QCQI (Nielsen and Chuang) Chapter Two Worked Solutions

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1 Problem 2.71

Exercise 2.71: (Criterion to decide if a state is mixed or pure) Let ρ be a density operator. Show that $\mathrm{tr}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

First, we define a mixed state density operator (ρ), which is a mixture of pure states with some probability (p),

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$
$$\rho^2 = \sum_l \sum_n p_l p_n \langle\psi_l|\psi_n\rangle |\psi_l\rangle\langle\psi_n|.$$

$$Tr(\rho^2) = \sum_m \langle e_m|\rho^2|e_m\rangle = \sum_m \sum_l \sum_n p_l p_n \langle\psi_l|\psi_n\rangle\langle e_m|\psi_l\rangle\langle\psi_n|e_m\rangle =$$
$$\sum_m \sum_l \sum_n p_l p_n \langle\psi_l|\psi_n\rangle\langle\psi_n|e_m\rangle\langle e_m|\psi_l\rangle = \sum_l \sum_n p_l p_n \langle\psi_l|\psi_n\rangle\langle\psi_n|(\sum_m |e_m\rangle\langle e_m|)|\psi_l\rangle =$$
$$\sum_l \sum_n p_l p_n \langle\psi_l|\psi_n\rangle\langle\psi_n|\psi_l\rangle = \sum_l \sum_n p_l p_n |\langle\psi_l|\psi_n\rangle|^2 = \sum_l p_l \sum_n p_n |\langle\psi_l|\psi_n\rangle|^2 =$$
$$= \sum_l \sum_m |\langle\psi_l|\psi_n\rangle|^2 \leq 1.$$

This achieves equality only when $n = l$, i.e. when ρ is a pure state.

2 Problem 2.72

Exercise 2.72: (Bloch sphere for mixed states) The Bloch sphere picture for pure states of a single qubit was introduced in Section 1.2. This description has an important generalization to mixed states as follows.

(1) Show that an arbitrary density matrix for a mixed state qubit may be written as

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}, \tag{2.175}$$

where $\vec{r}$ is a real three-dimensional vector such that $\|\vec{r}\| \leq 1$. This vector is known as the *Bloch vector* for the state ρ.

(2) What is the Bloch vector representation for the state $\rho = I/2$?

(3) Show that a state ρ is pure if and only if $\|\vec{r}\| = 1$.

(4) Show that for pure states the description of the Bloch vector we have given coincides with that in Section 1.2.

2.1 1

Any 2×2 Hermitian matrix can be decomposed into the identity and Pauli matrices as follows,

$$M = r_i I + r_x X + r_y Y + r_z Z.$$

$$M = r_i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + r_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + r_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} r_i + r_z & r_x - ir_y \\ r_x + ir_y & r_i - r_z \end{bmatrix}$$

$$Tr(M) = 2r_i \leq 1 \rightarrow r_i = \tfrac{1}{2},$$

where we have changed the $\leq$ to equality because arbitrary states in the bloch sphere are represented by the matrix

$$\begin{bmatrix} cos^2\frac{\theta}{2} & e^{-i\phi} sin\frac{\theta}{2} cos\frac{\theta}{2} \\ e^{i\phi} sin\frac{\theta}{2} cos\frac{\theta}{2} & sin^2\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} 1 + cos\theta & e^{-i\phi} sin\frac{\theta}{2} cos\frac{\theta}{2} \\ e^{i\phi} sin\frac{\theta}{2} cos\frac{\theta}{2} & 1 + sin\theta \end{bmatrix} = \tfrac{1}{2}(I + r_x X + r_y Y + r_z Z),$$

which has a trace of one. Therefore,

$$M = \tfrac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + r_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + r_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} r_i + r_z & r_x - ir_y \\ r_x + ir_y & r_i - r_z \end{bmatrix} = \tfrac{I + \vec{r} \cdot \vec{\sigma}}{2},$$

where $\vec{r} = (r_x, r_y, r_z)$.

Finally, we must show that $\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$ is positive definite. First,

$$\rho = \rho^\dagger.$$

Now we must verify that the eigenvalues are real and positive.

$$det|\rho - \lambda I| = \tfrac{1}{2} det \begin{vmatrix} 1 + r_z - \lambda & r_x - ir_y \\ r_x + ir_y & 1 - r_z - \lambda \end{vmatrix} = 0.$$

$$4\lambda^2 - 4\lambda - \|\vec{r}\|^2 + 1 = 0$$

Using the quadratic formula

$$\lambda = \tfrac{4 \pm 4\|\vec{r}\|}{8} = \tfrac{1}{2} \pm \tfrac{\|\vec{r}\|}{2},$$

which are real and positive. Therefore $\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$.

2.2 2

For $\rho = \frac{I}{2}$,

$$\vec{r} = (sin\theta cos\phi, sin\theta sin\phi, cos\theta) = (0, 0, 0) \rightarrow \theta = \tfrac{\pi}{2}, \phi = \tfrac{\pi}{2}.$$

Therefore, the Bloch sphere representation of ρ is

$$|\psi\rangle = \tfrac{\sqrt{2}}{2}|0\rangle + i\tfrac{\sqrt{2}}{2}|1\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + i\tfrac{1}{\sqrt{2}}|1\rangle.$$

2.3 3

Consider the pure state

$$\rho = |\psi\rangle\langle\psi| = \tfrac{1}{2}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}.$$

$$\rho^2 = \tfrac{1}{4}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}$$

$$Tr(\rho^2) = \tfrac{1}{4}((1+r_z)^2 + (r_x - ir_y)(r_x + ir_y) + (1-r_z)^2 + (r_x - ir_y)(r_x + ir_y)) =$$

$$\tfrac{1}{4}(2 + 2r_x^2 + 2r_y^2 + 2r_z^2) = \tfrac{1}{4}(2 + 2\|\vec{r}\|) = 1$$

$$\rightarrow \|\vec{r}\| = 1.$$

Therefore, in order to be a pure state $\|\vec{r}\| = 1$.

2.4 4

From earlier the Bloch sphere representation is

$$\begin{bmatrix} cos^2\frac{\theta}{2} & e^{-i\phi}sin\frac{\theta}{2}cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2}cos\frac{\theta}{2} & sin^2\frac{\theta}{2} \end{bmatrix} = \tfrac{1}{2}(I + r_xX + r_yY + r_zZ),$$

which is equivalent to the density matrix representation

$$\rho = cos^2\tfrac{\theta}{2}|0\rangle\langle 0| + e^{-i\phi}cos\tfrac{\theta}{2}sin\tfrac{\theta}{2}|0\rangle\langle 1| + e^{i\phi}cos\tfrac{\theta}{2}sin\tfrac{\theta}{2}|1\rangle\langle 0| + sin^2\tfrac{\theta}{2}|1\rangle\langle 1|,$$

since the sum of the outproducts is the identity with each element weighted by the coefficients in the expression of ρ.

3 Problem 2.74

Exercise 2.74: Suppose a composite of systems A and B is in the state $|a\rangle|b\rangle$, where $|a\rangle$ is a pure state of system A, and $|b\rangle$ is a pure state of system B. Show that the reduced density operator of system A alone is a pure state.

$$\rho^{AB} = |a\rangle\langle a| \otimes |b\rangle\langle b|$$

$$\rho^{A} = Tr_b(\rho^{AB}) = |a\rangle\langle a| \otimes Tr(|b\rangle\langle b|) = |a\rangle\langle a|,$$

which is the representation of a pure state, namely $|a\rangle$.

4 Problem 2.75

Exercise 2.75: For each of the four Bell states, find the reduced density operator for each qubit.

$$\rho_{\beta_{00}} = (\tfrac{|00\rangle+|11\rangle}{\sqrt{2}})(\tfrac{\langle 00|+\langle 11|}{\sqrt{2}}) = \tfrac{|00\rangle\langle 00|+|00\rangle\langle 11|+|11\rangle\langle 00|+|11\rangle\langle 11|}{2}$$

$$\rho^1_{\beta_{00}} = \rho^2_{\beta_{00}} = Tr_1(\rho_{\beta_{00}}) = \tfrac{\langle 0|0\rangle|0\rangle\langle 0|+\langle 1|0\rangle|0\rangle\langle 1|+\langle 0|1\rangle|1\rangle\langle 0|+\langle 1|1\rangle|1\rangle\langle 1|}{2} = \tfrac{|0\rangle\langle 0|+|1\rangle\langle 1|}{2} = \tfrac{I}{2}.$$

$$\rho_{\beta_{01}} = (\tfrac{|01\rangle+|10\rangle}{\sqrt{2}})(\tfrac{\langle 01|+\langle 10|}{\sqrt{2}}) = \tfrac{|01\rangle\langle 01|+|01\rangle\langle 10|+|10\rangle\langle 01|+|10\rangle\langle 10|}{2}$$

$$\rho^1_{\beta_{01}} = \rho^2_{\beta_{01}} = Tr_1(\rho_{\beta_{01}}) = \tfrac{|0\rangle\langle 0|+|1\rangle\langle 1|}{2} = \tfrac{I}{2}$$

$$\rho_{\beta_{10}} = (\frac{|00\rangle - |11\rangle}{\sqrt{2}})(\frac{\langle 00| - \langle 11|}{\sqrt{2}}) = \frac{|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|}{2}$$
$$Tr^1_{\beta 10} = Tr^2_{\beta 10} = \frac{I}{2}.$$

Similarly,

$$Tr^1_{\beta 11} = Tr^2_{\beta 11} = \frac{I}{2}.$$