QCQI (Nielsen and Chuang) Chapter Two Worked Solutions

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1 2.1

Three 2D vectors cannot all be linearly independent and to show this we can prove that they are linear combinations of each other.

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \text{therefore,}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

2 2.2

$$A = \begin{bmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{bmatrix} = \begin{bmatrix} \langle 0|0\rangle & \langle 0|1\rangle \\ \langle 1|0\rangle & \langle 1|0\rangle \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|.$$

We can construct a different but equivalent representation of A by constructing a new basis made by linear combinations of the original basis vectors. Let's define a new basis

$$(|a\rangle, |b\rangle) = (|0\rangle + |1\rangle, |0\rangle - |1\rangle)$$

and a change of basis matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix} = |0\rangle_{a,b}$$

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = |1\rangle_{a,b}. \text{ Therefore,}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

is the change of basis matrix. Finally,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = A_{|a\rangle,|b\rangle}.$$

3 Problem 2.66

Exercise 2.66: Show that the average value of the observable X_1Z_2 for a two qubit system measured in the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is zero.

The quantum average is called the expectation value, defined as $\langle\psi|B|\psi\rangle$, where B is some observable.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$X_1Z_2|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$$

$$\langle\psi|X_1Z_2|\psi\rangle = 0.$$

4 Problem 2.71

Exercise 2.71: (Criterion to decide if a state is mixed or pure) Let ρ be a density operator. Show that $\mathrm{tr}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

First, we define a mixed state density operator (ρ), which is a mixture of pure states with some probability (p),

$$\rho = \sum_n p_n|\psi_n\rangle\langle\psi_n|$$
$$\rho^2 = \sum_l\sum_n p_lp_n\langle\psi_l|\psi_n\rangle|\psi_l\rangle\langle\psi_n|.$$

$$Tr(\rho^2) = \sum_m\langle e_m|\rho^2|e_m\rangle = \sum_m\sum_l\sum_n p_lp_n\langle\psi_l|\psi_n\rangle\langle e_m|\psi_l\rangle\langle\psi_n|e_m\rangle =$$
$$\sum_m\sum_l\sum_n p_lp_n\langle\psi_l|\psi_n\rangle\langle\psi_n|e_m\rangle\langle e_m|\psi_l\rangle = \sum_l\sum_n p_lp_n\langle\psi_l|\psi_n\rangle\langle\psi_n|(\sum_m |e_m\rangle\langle e_m|)|\psi_l\rangle =$$
$$\sum_l\sum_n p_lp_n\langle\psi_l|\psi_n\rangle\langle\psi_n|\psi_l\rangle = \sum_l\sum_n p_lp_n|\langle\psi_l|\psi_n\rangle|^2 = \sum_l p_l\sum_n p_n|\langle\psi_l|\psi_n\rangle|^2 =$$
$$= \sum_l\sum_m |\langle\psi_l|\psi_n\rangle|^2 \leq 1.$$

This achieves equality only when $n = l$, i.e. when ρ is a pure state.

5 Problem 2.72

Exercise 2.72: (Bloch sphere for mixed states) The Bloch sphere picture for pure states of a single qubit was introduced in Section 1.2. This description has an important generalization to mixed states as follows.

(1) Show that an arbitrary density matrix for a mixed state qubit may be written as

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}, \qquad (2.175)$$

where $\vec{r}$ is a real three-dimensional vector such that $\|\vec{r}\| \leq 1$. This vector is known as the *Bloch vector* for the state ρ.

(2) What is the Bloch vector representation for the state $\rho = I/2$?

(3) Show that a state ρ is pure if and only if $\|\vec{r}\| = 1$.

(4) Show that for pure states the description of the Bloch vector we have given coincides with that in Section 1.2.

5.1 1

Any 2×2 Hermitian matrix can be decomposed into the identity and Pauli matrices as follows,

$$M = r_i I + r_x X + r_y Y + r_z Z.$$

$$M = r_i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + r_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + r_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} r_i + r_z & r_x - ir_y \\ r_x + ir_y & r_i - r_z \end{bmatrix}$$
$$Tr(M) = 2r_i \leq 1 \to r_i = \tfrac{1}{2},$$

where we have changed the $\leq$ to equality because arbitrary states in the bloch sphere are represented by the matrix

$$\begin{bmatrix} cos^2\frac{\theta}{2} & e^{-i\phi} sin\frac{\theta}{2} cos\frac{\theta}{2} \\ e^{i\phi} sin\frac{\theta}{2} cos\frac{\theta}{2} & sin^2\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} 1 + cos\theta & e^{-i\phi} sin\frac{\theta}{2} cos\frac{\theta}{2} \\ e^{i\phi} sin\frac{\theta}{2} cos\frac{\theta}{2} & 1 + sin\theta \end{bmatrix} = \tfrac{1}{2}(I + r_x X + r_y Y + r_z Z),$$

which has a trace of one. Therefore,

$$M = \tfrac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + r_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + r_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} r_i + r_z & r_x - ir_y \\ r_x + ir_y & r_i - r_z \end{bmatrix} = \tfrac{I + \vec{r}\cdot\vec{\sigma}}{2},$$
where $\vec{r} = (r_x, r_y, r_z)$.

Finally, we must show that $\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$ is positive definite. First,

$$\rho = \rho^\dagger.$$

Now we must verify that the eigenvalues are real and positive.

$$det|\rho - \lambda I| = \tfrac{1}{2} det \begin{vmatrix} 1 + r_z - \lambda & r_x - ir_y \\ r_x + ir_y & 1 - r_z - \lambda \end{vmatrix} = 0.$$
$$4\lambda^2 - 4\lambda - \|\vec{r}\|^2 + 1 = 0$$

Using the quadratic formula

$$\lambda = \tfrac{4 \pm 4\|\vec{r}\|}{8} = \tfrac{1}{2} \pm \tfrac{\|\vec{r}\|}{2},$$

which are real and positive. Therefore $\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$.

5.2 2

For $\rho = \frac{I}{2}$,

$$\vec{r} = (sin\theta cos\phi, sin\theta sin\phi, cos\theta) = (0, 0, 0) \to \theta = \tfrac{\pi}{2}, \phi = \tfrac{\pi}{2}.$$

Therefore, the Bloch sphere representation of ρ is

$$|\psi\rangle = \tfrac{\sqrt{2}}{2}|0\rangle + i\tfrac{\sqrt{2}}{2}|1\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + i\tfrac{1}{\sqrt{2}}|1\rangle.$$

5.3 3

Consider the pure state

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}.$$

$$\rho^2 = \frac{1}{4}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}$$

$$Tr(\rho^2) = \frac{1}{4}((1+r_z)^2 + (r_x - ir_y)(r_x + ir_y) + (1-r_z)^2 + (r_x - ir_y)(r_x + ir_y)) =$$

$$\frac{1}{4}(2 + 2r_x^2 + 2r_y^2 + 2r_z^2) = \frac{1}{4}(2 + 2\|\vec{r}\|) = 1$$

$$\rightarrow \|\vec{r}\| = 1.$$

Therefore, in order to be a pure state $\|\vec{r}\| = 1$.

5.4 4

From earlier the Bloch sphere representation is

$$\begin{bmatrix} cos^2\frac{\theta}{2} & e^{-i\phi}sin\frac{\theta}{2}cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2}cos\frac{\theta}{2} & sin^2\frac{\theta}{2} \end{bmatrix} = \frac{1}{2}(I + r_xX + r_yY + r_zZ),$$

which is equivalent to the density matrix representation

$$\rho = cos^2\frac{\theta}{2}|0\rangle\langle 0| + e^{-i\phi}cos\frac{\theta}{2}sin\frac{\theta}{2}|0\rangle\langle 1| + e^{i\phi}cos\frac{\theta}{2}sin\frac{\theta}{2}|1\rangle\langle 0| + sin^2\frac{\theta}{2}|1\rangle\langle 1|,$$

since the sum of the outproducts is the identity with each element weighted by the coefficients in the expression of ρ.

6 Problem 2.74

Exercise 2.74: Suppose a composite of systems A and B is in the state $|a\rangle|b\rangle$, where $|a\rangle$ is a pure state of system A, and $|b\rangle$ is a pure state of system B. Show that the reduced density operator of system A alone is a pure state.

$$\rho^{AB} = |a\rangle\langle a| \otimes |b\rangle\langle b|$$

$$\rho^A = Tr_b(\rho^{AB}) = |a\rangle\langle a| \otimes Tr(|b\rangle\langle b|) = |a\rangle\langle a|,$$

which is the representation of a pure state, namely $|a\rangle$.

7 Problem 2.75

Exercise 2.75: For each of the four Bell states, find the reduced density operator for each qubit.

$$\rho_{\beta_{00}} = (\frac{|00\rangle+|11\rangle}{\sqrt{2}})(\frac{\langle 00|+\langle 11|}{\sqrt{2}}) = \frac{|00\rangle\langle 00|+|00\rangle\langle 11|+|11\rangle\langle 00|+|11\rangle\langle 11|}{2}$$

$$\rho^1_{\beta_{00}} = \rho^2_{\beta_{00}} = Tr_1(\rho_{\beta_{00}}) = \frac{\langle 0|0\rangle|0\rangle\langle 0|+\langle 1|0\rangle|0\rangle\langle 1|+\langle 0|1\rangle|1\rangle\langle 0|+\langle 1|1\rangle|1\rangle\langle 1|}{2} = \frac{|0\rangle\langle 0|+|1\rangle\langle 1|}{2} = \frac{I}{2}.$$

$$\rho_{\beta_{01}} = (\frac{|01\rangle+|10\rangle}{\sqrt{2}})(\frac{\langle 01|+\langle 10|}{\sqrt{2}}) = \frac{|01\rangle\langle 01|+|01\rangle\langle 10|+|10\rangle\langle 01|+|10\rangle\langle 10|}{2}$$

$$\rho^1_{\beta_{01}} = \rho^2_{\beta_{01}} = Tr_1(\rho_{\beta_{01}}) = \frac{|0\rangle\langle 0|+|1\rangle\langle 1|}{2} = \frac{I}{2}$$

$$\rho_{\beta_{10}} = (\frac{|00\rangle - |11\rangle}{\sqrt{2}})(\frac{\langle 00| - \langle 11|}{\sqrt{2}}) = \frac{|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|}{2}$$
$$Tr^1_{\beta 10} = Tr^2_{\beta 10} = \frac{I}{2}.$$

Similarly,

$$Tr^1_{\beta 11} = Tr^2_{\beta 11} = \frac{I}{2}.$$

8 Problem 2.78

Exercise 2.78: Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1. Prove that $|\psi\rangle$ is a product state if and only if ρ^A (and thus ρ^B) are pure states.

Using the Schmidt decomposition of a composite system (Theorem 2.7) we can write the density operator of composite system AB as

$$|\psi\rangle = \sum_i \lambda_i^2 |i_A\rangle\langle i_B| \otimes |i_B\rangle\langle i_B|.$$

If we carry out this sum then $|\psi\rangle$ will evidently not be a product state. Therefore, the "sum" has only one term, namely when $i = 1$.

$$\sum_1 \lambda_1^2 = 1 \rightarrow \lambda_1 = 1$$
$$\text{and therefore, } \sum_{i=2} \lambda_i^2 = 0.$$

By the Schmidt decomposition the density operators are

$$\rho^A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A| \text{ and } \rho^B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|.$$

By the previous result the state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1. Therefore,

$$\rho^A = |i_A\rangle\langle i_A| \text{ and } \rho^B = |i_B\rangle\langle i_B|,$$

which are pure states.