How to use the Kelly Criterion on Favourable Bets

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December 2022

Abstract

In this paper we explore the application of the Kelly Criterion to a variety of different gambling games, all with mutually exclusive, unknown outcomes, and rounds played repeatedly, to find the optimal wagering strategy. We initially derive and explain the original Kelly Criterion using a game from Edward Thorp's paper titled, "The Kelly Criterion in Blackjack, Sports Betting, and the Stock Market". We then examine how to apply the Kelly Criterion to a variety of games each increasing in their complexity, including the explicit answer to a problem posed by Kelly himself in his original paper, concerning the generalization of the criterion to an instance with bets on multiple outcomes, like that of a horse race. The solution was presented by Peter Smoczynski and Dave Tomkins in their 2010 paper, "An explicit solution to the problem of optimizing the allozations of a bettor's wealth when wagering on horse races". Finally, we present and explore a new modified version of the model presented by Thorp. Overall this paper aims to demonstrate to readers how to apply the Kelly criterion to take advantage of favourable bets if they have the opportunity.

1 Introduction

(intro draft idea):

Gambling is not a deterministic enterprise; wagers are often placed with great uncertainty on the outcomes of the games. For most people, this makes placing a wager an extremely confusing event. Considering probabilities, bankroll, and payoffs is a complicated task. This was the inspiration behind a study by Dewey at all, who looked at how university students bet on a biased coin (heads comes up with probability 0.6) given a bankroll of 25 dollars and even pay-outs. Over 25 percent of the students went completely bust, while about 30 percent ended up with less than the original 25 dollar bankroll. (Even more interesting perhaps is the fact that 67 percent of participants bet on tails once, a clear fallacy given the parameters).If you are ever given the opportunity to make a wager on a favourable bet like above, an extremely useful strategy to employ, is the Kelly Criterion.

Gambling is a not a scientific/sure/certainty game but there is a lot of math in gambling/betting. This begs the question if there is a mathematical "solution" to betting, our exploration of this lead us to the paradox from Bertrand Russel. In this paper we will go through/follow/explain a set of mathematical problems and solutions that span a large portion of history. In the beginning, Bernoulli poses the lottery paradox thing. Hundreds of years later, John Kelly introduces the logarithm function as a utility function to represent the worth/value/utility of money, (this will later be adopted in economics as very common policy). Now that he has a way to express the value/utility of money mathematically, he sets out to define a mathematically proper/derived/based/verified/recommended/OPTIMAL way to gamble. However, gambling can be done in many ways so he originally defines his criterion on a two outcome biased coin flip scenario. In his original paper he also poses the question about expanding it to a more than two outcome mutually exclusive endings (only 1 horse can win a round/race). In 2011(i think), his original question was officially answered by (whoever tf wrote HORSELITERATURE) and the Kelly Criterion were expanded. In this paper, we will explain the Kelly Criterion on the original 2 outcome biased coin case, as well as its solution/generalization to the multiple outcome case (horse betting) and then introduce/explore a new twist/variable/complicating factor on the Kelly model with a new variation, jelly bean game.

2 Model Derivation

The model was originally derived by John Kelly of Bell Laboratories in 1956. The following derivation is based on a later paper written by Thorp in 2007 which expands on the findings of Kelly and gives the example of the biased coin example which we will show later on.

The concept of utility of wealth proves to be useful when examining continuous betting games, specifically logarithmic utility. Utility is term frequently used in economics to describe the value of something to a person. Heuristically, the reason we are concerned with the utility of wealth versus just outright wealth is the fact that the magnitude of a loss affects players of different wealth statutes differently. That is, a win of 10 dollars for someone who's wealth is 100 dollars is much greater in terms of "utility" than that of someones who's wealth is 10,000 dollars. This is also goes for for who the loss of a fixed value bet affects individuals of different wealth statuses differently. Kelly was concerned with the utility function that was the logarithm of the wealth, $ln(X_n)$ of a player. More specifically, he sought to maximize the expected value, $E[ln(X_n)]$ of a player. With this in mind the model derivation for a two outcome game follows below.

$X_0 =$ Initial Wealth

$X_n =$ Wealth after n trials

$p =$ probability of successful trial

$q = 1 - p =$ probability of unsuccessful trial

$f =$ fraction of wealth bet

$a =$ fraction of bet won from a successful trial

$b =$ fraction of bet lost from an unsuccessful trial

$S =$ Successful trial

$F =$ Unsuccessful Trial

$n = S + F =$ number of trials

As mentioned above, the result of the Kelly Criterion is a fixed fraction f of your wealth that you should bet on each trial of the game. After a successful trial our wealth will increase by $(1 + af)$ and after an unsuccessful trial our wealth will decrease by $(1 - bf)$. So our wealth after n trials, X_n, is given by: $X_n = X_0(1 + af)^S(1 - bf)^F$.
Here the identity $e^{n\ln(\frac{X_n}{X_0})^{1/n}} = \frac{X_n}{X_0}$ is noted as it shows that $\ln(X_n/X_0)^{1/n}$ is a measure of the exponential rate of increase in utility of each trial.
After manipulating the equation for X_n shown above we get.

$X_n/X_0 = (1 + af)^S(1 - bf)^F$ which implies,

$(X_n/X_0)^{1/n} = (1 + af)^{S/n}(1 - bf)^{F/n}$

$\ln(X_n/X_0)^{1/n} = (S/n)\ln(1 + af) + (F/n)\ln(1 - bf)$

The above is measure of the rate of utility increase per trial. In his original paper, Kelly maximizes the expected value of this equation, which we will now call $g(f)$, shown below.

$g(f) = E[\ln(X_n/X_0)^{1/n}] = E[(S/n)\ln(1 + af)] + E[(F/n)\ln(1 - bf)]$

What is important to notice about $g(f)$ above is how for fixed n and X_0, it relates back to our original discussion about utility. $g(f) = E[\ln(X_n/X_0)^{1/n}]$ can be re-arranged to show that $g(f) = (1/n)E[\ln(X_n)] - (1/n)\ln X_0$. So by maximizing $g(f)$ we are also maximizing $E[\ln(X_n)]$, the expected value of the logarithm of wealth. After pulling the constants out of the expected value functions above the derivation continuous as follows:

$g(f) = (1/n)\ln(1 + af)E[S] + (1/n)\ln(1 - bf)E[F]$

S and F are Binomial random variables. Which means $E[S] = np$ and $E[F] = nq$ respectively

$g(f) = (1/n)\ln(1 + af)(np) + (1/n)\ln(1 - bf)(nq)$

$g(f) = p\ln(1 + af) + q\ln(1 - bf)$

We want to find the fraction of wealth f that will maximizes $g(f)$. So we take the derivative.

$g'(f) = \frac{ap}{1+af} - \frac{bq}{1-fb} = 0$

$\frac{ap}{1+af} = \frac{bq}{1-bf}$

$ap(1 - bf) = bq(1 + af)$

$pa - pbaf = bq + qbaf$

$pa - bq = f(pab + qab)$ which leads to

$f(pab + (1 - p)(ab)) = pa - bq$

$f(ab) = pa - bq$

$f = \frac{p}{b} - \frac{q}{a}$

2.1 Analysis of Bias Coin

Lets take a look at the bias coined example that Thorp examined in his paper.
We have a biased coin where the probability of getting heads is 0.6. There is even payouts, which means that the value of $a = b = 1$. A player is given a 25 dollar bank roll to start out with. It follows that the optimal bet under the Kelly Criterion would be to place $0.6 - 0.4 = 0.2$ fraction of your wealth on each turn. This gives a utility (logarithm of wealth) growth rate of $g(0.2) = 0.6\ln(1.2) + 0.4\ln(0.8) = 0.0201$. Recall that $g(f)$ is also a measure of the logarithm of a players wealth increase. This means that each flip will give a player the dollar equivalent increase in utility of $e^{g(f)} = e^{0.0201} = 1.02034$ So if we were to flip the coin 300 times, the dollar equivalent would be be $25(1.02034)^{300} = 10,504$ ** . Even though the formula for the result is quite simple, we can scale the Kelly Criterion easily for games with many different mutually exclusive outcomes and uneven pay outs. The "Mad Marble" game below this. [** calculation double checked with the paper by Dewey] **INSERT GRAPHS**

3 New Game

The following game was inspired by a stack exchange forum cited in the bibliography. We analyze a new game where a player is given the opportunity to place a single bet on the result of multiple mutually exclusive

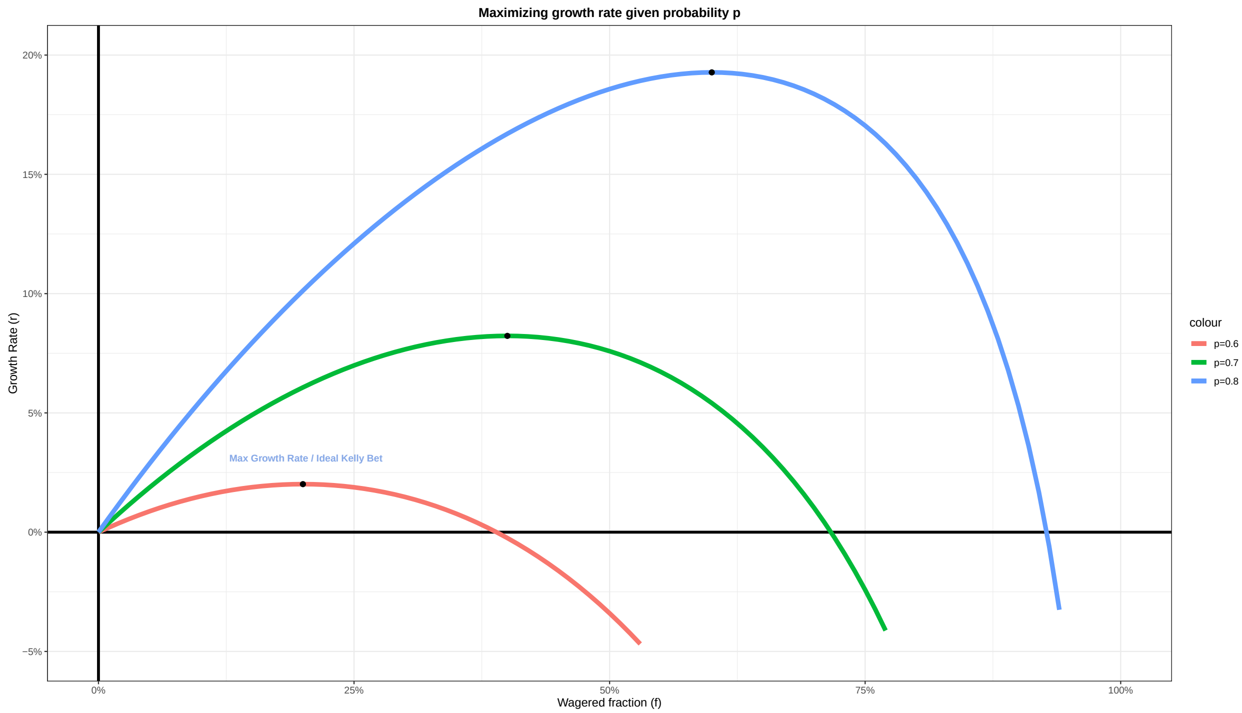

Figure 1: Analysis of multiple biased coins

outcomes. There is a bag with 10 marbles, 7 of which are red, 2 which are blue, and 1 that is green. They all have different payouts and the player is tasked with placing a wager and than pulling a marble. That is, the player is not betting on a specific colour marble coming up, they are simply placing a wager on the outcome of the pull. Let $i \in (R = Red, B = Blue, G = Green)$
The difference between this game and the bias coin game is twofold. We are now calculating the optimal Kelly Bet given multiple mu tally exclusive outcomes all with different payouts.

3.1 Marble Model Derivation

$(p_R, p_B, p_G) = (0.7, 0.2, 0.1) =$ the probability of each colour marble coming up.

$(a_R, a_B, a_G) = (-1, 2, 10) =$ the payouts of each colour marble coming up.

$(R_n, B_n, G_n) =$ number times a given marble is pulled after n trials.

We proceed in a very similar manner shown in section 2.

$X_n = X_0(1 - a_R f)^{R_n}(1 + a_B f)^{B_n}(1 + a_G)^{G_n}$

$(X_n/X_0)^{1/n} = (1 - a_R f)^{R_n/n}(1 + a_B f)^{B_n/n}(1 + a_G)^{G_n/n}$

After performing the same derivation as in 2.1, we take the logarithm of each side and take the expected value of the growth rate to obtain: (see appendix)

$$g(f) = 0.7\ln(1 - f) + 0.2\ln(1 + 2f) + 0.1\ln(1 + 10f)$$

3.2 Marble Game Analysis

Plugging into Wolfram Alpha we find the max f to be 0.1148. Meaning we should bet 0.1148 of our wealth on the outcome of this game. This leads to an expected utility increase of $g(0.1148) = 0.0324$ per wager. This will lead to a wealth increase of $e^{0.0324} = 1.033$ per trial, so after 300 turns, if we were to start with a 25 dollar bank roll, our dollar equivalent of utility will be $25(1.033)^{300} = 424,655$ dollars !

This example illustrates why the definition of a ”favourable bet” includes more than just the probability of that event being realized. This is why the Kelly Criterion is such a powerful tool. It combines the betting information of probability and payout to give a strategy. In the above example, your gonna loss your wager with 0.7 probability. On the surface this may seem like a bad game. However, since the other 0.3 marbles have a payout much larger than the loss incurred on the red marbles, the act of continuous betting on the marbles becomes a profitable one (If one is using the Kelly Criterion of course). However, in this game a player should expect to experience quite a lot of volatility. Every time they may make a big increase in wealth, they should should expect to immediately loss a fraction of it on the next trails. However, given a large sample size of 300 like above, their will likely be an opportunity where you pull two green marbles back to back and experience a 100x return within two trials.

4 Horse Betting

5 Bibliography

6 Appendix

Insert graphs and the marble game derivation