constant: $r_m C_m$

Passive Membrane

Linear Cable Equation with Impulse Current Injection:

$$\frac{\partial v(x,t)}{\partial t} - \frac{\partial^2 v(x,t)}{\partial x^2} + v(x,t) = \delta(t - t_0)\delta(x - x_0)$$

$$x \in (-\infty, \infty),\ \ t \in (0, \infty),\ \ x_0 = 0,\ \ t_0 = 0$$

Boundary Conditions:

$$\lim_{x \to \pm\infty} v(x,t) = 0$$

Green's Function for Linear Infinite Cable Equation

Fourier transform → 2 ODE's → Inverse Fourier transform

$$v(x,t) = \frac{\mathcal{H}(x,t)}{\sqrt{4\pi t}} e^{-t-\frac{x^2}{4t}}$$

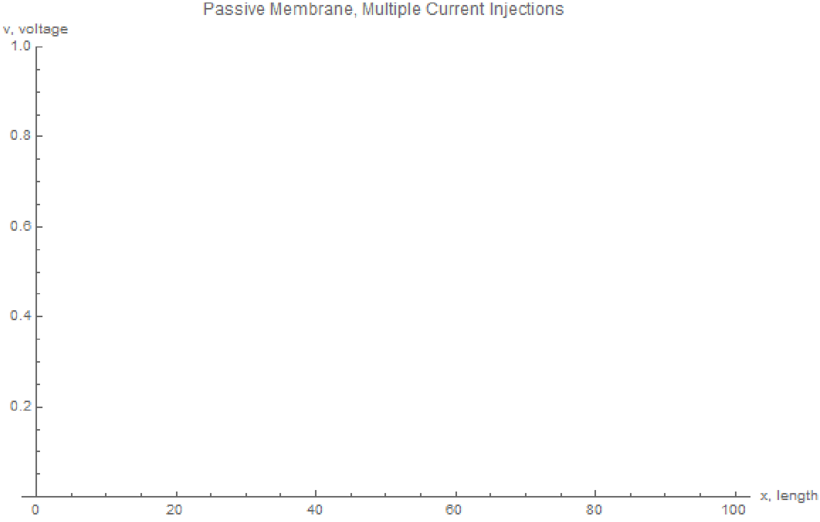
$$G_\infty(x-x_0, t-t_0) = \frac{\mathcal{H}(x-x_0, t-t_0)}{\sqrt{4\pi(t-t_0)}} e^{-(t-t_0)-\frac{(x-x_0)^2}{4(t-t_0)}}$$

Fundamental Solution for Arbitrary Forcing Term, $J_{ext}(x,t)$:

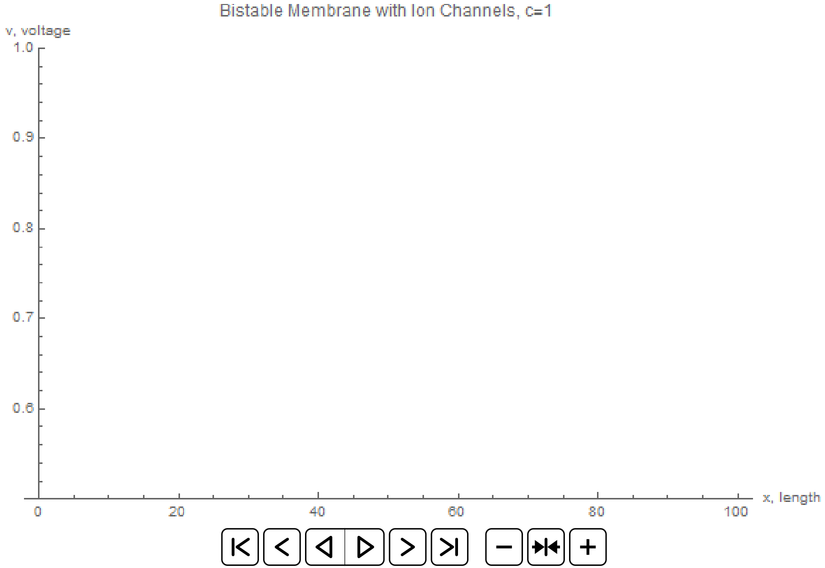
$$\hat{v}(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_\infty(x-x_0, t-t_0) J_{ext}(x_0, t_0) dx_0 dt_0$$

Multiple Current Injections:

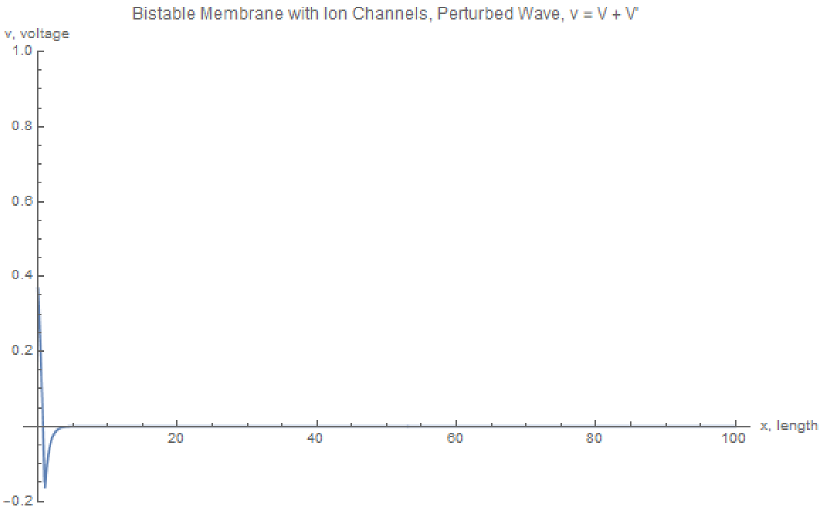
$$(x^*, t^*, A^*) = (1, 0.3, 0.3), (10, 1.1, 1), (30, 0, 0.5)$$



animation of traveling wave solution



animation of perturbed wave solution



Numerical Solutions

- Finite Difference Method
- Plugging approximations for the partial derivatives into the original PDE gives:

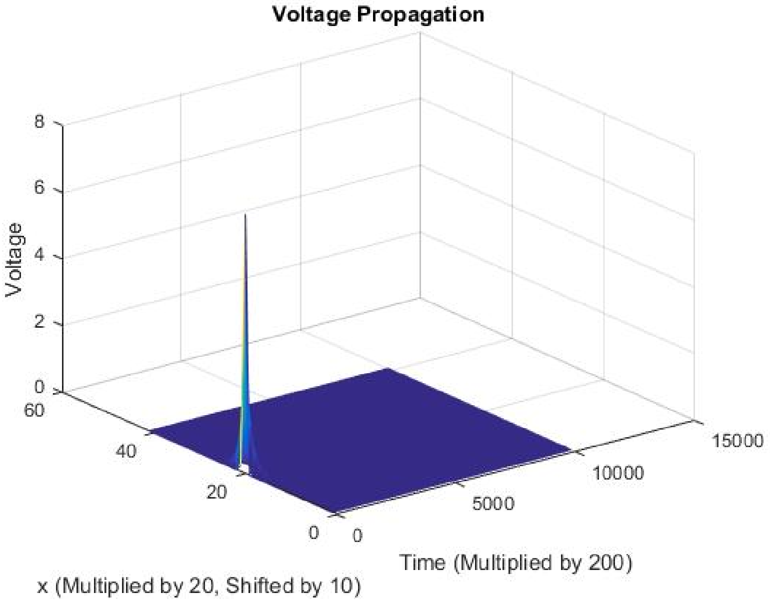
$$\frac{v_i^{j+1} - v_i^j}{\Delta t} = \frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta x)^2} - v_i^j + J_{ext}(x,t)$$

- Solving for the unknown term:

$$v_i^{j+1} = \frac{\Delta t}{(\Delta x)^2}(v_{i+1}^j + v_{i-1}^j) + (1 - \frac{\Delta t(2 + (\Delta x)^2)}{(\Delta x)^2})v_i^j + \Delta t * J_{ext}(x,t)$$

- Stability

Numerical Solutions Results



Traveling Wave Solutions

lalalala

Speed of Traveling Wave

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Stability of Traveling Wave

lalalala

Numerical Solutions for Traveling Wave

- Finite Difference Method–Similar to Previous Numerical Solution
- Resultant Equation (solved for the unknown term):

$$v_i^{j+1} = \frac{\Delta t}{(\Delta x)^2}(v_{i+1}^j + v_{i-1}^j) + (1 - \frac{\Delta t(2 + (\Delta x)^2)}{(\Delta x)^2})v_i^j + \Delta t * H(v_i^j - \theta)$$

Results

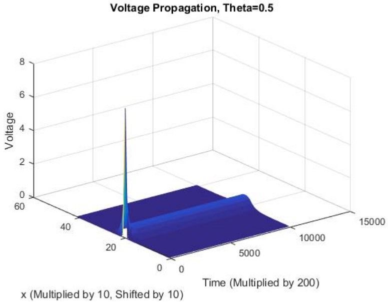

Figure: $\theta = 0.5$

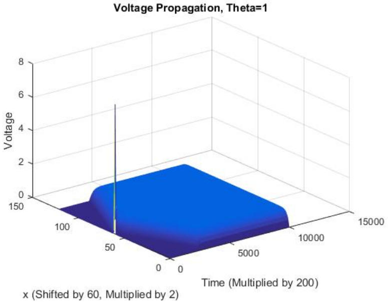

Figure: $\theta = 0.1$

- Numerically Solving for Speeds of the Waves
- Comparison with Analytic Results

Periodically Varying Threshold, Numerical Methods

- What is a periodically varying threshold?
- Governing Equation:

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} - v + H(v - \theta(1 + 0.5\cos(x))) + J_{ext}(x,t)$$

- Numerical Solution:

$$v_i^{j+1} = \frac{\Delta t}{(\Delta x)^2}(v_{i+1}^j + v_{i-1}^j) + (1 - \frac{\Delta t(2 + (\Delta x)^2)}{(\Delta x)^2})v_i^j$$

$$+\Delta t * H(v_i^j - \theta(1 + C\cos(x)))$$

Results

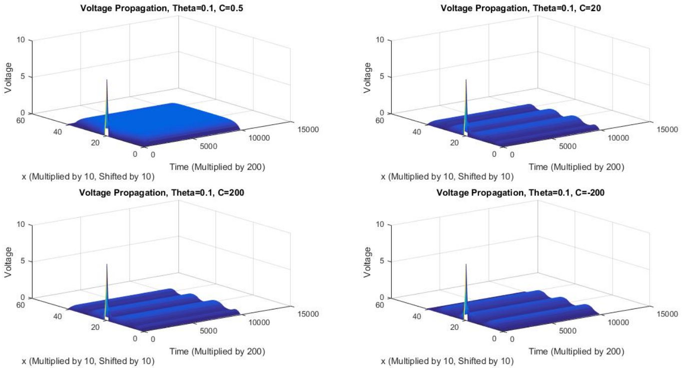

Figure: Varying C, $\theta = 0.1$

Conclusion: Comparison to full Hodgkin-Huxley Model

Three gating variables $x = m, n, h$, each satisfying ODE's:

$$C\frac{\partial v}{\partial t} = g_{\mathrm{Na}} m^3 h(v - E_{\mathrm{Na}}) + g_{\mathrm{K}} n^4 (v - E_{\mathrm{K}}) + g_{\mathrm{L}}(v - E_{\mathrm{L}}) + J_{ext}$$

$$\frac{dx}{dt} = -\frac{1}{\tau_x(v)}\left[x - x_0(v)\right]$$

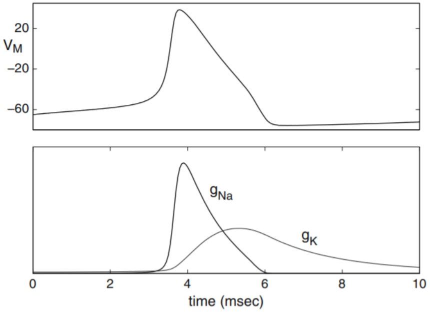

Figure: from Ermentrout & Terman, Math. Foundations of Neuroscience