Kelly Criterion
Identifying the most favourable bet

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Abstract

In this paper we explore the applications of the Kelly Criterion, a model created by John L. Kelly Jr [1] that attempts to derive the most optimal size of a 'wager' to maximize returns in a given trial-based betting game. While intuitively, the core model has the specific interpretation for optimizing outcomes in 'betting games', it has ever-reaching applications in the world of finance and decision-making. It is quite clear that these applications alone give us serious incentive to explore John Kelly's work. We begin by deriving the fundamental, simplest case model of the Kelly Criterion and it's interpretations. We than examine how to adapt the initial model to a few more complicated scenarios and analyze it's strengths and weaknesses. Overall, this paper aims to demonstrate to readers the math and intuition behind the idea of favourable bets and how they maximize profit/returns.

1 Introduction

Gambling is not a deterministic game; it often involves placing wagers on events which have a high degree of uncertainty with regards to their possible outcomes. However, while it is well within the realm of possibility to find what could be defined as a 'favourable wager' when considering probabilities, bankroll, payoffs, etc, taking advantage of such information can become overwhelming in complexity and/or marred in nuance. If one fails to consider the relevant variables in these scenarios (possibly in favor of personal, scientifically unfounded biases), then their decisions may not only be relatively sub-optimal, but even financially deleterious. This is observed in Dewey and Haghani's study which involves participants betting repeatedly on a biased coin [2]. Participants were given a 25$ bankroll and asked to place a wager on a coin flip, with 60% chance of heads occurring, continuously for 30 minutes. The payouts were 1:1, which meant that each participant could only either win or lose the amount they bet. 28% of participants ended up going losing their entire bankroll while only 21% ended up reaching the maximum payout within the 30 minutes. Given that the authors designed the game such that there exists a dominating strategy, it is quite compelling that very few participants managed to find and utilize such a strategy to maximize profit. It is even more compelling that over a fifth of participants managed to lose all their starting wealth. It is a reasonable assumption that those not well-acquainted in disciplines relating to probabilities and finance may find identifying the correct decision in similar scenarios to be quite unintuitive and difficult. Financial literacy is an extremely important skill for any adult in the modern world and because of this, there is clear motivation for developing and using mathematical tools that help analyze such decisions.

One such mathematical tool is the Kelly Criterion model, which evaluates what the optimal size of a bet when probabilities and payouts are known and fixed, which are assumptions that have real practicality and carryover to the real world. As an exercise in demonstrating the practicality of this model, there are two scenarios that we will derive an appropriate application of the criterion for and subsequently analyze: wagering on the outcome of a biased coin, and wagering on an event with three outcomes with unequal probabilities. We will also briefly do an overview on a more advanced application of the Kelly Criterion involving optimizing betting on horses, which involves multiple individual wagers on multiple outcomes [3].

2 Simple Coin Game

Fundamentally, Kelly's model maximizes the expected value of a player's logarithm of wealth function. The logarithm of wealth function is essentially a utility function, which is a definition commonly used in economics to measure the value of some 'commodity' to an individual [4]. For example, the 'utility' of 10$ for a given person depends on their respective wealth. If the person initially has $1,000,000, then $10 has much less 'utility' than if they initially had $100. Maximizing the expected value of this utility function also allows us to avoid a trivial solution which is explained in detail later.

2.1 Coin Game Model Derivation

We begin by following Edward Thorp's derivation for a model of a simple biased coin game with $p > 1 - p$ [5].
Let X_0 and X_n be a player's initial wealth and wealth at round n, respectively. To represent wins and losses, let $T_k = 1, -1$, which represents the win or loss on the k-th round.
Let B_k be the amount of wealth wagered on the k-th round. Therefore, the model for the game is as such:

$$X_n = X_0 + \sum_{k=1}^{n} T_k B_k$$

Since $p - q > 0$, there is a positive expected value. From derivation from source

$$E[X_n] = X_0 + \sum_{k=1}^{n} E[T_k B_k]$$

$$E[T_k] = 1 \cdot P(T_k = 1) + (-1 \cdot P(T_k = -1) = p - (1 - p) = p - q$$

$$= E[X_n] = X_0 + \sum_{k=1}^{n} (p - q) E[B_k]$$

If we want to maximize the player's wealth, $E[X_n]$, we would have to maximize $E[B_k]$, which involves betting everything every round. Intuitively, this is a very bad strategy as ruin (' going bust') would be achieved very quickly given that

$$\lim_{n \to \infty} (1 - p)^n = 1$$

If instead the player chose to bet a much smaller amount, they will avoid ruin but the rate of growth for the player's wealth will be quite small. We want to explore what is the most optimal fraction of wealth to be wagered each round. Therefore, we want to introduce a model that considers betting a fraction of our wealth f.

2.1.1 Simple Coin Game: Introducing fractional bets

Let's consider adjusting B_k such that $B_k = f X_{i-1}$ where $0 \leq f \leq 1$, representing the fraction of X_{i-1} to bet on any given round of wagering. We can change the previous model as follows:

$$X_n = X_0 + (1 + af)^S \cdot (1 - bf)^F$$

where S and F represents the number of successes and failures, respectively, and $S + F = n$. We've also added a and b to represent multipliers of how much is won or lost from a wager. For example, $a = 10$ and

$b = 2$ would represent a 10:2 payout. There is also the trivial case where $a = b = 1$ such that the payout is 1:1. Our main objective for this model is to have the largest rate of increase for

$$\frac{X_n}{X_0}$$

Therefore, we have this identity:

$$e^{nlog\frac{X_n}{X_0}^{\frac{1}{n}}} = \frac{X_n}{X_0}$$

This identity is necessary for deriving the utility function defined earlier.

By substituting and simplifying, we are given the "exponential rate of increase per trial" [5].

$$G_n(f) = \log\frac{X_n}{X_0}^{\frac{1}{n}} = \frac{S}{n}\log(1 + af) + \frac{F}{n}\log(1 - bf)$$

We now choose to maximize the expected value of the equation above. Maximizing the equation as is would give us the same outcome from our previous model such that to maximize the rate of increase would result in a trivial solution of winning every wager. It is also important that we maximize the utility function rather than the original derivation $X_n = X_0 + (1 + af)^S \cdot (1 - bf)^F$ as it would also lead to the same trivial solution of betting all wealth every round. Therefore, maximizing the expected value is the appropriate choice as it will take into account the probability of success and failure.

$$g(f) = E[\log\frac{X_n}{X_0}^{\frac{1}{n}}] = E[\frac{S}{n}\log(1 + af) + \frac{F}{n}\log(1 - bf)]$$

Since S and F are Binomial random variables, we have:

$$g(f) = p\log(1 + f) + q\log(1 - f)$$

Using the advanced mathematical technique of taking the first derivative and evaluating at 0, and a little bit of tedious algebra, we achieve the following,

$$g'(f) = \frac{ap}{1 + af} - \frac{bq}{1 - fb} = 0$$

$$\frac{ap}{1 + af} = \frac{bq}{1 - bf}$$

$$ap(1 - bf) = bq(1 + af)$$

$$pa - pbaf = bq + qbaf$$

$$pa - bq = f(pab + qab)$$

$$f(pab + (1 - p)(ab)) = pa - bq$$

$$f(ab) = pa - bq$$

$$f = \frac{p}{b} - \frac{q}{a}$$

The above result represents the most optimal fraction of wealth to bet on any given round. We can prove that this result is the only maximum by performing the same advanced mathematical technique but for the second derivative.

$$g''(f) = -\frac{p}{(1 + af)^2} - \frac{q}{(1 - bf)^2} < 0$$

Thus, since the function shows to be monotonically decreasing, the above holds statement holds true.

2.2 Analysis of Bias Coin

Lets take a look at the bias coined example that Thorp examined in his paper.
We have a biased coin where the probability of getting heads is 0.6. There is even payouts, which means that the value of $a = b = 1$. A player is given a 25 dollar bank roll to start out with. It follows that the optimal bet under the Kelly Criterion would be to place $0.6 - 0.4 = 0.2$ fraction of your wealth on each turn. This gives a utility (logarithm of wealth) growth rate of $g(0.2) = 0.6\ln(1.2) + 0.4\ln(0.8) = 0.0201$. This means that each flip will give a player the dollar equivalent increase in utility of $e^{g(f)} = e^{0.0201} = 1.02034$ So if we were to flip the coin 300 times, the dollar equivalent would be be $25(1.02034)^{300} = 10,504$ [2]. Even though the formula for the result is quite simple, we can scale the Kelly Criterion easily for games with many different mutually exclusive outcomes and uneven pay outs.

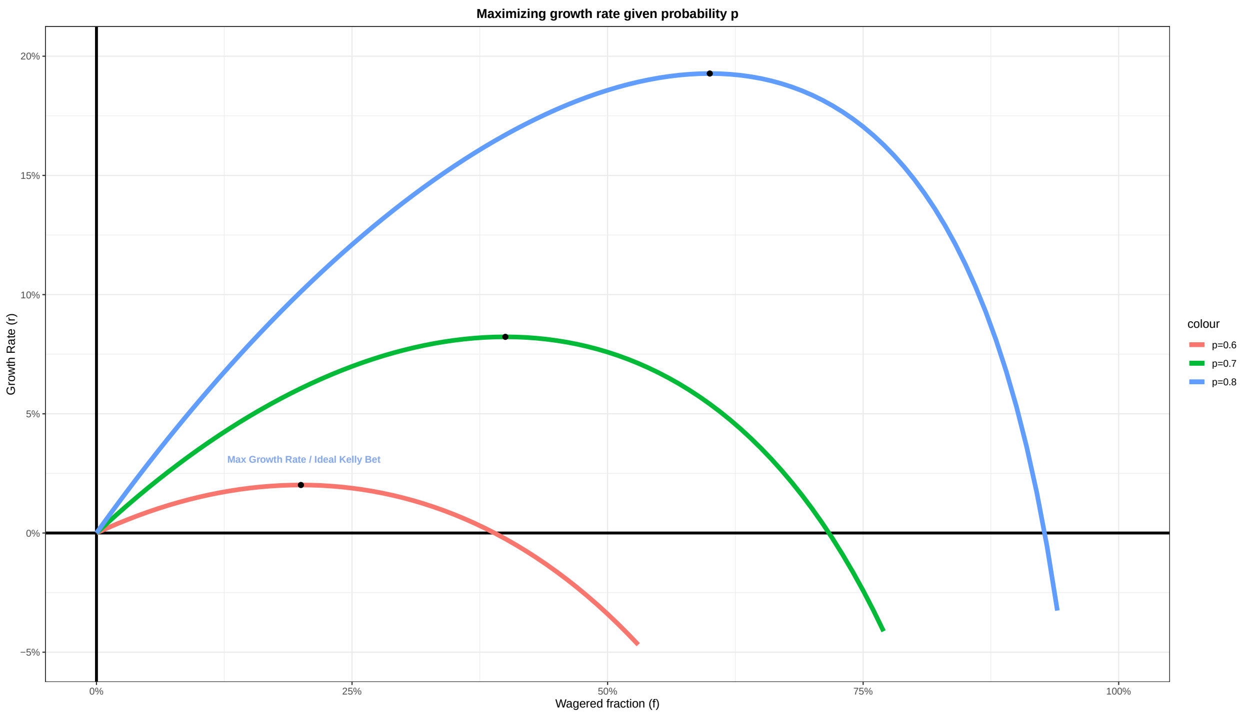

Figure 1: Analysis on multiple biased coins. a=b=1

3 Marble Game

The following game was inspired by a discussion found online [6]. We analyze a new game where a player is given the opportunity to place a single bet on the result of multiple mutually exclusive outcomes. There is a bag with 10 marbles, 7 of which are red, 2 which are blue, and 1 that is green. They all have different payouts and the player is tasked with placing a wager and than pulling a marble. That is, the player is not betting on a specific colour marble coming up, they are simply placing a wager on the outcome of the pull.
Let $i \in (R = Red, B = Blue, G = Green)$
The difference between this game and the bias coin game is twofold. We are now calculating the optimal Kelly Bet given multiple mutually exclusive outcomes all with different payouts.

3.1 Marble Model Derivation

Observe that this model is essentially a linear extension of the previous.
$(p_R, p_B, p_G) = (0.7, 0.2, 0.1) =$ the probability of each colour marble coming up.

$(a_R, a_B, a_G) = (-1, 2, 10) =$ the payouts of each colour marble coming up.

$(R_n, B_n, G_n) =$ number times a given marble is pulled after n trials.

We proceed in a very similar manner shown in section 2.

$X_n = X_0(1 - a_R f)^{R_n}(1 + a_B f)^{B_n}(1 + a_G f)^{G_n}$

$(X_n/X_0)^{1/n} = (1 - a_R f)^{R_n/n}(1 + a_B f)^{B_n/n}(1 + a_G f)^{G_n/n}$

After performing the same derivation as in 2.1.1. (see 'Marble Derivation' in appendix), we take the logarithm of each side and take the expected value of the growth rate to obtain:

$$g(f) = 0.7\ln(1 - f) + 0.2\ln(1 + 2f) + 0.1\ln(1 + 10f)$$

3.2 Marble Game Analysis

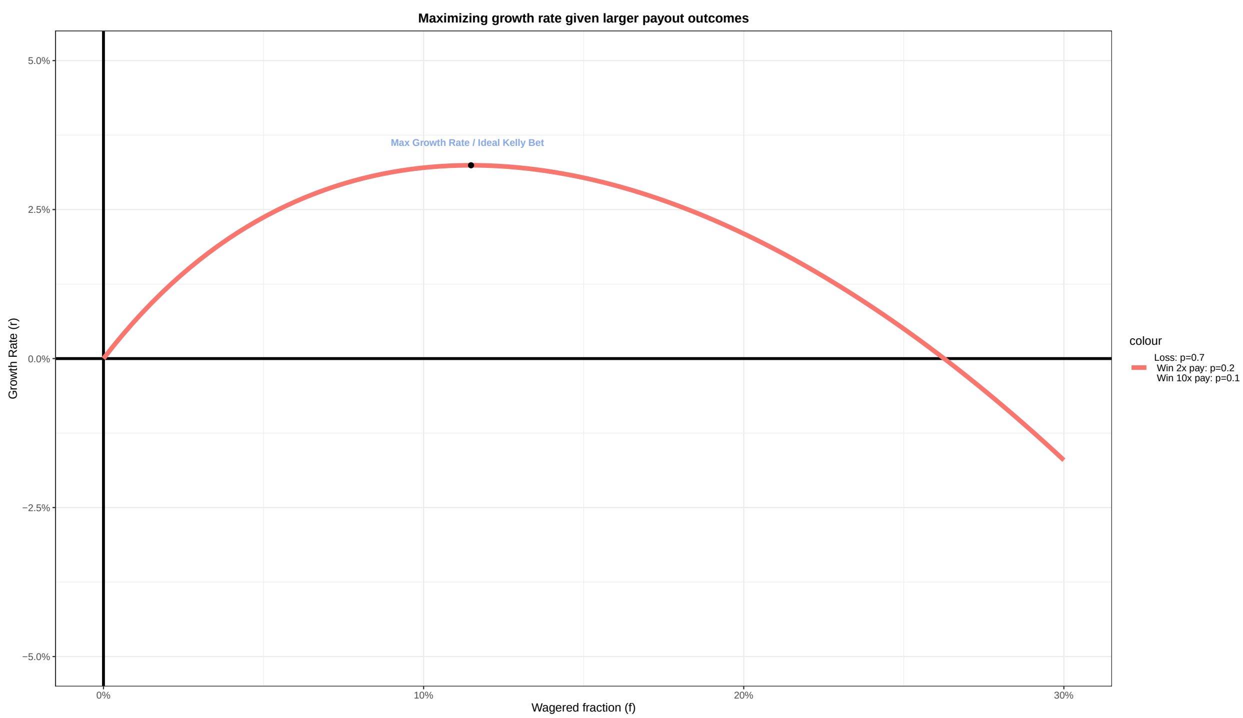

Figure 2: Analysis of Marble Game

Plugging into Wolfram Alpha we find the max f to be 0.1148. Meaning we should bet 0.1148 of our wealth on the outcome of this game. This leads to an expected utility increase of $g(0.1148) = 0.0324$ per wager. This will lead to a wealth increase of $e^{0.0324} = 1.033$ per trial, so after 300 turns, if we were to start with a 25 dollar bank roll again, we expect our wealth to increase to $25(1.033)^{300} = 424,655$ dollars !

This example illustrates why the term ‘favourable bet’ was chosen as it includes more than just the probability of that bet being realized. This is why the Kelly Criterion is such a powerful tool; it combines the betting information of probability and, equally as important, payout to give a strategy. In the above example, there is a 70% chance that you lose your wager, which, immediately may seem like a bad decision to gamble. However, since the remaining outcomes have a payout much larger than the loss incurred from the red marble outcome, a positive growth in wealth by repeat bets is determined by this model.

4 Horse Betting Problem

We take a look at a special model developed by Tomkins and Smoczynski to analyze horse betting [7]. To begin understanding the model, we must first explicitly define the rules and conditions of the game. In this scenario, we are considering a horse race, in which a set of horses denoted S, where $S = \{1, 2, \cdots, n-1, n\}$, all race together, and the model assumes races are repeated an infinite number of times. It is important to note that “repeating” the races does not mean that each raced will have the same results. The probabilities of each individual horse, k, placing first in a race, denoted π_k, and the fraction of initial wealth bet on a given horse, denoted f_k, (where $0 \leq f_k \leq 1$), will always remain the same. The outcome and winner of each race are based on the win probabilities; however, while the probabilities remain fixed, the winners of each race vary (i.e., it is not an infinite number of identical races, the winners vary). We also assume that there will always be exactly one winning horse per round, in other words, there are no ties between horses and no unfinished races.

4.1 Model Derivation

Before each round, bettors have the option to place bets on one or more horse(s). Let the fraction of a bettor’s initial wealth that they choose not to bet each race, be known as the reserve rate, denoted $R(S)$, where S is the set of horses racing that the player wishes to bet on. Then,

$$R(S) = \sum_{k=1}^{k=n} f_k$$

$$0 \leq R(S) \leq 1.$$

After betting is closed for each round, track management will take a specific portion of the total money bet for the house, denoted tt (for track take). The remaining money wagered, denoted D,

$D = (1 - tt)$ and $0 \leq D \leq 1$

is to be divided among all bettors who placed a bet on the winning horse k, the rest of the bettors receive nothing. Each player who placed a winning bet on horse k will receive winnings proportional to the fraction of the total money put on k, that their bet constituted. In this model we will not allow the player to hedge/scam bets in any way, or to bet more money than they currently have. Finally, we assume the bettor has made adequate assessments of race odds and is not betting values large enough to affect the estimates and bets of the other players. To place a bet a player needs two pieces of information; the horse(s) they wish to place the bet on, and the amount(s) they wish to bet. Let $b = [b_1, \cdots, b_n] \in R^n$ be a vector with each of its components, b_k, equal to the total amount of money bet by all bettors on the kth horse in a given round.

Thus, the total amount of money bet equals $\sum_m b_m$ where $\sum_m b_m = \sum_{m=1}^{m=n} b_m$, Then consider the vector,

$$\beta = \frac{b}{\sum_m b_m}$$

This new vector β has a weight of 1. Each of its components is a value from

$$[0, 1]$$

representing the amount bet on horse k divided by the total amount bet on all horses. Consider a player betting on a horse as an expression of their belief that that horse will win. Then, β can also thought of as the belief of all bettors collectively of the win probabilities of each horse. From here, we will define the revenue rate for a particular horse k as the amount received by a bettor who placed a bet on the winning horse k,

$$r_k = D/b_k \sum_m = \frac{D}{\beta_k} = Q_k + 1$$

where Q_k is the profit (in \$) received by a player when a \$1 bet is placed on the winning horse, k (also known as odds of horse k). From this, we can define the expected revenues as,

$$er_k = \frac{D\pi_k}{\beta_k} = \pi_k(Q_k + 1)$$

the expected value of a random variable is the sum over all outcomes of the product of the payoff of each outcome and the probability of each outcome. We define the expected revenue of a horse in a similar vein. The expected revenue for any horse k is the amount of money to be split among winning bettors (D) , multiplied by the probability of horse k winning π_k and the total amount of money bet on horse k (b_k), all divided by the total amount bet on all horses; or, the probability π_k multiplied by the sum of the revenue on a \$1 bet on horse k and the original \$1 bet. Therefore, in each round, a bettor can get one of two types of outcomes. Either, the player was incorrect and did not place a bet on the winning horse and thus loses all bets made that round, and gains nothing, or they did place a bet on the winning horse, and they receive their original bet back, plus a portion of the winnings relative to the fraction of the total money wagered on k that their bet made up, (where k is the winning horse of that round), they also lose any additional bets made on other horses. Thus, each round, the players wealth changes by a factor of $(1 - \sum_m f_m + (r_k)(f_k))$. Since this scenario deals with infinitely repeated races, we can derive an equation to represent the growth rate of a bettor's wealth after W rounds of horse betting, instead of calculating each rounds winnings or losses individually. Let w_k denote the number of rounds won by horse k, if W rounds have been completed. Then, we can express the fraction of wealth held by the player after W rounds with respect to their initial wealth; where $f = [f_1, \cdots, f_n]$ represent the fraction of the players initial wealth that is bet each round on each of the n horses, by,

$$\prod_{k=1}^{k=n} ((1 - \sum_m f_m + D\frac{f_k}{\beta_k})^{\frac{w_k}{W}})^W$$

where $W = \sum_k w_k$, since exactly one horse will win each race. This equation is the model for the return rate of W rounds of Kelly betting. This equation represents the initial wealth, minus all bets made, plus all winnings from horse k, raised to the exponent equal to fraction of the W races completed that horse k has won, multiplied across all n horses, then all raised to the power of W representing the number of rounds completed. Now, consider taking a set of n independent, identically distributed random variables, each with values $\ln(1 - \sum_m f_m + D\frac{f_k}{\beta_k})$ and respective probabilities π_k. The values of the variables are the natural log of the growth rate per round, this employs the function $log(x)$ as a utility function for changes in wealth, which was presented by Kelly in his original paper. Let $L_W(f)$ denote the average of these random variables,

$$L_W(f) = \sum_k \frac{w_k}{W} \ln(1 - \sum_m f_m + D\frac{f_k}{\beta_k})$$

Then, by the Law of Large Numbers, $L_W(f)$ converges to,

$$L(f) = \sum_k \pi_k \ln(1 - \sum_m f_m + D\frac{f_k}{\beta_k})$$

Further,

$$\prod_{k=1}^{k=n} (1 - \sum_m f_m + (D)\frac{f_k}{\beta_k})^{\frac{w_k}{W}} = e^{L_W(f)} \text{ converges to } e^{L(f)}$$

It is known that $L(f)$ is a continuous, concave down function, on the interval $(-\infty, \sum_k \pi_k \ln(1 + \frac{D}{\beta_k}))$, which must contain $L(f)$ since,

$$L(f) = \sum_k \pi_k \ln(1 - \sum_m f_m + D\frac{f_k}{\beta_k}) \leq \sum_k \pi_k \ln(1 + \frac{D}{\beta_k}),$$

under the current constraints on f. Therefore, the player is interested in finding the values such that $L(f)$ is maximized.

4.2 Horse Betting Analysis

From the various constraints put on f by the rules of the game, it is possible to create a system of optimally equations to locate the maximum value, as well as prove that the strategy f is indeed the optimal strategy for this game. The derivation and subsequent solving of this system employs KKT Theory and will be omitted from this explanation for ease of understanding. Instead, we will focus on the important formulas determined from these operations and the simple resulting algorithm which dictates how to apply the formulas to create the betting strategy. The formula for the optimal fraction of initial wealth to be bet on a particular horse k, denoted f_k^{opt}, is:

$$\begin{aligned} f_k^{opt} &= \pi_k - \beta_k \frac{\sum_{i \notin S} \pi_k}{D - \sum_{i \in S} \beta_i}, && \text{if } k \in S \\ f_k^{opt} &= 0 && \text{if } k \notin S. \end{aligned}$$

The reserve rate can be found using,

$$R(S) = (1 - \sum_{m \in S} f_m) = \frac{\sum_{i \notin S} \pi_i}{(D - \sum_{i \in S} \beta_i)}, \qquad \text{if } S \neq \emptyset \text{and } R(\emptyset) = 1.$$

From here, we can rewrite our initial equation for L(f), using the two results above, to get,

$$\ln(G^{opt}) = L(f^{opt}) = \sum_{k \in S} \pi_k \ln(\frac{D\pi_k}{\beta_k}) + \sum_{k \notin S} \pi_k \ln(R(S)).$$

where G^{opt} denotes the optimal growth rate for the bettor. Now that all necessary results have been established, we can finally detail the algorithm for finding the optimal set of horses and bets. The general rule for a set $S \leq [n]$,$[n] = 1, 2, \cdot, n-1, n$, if $\forall k \in S, er_k > R(S)$ and $\forall j \notin S, er_j \leq R(S)$, then $S = S^{opt}$. Therefore, there is always a single optimal set for each round, containing all horses that have an expected revenue larger than the current reserve rate; note that it is possible for $S = \emptyset$, in this case the best course of action is to not place any bets. The following algorithm details the steps for finding S^opt using the formulas provided.

Step 1: Calculate expected revenue rates for each of the n horses.
$er_k = \frac{(D\pi_k)}{\beta_k} = \pi_k(Q_{k+1})$
Step 2: Order the horses from largest to smallest(non-increasing) expected revenue er_k

Step 3: Set $S = \emptyset, i = 1, R(S) = 1$.

Step 4: Repeat these steps for each horse $i \in n$.
if $er_i > R(S)$:
Add horse i to S, and recalculate R(S) using,
$R(S) = (1 - \sum_{m \in S} f_m)$
else: move on to horse $i + 1$.

Once all repetitions are finished and there are no more horses that have an expected revenue greater than the current reserve rate, then $S = S^{opt}$, the optimal set of horses to bet on. With the set of horses now known, the player may then determine the optimal allocation of their wealth to bet on each of the horses in S using the formula,
$f_k^{opt} = \pi_k - \beta_k \frac{\sum_{i \notin S} \pi_i}{(D - \sum_{i \in S} \beta_i)}$, if $k \notin S$
$f_k^{opt} = 0$, if $k \notin S$.
The maximum of the average of the logarithm of the growth rate can be found using,

$$\ln(G^{opt}) = L(f^{opt}) = \sum_{k \in S} \pi_k \ln \frac{D\pi_k}{\beta_k} + \sum_{k \notin S} \pi_k \ln(R(S))$$

5 Assessments

5.1 Biased Coin / Marble Game

It is clear that the Kelly Criterion is a powerful tool that provides players with a way to balance the risk and reward of their wagers. As demonstrated by the examples above, it can be used in an efficient and simple manner to achieve long-term growth when favourable bets are allowed to be acted upon continuously. However, it does have some clear problems. For one, the Kelly Criterion is derived using expected values that depend on known probabilities. Though this may be useful when thinking about theoretical games, when moving into an environment where the probabilities are not so clear, such as in the stock market or horse races, the practical use of the Kelly Criterion becomes less clear. This hasn't stopped investors such as Warren Buffet, Edward Thorp, and Bill Gross from using some version of the Kelly Criterion in their own work however [8]. Another problem with the Kelly Criterion is the volatility that a player may experience when deploying it. Jane Hung in their paper 'Betting with the Kelly Criterion' [9], ran simulations using the Kelly criterion on a game similar to the bias coin example above. Though the exponential growth that is expected of the Kelly Criterion was met, there was plenty of volatility in the players wealth along the way. This poses a problem for the bettor deploying the Kelly Criterion as they will have to deal with the psychological swings of seeing their wealth fluctuate dramatically.

5.2 Horse Betting

There are several key aspects of the model that affect its utility and application. This model not only provides the derivation, solution, and proof of the strategy, but it also has a relatively easy to use algorithm for applying its results. The values needed and the given formulas can be easily implemented in a few lines of code in almost any programming language without any of the complex math required for the proof, making it possible to use in online betting or bringing with you on a computer to the track. This model is the result of the generalization and application of the Kelly criterion in a larger, more complex scenario. In the original, two outcome scenario, the Kelly criterion shows that wagers on outcomes with lower or less favourable probabilities should only be placed if the expected revenue is significantly large enough to outweigh the probable loss. Therefore, when intuiting how to extend this to an n outcome situation, we would wish to retain this condition and apply it to all outcomes that satisfy it; this is because betting on multiple horses increases the chance that the winning horse will be one that the player placed a bet on. Thus, the guiding ideas behind the model and its derivation, when considering the simpler case, logically and intuitively make sense, this is another benefit of this model. However, there are several assumptions and aspects of the model that make it difficult to apply to real life. The first, and most obvious, is the fact that horse races are not repeated infinitely in reality. However, while the model considers an infinite number of races, the optimal strategy proposed by the model does not require the player to continue betting for an infinite number of rounds. It allows the bettor to calculate the point at which maximum value occurs, this is when the player should stop betting, continuing beyond this point is unwise as it will eventually lead to ruin.

It is also an important distinction to make that, while the model does provide an explicit solution to the problem posed by Kelly in 1952 [1], that does not mean that this strategy is guaranteed or wins every time. As with all gambling, you are making an estimate on a future outcome. An estimate, no matter how much mathematics or data it may be based upon, will never be a certainty, as there is always a possibility of loss. A certain strategy being optimal does not mean that it cannot lose; if this were the case, then technically, the “best” strategy for this game would to be to bet once each round, on only the winning horse; however, since the winning horse is undetermined until the race occurs, there is no way for an individual to know this value beforehand when placing their bet. In other words, consider an extremely lucky player who guesses randomly each round and places a bet only on one horse, which then, by sheer luck, ends up being the winner of that rounds race. This hypothetical player could technically outperform another hypothetical player using the algorithm derived above on the same set of horses and races; however, this is extremely unlikely and cannot be guaranteed by the player, so it is ultimately a useless strategy. We label a strategy as optimal, if it produces the maximum expected average value over many iterations and when compared to the expected average value of any other strategy over the same period, will result in a larger value for the player. This is a common technique of analysis when dealing with unknown or randomized outcomes; however, it still creates a discrepancy between the theoretical viability of a strategy versus its success in practice when applied to bets in the real world.

Another criticism of this model lies within its assumptions. When deriving the model, we assumed that the individual horse player has made accurate assumptions of the probabilities of the horses. While typically odds are calculated and displayed by the racetrack for bettors to view and consider, the probabilities a player ultimately uses to place their bets are decided on solely by that player. As a result, in practice, it is less likely that probabilities estimated by the player are accurate to the true win probabilities. The Kelly bet frequently recommends the player place bets on multiple horses, not because all will win, but to reduce the chances of a total loss and increase the chance of a partial win, this practice lessens the effect that inaccurate estimates may have on the efficacy of the strategy. These discrepancies between the theoretical values and the values in practice do not invalidate the model entirely, it just means that, when applied to real world situations, the model is used more as a set of guidelines to advise the player or assess prospective wagers and strategies, as opposed to a direct, explicit set of instructions to follow.

Conclusion

In conclusion, it is clear that the Kelly Criterion is a powerful method for balancing risk and reward in gambling situations. It shows that by maximizing the expected value of a player's logarithm of wealth, one can find the optimal betting strategy for a game. As demonstrated by the coin and marble examples above, it can be used in an efficient and simple manner to achieve long-term growth when favourable bets are allowed to be acted upon continuously. Furthermore, as seen from the horse betting example, the Kelly criterion, provides a strategy of fractional betting for games with repeated rounds and independent mutually exclusive outcomes, of fixed probabilities. It details that a optimal bet is one placed on an outcome that has an expected revenue rate larger than the reserve rate. Although there are several discrepancies when applying the criterion to the real world, such as volatility and unknown probabilities, it still serves as a helpful guideline when analyzing potential wagers.

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Appendix: R Code

Figures 1 and 2

```
#Instantiating the functions we're graphing below into objects to keep code pretty
kellyfunc <- function(x){(0.6 * log(1+x)) + (0.4 * log(1-x))}
second <- function(x){(0.7 * log(1+x)) + (0.3 * log(1-x))}
third <- function(x){(0.8 * log(1+x)) + (0.2 * log(1-x))}
jellybeanbet <- function(x){(0.7 * log(1-x)) + (0.2 * log(1+2*x)) + (0.1 * log(1+10*x))}

# Solving for the maximum of the kellyfunc
max <- data.frame(optimise(kellyfunc,c(0,1),maximum = T))
maxsecond <-data.frame(optimise(second,c(0,1),maximum = T))
maxthird <-data.frame(optimise(third,c(0,1),maximum = T))
maxjelly <- data.frame(optimise(jellybeanbet,c(0,1),maximum = T))

# Setting a ggplot theme to make the graph a little prettier by default
theme_set(theme_bw())

# Building the Graph

# Initialize graph
graph <- ggplot() +
  # Add x and y intercepts
  geom_hline(yintercept = 0,linewidth = 1.25) +
  geom_vline(xintercept = 0,linewidth = 1.25) +

  # Adding geometry for our functions we want - the aes piece names them in the legend
  geom_function(fun = kellyfunc, linewidth = 2,aes(col = "p=0.6"))+
  geom_function(fun = second, linewidth = 2,aes(col = "p=0.7")) +
  geom_function(fun = third, linewidth = 2,aes(col = "p=0.8")) +
  #geom_function(fun = jellybeanbet, linewidth = 2,aes(col = "Loss: p=0.7
  #\n Win 2x pay: p=0.2 \n Win 10x pay: p=0.1")) +

  # Adds on the point representing the maximum for the kelly function
  geom_point(data = max,aes(x = maximum, y = objective),size = 2) +
  geom_point(data = maxsecond,aes(x = maximum, y = objective),size = 2) +
  geom_point(data = maxthird,aes(x = maximum, y = objective),size = 2) +
  #geom_point(data = maxjelly,aes(x = maximum, y = objective),size = 2) +
  # this is made irrelevant by the theme set above,  but I'm leaving it in
  # to save the color codes and syntax in case it's more desirable later
  #scale_colour_manual(values= c('#86A8E7','#5FFBF1'),name = 'Function') +

  # Rescales the labels so they are percentages and zooms to the relevant part of the graph
  scale_y_continuous(labels = scales::percent,limits = c(-0.05,0.25)) +
  scale_x_continuous(labels = scales::percent,limits = c(0,1))


# Defines a grob object from the grid package that
#lets us put an annotation straight onto the graph at a scaled location
kellylabel <- grobTree(textGrob('Max␣Growth␣Rate␣/␣Ideal␣Kelly␣Bet',
                                x=0.23,
                                y=0.3,
                                gp=gpar(col='#86A8E7',fontsize=9,fontface="bold")))

# Adding more stuff to the graph...
graph <- graph +
  # Cleaning up X and Y labels
  labs(x = 'Wagered␣fraction␣(f)', y = 'Growth␣Rate␣(r)')  +

  # Adding a title, formatting, and centering it
  ggtitle('Maximizing␣growth␣rate␣given␣larger␣payout␣outcomes') +
  theme(plot.title = element_text(size=12, face='bold',hjust = 0.5)) +
```

```
  # Add the label defined above onto the graph
  annotation_custom(kellylabel)
# Display it
graph
```

Appendix: Marble Game Derivation

We start off we're we left off in section 3.1

$(X_n/X_0)^{1/n} = (1 + a_R f)^{R_n/n}(1 + a_B f)^{B_n/n}(1 + a_G f)^{G_n/n}$

We take the logarithm of each side in order to get the "exponential rate of increase per trial" just as in section 2.1.1, let this function be called $G(f)$ once again:

$G(f) = \frac{R_n}{n} log(1 + a_R f) + \frac{B_n}{n} log(1 + a_B f) + \frac{G_n}{n} log(1 + a_G f)$

Once again, as per the Kelly Criterion, we want to be maximizing the expected growth rate $g(f)$

$g(f) = E[\frac{R_n}{n} log(1 + a_R f) + \frac{B_n}{n} log(1 + a_B f) + \frac{G_n}{n} log(1 + a_G f)]$

$g(f) = p_R log(1 + a_R f) + p_B log(1 + a_B f) + p_G log(1 + a_G f)$

We plug in the given values and continue.