# Cable Project

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# 1 Perturbation

## 1.1 PDE for perturbation function

$$\frac{\partial \psi}{\partial t} = c\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial \psi}{\partial \xi} - (1 - \delta(V(\xi) - \theta))\psi \tag{1}$$

## 1.2 Separation of Variables

We assume a separation of variables, $\psi(\xi, t) = S(\xi)T(t)$, which leads to

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{S}\frac{\partial^2 S}{\partial \xi^2} + \frac{c}{S}\frac{\partial S}{\partial \xi} - 1 + \delta(V(\xi) - \theta) = \lambda$$

The time-dependent equation is a homogeneous ODE:

$$\frac{1}{T}\frac{dT}{dt} = \lambda \implies \int \frac{dT}{T} = \int \lambda dt \implies T(t) = e^{\lambda t} \tag{2}$$

For the spatial dependence, we split the domain of the equation into three regions, depending on the value of $\delta(V(\xi) - \theta)$, with the boundary condition that $\lim_{\xi \to \pm\infty} \psi = 0$:

1) $\xi < 0$ and $\delta(V(\xi) - \theta) = 0$
2) $\xi > 0$ and $\delta(V(\xi) - \theta) = 0$
3) $\xi = 0$ and $\delta(V(\xi) - \theta) = \infty$

### 1.2.1 Region 1 and 2

In Region 1 and 2, we solve the equation:

$$\frac{d^2 S}{d\xi^2} + c\frac{dS}{d\xi} - (\lambda + 1)S = 0 \tag{3}$$

The boundary conditions are in Region 1: $\lim_{\xi \to -\infty} S_1 = 0$; and in Region 2: $\lim_{\xi \to \infty} S_2 = 0$. The solutions have the form:

$$S_1(\xi) = c_1 e^{\frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}\xi} S_2(\xi) = c_2 e^{\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2}\xi} \tag{4}$$

### 1.2.2 Region 3

At Region 3, we integrate the equation at a small interval around the point of discontinuity to examine the effect $\delta(V(\xi)-\theta)$ has on the solution. After separation of variables, the inhomogeneous PDE 1 becomes

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{S}\frac{\partial^2 S}{\partial \xi^2} + \frac{c}{S}\frac{\partial S}{\partial \xi} - 1 + \delta(V(\xi)-\theta) = \lambda$$

The spatial equation is

$$\frac{d^2 S}{d\xi^2} + c\frac{dS}{d\xi} - [\lambda + 1 + \delta(V(\xi)-\theta)]S = 0 \tag{5}$$

Since there is a discontinuity due to the dirac delta function here at $V(\xi) = \theta$, $\xi = 0$, we integrate a small interval $(+\epsilon, -\epsilon)$ around the discontinuity. Then, we taking the limit $\epsilon \to 0$ gives us information as to how the solutions in Region 1 and 2 (4) connect in Region 3.

$$\int_{-\epsilon}^{+\epsilon} \frac{d^2 S}{d\xi^2} d\xi + c\int_{-\epsilon}^{+\epsilon} \frac{dS}{d\xi} d\xi - \int_{-\epsilon}^{+\epsilon} (\lambda+1)S d\xi - \int_{-\epsilon}^{+\epsilon} \delta(V(\xi)-\theta)S d\xi = 0$$

For the first two terms, the fundamental theorem of calculus gives us the first derivative evaluated at the endpoints and the function evaluated at the endpoints.

$$\frac{dS}{d\xi}(+\epsilon) - \frac{dS}{d\xi}(-\epsilon) + c[S(+\epsilon) - S(-\epsilon)] - \int_{-\epsilon}^{+\epsilon} (\lambda+1)S d\xi - \int_{-\epsilon}^{+\epsilon} \delta(V(\xi)-\theta)S d\xi = 0$$

Now we take the limit:

$$\lim_{\xi\to 0}\left[\frac{dS}{d\xi}(+\epsilon) - \frac{dS}{d\xi}(-\epsilon) + c[S(+\epsilon) - S(-\epsilon)] - \int_{-\epsilon}^{+\epsilon} (\lambda+1)S d\xi - \int_{-\epsilon}^{+\epsilon} \delta(V(\xi)-\theta)S d\xi\right] = 0$$

Of the remaining two integrals, the first one goes to zero as the bounds shrink the zero since the integral is continuous, even if $S$ is not. The second integral involving the dirac delta function evaluates to $S(0)$ by the *sampling property*, since we are specifically looking at the small region centered around where $V(\xi) = \theta$. Therefore we get:

$$\lim_{\xi\to 0}\left[\frac{dS}{d\xi}(+\epsilon) - \frac{dS}{d\xi}(-\epsilon) + c[S(+\epsilon) - S(-\epsilon)\right] = S(0)$$

$S$ must be continuous if we are assuming that the derivative $\frac{dS}{d\xi}$ exists so $S(0^+) = S(0^-)$ and the constants in front of the homogeneous solutions (4) are equal, $c_1 = c_2 = K$, revealing a relationship between the value of the spatial term and the discontinuity between its derivative at $\xi = 0$.

$$S(0) = \lim_{\xi\to 0}\left[K\left(\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2}e^{\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2}} - \frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}e^{\frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}}\right)\right]$$

Taking the limit, we find the following, where $S(0) \in \mathbb{R}$ and $K \in \mathbb{R}$ since this is a physical system and we assume the solutions are real-valued.

$$S(0) = K\left(\frac{-c-\sqrt{c^2+4(\lambda+1)}}{2} - \frac{-c+\sqrt{c^2+4(\lambda+1)}}{2}\right) \tag{6}$$

$S$ is the spatial part of some arbitrary perturbation function that satisfies $\lim_{\xi\to\pm\infty} S(\xi) = 0$, and its value at zero could be either non-zero, or it could be 0. Since the boundary conditions provided do not fix $S(0)$ or $K$, we will examine both cases. In the latter case, if $S(0) = 0$ then $K = c_1 = 0$ and we find the trivial solution of $S = 0$ which gives us $\psi(\xi, t) = 0$. In the following section, we explore the former case.

## 1.3 Eigenvalues

Now to examine the family of solutions that depend on $\lambda$, we start with the $\lambda = 0$ case. Assuming we can rescale $S(0)$ to unity, eq (6) simplifies to:

$$1 = K\left(\frac{-c-\sqrt{c^2+4)}}{2} - \frac{-c+\sqrt{c^2+4)}}{2}\right) = -K\sqrt{c^2+4}$$

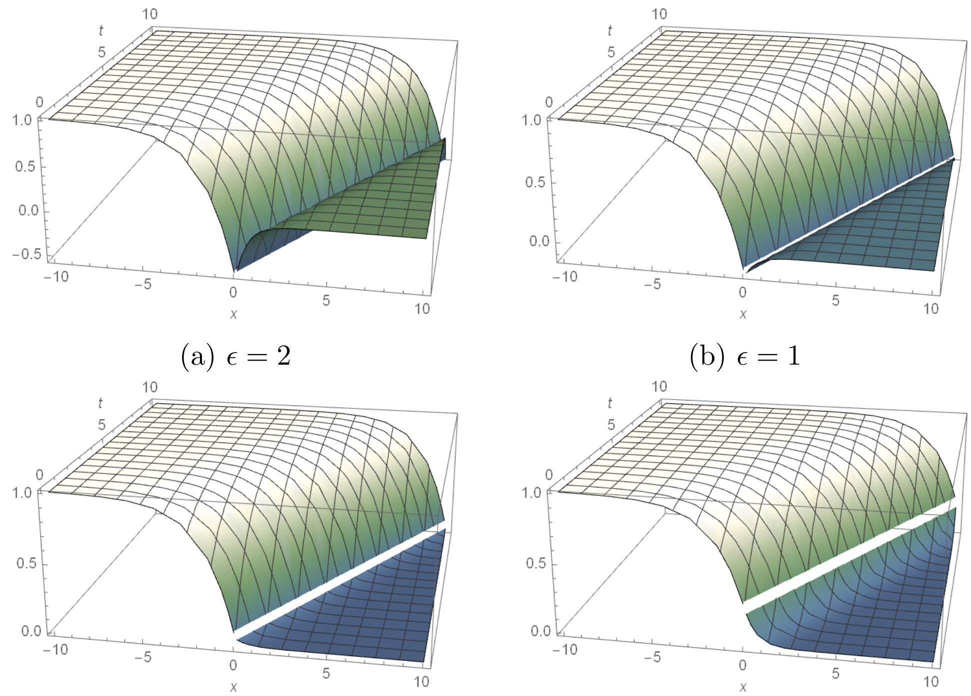
or

$$K = -\frac{1}{\sqrt{c^2+4}}$$

Plugging this constant back into our solutions for Region 1 and 2 (4) and multiplying it by the time-dependent solution (2), we find a solution for $\psi(\xi, t)$:

$$\psi(\xi,t) = \begin{cases} \frac{-1}{\sqrt{c^2+4}} e^{\frac{-c+\sqrt{c^2+4}}{2}}, & \xi < 0 \\ \frac{-1}{\sqrt{c^2+4}} e^{\frac{-c-\sqrt{c^2+4}}{2}}, & \xi > 0 \end{cases} \tag{7}$$

Recalling our traveling wave solution $V(\xi)$ (**??**), we note that $\psi(\xi, t) = V'(\xi)$. This solution corresponds to $v(x,t) = V(\xi) + \epsilon V'(\xi)$, a Taylor expansion linearization around the traveling wave solution. This perturbation is independent of time. We plot $v(x, t)$ with various values of $\epsilon$ to show its effect on $V(\xi)$.




(a) $\epsilon = 2$

(b) $\epsilon = 1$

(c) $\epsilon = 0.5$

(d) $\epsilon = 0.01$

Figure 1: $v(x,t) = V(\xi) + \epsilon V'(\xi)$ with various $\epsilon$ values

# References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