Weighted posets and the enriched monomial basis of QSym [slides]

Darij Grinberg and Ekaterina Vassilieva

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these slides: https://github.com/darijgr/fpsac21eta/raw/main/fps21eta-talk.pdf
paper: https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2021/58Grinberg.pdf

Summary of our work

Hsiao defines in [4] the **monomial peak functions** η_α: a class of quasisymmetric functions indexed by **odd compositions** of n. They provide a monomial-like basis to Stembridge's algebra of peaks [5] and are related to Stembridge peak functions K_α through

$$K_\alpha = \sum_{\substack{\beta \in \operatorname{Odd}(n);\\ \operatorname{Peak}(\beta) \subseteq \operatorname{Peak}(\alpha)}} \eta_\beta .$$

In the present work:

- We show that monomial peak functions may be extended to **a basis of (the whole) QSym**. We name this new basis the **enriched monomial basis** of QSym.
- We relate it to other bases of QSym, compute its **antipode** and **coproduct**.
- We introduce **weighted posets** and their **enriched P-partitions** (generalizing both the weighted posets of [1] and the enriched P-partitions of [5]), whose generating functions give a **universal framework** for many types of quasisymmetric functions.

$$\begin{tikzcd}
 & 2, \epsilon(2)=5 & & \\
3, \epsilon(3)=2 \arrow[ur] & & 1, \epsilon(1)=1 \arrow[ul] \arrow[r] & 4, \epsilon(4)=12 \\
 & 5, \epsilon(5)=2 \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

- We use our framework to compute the **product** of two enriched monomials:

$$\eta_\alpha \eta_\beta = \sum_{\substack{\gamma \in \alpha \sqcup\!\sqcup \beta;\\ I \subseteq \left(S_\beta(\gamma) \setminus (S_\beta(\gamma)-1)\right) \setminus \{1\}}} (-1)^{|I|} \eta_{\gamma \downarrow\downarrow I} .$$

1. Notation and basic definitions

1.1. Compositions and permutations statistics

- $\mathbb{P} = \{1, 2, \ldots\}$, $\mathbb{N} = \{0, 1, \ldots\}$, $[n] = \{1, 2, \ldots, n\}$, S_n the symmetric group on $[n]$
- $\operatorname{Comp}(n)$ is the set of *compositions* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ of n. $\operatorname{Odd}(n)$ is the set of *odd compositions* of n containing only odd integers. Let $\ell(\alpha) := p$, $|\alpha| := \sum_i \alpha_i = n$.
- The *descent set* $\operatorname{Des}(\pi)$ of $\pi \in S_n$ is $\operatorname{Des}(\pi) = \{1 \le i \le n-1 | \pi(i) > \pi(i+1)\}$.
- The *peak set* $\operatorname{Peak}(\pi)$ of π is $\operatorname{Peak}(\pi) = \{2 \le i \le n-1 | \pi(i-1) < \pi(i) > \pi(i+1)\}$.
- Compositions of n are in bijection with descent sets of permutations in S_n while odd compositions of n, are in one-to-one correspondence with peak sets. Denote *$\operatorname{Des}(\alpha)$* (*$\operatorname{Peak}(\alpha)$*) the descent set (peak set) assiociated with (odd) composition α.

1.2. Quasisymmetric functions

- For any $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \operatorname{Comp}(n)$, define the *monomial quasisymmetric function* M_α and the *fundamental quasisymmetric function* L_α by

$$M_\alpha = \sum_{i_1 < \cdots < i_p} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p}, \qquad L_\alpha = \sum_{\substack{i_1 \le \cdots \le i_n;\\ j \in \operatorname{Des}(\alpha) \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n} .$$

- As an example, for $n = 3$, we have

$$M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_2^2 x_4 + x_3^2 x_4 + \ldots,$$
$$L_{(2,1)} = \sum_{i \le j < k} x_i x_j x_k = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_1 x_2 x_4 + x_2^2 x_4 + \ldots .$$

- The sets $\{M_\alpha\}_{n \in \mathbb{N},\ \alpha \in \operatorname{Comp}(n)}$ and $\{L_\alpha\}_{n \in \mathbb{N},\ \alpha \in \operatorname{Comp}(n)}$ are two bases of the $\mathbf{k}$-module QSym. They are related through

$$L_\alpha = \sum_{\substack{\beta \in \operatorname{Comp}(n);\\ \operatorname{Des}(\alpha) \subseteq \operatorname{Des}(\beta)}} M_\beta . \tag{1}$$

1.3. Peak and monomial peak functions

- In [5], Stembridge introduces *peak* quasisymmetric functions. Given $n \in \mathbb{N}$ and $\alpha \in \operatorname{Odd}(n)$,

$$K_\alpha = \sum_{\substack{i_1 \le \cdots \le i_n;\\ j \in \operatorname{Peak}(\alpha) \Rightarrow i_{j-1} < i_{j+1}}} 2^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1} x_{i_2} \cdots x_{i_n} .$$

- Hsiao defines in [4] the *monomial peak functions*. For any odd composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \operatorname{Odd}(n)$, let

$$\eta_\alpha = (-1)^{(n-\ell(\alpha))/2} \sum_{i_1 \le \cdots \le i_p} 2^{|\{i_1, i_2, \ldots, i_p\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p} . \tag{2}$$

- An identity similar to Equation (1) relates peak and monomial peak functions:

$$K_\alpha = \sum_{\substack{\beta \in \operatorname{Odd}(n);\\ \operatorname{Peak}(\beta) \subseteq \operatorname{Peak}(\alpha)}} \eta_\beta . \tag{3}$$

1.4. Posets and P-partitions

- A *labelled poset* $P = ([n], <_P)$ is an arbitrary partial order $<_P$ on the set $[n]$.
- Let $P = ([n], <_P)$ be a labelled poset. A *P-partition* is a map $f : [n] \longrightarrow \mathbb{P}$ that satisfies the two following conditions:
  (i) If $i <_P j$, then $f(i) \le f(j)$.
  (ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$.
- Let $\mathbb{P}^\pm = \{-, +\} \times \mathbb{P}$. We equip the set $\mathbb{P}^\pm$ with a total order given by $-1 < 1 < -2 < 2 < -3 < \ldots$. Let $P = ([n], <_P)$ be a labelled poset. An *enriched P-partition* is a map $f : [n] \longrightarrow \mathbb{P}^\pm$ that satisfies the following two conditions:
  (i) If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in \mathbb{P}$.
  (ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -\mathbb{P}$.
- A more general concept was defined in [2]: Let $\mathcal{Z}$ be a subset of the totally ordered set $\mathbb{P}^\pm$ and $P = ([n], <_P)$ be a labelled poset. A *$\mathcal{Z}$-enriched P-partition* is an enriched P-partition $f : [n] \longrightarrow \mathbb{P}^\pm$ with $f([n]) \subseteq \mathcal{Z}$. Let $\mathcal{L}_\mathcal{Z}(P)$ denote the set of $\mathcal{Z}$-enriched P-partitions.
- Let $X = \{x_1, x_2, x_3, \ldots\}$, $P = ([n], <_P)$, and $\mathcal{Z} \subseteq \mathbb{P}^\pm$. Define the *$\mathcal{Z}$-generating function of P* as the formal power series

$$\Gamma_\mathcal{Z}([n], <_P) = \sum_{f \in \mathcal{L}_\mathcal{Z}([n], <_P)} \prod_{1 \le i \le n} x_{|f(i)|} . \tag{4}$$

- Given $\pi \in S_n$, let $P_\pi = ([n], <_\pi)$ denote the labelled poset where the order relation $<_\pi$ is such that $\pi_i <_\pi \pi_j$ if and only if $i < j$ (see Figure 1).

$$\pi_1 \longrightarrow \pi_2 \longrightarrow \ldots \longrightarrow \pi_n$$

Figure 1: The labelled poset associated to permutation π.

- Set $L_\pi = \Gamma_\mathbb{P}([n], <_\pi)$ and $K_\pi = \Gamma_{\mathbb{P}^\pm}([n], <_\pi)$. The function L_π is equal to the fundamental quasisymmetric function L_α indexed by the unique composition α such that $\operatorname{Des}(\alpha) = \operatorname{Des}(\pi)$. Similarly, K_π is equal to the peak function K_α indexed by the unique odd composition α such that $\operatorname{Peak}(\alpha) = \operatorname{Peak}(\pi)$.
- Given two permutations $\pi \in S_n$ and $\sigma \in S_m$

$$\Gamma_\mathcal{Z}([n], <_\pi) \Gamma_\mathcal{Z}([m], <_\sigma) = \sum_{\gamma \in \pi \sqcup\!\sqcup \sigma} \Gamma_\mathcal{Z}([n+m], <_\gamma) . \tag{5}$$

$$L_\pi L_\sigma = \sum_{\gamma \in \pi \sqcup\!\sqcup \sigma} L_\gamma, \qquad K_\pi K_\sigma = \sum_{\gamma \in \pi \sqcup\!\sqcup \sigma} K_\gamma . \tag{6}$$

2. The enriched monomial basis of QSym

2.1. The enriched monomial functions

Definition 1: Enriched monomials

For any $n \in \mathbb{N}$ and any composition $\alpha \in \operatorname{Comp}(n)$, we define a quasisymmetric function $\eta_\alpha \in \mathrm{QSym}$ by

$$\eta_\alpha = \sum_{\substack{\beta \in \operatorname{Comp}(n);\\ \operatorname{Des}(\beta) \subseteq \operatorname{Des}(\alpha)}} 2^{\ell(\beta)} M_\beta . \tag{7}$$

Examples

- **(a)** Setting $n = 5$ and $\alpha = (1, 3, 1)$ in this definition, we obtain

$$\eta_{(1,3,1)} = \sum_{\substack{\beta \in \operatorname{Comp}(5);\\ \operatorname{Des}(\beta) \subseteq \operatorname{Des}(1,3,1)}} 2^{\ell(\beta)} M_\beta = \sum_{\substack{\beta \in \operatorname{Comp}(5);\\ \operatorname{Des}(\beta) \subseteq \{1,4\}}} 2^{\ell(\beta)} M_\beta \qquad (\operatorname{Des}(1,3,1) = \{1,4\})$$
$$= 2^{\ell(5)} M_{(5)} + 2^{\ell(1,4)} M_{(1,4)} + 2^{\ell(4,1)} M_{(4,1)} + 2^{\ell(1,3,1)} M_{(1,3,1)}$$
$$= 2M_{(5)} + 4M_{(1,4)} + 4M_{(4,1)} + 8M_{(1,3,1)} .$$

- **(b)** For any positive integer n, we have $\eta_{(n)} = 2M_{(n)}$ (since the composition (n) satisfies $\operatorname{Des}(n) = \varnothing$). Likewise, the empty composition $\varnothing = ()$ satisfies $\eta_\varnothing = M_\varnothing$.

Proposition 1: Power series expansion

Let $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \operatorname{Comp}(n)$. Then,

$$\eta_\alpha = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n;\\ i_j = i_{j+1} \text{ for each } j \in [n-1] \setminus \operatorname{Des}(\alpha)}} 2^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1} x_{i_2} \cdots x_{i_n},$$

$$\eta_\alpha = \sum_{i_1 \le i_2 \le \cdots \le i_p} 2^{|\{i_1, i_2, \ldots, i_p\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p} .$$

Proposition 2: Relation to fundamental basis

Let n be a positive integer. Let $\alpha \in \operatorname{Comp}(n)$. Then,

$$\eta_\alpha = 2 \sum_{\gamma \in \operatorname{Comp}(n)} (-1)^{|\operatorname{Des}(\gamma) \setminus \operatorname{Des}(\alpha)|} L_\gamma .$$

2.2. The η_α as a basis, antipode and coproduct

Theorem 1: Enriched monomials are a basis of QSym

Assume that 2 is invertible in $\mathbf{k}$. Then, the family $(\eta_\alpha)_{n \in \mathbb{N},\ \alpha \in \operatorname{Comp}(n)}$ **is a basis** of the $\mathbf{k}$-module QSym. Furthermore, for $n \in \mathbb{N}$ and $\beta \in \operatorname{Comp}(n)$,

$$2^{\ell(\beta)} M_\beta = \sum_{\substack{\alpha \in \operatorname{Comp}(n);\\ \operatorname{Des}(\alpha) \subseteq \operatorname{Des}(\beta)}} (-1)^{\ell(\beta) - \ell(\alpha)} \eta_\alpha .$$

Proposition 3: Antipode of enriched monomials

Let $n \in \mathbb{N}$ and $\alpha \in \operatorname{Comp}(n)$. Then, the antipode S of QSym satisfies

$$S(\eta_\alpha) = (-1)^{\ell(\alpha)} \eta_{\operatorname{rev} \alpha} .$$

Theorem 2: Coproduct of enriched monomials

Consider the coproduct $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ of the Hopf algebra QSym (see [3, §5.1]). Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ be a composition. Then,

$$\Delta(\eta_\alpha) = \sum_{k=0}^{p} \eta_{(\alpha_1, \alpha_2, \ldots, \alpha_k)} \otimes \eta_{(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_p)} .$$

3. The product rule for the enriched monomial basis

3.1. Weighted posets

Definition 2: Labelled weighted posets

A **labelled weighted poset** is a triple $P = ([n], <_P, \epsilon)$ where $([n], <_P)$ is a labelled poset and $\epsilon : [n] \longrightarrow \mathbb{P}$ is a map (called the **weight function**). In a labelled weighted poset each node is marked with two numbers: its label $i \in [n]$ and its weight $\epsilon(i)$.

$$\begin{tikzcd}
 & 2, \epsilon(2)=5 & & \\
3, \epsilon(3)=2 \arrow[ur] & & 1, \epsilon(1)=1 \arrow[ul] \arrow[r] & 4, \epsilon(4)=12 \\
 & 5, \epsilon(5)=2 \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

For any set $\mathcal{Z} \subseteq \mathbb{P}^\pm$, we define the generating function $\Gamma_\mathcal{Z}([n], <_P, \epsilon)$ of the labelled weighted poset $([n], <_P, \epsilon)$ by

$$\Gamma_\mathcal{Z}([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_\mathcal{Z}([n], <_P)} \prod_{1 \le i \le n} x_{|f(i)|}^{\epsilon(i)} . \tag{8}$$

3.2. Universal quasisymmetric functions

Definition 3: Universal quasisymmetric functions

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a composition. Let $\pi = \pi_1 \ldots \pi_n$ be a permutation in S_n. Let $P_{\pi,\alpha} = ([n], <_\pi, \alpha)$ denote the labelled weighted poset composed of the labelled poset $([n], <_\pi)$ and the weight function sending the vertex labelled π_i to α_i.

$$\pi_1, \alpha_1 \longrightarrow \pi_2, \alpha_2 \longrightarrow \ldots \longrightarrow \pi_n, \alpha_n$$

We define the **universal quasisymmetric function** $U_{\pi,\alpha}^\mathcal{Z}$ as the generating function

$$U_{\pi,\alpha}^\mathcal{Z} = \Gamma_\mathcal{Z}([n], <_\pi, \alpha) . \tag{9}$$

Proposition 4: Specialisation of universal quasisymmetric functions

Let $n \in \mathbb{N}$. Let id_n and $\overline{id_n}$ denote the two permutations in S_n given by $id_n = 1\,2\,3 \ldots n$ and $\overline{id_n} = n\; n-1\; n-2 \ldots 1$ (in one-line notation). Denote further (1^n) the composition of n with n entries equal to 1. Let $\pi \in S_n$. Then,

$$U_{\pi,(1^n)}^{\mathbb{P}} = L_\pi, \qquad U_{\pi,(1^n)}^{\mathbb{P}^\pm} = K_\pi, \qquad U_{\overline{id_n},\alpha}^{\mathbb{P}} = M_\alpha, \qquad U_{id_n,\alpha}^{\mathbb{P}^\pm} = \eta_\alpha . \tag{10}$$

3.3. Product rule

Theorem 3: Product of universal quasisymmetric functions

Let $\mathcal{Z}$ be a subset of $\mathbb{P}^\pm$. Let π and σ be two permutations in S_n and S_m, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be two compositions with n and m entries. The product of two universal quasisymmetric functions is given by

$$U_{\pi,\alpha}^\mathcal{Z} U_{\sigma,\beta}^\mathcal{Z} = \sum_{(\tau,\gamma) \in (\pi,\alpha) \sqcup\!\sqcup (\sigma,\beta)} U_{\tau,\gamma}^\mathcal{Z} . \tag{11}$$

Proof. The proof is iterative and uses the following decomposition of posets. □

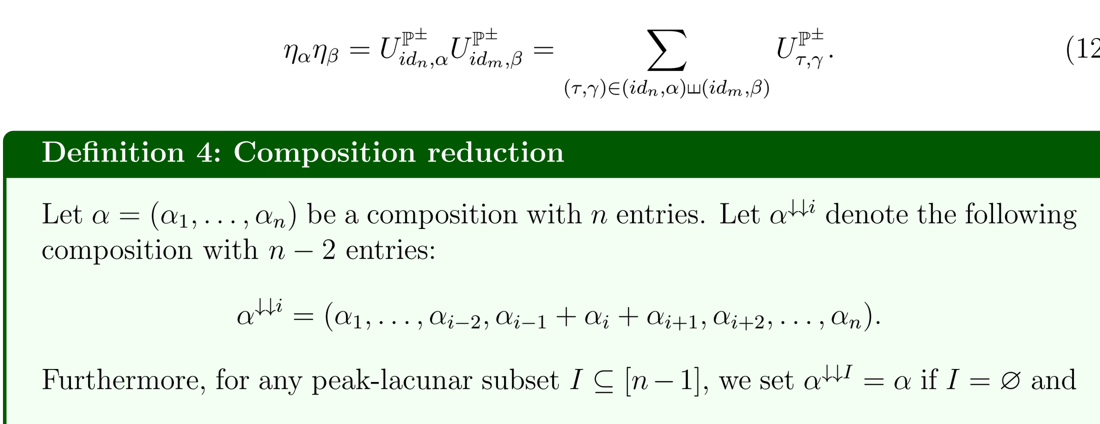

Figure 2: Decomposition of a double chain weighted poset into two posets with one incomparable pair less.

3.4. Product of enriched monomials

Let α and β be two compositions with n and m entries. Equations (10) and (11) imply:

$$\eta_\alpha \eta_\beta = U_{id_n,\alpha}^{\mathbb{P}^\pm} U_{id_m,\beta}^{\mathbb{P}^\pm} = \sum_{(\tau,\gamma) \in (id_n,\alpha) \sqcup\!\sqcup (id_m,\beta)} U_{\tau,\gamma}^{\mathbb{P}^\pm} . \tag{12}$$

Definition 4: Composition reduction

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a composition with n entries. Let $\alpha^{\downarrow\downarrow i}$ denote the following composition with $n - 2$ entries:

$$\alpha^{\downarrow\downarrow i} = (\alpha_1, \ldots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n) .$$

Furthermore, for any peak-lacunar subset $I \subseteq [n-1]$, we set $\alpha^{\downarrow\downarrow I} = \alpha$ if $I = \varnothing$ and

$$\alpha^{\downarrow\downarrow I} = \left(\left(\cdots \left(\alpha^{\downarrow i_k} \right) \cdots \right)^{\downarrow i_2} \right)^{\downarrow i_1},$$

where $i_1, i_2, \ldots, i_k$ are the elements of $I \neq \varnothing$ in increasing order. As an example, let $\alpha = (2, 1, 4, 3, 2)$. We have $\alpha^{\downarrow\downarrow 3} = (2, 8, 2)$ and $\alpha^{\downarrow\downarrow \{2,4\}} = (12)$.

Theorem 4: Universal quasisymmetric functions to enriched monomials

Let α be a composition with n entries and π a permutation in S_n. We have

$$U_{\pi,\alpha}^{\mathbb{P}^\pm} = \sum_{I \subseteq \operatorname{Peak}(\pi)} (-1)^{|I|} \eta_{\alpha \downarrow\downarrow I} . \tag{13}$$

Proof (sketch). From the definition of $U_{\pi,\alpha}^{\mathbb{P}^\pm}$, one can obtain without too much trouble that

$$U_{\pi,\alpha}^{\mathbb{P}^\pm} = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n;\\ j \in \operatorname{Peak}(\pi) \Rightarrow \neg(i_{j-1} = i_j = i_{j+1})}} 2^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_n}^{\alpha_n} .$$

By the inclusion-exclusion principle, this can be rewritten as

$$U_{\pi,\alpha}^{\mathbb{P}^\pm} = \sum_{I \subseteq \operatorname{Peak}(\pi)} (-1)^{|I|} \underbrace{\sum_{\substack{i_1 \le i_2 \le \cdots \le i_n;\\ j \in I \Rightarrow i_{j-1} = i_j = i_{j+1}}} 2^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_n}^{\alpha_n}}_{= \eta_{\alpha \downarrow\downarrow I}} . \qquad \square$$

Theorem 5: Product rule for enriched monomials

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be two compositions. Given a composition γ obtained by shuffling α and β (we shall denote this by $\gamma \in \alpha \sqcup\!\sqcup \beta$), let $S_\beta(\gamma)$ be the set of the positions of the entries of β in γ. Denote further $S_\beta(\gamma) - 1 = \{i - 1 | i \in S_\beta(\gamma)\}$. Then,

$$\eta_\alpha \eta_\beta = \sum_{\substack{\gamma \in \alpha \sqcup\!\sqcup \beta;\\ I \subseteq \left(S_\beta(\gamma) \setminus (S_\beta(\gamma)-1)\right) \setminus \{1\}}} (-1)^{|I|} \eta_{\gamma \downarrow\downarrow I} . \tag{14}$$

The sum ranges not over compositions γ but over ways to shuffle α with β. Thus, the same γ can appear in several addends of the sum.
As an example, one has

$$\eta_{(1,1)} \eta_{(2,3)} = \eta_{(1,1,2,3)} + \eta_{(1,2,1,3)} - \eta_{(4,3)} + \eta_{(2,1,1,3)} + \eta_{(1,2,3,1)}$$
$$- \eta_{(1,6)} + \eta_{(2,1,3,1)} - \eta_{(2,5)} + \eta_{(2,3,1,1)} - \eta_{(6,1)},$$
$$\eta_{(1,2)} \eta_{(2)} = \eta_{(2,1,2)} + 2\eta_{(1,2,2)} - \eta_{(5)} .$$

Proof (sketched). Recall (12), and rewrite the right-hand side using (13). Let (τ, γ) be a coshuffle in $(id_n, \alpha) \sqcup\!\sqcup (id_m, \beta)$. The index i belongs to $\operatorname{Peak}(\tau)$ if and only if τ_i is a letter of $n + id_m$ and τ_{i+1} is a letter of id_n and we have $i > 1$. That is, if and only if i is the index of an entry of β in γ and $i + 1$ is the index of an entry of α. Thus, $\operatorname{Peak}(\tau) = (S_\beta(\gamma) \setminus (S_\beta(\gamma) - 1)) \setminus \{1\}$. Therefore, we obtain (14). □

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