Refined dual stable Grothendieck polynomials and generalized Bender-Knuth involutions

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1. Introduction

Thomas Lam and Pavlo Pylyavskyy, in [LamPyl07, §9.1], (and earlier Mark Shimozono and Mike Zabrocki in unpublished work of 2003) studied *dual stable Grothendieck polynomials*, a deformation (in a sense) of the Schur functions. Let us briefly recount their definition.

Let λ/μ be a skew partition. The Schur function $s_{\lambda/\mu}$ is a multivariate generating function for the semistandard tableaux of shape λ/μ. In the same vein, the dual stable Grothendieck polynomial $g_{\lambda/\mu}$ is a generating function for the reverse plane partitions of shape λ/μ; these, unlike semistandard tableaux, are only required to have their entries increase *weakly* down columns (and along rows). More precisely, $g_{\lambda/\mu}$ is a formal power series in countably many commuting indeterminates $x_1, x_2, x_3, \ldots$ defined by

$$g_{\lambda/\mu} = \sum_{\substack{T \text{ is a reverse plane} \\ \text{partition of shape } \lambda/\mu}} \mathbf{x}^{\mathrm{ircont}(T)},$$

where $\mathbf{x}^{\mathrm{ircont}(T)}$ is the monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ whose i-th exponent a_i is the number of columns of T containing the entry i. As proven in [LamPyl07, §9.1], this power series $g_{\lambda/\mu}$ is a symmetric function (albeit, unlike $s_{\lambda/\mu}$, an inhomogeneous one in general). Lam and Pylyavskyy connect the $g_{\lambda/\mu}$ to the (more familiar) *stable Grothendieck polynomials* $G_{\lambda/\mu}$ (via a duality between the symmetric functions and their completion, which explains the name of the $g_{\lambda/\mu}$; see [LamPyl07, §9.4]) and to the *K*-theory of Grassmannians ([LamPyl07, §9.5]).

We devise a common generalization of the dual stable Grothendieck polynomial $g_{\lambda/\mu}$ and the classical skew Schur function $s_{\lambda/\mu}$. Namely, if $t_1, t_2, t_3, \ldots$ are

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countably many indeterminates, then we set

$$\widetilde{g}_{\lambda/\mu} = \sum_{\substack{T \text{ is a reverse plane} \\ \text{partition of shape } \lambda/\mu}} \mathbf{t}^{\mathrm{ceq}(T)} \mathbf{x}^{\mathrm{ircont}(T)},$$

where $\mathbf{t}^{\mathrm{ceq}(T)}$ is the product $t_1^{b_1} t_2^{b_2} t_3^{b_3} \cdots$ whose i-th exponent b_i is the number of cells in the i-th row of T whose entry equals the entry of their neighbor cell directly below them. This $\widetilde{g}_{\lambda/\mu}$ becomes $g_{\lambda/\mu}$ when all the t_i are set to 1, and becomes $s_{\lambda/\mu}$ when all the t_i are set to 0.

Our main result, Theorem 3.4, states that $\widetilde{g}_{\lambda/\mu}$ is a symmetric function (in the $x_1, x_2, x_3, \ldots$).

We prove this result (thus obtaining a new proof of [LamPyl07, Theorem 9.1]) first using an elaborate generalization of the classical Bender-Knuth involutions to reverse plane partitions.

TODO 1.1. Pavel: advertise your section here? It sort-of gives some justification for why the t_i are a natural thing to consider too.

The present paper is organized as follows: In Section 2, we recall classical definitions and introduce notations pertaining to combinatorics and symmetric functions. In Section 3, we define the dual stable Grothendieck polynomials $\widetilde{g}_{\lambda/\mu}$, state our main result (that they are symmetric functions), and do the first steps of its proof (by reducing it to a purely combinatorial statement about the existence of an involution with certain properties). In Section 5, we prove our main result by constructing the required involution. In Section 6, we recapitulate the definition of the classical Bender-Knuth involution, and sketch the proof that our involution is a generalization of the latter.

TODO 1.2. Pavel: your last section should also be summarized here.

TODO 1.3. Rewrite this after the shortening process is finished.

TODO 1.4. What is missing from the above introduction? Specifically:

- Are there any more results from the paper that should be advertised here?

- More reasons why care about dual stable Grothendieck polynomials? (Buch, Knutson, Postnikov could know some.)

- What I wrote about K-theory is rather shallow. More details?

  More specifically, and interestingly, I am wondering if our $\widetilde{g}_{\lambda/\mu}$ aren't K-theoretical classes of something multigraded (toric structure on the Grassmannian? there are two sides from which we can multiply a matrix by a diagonal matrix, and even if we "use up" one for taking "characters", the other one is still there).

1.1. Acknowledgments

We owe our familiarity with dual stable Grothendieck polynomials to Richard Stanley. We thank Alexander Postnikov for providing context and motivation.

TODO 1.5. Keep this up to date.

2. Notations and definitions

Let us begin by defining our notations (including some standard conventions from algebraic combinatorics).

2.1. Partitions and tableaux

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$.

A sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ of nonnegative integers is called a *weak composition* if the sum of its entries (denoted $|\alpha|$) is finite. We shall always write α_i for the i-th entry of a weak composition α.

A *partition* is a weak composition $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ satisfying $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$. As usual, we often omit trailing zeroes when writing a partition (e.g., the partition $(5, 2, 1, 0, 0, 0, \ldots)$ can thus be written as $(5, 2, 1)$).

TODO 2.1. Come up with a better notation for ircont, stagnant.

We identify each partition λ with the subset $\{(i, j) \in \mathbb{N}_+^2 \mid j \leq \lambda_i\}$ of $\mathbb{N}_+^2$ (called *the Young diagram of λ*). We draw this subset as a Young diagram (which is a left-aligned table of empty boxes, where the box $(1, 1)$ is in the top-left corner while the box $(2, 1)$ is direcly below it; this is the *English notation*, also known as the *matrix notation*).

TODO 2.2. What is the easiest place to refer the reader to for Young diagram basics, which uses notations compatible with ours (such as "filling" and "skew partition")?

A *skew partition λ/μ* is a pair of partitions satisfying $\mu \subset \lambda$ (as subsets of the plane). In this case, we shall also often use the notation λ/μ for the set-theoretic difference of λ and μ.

If λ/μ is a skew partition, then a *filling* of λ/μ means a map $T : \lambda/\mu \to \mathbb{N}_+$. It is visually represented by drawing λ/μ and filling each box c with the entry $T(c)$. Three examples of a filling can be found on Figure 1.

A filling $T : \lambda/\mu \to \mathbb{N}_+$ of λ/μ is called a *reverse plane partition of shape λ/μ* if its values increase weakly in each row of λ/μ from left to right and in each column of λ/μ from top to bottom. If, in addition, the values of T increase strictly down each column, then T is called a *semistandard tableau of shape λ/μ*.

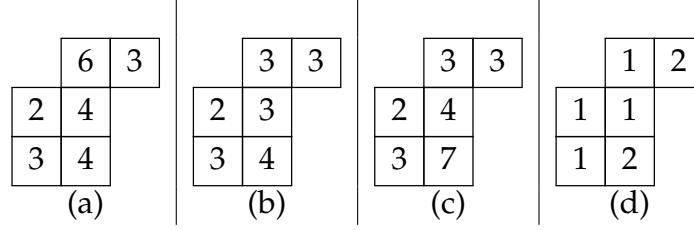|	6	3			3	3			3	3			1	2
2	4			2	3			2	4			1	1	
3	4			3	4			3	7			1	2	
	(a)				(b)				(c)				(d)	

Figure 1: Fillings of $(3,2,2)/(1)$: (a) is not an rpp as it has a 4 below a 6, (b) is an rpp but not a semistandard tableau as it has a 3 below a 3, (c) is a semistandard tableau (and hence also an rpp), (d) is a 12-rpp.

(See [Fulton97] for an exposition of properties and applications of semistandard tableaux.) We denote the set of all reverse plane partitions of shape λ/μ by RPP (λ/μ). We abbreviate reverse plane partitions as *rpps*. We define a *12-rpp* to be an rpp whose entries all belong to the set $\{1,2\}$ and denote the set of all 12-rpps of shape λ/μ by RPP^{12} (λ/μ).

> **TODO 2.3.** The notion of a 12-rpp is technical, at least in my part of the paper. Should we really introduce it here in this section?

Examples of an rpp, a 12-rpp and of a semistandard tableau can be found on Figure 1.

2.2. Symmetric functions

> **TODO 2.4.** Decide whether we want to work over $\mathbb{Z}$ or over an arbitrary commutative field **k** with unity.

A *symmetric function* is defined to be a bounded-degree[1] power series in $x_1, x_2, \ldots$ that is invariant under (finite) permutations[2] of $x_1, x_2, \ldots$.

> **TODO 2.5.** Make ground ring explicit (even if it is $\mathbb{Z}$).

The symmetric functions form a ring, which is called the *ring of symmetric functions* and denoted by Λ. (In [LamPyl07] this ring is denoted by Sym, while the notation Λ is reserved for the set of all partitions.) Symmetric functions are a classical field of research, and are closely related to Young diagrams and tableaux; see [Stan99, Chapter 7], [Macdon95] and [GriRei15, Chapter 2] for expositions.

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[1]A power series is said to be *bounded-degree* if there is an $N \in \mathbb{N}$ such that only monomials of degree $\leq N$ appear in the series.

[2]A permutation is *finite* if it fixes all but finitely many positive elements.

TODO 2.6. Update Grinberg-Reiner reference to whatever is most recent.

Given a filling T of a skew partition λ/μ, its *content* is a weak composition $\operatorname{cont}(T) = (r_1, r_2, \dots)$ where $r_i = \left|T^{-1}(i)\right|$ is the number of entries of T equal to i. For a skew partition λ/μ, we define the *Schur function* $s_{\lambda/\mu}$ to be the formal power series

$$s_{\lambda/\mu}(x_1, x_2, \dots) = \sum_{\substack{T \text{ is a semistandard} \\ \text{tableau of shape } Y(\lambda/\mu)}} \mathbf{x}^{\operatorname{cont}(T)}.$$

A nontrivial property of these Schur functions is that they are symmetric:

Proposition 2.7. We have $s_{\lambda/\mu} \in \Lambda$ for every skew partition λ/μ.

This result appears, e.g., in [Stan99, Theorem 7.10.2] and [GriRei15, Proposition 2.11]; it is commonly proven bijectively using the so-called *Bender-Knuth involutions*. We shall recall the definitions of these involutions in Section 6.

Replacing "semistandard tableau" by "rpp" in the definition of a Schur function in general gives a non-symmetric function. Nevertheless, Lam and Pylyavskyy [LamPyl07, §9] have noticed that it is possible to define symmetric functions from rpps, albeit it requires replacing the content $\operatorname{cont}(T)$ by a subtler construction.

Namely, for a filling T of a skew partition λ/μ, we define its *irredundant content* as a weak composition $\operatorname{ircont}(T) = (r_1, r_2, \dots)$ where r_i is the number of *columns* (rather than cells) of T that contain an entry equal to i. For instance, if T_a, T_b, T_c and T_d are the fillings from Figure 1, then their irredundant contents are given by

$$\operatorname{ircont}(T_a) = (0, 1, 2, 1, 0, 1), \qquad \operatorname{ircont}(T_b) = (0, 1, 3, 1),$$
$$\operatorname{ircont}(T_c) = (0, 1, 3, 1, 0, 0, 1), \qquad \operatorname{ircont}(T_d) = (2, 2),$$

because, for example, T_a has one column with a 4 in it (so $(\operatorname{ircont} T_a)_4 = 1$) and T_b contains three columns with a 3 (so $(\operatorname{ircont} T_b)_3 = 3$).

Notice that if T is a semistandard tableau, then $\operatorname{cont}(T)$ and $\operatorname{ircont}(T)$ coincide.

For the rest of this section, we fix a skew partition λ/μ. Now, the *dual stable Grothendieck polynomial* $g_{\lambda/\mu}$ is defined to be the formal power series

$$\sum_{\substack{T \text{ is an rpp} \\ \text{of shape } Y(\lambda/\mu)}} \mathbf{x}^{\operatorname{ircont}(T)}.$$

Unlike the Schur function $s_{\lambda/\mu}$, it is (in general) not homogeneous, because whenever a column of an rpp T contains an entry several times, the corresponding monomial $\mathbf{x}^{\operatorname{ircont}(T)}$ "counts" this entry only once. It is fairly clear that the highest-degree homogeneous component of $g_{\lambda/\mu}$ is $s_{\lambda/\mu}$ (the component of degree $|\lambda| - |\mu|$). Therefore, $g_{\lambda/\mu}$ can be regarded as an inhomogeneous deformation of the Schur function $s_{\lambda/\mu}$.

Lam and Pylyavskyy, in [LamPyl07, §9.1], have shown the following fact:

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Proposition 2.8. We have $g_{\lambda/\mu} \in \Lambda$ for every skew partition λ/μ.

They prove this proposition using generalized plactic algebras [FomGre06, Lemma 3.1] (and also give a second, combinatorial proof for the case $\mu = \varnothing$ by explicitly expanding $g_{\lambda/\varnothing}$ as a sum of Schur functions).

In the next section, we shall introduce a refinement of these $g_{\lambda/\mu}$, and later we will reprove Proposition 2.8 in a bijective and elementary way.

3. Refined dual stable Grothendieck polynomials

3.1. Definition

Let $\mathbf{t} = (t_1, t_2, t_3, \ldots)$ be a sequence of indeterminates. For any weak composition α, we define $\mathbf{t}^\alpha$ to be the monomial $t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \cdots$.

If T is a filling of a skew partition λ/μ, then a *stagnant cell* of T is a cell of λ/μ whose entry is equal to the entry directly below it. That is, a cell (i, j) of λ/μ is stagnant if $(i+1, j)$ is also a cell of λ/μ and $T(i, j) = T(i+1, j)$. Notice that a semistandard tableau is the same thing as an rpp which has no stagnant cells.

If T is a filling of λ/μ, then we define the *column equalities vector* of T to be a weak composition $\operatorname{ceq}(T) = (c_1, c_2, c_3, \ldots)$ where c_i is the number of $j \in \mathbb{N}_+$ such that (i, j) is a stagnant cell of T. Visually speaking, $(\operatorname{ceq}(T))_i$ is the number of columns of T whose entry in the i-th row equals their entry in the $(i+1)$-th row. For instance, for fillings T_a, T_b, T_c, T_d from Figure 1 we have $\operatorname{ceq}(T_a) = (0, 1)$, $\operatorname{ceq}(T_b) = (1, 0)$, $\operatorname{ceq}(T_c) = (0)$ and $\operatorname{ceq}(T_d) = (1, 1)$.

Notice that $|\operatorname{ceq}(T)|$ is the number of stagnant cells in T, so we have

$$|\operatorname{ceq}(T)| + |\operatorname{ircont}(T)| = |\lambda/\mu| \tag{1}$$

for all rpps T of shape λ/μ.

Let now λ/μ be a skew partition. We set

$$\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = \sum_{\substack{T \text{ is an rpp} \\ \text{of shape } \lambda/\mu}} \mathbf{t}^{\operatorname{ceq}(T)} \mathbf{x}^{\operatorname{ircont}(T)}.$$

Let us give some examples of $\widetilde{g}_{\lambda/\mu}$.

Example 3.1. (a) If λ/μ is a single row with n cells, then for each rpp T of shape λ/μ we have $\operatorname{ceq}(T) = (0, 0, \ldots)$ and $\operatorname{ircont}(T) = \operatorname{cont}(T)$ (in fact, any rpp of shape λ/μ is a semistandard tableau in this case). Therefore we get

$$\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = h_n(\mathbf{x}) = \sum_{a_1 \le a_2 \le \cdots \le a_n} x_{a_1} x_{a_2} \ldots x_{a_n}.$$

Here $h_n(\mathbf{x})$ is the n-th complete homogeneous symmetric function.

(b) If λ/μ is a single column with n cells then by (1) for all rpps T of shape λ/μ we have $|\mathrm{ceq}(T)| + |\mathrm{ircont}(T)| = n$ so in this case

$$\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = e_n(t_1, t_2, \ldots, t_{n-1}, x_1, x_2, \ldots) = \sum_{k=0}^{n} e_k(t_1, t_2, \ldots, t_{n-1}) \, e_{n-k}(x_1, x_2, \ldots),$$

where $e_i(\xi_1, \xi_2, \xi_3, \ldots)$ denotes the i-th elementary symmetric function in the indeterminates $\xi_1, \xi_2, \xi_3, \ldots$.

> **TODO 3.2.** Do we really need the third example here? (the one with $\lambda = (2, 1)$) DG: I expected both of you to care more about these examples than I did.

The power series $\widetilde{g}_{\lambda/\mu}$ generalize the power series $g_{\lambda/\mu}$ and $s_{\lambda/\mu}$ studied before. The following proposition is clear:

Proposition 3.3. Let λ/μ be a skew partition.
   (a) Specifying $\mathbf{t} = (1, 1, 1, \ldots)$ yields $\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = g_{\lambda/\mu}(\mathbf{x})$.
   (b) Specifying $\mathbf{t} = (0, 0, 0, \ldots)$ yields $\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = s_{\lambda/\mu}(\mathbf{x})$.

$\square$

3.2. The symmetry statement

Our main result is now the following:

Theorem 3.4. Let λ/μ be a skew partition. Then $\widetilde{g}_{\lambda/\mu}(\mathbf{x}, \mathbf{t})$ is symmetric in $\mathbf{x}$.

Clearly, this implies the symmetry of $g_{\lambda/\mu}$ and $s_{\lambda/\mu}$ due to Proposition 3.3.
   We shall prove Theorem 3.4 bijectively. The core of our proof will be the following restatement of Theorem 3.4:

Theorem 3.5. Let λ/μ be a skew partition and let $i \in \mathbb{N}_+$. Then, there exists an involution $\mathbf{B}_i : \mathrm{RPP}(\lambda/\mu) \to \mathrm{RPP}(\lambda/\mu)$ which preserves the ceq statistics and acts on ircont by the transposition of its i-th and $i+1$-th entries.

This involution $\mathbf{B}_i$ is a generalization of the i-th Bender-Knuth involution defined for semistandard tableaux (see, e.g., [GriRei15, proof of Proposition 2.11]), but its definition is more complicated than that of the latter.[3] Defining it and proving its properties will take a significant part of this paper.

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[3]We will compare our involution $\mathbf{B}_i$ with the i-th Bender-Knuth involution in Section 6.

3.3. Reduction to 12-rpps

We shall make one further simplification before we step to the actual proof of Theorem 3.5.

> **Lemma 3.6.** There exists an involution $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ (defined canonically in terms of λ/μ) which preserves the ceq statistics and switches the number of columns containing a 1 with the number of columns containing a 2 (i.e. acts as a transposition of 1 and 2 on ircont).

This Lemma implies Theorem 3.5: for any $i \in \mathbb{N}_+$ and for T an rpp of shape λ/μ, we construct $\mathbf{B}_i(T)$ as follows:

- Ignore all entries of T not equal to i or $i+1$.

- Replace all occurrences of i by 1 and all occurrences of $i+1$ by 2. We get a 12-rpp T' of some smaller shape (which is still a skew partition).

- Replace T' by $\mathbf{B}(T')$.

- In $\mathbf{B}(T')$, replace back all occurrences of 1 by i and all occurrences of 2 by $i+1$.

- Finally, restore the remaining entries of T that were ignored on the first step.

It is clear that this operation acts on $\mathrm{ircont}(T)$ by a transposition of the i-th and $i+1$-th entries. The fact that it does not change $\mathrm{ceq}(T)$ is also not hard to show: The set of stagnant cells remains the same.

4. Construction of B

In this section we are going to sketch the definition of **B** and state some of its properties. We postpone the proofs until the next section.

For the whole Sections 4 and 5, we shall be working in the situation of Lemma 3.6. In particular, we fix a skew partition λ/μ.

A *12-table* means a filling $T : \lambda/\mu \to \{1,2\}$ of λ/μ such that the entries of T are weakly increasing down columns. (We do not require them to be weakly increasing along rows.) Every column of a 12-table is a sequence of the form $(1,1,\ldots,1,2,2,\ldots,2)$. We say that such a sequence is

- *1-pure* if it is nonempty and consists purely of 1's,

- *2-pure* if it is nonempty and consists purely of 2's,

- *mixed* if it contains both 1's and 2's.

Definition 4.1. For a 12-table T, we define $\mathrm{flip}(T)$ to be the 12-table obtained from T by changing each column of T as follows:

- **If** this column is 1-pure, we replace all its entries by 2's (so that it becomes 2-pure).

  Otherwise, if this column is 2-pure, we replace all its entries by 1's (so that it becomes 1-pure).

  Otherwise (i.e., if this column is mixed), we do not change it.

If T is a 12-rpp then $\mathrm{flip}(T)$ need not be a 12-rpp, because it can contain a 2 to the left of a 1 in some row. We say that a positive integer k is a *descent* of a 12-table P if there is a 2 in the column k and there is a 1 to the right of it in the column $k+1$. We will encounter three possible kinds of descents depending on the types of columns k and $k+1$:

(M1) The k-th column of P is mixed and the $(k+1)$-th column of P is 1-pure.

(2M) The k-th column of P is 2-pure and the $(k+1)$-th column of P is mixed.

(21) The k-th column of P is 2-pure and the $(k+1)$-th column of P is 1-pure.

For an arbitrary 12-table it can happen also that two mixed columns form a descent, but such a descent will never arise in our process.

For each of the three types of descents we will define the corresponding *descent-resolution step* that can be informally explained by Figure 2:
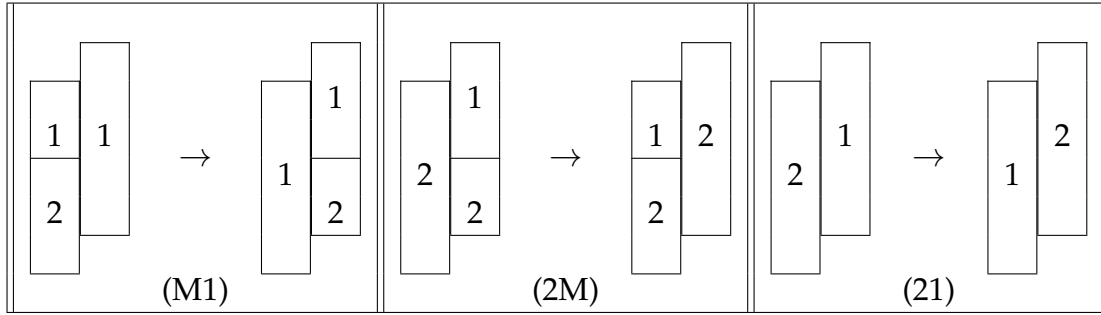
Figure 2: The three descent-resolution steps

For example, if k is a descent of type (M1) in a 12-table P, then we define the table $\mathrm{res}_k P$ as follows: the k-th column of $\mathrm{res}_k P$ is 1-pure; the $(k+1)$-th column of $\mathrm{res}_k P$ is mixed and the highest 2 in it is in the same row as the highest 2 in the k-th column of P; all other columns of $\mathrm{res}_k P$ are copied over from P unchanged.

Now the map **B** is defined as follows: take any 12-rpp T and apply flip to it to get a 12-table $\mathrm{flip}(T)$. Next, apply descent-resolution steps to $\mathrm{flip}(T)$ in arbitrary

order until we get a 12-table with no descents left. Put $\mathbf{B}(T) := P$. (A rigorous statement of this is Definition 5.12.)

In the next section we will see that $\mathbf{B}(T)$ is well-defined (that is, the process terminates after a finite number of descent-resolution steps, and the result does not depend on the order of steps). We will also see that $\mathbf{B}$ is an involution $\mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ that satisfies the assumptions of Lemma 3.6. An alternative proof of all these facts can be also found in Section 7.

5. Proof of Lemma 3.6

We now come to the actual proof of Lemma 3.6.

Recall that every column of a 12-table is a sequence of the form $(1, 1, \ldots, 1, 2, 2, \ldots, 2)$. If s is a sequence of the form $(1, 1, \ldots, 1, 2, 2, \ldots, 2)$, then we define the *signature* $\mathrm{sig}(s)$ of s to be

$$\mathrm{sig}(s) = \begin{cases} 0, & \text{if } s \text{ is 2-pure or empty;} \\ 1, & \text{if } s \text{ is mixed;} \\ 2, & \text{if } s \text{ is 1-pure} \end{cases}.$$

Definition 5.1. For any 12-table T, we define a nonnegative integer $\ell(T)$ by

$$\ell(T) = \sum_{h \in \mathbb{N}_+} h \cdot \mathrm{sig}(\text{the } h\text{-th column of } T).$$

For instance, if T is the 12-table

$$\begin{array}{|c|c|c|c|}
\hline
 & & 1 & 2 & 1 & 2 \\
\hline
\end{array}$$

		1	2	1	2
	1	1	2		
2	1	1	2		
2	2				

(2)

then $\ell(T) = 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 0 + 5 \cdot 2 + 6 \cdot 0 + 7 \cdot 0 + 8 \cdot 0 + \cdots = 18$.

5.1. Descents, separators, and benign 12-tables

If T is a 12-table, then we define a *descent* of T to be a positive integer i such that there exists an $r \in \mathbb{N}_+$ satisfying $(r, i) \in \lambda/\mu$, $(r, i+1) \in \lambda/\mu$, $T(r, i) = 2$ and $T(r, i+1) = 1$. For instance, the descents of the 12-table shown in (2) are 1 and 4. Clearly, a 12-rpp of shape λ/μ is the same as a 12-table which has no descents.

If T is a 12-table, and if $k \in \mathbb{N}_+$ is such that the k-th column of T is mixed, then we define $\mathrm{sep}_k T$ to be the smallest $r \in \mathbb{N}_+$ such that $(r, k) \in \lambda/\mu$ and

$T(r, k) = 2$. Thus, every 12-table T, every $r \in \mathbb{N}_+$ and every $k \in \mathbb{N}_+$ such that the k-th column of T is mixed and such that $(r, k) \in \lambda/\mu$ satisfy

$$T(r, k) = \begin{cases} 1, & \text{if } r < \text{sep}_k T; \\ 2, & \text{if } r \geq \text{sep}_k T. \end{cases} \tag{3}$$

If T is a 12-table, then we let seplist T denote the list of all values $\text{sep}_k T$ (in the order of increasing k), where k ranges over all positive integers for which the k-th column of T is mixed. For instance, if T is

		1	1	1
	2	1	1	2
1	2	1		
2	2	2		

then $\text{sep}_1 T = 4$, $\text{sep}_3 T = 4$, and $\text{sep}_5 T = 2$ (and there are no other k for which $\text{sep}_k T$ is defined), so that seplist $T = (4, 4, 2)$.

We say that a 12-table T is *benign* if the list seplist T is weakly decreasing.[4] Notice that 12-rpps are benign 12-tables, but the converse is not true. If T is a benign 12-table, then

> there exists no descent k of T such that both the k-th column of T
> and the $(k+1)$-th column of T are mixed. $\tag{4}$

Let $\text{BT}^{12}(\lambda/\mu)$ denote the set of all benign 12-tables; we have $\text{RPP}^{12}(\lambda/\mu) \subseteq \text{BT}^{12}(\lambda/\mu)$.

If $T \in \text{BT}^{12}(\lambda/\mu)$ then $\text{flip}(T) \in \text{BT}^{12}(\lambda/\mu)$ as well because T and $\text{flip}(T)$ have the same mixed columns. Thus the map flip defined for 12-tables in Definition 4.1 restricts to a map $\text{BT}^{12}(\lambda/\mu) \to \text{BT}^{12}(\lambda/\mu)$ which we will also denote flip.

> **Remark 5.2.** It is clear that flip is an involution on $\text{BT}^{12}(\lambda/\mu)$ that preserves ceq and seplist but acts as a transposition s_1 on ircont (that is, we have $\text{ircont}(\text{flip}(T)) = s_1 \cdot \text{ircont}(T)$ for every $T \in \text{BT}^{12}(\lambda/\mu)$).

5.2. Plan of the proof

Let us now briefly sketch the ideas behind the rest of the proof before we go into them in detail. The map flip : $\text{BT}^{12}(\lambda/\mu) \to \text{BT}^{12}(\lambda/\mu)$ does not generally send 12-rpps to 12-rpps (i.e., it does not restrict to a map $\text{RPP}^{12}(\lambda/\mu) \to \text{RPP}^{12}(\lambda/\mu)$). However, we shall amend this by defining a way to transform any

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[4]For example, the 12-table in (2) is benign, but replacing its third column by $(1, 2, 2)$ and its fourth column by $(1, 1, 2)$ would yield a 12-table which is not benign.

benign 12-table into a 12-rpp by what we call "resolving descents". The process of "resolving descents" will be a stepwise process, and will be formalized in terms of a binary relation $\Rightarrow$ on the set $\mathrm{BT}^{12}(\lambda/\mu)$ which we will soon introduce. The intuition behind saying "$P \Rightarrow Q$" is that the benign 12-table P has a descent, resolving which yields the benign 12-table Q. Starting with a benign 12-table P, we can repeatedly resolve descents until this is no longer possible. We have some freedom in performing this process, because at any step there can be a choice of several descents to resolve; but we will see that the final result does not depend on the process. Hence, the final result can be regarded as a function of P. We will denote it by $\mathrm{norm}\, P$, and we will see that it is a 12-rpp. We will then define a map $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ by $\mathbf{B}(T) = \mathrm{norm}(\mathrm{flip}\, T)$, and show that it is an involution satisfying the properties that we want it to satisfy.

5.3. Resolving descents

Now we come to the details.

Let $k \in \mathbb{N}_+$. Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$. Assume (for the whole Subsection 5.3) that k is a descent of P. Thus, the k-th column of P must contain at least one 2. Hence, the k-th column of P is either mixed or 2-pure. Similarly, the $(k+1)$-th column of P is either mixed or 1-pure. But the k-th and the $(k+1)$-th columns of P cannot both be mixed (by (4), because P is benign). Thus, exactly one of the following three statements holds:

(M1) The k-th column of P is mixed and the $(k+1)$-th column of P is 1-pure.

(2M) The k-th column of P is 2-pure and the $(k+1)$-th column of P is mixed.

(21) The k-th column of P is 2-pure and the $(k+1)$-th column of P is 1-pure.

Now, we define a new 12-table $\mathrm{res}_k P$ as follows (see Figure 2 for illustration):

- If we have (M1), then $\mathrm{res}_k P$ is the 12-table defined as follows: The k-th column of $\mathrm{res}_k P$ is 1-pure; the $(k+1)$-th column of $\mathrm{res}_k P$ is mixed and satisfies $\mathrm{sep}_{k+1}(\mathrm{res}_k P) = \mathrm{sep}_k P$; all other columns of $\mathrm{res}_k P$ are copied over from P unchanged.[5]

- If we have (2M), then $\mathrm{res}_k P$ is the 12-table defined as follows: The k-th column of $\mathrm{res}_k P$ is mixed and satisfies $\mathrm{sep}_k(\mathrm{res}_k P) = \mathrm{sep}_{k+1} P$; the $(k+1)$-th column of $\mathrm{res}_k P$ is 2-pure; all other columns of $\mathrm{res}_k P$ are copied over from P unchanged.

- If we have (21), then $\mathrm{res}_k P$ is the 12-table defined as follows: The k-th column of $\mathrm{res}_k P$ is 1-pure; the $(k+1)$-th column of $\mathrm{res}_k P$ is 2-pure; all other columns of $\mathrm{res}_k P$ are copied over from P unchanged.

---

[5]The reader should check that this 12-table is well-defined.

In either case, $\operatorname{res}_k P$ is a well-defined 12-table. It is furthermore clear that $\operatorname{seplist}(\operatorname{res}_k P) = \operatorname{seplist} P$. Thus, $\operatorname{res}_k P$ is benign (since P is benign); that is, $\operatorname{res}_k P \in \mathrm{BT}^{12}(\lambda/\mu)$. We say that $\operatorname{res}_k P$ is the 12-table obtained by *resolving* the descent k in P.

Sometimes we can resolve one of the several descents. For example:

Example 5.3. Let P be the 12-table on the left:

$$
\begin{array}{cccc}
P & \operatorname{res}_1 P & \operatorname{res}_2 P & \operatorname{res}_4 P
\end{array}
$$

P: rows $\;1\,2\,1\,/\,1\,1\,2\,/\,2\,1\,1\,/\,2\,2\,1\,/\,2$	$\operatorname{res}_1 P$: rows $\;1\,2\,1\,/\,2\,1\,2\,/\,1\,2\,1\,/\,2\,2\,1\,/\,2$	$\operatorname{res}_2 P$: rows $\;1\,2\,1\,/\,1\,1\,2\,/\,2\,1\,1\,/\,2\,1\,2\,/\,2$	$\operatorname{res}_4 P$: rows $\;1\,1\,2\,/\,1\,1\,1\,/\,2\,1\,1\,/\,2\,2\,1\,/\,2$

Then P is a benign 12-table, and its descents are 1, 2 and 4. We have $\operatorname{sep}_2 P = 4$.

If we set $k = 1$ then we have (2M), if we set $k = 2$ then we have (M1), and if we set $k = 4$ then we have (21). Resolving each of the conflicts results in the three 12-tables on the right.

We notice that each of the three 12-tables $\operatorname{res}_1 P$, $\operatorname{res}_2 P$ and $\operatorname{res}_4 P$ still has descents. In order to get a 12-rpp from P, we will have to keep resolving these descents until none remain.

We now observe some further properties of $\operatorname{res}_k P$:

Proposition 5.4. Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$ and $k \in \mathbb{N}_+$ be such that k is a descent of P.

(a) The 12-table $\operatorname{res}_k P$ differs from P only in columns k and $k+1$.

(b) The k-th and the $(k+1)$-th columns of $\operatorname{res}_k P$ depend only on the k-th and the $(k+1)$-th columns of P.

(c) We have
$$\operatorname{ceq}(\operatorname{res}_k P) = \operatorname{ceq}(P).$$

(d) We have
$$\operatorname{ircont}(\operatorname{res}_k P) = \operatorname{ircont}(P).$$

(e) We have that k is a descent of $\operatorname{flip}(\operatorname{res}_k P)$, and
$$\operatorname{res}_k(\operatorname{flip}(\operatorname{res}_k P)) = \operatorname{flip}(P).$$

(f) Recall that we defined a nonnegative integer $\ell(T)$ for every 12-table T in Definition 5.1. We have

$$\ell(P) > \ell(\mathrm{res}_k P).$$

Proof of Proposition 5.4. All parts of Proposition 5.4 follow from straightforward arguments using the definitions of res_k and flip and (3). $\square$

5.4. The descent-resolution relation $\Rightarrow$

Definition 5.5. Let us now define a binary relation $\Rightarrow$ on the set $\mathrm{BT}^{12}(\lambda/\mu)$ as follows: Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$ and $Q \in \mathrm{BT}^{12}(\lambda/\mu)$. If $k \in \mathbb{N}_+$, then we write $P \underset{k}{\Rightarrow} Q$ if k is a descent of P and we have $Q = \mathrm{res}_k P$. We write $P \Rightarrow Q$ if there exists a $k \in \mathbb{N}_+$ such that $P \underset{k}{\Rightarrow} Q$.

Proposition 5.4 translates into the following properties of this relation $\Rightarrow$:

Lemma 5.6. Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$ and $Q \in \mathrm{BT}^{12}(\lambda/\mu)$ be such that $P \Rightarrow Q$. Then:
    (a) We have $\mathrm{ceq}(Q) = \mathrm{ceq}(P)$.
    (b) We have $\mathrm{ircont}(Q) = \mathrm{ircont}(P)$.
    (c) The benign 12-tables $\mathrm{flip}(P)$ and $\mathrm{flip}(Q)$ have the property that $\mathrm{flip}(Q) \Rightarrow \mathrm{flip}(P)$.
    (d) We have $\ell(P) > \ell(Q)$.

We now define $\overset{*}{\Rightarrow}$ to be the reflexive-and-transitive closure of the relation $\Rightarrow$. [6] This relation $\overset{*}{\Rightarrow}$ is reflexive and transitive, and extends the relation $\Rightarrow$. Lemma 5.6 thus yields:

Lemma 5.7. Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$ and $Q \in \mathrm{BT}^{12}(\lambda/\mu)$ be such that $P \overset{*}{\Rightarrow} Q$. Then:
    (a) We have $\mathrm{ceq}(Q) = \mathrm{ceq}(P)$.
    (b) We have $\mathrm{ircont}(Q) = \mathrm{ircont}(P)$.
    (c) The benign 12-tables $\mathrm{flip}(P)$ and $\mathrm{flip}(Q)$ have the property that $\mathrm{flip}(Q) \overset{*}{\Rightarrow} \mathrm{flip}(P)$.
    (d) We have $\ell(P) \geq \ell(Q)$.

---

[6]Explicitly, this means that $\overset{*}{\Rightarrow}$ is defined as follows: For two elements $P \in \mathrm{BT}^{12}(\lambda/\mu)$ and $Q \in \mathrm{BT}^{12}(\lambda/\mu)$, we have $P \overset{*}{\Rightarrow} Q$ if and only if there exists a sequence $(a_0, a_1, \ldots, a_n)$ of elements of $\mathrm{BT}^{12}(\lambda/\mu)$ such that $a_0 = P$ and $a_n = Q$ and such that every $i \in \{0, 1, \ldots, n-1\}$ satisfies $a_i \Rightarrow a_{i+1}$. (Note that n is allowed to be 0.)

We now state the following crucial lemma:

Lemma 5.8. Let A, B and C be three elements of $\mathrm{BT}^{12}\left(\lambda/\mu\right)$ satisfying $A \Rightarrow B$ and $A \Rightarrow C$. Then, there exists a $D \in \mathrm{BT}^{12}\left(\lambda/\mu\right)$ such that $B \overset{*}{\Rightarrow} D$ and $C \overset{*}{\Rightarrow} D$.

Proof of Lemma 5.8. If $B = C$, then we can simply choose $D = B = C$; thus, we assume that $B \neq C$.

Let u, $v \in \mathbb{N}_+$ be such that $A \underset{u}{\Rightarrow} B$ and $A \underset{v}{\Rightarrow} C$. Since $B \neq C$, we have $u \neq v$. Without loss of generality, assume that $u < v$. We are in one of the following two cases:

Case 1: We have $u = v - 1$.

Case 2: We have $u < v - 1$.

Let us deal with Case 2 first. In this case, $\{u, u+1\} \cap \{v, v+1\} = \varnothing$. It follows that $\mathrm{res}_v\left(\mathrm{res}_u A\right)$ and $\mathrm{res}_u\left(\mathrm{res}_v A\right)$ are well-defined and $\mathrm{res}_u\left(\mathrm{res}_v A\right) = \mathrm{res}_v\left(\mathrm{res}_u A\right)$. Setting $D = \mathrm{res}_u\left(\mathrm{res}_v A\right) = \mathrm{res}_v\left(\mathrm{res}_u A\right)$ completes the proof in this case.

Now, let us consider Case 1. The v-th column of A must contain a 1 (since $v - 1$ is a descent of A) and a 2 (since v is a descent of A). Hence, the v-th column of A is mixed. Since A is benign, the $(v-1)$-th column of A is 2-pure and the $(v+1)$-th column of A is 1-pure. We can represent the relevant portion (that is, the $(v-1)$-th, v-th and $(v+1)$-th columns) of the 12-table A as follows:

$$A = \begin{array}{|c|c|c|} \hline & & 1 \\ \cline{3-3} & 1 & 1 \\ \hline 2 & & \\ \cline{1-1} & 2 & \\ \hline & & \end{array} \tag{5}$$

Notice that the separating line which separates the 1's from the 2's in column v is lower than the upper border of the $(v-1)$-th column (since $v - 1$ is a descent of A), and higher than the lower border of the $(v+1)$-th column (since v is a descent of A).

Let $s = \mathrm{sep}_v A$. Now, $B = \mathrm{res}_{v-1} A$, so B is represented as follows:

$$B = \begin{array}{|c|c|c|} \hline & & 1 \\ \cline{3-3} 1 & 2 & \\ \hline 2 & & \\ \hline & & \end{array} ,$$

where $\mathrm{sep}_{v-1} B = s$. Now, v is a descent of B. Resolving this descent yields a 12-table $\mathrm{res}_v B$ which is represented as follows:

$$\mathrm{res}_v B = \begin{array}{c} \boxed{\begin{array}{cc} 1 & 1 \end{array}} \\ \end{array} \ .$$

This, in turn, shows that $v - 1$ is a descent of $\mathrm{res}_v B$. Resolving this descent yields a 12-table $\mathrm{res}_{v-1}\left(\mathrm{res}_v B\right)$ which is represented as follows:

$$\mathrm{res}_{v-1}\left(\mathrm{res}_v B\right) = \quad , \tag{6}$$

where $\mathrm{sep}_v\left(\mathrm{res}_{v-1}\left(\mathrm{res}_v B\right)\right) = s$.

On the other hand, $C = \mathrm{res}_v A$. We can apply a similar argument as above to show that the 12-table $\mathrm{res}_v\left(\mathrm{res}_{v-1} C\right)$ is well-defined, and is exactly equal to the 12-table in (6). Hence, $\mathrm{res}_{v-1}\left(\mathrm{res}_v B\right) = \mathrm{res}_v\left(\mathrm{res}_{v-1} C\right)$, and setting D equal to this 12-table completes the proof in Case 1. □

TODO 5.9. I am not sure how much I like the proof in Case 1 above. I introduced "core squares" because I wanted to avoid an argument where it is unclear how much of the proof comes "from the picture". They give a clear, explicit reason why $v - 1$ is a descent of $\mathrm{res}_v B$ etc. In the current version, this is not obvious. Do you have a quick way to explain it?

[Pasha] I like the current proof of Case 1 very much. I think it is rigorous enough and does not need any further explanations.

5.5. The normalization map

The following proposition is the most important piece in our puzzle:

Proposition 5.10. For every $T \in \mathrm{BT}^{12}\left(\lambda/\mu\right)$, there exists a unique $N \in \mathrm{RPP}^{12}\left(\lambda/\mu\right)$ such that $T \stackrel{*}{\Rightarrow} N$.

Proof of Proposition 5.10. For every $T \in \mathrm{BT}^{12}(\lambda/\mu)$, let $\mathrm{Norm}(T)$ denote the set

$$\left\{ N \in \mathrm{RPP}^{12}(\lambda/\mu) \mid T \stackrel{*}{\Rightarrow} N \right\}.$$

Thus, in order to prove Proposition 5.10, we need to show that for every $T \in \mathrm{BT}^{12}(\lambda/\mu)$ this set $\mathrm{Norm}(T)$ is a one-element set.

We shall prove this by strong induction on $\ell(T)$. Fix some $T \in \mathrm{BT}^{12}(\lambda/\mu)$, and assume that

$$\mathrm{Norm}(S) \text{ is a one-element set for every } S \in \mathrm{BT}^{12}(\lambda/\mu) \text{ satisfying } \ell(S) < \ell(T). \tag{7}$$

We then need to prove that $\mathrm{Norm}(T)$ is a one-element set.

Let $\mathbf{Z} = \left\{ S \in \mathrm{BT}^{12}(\lambda/\mu) \mid T \Rightarrow S \right\}$. In other words, $\mathbf{Z}$ is the set of all benign 12-tables S which can be obtained from T by resolving one descent. If $\mathbf{Z}$ is empty, then $T \in \mathrm{RPP}^{12}(\lambda/\mu)$, so that $\mathrm{Norm}(T) = \{T\}$ and we are done. Hence, we can assume that $\mathbf{Z}$ is nonempty. Therefore $T \notin \mathrm{RPP}^{12}(\lambda/\mu)$.

Now, we have

$$\left\{ N \in \mathrm{BT}^{12}(\lambda/\mu) \mid T \stackrel{*}{\Rightarrow} N \right\} = \{T\} \cup \bigcup_{Z \in \mathbf{Z}} \left\{ N \in \mathrm{BT}^{12}(\lambda/\mu) \mid Z \stackrel{*}{\Rightarrow} N \right\}.$$

Intersecting both sides of this identity with the set $\mathrm{RPP}^{12}(\lambda/\mu)$, we obtain

$$\left\{ N \in \mathrm{RPP}^{12}(\lambda/\mu) \mid T \stackrel{*}{\Rightarrow} N \right\} = \underbrace{\left(\{T\} \cap \mathrm{RPP}^{12}(\lambda/\mu) \right)}_{\substack{=\varnothing \\ \text{(since } T \notin \mathrm{RPP}^{12}(\lambda/\mu))}} \cup \bigcup_{Z \in \mathbf{Z}} \underbrace{\left\{ N \in \mathrm{RPP}^{12}(\lambda/\mu) \mid Z \stackrel{*}{\Rightarrow} N \right\}}_{\substack{=\mathrm{Norm}(Z) \\ \text{(by the definition} \\ \text{of } \mathrm{Norm}(Z))}}$$

$$= \bigcup_{Z \in \mathbf{Z}} \mathrm{Norm}(Z).$$

Thus,

$$\mathrm{Norm}(T) = \left\{ N \in \mathrm{RPP}^{12}(\lambda/\mu) \mid T \stackrel{*}{\Rightarrow} N \right\} = \bigcup_{Z \in \mathbf{Z}} \mathrm{Norm}(Z). \tag{8}$$

Let us now notice that:

- By Lemma 5.6 **(d)** and (7), for every $Z \in \mathbf{Z}$, the set $\mathrm{Norm}(Z)$ is a one-element set.

- By Lemma 5.8, for every $B \in \mathbf{Z}$ and $C \in \mathbf{Z}$, we have $\mathrm{Norm}(B) \cap \mathrm{Norm}(C) \neq \varnothing$. [7]

---

[7] In more detail: By Lemma 5.8 there exists a $D \in \mathrm{BT}^{12}(\lambda/\mu)$ such that $B \stackrel{*}{\Rightarrow} D$ and $C \stackrel{*}{\Rightarrow} D$. By induction, $\mathrm{Norm}(D)$ is a one-element set. Its unique element clearly lies in both $\mathrm{Norm}(B)$ and $\mathrm{Norm}(C)$, so $\mathrm{Norm}(B) \cap \mathrm{Norm}(C) \neq \varnothing$.

Hence, (8) shows that $\mathrm{Norm}\,(T)$ is a union of one-element sets, any two of which have a nonempty intersection. Moreover, this union is nonempty (since $\mathbf{Z}$ is nonempty). Hence, $\mathrm{Norm}\,(T)$ itself is a one-element set. This completes our induction. $\qquad\square$

Definition 5.11. Let $T \in \mathrm{BT}^{12}\,(\lambda/\mu)$. Proposition 5.10 shows that there exists a unique $N \in \mathrm{RPP}^{12}\,(\lambda/\mu)$ such that $T \overset{*}{\Rightarrow} N$. We define $\mathrm{norm}\,(T)$ to be this N.

5.6. Definition of B

Definition 5.12. Let us define a map $\mathbf{B} : \mathrm{RPP}^{12}\,(\lambda/\mu) \to \mathrm{RPP}^{12}\,(\lambda/\mu)$ as follows:

Let $T \in \mathrm{RPP}^{12}\,(\lambda/\mu)$. We define $\mathbf{B}\,(T)$ to be $\mathrm{norm}\,(\mathrm{flip}\,(T))$.

In order to complete the proof of Lemma 3.6, we need to show that $\mathbf{B}$ is an involution, preserves the ceq statistic, and switches the number of columns containing 1 with the number of columns containing 2. At this point, all of this is easy:

$\mathbf{B}$ *is an involution.* Let $T \in \mathrm{RPP}^{12}\,(\lambda/\mu)$. We have $\mathrm{flip}\,(T) \overset{*}{\Rightarrow} \mathrm{norm}\,(\mathrm{flip}\,(T)) = \mathbf{B}\,(T)$. Lemma 5.7 **(c)** thus yields $\mathrm{flip}\,(\mathbf{B}\,(T)) \overset{*}{\Rightarrow} \mathrm{flip}\,(\mathrm{flip}\,T) = T$.

But $\mathbf{B}(\mathbf{B}(T)) = \mathrm{norm}\,(\mathrm{flip}\,(\mathbf{B}\,(T)))$ is the unique $N \in \mathrm{RPP}^{12}\,(\lambda/\mu)$ such that $\mathrm{flip}\,(\mathbf{B}\,(T)) \overset{*}{\Rightarrow} N$. Since $T \in \mathrm{RPP}^{12}\,(\lambda/\mu)$, we have $\mathbf{B}(\mathbf{B}(T)) = T$, as desired. $\quad\square$

$\mathbf{B}$ *preserves* ceq. Let $T \in \mathrm{RPP}^{12}\,(\lambda/\mu)$. As above, $\mathrm{flip}\,(T) \overset{*}{\Rightarrow} \mathbf{B}\,(T)$. Lemma 5.7 **(a)** and Remark 5.2 thus yield

$$\mathrm{ceq}\,(\mathbf{B}\,(T)) = \mathrm{ceq}\,(\mathrm{flip}\,(T)) = \mathrm{ceq}\,(T)\,.$$

$\square$

$\mathbf{B}$ *switches the numbers of columns containing 1 and 2.* Let $T \in \mathrm{RPP}^{12}\,(\lambda/\mu)$. As above, $\mathrm{flip}\,(T) \overset{*}{\Rightarrow} \mathbf{B}\,(T)$. Lemma 5.7 **(b)** and Remark 5.2 thus yield

$$\mathrm{ircont}\,(\mathbf{B}\,(T)) = \mathrm{ircont}\,(\mathrm{flip}\,(T)) = s_1 \cdot \mathrm{ircont}\,(T)\,.$$

$\square$

Lemma 3.6 is now proven.

6. The classical Bender-Knuth involutions

6.1. Recalling the definition of $\mathbf{B}_i$

We fix a skew partition λ/μ and a positive integer i for the whole Section 6.

Theorem 3.5 merely claims the existence of an involution $\mathbf{B}_i : \mathrm{RPP}(\lambda/\mu) \to \mathrm{RPP}(\lambda/\mu)$ satisfying certain properties. Such an involution, per se, needs not be unique. However, if we trace back the proof of Theorem 3.5 (and the proofs of the lemmas that were used in this proof), we notice that this proof constructs a specific involution $\mathbf{B}_i$. This construction is spread across various proofs; we can summarize it as follows:

- The main step of the construction was the construction of the involution $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ in Lemma 3.6 (for a given finite convex subset Z of $\mathbb{N}_+^2$). This is an involution which sends 12-rpps of shape Z to 12-rpps of the same shape Z, and it was constructed as follows: Given a 12-rpp T of shape Z, we set $\mathbf{B}(T) = \mathrm{norm}(\mathrm{flip}(T))$. (Recall that $\mathrm{flip}(T)$ fills all the 1-pure columns of T with 2's while simultaneously filling all the 2-pure columns of T with 1's. Recall furthermore that $\mathrm{norm}(\mathrm{flip}(T))$ is obtained from $\mathrm{flip}(T)$ by repeatedly resolving descents until no descents remain.)

- Having constructed this map $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$, we can construct the involution $\mathbf{B}_Z : \mathrm{RPP}^{12}(\lambda/\mu)_Z \to \mathrm{RPP}^{12}(\lambda/\mu)_Z$ in Lemma **??** (for a given finite convex subset Z of $\mathbb{N}_+^2$) as follows: Given an rpp S of shape Z whose entries are i's and $(i+1)$'s, we first replace these entries by 1's and 2's (respectively), so that we obtain a 12-rpp; then, we apply the involution $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ to this 12-rpp; and then, in the resulting 12-rpp, we change the 1's and 2's back into i's and $(i+1)$'s. The resulting rpp is $\mathbf{B}_Z(S)$.

- Finally, we can construct the involution $\mathbf{B}_i : \mathrm{RPP}(\lambda/\mu) \to \mathrm{RPP}(\lambda/\mu)$. To wit, if we are given an rpp $S \in \mathrm{RPP}(\lambda/\mu)$, then we can restrict our attention to the cells of S which contain the entries i and $i+1$. These cells form an rpp of some shape Z. We then apply the involution $\mathbf{B}_Z$ to this new rpp, while leaving all the remaining entries of S unchanged. The result is an rpp of shape $Y(\lambda/\mu)$ again; this rpp is $\mathbf{B}_i(S)$.

In the following, whenever we will be talking about the involution $\mathbf{B}_i$, we will always mean this particular involution $\mathbf{B}_i$, rather than an arbitrary involution $\mathbf{B}_i$ which satisfies the claims of Theorem 3.5.

6.2. The Bender-Knuth involutions

We claimed that our involution $\mathbf{B}_i : \mathrm{RPP}(\lambda/\mu) \to \mathrm{RPP}(\lambda/\mu)$ is a generalization of the i-th Bender-Knuth involution defined for semistandard tableaux. Let us

now elaborate on this claim. First, we shall define the i-th Bender-Knuth involution (following [GriRei15, proof of Proposition 2.11] and [Stan99, proof of Theorem 7.10.2]).

Let SST (λ/μ) denote the set of all semistandard tableaux of shape $Y(\lambda/\mu)$. We define a map $B_i : \text{SST}(\lambda/\mu) \to \text{SST}(\lambda/\mu)$ as follows:[8]

Let $T \in \text{SST}(\lambda/\mu)$. Then, T is a semistandard tableau, so that every column of T contains at most one i and at most one $i+1$. We shall ignore all columns of T which contain both an i and an $i+1$; that is, we mark all the entries of all such columns as "ignored". Now, let $k \in \mathbb{N}_+$. The k-th row of T is a weakly increasing sequence of positive integers; thus, it contains a (possibly empty) string of i's followed by a (possibly empty) string of $(i+1)$'s. These two strings together form a substring of the k-th row which looks as follows:

$$(i, i, \ldots, i, i+1, i+1, \ldots, i+1)$$

[9]. Some of the entries of this substring are "ignored"; it is easy to see that the "ignored" i's are gathered at the left end of the substring whereas the "ignored" $(i+1)$'s are gathered at the right end of the substring. So the substring looks as follows:

$$\left(\underbrace{i, i, \ldots, i}_{\substack{a \text{ many } i\text{'s which} \\ \text{are "ignored"}}} , \underbrace{i, i, \ldots, i}_{\substack{r \text{ many } i\text{'s which} \\ \text{are not "ignored"}}} , \underbrace{i+1, i+1, \ldots, i+1}_{\substack{s \text{ many } (i+1)\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{b \text{ many } (i+1)\text{'s which} \\ \text{are "ignored"}}} \right)$$

for some $a, r, s, b \in \mathbb{N}$. Now, we change this substring into

$$\left(\underbrace{i, i, \ldots, i}_{\substack{a \text{ many } i\text{'s which} \\ \text{are "ignored"}}} , \underbrace{i, i, \ldots, i}_{\substack{s \text{ many } i\text{'s which} \\ \text{are not "ignored"}}} , \underbrace{i+1, i+1, \ldots, i+1}_{\substack{r \text{ many } (i+1)\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{b \text{ many } (i+1)\text{'s which} \\ \text{are not "ignored"}}} \right) .$$

And we do this for every $k \in \mathbb{N}_+$ (simultaneously or consecutively – it does not matter). At the end, we have obtained a new semistandard tableau of shape $Y(\lambda/\mu)$. We define $B_i(T)$ to be this new tableau.

---

[8]We refer to Example 6.1 below for illustration.

[9]Of course, this might contain no i's or no $(i+1)$'s.

Example 6.1. Let us give an example of this construction of B_i. Namely, let $i = 2$, let $\lambda = (7, 6, 4, 1)$, and let $\mu = (3)$. Let T be the semistandard tableau

$$
\begin{array}{cccccccc}
 & & & & 1 & 1 & 2 & 2 \\
 & 1 & 2 & 2 & 2 & 3 & 3 & \\
 & 3 & 3 & 5 & 6 & & & \\
 & 4 & & & & & &
\end{array}
$$

of shape $Y(\lambda/\mu)$. We want to find $B_i(T)$.

The columns that contain both an i and an $i + 1$ (that is, both a 2 and a 3) are the second and the sixth columns. So we mark all entries of these two columns as "ignored". Now, the substring of the 2-nd row of T formed by the i's and the $(i + 1)$'s looks as follows:

$$
\left(\underbrace{2}_{\substack{\text{1 many 2's which} \\ \text{are "ignored"}}} , \underbrace{2, 2}_{\substack{\text{2 many 2's which} \\ \text{are not "ignored"}}} , \underbrace{3}_{\substack{\text{1 many 3's which} \\ \text{are not "ignored"}}} , \underbrace{3}_{\substack{\text{2 many 3's which} \\ \text{are not "ignored"}}} \right).
$$

So we change it into

$$
\left(\underbrace{2}_{\substack{\text{1 many 2's which} \\ \text{are "ignored"}}} , \underbrace{2}_{\substack{\text{1 many 2's which} \\ \text{are not "ignored"}}} , \underbrace{3, 3}_{\substack{\text{2 many 3's which} \\ \text{are not "ignored"}}} , \underbrace{3}_{\substack{\text{2 many 3's which} \\ \text{are not "ignored"}}} \right).
$$

Similarly, we change the substring $(2, 2)$ of the 1-st row of T into $(2, 3)$ (because its first 2 is "ignored" but its second 2 is not), and we change the substring $(3, 3)$ of the 3-rd row of T into $(2, 3)$ (because its first 3 is not "ignored" but its second 3 is). The substring of the 4-th row, of the 5-th row, of the 6-th row, etc., formed by the i's and $(i + 1)$'s are empty (because these rows have neither i's nor $(i + 1)$'s), and thus we do not make any changes on them. Now, $B_i(T)$ is defined to be the tableau that results from all of these changes; thus,

$$
B_i(T) =
\begin{array}{cccccccc}
 & & & & 1 & 1 & 2 & 3 \\
 & 1 & 2 & 2 & 3 & 3 & 3 & \\
 & 2 & 3 & 5 & 6 & & & \\
 & 4 & & & & & &
\end{array}.
$$

Proposition 6.2. The map $B_i : \mathrm{SST}(\lambda/\mu) \to \mathrm{SST}(\lambda/\mu)$ thus defined is an involution. It is known as the *i-th Bender-Knuth involution*.

Proposition 6.2 is easy to prove (and is usually proven in less or more detail everywhere the map B_i is defined).

Now, every semistandard tableau of shape $Y(\lambda/\mu)$ is also an rpp of shape $Y(\lambda/\mu)$. In other words, $\mathrm{SST}(\lambda/\mu) \subseteq \mathrm{RPP}(\lambda/\mu)$. Hence, $\mathbf{B}_i(T)$ is defined for every $T \in \mathrm{SST}(\lambda/\mu)$. Now, the claim that we want to make (that our involution $\mathbf{B}_i$ is a generalization of the i-th Bender-Knuth involution B_i) can be stated as follows:

Proposition 6.3. For every $T \in \mathrm{SST}(\lambda/\mu)$, we have $B_i(T) = \mathbf{B}_i(T)$.

Proof of Proposition 6.3 (sketched). We shall abbreviate "semistandard tableau" as "sst". We define a *12-sst* to be an sst whose entries all belong to the set $\{1, 2\}$.

Let Z be a finite convex subset of $\mathbb{N}_+^2$. Let R denote the set of all 12-ssts of shape Z. We define a map $B : R \to R$ in the same way as we defined the map $B_i : \mathrm{SST}(\lambda/\mu) \to \mathrm{SST}(\lambda/\mu)$, with the only differences that we replace every appearance of "$\mathrm{SST}(\lambda/\mu)$", of "$i$" and of "$i+1$" by "$R$", "1" and "2", respectively. Then, this map $B : R \to R$ is an involution.

Now let us forget that we fixed Z. We thus have constructed a map $B : R \to R$ for every finite convex subset Z of $\mathbb{N}_+^2$. Now, recall how the map $\mathbf{B}_i : \mathrm{RPP}(\lambda/\mu) \to \mathrm{RPP}(\lambda/\mu)$ was constructed from the maps $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ for every finite convex subset Z of $\mathbb{N}_+^2$ (essentially by forgetting all entries of an rpp except for the entries i and $i+1$ and relabelling these entries i and $i+1$ as 1 and 2). Similarly, the map $B_i : \mathrm{SST}(\lambda/\mu) \to \mathrm{SST}(\lambda/\mu)$ can be constructed from the maps $B : R \to R$ for every finite convex subset Z of $\mathbb{N}_+^2$ (essentially by forgetting all entries of an sst except for the entries i and $i+1$ and relabelling these entries i and $i+1$ as 1 and 2). Thus, in order to prove that $B_i(T) = \mathbf{B}_i(T)$ for every $T \in \mathrm{SST}(\lambda/\mu)$, it suffices to show that $B(T) = \mathbf{B}(T)$ for every finite convex subset Z of $\mathbb{N}_+^2$ and any 12-sst T of shape Z.

TODO 6.4. Is this readable?

So let Z be any finite convex subset of $\mathbb{N}_+^2$, and let T be a 12-sst of shape Z. We need to prove that $B(T) = \mathbf{B}(T)$.

Example 6.5. Here is an example of a 12-sst:

$$T = \qquad\qquad \begin{array}{ccccc} & & & & \boxed{1} \\ & & \boxed{1} & \boxed{1} & \boxed{2} \\ & \boxed{1}\,\boxed{2}\,\boxed{2}\,\boxed{2} & & & \\ \boxed{1}\,\boxed{2} & & & & \\ \boxed{2} & & & & \end{array}. \tag{9}$$

It satisfies

$$B(T) =$$

						1
			1	2	2	
	1	1	2	2		

1	1
2	

(10)

and

$$\text{flip}(T) =$$

						1
			1	2	2	
	2	1	1	2		

1	1
2	

(11)

(where flip (T) is defined as in the construction of $\mathbf{B}(T)$).

We make a few basic observations: The columns which are ignored in the construction of $B(T)$ are the columns which contain both a 1 and a 2. [10] These columns contain exactly two entries each (because a column of a 12-sst can only contain at most one 1, at most one 2 and no other entries), while every other column is either empty or contains only one entry. As a consequence, every entry of T which is not "ignored" in the construction of $B(T)$ is alone in its column.

Let us compare the basic ideas of the constructions of $B(T)$ and $\mathbf{B}(T)$:

- To construct $B(T)$, we ignore all columns of T which contain both a 1 and a 2; that is, we mark all entries in these columns as "ignored". Then, in every row, we let r be the number of 1's which are not "ignored", and let s be the number of 2's which are not "ignored". We replace these r many 1's and s many 2's by s many 1's and r many 2's. This we do for every row; the resulting 12-sst is $B(T)$.

- To construct $\mathbf{B}(T)$, we consider T as a 12-rpp, and we identify which of its columns are 1-pure, which are 2-pure and which are mixed. Then, we replace all entries of all 1-pure columns by 2's, while simultaneously replacing all entries of all 2-pure columns by 1's. The resulting 12-table is denoted flip (T). Then, we repeatedly resolve descents in flip (T) until no more descents remain. The resulting 12-table norm (flip (T)) is a 12-rpp, and is denoted $\mathbf{B}(T)$.

If we compare the two constructions just described, we first notice that the columns ignored in the construction of $B(T)$ are precisely the mixed columns of

---

[10]For instance, in the 12-sst (9), the ignored columns are the 1-st, the 7-th and the 9-th columns.

T. Thus, the 12-table flip (T) can be obtained from T by replacing all 1's which are not "ignored" by 2's while simultaneously replacing all 2's which are not "ignored" by 1's. Thus, for any given $r \in \mathbb{N}_+$, if the r-th row of T contains r many 1's which are not "ignored" and s many 2's which are not "ignored", then the r-th row of flip (T) contains r many 2's which are not "ignored" and s many 1's which are not "ignored" (while the "ignored" entries in T appear in flip (T) unchanged). So we can restate the construction of flip (T) as follows:

- To construct flip (T) from T, do the following: In every row of T, let r be the number of 1's which are not "ignored", and let s be the number of 2's which are not "ignored". We replace these r many 1's and s many 2's by r many 2's and s many 1's (in this order). This we do for every row; the resulting 12-table is flip (T).

Compare this to our construction of $B(T)$:

- To construct $B(T)$ from T, do the following: In every row of T, let r be the number of 1's which are not "ignored", and let s be the number of 2's which are not "ignored". We replace these r many 1's and s many 2's by s many 1's and r many 2's (in this order). This we do for every row; the resulting 12-sst is $B(T)$.

Comparing these two constructions makes it clear that each row of $B(T)$ differs from the corresponding row of flip (T) merely in the order in which the non-"ignored" entries appear: In $B(T)$, the non-"ignored" 1's appear before the non-"ignored" 2's (as they must, $B(T)$ being an sst), whereas in flip (T) they appear in the opposite order. Hence, $B(T)$ can be obtained from flip (T) by sorting all non-"ignored" entries into increasing order in each row.

Now, let us notice that every pair of a non-"ignored" 2 and a non-"ignored" 1 lying in the same row of flip (T) cause a descent[11]. Conversely, all descents of flip (T) are caused by a non-"ignored" 2 and a non-"ignored" 1 lying in the same row (because all "ignored" entries are carried over from T without change and thus cannot take part in descents). We can resolve these descents one after the other (starting with the 2 and the 1 that are adjacent to each other), until none are left. The result is a 12-rpp. What is this 12-rpp?

- On the one hand, this 12-rpp is norm (flip (T)), because norm (flip (T)) is defined as what results when all descents of flip (T) are resolved.

- On the other hand, this 12-rpp is $B(T)$. In fact, resolving a descent caused by a non-"ignored" 2 and a non-"ignored" 1 lying in the same row results in this 2 getting switched with the 1 (while no other entries get moved[12]).

---

[11]More precisely: If $r \in \mathbb{N}_+$, $i \in \mathbb{N}_+$ and $j \in \mathbb{N}_+$ are such that (flip T) (r, i) is a non-"ignored" 2 and that (flip T) (r, j) is a non-"ignored" 1, then (i, j) is a descent of flip T.

[12]This is because every non-"ignored" entry is alone in its column.

Hence, when we resolve the descents, we just sort all non-"ignored" entries into increasing order in each row. But as we know, the 12-table obtained from flip (T) by sorting all non-"ignored" entries into increasing order in each row is $B(T)$.

So we have found a 12-rpp which equals both norm $(\text{flip}(T))$ and $B(T)$. Thus, $B(T) = \text{norm}(\text{flip}(T)) = \mathbf{B}(T)$. This completes our proof of Proposition 6.3.

$\square$

TODO 6.6. Can you follow the above argument? (Preferrably at 3AM and/or under the influence.)

7. The structure of 12-rpps

In this section, we let $\mathbf{k}$ be the polynomial ring $\mathbb{Z}[t_1, t_2, t_3, \ldots]$ in countably many indeterminates, and we restrict ourselves to the two-variable dual stable Grothendieck polynomial $\widetilde{g}_{\lambda/\mu}(x_1, x_2, 0, 0, \ldots; \mathbf{t})$ defined as the result of substituting $0, 0, 0, \ldots$ for $x_3, x_4, x_5, \ldots$ in $\widetilde{g}_{\lambda/\mu}$. We can represent it as a polynomial in $\mathbf{t}$ with coefficients in $\mathbb{Z}[x_1, x_2]$:

$$\widetilde{g}_{\lambda/\mu}(x_1, x_2, 0, 0, \ldots; \mathbf{t}) = \sum_{\alpha \in \mathbb{N}^{\mathbb{N}_+}} \mathbf{t}^\alpha Q_\alpha(x_1, x_2),$$

where the sum ranges over all weak compositions α, and all but finitely many $Q_\alpha(x_1, x_2)$ are 0.

We shall show that each $Q_\alpha(x_1, x_2)$ is either zero or has the form

$$Q_\alpha(x_1, x_2) = (x_1 x_2)^M P_{n_0}(x_1, x_2) P_{n_1}(x_1, x_2) \cdots P_{n_r}(x_1, x_2), \tag{12}$$

where M, r and $n_0, n_1, \ldots, n_r$ are nonnegative integers naturally associated to α and λ/μ and

$$P_n(x_1, x_2) = (x_1^{n+1} - x_2^{n+1})/(x_1 - x_2) = x_1^n + x_1^{n-1} x_2 + \cdots + x_1 x_2^{n-1} + x_2^n.$$

We fix the skew partition λ/μ throughout the whole section. We will have a running example with $\lambda = (7, 7, 7, 4, 4)$ and $\mu = (5, 3, 2)$.

7.1. Irreducible components

We recall that a *12-rpp* means an rpp whose entries all belong to the set $\{1, 2\}$.

Given a 12-rpp T, consider the set $\text{NS}(T)$ of all cells $(i, j) \in \lambda/\mu$ such that $T(i, j) = 1$ but $(i+1, j) \in \lambda/\mu$ and $T(i+1, j) = 2$. (In other words, $\text{NS}(T)$ is the set of all non-stagnant cells in T which are filled with a 1 and which are not the lowest cells in their columns.) Clearly, $\text{NS}(T)$ contains at most one cell from each column; thus, let us write $\text{NS}(T) = \{(i_1, j_1), (i_2, j_2), \ldots, (i_s, j_s)\}$
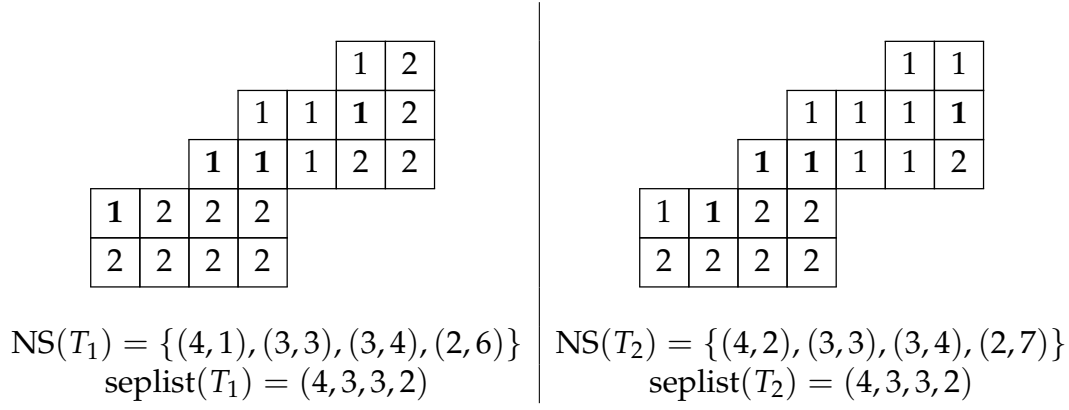
$$\mathrm{NS}(T_1) = \{(4,1),(3,3),(3,4),(2,6)\} \qquad \mathrm{NS}(T_2) = \{(4,2),(3,3),(3,4),(2,7)\}$$
$$\mathrm{seplist}(T_1) = (4,3,3,2) \qquad\qquad \mathrm{seplist}(T_2) = (4,3,3,2)$$

Figure 3: Two 12-rpps of the same shape and with the same seplist-partition.

with $j_1 < j_2 < \cdots < j_s$. Because T is a 12-rpp, it follows that the numbers $i_1, i_2, \ldots, i_s$ decrease weakly, therefore they form a partition which we called *the seplist-partition of T* and denoted

$$\mathrm{seplist}(T) := (i_1, i_2, \ldots, i_s)$$

in Section 5.1. An example of calculation of $\mathrm{seplist}(T)$ and $\mathrm{NS}(T)$ is illustrated on Figure 3.

We would like to answer the following question: for which partitions $\nu = (i_1 \geq \cdots \geq i_s > 0)$ does there exist a 12-rpp T of shape λ/μ such that $\mathrm{seplist}(T) = \nu$?

A trivial necessary condition for this to happen is that there should exist some numbers $j_1 < j_2 < \cdots < j_s$ such that

$$(i_1, j_1), (i_1 + 1, j_1), (i_2, j_2), (i_2 + 1, j_2), \ldots, (i_s, j_s), (i_s + 1, j_s) \in \lambda/\mu. \qquad (13)$$

We say that a partition ν is *admissible* if such $j_1 < j_2 < \cdots < j_s$ exist.

Until the end of Section 7, we make one further assumption: namely, that the skew partition λ/μ is connected as a subgraph of $\mathbb{Z}^2$. This is a harmless assumption, since every skew partition λ/μ can be written as a disjoint union of such connected skew partitions and the corresponding seplist-partition splits into several independent parts, the polynomials $\widetilde{g}_{\lambda/\mu}$ get multiplied and the right hand side of (12) changes accordingly.

For each integer i the set of all integers j such that $(i, j), (i+1, j) \in \lambda/\mu$ is just an interval $[\mu_i + 1, \lambda_{i+1}]$, which we call *the support of i* and denote $\mathrm{supp}(i) := [\mu_i + 1, \lambda_{i+1}]$.

Assume that $\nu = (i_1 \geq \cdots \geq i_s > 0)$ is an admissible partition. Then, $\mathrm{supp}(i_k)$ is nonempty for each k. For two integers $a < b$, by $\nu|_{\subseteq[a,b)}$ we denote the subpartition $(i_r, i_{r+1}, \ldots, i_{r+q})$ of ν such that for $r \leq k \leq r + q$ we have $\mathrm{supp}(i_k) \subseteq [a, b)$. In this case, we put $\#\nu|_{\subseteq[a,b)} := q + 1$ which is just the number of entries in $\nu|_{\subseteq[a,b)}$. Similarly, we put $\nu|_{\cap[a,b)}$ to be the subpartition $(i_r, i_{r+1}, \ldots, i_{r+q})$ of ν

such that for $r \leq k \leq r + q$ we have $\mathrm{supp}(i_k) \cap [a,b) \neq \varnothing$. For example, for $v = (4,3,3,2)$ and λ/μ as on Figure 3, we have

$$\mathrm{supp}(3) = [3,4], \ \mathrm{supp}(2) = [4,7], \ \mathrm{supp}(4) = [1,4],$$

$$v|_{\subseteq [2,7)} = (3,3), \ v|_{\subseteq [2,8)} = (3,3,2), \ v|_{\subseteq [4,8)} = (2), \ v|_{\cap [4,5)} = (4,3,3,2), \ \#v|_{\subseteq [2,7)} = 2.$$

We introduce several definitions: An admissible partition $v = (i_1 \geq \cdots \geq i_s > 0)$ is called

- *non-representable*   if for some $a < b$ we have $\#v|_{\subseteq [a,b)} > b - a$;
- *representable*   if for all $a < b$ we have $\#v|_{\subseteq [a,b)} \leq b - a$;

a representable partition v is called

- *irreducible*   if for all $a < b$ we have $\#v|_{\subseteq [a,b)} < b - a$;
- *reducible*   if for some $a < b$ we have $\#v|_{\subseteq [a,b)} = b - a$.

For example, $v = (4,3,3,2)$ is representable but reducible because we have $v|_{\subseteq [3,5)} = (3,3)$ so $\#v|_{\subseteq [3,5)} = 2 = 5 - 3$.

Note that these notions depend on the skew partition; thus, when we want to use a skew partition $\widetilde{\lambda/\mu}$ rather than λ/μ, we will write that v is non-representable/irreducible/etc. *with respect to* $\widetilde{\lambda/\mu}$, and we denote the corresponding partitions by $v|_{\subseteq [a,b)}^{\widetilde{\lambda/\mu}}$.

These definitions can be motivated as follows. Suppose that a partition v is non-representable, so there exist integers $a < b$ such that $\#v|_{\subseteq [a,b)} > b - a$. Recall that $v|_{\subseteq [a,b)} =: (i_r, i_{r+1}, \ldots, i_{r+q})$ contains all entries of v whose support is a subset of $[a,b)$. Thus in order for condition (13) to be true there must exist some integers $j_r < j_{r+1} < \cdots < j_{r+q}$ such that

$$(i_r, j_r), (i_r + 1, j_r), \ldots, (i_{r+q}, j_{r+q}), (i_{r+q} + 1, j_{r+q}) \in \lambda/\mu.$$

On the other hand, by the definition of the support, we must have $j_k \in \mathrm{supp}(i_k) \subseteq [a,b)$ for all $r \leq k \leq r + q$. Therefore we get $q + 1$ distinct elements of $[a,b)$ which is impossible if $q + 1 = \#v|_{\subseteq [a,b)} > b - a$. It means that a non-representable partition v is never a seplist-partition of a 12-rpp T.

Suppose now that a partition v is reducible, so for some $a < b$ we get an equality $\#v|_{\subseteq [a,b)} = b - a$. Then these integers $j_r < \cdots < j_{r+q}$ should still all belong to $[a,b)$ and there are exactly $b - a$ of them, hence

$$j_r = a, \ j_{r+1} = a + 1, \ \ldots, \ j_{r+q} = a + q = b - 1. \tag{14}$$

Because $\mathrm{supp}(i_r) \subseteq [a,b)$ but $\mathrm{supp}(i_r) \neq \varnothing$ (since v is admissible), we have $(i_r, a - 1) \notin \lambda/\mu$. Thus, placing a 1 into (i_r, a) and 2's into $(i_r + 1, a), (i_r + 2, a), \ldots$ does not put any restrictions on entries in columns $1, \ldots, a - 1$. And the same is true for columns $b, b + 1, \ldots$ when we place a 2 into $(i_{r+q} + 1, b - 1)$ and 1's into all cells above. Thus, if a partition v is reducible, then the filling of columns

$a, a+1, \ldots, b-1$ is uniquely determined (by (14)), and the filling of the rest can be arbitrary – the problem of existence of a 12-rpp T such that $\text{seplist}(T) = v$ reduces to two smaller independent problems of the same kind (one for the columns $1, 2, \ldots, a-1$, the other for the columns[13] $b, b+1, \ldots, \lambda_1$). One can continue this reduction process and end up with several independent irreducible components separated from each other by mixed columns. An illustration of this phenomenon can be seen on Figure 3: the columns 3 and 4 must be mixed for any 12-rpps T with $\text{seplist}(T) = (4, 3, 3, 2)$.

More explicitly, we have thus shown that every interval $[a, b) \subseteq [1, \lambda_1 + 1)$ satisfying $\#v\big|_{\subseteq[a,b)} = b - a$ splits our problem into two independent subproblems. But if two such intervals $[a, b)$ and $[c, d)$ intersect, then their union is another such interval. Hence, the maximal (with respect to inclusion) among all such intervals are pairwise disjoint and separated from each other by at least a distance of 1. This yields part **(a)** of the following lemma:

Lemma 7.1. Let v be a representable partition.

(a) There exist unique integers $(1 = b_0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r \leq a_{r+1} = \lambda_1 + 1)$ satisfying the following two conditions:

1. For all $1 \leq k \leq r$, we have $\#v\big|_{\subseteq[a_k, b_k)} = b_k - a_k$.

2. The set $\bigcup_{k=0}^{r}[b_k, a_{k+1})$ is minimal (with respect to inclusion) among all sequences $(1 = b_0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r \leq a_{r+1} = \lambda_1 + 1)$ satisfying property 1.

Furthermore, for these integers, we have:
(b) The partition v is the concatenation

$$\left(v\big|_{\cap[b_0, a_1)}\right)\left(v\big|_{\subseteq[a_1, b_1)}\right)\left(v\big|_{\cap[b_1, a_2)}\right)\left(v\big|_{\subseteq[a_2, b_2)}\right)\cdots\left(v\big|_{\cap[b_r, a_{r+1})}\right)$$

(where we regard a partition as a sequence of positive integers, with no trailing zeroes).

(c) The partitions $v\big|_{\cap[b_k, a_{k+1})}$ are irreducible with respect to $\lambda/\mu\big|_{[b_k, a_{k+1})}$, which is the skew partition λ/μ with columns $1, 2, \ldots, b_k - 1, a_{k+1}, a_{k+1} + 1, \ldots$ removed.

Proof. Part **(a)** has already been proven.

(b) Let $v = (i_1 \geq \cdots \geq i_s > 0)$. If $\text{supp}(i_r) \subset [a_k, b_k)$ for some k then i_r appears in exactly one of the concatenated partitions, namely, $v\big|_{\subseteq[a_k, b_k)}$. Otherwise there is an integer k such that $\text{supp}(i_r) \cap [b_k, a_{k+1}) \neq \varnothing$. It remains to show that such k is unique, that is, that $\text{supp}(i_r) \cap [b_{k+1}, a_{k+2}) = \varnothing$. But the interval $[a_{k+1}, b_{k+1})$ is non-empty, therefore there is an element i in v with $\text{supp}(i) \subset [a_{k+1}, b_{k+1})$. It remains to note that we get a contradiction: we get two numbers i, i_r with $\text{supp}(i_r)$ being both to the left and to the right of $\text{supp}(i)$.

---

[13]Recall that a 12-rpp of shape λ/μ cannot have any nonempty column beyond the λ_1'th one.

(c) To verify the last claim, we need to show that if $[c, d)$ is a nonempty interval contained in $[b_k, a_{k+1})$ for some k, then $\#\nu'|^J_{\subseteq[c,d)} < d - c$, where J denotes the restricted skew partition $\lambda/\mu|_{[b_k, a_{k+1})}$, and where $\nu' = \nu|_{\cap[b_k, a_{k+1})}$. If $c > b_k$ and $d < a_{k+1}$ then this follows from $\nu|_{\subseteq[c,d)} < d - c$, so suppose $c = b_k$ and $\#\nu'|^J_{\subseteq[c,d)} = d - c$. This immediately implies that $\#\nu|_{\subseteq[a_k,d)} = d - a_k$ which contradicts the minimality of $[b_k, a_{k+1})$. The case $d = a_{k+1}$ is completely analogous. $\square$

Definition 7.2. In the context of Lemma 7.1, for $0 \leq k \leq r$ the subpartitions $\nu|_{\cap[b_k, a_{k+1})}$ are called *the irreducible components of ν* and the numbers $n_k := a_{k+1} - b_k - \#\nu|_{\cap[b_k, a_{k+1})}$ are called their *degrees* (for T with $\mathrm{seplist}(T) = \nu$, the k-th degree n_k is equal to the number of pure columns of T inside the corresponding k-th irreducible component).

Example 7.3. For $\nu = (4, 3, 3, 2)$ we have $r = 1, b_0 = 1, a_1 = 3, b_1 = 5, a_2 = 8$. The irreducible components of ν are (4) and (2) and their degrees are $3 - 1 - 1 = 1$ and $8 - 5 - 1 = 2$ respectively. We have $\nu|_{\cap[1,3)} = (4), \nu|_{\subseteq[3,5)} = (3,3), \nu|_{\cap[5,8)} = (2)$.

7.2. The structural theorem and its applications

It is easy to see that for a 12-rpp T, the number $\#\mathrm{seplist}(T)$ is equal to the number of mixed columns in T.

Let $\mathrm{RPP}^{12}(\lambda/\mu)$ denote the set of all 12-rpps T of shape λ/μ, and let $\mathrm{RPP}^{12}(\lambda/\mu; \nu)$ denote its subset that contains only 12-rpps T with $\mathrm{seplist}(T) = \nu$. Now we are ready to state a theorem that completely describes the structure of irreducible components:

Theorem 7.4. Let ν be an irreducible partition. Then for all $0 \leq m \leq \lambda_1 - \#\nu$ there is exactly one 12-rpp $T \in \mathrm{RPP}^{12}(\lambda/\mu; \nu)$ with $\#\nu$ mixed columns, m 1-pure columns and $(\lambda_1 - \#\nu - m)$ 2-pure columns. Moreover, these are the only elements of $\mathrm{RPP}^{12}(\lambda/\mu; \nu)$. In other words, for an irreducible partition ν we have

$$\sum_{T \in \mathrm{RPP}^{12}(\lambda/\mu; \nu)} \mathbf{x}^{\mathrm{ircont}(T)} = (x_1 x_2)^{\#\nu} P_{\lambda_1 - \#\nu}(x_1, x_2). \tag{15}$$

Example 7.5. Each of the two 12-rpps on Figure 3 has two irreducible components. One of them is supported on the first two columns and the other one is supported on the last three columns. Here are all possible 12-rpps for each component:

$$\begin{array}{|c|c|}\hline 1 & \mathbf{1} \\\hline 1 & 2 \\\hline\end{array} \quad \begin{array}{|c|c|}\hline \mathbf{1} & 2 \\\hline 2 & 2 \\\hline\end{array}$$

$\lambda = (2,2);\ \mu = ();\ \nu = (1)$

$$\begin{array}{cc} & \begin{array}{|c|c|}\hline 1 & 1 \\\hline\end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 1 & \mathbf{1} \\\hline 1 & 1 & 2 \\\hline\end{array} \end{array}$$

$\lambda = (3,3,3);\ \mu = (1);\ \nu = (2).$

After decomposing into irreducible components, we can obtain a formula for general representable partitions:

Corollary 7.6. Let ν be a representable partition. Then

$$\sum_{T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)} \mathbf{x}^{\mathrm{ircont}(T)} = (x_1 x_2)^M P_{n_0}(x_1,x_2) P_{n_1}(x_1,x_2) \ldots P_{n_r}(x_1,x_2), \quad (16)$$

where the numbers $M, r, n_0, \ldots, n_r$ are defined above: $M = \#\nu$, $r+1$ is the number of irreducible components of ν and $n_0, n_1, \ldots, n_r$ are their degrees.

For a 12-rpp T, the vectors $\mathrm{seplist}(T)$ and $\mathrm{ceq}(T)$ uniquely determine each other: if $(\mathrm{ceq}(T))_i = h$ then $\mathrm{seplist}(T)$ contains exactly $\lambda_{i+1} - \mu_i - h$ entries equal to i, and this correspondence is one-to-one. Therefore, the polynomials on both sides of (16) are equal to $Q_\alpha(x_1, x_2)$ where the vector α is the one that corresponds to ν.

Note that the polynomials $P_n(x_1, x_2)$ are symmetric for all n. Since the question about the symmetry of $\widetilde{g}_{\lambda/\mu}$ can be reduced to the two-variable case, Theorem 7.4 gives an alternative proof of the symmetry of $\widetilde{g}_{\lambda/\mu}$:

Corollary 7.7. The polynomials $\widetilde{g}_{\lambda/\mu} \in \mathbf{k}[x_1, x_2, x_3, \ldots]$ are symmetric.

This holds for any $\mathbf{k}$ and any $t_1, t_2, t_3, \ldots \in \mathbf{k}$, since the case we have considered (where $t_1, t_2, t_3, \ldots$ are polynomial indeterminates over $\mathbb{Z}$) is universal.[14]

Another application of Theorem 7.4 is a complete description of Bender-Knuth involutions on rpps.

Corollary 7.8. Let ν be an irreducible partition. Then there is a unique map $b : \mathrm{RPP}^{12}(\lambda/\mu;\nu) \to \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ such that for all $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ we have

$$\mathrm{ircont}(T) = (c_1, c_2, 0, 0, \ldots) \implies \mathrm{ircont}(b(T)) = (c_2, c_1, 0, 0, \ldots).$$

In this case, such a map is an involution on $\mathrm{RPP}^{12}(\lambda/\mu;\nu)$. So, for irreducible partition ν the corresponding Bender-Knuth involution exists and is unique.

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[14]Of course, our standing assumption that λ/μ is connected can be lifted here, because in general, $\widetilde{g}_{\lambda/\mu}$ is the product of the analogous power series corresponding to the connected components of λ/μ. So we have obtained a new proof of Theorem 3.4.

Take any 12-rpp $T \in \text{RPP}^{12}(\lambda/\mu; \nu)$ and recall that a 12-table flip(T) is obtained from T by simultaneously replacing all entries in 1-pure columns by 2 and all entries in 2-pure columns by 1.

> **Corollary 7.9.** If ν is an irreducible partition, then no matter in which order one resolves descents in flip(T), the resulting tableau T' will be the same. The map $T \to T'$ is the unique Bender-Knuth involution on $\text{RPP}^{12}(\lambda/\mu; \nu)$.

Proof of Corollary 7.9. Descent-resolution steps applied to flip(T) in any order eventually give an element of $\text{RPP}^{12}(\lambda/\mu; \nu)$ with the desired ircont. There is only one such element. So we get a map $\text{RPP}^{12}(\lambda/\mu; \nu) \to \text{RPP}^{12}(\lambda/\mu; \nu)$ that satisfies the assumptions of Corollary 7.8. $\square$

Finally, notice that for a general representable partition ν descents may only occur inside each irreducible component independently, so we conclude the chain of corollaries by stating that our constructed involutions are canonical in the following sense:

> **Corollary 7.10.** For a representable partition, the map $\mathbf{B} : \text{RPP}^{12}(\lambda/\mu; \nu) \to \text{RPP}^{12}(\lambda/\mu; \nu)$ is the unique involution that interchanges the number of 1-pure columns with the number of 2-pure columns inside each irreducible component.

7.3. The proof

Let $\nu = (i_1, \ldots, i_s)$ be an irreducible partition. We start with the following simple observation:

> **Lemma 7.11.** Let $T \in \text{RPP}^{12}(\lambda/\mu; \nu)$. Then any 1-pure column of T is to the left of any 2-pure column of T.

Proof of Lemma 7.11. Suppose it is false and we have a 1-pure column to the right of a 2-pure column. Consider the closest pair of columns a and b such that a is 2-pure and b is 1-pure, then the columns $a+1, \ldots, b-1$ must all be mixed. Therefore the set $\text{NS}(T)$ contains $\{(i_1, a+1), (i_2, a+2), \ldots, (i_{b-1-a}, b-1)\}$. And because a is 2-pure and b is 1-pure, the support of any i_k for $k = 1 \ldots b-1-a$ is a subset of $[a+1, b)$ which contradicts the irreducibility of ν. $\square$

Proof of Theorem 7.4. We proceed by induction on the number of columns in λ/μ. If the number of columns is one then the statement of Theorem 7.4 is obvious. Suppose that we have proven that for all skew partitions $\widetilde{\lambda/\mu}$ with less than λ_1 columns and for all partitions $\widetilde{\nu}$ irreducible with respect to $\widetilde{\lambda/\mu}$ and for all $0 \leq \widetilde{m} \leq \widetilde{\lambda_1}) - \#\widetilde{\nu}$, there is exactly one 12-rpp $\widetilde{T}$ of shape $\widetilde{\lambda/\mu}$ with exactly $\widetilde{m}$ 1-pure columns, exactly $\#\widetilde{\nu}$ mixed columns and exactly $(\widetilde{\lambda_1} - \#\widetilde{\nu} - \widetilde{m})$ 2-pure columns. Now we want to prove the same for λ/μ.

Take any 12-rpp $T \in \mathrm{RPP}^{12}\left(\lambda/\mu; \nu\right)$ with $\mathrm{seplist}(T) = \nu$ and with m 1-pure columns for $0 \le m \le \lambda_1 - \#\nu$. Suppose first that $m > 0$. Then there is at least one 1-pure column in T. Let $q + 1$ be the leftmost such column for some $q \ge 0$. Then by Lemma 7.11 the columns $1, 2, \ldots, q$ are mixed. If $q > 0$ then the supports of $i_1, i_2, \ldots, i_q$ are all contained inside $[1, q+1)$ and we get a contradiction with the irreducibility of ν. The only remaining case is that $q = 0$ and the first column of T is 1-pure. Let $\widetilde{\lambda/\mu}$ denote λ/μ with the first column removed. Then ν may not be irreducible with respect to $\widetilde{\lambda/\mu}$, because it may happen that $\#\nu\big|_{\subseteq[1,b)}^{\widetilde{\lambda/\mu}} = b - 1$ for some b. In any case ν has only one irreducible component with respect to $\widetilde{\lambda/\mu}$ to which we can apply the induction hypothesis. We are done with the case $m > 0$. If $m < \lambda_1 - \#\nu$ then we can apply a mirrored argument to the last column, and it remains to note that for $\lambda_1 - \#\nu > 0$ the cases $m > 0$ and $m < \lambda_1 - \#\nu$ cover everything and for $\lambda_1 = \#\nu$ we get that T is a 12-rpp with all mixed columns, but there is obviously a unique such 12-rpp.

Note that this inductive proof shows both existence and uniqueness of the 12-rpp with desired properties. $\qquad\square$

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