HBSS_CONT

https://github.com/dario-anastasio/HBSS_Cont

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HBSS_Cont is a Matlab toolbox for computing nonlinear frequency response curves (NFRCs) of periodically forced systems.

It is built around three core ideas:

1. **Harmonic Balance (HB)** to represent steady-state periodic responses.
2. **Extended state-space modeling**, where nonlinearities are treated as feedback forces. This makes it compatible with nonlinear system identification schemes such as NSI (nonlinear subspace identification [1] and NFR-ID (nonlinear frequency response identification [2].
3. **Pseudo-Arc-Length Continuation** to robustly follow solution branches, including folds and unstable regions.

The toolbox is intended for **research and educational use**.

This document provides a tutorial-style description of the method and the main settings. Detailed examples are described separately.

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Problem statement and assumptions

The problem is formulated as a nonlinear mechanical system in the *extended* state-space framework. The state-space formulation can be obtained by starting from following equation of motion in the physical domain:

$$\boldsymbol{M}\ddot{\boldsymbol{y}}(t) + \boldsymbol{C}_v\dot{\boldsymbol{y}}(t) + \boldsymbol{K}\boldsymbol{y}(t) + \boldsymbol{f}^{\mathrm{nl}}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t) = \boldsymbol{f}(t),$$

where $\boldsymbol{M}$, $\boldsymbol{C}_v$ and $\boldsymbol{K}$ are the mass, viscous damping, and stiffness matrices respectively, while $\boldsymbol{y}(t)$ and $\boldsymbol{f}(t)$ are the generalized displacement and external force vectors. The nonlinear part of the equation is described by the term $\boldsymbol{f}^{nl}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t)$, and generally depends on displacements and/or velocities. It is assumed that $\boldsymbol{f}^{nl}$ can be decomposed into J distinct nonlinear contributions using a linear-in-the-parameters model, thus yielding:

$$\boldsymbol{f}^{\mathrm{nl}}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t) = \sum_{j=1}^{J} \mu_j \boldsymbol{L}_{\mathrm{nl}_j} g_j(\boldsymbol{y}, \dot{\boldsymbol{y}}, t)$$

Each term of the summation is defined by a coefficient μ_j, a nonlinear basis function $g_j(\boldsymbol{y}, \dot{\boldsymbol{y}}, t)$ and a location vector $\boldsymbol{L}_{\mathrm{nl}_j} \in \mathbb{R}^N$. The elements of $\boldsymbol{L}_{\mathrm{nl}_j}$ can assume the values -1, 1 or 0 and define the position of the j^{th} nonlinearity. Defining now the extended-input vector $\boldsymbol{f}^e$ as

$$\boldsymbol{f}^{\mathrm{e}}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t) = \left[\boldsymbol{f}(t)^{\mathrm{T}}, \quad -g_1(\boldsymbol{y}, \dot{\boldsymbol{y}}, t), \quad \dots, \quad -g_J(\boldsymbol{y}, \dot{\boldsymbol{y}}, t)\right]^{\mathrm{T}}$$

and introducing the state vector $\boldsymbol{x} = [\boldsymbol{y}^{\mathrm{T}}, \dot{\boldsymbol{y}}^{\mathrm{T}}]^{\mathrm{T}}$, a continuous state-space formulation can be retrieved:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}^{\mathrm{e}}\boldsymbol{f}^{\mathrm{e}}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t) \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}^{\mathrm{e}}\boldsymbol{f}^{\mathrm{e}}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t) \end{cases}$$

The matrices $\boldsymbol{A}, \boldsymbol{B}^{\mathrm{e}}, \boldsymbol{C}, \boldsymbol{D}^{\mathrm{e}}$ are the state, extended input, output and extended direct feedthrough matrices, respectively. The "continuous" formulation is adopted in this document, but the software can handle the discrete case too, which is the most common one when estimating these matrices from experimental measurements.

Assuming that the state-space model is driven by a periodic excitation and that the system response has the same period as the excitation, the time-domain vectors can be expressed as Fourier series up to a user-defined order H. This is the starting point of HBSS_Cont. The reader is referred to [3] for the detailed steps of the method.

Assumptions

- The system admits a time-periodic steady-state response under harmonic forcing.
- The forcing term contains a single input. The force applied to the system to generate the NFRS is described by a fundamental harmonic contribution.
- The steady-state response can be adequately approximated by a truncated Fourier series with a finite number of harmonics.
- The nonlinear part of the equation of motion can be written as a linear-in-the-parameters model with a user-defined number of nonlinear basis functions. The derivatives of such functions with respect to the model outputs are either given by the user or computed numerically by the algorithm.
- The model takes the form of an extended state-space formulation. The output of the state-space model is in terms of displacement. Acceleration-based models are currently not supported.

Obtaining the state-space matrices

From an analytical model

This section describes how to construct the state-space matrices required by HBSS_Cont starting from an analytical (physics-based) mechanical model, consistently with the assumptions adopted by the toolbox.

The workflow described here is implemented in the helper function `hbss_build_extended_ss`, which can be easily called to directly obtain the matrices.

The continuous-time state-space model is derived considering the state vector $\boldsymbol{x}$ defined in previous section, and output vector equal to $\boldsymbol{y}$. The following matrices can be obtained:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C}_{\mathrm{v}} \end{bmatrix}, \qquad \boldsymbol{B}^{\mathrm{e}} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\left[\boldsymbol{L}_{\mathrm{f}} \quad \mu_1 \boldsymbol{L}_{\mathrm{nl}_1} \dots \mu_J \boldsymbol{L}_{\mathrm{nl}_J}\right] \end{bmatrix}, \qquad \boldsymbol{C} = [\boldsymbol{I} \quad \boldsymbol{0}], \qquad \boldsymbol{D}^{e} = \boldsymbol{0}.$$

Here:

- $\boldsymbol{L}_{\mathrm{f}}$ is a logical vector of size $n_y \times 1$ having all entries equal to zero, except for the location of the external forcing term, which is equal to 1.
- $\boldsymbol{L}_{\mathrm{nl}_j}$ is the logical vector previously defined, of size $n_y \times 1$, defining the DOFs connections of the j^{th} nonlinearity.

From experimental data

Experimental methods such as NSI [1,3–5] and NFR-ID [2] allow estimating nonlinear state-space models directly from experimental data, either in the time or frequency domain. The reader is referred to the cited references for detailed descriptions of these identification techniques.

In these cases, the identified model is typically provided in discrete-time state-space form, associated with a given sampling period T_s.

HBSS_Cont accepts discrete-time state-space matrices as input and automatically converts them to an equivalent continuous-time representation assuming a zero-order hold (ZOH) on the inputs. The ZOH assumption implies that the input signal is considered piecewise constant over each sampling interval T_s. Under this assumption, the discrete-time state-space model is interpreted as the exact discretization of an underlying continuous-time system subject to zero-order-hold inputs. The continuous-time matrices are obtained internally using standard ZOH-based discretization inversion, which is exact for linear systems under piecewise-constant excitation.

This assumption is generally appropriate when the sampling frequency is sufficiently higher than the highest frequency of interest in the system response, and the identified discrete-time model is intended to represent an underlying continuous-time dynamics.

However, it should be considered that the ZOH assumption introduces an implicit modeling hypothesis on the inter-sample behavior. Also, very low sampling rates or strongly nonlinear inter-sample dynamics may reduce the accuracy of the continuous-time approximation.

User-defined parameters

HBSS_Cont is configured through a single Matlab struct, hereafter called HB. The user is expected to populate HB with the state-space model, the forcing definition, and a set of numerical options controlling Harmonic Balance, continuation, stability analysis and plotting.

Mandatory fields

- `HB.A` : state matrix
- `HB.Be` : extended input matrix
- `HB.C` : output matrix mapping states to measured outputs
- `HB.De` : extended direct feedthrough matrix
- `HB.Ts` : sampling time (s). Use `Ts = 0` for continuous-time models; `Ts > 0` for discrete-time models (automatically converted to continuous time internally)
- `HB.fLim` : frequency range for the continuation, in Hz, as a two-element vector `[fMin fMax]`
- `HB.Harmonics` : vector of harmonics included in HB. It must start from 0 (DC).
- `HB.NL.fNL`: cell array of nonlinear laws.
  Each element is a function handle `fNL{i}(y, yd)` returning the i^{th} nonlinear force over one period in AFT.
- `HB.F` : external forcing fundamental Fourier coefficient (one-sided).
  It can be a single value, or a function of the angular frequency (in rad/s).

Optional fields and recommended settings

- `HB.nSamples`: number of time samples used by the AFT operator over one period (default: 512). It should be sufficiently larger to avoid aliasing in the FFT/IFFT steps.
- `HB.scaleY` : scaling factor applied internally to the unknown harmonic coefficients to improve conditioning. It can be a numeric scalar or the string `'auto'` (default), which estimates a scaling based on the linear FRF over the selected frequency range.
- `HB.solver` : struct controlling the nonlinear corrector (Newton) solver.
- `HB.solver.opts`: solver options. For `fsolve`, it must be an `optimoptions` object.
- `HB.cont.hmax`: maximum continuation step size (default: 1).
- `HB.cont.hmin`: minimum continuation step size (default: 1e-5).
- `HB.cont.maxIterCont`: maximum number of continuation iterations (default: 10).
- `HB.plot.enable`: enable/disable plotting (default: true).
- `HB.plot.chOut`: output channel index used for the NFRC plot (default: 1).
- `HB.plot.updateEvery`: refresh rate (in continuation points) for plot updates (default: 10).

Optional analytical derivatives of nonlinearities

To speed up and improve robustness of the stability analysis (Floquet multipliers), the user can optionally provide analytical partial derivatives of the nonlinear forces with respect to displacement and velocity. If they are not provided, finite differences are used internally.

- `HB.NL.dFNL`: cell array. Element `dFNL{i,j}(y, yd)` must return $\partial f_{\mathrm{nl},i}/\partial y_j$ evaluated at a single time sample
- `HB.NL.dFNLvel`: cell array. Element `dFNLvel{i,j}(y, yd)` must return $\partial f_{\mathrm{nl},i}/\partial \dot{y}_j$ evaluated at a single time sample.

Examples provided in the repository

The examples folder contains a set of MATLAB scripts illustrating the use of the HBSS_Cont toolbox on representative nonlinear dynamical systems.

Duffing oscillator (analytical)

This example considers a classical single-degree-of-freedom Duffing oscillator, which is commonly used as a benchmark in nonlinear vibration analysis.

The system includes a cubic stiffness nonlinearity and may optionally include a quadratic stiffness term.

The state-space matrices are constructed analytically starting from the physical parameters of the system, namely mass, damping, and linear stiffness.

Five-degree-of-freedom chain with cubic stiffness (analytical)

This example considers a mechanical system composed of five degrees of freedom connected through linear springs and dampers, with additional cubic stiffness nonlinearities acting between all adjacent degrees of freedom, except between the first mass and ground.

The model is constructed analytically from the physical mass, damping, and stiffness matrices, resulting in a multi-degree-of-freedom nonlinear system.

This example is taken from [6].

Cantilever beam with motion limiters (experimental)

In this example, the system matrices are obtained experimentally using the NFR-ID (Nonlinear Frequency Response Identification) method.

The physical system consists of a cantilever beam with motion limiters located at the free end.

The motion limiters introduce a discontinuous nonlinearity, modeled as the sum of multiple piecewise-linear functions characterized by different gap values.

Double-well oscillator identified (experimental)

This example is based on an experimentally identified double-well oscillator, with system matrices obtained through Nonlinear Subspace Identification (NSI) from random excitation measurements.

The physical system exhibits bi-stable behavior; however, the available experimental data correspond to in-well oscillations only. As a result, the identified model captures the local dynamics within a single potential well and does not describe cross-well motion.

The system is characterized by a softening nonlinearity and exhibits period-doubling bifurcations under harmonic excitation.

In this example, the excitation amplitude used in the experiments varies proportionally to the square of the excitation frequency. Accordingly, the forcing amplitude is defined as a function of the excitation frequency, with coefficients obtained from experimental identification.

References

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