

# Weighted posets and the enriched monomial basis of QSym [slides]

Darij Grinberg and Ekaterina Vassilieva

2022-01-17

these slides: <https://github.com/darijgr/fpsac21eta/raw/main/fps21eta-talk.pdf>

paper: <https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2021/58Grinberg.pdf>

## Summary of our work

Hsiao defines in [4] the **monomial peak functions**  $\eta_\alpha$ : a class of quasisymmetric functions indexed by **odd compositions** of  $n$ .

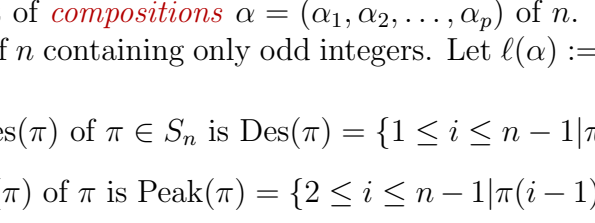
$$\eta_\alpha = (-1)^{(n-\ell(\alpha))/2} \sum_{i_1 \leq \dots \leq i_p} 2^{\{i_1, i_2, \dots, i_p\}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p}.$$

They provide a monomial-like basis to Stembridge's algebra of peaks [5] and are related to Stembridge peak functions  $K_\alpha$  through

$$K_\alpha = \sum_{\substack{\beta \in \text{Odd}(n); \\ \text{Peak}(\beta) \subseteq \text{Peak}(\alpha)}} \eta_\beta.$$

In the present work:

- We show that monomial peak functions may be extended to a **basis of (the whole) QSym**. We name this new basis the **enriched monomial basis** of QSym.
- We relate it to other bases of QSym, compute its **antipode** and **coproduct**.
- We introduce **weighted posets** and their **enriched P-partitions** (generalizing both the weighted posets of [1] and the enriched P-partitions of [5]), whose generating functions give a **universal framework** for many types of quasisymmetric functions.



- We use our framework to compute the **product** of two enriched monomials:

$$\eta_\alpha \eta_\beta = \sum_{\substack{\gamma \in \text{owb}; \\ I \subseteq (S_\beta(\gamma) \setminus (S_\beta(\gamma) - 1)) \setminus \{1\}}} (-1)^{|I|} \eta_{\gamma \uparrow \downarrow I}.$$

## 1. Notation and basic definitions

### 1.1. Compositions and permutations statistics

- $\mathbb{P} = \{1, 2, \dots\}$ ,  $\mathbb{N} = \{0, 1, \dots\}$ ,  $[n] = \{1, 2, \dots, n\}$ ,  $S_n$  the symmetric group on  $[n]$ .
- $\text{Comp}(n)$  is the set of **compositions**  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$  of  $n$ .  $\text{Odd}(n)$  is the set of **odd compositions** of  $n$  containing only odd integers. Let  $\ell(\alpha) := p$ ,  $|\alpha| := \sum_i \alpha_i = n$ .
- The **descent set**  $\text{Des}(\pi)$  of  $\pi \in S_n$  is  $\text{Des}(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$ .
- The **peak set**  $\text{Peak}(\pi)$  of  $\pi$  is  $\text{Peak}(\pi) = \{2 \leq i \leq n-1 \mid \pi(i-1) < \pi(i) > \pi(i+1)\}$ .
- Compositions of  $n$  are in bijection with descent sets of permutations in  $S_n$ , while odd compositions of  $n$  are in (a more complicated) bijection with peak sets. Denote  $\text{Des}(\alpha)$  (resp.  $\text{Peak}(\alpha)$ ) the descent set (resp., peak set) associated with composition (resp., odd composition)  $\alpha$ .

### 1.2. Quasisymmetric functions

- For any  $n \in \mathbb{N}$  and  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \text{Comp}(n)$ , define the **monomial quasisymmetric function**  $M_\alpha$  and the **fundamental quasisymmetric function**  $L_\alpha$  by

$$M_\alpha = \sum_{i_1 < \dots < i_p} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p}, \quad L_\alpha = \sum_{\substack{i_1 \leq \dots \leq i_n; \\ j \in \text{Des}(\alpha) \implies i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

- As an example, for  $n = 3$ , we have

$$M_{(2,1)} = \sum_{i < j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_2^2 x_4 + x_3^2 x_4 + \dots,$$

$$L_{(2,1)} = \sum_{i \leq j < k} x_i x_j x_k = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_1 x_2 x_4 + x_2^2 x_4 + \dots$$

- The sets  $\{M_\alpha\}_{n \in \mathbb{N}, \alpha \in \text{Comp}(n)}$  and  $\{L_\alpha\}_{n \in \mathbb{N}, \alpha \in \text{Comp}(n)}$  are two bases of the **k-module** QSym of quasisymmetric functions. They are related through

$$L_\alpha = \sum_{\substack{\beta \in \text{Comp}(n); \\ \text{Des}(\alpha) \subseteq \text{Des}(\beta)}} M_\beta. \quad (1)$$

### 1.3. Peak and monomial peak functions

- In [5], Stembridge introduces **peak quasisymmetric functions**. Given  $n \in \mathbb{N}$  and  $\alpha \in \text{Odd}(n)$ , the corresponding function is

$$K_\alpha = \sum_{\substack{i_1 \leq \dots \leq i_n; \\ j \in \text{Peak}(\alpha) \implies i_{j-1} < i_j < i_{j+1}}} 2^{\{i_1, i_2, \dots, i_n\}} x_{i_1} x_{i_2} \dots x_{i_n} \in \text{QSym}.$$

- Hsiao defines in [4] the **monomial peak functions**. For any odd composition  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \text{Odd}(n)$ , the corresponding function is

$$\eta_\alpha = (-1)^{(n-\ell(\alpha))/2} \sum_{i_1 \leq \dots \leq i_p} 2^{\{i_1, i_2, \dots, i_p\}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p} \in \text{QSym}. \quad (2)$$

- An identity similar to Equation (2) relates peak and monomial peak functions:

$$K_\alpha = \sum_{\substack{\beta \in \text{Odd}(n); \\ \text{Peak}(\beta) \subseteq \text{Peak}(\alpha)}} \eta_\beta. \quad (3)$$

### 1.4. Posets and P-partitions

- A **labelled poset**  $P = ([n], <_P)$  is an arbitrary partial order  $<_P$  on the set  $[n]$ .
- Let  $P = ([n], <_P)$  be a labelled poset. A **P-partition** is a map  $f : [n] \rightarrow \mathbb{P}$  that satisfies the two following conditions:
  - If  $i <_P j$ , then  $f(i) \leq f(j)$ .
  - If  $i <_P j$  and  $i > j$ , then  $f(i) < f(j)$ .
- Let  $\mathbb{P}^\pm$  be the (unusually) totally ordered set

$$\mathbb{P}^\pm = \{-1 < 1 < -2 < 2 < -3 < 3 < \dots\} = \mathbb{P} \cup (-\mathbb{P}).$$

Let  $P = ([n], <_P)$  be a labelled poset. An **enriched P-partition** is a map  $f : [n] \rightarrow \mathbb{P}^\pm$  that satisfies the following two conditions:

- If  $i <_P j$  and  $i < j$ , then  $f(i) < f(j)$  or  $f(i) = f(j) \in \mathbb{P}$ .
- If  $i <_P j$  and  $i > j$ , then  $f(i) < f(j)$  or  $f(i) = f(j) \in -\mathbb{P}$ .

- A more general concept was defined in [2]:

Let  $\mathcal{Z}$  be a subset of  $\mathbb{P}^\pm$ , and  $P = ([n], <_P)$  be a labelled poset. A  **$\mathcal{Z}$ -enriched P-partition** is an enriched P-partition  $f : [n] \rightarrow \mathbb{P}^\pm$  with  $f([n]) \subseteq \mathcal{Z}$ . Let  $\mathcal{L}_\mathcal{Z}(P)$  denote the set of  $\mathcal{Z}$ -enriched P-partitions.

- Let  $X = \{x_1, x_2, x_3, \dots\}$ ,  $P = ([n], <_P)$ , and  $\mathcal{Z} \subseteq \mathbb{P}^\pm$ . Define the  **$\mathcal{Z}$ -generating function of  $P$**  as the formal power series

$$\Gamma_\mathcal{Z}([n], <_P) = \sum_{f \in \mathcal{L}_\mathcal{Z}([n], <_P)} \prod_{1 \leq i \leq n} x_{f(i)}. \quad (4)$$

- Given  $\pi \in S_n$ , let  $P_\pi = ([n], <_\pi)$  denote the labelled poset where the order relation  $<_\pi$  is such that  $\pi_i <_\pi \pi_j$  if and only if  $i < j$  (see Figure 1).

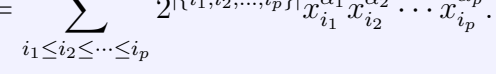


Figure 1: The labelled poset associated to permutation  $\pi$ .

- Set  $L_\pi = \Gamma_\mathbb{P}([n], <_\pi)$  and  $K_\pi = \Gamma_{\mathbb{P}^\pm}([n], <_\pi)$ . The function  $L_\pi$  is equal to the fundamental quasisymmetric function  $L_\alpha$  indexed by the unique composition  $\alpha$  such that  $\text{Des}(\alpha) = \text{Des}(\pi)$ . Similarly,  $K_\pi$  is equal to the peak function  $K_\alpha$  indexed by the unique odd composition  $\alpha$  such that  $\text{Peak}(\alpha) = \text{Peak}(\pi)$ .
- Given two permutations  $\pi \in S_n$  and  $\sigma \in S_m$ , we have

$$\Gamma_\mathcal{Z}([n], <_\pi) \cdot \Gamma_\mathcal{Z}([m], <_\sigma) = \sum_{\gamma \in \pi \cup \sigma} \Gamma_\mathcal{Z}([n+m], <_\gamma). \quad (5)$$

This recovers the known facts that

$$L_\pi L_\sigma = \sum_{\gamma \in \pi \cup \sigma} L_\gamma, \quad K_\pi K_\sigma = \sum_{\gamma \in \pi \cup \sigma} K_\gamma.$$

## 2. The enriched monomial basis of QSym

### 2.1. The enriched monomial functions

#### Definition 1: Enriched monomials

For any  $n \in \mathbb{N}$  and any composition  $\alpha \in \text{Comp}(n)$ , we define a quasisymmetric function  $\eta_\alpha \in \text{QSym}$  (called an **enriched monomial quasisymmetric function**) by

$$\eta_\alpha = \sum_{\substack{\beta \in \text{Comp}(n); \\ \text{Des}(\beta) \subseteq \text{Des}(\alpha)}} 2^{\ell(\beta)} M_\beta. \quad (6)$$

#### Examples

- Setting  $n = 5$  and  $\alpha = (1, 3, 1)$  in this definition, we obtain

$$\begin{aligned} \eta_{(1,3,1)} &= \sum_{\substack{\beta \in \text{Comp}(5); \\ \text{Des}(\beta) \subseteq \{1,4\}}} 2^{\ell(\beta)} M_\beta \quad (\text{since } \text{Des}(1,3,1) = \{1,4\}) \\ &= 2^{\ell(5)} M_{(5)} + 2^{\ell(1,4)} M_{(1,4)} + 2^{\ell(4,1)} M_{(4,1)} + 2^{\ell(1,3,1)} M_{(1,3,1)} \\ &= 2M_{(5)} + 4M_{(1,4)} + 4M_{(4,1)} + 8M_{(1,3,1)}. \end{aligned}$$

- For any positive integer  $n$ , we have  $\eta_{(n)} = 2M_{(n)}$  (since the composition  $(n)$  satisfies  $\text{Des}(n) = \emptyset$ ). Likewise, the empty composition  $\emptyset = ()$  satisfies  $\eta_\emptyset = M_\emptyset$ .

#### Proposition 1: Power series expansion

Let  $n \in \mathbb{N}$  and  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \text{Comp}(n)$ . Then,

$$\begin{aligned} \eta_\alpha &= \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ j \notin \text{Des}(\alpha) \implies i_j = i_{j+1}}} 2^{\{i_1, i_2, \dots, i_n\}} x_{i_1} x_{i_2} \dots x_{i_n} \\ &= \sum_{i_1 \leq i_2 \leq \dots \leq i_p} 2^{\{i_1, i_2, \dots, i_p\}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p}. \end{aligned}$$

#### Proposition 2: Relation to fundamental basis

Let  $n$  be a positive integer. Let  $\alpha \in \text{Comp}(n)$ . Then,

$$\eta_\alpha = 2 \sum_{\gamma \in \text{Comp}(n)} (-1)^{|\text{Des}(\gamma) \setminus \text{Des}(\alpha)|} L_\gamma.$$

#### Proposition 3: Expansion of monomial functions in enriched monomial functions

For any  $n \in \mathbb{N}$  and  $\beta \in \text{Comp}(n)$ , we have

$$2^{\ell(\beta)} M_\beta = \sum_{\substack{\alpha \in \text{Comp}(n); \\ \text{Des}(\alpha) \subseteq \text{Des}(\beta)}} (-1)^{\ell(\beta) - \ell(\alpha)} \eta_\alpha.$$

### 2.2. The $\eta_\alpha$ as a basis, antipode and coproduct

#### Theorem 1: Enriched monomials are a basis of QSym

Assume that 2 is invertible in **k**. Then, the family  $(\eta_\alpha)_{n \in \mathbb{N}, \alpha \in \text{Comp}(n)}$  is a **basis** of the **k-module** QSym.

The **k-module** QSym is a Hopf algebra; let  $S$  be its antipode.

#### Proposition 4: Antipode of enriched monomials

Let  $n \in \mathbb{N}$  and  $\alpha \in \text{Comp}(n)$ . Then, the antipode  $S$  of QSym satisfies

$$S(\eta_\alpha) = (-1)^{\ell(\alpha)} \eta_{\text{rev } \alpha}.$$

Here, if  $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_p)$ , then  $\text{rev } \alpha := (\alpha_p, \alpha_{p-1}, \dots, \alpha_1)$ .

#### Theorem 2: Coproduct of enriched monomials

Consider the coproduct  $\Delta : \text{QSym} \rightarrow \text{QSym} \otimes \text{QSym}$  of the Hopf algebra QSym (see [3, §5.1]). Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$  be a composition. Then,

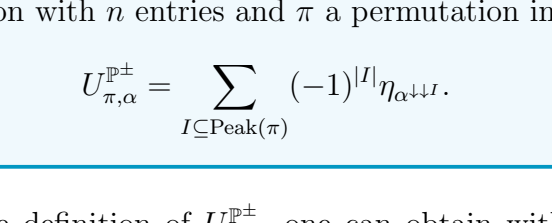
$$\Delta(\eta_\alpha) = \sum_{k=0}^p \eta_{(\alpha_1, \alpha_2, \dots, \alpha_k)} \otimes \eta_{(\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_p)}.$$

## 3. The product rule for the enriched monomial basis

### 3.1. Weighted posets

#### Definition 2: Labelled weighted posets

A **labelled weighted poset** is a triple  $P = ([n], <_P, \epsilon)$  where  $([n], <_P)$  is a labelled poset and  $\epsilon : [n] \rightarrow \mathbb{P}$  is a map (called the **weight function**). In a labelled weighted poset each node is marked with two numbers: its label  $i \in [n]$  and its weight  $\epsilon(i)$ .



For any set  $\mathcal{Z} \subseteq \mathbb{P}^\pm$ , we define the generating function  $\Gamma_\mathcal{Z}([n], <_P, \epsilon)$  of the labelled weighted poset  $([n], <_P, \epsilon)$  by

$$\Gamma_\mathcal{Z}([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_\mathcal{Z}([n], <_P)} \prod_{1 \leq i \leq n} x_{f(i)}^{\epsilon(i)}. \quad (7)$$

### 3.2. Universal quasisymmetric functions

#### Definition 3: Universal quasisymmetric functions

Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  be a composition. Let  $\pi = \pi_1 \dots \pi_n$  be a permutation in  $S_n$ . Let  $P_{\pi, \alpha} = ([n], <_\pi, \alpha)$  denote the labelled weighted poset composed of the labelled poset  $([n], <_\pi)$  and the weight function sending the vertex labelled  $\pi_i$  to  $\alpha_i$ .

$$\pi_1, \alpha_1 \longrightarrow \pi_2, \alpha_2 \longrightarrow \dots \longrightarrow \pi_n, \alpha_n$$

We define the **universal quasisymmetric function**  $U_{\pi, \alpha}^\mathcal{Z}$  as the generating function

$$U_{\pi, \alpha}^\mathcal{Z} = \Gamma_\mathcal{Z}([n], <_\pi, \alpha). \quad (8)$$

#### Proposition 5: Specialisation of universal quasisymmetric functions

Let  $n \in \mathbb{N}$ . Let  $id_n$  and  $\overline{id}_n$  denote the two permutations in  $S_n$  given by  $id_n = 1 \ 2 \ 3 \dots n$  and  $\overline{id}_n = n-1 \ n-2 \dots 1$  (in one-line notation). Denote further  $(1^n)$  the composition of  $n$  with  $n$  entries equal to 1. Let  $\pi \in S_n$ . Then,

$$U_{\pi, (1^n)}^\mathbb{P} = L_\pi, \quad U_{\pi, (1^n)}^\mathbb{P} = K_\pi, \quad U_{id_n, \alpha}^\mathbb{P} = M_\alpha, \quad U_{\overline{id}_n, \alpha}^\mathbb{P} = \eta_\alpha. \quad (9)$$

### 3.3. Product rule

#### Theorem 3: Product of universal quasisymmetric functions

Let  $\mathcal{Z}$  be a subset of  $\mathbb{P}^\pm$ . Let  $\pi$  and  $\sigma$  be two permutations in  $S_n$  and  $S_m$ , and let  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_m)$  be two compositions with  $n$  and  $m$  entries. The product of two universal quasisymmetric functions is given by

$$U_{\pi, \alpha}^\mathcal{Z} U_{\sigma, \beta}^\mathcal{Z} = \sum_{(\tau, \gamma) \in (id_n, \alpha) \cup (id_m, \beta)} U_{\tau, \gamma}^\mathcal{Z}. \quad (10)$$

*Proof.* The proof is iterative and uses the following decomposition of posets.  $\square$



Figure 2: Decomposition of a double chain weighted poset into two posets with one incomparable pair less.

### 3.4. Product of enriched monomials

Let  $\alpha$  and  $\beta$  be two compositions with  $n$  and  $m$  entries. Equations (10) and (11) imply:

$$\eta_\alpha \eta_\beta = U_{id_n, \alpha}^\mathbb{P} U_{id_m, \beta}^\mathbb{P} = \sum_{(\tau, \gamma) \in (id_n, \alpha) \cup (id_m, \beta)} U_{\tau, \gamma}^\mathbb{P}. \quad (11)$$

#### Definition 4: Composition reduction

Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  be a composition with  $n$  entries. Let  $\alpha^{i \downarrow i}$  denote the following composition with  $n-2$  entries:

$$\alpha^{i \downarrow i} = (\alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_n).$$

Furthermore, for any peak-lacunar subset  $I \subseteq [n-1]$ , we set  $\alpha^{i \downarrow I} = \alpha$  if  $I = \emptyset$  and

$$\alpha^{i \downarrow I} = \left( (\dots (\alpha^{i \downarrow i_k}) \dots)^{i \downarrow i_{k-1}} \right)^{i \downarrow i_1},$$

where  $i_1, i_2, \dots, i_k$  are the elements of  $I \neq \emptyset$  in increasing order. As an example, let  $\alpha = (2, 1, 4, 3, 2)$ . We have  $\alpha^{i \downarrow \{3\}} = (2, 8, 2)$  and  $\alpha^{i \downarrow \{2, 4\}} = (12)$ .

#### Theorem 4: Universal quasisymmetric functions to enriched monomials

Let  $\alpha$  be a composition with  $n$  entries and  $\pi$  a permutation in  $S_n$ . We have

$$U_{\pi, \alpha}^\mathbb{P} = \sum_{I \subseteq \text{Peak}(\pi)} (-1)^{|I|} \eta_{\alpha^{i \downarrow I}}. \quad (12)$$

*Proof (sketch).* From the definition of  $U_{\pi, \alpha}^\mathbb{P}$ , one can obtain without too much trouble that

$$U_{\pi, \alpha}^\mathbb{P} = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ j \in \text{Peak}(\pi) \implies -(i_{j-1} = i_j = i_{j+1})}} 2^{\{i_1, i_2, \dots, i_n\}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_n}^{\alpha_n}.$$

By the inclusion-exclusion principle, this can be rewritten as

$$U_{\pi, \alpha}^\mathbb{P} = \sum_{I \subseteq \text{Peak}(\pi)} (-1)^{|I|} \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ j \in I \implies i_{j-1} = i_j = i_{j+1}}} 2^{\{i_1, i_2, \dots, i_n\}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_n}^{\alpha_n}.$$

$\square$

#### Theorem 5: Product rule for enriched monomials

Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_m)$  be two compositions. Given a composition  $\gamma$  obtained by shuffling  $\alpha$  and  $\beta$  (we shall denote this by  $\gamma \in \alpha \cup \beta$ ), let  $S_\beta(\gamma)$  be the set of the positions of the entries of  $\beta$  in  $\gamma$ . Denote further <

*Proof (sketched).* Recall (12), and rewrite the right-hand side using (13). Let  $(\tau, \gamma)$  be a coshuffle in  $(id_n, \alpha) \sqcup (id_m, \beta)$ . The index  $i$  belongs to  $\text{Peak}(\tau)$  if and only if  $\tau_i$  is a letter of  $n + id_m$  and  $\tau_{i+1}$  is a letter of  $id_n$  and we have  $i > 1$ . That is, if and only if  $i$  is the index of an entry of  $\beta$  in  $\gamma$  and  $i + 1$  is the index of an entry of  $\alpha$ . Thus,  $\text{Peak}(\tau) = (S_\beta(\gamma) \setminus (S_\beta(\gamma) - 1)) \setminus \{1\}$ . Therefore, we obtain (14).  $\square$

## References

- [1] D. Grinberg. “Double posets and the antipode of QSym”. In: *The Electronic Journal of Combinatorics* 24.2 (2017), P22.
- [2] D. Grinberg. “Shuffle-compatible permutation statistics II: the exterior peak set”. In: *Electronic Journal of Combinatorics* 25.4 (2018), P4.17.
- [3] D. Grinberg and V. Reiner. *Hopf Algebras in Combinatorics*. arXiv:1409.8356v7. 2020. URL: <http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf>.
- [4] S. K. Hsiao. “Structure of the peak Hopf algebra of quasisymmetric functions”. 2007.
- [5] J. Stembridge. “Enriched  $P$ -partitions.” In: *Trans. Amer. Math. Soc.* 349.2 (1997), pp. 763–788.