QCQI (Nielsen and Chuang) Chapter Four Worked Solutions

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1 Problem 8.13

First let's define a unitary transform of ρ

$$A = U\rho U^\dagger = \tfrac{1}{2}UIU^\dagger + \tfrac{1}{2}U\vec{r}\cdot\vec{\sigma}U^\dagger.$$

To show that the unitary transformation of ρ in the Bloch sphere is just a rotation we must first show that A returns a valid density matrix. First, the trace condition

$$Tr(A) = Tr(U\rho U^\dagger) = Tr(UU^\dagger\rho) = Tr(I\rho) = Tr(\rho).$$

The trace of ρ i preserved and the trace condition is satisfied. Next is the positivity condition. With any vector $|\phi\rangle$ in the Bloch sphere

$$\langle\phi|A|\phi\rangle = \tfrac{1}{2}\langle\phi|I|\phi\rangle + \tfrac{1}{2}\langle\phi|U\vec{r}\cdot\vec{\sigma}U^\dagger|\phi\rangle = \tfrac{1}{2} + \tfrac{1}{2}\langle\phi|\gamma\rangle \geq 0,$$
$$\text{where } |\gamma\rangle = U\vec{r}\cdot\vec{\sigma}U^\dagger|\phi\rangle.$$

The quantity is indeed greater than or equal to zero because $|\gamma\rangle$ is a vector that lies inside the Bloch sphere since $U\vec{r}\cdot\vec{\sigma}U^\dagger$, from the definition of $|\gamma\rangle$, is a mapping from the Bloch sphere to itself. Therefore, $-1 \leq \langle\phi|\gamma\rangle \leq 1$ and $A \geq 0$. So, the positivity condition is satisfied. Finally, we must show that if A, the unitary transformation, is just a rotation of ρ in the Bloch sphere, then the length of ρ is preserved.

$$|\langle A|A\rangle| = (U\rho U^\dagger)^\dagger(U\rho U^\dagger) = (U\rho^\dagger U^\dagger)(U\rho U^\dagger) = U\rho\rho^\dagger U^\dagger = U\langle\rho|\rho\rangle U^\dagger = |\rho|UU^\dagger = |\rho|,$$

where $|\rho|$ denotes the norm of ρ. In conclusion, a unitary transformation in the Bloch sphere produces a valid density matrix and does not change its magnitude, therefore, it must be a rotation in the Bloch sphere.

2 Problem 8.19

Using the fact that $Tr\rho = 1$ we can write $\epsilon(\rho) = \frac{p}{d}I + (1-p)\rho$ as

$$\epsilon(\rho) = \tfrac{p}{d}ITr\rho + (1-p)\rho = \textstyle\sum_i = \tfrac{p}{d}|i\rangle\langle i|\langle i|\rho|i\rangle + (1-p)\rho = \sum_i = \tfrac{p}{d}|i\rangle\langle i|\rho|i\rangle\langle i| + (1-p)\rho.$$

This is the generalized operator-sum representation with operation elements $\{\sqrt{\frac{p}{d}}|i\rangle\langle i|, \sqrt{1-p}\}$.

3 Problem 8.22

This is a simple exercise in matrix multiplication.

$$\epsilon_{AD}(\rho) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix} =$$
$$\begin{bmatrix} a & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} + \begin{bmatrix} c\gamma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a + c\gamma & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}$$

We also have

$$Tr\rho = a + c = 1 \rightarrow c = 1 - a$$
$$\epsilon_{AD}(\rho) = \begin{bmatrix} a + (1-a)\gamma & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} = \begin{bmatrix} 1 - (1-\gamma)(1-a) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}.$$