Refined dual stable Grothendieck polynomials and generalized Bender-Knuth involutions

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1. Introduction

Thomas Lam and Pavlo Pylyavskyy, in [LamPyl07, §9.1], (and earlier Mark Shimozono and Mike Zabrocki in unpublished work of 2003) studied *dual stable Grothendieck polynomials*, a deformation (in a sense) of the Schur functions. Let us briefly recount their definition.

Let λ/μ be a skew partition. The Schur function $s_{\lambda/\mu}$ is a multivariate generating function for the semistandard tableaux of shape λ/μ. In the same vein, the dual stable Grothendieck polynomial $g_{\lambda/\mu}$ is a generating function for the reverse plane partitions of shape λ/μ; these, unlike semistandard tableaux, are only required to have their entries increase *weakly* down columns (and along rows). More precisely, $g_{\lambda/\mu}$ is a formal power series in countably many commuting indeterminates $x_1, x_2, x_3, \ldots$ defined by

$$g_{\lambda/\mu} = \sum_{\substack{T \text{ is a reverse plane} \\ \text{partition of shape } \lambda/\mu}} \mathbf{x}^{\operatorname{ircont}(T)},$$

where $\mathbf{x}^{\operatorname{ircont}(T)}$ is the monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ whose i-th exponent a_i is the number of columns of T containing the entry i. As proven in [LamPyl07, §9.1], this power series $g_{\lambda/\mu}$ is a symmetric function (albeit, unlike $s_{\lambda/\mu}$, an inhomogeneous one in general). Lam and Pylyavskyy connect the $g_{\lambda/\mu}$ to the (more familiar) *stable Grothendieck polynomials* $G_{\lambda/\mu}$ (via a duality between the symmetric functions and their completion, which explains the name of the $g_{\lambda/\mu}$; see [LamPyl07, §9.4]) and to the K-theory of Grassmannians ([LamPyl07, §9.5]).

We devise a common generalization of the dual stable Grothendieck polynomial $g_{\lambda/\mu}$ and the classical skew Schur function $s_{\lambda/\mu}$. Namely, if $t_1, t_2, t_3, \ldots$ are countably many indeterminates, then we set

$$\widetilde{g}_{\lambda/\mu} = \sum_{\substack{T \text{ is a reverse plane} \\ \text{partition of shape } \lambda/\mu}} \mathbf{t}^{\operatorname{ceq}(T)} \mathbf{x}^{\operatorname{ircont}(T)},$$

where $\mathbf{t}^{\operatorname{ceq}(T)}$ is the product $t_1^{b_1} t_2^{b_2} t_3^{b_3} \cdots$ whose i-th exponent b_i is the number of cells in the i-th row of T whose entry equals the entry of their neighbor cell directly below them. This $\widetilde{g}_{\lambda/\mu}$ becomes $g_{\lambda/\mu}$ when all the t_i are set to 1, and becomes $s_{\lambda/\mu}$ when all the t_i are set to 0.

Our main result, Theorem 3.4, states that $\widetilde{g}_{\lambda/\mu}$ is a symmetric function (in the $x_1, x_2, x_3, \ldots$).

We prove this result (thus obtaining a new proof of [LamPyl07, Theorem 9.1]) first using an elaborate generalization of the classical Bender-Knuth involutions to reverse plane partitions.

TODO 1.1. Pavel: advertise your section here? It sort-of gives some justification for why the t_i are a natural thing to consider too.

The present paper is organized as follows: In Section 2, we recall classical definitions and introduce notations pertaining to combinatorics and symmetric functions. In Section 3, we define the dual stable Grothendieck polynomials $\widetilde{g}_{\lambda/\mu}$, state our main result (that they are symmetric functions), and do the first steps of its proof (by reducing it to a purely combinatorial statement about the existence of an involution with certain properties). In Section 4, we prove our main result by constructing the required involution. In Section 5, we recapitulate the definition of the classical Bender-Knuth involution, and sketch the proof that our involution is a generalization of the latter.

TODO 1.2. Pavel: your last section should also be summarized here.

TODO 1.3. Rewrite this after the shortening process is finished.

TODO 1.4. What is missing from the above introduction? Specifically:

- Are there any more results from the paper that should be advertised here?
- More reasons why care about dual stable Grothendieck polynomials? (Buch, Knutson, Postnikov could know some.)
- What I wrote about K-theory is rather shallow. More details?

  More specifically, and interestingly, I am wondering if our $\widetilde{g}_{\lambda/\mu}$ aren't K-theoretical classes of something multigraded (toric structure on the Grassmannian? there are two sides from which we can multiply a matrix by a diagonal matrix, and even if we "use up" one for taking "characters", the other one is still there).

1.1. Acknowledgments

We owe our familiarity with dual stable Grothendieck polynomials to Richard Stanley. We thank Alexander Postnikov for providing context and motivation.

TODO 1.5. Keep this up to date.

2. Notations and definitions

Let us begin by defining our notations (including some standard conventions from algebraic combinatorics).

2.1. Partitions and tableaux

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$.

A sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ of nonnegative integers is called a *weak composition* if the sum of its entries (denoted $|\alpha|$) is finite. We shall always write α_i for the i-th entry of a weak composition α.

A *partition* is a weak composition $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ satisfying $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$. As usual, we often omit trailing zeroes when writing a partition (e.g., the partition $(5, 2, 1, 0, 0, 0, \ldots)$ can thus be written as $(5, 2, 1)$).

TODO 2.1. Come up with a better notation for ircont, stagnant.

We identify each partition λ with the subset $\left\{(i, j) \in \mathbb{N}_+^2 \mid j \leq \lambda_i\right\}$ of $\mathbb{N}_+^2$ (called *the Young diagram of* λ). We draw this subset as a Young diagram (which is a left-aligned table of empty boxes, where the box $(1, 1)$ is in the top-left corner while the box $(2, 1)$ is direcly below it; this is the *English notation*, also known as the *matrix notation*).

TODO 2.2. What is the easiest place to refer the reader to for Young diagram basics, which uses notations compatible with ours (such as "filling" and "skew partition")?

A *skew partition* λ/μ is a pair of partitions satisfying $\mu \subset \lambda$ (as subsets of the plane). In this case, we shall also often use the notation λ/μ for the set-theoretic difference of λ and μ.

If λ/μ is a skew partition, then a *filling* of λ/μ means a map $T : \lambda/\mu \to \mathbb{N}_+$. It is visually represented by drawing λ/μ and filling each box c with the entry $T(c)$. Three examples of a filling can be found on Figure 1.

A filling $T : \lambda/\mu \to \mathbb{N}_+$ of λ/μ is called a *reverse plane partition of shape* λ/μ if its values increase weakly in each row of λ/μ from left to right and in each column of λ/μ from top to bottom. If, in addition, the values of T increase strictly down each column, then T is called a *semistandard tableau of shape* λ/μ.

$$\begin{array}{c|c|c|c}
\begin{array}{|c|c|c|}\hline \cdot & 6 & 3 \\ \hline 2 & 4 & \\ \hline 3 & 4 & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline \cdot & 3 & 3 \\ \hline 2 & 3 & \\ \hline 3 & 4 & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline \cdot & 3 & 3 \\ \hline 2 & 4 & \\ \hline 3 & 7 & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline \cdot & 1 & 2 \\ \hline 1 & 1 & \\ \hline 1 & 2 & \\ \hline \end{array} \\
\text{(a)} & \text{(b)} & \text{(c)} & \text{(d)}
\end{array}$$

Figure 1: Fillings of $(3,2,2)/(1)$: (a) is not an rpp as it has a 4 below a 6, (b) is an rpp but not a semistandard tableau as it has a 3 below a 3, (c) is a semistandard tableau (and hence also an rpp), (d) is a 12-rpp.

(See [Fulton97] for an exposition of properties and applications of semistandard tableaux.) We denote the set of all reverse plane partitions of shape λ/μ by $\operatorname{RPP}(\lambda/\mu)$. We abbreviate reverse plane partitions as *rpps*. We define a *12-rpp* to be an rpp whose entries all belong to the set $\{1,2\}$ and denote the set of all 12-rpps of shape λ/μ by $\operatorname{RPP}^{12}(\lambda/\mu)$.

TODO 2.3. The notion of a 12-rpp is technical, at least in my part of the paper. Should we really introduce it here in this section?

Examples of an rpp, a 12-rpp and of a semistandard tableau can be found on Figure 1.

2.2. Symmetric functions

TODO 2.4. Decide whether we want to work over $\mathbb{Z}$ or over an arbitrary commutative field $\mathbf{k}$ with unity.

A *symmetric function* is defined to be a bounded-degree power series in $x_1, x_2, \ldots$ that is invariant under (finite) permutations of $x_1, x_2, \ldots$.

TODO 2.5. Define bounded-degree or refer to some place (e.g., GriRei15)? Make ground ring explicit (even if it is $\mathbb{Z}$.) Define finite permutations.

The symmetric functions form a ring, which is called the *ring of symmetric functions* and denoted by Λ. (In [LamPyl07] this ring is denoted by Sym, while the notation Λ is reserved for the set of all partitions.) Symmetric functions are a classical field of research, and are closely related to Young diagrams and tableaux; see [Stan99, Chapter 7], [Macdon95] and [GriRei15, Chapter 2] for expositions.

TODO 2.6. Update Grinberg-Reiner reference to whatever is most recent.

Given a filling T of a skew partition λ/μ, its *content* is a weak composition $\operatorname{cont}(T) = (r_1, r_2, \dots)$ where $r_i = \left|T^{-1}(i)\right|$ is the number of entries of T equal to i. For a skew partition λ/μ, we define the *Schur function* $s_{\lambda/\mu}$ to be the formal power series

$$s_{\lambda/\mu}(x_1, x_2, \dots) = \sum_{\substack{T \text{ is a semistandard} \\ \text{tableau of shape } Y(\lambda/\mu)}} \mathbf{x}^{\operatorname{cont}(T)}.$$

A nontrivial property of these Schur functions is that they are symmetric:

Proposition 2.7. We have $s_{\lambda/\mu} \in \Lambda$ for every skew partition λ/μ.

This result appears, e.g., in [Stan99, Theorem 7.10.2] and [GriRei15, Proposition 2.11]; it is commonly proven bijectively using the so-called *Bender-Knuth involutions*. We shall recall the definitions of these involutions in Section 5.

Replacing "semistandard tableau" by "rpp" in the definition of a Schur function in general gives a non-symmetric function. Nevertheless, Lam and Pylyavskyy [LamPyl07, §9] have noticed that it is possible to define symmetric functions from rpps, albeit it requires replacing the content $\operatorname{cont}(T)$ by a subtler construction.

Namely, for a filling T of a skew partition λ/μ, we define its *irredundant content* as a weak composition $\operatorname{ircont}(T) = (r_1, r_2, \dots)$ where r_i is the number of *columns* (rather than cells) of T that contain an entry equal to i. For instance, if T_a, T_b, T_c and T_d are the fillings from Figure 1, then their irredundant contents are given by

$$\operatorname{ircont}(T_a) = (0,1,2,1,0,1), \qquad \operatorname{ircont}(T_b) = (0,1,3,1),$$
$$\operatorname{ircont}(T_c) = (0,1,3,1,0,0,1), \qquad \operatorname{ircont}(T_d) = (2,2),$$

because, for example, T_a has one column with a 4 in it (so $(\operatorname{ircont} T_a)_4 = 1$) and T_b contains three columns with a 3 (so $(\operatorname{ircont} T_b)_3 = 3$).

Notice that if T is a semistandard tableau, then $\operatorname{cont}(T)$ and $\operatorname{ircont}(T)$ coincide.

For the rest of this section, we fix a skew partition λ/μ. Now, the *dual stable Grothendieck polynomial* $g_{\lambda/\mu}$ is defined to be the formal power series

$$\sum_{\substack{T \text{ is an rpp} \\ \text{of shape } Y(\lambda/\mu)}} \mathbf{x}^{\operatorname{ircont}(T)}.$$

Unlike the Schur function $s_{\lambda/\mu}$, it is (in general) not homogeneous, because whenever a column of an rpp T contains an entry several times, the corresponding monomial $\mathbf{x}^{\operatorname{ircont}(T)}$ "counts" this entry only once. It is fairly clear that the highest-degree homogeneous component of $g_{\lambda/\mu}$ is $s_{\lambda/\mu}$ (the component of degree $|\lambda| - |\mu|$). Therefore, $g_{\lambda/\mu}$ can be regarded as an inhomogeneous deformation of the Schur function $s_{\lambda/\mu}$.

Lam and Pylyavskyy, in [LamPyl07, §9.1], have shown the following fact:

Proposition 2.8. We have $g_{\lambda/\mu} \in \Lambda$ for every skew partition λ/μ.

They prove this proposition using generalized plactic algebras [FomGre06, Lemma 3.1] (and also give a second, combinatorial proof for the case $\mu = \varnothing$ by explicitly expanding $g_{\lambda/\varnothing}$ as a sum of Schur functions).

In the next section, we shall introduce a refinement of these $g_{\lambda/\mu}$, and later we will reprove Proposition 2.8 in a bijective and elementary way.

3. Refined dual stable Grothendieck polynomials

3.1. Definition

Let $\mathbf{t} = (t_1, t_2, t_3, \dots)$ be a sequence of indeterminates. For any weak composition α, we define $\mathbf{t}^\alpha$ to be the monomial $t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \cdots$.

If T is a filling of a skew partition λ/μ, then a *stagnant cell* of T is a cell of λ/μ whose entry is equal to the entry directly below it. That is, a cell (i,j) of λ/μ is stagnant if $(i+1,j)$ is also a cell of λ/μ and $T(i,j) = T(i+1,j)$. Notice that a semistandard tableau is the same thing as an rpp which has no stagnant cells.

If T is a filling of λ/μ, then we define the *column equalities vector* of T to be a weak composition $\operatorname{ceq}(T) = (c_1, c_2, \dots)$ where c_i is the number number of $j \in \mathbb{N}_+$ such that (i,j) is a stagnant cell of T. Visually speaking, $(\operatorname{ceq}(T))_i$ is the number of columns of T whose entry in the i-th row equals their entry in the $(i+1)$-th row. For instance, for fillings T_a, T_b, T_c, T_d from Figure 1 we have $\operatorname{ceq}(T_a) = (0,1)$, $\operatorname{ceq}(T_b) = (1,0)$, $\operatorname{ceq}(T_c) = (0)$ and $\operatorname{ceq}(T_d) = (1,1)$.

Notice that $|\operatorname{ceq}(T)|$ is the number of stagnant cells in T, so we have

$$|\operatorname{ceq}(T)| + |\operatorname{ircont}(T)| = |\lambda/\mu| \tag{1}$$

for all rpps T of shape λ/μ.

Let now λ/μ be a skew partition. We set

$$\widetilde{g}_{\lambda/\mu}(\mathbf{x};\mathbf{t}) = \sum_{\substack{T \text{ is an rpp} \\ \text{of shape } \lambda/\mu}} \mathbf{t}^{\operatorname{ceq}(T)} \mathbf{x}^{\operatorname{ircont}(T)}.$$

Let us give some examples of $\widetilde{g}_{\lambda/\mu}$.

Example 3.1. (a)

If λ/μ is a single row with n cells, then for each rpp T of shape λ/μ we have $\operatorname{ceq}(T) = (0,0,\dots)$ and $\operatorname{ircont}(T) = \operatorname{cont}(T)$ (in fact, any rpp of shape λ/μ is a semistandard tableau in this case). Therefore we get

$$\widetilde{g}_{\lambda/\mu}(\mathbf{x};\mathbf{t}) = h_n(\mathbf{x}) = \sum_{a_1 \le a_2 \le \cdots \le a_n} x_{a_1} x_{a_2} \dots x_{a_n}.$$

Here $h_n(\mathbf{x})$ is the n-th complete homogeneous symmetric function.

(b) If λ/μ is a single column with n cells then by (1) for all rpps T of shape λ/μ we have $|\text{ceq}(T)| + |\text{ircont}(T)| = n$ so in this case

$$\widetilde{g}_{\lambda/\mu}(\mathbf{x};\mathbf{t}) = e_n(t_1, t_2, \ldots, t_{n-1}, x_1, x_2, \ldots) = \sum_{k=0}^{n} e_k(t_1, t_2, \ldots, t_{n-1})\, e_{n-k}(x_1, x_2, \ldots),$$

where $e_i(\xi_1, \xi_2, \xi_3, \ldots)$ denotes the i-th elementary symmetric function in the indeterminates $\xi_1, \xi_2, \xi_3, \ldots$.

TODO 3.2. Do we really need the third example here? (the one with $\lambda = (2,1)$) DG: I expected both of you to care more about these examples than I did.

The power series $\widetilde{g}_{\lambda/\mu}$ generalize the power series $g_{\lambda/\mu}$ and $s_{\lambda/\mu}$ studied before. The following proposition is clear:

Proposition 3.3. Let λ/μ be a skew partition.
(a) Specifying $\mathbf{t} = (1, 1, 1, \ldots)$ yields $\widetilde{g}_{\lambda/\mu}(\mathbf{x};\mathbf{t}) = g_{\lambda/\mu}(\mathbf{x})$.
(b) Specifying $\mathbf{t} = (0, 0, 0, \ldots)$ yields $\widetilde{g}_{\lambda/\mu}(\mathbf{x};\mathbf{t}) = s_{\lambda/\mu}(\mathbf{x})$.

$\square$

3.2. The symmetry statement

Our main result is now the following:

Theorem 3.4. Let λ/μ be a skew partition. Then $\widetilde{g}_{\lambda/\mu}(\mathbf{x},\mathbf{t})$ is symmetric in $\mathbf{x}$.

Clearly, this implies the symmetry of $g_{\lambda/\mu}$ and $s_{\lambda/\mu}$ due to Proposition 3.3.

We shall prove Theorem 3.4 bijectively. The core of our proof will be the following restatement of Theorem 3.4:

Theorem 3.5. Let λ/μ be a skew partition and let $i \in \mathbb{N}_+$. Then, there exists an involution $\mathbf{B}_i : \text{RPP}(\lambda/\mu) \to \text{RPP}(\lambda/\mu)$ which preserves the ceq statistics and acts on ircont by the transposition of its i-th and $i+1$-th entries.

This involution $\mathbf{B}_i$ is a generalization of the i-th Bender-Knuth involution defined for semistandard tableaux (see, e.g., [GriRei15, proof of Proposition 2.11]), but its definition is more complicated than that of the latter.[1] Defining it and proving its properties will take a significant part of this paper.

[1]We will compare our involution $\mathbf{B}_i$ with the i-th Bender-Knuth involution in Section 5.

3.3. Reduction to 12-rpps

We shall make one further simplification before we step to the actual proof of Theorem 3.5.

Lemma 3.6. There exists an involution $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ (defined canonically in terms of λ/μ) which preserves the ceq statistics and switches the number of columns containing a 1 with the number of columns containing a 2 (i.e. acts as a transposition of 1 and 2 on ircont).

This Lemma implies Theorem 3.5: for any $i \in \mathbb{N}_+$ and for T an rpp of shape λ/μ, we construct $\mathbf{B}_i(T)$ as follows:

- Ignore all entries of T not equal to i or $i+1$.
- Replace all occurences of i by 1 and all occurences of $i+1$ by 2. We get a 12-rpp T' of some smaller shape.
- Replace T' by $\mathbf{B}(T')$.
- In T', replace back all occurences of 1 by i and all occurences of 2 by $i+1$.
- Finally, restore the remaining entries of T that were ignored on the first step.

It is clear that this operation acts on $\mathrm{ircont}(T)$ by a transposition of the i-th and $i+1$-th entries. The fact that it does not change $\mathrm{ceq}(T)$ is also not hard to show: the number column equalities with both entries equal to i or with both entries equal to $i+1$ in each row does not change by the properties of $\mathbf{B}$ while the number of column equalities with both entries equal to something other than i or $i+1$ in each row does not change because the values in the corresponding cells remain unchanged.

4. Proof of Lemma 3.6

We now come to the actual proof of Lemma 3.6. For the whole Section 4, we shall be working in the situation of Lemma 3.6.

4.1. 12-tables and the four types of their columns

Let Z be a finite convex subset of $\mathbb{N}_+^2$. We shall keep Z fixed for the rest of Section 4. Let $\mathbf{R}$ denote the set of all 12-rpps of shape Z.

A *12-table* will mean a map $T : Z \to \{1,2\}$ such that the entries of T are weakly increasing down columns. (We do not require them to be weakly increasing along rows.) Every column of a 12-table is a sequence of the form $(1,1,\ldots,1,2,2,\ldots,2)$. We say that such a sequence is

- *1-pure* if it is nonempty and consists purely of 1's,
- *2-pure* if it is nonempty and consists purely of 2's,
- *mixed* if it contains both 1's and 2's.

If s is a sequence of the form $(1, 1, \ldots, 1, 2, 2, \ldots, 2)$, then we define the *signature* $\operatorname{sig}(s)$ of s to be

$$\operatorname{sig}(s) = \begin{cases} 0, & \text{if } s \text{ is 2-pure or empty;} \\ 1, & \text{if } s \text{ is mixed;} \\ 2, & \text{if } s \text{ is 1-pure} \end{cases}$$

For any 12-table T, we define a nonnegative integer $\ell(T)$ by

$$\ell(T) = \sum_{h \in \mathbb{N}_+} h \cdot \operatorname{sig}(\text{the } h\text{-th column of } T).$$

For instance, if T is the 12-table

$$\begin{array}{|c|c|c|c|c|} \hline & & 1 & 2 & 1 & 2 \\ \hline & 1 & 1 & 2 & & \\ \hline 2 & 1 & 1 & 2 & & \\ \hline 2 & 2 & & & & \\ \hline \end{array}$$

then $\ell(T) = 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 0 + 5 \cdot 2 + 6 \cdot 0 + 7 \cdot 0 + 8 \cdot 0 + \cdots = 18$.

4.2. Descents, separators, and benign 12-tables

If T is a 12-table, then we define a *descent* of T to be a positive integer i such that there exists an $r \in \mathbb{N}_+$ satisfying $(r, i) \in Z$, $(r, i+1) \in Z$, $T(r, i) = 2$ and $T(r, i+1) = 1$. For instance, the descents of the 12-table

$$\begin{array}{|c|c|c|c|c|} \hline & & 1 & 2 & 1 & 2 \\ \hline & 1 & 1 & 2 & & \\ \hline 2 & 1 & 1 & 2 & & \\ \hline 2 & 2 & & & & \\ \hline \end{array}$$

are 1 and 4. Clearly, a 12-rpp of shape Z is the same as a 12-table which has no descents.

If T is a 12-table, and if $k \in \mathbb{N}_+$ is such that the k-th column of T is mixed, then we define $\operatorname{sep}_k T$ to be the smallest $r \in \mathbb{N}_+$ such that $(r, k) \in Z$ and $T(r, k) = 2$. Thus, every 12-table T, every $r \in \mathbb{N}_+$ and every $k \in \mathbb{N}_+$ such that the k-th column of T is mixed and such that $(r, k) \in Z$ satisfy

$$T(r, k) = \begin{cases} 1, & \text{if } r < \operatorname{sep}_k T; \\ 2, & \text{if } r \geq \operatorname{sep}_k T. \end{cases} \tag{2}$$

If T is a 12-table, then we let $\operatorname{seplist} T$ denote the list of all values $\operatorname{sep}_k T$ (in the order of increasing k), where k ranges over all positive integers for which the k-th column of T is mixed. For instance, if T is

		1	1	1
	2	1	1	2
1	2	1		
2	2	2		

then $\operatorname{sep}_1 T = 4$, $\operatorname{sep}_3 T = 4$, and $\operatorname{sep}_5 T = 2$ (and there are no other k for which $\operatorname{sep}_k T$ is defined), so that $\operatorname{seplist} T = (4, 4, 2)$.

We say that a 12-table T is *benign* if the list $\operatorname{seplist} T$ is weakly decreasing. Notice that 12-rpps are benign 12-tables, but the converse is not true. If T is a benign 12-table, then

$$\text{there exists no descent } k \text{ of } T \text{ such that both the } k\text{-th column of } T \text{ and the } (k+1)\text{-th column of } T \text{ are mixed.} \tag{3}$$

Let $\mathbf{S}$ denote the set of all benign 12-tables; we have $\mathbf{R} \subseteq \mathbf{S}$.

4.3. The flip map on benign 12-tables

We define a map $\operatorname{flip} : \mathbf{S} \to \mathbf{S}$ as follows: Let $T \in \mathbf{S}$. For every nonempty column of T, we transform the column as follows:

- If the column is 1-pure, we replace all its entries by 2's.
- If the column is 2-pure, we replace all its entries by 1's.
- Otherwise we do not change it.

Once these transformations are made, the resulting filling of Z is a 12-table which is still benign. We define $\operatorname{flip}(T)$ to be this resulting 12-table. Thus, the map $\operatorname{flip} : \mathbf{S} \to \mathbf{S}$ is defined.

For example, if T is

	1	1	2	1
1	1	1		
1	2	1		
1	2			
2				

then $\operatorname{flip}(T)$ is

$$\begin{array}{|c|c|c|c|c|}
\cline{2-5}
\multicolumn{1}{c|}{} & 1 & 2 & 1 & 2 \\
\hline
1 & 1 & 2 & \multicolumn{2}{c}{} \\
\cline{1-3}
1 & 2 & 2 & \multicolumn{2}{c}{} \\
\cline{1-3}
1 & 2 & \multicolumn{3}{c}{} \\
\cline{1-2}
2 & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array}$$

The following proposition gathers some easy properties of flip:

Proposition 4.1. **(a)** We have $\operatorname{flip} \circ \operatorname{flip} = \operatorname{id}$.

(b) Let T be a benign 12-table. When T is transformed into $\operatorname{flip}(T)$, the 1-pure columns of T become 2-pure columns of $\operatorname{flip}(T)$, and the 2-pure columns of T become 1-pure columns of $\operatorname{flip}(T)$, while the mixed columns and the empty columns do not change.

(c) For every benign 12-table T, we have

$$\operatorname{ceq}(\operatorname{flip}(T)) = \operatorname{ceq}(T) \tag{4}$$

and

$$\operatorname{ircont}(\operatorname{flip}(T)) = s_1 \cdot \operatorname{ircont}(T). \tag{5}$$

Proof of Proposition 4.1. All of Proposition 4.1 is straightforward to prove. $\square$

We notice that, when the map flip acts on a benign 12-table T, it transforms every column of T independently. Thus, we have the following:

Remark 4.2. If P and Q are two benign 12-tables, and if $i \in \mathbb{N}_+$ is such that

$$(\text{the } i\text{-th column of } P) = (\text{the } i\text{-th column of } Q),$$

then

$$(\text{the } i\text{-th column of } \operatorname{flip}(P)) = (\text{the } i\text{-th column of } \operatorname{flip}(Q)).$$

4.4. Plan of the proof

Let us now briefly sketch the ideas behind the rest of the proof before we go into them in detail. The map $\operatorname{flip} : \mathbf{S} \to \mathbf{S}$ does not generally send 12-rpps to 12-rpps (i.e., it does not restrict to a map $\mathbf{R} \to \mathbf{R}$). However, we shall amend this by defining a way to transform any benign 12-table into a 12-rpp by what we call "resolving descents". The process of "resolving descents" will be a stepwise process, and will be formalized in terms of a binary relation $\Rrightarrow$ on the set $\mathbf{S}$ which we will soon introduce. The intuition behind saying "$P \Rrightarrow Q$" is that the benign 12-table P has a descent, resolving which yields the benign 12-table Q. Starting with a benign 12-table P, we can repeatedly resolve descents until this is no longer possible. We have some freedom in performing this process, because at any step there can be a choice of several descents to resolve; but we will see that the final result does not depend on the process. Hence, the final result can be regarded as a function of P. We will denote it by $\operatorname{norm} P$, and we will see that it is a 12-rpp. We will then define a map $\mathbf{B} : \mathbf{R} \to \mathbf{R}$ by $\mathbf{B}(T) = \operatorname{norm}(\operatorname{flip} T)$, and show that it is an involution satisfying the properties that we want it to satisfy.

4.5. Resolving descents

Now we come to the details.

Let $k \in \mathbb{N}_+$. Let $P \in \mathbf{S}$. Assume (for the whole Subsection 4.5) that k is a descent of P. Thus, the k-th column of P must contain at least one 2. Hence, the k-th column of P is either mixed or 2-pure. Similarly, the $(k+1)$-th column of P is either mixed or 1-pure. But the k-th and the $(k+1)$-th columns of P cannot both be mixed because P is benign. Thus we introduce the following notations:

- We say that the 12-table P has *k-type M1* if the k-th column of P is mixed and the $(k+1)$-th column of P is 1-pure.
- We say that the 12-table P has *k-type 2M* if the k-th column of P is 2-pure and the $(k+1)$-th column of P is mixed.
- We say that the 12-table P has *k-type 21* if the k-th column of P is 2-pure and the $(k+1)$-th column of P is 1-pure.

Now, we define a new 12-table $\operatorname{res}_k P$ as follows:

- If P has k-type M1, then we let $\operatorname{res}_k P$ be the 12-table defined as follows: The k-th column of $\operatorname{res}_k P$ is 1-pure; the $(k+1)$-th column of $\operatorname{res}_k P$ is mixed and satisfies $\operatorname{sep}_{k+1}(\operatorname{res}_k P) = \operatorname{sep}_k P$; all other columns of $\operatorname{res}_k P$ are copied over from P unchanged.[2]
- If P has k-type 2M, then we let $\operatorname{res}_k P$ be the 12-table defined as follows: The k-th column of $\operatorname{res}_k P$ is mixed and satisfies $\operatorname{sep}_k(\operatorname{res}_k P) = \operatorname{sep}_{k+1} P$; the $(k+1)$-th column of $\operatorname{res}_k P$ is 2-pure (i.e., it is filled with 2's); all other columns of $\operatorname{res}_k P$ are copied over from P unchanged.
- If P has k-type 21, then we let $\operatorname{res}_k P$ be the 12-table defined as follows: The k-th column of $\operatorname{res}_k P$ is 1-pure; the $(k+1)$-th column of $\operatorname{res}_k P$ is 2-pure; all other columns of $\operatorname{res}_k P$ are copied over from P unchanged.

[2]The reader should check that this definition is well-defined.

In either case, $\operatorname{res}_k P$ is a well-defined 12-table. It is furthermore clear that $\operatorname{seplist}(\operatorname{res}_k P) = \operatorname{seplist} P$. Thus, $\operatorname{res}_k P$ is benign (since P is benign); that is, $\operatorname{res}_k P \in \mathbf{S}$. We say that $\operatorname{res}_k P$ is the 12-table obtained by *resolving* the descent k in P. Let us give some examples:

Example 4.3. Let P be

$$\begin{array}{ccccc} & & 1 & 2 & 1 \\ & 1 & 1 & 2 & \\ 2 & 1 & 1 & & \\ 2 & 2 & 1 & & \\ 2 & & & & \end{array}$$

Then P is a benign 12-table, and its descents are $1, 2$ and 4. We have $\operatorname{sep}_2 P = 4$.

If we set $k = 1$, then P has k-type 2M, and resolving the descent $k = 1$ gives us the 12-table $\operatorname{res}_1 P$:

$$\begin{array}{ccccc} & & 1 & 2 & 1 \\ & 2 & 1 & 2 & \\ 1 & 2 & 1 & & \\ 2 & 2 & 1 & & \\ 2 & & & & \end{array}$$

If we instead set $k = 2$, then P has k-type M1, and resolving the descent $k = 2$ gives us the 12-table $\operatorname{res}_2 P$:

$$\begin{array}{ccccc} & & 1 & 2 & 1 \\ & 1 & 1 & 2 & \\ 2 & 1 & 1 & & \\ 2 & 1 & 2 & & \\ 2 & & & & \end{array}$$

If we instead set $k = 4$, then P has k-type 21, and resolving the descent $k = 4$ gives us the 12-table $\operatorname{res}_4 P$:

$$\begin{array}{ccccc} & & 1 & 1 & 2 \\ & 1 & 1 & 1 & \\ 2 & 1 & 1 & & \\ 2 & 2 & 1 & & \\ 2 & & & & \end{array}$$

We notice that each of the three 12-tables $\operatorname{res}_1 P$, $\operatorname{res}_2 P$ and $\operatorname{res}_4 P$ still has descents. In order to get a 12-rpp from P, we will have to keep resolving these descents until none remain.

We now observe some further properties of $\operatorname{res}_k P$:

Proposition 4.4. Let $P \in \mathbf{S}$ and $k \in \mathbb{N}_+$ be such that k is a descent of P.

(a) The 12-table $\operatorname{res}_k P$ differs from P only in columns k and $k+1$. In other words,

$$(\text{the } h\text{-th column of } \operatorname{res}_k P) = (\text{the } h\text{-th column of } P) \tag{6}$$

for every $h \in \mathbb{N}_+ \setminus \{k, k+1\}$.

(b) The k-th and the $(k+1)$-th columns of $\operatorname{res}_k P$ depend only on the k-th and the $(k+1)$-th columns of P. In other words, if Q is a further benign 12-table satisfying

$$(\text{the } h\text{-th column of } Q) = (\text{the } h\text{-th column of } P) \qquad \text{for each } h \in \{k, k+1\},$$

then k is a descent of Q and we have

$$(\text{the } h\text{-th column of } \operatorname{res}_k Q) = (\text{the } h\text{-th column of } \operatorname{res}_k P) \qquad \text{for each } h \in \{k, k+1\}. \tag{7}$$

(c) We have

$$\operatorname{ceq}(\operatorname{res}_k P) = \operatorname{ceq}(P). \tag{8}$$

(d) We have

$$\begin{aligned}&(\text{the number of } h \in \mathbb{N}_+ \text{ such that the } h\text{-th column of } \operatorname{res}_k P \text{ is mixed})\\ &= (\text{the number of } h \in \mathbb{N}_+ \text{ such that the } h\text{-th column of } P \text{ is mixed}),\end{aligned} \tag{9}$$

$$\begin{aligned}&(\text{the number of } h \in \mathbb{N}_+ \text{ such that the } h\text{-th column of } \operatorname{res}_k P \text{ is 1-pure})\\ &= (\text{the number of } h \in \mathbb{N}_+ \text{ such that the } h\text{-th column of } P \text{ is 1-pure}),\end{aligned} \tag{10}$$

$$\begin{aligned}&(\text{the number of } h \in \mathbb{N}_+ \text{ such that the } h\text{-th column of } \operatorname{res}_k P \text{ is 2-pure})\\ &= (\text{the number of } h \in \mathbb{N}_+ \text{ such that the } h\text{-th column of } P \text{ is 2-pure}),\end{aligned} \tag{11}$$

and

$$\operatorname{ircont}(\operatorname{res}_k P) = \operatorname{ircont}(P). \tag{12}$$

(e) For every $r \in \mathbb{N}_+$ and $i \in \mathbb{N}_+$ satisfying $(r, i) \in Z$ and $(r, s_k(i)) \in Z$, we have

$$P(r, i) = (\operatorname{res}_k P)(r, s_k(i)). \tag{13}$$

(g) The benign 12-tables $\operatorname{flip}(P)$ and $\operatorname{flip}(\operatorname{res}_k P)$ have the property that

$$\left(\begin{array}{c} k \text{ is a descent of } \operatorname{flip}(\operatorname{res}_k P), \\ \text{and we have } \operatorname{flip}(P) = \operatorname{res}_k(\operatorname{flip}(\operatorname{res}_k P)) \end{array}\right). \tag{14}$$

(h) Recall that we defined a nonnegative integer $\ell(T)$ for every 12-table T in Subsection 4.1. We have

$$\ell(P) > \ell(\operatorname{res}_k P). \tag{15}$$

Proof of Proposition 4.4. Most of Proposition 4.4 succumbs to straightforward arguments using the definitions of res_k and flip coupled with a thorough case analysis, with an occasional use of the convexity of Z and of the formula (2). Merely part **(c)** requires a bit more thinking.

(c) A cell (i,j) in Z will be called *good* if the cell $(i+1,j)$ also belongs to Z. Notice that every stagnant cell of P or of $\operatorname{res}_k P$ must be good.

In order to prove (8), we need to show that, for every $r \in \mathbb{N}_+$, the number of stagnant cells of P in row r equals the number of stagnant cells of $\operatorname{res}_k P$ in row r. Instead of comparing the numbers of stagnant cells, we can just as well compare the numbers of good cells that are not stagnant (because all stagnant cells are good, and because the total number of good cells clearly depends only on Z and not on the 12-table). So we need to show that, for every $r \in \mathbb{N}_+$, the number of good cells in row r that are not stagnant cells of P equals the number of good cells in row r that are not stagnant cells of $\operatorname{res}_k P$.

Fix $r \in \mathbb{N}_+$. The number of good cells in row r that are not stagnant cells of P is precisely the number of appearances of $r+1$ in the list $\operatorname{seplist} P$ (because the good cells that are not stagnant cells of P are precisely the cells of the form $(\operatorname{sep}_k P, k)$, where k is a positive integer such that the k-th column of P is mixed). Similarly, the number of good cells in row r that are not stagnant cells of $\operatorname{res}_k P$ is precisely the number of appearances of $r+1$ in the list $\operatorname{seplist}(\operatorname{res}_k P)$. These two numbers are equal, because $\operatorname{seplist}(\operatorname{res}_k P) = \operatorname{seplist} P$. As explained above, this completes the proof of (8). $\square$

4.6. The descent-resolution relation $\Rrightarrow$

Definition 4.5. Let us now define a binary relation $\Rrightarrow$ on the set $\mathbf{S}$ as follows: Let $P \in \mathbf{S}$ and $Q \in \mathbf{S}$. If $k \in \mathbb{N}_+$, then we write $P \underset{k}{\Rrightarrow} Q$ if and only if k is a descent of P and we have $Q = \operatorname{res}_k P$. (In other words, if $k \in \mathbb{N}_+$, then we write $P \underset{k}{\Rrightarrow} Q$ if and only if k is a descent of P and the 12-table Q is obtained from P by resolving this descent.) We write $P \Rrightarrow Q$ if and only if there exists a $k \in \mathbb{N}_+$ such that $P \underset{k}{\Rrightarrow} Q$. (In other words, we write $P \Rrightarrow Q$ if and only if the 12-table Q is obtained from P by resolving a descent.) Thus, the relation $\Rrightarrow$ is defined.

Some of what was shown above translates into properties of this relation $\Rrightarrow$:

Lemma 4.6. Let $P \in \mathbf{S}$ and $Q \in \mathbf{S}$ be such that $P \Rrightarrow Q$. Then:

(a) We have $\operatorname{ceq}(Q) = \operatorname{ceq}(P)$.

(b) We have $\operatorname{ircont}(Q) = \operatorname{ircont}(P)$.

(c) The benign 12-tables $\operatorname{flip}(P)$ and $\operatorname{flip}(Q)$ have the property that $\operatorname{flip}(Q) \Rrightarrow \operatorname{flip}(P)$.

(d) We have $\ell(P) > \ell(Q)$.

Proof of Lemma 4.6. We have $P \Rrightarrow Q$. In other words, there exists a $k \in \mathbb{N}_+$ such that $P \underset{k}{\Rrightarrow} Q$. Consider this k. We have $P \underset{k}{\Rrightarrow} Q$. In other words, k is a descent of P and we have $Q = \operatorname{res}_k P$.

(a) We have $\operatorname{ceq}\left(\underbrace{Q}_{=\operatorname{res}_k P}\right) = \operatorname{ceq}(\operatorname{res}_k P) = \operatorname{ceq}(P)$ (by (8)). This proves Lemma 4.6 **(a)**.

(b) This follows similarly from (12).

(c) From (14), we know that k is a descent of $\operatorname{flip}(\operatorname{res}_k P)$, and we have $\operatorname{flip}(P) = \operatorname{res}_k(\operatorname{flip}(\operatorname{res}_k P))$. In other words, $\operatorname{flip}(\operatorname{res}_k P) \underset{k}{\Rrightarrow} \operatorname{flip}(P)$. Thus, $\operatorname{flip}(\operatorname{res}_k P) \Rrightarrow \operatorname{flip}(P)$. In other words, $\operatorname{flip}(Q) \Rrightarrow \operatorname{flip}(P)$ (since $Q = \operatorname{res}_k P$). This proves Lemma 4.6 **(c)**.

(d) From (15), we have $\ell(P) > \ell\left(\underbrace{\operatorname{res}_k P}_{=Q}\right) = \ell(Q)$. This proves Lemma 4.6 **(d)**. □

We now define $\overset{*}{\Rrightarrow}$ to be the reflexive-and-transitive closure of the relation $\Rrightarrow$ [3]. This relation $\overset{*}{\Rrightarrow}$ is reflexive and transitive, and extends the relation $\Rrightarrow$. If $P \in \mathbf{S}$ and $Q \in \mathbf{S}$, then the relation "$P \overset{*}{\Rrightarrow} Q$" can be interpreted as "Q can be obtained from P by repeatedly resolving descents" (because $P \Rrightarrow Q$ holds if and only if Q is obtained from P by resolving a descent).

Lemma 4.7. Let $P \in \mathbf{S}$ and $Q \in \mathbf{S}$ be such that $P \overset{*}{\Rrightarrow} Q$. Then:

(a) We have $\operatorname{ceq}(Q) = \operatorname{ceq}(P)$.

(b) We have $\operatorname{ircont}(Q) = \operatorname{ircont}(P)$.

(c) The benign 12-tables $\operatorname{flip}(P)$ and $\operatorname{flip}(Q)$ have the property that $\operatorname{flip}(Q) \overset{*}{\Rrightarrow} \operatorname{flip}(P)$.

(d) We have $\ell(P) \geq \ell(Q)$.

Proof of Lemma 4.7. Recalling that $\overset{*}{\Rrightarrow}$ is the reflexive-and-transitive closure of the relation $\Rrightarrow$, we see that Lemma 4.7 follows by induction using Lemma 4.6. □

In Subsection 4.1, we defined a nonnegative integer $\ell(T)$ for every 12-table T. In particular, $\ell(T)$ is defined for every $T \in \mathbf{S}$. We thus have a map $\ell : \mathbf{S} \to \mathbb{N}$ which sends every $T \in \mathbf{S}$ to $\ell(T)$.

We now state the following crucial lemma:

[3] Explicitly, this means that $\overset{*}{\Rrightarrow}$ is defined as follows: For two elements $P \in \mathbf{S}$ and $Q \in \mathbf{S}$, we have $P \overset{*}{\Rrightarrow} Q$ if and only if there exists a sequence $(a_0, a_1, \ldots, a_n)$ of elements of $\mathbf{S}$ such that $a_0 = P$ and $a_n = Q$ and such that every $i \in \{0, 1, \ldots, n-1\}$ satisfies $a_i \Rrightarrow a_{i+1}$.

Lemma 4.8. Let A, B and C be three elements of $\mathbf{S}$ satisfying $A \Rrightarrow B$ and $A \Rrightarrow C$. Then, there exists a $D \in \mathbf{S}$ such that $B \overset{*}{\Rrightarrow} D$ and $C \overset{*}{\Rrightarrow} D$.

Proof of Lemma 4.8. If $B = C$, then we can simply choose $D = B = C$ and be done with it; thus, we WLOG assume that $B \neq C$.

We have $A \Rrightarrow B$. In other words, there exists a $k \in \mathbb{N}_+$ such that $A \underset{k}{\Rrightarrow} B$. Let us denote this k by u. Thus, $A \underset{u}{\Rrightarrow} B$. In other words, u is a descent of A and we have $B = \operatorname{res}_u A$ (due to the definition of "$A \underset{u}{\Rrightarrow} B$"). Similarly, we can find a $v \in \mathbb{N}_+$ such that v is a descent of A and we have $C = \operatorname{res}_v A$. Consider this v as well.

We have $\operatorname{res}_u A = B \neq C = \operatorname{res}_v A$ and thus $u \neq v$. Hence, either $u < v$ or $u > v$. We WLOG assume that $u < v$ (since otherwise, we can simply switch u with v). Hence, we are in one of the following two Cases:

Case 1: We have $u = v - 1$.

Case 2: We have $u < v - 1$.

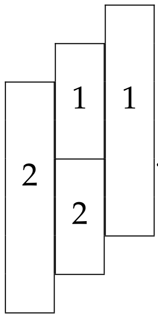
Let us deal with Case 2 first (since it is the simpler of the two). In this case, $u < v - 1$, so that $\{u, u+1\} \cap \{v, v+1\} = \varnothing$.

Now, the operation of resolving the descent u in A (that is, the passage from A to $\operatorname{res}_u A$) only affects the columns u and $u+1$, and thus it preserves the descent v (since $\{u, u+1\} \cap \{v, v+1\} = \varnothing$). Hence, $\operatorname{res}_v(\operatorname{res}_u A)$ is well-defined. Similarly, $\operatorname{res}_u(\operatorname{res}_v A)$ is well-defined.

Recall again that $\{u, u+1\} \cap \{v, v+1\} = \varnothing$. Thus, the operation of resolving the descent u and the operation of resolving the descent v "do not interact" (in the sense that the former only changes the columns u and $u+1$, and changes them in a way that does not depend on any of the other columns; and similarly for the latter). Therefore, the two operations can be applied one after the other in any order; the results will be the same. In other words, $\operatorname{res}_u(\operatorname{res}_v A) = \operatorname{res}_v(\operatorname{res}_u A)$. Now, set $D = \operatorname{res}_u(\operatorname{res}_v A) = \operatorname{res}_v(\operatorname{res}_u A)$. Then, $D = \operatorname{res}_u \underbrace{(\operatorname{res}_v A)}_{=C} = \operatorname{res}_u C$ and thus $C \underset{u}{\Rrightarrow} D$, so that $C \Rrightarrow D$, therefore $C \overset{*}{\Rrightarrow} D$. Similarly, $B \overset{*}{\Rrightarrow} D$. Hence, we have found a $D \in \mathbf{S}$ such that $B \overset{*}{\Rrightarrow} D$ and $C \overset{*}{\Rrightarrow} D$. This completes the proof of Lemma 4.8 in Case 2.

Now, let us consider Case 1. In this case, $u = v - 1$. Hence, $v - 1$ is a descent of A (since u is a descent of A), and we have $B = \operatorname{res}_u A = \operatorname{res}_{v-1} A$ (since $u = v - 1$).

The v-th column of A must contain a 1 (since $v - 1$ is a descent of A) and a 2 (since v is a descent of A). Hence, the v-th column of A is mixed. The $(v-1)$-th column of A is 2-pure[4], and the $(v+1)$-th column of A is 1-pure[5]. We can thus semiotically represent the 12-table A as follows:

$A =$  . (16)

In this representation, we only draw the $(v-1)$-th, the v-th and the $(v+1)$-th columns (since the remaining columns are neither used nor changed by res_{v-1} and res_v, and thus are irrelevant to our argument); we use a rectangle with a "1" inside to signify a string of 1's in a column[6], and we use a rectangle with a "2" inside to signify a string of 2's in a column.

Let $s = \mathrm{sep}_v A$. Then, (s, v) and $(s+1, v)$ belong to Z and satisfy $A(s, v) = 1$ and $A(s+1, v) = 2$ (by the definition of $\mathrm{sep}_v A$). Also, $(s, v-1)$ must belong to Z [7]. Hence, $(s+1, v-1)$ must belong to Z as well[8]. Similarly, $(s+1, v+1)$ and $(s, v+1)$ belong to Z. Altogether, we thus know that all six squares (s, v), $(s+1, v)$, $(s, v-1)$, $(s+1, v-1)$, $(s+1, v+1)$ and $(s, v+1)$ belong to Z. We shall denote these six squares as the "core squares". The restriction of A to the core squares is $\begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline 2 & 2 & 1 \\ \hline \end{array}$ [9].

Now, A has $(v-1)$-type 2M, and resolving the descent $v-1$ of A yields

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[4] *Proof.* Assume the contrary. Then, the $(v-1)$-th column of A contains a 2 (because $v-1$ is a descent of A) but is not 2-pure. Hence, this column is mixed. But A is benign. Hence, (3) (applied to $T = A$) yields that there exists no descent k of A such that both the k-th column of A and the $(k+1)$-th column of A are mixed. But $v-1$ is such a descent (since both the $(v-1)$-th and the v-th columns of A are mixed). This contradiction proves that our assumption was wrong, qed.

[5] This follows similarly.

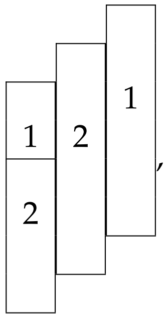
[6] The length of the rectangle is immaterial; it does not say anything about the number of 1's.

[7] *Proof.* We know that $v-1$ is a descent of A. Hence, there exists an $r \in \mathbb{N}_+$ such that $(r, v-1) \in Z$, $(r, v) \in Z$, $A(r, v-1) = 2$ and $A(r, v) = 1$. Consider this r. If we had $s+1 \leq r$, then we would have $A(s+1, v) \leq A(r, v)$ (since the entries of A are weakly decreasing down columns), which would contradict $A(r, v) = 1 < 2 = A(s+1, v)$. Therefore, we cannot have $s+1 \leq r$. Hence, $r < s+1$, so that $r \leq s$. Hence, (**??**) (applied to r, s, s, $v-1$, $v-1$ and v instead of i, i', i'', j, j' and j'') yields $(s, v-1) \in Z$, qed.

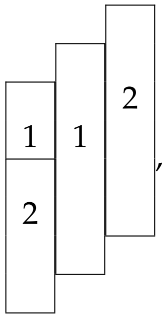
[8] by (**??**) (applied to s, $s+1$, $s+1$, $v-1$, $v-1$ and v instead of i, i', i'', j, j' and j'')

[9] Indeed, the two core squares in the v-th column have entries $A(s, v) = 1$ and $A(s+1, v) = 2$; the two core squares in the $(v-1)$-th column have entries 2 (since the $(v-1)$-th column of A is 2-pure); and the two core squares in the $(v+1)$-th column have entries 1 (since the $(v+1)$-th column of A is 1-pure).

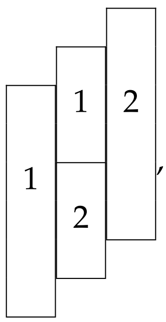
$\operatorname{res}_{v-1} A = B$. Hence, B is represented semiotically as follows:

$B =$  ,

and the restriction of B to the core squares is $\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 2 & 1 \\ \hline\end{array}$. This shows that v is a descent of B, and that B has v-type 21. Hence, resolving this descent in B yields a 12-table $\operatorname{res}_v B$ which is represented semiotically as follows:

$\operatorname{res}_v B =$  ,

and the restriction of $\operatorname{res}_v B$ to the core squares is $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 1 & 2 \\ \hline\end{array}$. This, in turn, shows that $v-1$ is a descent of $\operatorname{res}_v B$, and that $\operatorname{res}_v B$ has $(v-1)$-type M1. Thus, resolving this descent in $\operatorname{res}_v B$ yields a 12-table $\operatorname{res}_{v-1}(\operatorname{res}_v B)$ which is represented semiotically as follows:

$\operatorname{res}_{v-1}(\operatorname{res}_v B) =$  , (17)

and the restriction of $\operatorname{res}_{v-1}(\operatorname{res}_v B)$ to the core squares is $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 1 & 2 & 2 \\ \hline\end{array}$.

On the other hand, A has v-type M1. Resolving the descent v of A yields $\operatorname{res}_v A = C$. Thus, we can represent C semiotically and find its restriction to the core squares. This shows us that C has $v-1$ as a descent and has $(v-1)$-type 21. Resolving this descent yields a 12-table $\operatorname{res}_{v-1} C$ which we can again represent semiotically and find its restriction to the core squares. Doing this, we observe that $\operatorname{res}_{v-1} C$ has v as a descent and has v-type 2M. Resolving this descent yields a 12-table $\operatorname{res}_v (\operatorname{res}_{v-1} C)$ whose semiotic representation and restriction to the core squares can again be found. We leave the details of this argument to the reader, but we state its result: The 12-table $\operatorname{res}_v (\operatorname{res}_{v-1} C)$ is well-defined and has the same semiotic representation and the same restriction to the core squares as the 12-table $\operatorname{res}_{v-1} (\operatorname{res}_v B)$. Consequently, the 12-tables $\operatorname{res}_v (\operatorname{res}_{v-1} C)$ and $\operatorname{res}_{v-1} (\operatorname{res}_v B)$ are equal[10].

Hence, we can set $D = \operatorname{res}_v (\operatorname{res}_{v-1} C) = \operatorname{res}_{v-1} (\operatorname{res}_v B)$. Consider this D. We have $C \Rrightarrow \operatorname{res}_{v-1} C$ (since $C \underset{v-1}{\Rrightarrow} \operatorname{res}_{v-1} C$) and $\operatorname{res}_{v-1} C \Rrightarrow \operatorname{res}_v (\operatorname{res}_{v-1} C)$ (since $\operatorname{res}_{v-1} C \underset{v}{\Rrightarrow} \operatorname{res}_v (\operatorname{res}_{v-1} C)$). Combining these two relations, we obtain $C \overset{*}{\Rrightarrow} \operatorname{res}_v (\operatorname{res}_{v-1} C)$ (since $\overset{*}{\Rrightarrow}$ is the reflexive-and-transitive closure of the relation $\Rrightarrow$). In other words, $C \overset{*}{\Rrightarrow} D$ (since $D = \operatorname{res}_v (\operatorname{res}_{v-1} C)$). Similarly, $B \overset{*}{\Rrightarrow} D$. Thus, we have found a $D \in \mathbf{S}$ such that $B \overset{*}{\Rrightarrow} D$ and $C \overset{*}{\Rrightarrow} D$. This completes the proof of Lemma 4.8 in Case 1. $\square$

4.7. The normalization map

The following proposition is the most important piece in our puzzle:

[10] *Proof.* To see this, we need to show that for every $h \in \mathbb{N}_+$, the h-th column of $\operatorname{res}_v (\operatorname{res}_{v-1} C)$ equals the h-th column of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$.

For $h \notin \{v-1, v, v+1\}$, this is obvious (because for $h \notin \{v-1, v, v+1\}$, the h-th column of a 12-table never changes under res_v or res_{v-1}).

For $h = v-1$, this is again obvious (because the semiotic representation of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$ given in (17) shows that the $(v-1)$-th column of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$ is 1-pure, and the same can be said of the $(v-1)$-th column of $\operatorname{res}_v (\operatorname{res}_{v-1} C)$).

For $h = v+1$, this is also obvious (because the semiotic representation of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$ given in (17) shows that the $(v+1)$-th column of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$ is 2-pure, and the same can be said of the $(v+1)$-th column of $\operatorname{res}_v (\operatorname{res}_{v-1} C)$).

It thus only remains to deal with the case of $h = v$. In other words, we need to prove that the v-th column of $\operatorname{res}_v (\operatorname{res}_{v-1} C)$ equals the v-th column of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$.

We know from (17) that the v-th column of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$ is mixed. Moreover, the restriction of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$ to the core squares is $\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 1 & 2 & 2 \\ \hline \end{array}$; therefore, the last 1 and the first 2 in the v-th column of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$ are in the cells (s, v) and $(s+1, v)$, respectively. But the same can be said about the v-th column of $\operatorname{res}_v (\operatorname{res}_{v-1} C)$. Hence, the v-th column of $\operatorname{res}_v (\operatorname{res}_{v-1} C)$ and the v-th column of $\operatorname{res}_{v-1} (\operatorname{res}_v B)$ both are mixed, and the cell containing the last 1 is the same for both of these columns. This yields that these columns must be equal. As we know, this finishes our proof.

Proposition 4.9. For every $T \in \mathbf{S}$, there exists a unique $N \in \mathbf{R}$ such that $T \overset{*}{\Rrightarrow} N$.

Before we come to the proof of this proposition, let us translate it into words in order to illuminate its meaning. Recall that $\mathbf{R}$ is the set of all 12-rpps (of shape Z), that $\mathbf{S}$ is the set of all benign 12-tables, and that the relation "$P \overset{*}{\Rrightarrow} Q$" can be interpreted as "Q can be obtained from P by repeatedly resolving descents". Hence, Proposition 4.9 says that, for every benign 12-table T, there exists a unique 12-rpp N which can be obtained from T by repeatedly resolving descents. We can actually make a slightly stronger claim[11]: If we start with a 12-table T and repeatedly resolve descents until no more descents remain, then the process of resolving descents eventually stops[12], and the result is a 12-rpp, which is independent of the choices made in the process (the choices of what descents we resolve). This is actually a familiar instance of what is called *confluence* of a rewriting system, and our proof uses a tactic known as "diamond lemma" or "Newman lemma" (see, e.g., [BezCoq03], or [BaaNip98, Lemma 2.7.2 + Fact 2.1.7]); however, the tactic is simple enough that we found it easier to integrate it into the proof rather than state it as a lemma.

Proof of Proposition 4.9. For every $T \in \mathbf{S}$, let $\operatorname{Norm}(T)$ denote the set

$$\left\{ N \in \mathbf{R} \ \mid \ T \overset{*}{\Rrightarrow} N \right\}.$$

Thus, in order to prove Proposition 4.9, it is clearly enough to show that, for every $T \in \mathbf{S}$, this set $\operatorname{Norm}(T)$ is a one-element set.

We shall prove this by strong induction on $\ell(T)$. So we fix some $T \in \mathbf{S}$, and we assume that we already know that

$$\operatorname{Norm}(S) \text{ is a one-element set for every } S \in \mathbf{S} \text{ satisfying } \ell(S) < \ell(T). \qquad (18)$$

We then need to prove that $\operatorname{Norm}(T)$ is a one-element set.

Let $\mathbf{Z}$ denote the set $\{S \in \mathbf{S} \ \mid \ T \Rrightarrow S\}$. In other words, $\mathbf{Z}$ is the set of all benign 12-tables S which can be obtained from T by resolving one single descent. If $\mathbf{Z}$ is empty, then we are done[13]. Hence, we WLOG assume that $\mathbf{Z}$ is nonempty.

[11] which will fall out of the proof of Proposition 4.9 given below

[12] This is fairly clear: Every time we resolve a descent, the nonnegative integer $\ell(T)$ decreases (due to (15)); but a nonnegative integer cannot keep decreasing forever.

[13] *Proof.* Assume that $\mathbf{Z}$ is empty. Thus, the benign 12-table S has no descent (since any descent could be resolved and then would yield a 12-table in $\mathbf{Z}$). In other words, S is a 12-rpp (since a 12-rpp of shape Z is the same as a 12-table which has no descents). In other words, $S \in \mathbf{R}$. Hence, the set $\operatorname{Norm}(T) = \left\{ N \in \mathbf{R} \ \mid \ T \overset{*}{\Rrightarrow} N \right\}$ has **at least** one element (namely, S). This set also clearly has **at most** one element (since S has no descent, and thus the only 12-table that can be obtained from S by resolving descents is S itself). Hence, this set has exactly one element. In other words, $\operatorname{Norm}(T)$ is a one-element set. This is exactly what we wanted to prove.

Therefore, the 12-table T has at least one descent. Consequently, T is not a 12-rpp[14]. In other words, $T \notin \mathbf{R}$.

Now, we recall that $\overset{*}{\Rrightarrow}$ is the reflexive-and-transitive closure of $\Rrightarrow$. Hence,

$$\left\{ N \in \mathbf{S} \ \mid \ T \overset{*}{\Rrightarrow} N \right\} = \{T\} \cup \bigcup_{Z \in \mathbf{Z}} \left\{ N \in \mathbf{S} \ \mid \ Z \overset{*}{\Rrightarrow} N \right\}.$$

Intersecting both sides of this identity with the set $\mathbf{R}$, we obtin

$$\left\{ N \in \mathbf{R} \ \mid \ T \overset{*}{\Rrightarrow} N \right\} = \underbrace{(\{T\} \cap \mathbf{R})}_{\substack{=\varnothing \\ (\text{since } T \notin \mathbf{R})}} \cup \bigcup_{Z \in \mathbf{Z}} \underbrace{\left\{ N \in \mathbf{R} \ \mid \ Z \overset{*}{\Rrightarrow} N \right\}}_{\substack{=\operatorname{Norm}(Z) \\ (\text{by the definition of } \operatorname{Norm}(Z))}}$$
$$= \bigcup_{Z \in \mathbf{Z}} \operatorname{Norm}(Z).$$

Thus,

$$\operatorname{Norm}(T) = \left\{ N \in \mathbf{R} \ \mid \ T \overset{*}{\Rrightarrow} N \right\} = \bigcup_{Z \in \mathbf{Z}} \operatorname{Norm}(Z). \qquad (19)$$

Let us now notice that:

- For every $Z \in \mathbf{Z}$, the set $\operatorname{Norm}(Z)$ is a one-element set[15].
- For every $B \in \mathbf{Z}$ and $C \in \mathbf{Z}$, we have $\operatorname{Norm}(B) \cap \operatorname{Norm}(C) \neq \varnothing$ [16].

Hence, (19) shows that $\operatorname{Norm}(T)$ is a union of one-element sets, any two of which have a nonempty intersection (i.e., are identical, because they are one-element sets). Moreover, this union is nonempty (since $\mathbf{Z}$ is nonempty). Hence, $\operatorname{Norm}(T)$ itself is a one-element set. This completes our induction. $\square$

[14] since a 12-rpp of shape Z is the same as a 12-table which has no descents

[15] *Proof.* Let $Z \in \mathbf{Z}$. Then, $Z \in \mathbf{Z} = \{S \in \mathbf{S} \ \mid \ T \Rrightarrow S\}$, so that $T \Rrightarrow Z$. Lemma 4.6 **(d)** (applied to $P = T$ and $Q = Z$) thus yields $\ell(T) > \ell(Z)$. Hence, (18) (applied to $S = Z$) yields that $\operatorname{Norm}(Z)$ is a one-element set, qed.

[16] *Proof.* Let $B \in \mathbf{Z}$ and $C \in \mathbf{Z}$. We have $B \in \mathbf{Z} = \{S \in \mathbf{S} \ \mid \ T \Rrightarrow S\}$, so that $T \Rrightarrow B$. Lemma 4.6 **(d)** (applied to $P = T$ and $Q = B$) thus yields $\ell(T) > \ell(B)$.

We have $T \Rrightarrow C$ (similarly to $T \Rrightarrow B$). Hence, Lemma 4.8 (applied to $A = T$) yields that there exists a $D \in \mathbf{S}$ such that $B \overset{*}{\Rrightarrow} D$ and $C \overset{*}{\Rrightarrow} D$. Consider this D. Lemma 4.7 **(d)** (applied to $P = B$ and $Q = D$) yields $\ell(B) \geq \ell(D)$. Thus, $\ell(T) > \ell(B) \geq \ell(D)$. Hence, $\operatorname{Norm}(D)$ is a one-element set (by (18), applied to $S = D$).

But $B \overset{*}{\Rrightarrow} D$. Thus, every $N \in \mathbf{R}$ satisfying $D \overset{*}{\Rrightarrow} N$ also satisfies $B \overset{*}{\Rrightarrow} N$ (since the relation $\overset{*}{\Rrightarrow}$ is transitive). In other words, every element of $\operatorname{Norm}(D)$ is also an element of $\operatorname{Norm}(B)$. In other words, $\operatorname{Norm}(D) \subseteq \operatorname{Norm}(B)$. Similarly, $\operatorname{Norm}(D) \subseteq \operatorname{Norm}(C)$. Hence, $\operatorname{Norm}(D) \subseteq \operatorname{Norm}(B) \cap \operatorname{Norm}(C)$. Hence, $\operatorname{Norm}(B) \cap \operatorname{Norm}(C) \neq \varnothing$ (since $\operatorname{Norm}(D)$ is a one-element set), qed.

Definition 4.10. We now define a map norm : $\mathbf{S} \to \mathbf{R}$ as follows:

Let $T \in \mathbf{S}$. Proposition 4.9 shows that there exists a unique $N \in \mathbf{R}$ such that $T \overset{*}{\Rrightarrow} N$. We define $\operatorname{norm}(T)$ to be this N.

Thus, the map norm : $\mathbf{S} \to \mathbf{R}$ is defined, and satisfies

$$T \overset{*}{\Rrightarrow} \operatorname{norm}(T) \qquad \text{for every } T \in \mathbf{S}. \tag{20}$$

Example 4.11. Let us give an example of a computation of $\operatorname{norm}(T)$. For this example, let us take

$$T = \begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 2 & 1 \\ \cline{2-5} \multicolumn{1}{c|}{} & 1 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & 1 & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 2 & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Then, $\operatorname{norm}(T)$ is the unique $N \in \mathbf{R}$ such that $T \overset{*}{\Rrightarrow} N$. Thus, we can obtain $\operatorname{norm}(T)$ from T by repeatedly resolving descents until no more descents are left (because "$T \overset{*}{\Rrightarrow} N$" means "$N$ can be obtained from T by repeatedly resolving descents", and "$N \in \mathbf{R}$" is equivalent to the statement that N has no descents). The word "unique" here implies that, in whatever order we resolve descents, the result will always be the same. And the procedure will eventually come to an end because the nonnegative integer $\ell(T)$ decreases every time we resolve a descent in T (by Lemma 4.6 **(d)**).

Let us first resolve the descent 2 in T. This gives us the 12-table

$$\operatorname{res}_2 T = \begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 2 & 1 \\ \cline{2-5} \multicolumn{1}{c|}{} & 1 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & 1 & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 1 & 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

(In fact, we have seen this in Example 4.3 already, but we denoted the 12-table by P there.) Next, resolving the descent 4 in $\operatorname{res}_2 T$, we obtain the 12-table

$$\operatorname{res}_4(\operatorname{res}_2 T) = \begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 1 & 2 \\ \cline{2-5} \multicolumn{1}{c|}{} & 1 & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & 1 & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 1 & 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

We go on by resolving the descent 1 in $\operatorname{res}_4(\operatorname{res}_2 T)$, and thus obtain

$$\operatorname{res}_1(\operatorname{res}_4(\operatorname{res}_2 T)) = \begin{ytableau} \none & \none & 1 & 1 & 2 \\ \none & 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 \end{ytableau}.$$

Next, we resolve the descent 2 in $\operatorname{res}_1(\operatorname{res}_4(\operatorname{res}_2 T))$, and obtain

$$\operatorname{res}_2(\operatorname{res}_1(\operatorname{res}_4(\operatorname{res}_2 T))) = \begin{ytableau} \none & \none & 2 & 1 & 2 \\ \none & 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 \end{ytableau}.$$

Next, we resolve the descent 3 in $\operatorname{res}_2(\operatorname{res}_1(\operatorname{res}_4(\operatorname{res}_2 T)))$, and this leads us to

$$\operatorname{res}_3(\operatorname{res}_2(\operatorname{res}_1(\operatorname{res}_4(\operatorname{res}_2 T)))) = \begin{ytableau} \none & \none & 1 & 2 & 2 \\ \none & 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 \end{ytableau}.$$

Finally, we resolve the descent 2 in $\operatorname{res}_3(\operatorname{res}_2(\operatorname{res}_1(\operatorname{res}_4(\operatorname{res}_2 T))))$, and thus obtain

$$\operatorname{res}_2(\operatorname{res}_3(\operatorname{res}_2(\operatorname{res}_1(\operatorname{res}_4(\operatorname{res}_2 T))))) = \begin{ytableau} \none & \none & 1 & 2 & 2 \\ \none & 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 \end{ytableau}.$$

This 12-table $\operatorname{res}_2(\operatorname{res}_3(\operatorname{res}_2(\operatorname{res}_1(\operatorname{res}_4(\operatorname{res}_2 T)))))$ has no more descents, and thus is final. So $\operatorname{norm}(T) = \operatorname{res}_2(\operatorname{res}_3(\operatorname{res}_2(\operatorname{res}_1(\operatorname{res}_4(\operatorname{res}_2 T)))))$.

4.8. Definition of B

We can now finally prove Lemma 3.6.

Definition 4.12. Let us define a map $\mathbf{B} : \mathbf{R} \to \mathbf{R}$ as follows:

Let $T \in \mathbf{R}$. Then, $T \in \mathbf{R} \subseteq \mathbf{S}$. Hence, $\operatorname{flip}(T) \in \mathbf{S}$ is well-defined, and thus $\operatorname{norm}(\operatorname{flip}(T)) \in \mathbf{R}$ is well-defined. We define $\mathbf{B}(T)$ to be $\operatorname{norm}(\operatorname{flip}(T))$.

Thus, the map $\mathbf{B}$ is defined. In order to complete the proof of Lemma 3.6, we need to show that this map $\mathbf{B}$ is an involution and that, for every $S \in \mathbf{R}$, the equalities (**??**) and (**??**) hold. At this point, all of this is easy:

Proof that $\mathbf{B}$ *is an involution:* Let $T \in \mathbf{R}$. The definition of $\mathbf{B}$ yields $\mathbf{B}(T) = \operatorname{norm}(\operatorname{flip}(T))$. From (20) (applied to $\operatorname{flip}(T)$ instead of T), we have $\operatorname{flip}(T) \overset{*}{\Rrightarrow} \operatorname{norm}(\operatorname{flip}(T))$. This rewrites as $\operatorname{flip}(T) \overset{*}{\Rrightarrow} \mathbf{B}(T)$ (since $\mathbf{B}(T) = \operatorname{norm}(\operatorname{flip}(T))$). Lemma 4.7 **(c)** (applied to $P = \operatorname{flip}(T)$ and $Q = \mathbf{B}(T)$) thus yields $\operatorname{flip}(\mathbf{B}(T)) \overset{*}{\Rrightarrow} \operatorname{flip}(\operatorname{flip} T)$. Since $\operatorname{flip}(\operatorname{flip} T) = \underbrace{(\operatorname{flip} \circ \operatorname{flip})}_{\substack{=\operatorname{id} \\ \text{(by Proposition 4.1 (a))}}}(T) = T$, this rewrites as $\operatorname{flip}(\mathbf{B}(T)) \overset{*}{\Rrightarrow} T$.

But $\operatorname{norm}(T)$ is the unique $N \in \mathbf{R}$ such that $T \overset{*}{\Rrightarrow} N$ (by the definition of $\operatorname{norm}(T)$). Applying this to $\operatorname{flip}(\mathbf{B}(T))$ instead of T, we see that $\operatorname{norm}(\operatorname{flip}(\mathbf{B}(T)))$ is the unique $N \in \mathbf{R}$ such that $\operatorname{flip}(\mathbf{B}(T)) \overset{*}{\Rrightarrow} N$. Hence, every $N \in \mathbf{R}$ such that $\operatorname{flip}(\mathbf{B}(T)) \overset{*}{\Rrightarrow} N$ must satisfy $N = \operatorname{norm}(\operatorname{flip}(\mathbf{B}(T)))$. Applying this to $N = T$, we obtain $T = \operatorname{norm}(\operatorname{flip}(\mathbf{B}(T)))$ (since $T \in \mathbf{R}$ and $\operatorname{flip}(\mathbf{B}(T)) \overset{*}{\Rrightarrow} T$).

But $(\mathbf{B} \circ \mathbf{B})(T) = \mathbf{B}(\mathbf{B}(T)) = \operatorname{norm}(\operatorname{flip}(\mathbf{B}(T)))$ (by the definition of $\mathbf{B}(\mathbf{B}(T))$). Comparing this with $T = \operatorname{norm}(\operatorname{flip}(\mathbf{B}(T)))$, we obtain $(\mathbf{B} \circ \mathbf{B})(T) = T$.

Let us now forget that we fixed T. We thus have shown that $(\mathbf{B} \circ \mathbf{B})(T) = T$ for every $T \in \mathbf{R}$. In other words, $\mathbf{B} \circ \mathbf{B} = \operatorname{id}$. In other words, $\mathbf{B}$ is an involution.

*Proof of the equality (**??**) for every* $S \in \mathbf{R}$*:* Let $S \in \mathbf{R}$. The definition of $\mathbf{B}$ yields $\mathbf{B}(S) = \operatorname{norm}(\operatorname{flip}(S))$. But (20) (applied to $T = \operatorname{flip}(S)$) yields $\operatorname{flip}(S) \overset{*}{\Rrightarrow} \operatorname{norm}(\operatorname{flip}(S))$. This rewrites as $\operatorname{flip}(S) \overset{*}{\Rrightarrow} \mathbf{B}(S)$ (since $\mathbf{B}(S) = \operatorname{norm}(\operatorname{flip}(S))$). Lemma 4.7 **(a)** (applied to $P = \operatorname{flip}(S)$ and $Q = \mathbf{B}(S)$) thus yields

$$\operatorname{ceq}(\mathbf{B}(S)) = \operatorname{ceq}(\operatorname{flip}(S)) = \operatorname{ceq}(S)$$

(by (4), applied to $T = S$). This proves (**??**).

*Proof of the equality (**??**) for every* $S \in \mathbf{R}$*:* Let $S \in \mathbf{R}$. The definition of $\mathbf{B}$ yields $\mathbf{B}(S) = \operatorname{norm}(\operatorname{flip}(S))$. But (20) (applied to $T = \operatorname{flip}(S)$) yields $\operatorname{flip}(S) \overset{*}{\Rrightarrow} \operatorname{norm}(\operatorname{flip}(S))$. This rewrites as $\operatorname{flip}(S) \overset{*}{\Rrightarrow} \mathbf{B}(S)$ (since $\mathbf{B}(S) = \operatorname{norm}(\operatorname{flip}(S))$). Lemma 4.7 **(b)** (applied to $P = \operatorname{flip}(S)$ and $Q = \mathbf{B}(S)$) thus yields

$$\operatorname{ircont}(\mathbf{B}(S)) = \operatorname{ircont}(\operatorname{flip}(S)) = s_1 \cdot \operatorname{ircont}(S)$$

(by (5), applied to $T = S$). This proves (**??**).

We have thus shown that $\mathbf{B}$ is an involution, and that, for every $S \in \mathbf{R}$, the equalities (**??**) and (**??**) hold. This completes the proof of Lemma 3.6. Thus, Lemma **??** is proven (since we have proven it using Lemma 3.6), and consequently Theorem 3.5 is proven (since we have derived it from Lemma **??**). This, in turn, finishes the proof of Theorem 3.4 (since we have proven Theorem 3.4 using Theorem 3.5).

5. The classical Bender-Knuth involutions

5.1. Recalling the definition of $\mathbf{B}_i$

We fix a skew partition λ/μ and a positive integer i for the whole Section 5.

Theorem 3.5 merely claims the existence of an involution $\mathbf{B}_i : \operatorname{RPP}(\lambda/\mu) \to \operatorname{RPP}(\lambda/\mu)$ satisfying certain properties. Such an involution, per se, needs not be unique. However, if we trace back the proof of Theorem 3.5 (and the proofs of the lemmas that were used in this proof), we notice that this proof constructs a specific involution $\mathbf{B}_i$. This construction is spread across various proofs; we can summarize it as follows:

- The main step of the construction was the construction of the involution $\mathbf{B} : \mathbf{R} \to \mathbf{R}$ in Lemma 3.6 (for a given finite convex subset Z of $\mathbb{N}_+^2$). This is an involution which sends 12-rpps of shape Z to 12-rpps of the same shape Z, and it was constructed as follows: Given a 12-rpp T of shape Z, we set $\mathbf{B}(T) = \operatorname{norm}(\operatorname{flip}(T))$. (Recall that $\operatorname{flip}(T)$ fills all the 1-pure columns of T with 2's while simultaneously filling all the 2-pure columns of T with 1's. Recall furthermore that $\operatorname{norm}(\operatorname{flip}(T))$ is obtained from $\operatorname{flip}(T)$ by repeatedly resolving descents until no descents remain.)

- Having constructed this map $\mathbf{B} : \mathbf{R} \to \mathbf{R}$, we can construct the involution $\mathbf{B}_Z : \mathbf{R}_Z \to \mathbf{R}_Z$ in Lemma **??** (for a given finite convex subset Z of $\mathbb{N}_+^2$) as follows: Given an rpp S of shape Z whose entries are i's and $(i+1)$'s, we first replace these entries by 1's and 2's (respectively), so that we obtain a 12-rpp; then, we apply the involution $\mathbf{B} : \mathbf{R} \to \mathbf{R}$ to this 12-rpp; and then, in the resulting 12-rpp, we change the 1's and 2's back into i's and $(i+1)$'s. The resulting rpp is $\mathbf{B}_Z(S)$.

- Finally, we can construct the involution $\mathbf{B}_i : \operatorname{RPP}(\lambda/\mu) \to \operatorname{RPP}(\lambda/\mu)$. To wit, if we are given an rpp $S \in \operatorname{RPP}(\lambda/\mu)$, then we can restrict our attention to the cells of S which contain the entries i and $i+1$. These cells form an rpp of some shape Z. We then apply the involution $\mathbf{B}_Z$ to this new rpp, while leaving all the remaining entries of S unchanged. The result is an rpp of shape $Y(\lambda/\mu)$ again; this rpp is $\mathbf{B}_i(S)$.

In the following, whenever we will be talking about the involution $\mathbf{B}_i$, we will always mean this particular involution $\mathbf{B}_i$, rather than an arbitrary involution $\mathbf{B}_i$ which satisfies the claims of Theorem 3.5.

5.2. The Bender-Knuth involutions

We claimed that our involution $\mathbf{B}_i : \operatorname{RPP}(\lambda/\mu) \to \operatorname{RPP}(\lambda/\mu)$ is a generalization of the i-th Bender-Knuth involution defined for semistandard tableaux. Let us now elaborate on this claim. First, we shall define the i-th Bender-Knuth involution (following [GriRei15, proof of Proposition 2.11] and [Stan99, proof of Theorem 7.10.2]).

Let $\operatorname{SST}(\lambda/\mu)$ denote the set of all semistandard tableaux of shape $Y(\lambda/\mu)$. We define a map $B_i : \operatorname{SST}(\lambda/\mu) \to \operatorname{SST}(\lambda/\mu)$ as follows:[17]

Let $T \in \operatorname{SST}(\lambda/\mu)$. Then, T is a semistandard tableau, so that every column of T contains at most one i and at most one $i+1$. We shall ignore all columns of T which contain both an i and an $i+1$; that is, we mark all the entries of all such columns as "ignored". Now, let $k \in \mathbb{N}_+$. The k-th row of T is a weakly increasing sequence of positive integers; thus, it contains a (possibly empty) string of i's followed by a (possibly empty) string of $(i+1)$'s. These two strings together form a substring of the k-th row which looks as follows:

$$(i, i, \ldots, i, i+1, i+1, \ldots, i+1)$$

[18]. Some of the entries of this substring are "ignored"; it is easy to see that the "ignored" i's are gathered at the left end of the substring whereas the "ignored" $(i+1)$'s are gathered end of the substring. So the substring looks as follows:

$$\left(\underbrace{i, i, \ldots, i}_{\substack{a \text{ many } i\text{'s which} \\ \text{are "ignored"}}}, \underbrace{i, i, \ldots, i}_{\substack{r \text{ many } i\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{s \text{ many } (i+1)\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{b \text{ many } (i+1)\text{'s which} \\ \text{are not "ignored"}}} \right)$$

for some $a, r, s, b \in \mathbb{N}$. Now, we change this substring into

$$\left(\underbrace{i, i, \ldots, i}_{\substack{a \text{ many } i\text{'s which} \\ \text{are "ignored"}}}, \underbrace{i, i, \ldots, i}_{\substack{s \text{ many } i\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{r \text{ many } (i+1)\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{b \text{ many } (i+1)\text{'s which} \\ \text{are not "ignored"}}} \right).$$

And we do this for every $k \in \mathbb{N}_+$ (simultaneously or consecutively – it does not matter). At the end, we have obtained a new semistandard tableau of shape $Y(\lambda/\mu)$. We define $B_i(T)$ to be this new tableau.

[17] We refer to Example 5.1 below for illustration.
[18] Of course, this might contain no i's or no $(i+1)$'s.

Example 5.1. Let us give an example of this construction of B_i. Namely, let $i = 2$, let $\lambda = (7, 6, 4, 1)$, and let $\mu = (3)$. Let T be the semistandard tableau

$$\begin{array}{|c|c|c|c|c|c|c|} \cline{4-7} \multicolumn{3}{c|}{} & 1 & 1 & 2 & 2 \\ \hline 1 & 2 & 2 & 2 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-6} 3 & 3 & 5 & 6 & \multicolumn{3}{c}{} \\ \cline{1-4} 4 & \multicolumn{6}{c}{} \\ \cline{1-1} \end{array}$$

of shape $Y(\lambda/\mu)$. We want to find $B_i(T)$.

The columns that contain both an i and an $i+1$ (that is, both a 2 and a 3) are the second and the sixth columns. So we mark all entries of these two columns as "ignored". Now, the substring of the 2-nd row of T formed by the i's and the $(i+1)$'s looks as follows:

$$\left(\underbrace{2}_{\substack{\text{1 many 2's which} \\ \text{are "ignored"}}}, \underbrace{2, 2}_{\substack{\text{2 many 2's which} \\ \text{are not "ignored"}}}, \underbrace{3}_{\substack{\text{1 many 3's which} \\ \text{are not "ignored"}}}, \underbrace{3}_{\substack{\text{2 many 3's which} \\ \text{are not "ignored"}}} \right).$$

So we change it into

$$\left(\underbrace{2}_{\substack{\text{1 many 2's which} \\ \text{are "ignored"}}}, \underbrace{2}_{\substack{\text{1 many 2's which} \\ \text{are not "ignored"}}}, \underbrace{3, 3}_{\substack{\text{2 many 3's which} \\ \text{are not "ignored"}}}, \underbrace{3}_{\substack{\text{2 many 3's which} \\ \text{are not "ignored"}}} \right).$$

Similarly, we change the substring $(2, 2)$ of the 1-st row of T into $(2, 3)$ (because its first 2 is "ignored" but its second 2 is not), and we change the substring $(3, 3)$ of the 3-rd row of T into $(2, 3)$ (because its first 3 is not "ignored" but its second 3 is). The substring of the 4-th row, of the 5-th row, of the 6-th row, etc., formed by the i's and $(i+1)$'s are empty (because these rows have neither i's nor $(i+1)$'s), and thus we do not make any changes on them. Now, $B_i(T)$ is defined to be the tableau that results from all of these changes; thus,

$$B_i(T) = \begin{array}{|c|c|c|c|c|c|c|} \cline{4-7} \multicolumn{3}{c|}{} & 1 & 1 & 2 & 3 \\ \hline 1 & 2 & 2 & 3 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-6} 2 & 3 & 5 & 6 & \multicolumn{3}{c}{} \\ \cline{1-4} 4 & \multicolumn{6}{c}{} \\ \cline{1-1} \end{array}.$$

Proposition 5.2. The map $B_i : \operatorname{SST}(\lambda/\mu) \to \operatorname{SST}(\lambda/\mu)$ thus defined is an involution. It is known as the *i-th Bender-Knuth involution*.

Proposition 5.2 is easy to prove (and is usually proven in less or more detail everywhere the map B_i is defined).

Now, every semistandard tableau of shape $Y(\lambda/\mu)$ is also an rpp of shape $Y(\lambda/\mu)$. In other words, $\operatorname{SST}(\lambda/\mu) \subseteq \operatorname{RPP}(\lambda/\mu)$. Hence, $\mathbf{B}_i(T)$ is defined for every $T \in \operatorname{SST}(\lambda/\mu)$. Now, the claim that we want to make (that our involution $\mathbf{B}_i$ is a generalization of the i-th Bender-Knuth involution B_i) can be stated as follows:

Proposition 5.3. For every $T \in \operatorname{SST}(\lambda/\mu)$, we have $B_i(T) = \mathbf{B}_i(T)$.

Proof of Proposition 5.3 (sketched). We shall abbreviate "semistandard tableau" as "sst". We define a *12-sst* to be an sst whose entries all belong to the set $\{1,2\}$.

Let Z be a finite convex subset of $\mathbb{N}_+^2$. Let R denote the set of all 12-ssts of shape Z. We define a map $B : R \to R$ in the same way as we defined the map $B_i : \operatorname{SST}(\lambda/\mu) \to \operatorname{SST}(\lambda/\mu)$, with the only differences that we replace every appearance of "$\operatorname{SST}(\lambda/\mu)$", of "$i$" and of "$i+1$" by "$R$", "1" and "2", respectively. Then, this map $B : R \to R$ is an involution.

Now let us forget that we fixed Z. We thus have constructed a map $B : R \to R$ for every finite convex subset Z of $\mathbb{N}_+^2$. Now, recall how the map $\mathbf{B}_i : \operatorname{RPP}(\lambda/\mu) \to \operatorname{RPP}(\lambda/\mu)$ was constructed from the maps $\mathbf{B} : \mathbf{R} \to \mathbf{R}$ for every finite convex subset Z of $\mathbb{N}_+^2$ (essentially by forgetting all entries of an rpp except for the entries i and $i+1$ and relabelling these entries i and $i+1$ as 1 and 2). Similarly, the map $B_i : \operatorname{SST}(\lambda/\mu) \to \operatorname{SST}(\lambda/\mu)$ can be constructed from the maps $B : R \to R$ for every finite convex subset Z of $\mathbb{N}_+^2$ (essentially by forgetting all entries of an sst except for the entries i and $i+1$ and relabelling these entries i and $i+1$ as 1 and 2). Thus, in order to prove that $B_i(T) = \mathbf{B}_i(T)$ for every $T \in \operatorname{SST}(\lambda/\mu)$, it suffices to show that $B(T) = \mathbf{B}(T)$ for every finite convex subset Z of $\mathbb{N}_+^2$ and any 12-sst T of shape Z.

TODO 5.4. Is this readable?

So let Z be any finite convex subset of $\mathbb{N}_+^2$, and let T be a 12-sst of shape Z. We need to prove that $B(T) = \mathbf{B}(T)$.

Example 5.5. Here is an example of a 12-sst:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & & & & & & & & 1 \\ \hline & & & & & 1 & 1 & 2 & \\ \hline & & & 1 & 2 & 2 & 2 & & \\ \hline 1 & 2 & & & & & & & \\ \hline 2 & & & & & & & & \\ \hline \end{array} . \qquad (21)$$

It satisfies

$$B\left(T\right) = \begin{array}{ccccccccc} & & & & & & & & 1 \\ & & & & & 1 & 2 & 2 & \\ & & & 1 & 1 & 2 & 2 & & \\ 1 & 1 & & & & & & & \\ 2 & & & & & & & & \end{array} \tag{22}$$

and

$$\operatorname{flip}\left(T\right) = \begin{array}{ccccccccc} & & & & & & & & 1 \\ & & & & & 1 & 2 & 2 & \\ & & & 2 & 1 & 1 & 2 & & \\ 1 & 1 & & & & & & & \\ 2 & & & & & & & & \end{array} \tag{23}$$

(where $\operatorname{flip}\left(T\right)$ is defined as in the construction of $\mathbf{B}\left(T\right)$).

We make a few basic observations: The columns which are ignored in the construction of $B\left(T\right)$ are the columns which contain both a 1 and a 2. [19] These columns contain exactly two entries each (because a column of a 12-sst can only contain at most one 1, at most one 2 and no other entries), while every other column is either empty or contains only one entry. As a consequence, every entry of T which is not "ignored" in the construction of $B\left(T\right)$ is alone in its column.

Let us compare the basic ideas of the constructions of $B\left(T\right)$ and $\mathbf{B}\left(T\right)$:

- To construct $B\left(T\right)$, we ignore all columns of T which contain both a 1 and a 2; that is, we mark all entries in these columns as "ignored". Then, in every row, we let r be the number of 1's which are not "ignored", and let s be the number of 2's which are not "ignored". We replace these r many 1's and s many 2's by s many 1's and r many 2's. This we do for every row; the resulting 12-sst is $B\left(T\right)$.

- To construct $\mathbf{B}\left(T\right)$, we consider T as a 12-rpp, and we identify which of its columns are 1-pure, which are 2-pure and which are mixed. Then, we replace all entries of all 1-pure columns by 1's, while simultaneously replacing all entries of all 2-pure columns by 2's. The resulting 12-table is denoted $\operatorname{flip}\left(T\right)$. Then, we repeatedly resolve descents in $\operatorname{flip}\left(T\right)$ until no more descents remain. The resulting 12-table $\operatorname{norm}\left(\operatorname{flip}\left(T\right)\right)$ is a 12-rpp, and is denoted $\mathbf{B}\left(T\right)$.

If we compare the two constructions just described, we first notice that the columns ignored in the construction of $B\left(T\right)$ are precisely the mixed columns of

[19]For instance, in the 12-sst (21), the ignored columns are the 1-st, the 7-th and the 9-th columns.

T. Thus, the 12-table flip (T) can be obtained from T by replacing all 1's which are not "ignored" by 2's while simultaneously replacing all 2's which are not "ignored" by 1's. Thus, for any given $r \in \mathbb{N}_+$, if the r-th row of T contains r many 1's which are not "ignored" and s many 2's which are not "ignored", then the r-th row of flip (T) contains r many 2's which are not "ignored" and s many 1's which are not "ignored" (while the "ignored" entries in T appear in flip (T) unchanged). So we can restate the construction of flip (T) as follows:

- To construct flip (T) from T, do the following: In every row of T, let r be the number of 1's which are not "ignored", and let s be the number of 2's which are not "ignored". We replace these r many 1's and s many 2's by r many 2's and s many 1's (in this order). This we do for every row; the resulting 12-table is flip (T).

Compare this to our construction of $B(T)$:

- To construct $B(T)$ from T, do the following: In every row of T, let r be the number of 1's which are not "ignored", and let s be the number of 2's which are not "ignored". We replace these r many 1's and s many 2's by s many 1's and r many 2's (in this order). This we do for every row; the resulting 12-sst is $B(T)$.

Comparing these two constructions makes it clear that each row of $B(T)$ differs from the corresponding row of flip (T) merely in the order in which the non-"ignored" entries appear: In $B(T)$, the non-"ignored" 1's appear before the non-"ignored" 2's (as they must, $B(T)$ being an sst), whereas in flip (T) they appear in the opposite order. Hence, $B(T)$ can be obtained from flip (T) by sorting all non-"ignored" entries into increasing order in each row.

Now, let us notice that every pair of a non-"ignored" 2 and a non-"ignored" 1 lying in the same row of flip (T) cause a descent[20]. Conversely, all descents of flip (T) are caused by a non-"ignored" 2 and a non-"ignored" 1 lying in the same row (because all "ignored" entries are carried over from T without change and thus cannot take part in descents). We can resolve these descents one after the other (starting with the 2 and the 1 that are adjacent to each other), until none are left. The result is a 12-rpp. What is this 12-rpp?

- On the one hand, this 12-rpp is norm (flip (T)), because norm (flip (T)) is defined as what results when all descents of flip (T) are resolved.
- On the other hand, this 12-rpp is $B(T)$. In fact, resolving a descent caused by a non-"ignored" 2 and a non-"ignored" 1 lying in the same row results in this 2 getting switched with the 1 (while no other entries get moved[21]).

[20]More precisely: If $r \in \mathbb{N}_+$, $i \in \mathbb{N}_+$ and $j \in \mathbb{N}_+$ are such that $(\operatorname{flip} T)(r, i)$ is a non-"ignored" 2 and that $(\operatorname{flip} T)(r, j)$ is a non-"ignored" 1, then (i, j) is a descent of flip T.

[21]This is because every non-"ignored" entry is alone in its column.

Hence, when we resolve the descents, we just sort all non-"ignored" entries into increasing order in each row. But as we know, the 12-table obtained from flip (T) by sorting all non-"ignored" entries into increasing order in each row is $B(T)$.

So we have found a 12-rpp which equals both norm (flip (T)) and $B(T)$. Thus, $B(T) = \operatorname{norm}(\operatorname{flip}(T)) = \mathbf{B}(T)$. This completes our proof of Proposition 5.3. □

TODO 5.6. Can you follow the above argument? (Preferrably at 3AM and/or under the influence.)

6. The structure of 12-rpps

In this section, we let $\mathbf{k}$ be the polynomial ring $\mathbb{Z}[t_1, t_2, t_3, \ldots]$ in countably many indeterminates, and we restrict ourselves to the two-variable dual stable Grothendieck polynomial $\widetilde{g}_{\lambda/\mu}(x_1, x_2, 0, 0, \ldots; \mathbf{t})$ defined as the result of substituting $0, 0, 0, \ldots$ for $x_3, x_4, x_5, \ldots$ in $\widetilde{g}_{\lambda/\mu}$. We can represent it as a polynomial in $\mathbf{t}$ with coefficients in $\mathbb{Z}[x_1, x_2]$:

$$\widetilde{g}_{\lambda/\mu}(x_1, x_2, 0, 0, \ldots; \mathbf{t}) = \sum_{\alpha \in \mathbb{N}^{\mathbb{N}_+}} \mathbf{t}^{\alpha} Q_{\alpha}(x_1, x_2),$$

where the sum ranges over all weak compositions α, and all but finitely many $Q_\alpha(x_1, x_2)$ are 0. (The $Q_\alpha(x_1, x_2)$ here belong to $\mathbb{Z}[x_1, x_2]$.)

We shall show that each $Q_\alpha(x_1, x_2)$ is either zero or has the form

$$Q_\alpha(x_1, x_2) = (x_1 x_2)^M P_{n_0}(x_1, x_2) P_{n_1}(x_1, x_2) \cdots P_{n_r}(x_1, x_2),$$

where M, r and $n_0, n_1, \ldots, n_r$ are nonnegative integers naturally associated to α and λ/μ and

$$P_n(x_1, x_2) = (x_1^{n+1} - x_2^{n+1})/(x_1 - x_2) = x_1^n + x_1^{n-1} x_2 + \cdots + x_1 x_2^{n-1} + x_2^n.$$

We fix the skew partition λ/μ throughout the whole section. Abusing notation, we shall abbreviate $Y(\lambda/\mu)$ as λ/μ. We will have a running example with $\lambda = (7, 7, 7, 4, 4)$ and $\mu = (5, 3, 2)$.

6.1. Irreducible components

We recall that a *12-rpp* means an rpp whose entries all belong to the set $\{1, 2\}$.

Given a 12-rpp T, consider the set $\mathrm{NS}(T)$ of all cells $(i, j) \in \lambda/\mu$ such that $T(i, j) = 1$ but $(i+1, j) \in \lambda/\mu$ and $T(i+1, j) = 2$. (In other words, $\mathrm{NS}(T)$ is the set of all non-stagnant cells in T which are filled with a 1 and which are not the lowest cells in their columns.) Clearly, $\mathrm{NS}(T)$ contains at most one

$$
\begin{array}{ccccccc}
 & & & & & 1 & 2 \\
 & & & 1 & 1 & \mathbf{1} & 2 \\
 & & \mathbf{1} & \mathbf{1} & 1 & 2 & 2 \\
\mathbf{1} & 2 & 2 & 2 & & & \\
2 & 2 & 2 & 2 & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & & & 1 & 1 \\
 & & & 1 & 1 & 1 & \mathbf{1} \\
 & & \mathbf{1} & \mathbf{1} & 1 & 1 & 2 \\
1 & \mathbf{1} & 2 & 2 & & & \\
2 & 2 & 2 & 2 & & &
\end{array}
$$

$\mathrm{NS}(T_1) = \{(4,1),(3,3),(3,4),(2,6)\}$
$\mathrm{seplist}(T_1) = (4,3,3,2)$

$\mathrm{NS}(T_2) = \{(4,2),(3,3),(3,4),(2,7)\}$
$\mathrm{seplist}(T_2) = (4,3,3,2)$

Figure 2: Two 12-rpps of the same shape and with the same seplist-partition.

cell from each column; thus, let us write $\mathrm{NS}(T) = \{(i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s)\}$ with $j_1 < j_2 < \cdots < j_s$. Because T is a 12-rpp, it follows that the numbers $i_1,i_2,\ldots,i_s$ decrease weakly, therefore they form a partition which we called *the seplist-partition of* T and denoted

$$\mathrm{seplist}(T) := (i_1,i_2,\ldots,i_s)$$

in Section 4.2. An example of calculation of $\mathrm{seplist}(T)$ and $\mathrm{NS}(T)$ is illustrated on Figure 2.

We would like to answer the following question: for which partitions $\nu = (i_1 \geq \cdots \geq i_s > 0)$ does there exist a 12-rpp T of shape λ/μ such that $\mathrm{seplist}(T) = \nu$?

A trivial necessary condition for this to happen is that there should exist some numbers $j_1 < j_2 < \cdots < j_s$ such that

$$(i_1,j_1),(i_1+1,j_1),(i_2,j_2),(i_2+1,j_2),\ldots,(i_s,j_s),(i_s+1,j_s) \in \lambda/\mu. \tag{24}$$

We say that a partition ν is *admissible* if such $j_1 < j_2 < \cdots < j_s$ exist.

Until the end of Section 6, we make one further assumption: namely, that the skew partition λ/μ is connected[22].

For each integer i the set of all integers j such that $(i,j),(i+1,j) \in \lambda/\mu$ is just an interval $[\mu_i+1,\lambda_{i+1}]$, which we call *the support of* i and denote $\mathrm{supp}(i) := [\mu_i+1,\lambda_{i+1}]$.

[22]By this, we mean that λ/μ has at least one cell, and cannot be represented as a disjoint union of two nonempty skew partitions α/β and γ/δ such that no cell of α/β is adjacent to any cell of γ/δ. This is a harmless assumption, since every skew partition λ/μ can be written as a disjoint union of such connected skew partitions, and these "connected components" do not interact when it comes to studying rpps: Choosing a 12-rpp of shape λ/μ is tantamount to choosing a 12-rpp for each of these components; and choosing a 12-rpp of shape λ/μ with seplist-partition equal to a given partition ν is tantamount to choosing a 12-rpp for each of the components with seplist-partition equal to the "appropriate piece" of ν. (What an "appropriate piece" is should be clear enough, since two distinct connected components are supported on different rows.)

Assume that $\nu = (i_1 \geq \cdots \geq i_s > 0)$ is an admissible partition. Then, $\mathrm{supp}(i_k)$ is nonempty for each k. For two integers $a < b$, by $\nu\big|_{\subseteq[a,b)}$ we denote the subpartition $(i_r, i_{r+1}, \dots, i_{r+q})$ of ν such that for $r \leq k \leq r+q$ we have $\mathrm{supp}(i_k) \subseteq [a,b)$. In this case, we put $\#\nu\big|_{\subseteq[a,b)} := q+1$ which is just the number of entries in $\nu\big|_{\subseteq[a,b)}$. Similarly, we put $\nu\big|_{\cap[a,b)}$ to be the subpartition $(i_r, i_{r+1}, \dots, i_{r+q})$ of ν such that for $r \leq k \leq r+q$ we have $\mathrm{supp}(i_k) \cap [a,b) \neq \emptyset$. For example, for $\nu = (4,3,3,2)$ and λ/μ as on Figure 2, we have

$$\mathrm{supp}(3) = [3,4],\ \mathrm{supp}(2) = [4,7],\ \mathrm{supp}(4) = [1,4],$$

$$\nu\big|_{\subseteq[2,7)} = (3,3),\ \nu\big|_{\subseteq[2,8)} = (3,3,2),\ \nu\big|_{\subseteq[4,8)} = (2),\ \nu\big|_{\cap[4,5)} = (4,3,3,2),\ \#\nu\big|_{\subseteq[2,7)} = 2.$$

We introduce several definitions: An admissible partition $\nu = (i_1 \geq \cdots \geq i_s > 0)$ is called

- *non-representable* if for some $a < b$ we have $\#\nu\big|_{\subseteq[a,b)} > b-a$;
- *representable* if for all $a < b$ we have $\#\nu\big|_{\subseteq[a,b)} \leq b-a$;

a representable partition ν is called

- *irreducible* if for all $a < b$ we have $\#\nu\big|_{\subseteq[a,b)} < b-a$;
- *reducible* if for some $a < b$ we have $\#\nu\big|_{\subseteq[a,b)} = b-a$.

For example, $\nu = (4,3,3,2)$ is representable but reducible because we have $\nu\big|_{\subseteq[3,5)} = (3,3)$ so $\#\nu\big|_{\subseteq[3,5)} = 2 = 5-3$.

Note that these notions depend on the skew partition; thus, when we want to use a skew partition $\widetilde{\lambda/\mu}$ rather than λ/μ, we will write that ν is non-representable/irreducible/etc. *with respect to* $\widetilde{\lambda/\mu}$, and we denote the corresponding partitions by $\nu\big|^{\widetilde{\lambda/\mu}}_{\subseteq[a,b)}$.

These definitions can be motivated as follows. Suppose that a partition ν is non-representable, so there exist integers $a < b$ such that $\#\nu\big|_{\subseteq[a,b)} > b-a$. Recall that $\nu\big|_{\subseteq[a,b)} =: (i_r, i_{r+1}, \dots, i_{r+q})$ contains all entries of ν whose support is a subset of $[a,b)$. Thus in order for condition (24) to be true there must exist some integers $j_r < j_{r+1} < \cdots < j_{r+q}$ such that

$$(i_r, j_r), (i_r+1, j_r), \dots, (i_{r+q}, j_{r+q}), (i_{r+q}+1, j_{r+q}) \in \lambda/\mu.$$

On the other hand, by the definition of the support, we must have $j_k \in \mathrm{supp}(i_k) \subseteq [a,b)$ for all $r \leq k \leq r+q$. Therefore we get $q+1$ distinct elements of $[a,b)$ which is impossible if $q+1 = \#\nu\big|_{\subseteq[a,b)} > b-a$. It means that a non-representable partition ν is never a seplist-partition of a 12-rpp T.

Suppose now that a partition ν is reducible, so for some $a < b$ we get an equality $\#\nu\big|_{\subseteq[a,b)} = b-a$. Then these integers $j_r < \cdots < j_{r+q}$ should still all belong to $[a,b)$ and there are exactly $b-a$ of them, hence

$$j_r = a,\ j_{r+1} = a+1,\ \dots,\ j_{r+q} = a+q = b-1. \tag{25}$$

Because $\operatorname{supp}(i_r) \subseteq [a,b)$ but $\operatorname{supp}(i_r) \neq \varnothing$ (since ν is admissible), we have $(i_r, a-1) \notin \lambda/\mu$. Thus, placing a 1 into (i_r, a) and 2's into $(i_r+1, a), (i_r+2, a), \ldots$ does not put any restrictions on entries in columns $1, \ldots, a-1$. And the same is true for columns $b, b+1, \ldots$ when we place a 2 into $(i_{r+q}+1, b-1)$ and 1's into all cells above. Thus, if a partition ν is reducible, then the filling of columns $a, a+1, \ldots, b-1$ is uniquely determined (by (25)), and the filling of the rest can be arbitrary – the problem of existence of a 12-rpp T such that $\operatorname{seplist}(T) = \nu$ reduces to two smaller independent problems of the same kind (one for the columns $1, 2, \ldots, a-1$, the other for the columns[23] $b, b+1, \ldots, \lambda_1$). One can continue this reduction process and end up with several independent irreducible components separated from each other by mixed columns. An illustration of this phenomenon can be seen on Figure 2: the columns 3 and 4 must be mixed for any 12-rpps T with $\operatorname{seplist}(T) = (4,3,3,2)$.

More explicitly, we have thus shown that every interval $[a,b) \subseteq [1, \lambda_1+1)$ satisfying $\#\nu\big|_{\subseteq[a,b)} = b-a$ splits our problem into two independent subproblems. But if two such intervals $[a,b)$ and $[c,d)$ intersect, then their union is another such interval[24]. Hence, the maximal (with respect to inclusion) among all such intervals are pairwise disjoint and separated from each other by at least a distance of 1. This yields part **(a)** of the following lemma:

Lemma 6.1. Let ν be a representable partition.

(a) There exist unique integers $(1 = b_0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r \leq a_{r+1} = \lambda_1 + 1)$ satisfying the following two conditions:

1. For all $1 \leq k \leq r$, we have $\#\nu\big|_{\subseteq[a_k,b_k)} = b_k - a_k$.
2. The set $\bigcup_{k=0}^{r} [b_k, a_{k+1})$ is minimal (with respect to inclusion) among all sequences $(1 = b_0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r \leq a_{r+1} = \lambda_1 + 1)$ satisfying property 1.

[23]Recall that a 12-rpp of shape λ/μ cannot have any nonempty column beyond the λ_1'th one.

[24]*Proof.* Assume that two intervals $[a,b)$ and $[c,d)$ satisfying $\#\nu\big|_{\subseteq[a,b)} = b-a$ and $\#\nu\big|_{\subseteq[c,d)} = d-c$ intersect. We need to show that their union is another such interval.

We WLOG assume that $a \leq c$. Then, $c \leq b$ (since the intervals intersect). If $b > d$, then the union of the two intervals is simply $[a,b)$, which makes our claim obvious. Hence, we WLOG assume that $b \leq d$. Thus, $a \leq c \leq b \leq d$. The union of the two intervals is therefore $[a,d)$, and we must show that $\#\nu\big|_{\subseteq[a,d)} = d-a$. A set of positive integers is a subset of both $[a,b)$ and $[c,d)$ if and only if it is a subset of $[c,b)$. On the other hand, a set of positive integers that is a subset of either $[a,b)$ or $[c,d)$ must be a subset of $[a,d)$ (but not conversely). Combining these two observations, we obtain $\#\nu\big|_{\subseteq[a,d)} \geq \#\nu\big|_{\subseteq[a,b)} + \#\nu\big|_{\subseteq[c,d)} - \#\nu\big|_{\subseteq[c,b)}$. Since ν is representable (or, when $b = c$, for obvious reasons), we have $\#\nu\big|_{\subseteq[c,b)} \leq b-c$. Thus,

$$\#\nu\big|_{\subseteq[a,d)} \geq \underbrace{\#\nu\big|_{\subseteq[a,b)}}_{=b-a} + \underbrace{\#\nu\big|_{\subseteq[c,d)}}_{=d-c} - \underbrace{\#\nu\big|_{\subseteq[c,b)}}_{\leq b-c} \geq (b-a) + (d-c) - (b-c) = d-a.$$

Combined with $\#\nu\big|_{\subseteq[a,d)} \leq d-a$ (since ν is representable), this yields $\#\nu\big|_{\subseteq[a,d)} = d-a$, qed.

Furthermore, for these integers, we have:
(b) The partition ν is the concatenation

$$\left(\nu\big|_{\cap[b_0,a_1)}\right)\left(\nu\big|_{\subseteq[a_1,b_1)}\right)\left(\nu\big|_{\cap[b_1,a_2)}\right)\left(\nu\big|_{\subseteq[a_2,b_2)}\right)\cdots\left(\nu\big|_{\cap[b_r,a_{r+1})}\right)$$

(where we regard a partition as a sequence of positive integers, with no trailing zeroes).

(c) The partitions $\nu\big|_{\cap[b_k,a_{k+1})}$ are irreducible with respect to $\lambda/\mu\big|_{[b_k,a_{k+1})}$, which is the skew partition λ/μ with columns $1, 2, \ldots, b_k - 1, a_{k+1}, a_{k+1}+1, \ldots$ removed.

Proof. Part **(a)** has already been proven.

(b) [Fill me in]

To verify the second claim, we need to show that if $[c, d)$ is a nonempty interval contained in $[b_k, a_{k+1})$ for some k, then $\#\nu'\big|^{J}_{\subseteq[c,d)} < d - c$, where J denotes the restricted skew partition $\lambda/\mu\big|_{[b_k,a_{k+1})}$, and where $\nu' = \nu\big|_{\cap[b_k,a_{k+1})}$. [Fill me in] $\square$

Definition 6.2. In the context of Lemma 6.1, for $0 \le k \le r$ the subpartitions $\nu\big|_{\cap[b_k,a_{k+1})}$ are called *the irreducible components of* ν and the numbers $n_k := a_{k+1} - b_k - \#\nu\big|_{\cap[b_k,a_{k+1})}$ are called their *degrees* (for T with $\operatorname{seplist}(T) = \nu$, the k-th degree n_k is equal to the number of pure columns of T inside the corresponding k-th irreducible component).

Example 6.3. For $\nu = (4, 3, 3, 2)$ we have $r = 1, b_0 = 1, a_1 = 3, b_1 = 5, a_2 = 8$. The irreducible components of ν are (4) and (2) and their degrees are $3 - 1 - 1 = 1$ and $8 - 5 - 1 = 2$ respectively. We have $\nu\big|_{\cap[1,3)} = (4), \nu\big|_{\subseteq[3,5)} = (3, 3), \nu\big|_{\cap[5,8)} = (2)$.

6.2. The structural theorem and its applications

It is easy to see that for a 12-rpp T, the number $\#\operatorname{seplist}(T)$ is equal to the number of mixed columns in T.

Let $\mathrm{RPP}^{12}(\lambda/\mu)$ denote the set of all 12-rpps T of shape λ/μ, and let $\mathrm{RPP}^{12}(\lambda/\mu; \nu)$ denote its subset that contains only 12-rpps T with $\operatorname{seplist}(T) = \nu$. Now we are ready to state a theorem that completely describes the structure of irreducible components:

Theorem 6.4. Let ν be an irreducible partition. Then for all $0 \le m \le \lambda_1 - \#\nu$ there is exactly one 12-rpp $T \in \mathrm{RPP}^{12}(\lambda/\mu; \nu)$ with $\#\nu$ mixed columns, m 1-pure columns and $(\lambda_1 - \#\nu - m)$ 2-pure columns. Moreover, these are the only elements of $\mathrm{RPP}^{12}(\lambda/\mu; \nu)$. In other words, for an irreducible partition ν we have

$$\sum_{T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)} \mathbf{x}^{\operatorname{ircont}(T)} = (x_1 x_2)^{\#\nu} P_{\lambda_1 - \#\nu}(x_1, x_2). \tag{26}$$

Example 6.5. Each of the two 12-rpps on Figure 2 has two irreducible components. One of them is supported on the first two columns and the other one is supported on the last three columns. Here are all possible 12-rpps for each component:

$$\begin{array}{|c|c|}\hline 1 & \mathbf{1} \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline \mathbf{1} & 2 \\ \hline 2 & 2 \\ \hline \end{array} \qquad \Bigg| \qquad \begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{1} & 1 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\mathbf{1}} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{1} & 2 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{\mathbf{1}} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{2} & 2 \\ \hline \multicolumn{1}{|c|}{\mathbf{1}} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

$\lambda = (2,2);\ \mu = ();\ \nu = (1)$ | $\lambda = (3,3,3);\ \mu = (1);\ \nu = (2)$.

After decomposing into irreducible components, we can obtain a formula for general representable partitions:

Corollary 6.6. Let ν be a representable partition. Then

$$\sum_{T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)} \mathbf{x}^{\mathrm{ircont}(T)} = (x_1x_2)^M P_{n_0}(x_1,x_2) P_{n_1}(x_1,x_2) \dots P_{n_r}(x_1,x_2), \tag{27}$$

where the numbers $M, r, n_0, \dots, n_r$ are defined above: $M = \#\nu$, $r+1$ is the number of irreducible components of ν and $n_0, n_1, \dots, n_r$ are their degrees.

For a 12-rpp T, the vectors $\mathrm{seplist}(T)$ and $\mathrm{ceq}(T)$ uniquely determine each other: if $(\mathrm{ceq}(T))_i = h$ then $\mathrm{seplist}(T)$ contains exactly $\lambda_{i+1} - \mu_i - h$ entries equal to i, and this correspondence is one-to-one. Therefore, the polynomials on both sides of (27) are equal to $Q_\alpha(x_1,x_2)$ where the vector α is the one that corresponds to ν.

Note that the polynomials $P_n(x_1,x_2)$ are symmetric for all n. Since the question about the symmetry of $\widetilde{g}_{\lambda/\mu}$ can be reduced to the two-variable case, Theorem 6.4 gives an alternative proof of the symmetry of $\widetilde{g}_{\lambda/\mu}$:

Corollary 6.7. The polynomials $\widetilde{g}_{\lambda/\mu} \in \mathbf{k}[x_1, x_2, x_3, \dots]$ are symmetric.

This holds for any $\mathbf{k}$ and any $t_1, t_2, t_3, \dots \in \mathbf{k}$, since the case we have considered (where $t_1, t_2, t_3, \dots$ are polynomial indeterminates over $\mathbb{Z}$) is universal.[25]

Another application of Theorem 6.4 is a complete description of Bender-Knuth involutions on rpps.

Corollary 6.8. Let ν be an irreducible partition. Then there is a unique map $b : \mathrm{RPP}^{12}(\lambda/\mu;\nu) \to \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ such that for all $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ we have

$$\mathrm{ircont}(T) = (c_1, c_2, 0, 0, \dots) \Longrightarrow \mathrm{ircont}(b(T)) = (c_2, c_1, 0, 0, \dots).$$

[25] Of course, our standing assumption that λ/μ is connected can be lifted here, because in general, $\widetilde{g}_{\lambda/\mu}$ is the product of the analogous power series corresponding to the connected components of λ/μ. So we have obtained a new proof of Theorem 3.4.

In this case, such a map is an involution on $\mathrm{RPP}^{12}(\lambda/\mu;\nu)$. So, for irreducible partition ν the corresponding Bender-Knuth involution exists and is unique.

Take any 12-rpp $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ and recall that a 12-table $\mathrm{flip}(T)$ is obtained from T by simultaneously replacing all entries in 1-pure columns by 2 and all entries in 2-pure columns by 1.

Corollary 6.9. If ν is an irreducible partition, then no matter in which order one resolves conflicts in $\mathrm{flip}(T)$, the resulting tableau T' will be the same. The map $T \to T'$ is the unique Bender-Knuth involution on $\mathrm{RPP}^{12}(\lambda/\mu;\nu)$.

Proof of Corollary 6.9. By Proposition **??**, conflict-resolving steps applied in any order give an element of $\mathrm{RPP}^{12}(\lambda/\mu;\nu)$. So we get a map $\mathrm{RPP}^{12}(\lambda/\mu;\nu) \to \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ that satisfies the assumptions of Corollary 6.8. □

Finally, notice that for a general representable partition ν conflicts may only occur inside each irreducible component independently, so we conclude the chain of corollaries by stating that our constructed involutions are canonical in the following sense:

Corollary 6.10. For a representable partition, the map $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu;\nu) \to \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ is the unique involution that interchanges the number of 1-pure columns with the number of 2-pure columns inside each irreducible component.

6.3. The proof

Let $\nu = (i_1, \dots, i_s)$ be an irreducible partition. We start with the following simple observation:

Lemma 6.11. Let $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$. Then any 1-pure column of T is to the left of any 2-pure column of T.

Proof of Lemma 6.11. Suppose it is false and we have a 1-pure column to the right of a 2-pure column. Consider the closest pair of columns a and b such that a is 2-pure and b is 1-pure, then the columns $a+1, \dots, b-1$ must all be mixed. Therefore the set $\mathrm{NS}(T)$ contains $\{(i_1, a+1), (i_2, a+2), \dots, (i_{b-1-a}, b-1)\}$. And because a is 2-pure and b is 1-pure, the support of any i_k for $k = 1 \dots b-1-a$ is a subset of $[a+1, b)$ which contradicts the irreducibility of ν. □

Proof of Theorem 6.4. We proceed by induction on the number of columns in λ/μ. If the number of columns is one then the statement of Theorem 6.4 is obvious. Suppose that we have proven that for all skew partitions $\widetilde{\lambda/\mu}$ with less than λ_1 columns and for all partitions $\widetilde{\nu}$ irreducible with respect to $\widetilde{\lambda/\mu}$ and for all

$0 \leq \widetilde{m} \leq \widetilde{\lambda}_1) - \#\widetilde{\nu}$, there is exactly one 12-rpp $\widetilde{T}$ of shape $\widetilde{\lambda/\mu}$ with exactly $\widetilde{m}$ 1-pure columns, exactly $\#\widetilde{\nu}$ mixed columns and exactly $(\widetilde{\lambda}_1 - \#\widetilde{\nu} - \widetilde{m})$ 2-pure columns. Now we want to prove the same for λ/μ.

Take any 12-rpp $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ with $\operatorname{seplist}(T) = \nu$ and with m 1-pure columns for $0 \leq m \leq \lambda_1 - \#\nu$. Suppose first that $m > 0$. Then there is at least one 1-pure column in T. Let $q+1$ be the leftmost such column for some $q \geq 0$. Then by Lemma 6.11 the columns $1, 2, \ldots, q$ are mixed. If $q > 0$ then the supports of $i_1, i_2, \ldots, i_q$ are all contained inside $[1, q+1)$ and we get a contradiction with the irreducibility of ν. The only remaining case is that $q = 0$ and the first column of T is 1-pure. Let $\widetilde{\lambda/\mu}$ denote λ/μ with the first column removed. Then ν may not be irreducible with respect to $\widetilde{\lambda/\mu}$, because it may happen that $\#\nu|_{\subseteq[1,b)}^{\widetilde{\lambda/\mu}} = b-1$ for some b. In this case we can ignore these first $b-1$ mixed columns and apply the induction hypothesis. We are done with the case $m > 0$. If $m < \lambda_1 - \#\nu$ then we can apply a mirrored argument to the last column, and it remains to note that for $\lambda_1 - \#\nu > 0$ the cases $m > 0$ and $m < \lambda_1 - \#\nu$ cover everything and for $\lambda_1 = \#\nu$ we get that T is a 12-rpp with all mixed columns, but there is obviously a unique such 12-rpp. □

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