QCQI (Nielsen and Chuang) Chapter Four Worked Solutions

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1 4.2

Exercise 4.2: Let x be a real number and A a matrix such that $A^2 = I$. Show that

$$exp(iAx) = cos(x)I + isin(x)A.$$

Use this result to verify Equations (4.4) through (4.6).

$$exp(iAx) = 1 + iAx + \frac{(iAx)^2}{2!} + ... = (1 + \frac{(iAx)^2}{2!} + \frac{(iAx)^4}{4!} + ...) + (iAx + \frac{(iAx)^3}{3!} + \frac{(iAx)^5}{5!} + ...) = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...)A = cos(x)I + isin(x)A \text{ (i)}.$$

By fitting equations (4.4) to the form of equation (i), $A = X$ and $x = -\frac{\theta}{2}$, therefore $R_x(\theta) = e^{-i\theta X/2} = cos\frac{\theta}{2}I - isin\frac{\theta}{2}X$. The same process is used for equations (4.5), where $A = Y$ and $x = -\frac{\theta}{2}$, and (4.6), where $A = Z$ and $x = -\frac{\theta}{2}$.

2 4.13

Exercise 4.13: (Circuit identities) It is useful to be able to simplify circuits by inspection, using well-known identities. Prove the following three identities:

$$HXH = Z;\ HYH = -Y;\ HZH = X.$$

$$HXH = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = Z.$$

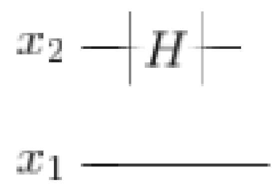
$$HYH = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}0 & i\\ -i & 0\end{bmatrix} = -Y.$$

Since the Hadamard matrix is hermitian $HH^\dagger = H^2 = I$, therefore

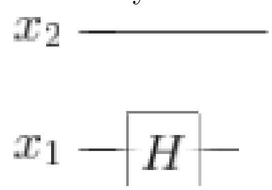
$$HXH = Z => H^\dagger HXHH^\dagger = H^\dagger ZH^\dagger => X = HZH.$$

3 4.16

Exercise 4.16: (Matrix representation of multi-qubit gates) What is the 4×4 unitary matrix for the circuit



in the computational basis? What is the unitary matrix for the circuit



in the computational basis?.

For the first circuit

$$(H \otimes I)(|x_2\rangle \otimes |x_1\rangle) = (H|x_2\rangle) \otimes (I|x_1\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

and for the second circuit

$$(I \otimes H)(|x_2\rangle \otimes |x_1\rangle) = (I|x_2\rangle) \otimes (H|x_1\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

4 4.17

Exercise 4.17: (Building CNOT from controlled-Z gates) Construct a CNOT gate from one controlled-Z gate, that is, the gate whose action in the computational basis is specified by the unitary matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and two Hadamard gates, specifying the control and target qubits.

The CNOT gate must flip the second (target) qubit if the first (control) qubit has the value one. We know that the Hadamard gate creates an equal superposition of the zero and one states and the controlled-Z gate will multiply the fourth qubit by negative one. The final Hadamard gate will then return the transformed qubits into the computational basis. From this reaosning it can be shown that the CNOT gate defined as $H(CZ)H$ will flip the second (target) qubit only if the first (control) qubit has a value one.

This can be checked by observing the actions of the Hadamard and controlled-Z gates on qubits in the computational basis.

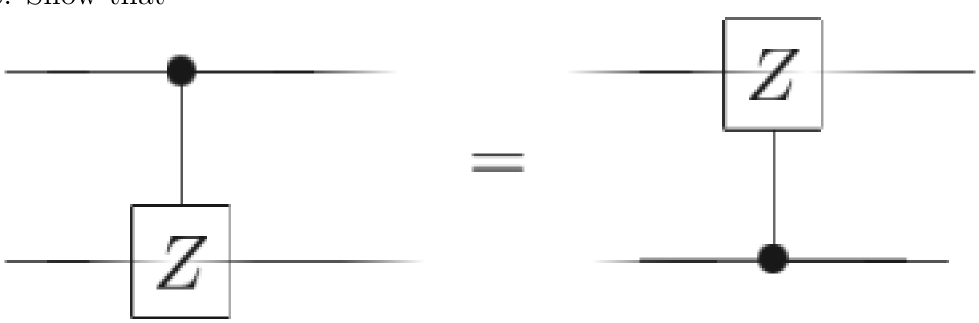
$$|10\rangle \to H \to \begin{cases} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \end{cases} \to CZ \to \begin{cases} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{cases} \to H \to \begin{cases} |1\rangle \\ |1\rangle \end{cases} = |11\rangle.$$

Similarly, $|00\rangle \to |00\rangle, |01\rangle \to |01\rangle$ and $|11\rangle \to |10\rangle$.
The definition $CNOT = H(CZ)H$ can also be checked by simple matrix multiplication.

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

5 4.18

Exercise 4.18: Show that



Observe that the action of the Z gate on a qubit is to flip it 180 about the z-axis,

$$Z|0\rangle = |0\rangle \text{ and } Z|1\rangle = -|1\rangle.$$

Therefore, if the control qubit has a value of one the Z gate will perform this action on the target qubit. Since the Z gate only changes $|1\rangle$ and is not applied if the control qubit is $|0\rangle$ both the control and target qubits must be $|1\rangle$; in this case the two circuits above are equivalent. Denoting the gate in the first circuit as CZ and the gate in the second circuit as

ZC,
$CZ|00\rangle = ZC|00\rangle = |00\rangle$, $CZ|01\rangle = ZC|01\rangle = |01\rangle$, $CZ|10\rangle = ZC|10\rangle = |10\rangle$, $CZ|11\rangle = ZC|11\rangle = -|11\rangle$.