Weighted posets and the enriched monomial basis of QSym [slides]

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these slides: https://github.com/darijgr/fpsac21eta/raw/main/fps21eta-talk.pdf
paper: https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2021/58Grinberg.pdf

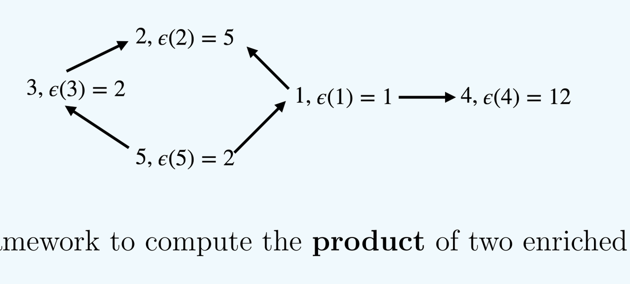
Summary of our work

Hsiao defines in [4] the **monomial peak functions** η_α: a class of quasisymmetric functions indexed by **odd compositions** of n. They provide a monomial-like basis to Stembridge's algebra of peaks [5] and are related to Stembridge peak functions K_α through

$$K_\alpha = \sum_{\substack{\beta\in \mathrm{Odd}(n);\\ \mathrm{Peak}(\beta)\subseteq \mathrm{Peak}(\alpha)}} \eta_\beta .$$

In the present work:

- We show that monomial peak functions may be extended to **a basis of (the whole) QSym**. We name this new basis the **enriched monomial basis** of QSym.
- We relate it to other bases of QSym, compute its **antipode** and **coproduct**.
- We introduce **weighted posets** and their **enriched P-partitions** (generalizing both the weighted posets of [1] and the enriched P-partitions of [5]), whose generating functions give a **universal framework** for many types of quasisymmetric functions.



- We use our framework to compute the **product** of two enriched monomials:

$$\eta_\alpha\eta_\beta = \sum_{\substack{\gamma\in\alpha ⧢ \beta;\\ I\subseteq \left(S_\beta(\gamma)\setminus(S_\beta(\gamma)-1)\right)\setminus\{1\}}} (-1)^{|I|}\eta_{\gamma^{\downarrow\downarrow I}}.$$

1. Notation and basic definitions

1.1. Compositions and permutations statistics

- $\mathbb{P}=\{1,2,\dots\}$, $\mathbb{N}=\{0,1,\dots\}$, $[n]=\{1,2,\dots,n\}$, S_n the symmetric group on $[n]$.
- $\mathrm{Comp}(n)$ is the set of *compositions* $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_p)$ of n. $\mathrm{Odd}(n)$ is the set of *odd compositions* of n containing only odd integers. Let $\ell(\alpha):=p$, $|\alpha|:=\sum_i \alpha_i = n$.
- The *descent set* $\mathrm{Des}(\pi)$ of $\pi\in S_n$ is $\mathrm{Des}(\pi)=\{1\le i\le n-1 \mid \pi(i)>\pi(i+1)\}$.
- The *peak set* $\mathrm{Peak}(\pi)$ of π is $\mathrm{Peak}(\pi)=\{2\le i\le n-1 \mid \pi(i-1)<\pi(i)>\pi(i+1)\}$.
- Compositions of n are in bijection with descent sets of permutations in S_n, while odd compositions of n are in (a more complicated) bijection with peak sets. Denote $\mathrm{Des}(\alpha)$ (resp. $\mathrm{Peak}(\alpha)$) the descent set (resp., peak set) associated with composition (resp., odd composition) α.

1.2. Quasisymmetric functions

- For any $n\in\mathbb{N}$ and $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_p)\in\mathrm{Comp}(n)$, define the *monomial quasisymmetric function* M_α and the *fundamental quasisymmetric function* L_α by

$$M_\alpha=\sum_{i_1<\cdots<i_p} x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_p}^{\alpha_p}, \qquad L_\alpha=\sum_{\substack{i_1\le\cdots\le i_n;\\ j\in\mathrm{Des}(\alpha)\Rightarrow i_j<i_{j+1}}} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

- As an example, for $n=3$, we have

$$M_{(2,1)}=\sum_{i<j}x_i^2x_j = x_1^2x_2+x_1^2x_3+x_2^2x_3+x_1^2x_4+x_2^2x_4+x_3^2x_4+\dots,$$
$$L_{(2,1)}=\sum_{i\le j<k}x_ix_jx_k = x_1^2x_2+x_1^2x_3+x_1x_2x_3+x_2^2x_3+x_1^2x_4+x_1x_2x_4+x_2^2x_4+\dots.$$

- The sets $\{M_\alpha\}_{n\in\mathbb{N},\ \alpha\in\mathrm{Comp}(n)}$ and $\{L_\alpha\}_{n\in\mathbb{N},\ \alpha\in\mathrm{Comp}(n)}$ are two bases of the $\mathbf{k}$-module QSym of quasisymmetric functions. They are related through

$$L_\alpha=\sum_{\substack{\beta\in\mathrm{Comp}(n);\\ \mathrm{Des}(\alpha)\subseteq\mathrm{Des}(\beta)}} M_\beta. \tag{1}$$

1.3. Peak and monomial peak functions

- In [5], Stembridge introduces *peak quasisymmetric functions*. Given $n\in\mathbb{N}$ and $\alpha\in\mathrm{Odd}(n)$, the corresponding function is

$$K_\alpha=\sum_{\substack{i_1\le\cdots\le i_n;\\ j\in\mathrm{Peak}(\alpha)\Rightarrow i_{j-1}<i_{j+1}}} 2^{|\{i_1,i_2,\dots,i_n\}|}x_{i_1}x_{i_2}\cdots x_{i_n}\in\mathrm{QSym}.$$

- Hsiao defines in [4] the *monomial peak functions*. For any odd composition $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_p)\in\mathrm{Odd}(n)$, the corresponding function is

$$\eta_\alpha=(-1)^{(n-\ell(\alpha))/2}\sum_{i_1\le\cdots\le i_p} 2^{|\{i_1,i_2,\dots,i_p\}|}x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_p}^{\alpha_p}\in\mathrm{QSym}. \tag{2}$$

- An identity similar to Equation (1) relates peak and monomial peak functions:

$$K_\alpha=\sum_{\substack{\beta\in\mathrm{Odd}(n);\\ \mathrm{Peak}(\beta)\subseteq\mathrm{Peak}(\alpha)}}\eta_\beta. \tag{3}$$

1.4. Posets and P-partitions

- A *labelled poset* $P=([n],<_P)$ is an arbitrary partial order $<_P$ on the set $[n]$.
- Let $P=([n],<_P)$ be a labelled poset. A *P-partition* is a map $f:[n]\longrightarrow\mathbb{P}$ that satisfies the two following conditions:
  (i) If $i<_P j$, then $f(i)\le f(j)$.
  (ii) If $i<_P j$ and $i>j$, then $f(i)<f(j)$.
- Let $\mathbb{P}^\pm$ be the (unusually) totally ordered set

$$\mathbb{P}^\pm=\{-1<1<-2<2<-3<3<\dots\}=\mathbb{P}\cup(-\mathbb{P}).$$

  Let $P=([n],<_P)$ be a labelled poset. An *enriched P-partition* is a map $f:[n]\longrightarrow\mathbb{P}^\pm$ that satisfies the following two conditions:
  (i) If $i<_P j$ and $i<j$, then $f(i)<f(j)$ or $f(i)=f(j)\in\mathbb{P}$.
  (ii) If $i<_P j$ and $i>j$, then $f(i)<f(j)$ or $f(i)=f(j)\in-\mathbb{P}$.
- A more general concept was defined in [2]:
  Let $\mathcal{Z}$ be a subset of $\mathbb{P}^\pm$, and $P=([n],<_P)$ be a labelled poset. A *$\mathcal{Z}$-enriched P-partition* is an enriched P-partition $f:[n]\longrightarrow\mathbb{P}^\pm$ with $f([n])\subseteq\mathcal{Z}$. Let $\mathcal{L}_{\mathcal{Z}}(P)$ denote the set of $\mathcal{Z}$-enriched P-partitions.
- Let $X=\{x_1,x_2,x_3,\dots\}$, $P=([n],<_P)$, and $\mathcal{Z}\subseteq\mathbb{P}^\pm$. Define the *$\mathcal{Z}$-generating function of P* as the formal power series

$$\Gamma_{\mathcal{Z}}([n],<_P)=\sum_{f\in\mathcal{L}_{\mathcal{Z}}([n],<_P)}\ \prod_{1\le i\le n} x_{|f(i)|}. \tag{4}$$

- Given $\pi\in S_n$, let $P_\pi=([n],<_\pi)$ denote the labelled poset where the order relation $<_\pi$ is such that $\pi_i<_\pi\pi_j$ if and only if $i<j$ (see Figure 1).

$$\pi_1\longrightarrow\pi_2\longrightarrow\dots\longrightarrow\pi_n$$

Figure 1: The labelled poset associated to permutation π.

- Set $L_\pi=\Gamma_{\mathbb{P}}([n],<_\pi)$ and $K_\pi=\Gamma_{\mathbb{P}^\pm}([n],<_\pi)$. The function L_π is equal to the fundamental quasisymmetric function L_α indexed by the unique composition α such that $\mathrm{Des}(\alpha)=\mathrm{Des}(\pi)$. Similarly, K_π is equal to the peak function K_α indexed by the unique odd composition α such that $\mathrm{Peak}(\alpha)=\mathrm{Peak}(\pi)$.
- Given two permutations $\pi\in S_n$ and $\sigma\in S_m$, we have

$$\Gamma_{\mathcal{Z}}([n],<_\pi)\cdot\Gamma_{\mathcal{Z}}([m],<_\sigma)=\sum_{\gamma\in\pi ⧢ \sigma}\Gamma_{\mathcal{Z}}([n+m],<_\gamma). \tag{5}$$

  This recovers the known facts that

$$L_\pi L_\sigma=\sum_{\gamma\in\pi ⧢ \sigma}L_\gamma,\qquad K_\pi K_\sigma=\sum_{\gamma\in\pi ⧢ \sigma}K_\gamma.$$

2. The enriched monomial basis of QSym

2.1. The enriched monomial functions

Definition 1: Enriched monomials

For any $n\in\mathbb{N}$ and any composition $\alpha\in\mathrm{Comp}(n)$, we define a quasisymmetric function $\eta_\alpha\in\mathrm{QSym}$ by

$$\eta_\alpha=\sum_{\substack{\beta\in\mathrm{Comp}(n);\\ \mathrm{Des}(\beta)\subseteq\mathrm{Des}(\alpha)}}2^{\ell(\beta)}M_\beta. \tag{6}$$

Examples

(a) Setting $n=5$ and $\alpha=(1,3,1)$ in this definition, we obtain

$$\begin{aligned}\eta_{(1,3,1)}&=\sum_{\substack{\beta\in\mathrm{Comp}(5);\\ \mathrm{Des}(\beta)\subseteq\mathrm{Des}(1,3,1)}}2^{\ell(\beta)}M_\beta=\sum_{\substack{\beta\in\mathrm{Comp}(5);\\ \mathrm{Des}(\beta)\subseteq\{1,4\}}}2^{\ell(\beta)}M_\beta \qquad (\mathrm{Des}(1,3,1)=\{1,4\})\\ &=2^{\ell(5)}M_{(5)}+2^{\ell(1,4)}M_{(1,4)}+2^{\ell(4,1)}M_{(4,1)}+2^{\ell(1,3,1)}M_{(1,3,1)}\\ &=2M_{(5)}+4M_{(1,4)}+4M_{(4,1)}+8M_{(1,3,1)}.\end{aligned}$$

(b) For any positive integer n, we have $\eta_{(n)}=2M_{(n)}$ (since the composition (n) satisfies $\mathrm{Des}(n)=\varnothing$). Likewise, the empty composition $\varnothing=()$ satisfies $\eta_\varnothing=M_\varnothing$.

Proposition 1: Power series expansion

Let $n\in\mathbb{N}$ and $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_p)\in\mathrm{Comp}(n)$. Then,

$$\eta_\alpha=\sum_{\substack{i_1\le i_2\le\cdots\le i_n;\\ i_j=i_{j+1}\text{ for each }j\notin\mathrm{Des}(\alpha)}}2^{|\{i_1,i_2,\dots,i_n\}|}x_{i_1}x_{i_2}\cdots x_{i_n}=\sum_{i_1\le i_2\le\cdots\le i_p}2^{|\{i_1,i_2,\dots,i_p\}|}x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_p}^{\alpha_p}.$$

Proposition 2: Relation to fundamental basis

Let n be a positive integer and $\alpha\in\mathrm{Comp}(n)$. Then,

$$\eta_\alpha=2\sum_{\substack{\beta\in\mathrm{Comp}(n);\\ \mathrm{Des}(\beta)\subseteq\mathrm{Des}(\alpha)}}(-1)^{|\mathrm{Des}(\alpha)\setminus\mathrm{Des}(\beta)|}L_\beta.$$

2.2. The η_α as a basis, antipode and coproduct

Theorem 1: Enriched monomials as a basis

Assume that 2 is invertible in $\mathbf{k}$. Then, the family $(\eta_\alpha)_{n\in\mathbb{N},\,\alpha\in\mathrm{Comp}(n)}$ is a basis of the $\mathbf{k}$-module QSym. Furthermore, for $n\in\mathbb{N}$ and $\beta\in\mathrm{Comp}(n)$,

$$2^{\ell(\beta)}M_\beta=\sum_{\substack{\alpha\in\mathrm{Comp}(n);\\ \mathrm{Des}(\alpha)\subseteq\mathrm{Des}(\beta)}}(-1)^{\ell(\beta)-\ell(\alpha)}\eta_\alpha.$$

Proposition 3: Antipode of enriched monomials

Let $n\in\mathbb{N}$ and $\alpha\in\mathrm{Comp}(n)$. Then, the antipode S of QSym satisfies

$$S(\eta_\alpha)=(-1)^{\ell(\alpha)}\eta_{\mathrm{rev}\,\alpha}.$$

Theorem 2: Coproduct of enriched monomials

Consider the coproduct $\Delta:\mathrm{QSym}\to\mathrm{QSym}\otimes\mathrm{QSym}$ of the Hopf algebra QSym (see [3, §5.1]). Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_p)$ be a composition. Then,

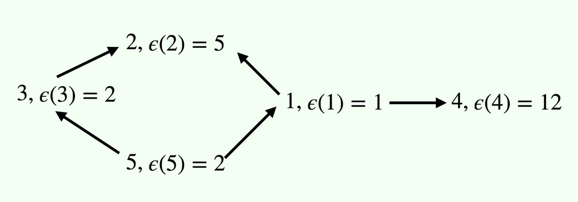
$$\Delta(\eta_\alpha)=\sum_{k=0}^{p}\eta_{(\alpha_1,\alpha_2,\dots,\alpha_k)}\otimes\eta_{(\alpha_{k+1},\alpha_{k+2},\dots,\alpha_p)}.$$

3. The product rule for the enriched monomial basis

3.1. Weighted posets

Definition 2: Labelled weighted posets

A **labelled weighted poset** is a triple $P=([n],<_P,\epsilon)$ where $([n],<_P)$ is a labelled poset and $\epsilon:[n]\to\mathbb{P}$ is a map (called the **weight function**). In a labelled weighted poset each node is marked with two numbers: its label $i\in[n]$ and its weight $\epsilon(i)$.



For any $\mathcal{Z}\subseteq\mathbb{P}^\pm$, we define the generating function $\Gamma_{\mathcal{Z}}([n],<_P,\epsilon)$ of the labelled weighted poset $([n],<_P,\epsilon)$ by

$$\Gamma_{\mathcal{Z}}([n],<_P,\epsilon)=\sum_{f\in\mathcal{L}_{\mathcal{Z}}([n],<_P)}\ \prod_{1\le i\le n}x_{|f(i)|}^{\epsilon(i)}. \tag{7}$$

3.2. Universal quasisymmetric functions

Definition 3: Universal quasisymmetric functions

Let $\alpha=(\alpha_1,\dots,\alpha_n)$ be a composition. Let $\pi=\pi_1\pi_2\dots\pi_n$ be a permutation in S_n. Let $P_{\pi,\alpha}=([n],<_\pi,\epsilon)$ denote the labelled weighted poset consisting of the labelled poset $([n],<_\pi)$ and the weight function sending the vertex labelled π_i to α_i.

$$\pi_1,\alpha_1\longrightarrow\pi_2,\alpha_2\longrightarrow\dots\longrightarrow\pi_n,\alpha_n$$

We define the universal quasisymmetric function $U^{\mathcal{Z}}_{\pi,\alpha}$ as the generating function

$$U^{\mathcal{Z}}_{\pi,\alpha}=\Gamma_{\mathcal{Z}}(P_{\pi,\alpha}). \tag{8}$$

Proposition 4: Specialisation of universal quasisymmetric functions

Let $n\in\mathbb{N}$. Let id_n and $\overline{id_n}$ denote the two permutations in S_n given by $1\,2\,3\dots n$ and $n\,n-1\dots2\,1$ (in one-line notation). Denote further (1^n) the composition of n with n entries equal to 1. Let $\pi\in S_n$. Then

$$U^{\mathbb{P}}_{\pi,(1^n)}=L_\pi,\qquad U^{\mathbb{P}^\pm}_{\pi,(1^n)}=K_\pi,\qquad U^{\mathbb{P}}_{id_n,\alpha}=M_\alpha,\qquad U^{\mathbb{P}^\pm}_{id_n,\alpha}=\eta_\alpha. \tag{9}$$

3.3. Product rule

Theorem 3: Product of universal quasisymmetric functions

Let $\mathcal{Z}$ be a subset of $\mathbb{P}^\pm$. Let π and σ be two permutations in S_n and S_m, and let $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_m)$ be two compositions with n and m entries. The product of two universal quasisymmetric functions is given by

$$U^{\mathcal{Z}}_{\pi,\alpha}U^{\mathcal{Z}}_{n+\sigma,\beta}=\sum_{(\tau,\gamma)\in(\pi,\alpha)⧢(n+\sigma,\beta)}U^{\mathcal{Z}}_{\tau,\gamma}. \tag{10}$$

Proof. The proof is iterative and uses the following decomposition of posets. □

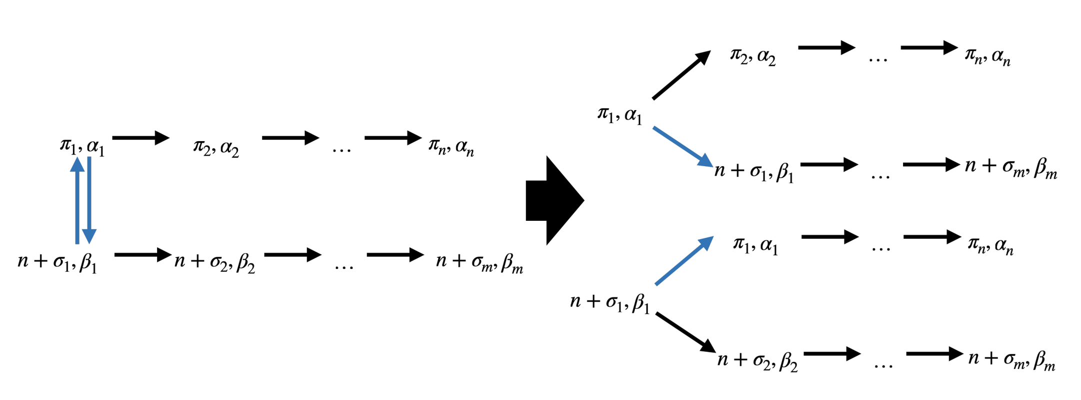


Figure 2: Decomposition of a double chain weighted poset into two posets with one incomparable pair less.

3.4. Product of enriched monomials

Let α and β be two compositions with n and m entries. Equations (9) and (10) imply:

$$\eta_\alpha\eta_\beta=U^{\mathbb{P}^\pm}_{id_n,\alpha}U^{\mathbb{P}^\pm}_{id_m,\beta}=\sum_{(\tau,\gamma)\in(id_n,\alpha)⧢(id_m,\beta)}U^{\mathbb{P}^\pm}_{\tau,\gamma}. \tag{11}$$

Definition 4: Composition reduction

Let $\alpha=(\alpha_1,\dots,\alpha_n)$ be a composition with n entries. Let $\alpha^{\downarrow\downarrow i}$ denote the following composition with $n-2$ entries:

$$\alpha^{\downarrow\downarrow i}=(\alpha_1,\dots,\alpha_{i-2},\alpha_{i-1}+\alpha_i+\alpha_{i+1},\alpha_{i+2},\dots,\alpha_n).$$

Furthermore, for any peak-lacunar subset $I\subseteq[n-1]$, we set $\alpha^{\downarrow\downarrow I}=\alpha$ if $I=\varnothing$ and

$$\alpha^{\downarrow\downarrow I}=\left(\left(\cdots\left(\alpha^{\downarrow i_k}\right)\cdots\right)^{\downarrow i_2}\right)^{\downarrow i_1},$$

where $i_1,i_2,\dots,i_k$ are the elements of $I\neq\varnothing$ in increasing order. As an example, let $\alpha=(2,1,4,3,2)$. We have $\alpha^{\downarrow\downarrow3}=(2,8,2)$ and $\alpha^{\downarrow\downarrow\{2,4\}}=(12)$.

Theorem 4: Universal quasisymmetric functions to enriched monomials

Let α be a composition with n entries and π a permutation in S_n. We have

$$U^{\mathbb{P}^\pm}_{\pi,\alpha}=\sum_{I\subseteq\mathrm{Peak}(\pi)}(-1)^{|I|}\eta_{\alpha^{\downarrow\downarrow I}}. \tag{12}$$

Proof (sketch). From the definition of $U^{\mathbb{P}^\pm}_{\pi,\alpha}$, one can obtain without too much trouble that

$$U^{\mathbb{P}^\pm}_{\pi,\alpha}=\sum_{\substack{i_1\le i_2\le\cdots\le i_n;\\ j\in\mathrm{Peak}(\pi)\Rightarrow\neg(i_{j-1}=i_j=i_{j+1})}}2^{|\{i_1,i_2,\dots,i_n\}|}x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\dots x_{i_n}^{\alpha_n}.$$

By the inclusion-exclusion principle, this can be rewritten as

$$U^{\mathbb{P}^\pm}_{\pi,\alpha}=\sum_{I\subseteq\mathrm{Peak}(\pi)}(-1)^{|I|}\underbrace{\sum_{\substack{i_1\le i_2\le\cdots\le i_n;\\ j\in I\Rightarrow i_{j-1}=i_j=i_{j+1}}}2^{|\{i_1,i_2,\dots,i_n\}|}x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\dots x_{i_n}^{\alpha_n}}_{=\eta_{\alpha^{\downarrow\downarrow I}}}.$$ □

Theorem 5: Product rule for enriched monomials

Let $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_m)$ be two compositions. Given a composition γ obtained by shuffling α and β (we shall denote this by $\gamma\in\alpha ⧢ \beta$), let $S_\beta(\gamma)$ be the set of the positions of the entries of β in γ. Denote further $S_\beta(\gamma)-1=\{i-1 \mid i\in S_\beta(\gamma)\}$. Then,

$$\eta_\alpha\eta_\beta=\sum_{\substack{\gamma\in\alpha ⧢ \beta;\\ I\subseteq\left(S_\beta(\gamma)\setminus(S_\beta(\gamma)-1)\right)\setminus\{1\}}}(-1)^{|I|}\eta_{\gamma^{\downarrow\downarrow I}}. \tag{13}$$

The sum ranges not over compositions γ but over ways to shuffle α with β. Thus, the same γ can appear in several addends of the sum.
As an example, one has

$$\begin{aligned}\eta_{(1,1)}\eta_{(2,3)}&=\eta_{(1,1,2,3)}+\eta_{(1,2,1,3)}-\eta_{(4,3)}+\eta_{(2,1,1,3)}+\eta_{(1,2,3,1)}\\ &\quad-\eta_{(1,6)}+\eta_{(2,1,3,1)}-\eta_{(2,5)}+\eta_{(2,3,1,1)}-\eta_{(6,1)},\\ \eta_{(1,2)}\eta_{(2)}&=\eta_{(2,1,2)}+2\eta_{(1,2,2)}-\eta_{(5)}.\end{aligned}$$

Proof (sketched). Recall (11), and rewrite the right-hand side using (12). Let (τ,γ) be a coshuffle in $(id_n,\alpha)⧢(id_m,\beta)$. The index i belongs to $\mathrm{Peak}(\tau)$ if and only if τ_i is a letter of $n+id_m$ and τ_{i+1} is a letter of id_n and we have $i>1$. That is, if and only if i is the index of an entry of β in γ and $i+1$ is the index of an entry of α. Thus, $\mathrm{Peak}(\tau)=(S_\beta(\gamma)\setminus(S_\beta(\gamma)-1))\setminus\{1\}$. Therefore, we obtain (13). □

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