# QCQI (Nielsen and Chuang) Chapter Five Worked Solutions

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**Exercise 5.1:** Give a direct proof that the linear transformation defined by Equation (5.2) is unitary.

If the transform $T$ is unitary, then $TT^\dagger = I$.

$$T = \frac{1}{\sqrt{N}} \sum_{k,j=0}^{N-1} e^{2\pi i jk/N} |k\rangle\langle j|$$

$$T^\dagger = \frac{1}{\sqrt{N}} \sum_{k',j'=0}^{N-1} e^{-2\pi i j'k'/N} |j'\rangle\langle k'|$$

$$TT^\dagger = \frac{1}{N} \sum_{k',j'=0}^{N-1} \sum_{k,j=0}^{N-1} e^{\frac{2\pi i}{N}(jk - j'k')} |k\rangle\langle j|j'\rangle\langle k'| = \frac{1}{N} \sum_{k',j'=0}^{N-1} \sum_{k,j=0}^{N-1} e^{\frac{2\pi i}{N}(jk - j'k')} \delta_{jj'} |k\rangle\langle k'| =$$

$$= \frac{1}{N} \sum_{k',j=0}^{N-1} \sum_{k,j=0}^{N-1} e^{\frac{2\pi i j}{N}(k-k')} |k\rangle\langle k'| = I$$

from the property that

$$\delta_{nm} = \frac{1}{N} \sum_{k=1}^{N} e^{\frac{2\pi i k}{N}(n-m)}.$$

Similarly,

$$T^\dagger T = \frac{1}{N} \sum_{k,j'=0}^{N-1} \sum_{k,j=0}^{N-1} e^{\frac{2\pi i k}{N}(j-j')} |j\rangle\langle j'| = I$$

Therefore, the transformation is unitary.

**Exercise 5.2:** Explicitly compute the Fourier transform of the n qubit state $|00...0\rangle$.

$$|00...0\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle.$$

**Exercise 5.10:** Show that the order of x = 5 modulo N = 21 is 6.

$$\begin{aligned}
5^0 &= 1(mod21)\\
5^1 &= 5(mod21)\\
5^2 &= 4(mod21)\\
5^3 &= 9(mod21)\\
5^4 &= 18(mod21)\\
5^5 &= 6(mod21)\\
5^6 &= 1(mod21)
\end{aligned}$$

**Problem 4:** Consider the $n$ Qbit state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|x_0\rangle_n + |x_0 + r\rangle_n)$$

where $x_0$ and $r$ are fixed non-negative integers and $0 < x_0 + r \leq 2^n - 1$. Apply that quantum Fourier gate given by

$$U_{FT}|x\rangle_n = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i xy/2^n} |y\rangle_n$$

to this state and determine the output state. After applying the Fourier gate and then measuring all $n$ Qbits, what is the probability that in the post measurement state all Qbits have the value 1. Write this probability in its simplest form.

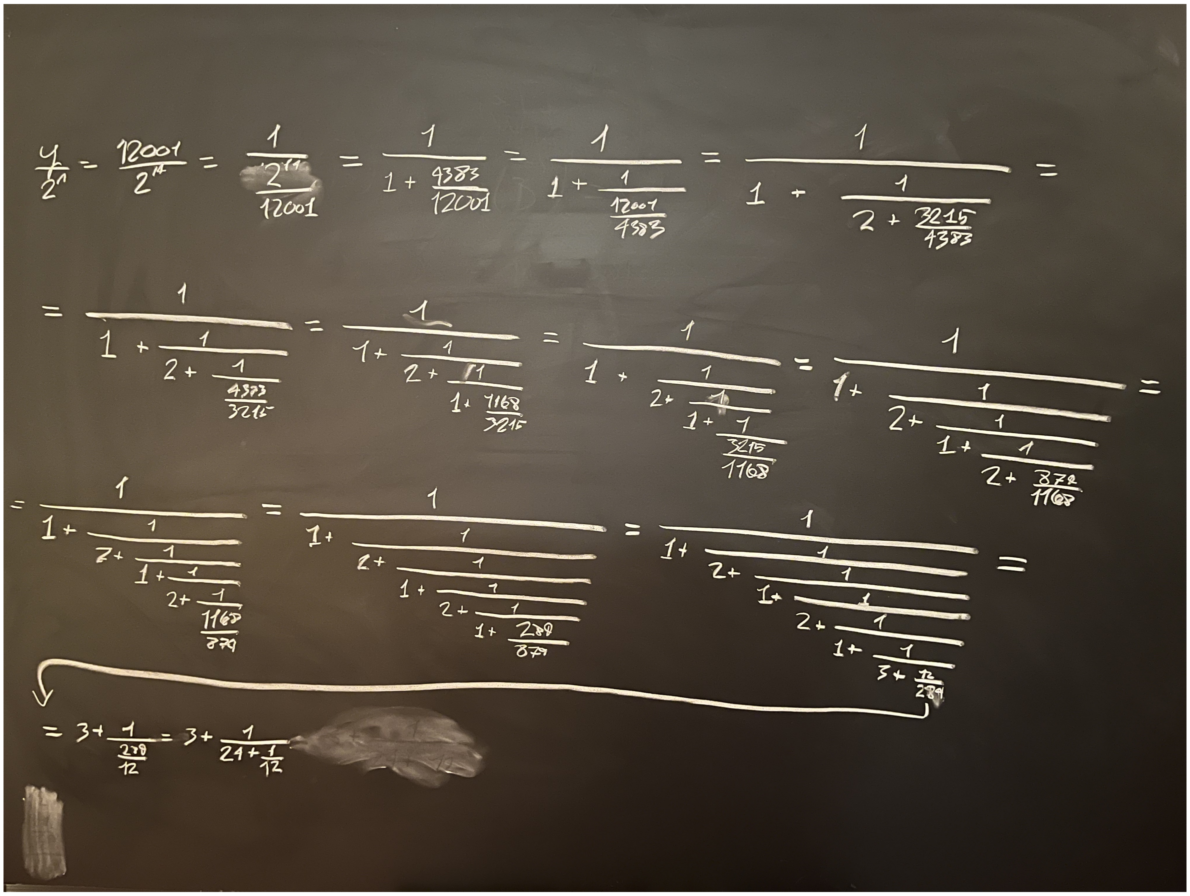
$$U_{FT}|\psi\rangle = \frac{1}{\sqrt{2^{n+1}}} \sum_{y=0}^{2^n-1} (e^{2\pi i x_0 y/2^n} + e^{2\pi i (x_0+r)y/2^n})|y\rangle_n = \frac{1}{\sqrt{2^{n+1}}} \sum_{y=0}^{2^n-1} (1 + e^{2\pi i r y/2^n})|y\rangle_n.$$

$$P(1) = |\langle 1|U_{FT}|\psi\rangle|^2 = \frac{1}{2^{n+1}}|1+e^{2\pi i r/2^n}|^2 = \frac{1}{2^{n+1}}(2+e^{2\pi i r/2^n}+e^{-2\pi i r/2^n}) = \frac{1}{2^n}(1+cos(2\pi r/2^n))$$

**Problem 6:** Consider the Shor algorithm and suppose the output register is has 7 Qbits and the input register has 14 Qbits. Measuring first the output register, then applying the Quantum Fourier transform and then measuring the input register yields the post measurement state $|12001\rangle$. Is there a number $r < 2^7$ and $1 \leq j < r$ such that

$$\left|\frac{12001}{2^{14}} - \frac{j}{r}\right| < \frac{1}{2r^2} \ ?$$

By expanding $\frac{12001}{2^{14}}$ via continued fractions we can find values of j and r. Each truncation of the continued fraction provides values of j and r that may or may not satisfy the above inequalities, so we pick the ones that do. Below is the continued fraction of $\frac{12001}{2^{14}}$:



Three different truncations (in order) show that three values of j and r satisfy the inequalities: j=3 and r=4, j=11 and r=15, and j=52 and r=71.