The cable project

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1 2

1.1 2a

1.2 2b

Boundary conditions:

$$\lim_{x\to\infty} \frac{\partial v(x,t)}{\partial x} = 0$$

and

$$\lim_{x\to\infty} \frac{\partial v(x,t)}{\partial x} = 0$$

Application of the change of variables $v(x,t) = V(\xi)$, where $\xi = x - ct$ to $\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} - v(x,t) + H(v-\theta) + J_{ext}(x,t)$ yields the following second order ordinary differential equation. Assume that the external current. $J_{ext})(x,t) = 0$.

$$-cV'(\xi) = V''(\xi) - V(\xi) + H(V(\xi) - \theta) \tag{1}$$

1.3 2c

The two ordinary differential equations are:

$$-cV_1'(\xi) = V_1''(\xi) - V_1(\xi) \tag{2}$$

for $\xi \in (0, \infty)$ and

$$-cV_2'(\xi) = V_2''(\xi) - V_2(\xi) + 1 \tag{3}$$

for $\xi \in (-\infty, 0)$

The boundary conditions are

$$\lim_{\xi \to -\infty} \frac{dV_1(\xi)}{d\xi} = 0$$

$$\lim_{\xi \to \infty} \frac{dV_2(\xi)}{d\xi} = 0$$

$$\lim_{\xi \to -\infty} V_1(\xi) = 1$$

$$\lim_{\xi \to \infty} V_2(\xi) = 0$$

$$V_1(0) = V_2(0)$$

$$\frac{dV_1}{d\xi}(0) = \frac{dV_2}{d\xi}(0)$$

1.4 2d

Equation 2 is a homogeneous ordinary differential equation. The corresponding characteristic equation, $r^2 + cr - 1 = 0$, has roots of $r = \frac{-c \pm \sqrt{c^2+4}}{2}$. Thus,

$$V_1 = c_1 e^{\frac{1}{2}(-c+\sqrt{c^2+4})\xi} + c_2 e^{-\frac{1}{2}(c+\sqrt{c^2+4})\xi}, \xi \in (0, \infty) \tag{4}$$

In order to solve equation 3, use the method of undetermined coefficients. First, determine the homogeneous solution of the differential equation by solving $cV_2'(\xi) = V_2''(\xi) - V_2(\xi)$ for V_2. The characteristic equation, $r^2 + cr - 1 = 0$, has roots of $r = \frac{-c \pm \sqrt{c^2+4}}{2}$. Thus, the homogeneous solution is

$$V_{2,h} = c_3 e^{\frac{1}{2}(-c+\sqrt{c^2+4})\xi} + c_4 e^{-\frac{1}{2}(c+\sqrt{c^2+4})\xi} \tag{5}$$

Guess a particular solution of the form $V_{2,p} = A$. Plugging $V_{2,p}$ into equation 3 yields

$$-c(0) = 0 - A + 1$$

$$A = 1$$

$$V_{2,p} = 1$$

Thus, the solution to equation 3 is

$$V_2 = c_3 e^{\frac{1}{2}(-c+\sqrt{c^2+4})\xi} + c_4 e^{-\frac{1}{2}(c+\sqrt{c^2+4})\xi} + 1, \xi \in (-\infty, 0) \tag{6}$$

Next, use the boundary conditions to eliminate the arbitrary coefficients.

$$\lim_{\xi\to\infty} \frac{dV_1(\xi)}{d\xi} = 0 \Rightarrow c_1 = 0$$

$$\lim_{\xi\to-\infty} \frac{dV_2(\xi)}{d\xi} = 0 \Rightarrow c_4 = 0$$

$$V_1(0) = V_2(0) \Rightarrow c_2 = c_3 + 1$$

$$\frac{dV_1}{d\xi}(0) = \frac{dV_2}{d\xi}(0) \Rightarrow -\frac{1}{2}(c+\sqrt{c^2+4})c_2 = \frac{1}{2}(-c+\sqrt{c^2+4})c_3$$

The solution to this linear system of equations is $c_1 = 0, c_2 = \frac{\frac{1}{2}(-c+\sqrt{c^2+4})}{\frac{1}{2}(-c+\sqrt{c^2+4})+\frac{1}{2}(c+\sqrt{c^2+4})} = \frac{-c+\sqrt{c^2+4}}{2\sqrt{c^2+4}}, c_3 = \frac{-\frac{1}{2}(c+\sqrt{c^2+4})}{\frac{1}{2}(-c+\sqrt{c^2+4})+\frac{1}{2}(c+\sqrt{c^2+4})} = \frac{-c-\sqrt{c^2+4}}{2\sqrt{c^2+4}}$, and $c_4 = 0$.
The solution to equation 2 is

$$V_1(\xi) = \frac{-c+\sqrt{c^2+4}}{2\sqrt{c^2+4}} e^{-\frac{1}{2}(c+\sqrt{c^2+4})\xi}, \xi \in (0, \infty). \tag{7}$$

The solution to equation 3 is

$$V_2(\xi) = \frac{-c-\sqrt{c^2+4}}{2\sqrt{c^2+4}} e^{\frac{1}{2}(-c+\sqrt{c^2+4})\xi} + 1, \xi \in (-\infty, 0). \tag{8}$$

1.5 2e

In order to understand the relationship between the speed of the traveling front and the threshold value θ, apply the threshold condition that $V(0) = \theta$. Application of this threshold condition to the traveling wave solutions gives $\frac{-c}{2\sqrt{c^2+4}} + \frac{1}{2} = \theta$.

$$\frac{-c}{\sqrt{c^2+4}} + 1 = 2\theta$$

$$\frac{-c}{\sqrt{c^2+4}} = 2\theta 1$$

$$\frac{c^2}{c^2+4} = (2\theta-1)^2$$

$$c^2(1-(2\theta-1)^2) = 4(2\theta-1)^2$$

$$c = \sqrt{\frac{4(2\theta - 1)^2}{1 - (2\theta - 1)^2}}$$

$$c = \sqrt{\frac{-(2\theta - 1)^2}{\theta^2 - \theta}}$$

1.6 2f

Partial differential equation:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v(x,t)) \tag{9}$$

where $f(v) = -v + H(v - \theta)$.

In order to analyze the stability of the traveling wave solution found, let $v(x,t) = V(\xi) + \epsilon\psi(\xi,t)$, where $0 < \epsilon << 1$ and $\psi(\xi,t)$ represents a small perturbation to the traveling wave solution.

$$\frac{\partial v}{\partial t} = \frac{\partial \xi}{\partial t}\frac{dV}{d\xi} + \epsilon(\frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \psi}{\partial t})$$

$$\Rightarrow \frac{\partial v}{\partial t} = -c\frac{dV}{d\xi} + \epsilon(-c\frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial t})$$

$$\frac{\partial v}{\partial x} = \frac{\partial \xi}{\partial x}\frac{dV}{d\xi} + \epsilon\frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial \xi}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{dV}{d\xi} + \epsilon\frac{\partial \psi}{\partial \xi}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial \xi}{\partial x}\frac{d^2 V}{d\xi^2} + \epsilon\frac{\partial^2 \psi}{\partial \xi^2}\frac{\partial \xi}{\partial x}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{d^2 V}{d\xi^2} + \epsilon\frac{\partial^2 \psi}{\partial \xi^2}$$

Equation 9 can be rewritten as

$$-c\frac{dV(\xi)}{d\xi} + \epsilon(-c\frac{\partial \psi(\xi,t)}{\partial \xi} + \frac{\partial \psi(\xi,t)}{\partial t}) = \frac{d^2 V(\xi)}{d\xi^2} + \epsilon\frac{\partial^2 \psi(\xi,t)}{\partial \xi^2} + f(V(\xi) + \epsilon\psi(\xi,t)) \tag{10}$$

where $f(V(\xi) + \epsilon\psi) = -(V(\xi) + \epsilon\psi(\xi,t)) + H(V(\xi) + \epsilon\psi(\xi,t) - \theta)$.

Analysis of equation 10 can be simplified by using the Taylor expansion of $f(V(\xi) + \epsilon\psi)$ with respect to ϵ, about $\epsilon = 0$.

$$f(V(\xi) + \epsilon\psi) \approx f(V(\xi) + \epsilon\psi)\Big|_{\epsilon=0} + \left(\frac{\partial f(V(\xi) + \epsilon\psi)}{\partial \epsilon}\Big|_{\epsilon=0}\right)\epsilon + O(\epsilon^2)$$

$$f(V(\xi)+\epsilon\psi) \approx f\big(V(\xi)\big) + \big(-\psi(\xi,t)+\psi(\xi,t)\delta(V(\xi)-\theta)\big)\epsilon$$
$$f(V(\xi)+\epsilon\psi) \approx -V(\xi) + H(V(\xi)-\theta) + \psi(\xi,t)\epsilon\big(-1+\delta(V(\xi)-\theta)\big)$$

Equation 10 becomes

$$-c\frac{dV(\xi)}{d\xi}+\epsilon(-c\frac{\partial\psi(\xi,t)}{\partial\xi}+\frac{\partial\psi(\xi,t)}{\partial t}) = \frac{d^2V(\xi)}{d\xi^2}+\epsilon\frac{\partial^2\psi(\xi,t)}{\partial\xi^2}-V(\xi)+H(V(\xi)-\theta)+\psi(\xi,t)\epsilon\big(-1+\delta(V(\xi)-\theta)\big)$$

In order to analyze the long term behavior of the disturbance, look only at the terms involving ψ and ϵ. The partial differential equation that models the behavior of the perturbation is:

$$\epsilon(-c\frac{\partial\psi(\xi,t)}{\partial\xi}+\frac{\partial\psi(\xi,t)}{\partial t}) = \epsilon\frac{\partial^2\psi(\xi,t)}{\partial\xi^2}+\psi(\xi,t)\epsilon\big(-1+\delta(V(\xi)-\theta)\big)$$
$$-c\frac{\partial\psi(\xi,t)}{\partial\xi}+\frac{\partial\psi(\xi,t)}{\partial t} = \frac{\partial^2\psi(\xi,t)}{\partial\xi^2}+\psi(\xi,t)\big(-1+\delta(V(\xi)-\theta)\big)$$

In order to further understand the behavior of the perturbation, solve the partial differential equation to the left and the right of $V(\epsilon)=\theta$. The governing equation for $V(\xi)\neq\theta$ is

$$-c\frac{\partial\psi(\xi,t)}{\partial\xi}+\frac{\partial\psi(\xi,t)}{\partial t} = \frac{\partial^2\psi(\xi,t)}{\partial\xi^2}-\psi(\xi,t) \tag{11}$$

since $\delta(V(\xi)-\theta)=0$ whenever $V(\xi)\neq\theta$. This equation can be solved using separation of variables, which means solutions should have the form of $\psi(\xi,t)=F(\xi)G(t)\neq 0$. This proposed solution must satisfy the partial differential equation in equation 11.

$$-cG(t)\frac{dF(\xi)}{d\xi}+F(\xi)\frac{dG(t)}{dt} = G(t)\frac{d^2F(\xi)}{d\xi^2}-F(\xi)G(t)$$

$$-c\frac{1}{F(\xi)}\frac{dF}{d\xi}+\frac{1}{G(t)}\frac{dG}{dt} = \frac{1}{F(\xi)}\frac{d^2F(\xi)}{D\xi^2}-1$$