

Weighted posets and the enriched monomial basis of QSym [slides]

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these slides: <https://github.com/darijgr/fpsac21eta/raw/main/fps21eta-talk.pdf>

paper: <https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2021/58Grinberg.pdf>

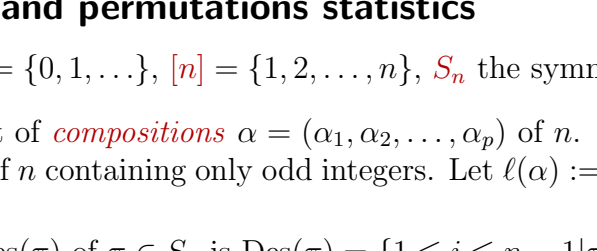
Summary of our work

Hsiao defines in [4] the **monomial peak functions** η_α : a class of quasisymmetric functions indexed by **odd compositions** of n . They provide a monomial-like basis to Stembridge's algebra of peaks [5] and are related to Stembridge peak functions K_α through

$$K_\alpha = \sum_{\substack{\beta \in \text{Odd}(n); \\ \text{Peak}(\beta) \subseteq \text{Peak}(\alpha)}} \eta_\beta.$$

In the present work:

- We show that monomial peak functions may be extended to a **basis of (the whole) QSym**. We name this new basis the **enriched monomial basis** of QSym.
- We relate it to other bases of QSym, compute its **antipode** and **coproduct**.
- We introduce **weighted posets** and their **enriched P-partitions** (generalizing both the weighted posets of [1] and the enriched P-partitions of [5]), whose generating functions give a **universal framework** for many types of quasisymmetric functions.



- We use our framework to compute the **product** of two enriched monomials:

$$\eta_\alpha \eta_\beta = \sum_{\substack{\gamma \in \text{ow}(\beta); \\ I \subseteq (S_\beta(\gamma) \setminus (S_\beta(\gamma) - 1)) \setminus \{1\}}} (-1)^{|I|} \eta_{\gamma \dot{+} I}.$$

1. Notation and basic definitions

1.1. Compositions and permutations statistics

- $\mathbb{P} = \{1, 2, \dots\}$, $\mathbb{N} = \{0, 1, \dots\}$, $[n] = \{1, 2, \dots, n\}$, S_n the symmetric group on $[n]$.
- $\text{Comp}(n)$ is the set of **compositions** $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$ of n . $\text{Odd}(n)$ is the set of **odd compositions** of n containing only odd integers. Let $\ell(\alpha) := p$, $|\alpha| := \sum_i \alpha_i = n$.
- The **descent set** $\text{Des}(\pi)$ of $\pi \in S_n$ is $\text{Des}(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$.
- The **peak set** $\text{Peak}(\pi)$ of π is $\text{Peak}(\pi) = \{2 \leq i \leq n-1 \mid \pi(i-1) < \pi(i) > \pi(i+1)\}$.
- Compositions of n are in bijection with descent sets of permutations in S_n , while odd compositions of n are in (a more complicated) bijection with peak sets. Denote $\text{Des}(\alpha)$ (resp. **Peak**(α)) the descent set (resp., peak set) associated with composition (resp., odd composition) α .

1.2. Quasisymmetric functions

- For any $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \text{Comp}(n)$, define the **monomial quasisymmetric function** M_α and the **fundamental quasisymmetric function** L_α by

$$M_\alpha = \sum_{i_1 < \dots < i_p} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p}, \quad L_\alpha = \sum_{\substack{i_1 \leq \dots \leq i_n; \\ j \in \text{Des}(\alpha) \implies i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

- As an example, for $n = 3$, we have

$$M_{(2,1)} = \sum_{i < j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_2^2 x_4 + x_2^2 x_4 + x_3^2 x_4 + \dots,$$

$$L_{(2,1)} = \sum_{i \leq j < k} x_i x_j x_k = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_1 x_2 x_4 + x_2^2 x_4 + \dots$$

- The sets $\{M_\alpha\}_{n \in \mathbb{N}, \alpha \in \text{Comp}(n)}$ and $\{L_\alpha\}_{n \in \mathbb{N}, \alpha \in \text{Comp}(n)}$ are two bases of the **k-module** QSym of quasisymmetric functions. They are related through

$$L_\alpha = \sum_{\substack{\beta \in \text{Comp}(n); \\ \text{Des}(\beta) \subseteq \text{Des}(\alpha)}} M_\beta. \quad (1)$$

1.3. Peak and monomial peak functions

- In [5], Stembridge introduces **peak quasisymmetric functions**. Given $n \in \mathbb{N}$ and $\alpha \in \text{Odd}(n)$, the corresponding function is

$$K_\alpha = \sum_{\substack{i_1 \leq \dots \leq i_n; \\ j \in \text{Peak}(\alpha) \implies i_{j-1} < i_j > i_{j+1}}} 2^{| \{i_1, i_2, \dots, i_n\} |} x_{i_1} x_{i_2} \dots x_{i_n} \in \text{QSym}.$$

- Hsiao defines in [4] the **monomial peak functions**. For any odd composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \text{Odd}(n)$, the corresponding function is

$$\eta_\alpha = (-1)^{(n-\ell(\alpha))/2} \sum_{i_1 \leq \dots \leq i_p} 2^{| \{i_1, i_2, \dots, i_p\} |} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p} \in \text{QSym}. \quad (2)$$

- An identity similar to Equation (1) relates peak and monomial peak functions:

$$K_\alpha = \sum_{\substack{\beta \in \text{Odd}(n); \\ \text{Peak}(\beta) \subseteq \text{Peak}(\alpha)}} \eta_\beta. \quad (3)$$

1.4. Posets and P-partitions

- A **labelled poset** $P = ([n], <_P)$ is an arbitrary partial order $<_P$ on the set $[n]$.
- Let $P = ([n], <_P)$ be a labelled poset. A **P-partition** is a map $f : [n] \rightarrow \mathbb{P}$ that satisfies the two following conditions:
 - If $i <_P j$, then $f(i) \leq f(j)$.
 - If $i <_P j$ and $i > j$, then $f(i) < f(j)$.
- Let \mathbb{P}^\pm be the (unusually) totally ordered set

$$\mathbb{P}^\pm = \{-1 < 1 < -2 < 2 < -3 < 3 < \dots\} = \mathbb{P} \cup (-\mathbb{P}).$$

Let $P = ([n], <_P)$ be a labelled poset. An **enriched P-partition** is a map $f : [n] \rightarrow \mathbb{P}^\pm$ that satisfies the following two conditions:

- If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in \mathbb{P}$.
- If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -\mathbb{P}$.

- A more general concept was defined in [2]:
Let \mathcal{Z} be a subset of \mathbb{P}^\pm , and $P = ([n], <_P)$ be a labelled poset. A **Z-enriched P-partition** is an enriched P-partition $f : [n] \rightarrow \mathbb{P}^\pm$ with $f([n]) \subseteq \mathcal{Z}$. Let $\mathcal{L}_Z(P)$ denote the set of Z-enriched P-partitions.
- Let $X = \{x_1, x_2, x_3, \dots\}$, $P = ([n], <_P)$, and $\mathcal{Z} \subseteq \mathbb{P}^\pm$. Define the **Z-generating function of P** as the formal power series

$$\Gamma_Z([n], <_P) = \sum_{f \in \mathcal{L}_Z([n], <_P)} \prod_{1 \leq i \leq n} x_{f(i)}. \quad (4)$$

- Given $\pi \in S_n$, let $P_\pi = ([n], <_\pi)$ denote the labelled poset where the order relation $<_\pi$ is such that $\pi_i <_\pi \pi_j$ if and only if $i < j$ (see Figure 1).

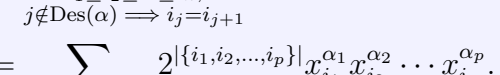


Figure 1: The labelled poset associated to permutation π .

- Set $L_\pi = \Gamma_P([n], <_\pi)$ and $K_\pi = \Gamma_{\mathbb{P}^\pm}([n], <_\pi)$. The function L_π is equal to the fundamental quasisymmetric function L_α indexed by the unique composition α such that $\text{Des}(\alpha) = \text{Des}(\pi)$. Similarly, K_π is equal to the peak function K_α indexed by the unique odd composition α such that $\text{Peak}(\alpha) = \text{Peak}(\pi)$.
- Given two permutations $\pi \in S_n$ and $\sigma \in S_m$, we have

$$\Gamma_Z([n], <_\pi) \cdot \Gamma_Z([m], <_\sigma) = \sum_{\gamma \in \pi \cup \sigma} \Gamma_Z([n+m], <_\gamma). \quad (5)$$

This recovers the known facts that

$$L_\pi L_\sigma = \sum_{\gamma \in \pi \cup \sigma} L_\gamma, \quad K_\pi K_\sigma = \sum_{\gamma \in \pi \cup \sigma} K_\gamma.$$

2. The enriched monomial basis of QSym

2.1. The enriched monomial functions

Definition 1: Enriched monomials

For any $n \in \mathbb{N}$ and any composition $\alpha \in \text{Comp}(n)$, we define a quasisymmetric function $\eta_\alpha \in \text{QSym}$ (called an **enriched monomial quasisymmetric function**) by

$$\eta_\alpha = \sum_{\substack{\beta \in \text{Comp}(n); \\ \text{Des}(\beta) \subseteq \text{Des}(\alpha)}} 2^{\ell(\beta)} M_\beta. \quad (6)$$

Examples

- Setting $n = 5$ and $\alpha = (1, 3, 1)$ in this definition, we obtain

$$\begin{aligned} \eta_{(1,3,1)} &= \sum_{\substack{\beta \in \text{Comp}(5); \\ \text{Des}(\beta) \subseteq \{1,4\}}} 2^{\ell(\beta)} M_\beta \quad (\text{since } \text{Des}(1,3,1) = \{1,4\}) \\ &= 2^{\ell(5)} M_{(5)} + 2^{\ell(1,4)} M_{(1,4)} + 2^{\ell(4,1)} M_{(4,1)} + 2^{\ell(1,3,1)} M_{(1,3,1)} \\ &= 2M_{(5)} + 4M_{(1,4)} + 4M_{(4,1)} + 8M_{(1,3,1)}. \end{aligned}$$

- For any positive integer n , we have $\eta_{(n)} = 2M_{(n)}$ (since the composition (n) satisfies $\text{Des}(n) = \emptyset$). Likewise, the empty composition $\emptyset = ()$ satisfies $\eta_\emptyset = M_\emptyset$.

Proposition 1: Power series expansion

Let $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \text{Comp}(n)$. Then,

$$\begin{aligned} \eta_\alpha &= \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ j \notin \text{Des}(\alpha) \implies i_j = i_{j+1}}} 2^{| \{i_1, i_2, \dots, i_n\} |} x_{i_1} x_{i_2} \dots x_{i_n} \\ &= \sum_{i_1 \leq i_2 \leq \dots \leq i_p} 2^{| \{i_1, i_2, \dots, i_p\} |} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p}. \end{aligned}$$

Proposition 2: Relation to fundamental basis

Let n be a positive integer. Let $\alpha \in \text{Comp}(n)$. Then,

$$\eta_\alpha = 2 \sum_{\gamma \in \text{Comp}(n)} (-1)^{|\text{Des}(\gamma) \setminus \text{Des}(\alpha)|} L_\gamma.$$

Proposition 3: Expansion of monomial functions in enriched monomial functions

For any $n \in \mathbb{N}$ and $\beta \in \text{Comp}(n)$, we have

$$2^{\ell(\beta)} M_\beta = \sum_{\substack{\alpha \in \text{Comp}(n); \\ \text{Des}(\alpha) \subseteq \text{Des}(\beta)}} (-1)^{\ell(\beta) - \ell(\alpha)} \eta_\alpha.$$

2.2. The η_α as a basis, antipode and coproduct

Theorem 1: Enriched monomials are a basis of QSym

Assume that 2 is invertible in **k**. Then, the family $(\eta_\alpha)_{n \in \mathbb{N}, \alpha \in \text{Comp}(n)}$ is a **basis** of the **k-module** QSym.

The **k-module** QSym is a Hopf algebra; let S be its antipode.

Proposition 4: Antipode of enriched monomials

Let $n \in \mathbb{N}$ and $\alpha \in \text{Comp}(n)$. Then, the antipode S of QSym satisfies

$$S(\eta_\alpha) = (-1)^{\ell(\alpha)} \eta_{\text{rev } \alpha}.$$

Here, if $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_p)$, then $\text{rev } \alpha := (\alpha_p, \alpha_{p-1}, \dots, \alpha_1)$.

Theorem 2: Coproduct of enriched monomials

Consider the coproduct $\Delta : \text{QSym} \rightarrow \text{QSym} \otimes \text{QSym}$ of the Hopf algebra QSym (see [3, §5.1]). Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$ be a composition. Then,

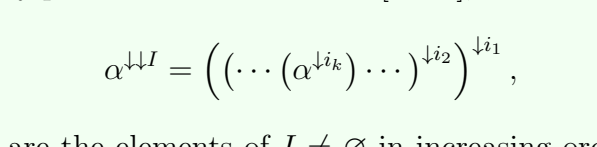
$$\Delta(\eta_\alpha) = \sum_{k=0}^p \eta_{(\alpha_1, \alpha_2, \dots, \alpha_k)} \otimes \eta_{(\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_p)}.$$

3. The product rule for the enriched monomial basis

3.1. Weighted posets

Definition 2: Labelled weighted posets

A **labelled weighted poset** is a triple $P = ([n], <_P, \epsilon)$ where $([n], <_P)$ is a labelled poset and $\epsilon : [n] \rightarrow \mathbb{P}$ is a map (called the **weight function**). In a labelled weighted poset each node is marked with two numbers: its label $i \in [n]$ and its weight $\epsilon(i)$.



For any set $\mathcal{Z} \subseteq \mathbb{P}^\pm$, we define the generating function $\Gamma_Z([n], <_P, \epsilon)$ of the labelled weighted poset $([n], <_P, \epsilon)$ by

$$\Gamma_Z([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_Z([n], <_P)} \prod_{1 \leq i \leq n} x_{f(i)}^{\epsilon(i)}. \quad (7)$$

3.2. Universal quasisymmetric functions

Definition 3: Universal quasisymmetric functions

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a composition. Let $\pi = \pi_1 \dots \pi_n$ be a permutation in S_n . Let $P_{\pi, \alpha} = ([n], <_\pi, \alpha)$ denote the labelled weighted poset composed of the labelled poset $([n], <_\pi)$ and the weight function sending the vertex labelled π_i to α_i .

$$\pi_1, \alpha_1 \longrightarrow \pi_2, \alpha_2 \longrightarrow \dots \longrightarrow \pi_n, \alpha_n$$

We define the **universal quasisymmetric function** $U_{\pi, \alpha}^\pm$ as the generating function

$$U_{\pi, \alpha}^\pm = \Gamma_Z([n], <_\pi, \alpha). \quad (8)$$

Proposition 5: Specialisation of universal quasisymmetric functions

Let $n \in \mathbb{N}$. Let id_n and \overline{id}_n denote the two permutations in S_n given by $id_n = 1 \ 2 \ 3 \dots n$ and $\overline{id}_n = n \ n-1 \ n-2 \dots 1$ (in one-line notation). Denote further (1^n) the composition of n with n entries equal to 1. Let $\pi \in S_n$. Then,

$$U_{\pi, (1^n)}^\pm = L_\pi, \quad U_{\pi, (1^n)}^\mp = K_\pi, \quad U_{id_n, \alpha}^\pm = M_\alpha, \quad U_{id_n, \alpha}^\mp = \eta_\alpha. \quad (9)$$

3.3. Product rule

Theorem 3: Product of universal quasisymmetric functions

Let \mathcal{Z} be a subset of \mathbb{P}^\pm . Let π and σ be two permutations in S_n and S_m , and let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_m)$ be two compositions with n and m entries. The product of two universal quasisymmetric functions is given by

$$U_{\pi, \alpha}^\pm U_{\sigma, \beta}^\pm = \sum_{(\tau, \gamma) \in (\pi, \alpha) \sqcup (\sigma, \beta)} U_{\tau, \gamma}^\pm. \quad (10)$$

Proof. The proof is iterative and uses the following decomposition of posets. \square

Figure 2: Decomposition of a double chain weighted poset into two posets with one incomparable pair less.

3.4. Product of enriched monomials

Let α and β be two compositions with n and m entries. Equations (9) and (10) imply:

$$\eta_\alpha \eta_\beta = U_{id_n, \alpha}^\mp U_{id_m, \beta}^\pm = \sum_{(\tau, \gamma) \in (id_n, \alpha) \sqcup (id_m, \beta)} U_{\tau, \gamma}^\pm. \quad (11)$$

Definition 4: Composition reduction

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a composition with n entries. Let $\alpha^{+i, i}$ denote the following composition with $n-2$ entries:

$$\alpha^{+i, i} = (\alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_n).$$

Furthermore, for any peak-lacunar subset $I \subseteq [n-1]$, we set $\alpha^{+I, I} = \alpha$ if $I = \emptyset$ and

$$\alpha^{+I, I} = \left((\dots (\alpha^{+i_k, i_k}) \dots)^{+i_2} \right)^{+i_1},$$

where i_1, i_2, \dots, i_k are the elements of $I \neq \emptyset$ in increasing order. As an example, let $\alpha = (2, 1, 4, 3, 2)$. We have $\alpha^{+1, 3} = (2, 8, 2)$ and $\alpha^{+1, 2, 4} = (12)$.

Theorem 4: Universal quasisymmetric functions to enriched monomials

Let α be a composition with n entries and π a permutation in S_n . We have

$$U_{\pi, \alpha}^\pm = \sum_{I \subseteq \text{Peak}(\pi)} (-1)^{|I|} \eta_{\alpha^{+I, I}}. \quad (12)$$

Proof (sketch). From the definition of $U_{\pi, \alpha}^\pm$, one can obtain without too much trouble that

$$U_{\pi, \alpha}^\pm = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ j \in \text{Peak}(\pi) \implies -i_{j-1} = i_j = i_{j+1}}} 2^{| \{i_1, i_2, \dots, i_n\} |} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_n}^{\alpha_n}.$$

By the inclusion-exclusion principle, this can be rewritten as

$$U_{\pi, \alpha}^\pm = \sum_{I \subseteq \text{Peak}(\pi)} (-1)^{|I|} \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ j \in I \implies i_{j-1} = i_j = i_{j+1}}} 2^{| \{i_1, i_2, \dots, i_n\} |} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_n}^{\alpha_n}. \quad \square$$

Theorem 5: Product rule for enriched monomials

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_m)$ be two compositions. Given a composition γ obtained by shuffling α and β (we shall denote this by $\gamma \in \alpha \sqcup \beta$), let $S_\beta(\gamma)$ be the set of the positions of the entries of β in γ . Denote further $S_\beta(\gamma) - 1 = \{i-1 \mid i \in S_\beta(\gamma)\}$. Then,

$$\eta_\alpha \eta_\beta = \sum_{\gamma \in \alpha \sqcup \beta; I \subseteq (S_\beta(\gamma) \setminus (S_\beta(\gamma) - 1)) \setminus \{1\}} (-1)^{|I|} \eta_{\gamma \dot{+} I}. \quad (13)$$

The sum ranges not over compositions γ but over ways to shuffle α with β . Thus, the same γ can appear in several addends of the sum.

As an example, one has

$$\begin{aligned} \eta_{(1,1)} \eta_{(2,3)} &= \eta_{(1,1,2,3)} + \eta_{(1,2,1,3)} - \eta_{(4,3)} + \eta_{(2,1,1,3)} + \eta_{(1,2,3,1)} \\ &\quad - \eta_{(1,6)} + \eta_{(2,1,3,1)} - \eta_{(2,5)} + \eta_{(2,3,1,1)} - \eta_{(6,1)}, \\ \eta_{(1,2)} \eta_{(2)} &= \eta_{(2,1,2)} + 2\eta_{(1,2,2)} - \eta_{(5)}. \end{aligned}$$

Proof (sketched). Recall (id_n) , and rewrite the right-hand side using

References

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