Robotics 1
Exercise Solver

November 14, 2023

Contents

1 DC motors

1.1 Electrical and mechanical balance

$$V_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_{emf}(t) \tag{1}$$

$$v_{emf}(t) = k_v \omega(t) \tag{2}$$

in control domain:

$$V_a = (R_a + sL_a)I_a + V_{emf} \tag{3}$$

$$V_{emf} = k_v \omega \tag{4}$$

where V_a is the voltage applied to the motor, R_a is the armature resistance, L_a is the armature inductance, i_a is the armature current, v_{emf} is the back emf, k_v is the back emf constant and ω is the angular velocity of the motor.

$$\tau_m(t) = I_m(t) \frac{d\omega(t)}{dt} + F_m \omega(t) + \tau_{load}(t) \tag{5}$$

$$\tau_m(t) = k_t i_a(t) \tag{6}$$

in control domain:

$$T_m = (sI_m + F_m)\omega + T_{load} \tag{7}$$

$$T_m = k_t I_a \tag{8}$$

where τ_m is the motor torque, I_m is the motor inertia, F_m is the motor friction, τ_{load} is the load torque and k_t is the torque constant. **Note:** $k_v = k_t$ **numerically!**

1.2 Reduction ratio

The reduction ratio of a the ransmission chain is the product of the reduction ratios of the single elements of the chain:

$$\eta = \sum_{i=1}^{n} \eta_i \tag{9}$$

1.2.1 Harmonic drives

$$\eta = \frac{\#theet_{FS}}{\#theet_{CS} - \#theet_{FS}} = \frac{\#theet_{FS}}{2} \tag{10}$$

$$\#theet_{FS} = \#theet_{CS} - 2 \tag{11}$$

1.2.2 Standard gears

Given two gears of radius r_1 and r_2 the reduction ratio is:

$$\eta = \frac{r_2}{r_1} \tag{12}$$

1.3 Optimal reduction ratio

$$\eta_{opt} = \sqrt{\frac{J_{load}}{J_{motor}}} \tag{13}$$

1.4 Optimal torque

We impose the relation between the angular acceleration of the load and the motor:

$$\ddot{\theta}_m = \eta \ddot{\theta}_l \tag{14}$$

$$\tau_m = J_m * \ddot{\theta}_m + \frac{1}{\eta}(J_l * \ddot{\theta}_l) \tag{15}$$

2 Encoders

2.1 Absolute encoders

The resolution of an absolute encoder is given by:

$$res = \frac{2\pi}{2^{N_t}} \tag{16}$$

where N_t is the number of bits of the encoder. **Note: the resolution changes from base to link end!**

$$res_{base} = res_{link}/L \tag{17}$$

where L is the length of the link.

2.2 Incremental encoders

The resolution of an incremental encoder is given by:

$$rse = \frac{2\pi}{2^{N_t}} \tag{18}$$

The number of bit of the encoder is given by:

$$N_t = \log_2(N_p) \tag{19}$$

where N_p is the number of pulses per turn of the encoder.

2.3 Multi-turn encoders

The number of bits to count the turns in a multi-turn encoder is given by:

$$N_t = \log_2(N_{turns}) \tag{20}$$

where N_{turns} is the number of turns of the encoder. The number of turns of the encoder is given by:

$$N_{turns} = \frac{\Delta\theta_{max} * n_r}{2\pi} \tag{21}$$

where $\delta\theta_{max}$ is the maximum angle of the encoder and n_r is the reduction ratio.

3 Rotation Matrices

3.1 Check if R is a rotation matrix

To check if R is a rotation matrix we have to check:

- det(R) = 1
- Orthogonality: $R^T R = I$
- Normality: for each column R_i of R, $||R_i|| = 1$

3.2 General Rotation

$$^{A}R_B = \begin{bmatrix} x_A x_B & y_A x_B & z_A x_B \\ x_A y_B & y_A y_B & z_A y_B \\ x_A z_B & y_A z_B & z_A z_B \end{bmatrix} \tag{22}$$

3.3 Rotation direct problem

To find R from θ and $\mathbf{r}$ we use the Rodrigues' rotation formula:

$$R(\theta, r) = rr^T + (I - rr^T)\cos(\theta) + (S(r))\sin(\theta) \tag{23}$$

where $S(r)$ is the skew-symmetric matrix of $\mathbf{r}$:

$$S(r) = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \tag{24}$$

3.4 Rotation inverse problem

To find θ and $\mathbf{r}$ from R we fisrt check if there is a singularity:

$$\sin(\theta) = \frac{1}{2}\left(\sqrt{(R_{23} - R_{32})^2 + (R_{13} - R_{31})^2 + (R_{12} - R_{21})^2}\right) \tag{25}$$

3.4.1 singularity (hence $\sin(\theta) = 0$)

If it is a singularity we can find $\mathbf{r}$ and θ: if θ is 0: there is no solution for r. if θ is $\pm\pi$:
we set $sin(\theta) = 0$, $\cos(\theta) = -1$ and we find $\mathbf{r}$:

$$\mathbf{r} = \begin{bmatrix} \pm\sqrt{\frac{R_{11}+1}{2}} \\ \pm\sqrt{\frac{R_{22}+1}{2}} \\ \pm\sqrt{\frac{R_{33}+1}{2}} \end{bmatrix} \tag{26}$$

To decide the signs of the elements of $\mathbf{r}$ we can use the following criteria:

- $r_x r_y = R_{12}/2$
- $r_x r_z = R_{13}/2$
- $r_y r_z = R_{23}/2$

3.4.2 not singularity

If the singularity is not present we can find theta and $\mathbf{r}$:
Note: we obtain two solutions for θ and cosequently r

$$\cos(\theta) = (R_{11} + R_{22} + R_{33} - 1) \tag{27}$$

$$\sin(\theta) = \pm\sqrt{(R_{32} - R_{23})^2 + (R_{13} - R_{31})^2 + (R_{21} - R_{12})^2} \tag{28}$$

$$\theta = \operatorname{atan2}(\sin\theta, \cos\theta) \in (-\pi, \pi] \tag{29}$$

$$\mathbf{r} = \frac{1}{2\sin(\theta)} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} \tag{30}$$

4 Euler

4.1 Euler direct problem

To find R from ϕ, θ and ψ around axis X,Y,Z we use the following formula:

$$R(\phi, \theta, \psi) = R_x(\phi)R_y(\theta)R_z(\psi) \tag{31}$$

4.2 Inverse Problem

Given a rotation matrix R we can find ϕ, θ and ψ: Fisrt check if there is a singularity (if $\theta = 0$ or $\pm\pi$).

4.2.1 singularity (hence $R_{13}^2 + R_{23}^2 = 0$)

If it is a singularity we can find $\phi + \psi$ and $\phi - \psi$

4.3 not singularity

If it is not a singularity we can find ϕ, θ and ψ:

$$\theta = \operatorname{atan2}\left(\pm\sqrt{R_{13}^2 + R_{23}^2}, R_{33}\right) \tag{32}$$

$$\phi = \operatorname{atan2}\left(R_{13}/\sin(\theta), -R_{23}/\sin(\theta)\right) \tag{33}$$

$$\psi = \operatorname{atan2}\left(R_{31}/\sin(\theta), R_{32}/\sin(\theta)\right) \tag{34}$$

5 Roll Pitch Yawn

5.1 RPY direct problem

To find R from ψ, θ and ϕ we use the following formula:

$$R(\psi, \theta, \phi) = R_z(\phi)R_y(\theta)R_x(\psi) \tag{35}$$

Note: the order of the angle is reversed!

5.2 Inverse Problem

Given a rotation matrix R we can find angles of rotation ψ, θ and ϕ: Fisrt we check if there is a singularity (if $R_{32}^2 + R_{33}^2 = 0$), we then have two cases:

- **No Singularity** — We can find all three parameters of the rotational matrix R

$$\theta = \operatorname{atan2}\left(-R_{31}, \pm\sqrt{R_{32}^2 + R_{33}^2}\right) \tag{36}$$

$$\phi = \operatorname{atan2}\left(R_{21}/\cos(\theta), R_{11}/\cos(\theta)\right) \tag{37}$$

$$\psi = \operatorname{atan2}\left(R_{32}/\cos(\theta), R_{33}/\cos(\theta)\right) \tag{38}$$

- **Singularity** — We cannot find all three angles, only θ and a combination of ϕ and ψ, the formula for these combinations is:

$$\begin{cases} \phi - \psi = \text{atan2}\left\{R_{2,3}, R_{1,3}\right\} & \text{if } \theta = \frac{\pi}{2} \\ \phi + \psi = \text{atan2}\left\{-R_{2,3}, R_{2,2}\right\} & \text{if } \theta = -\frac{\pi}{2} \end{cases} \tag{39}$$

6 DH frames

6.1 Assign axis

- z_i along the direction of joint i+1.
- x_i along the common normal between z_i and z_{i-1}.
- y_i completes the right-handed coordinate system.

6.2 DH table

- θ_i angle between x_{i-1} and x_i measured about z_{i-1}.
- d_i distance between x_{i-1} and x_i measured along z_{i-1}.
- a_i distance between z_{i-1} and z_i measured along x_i.
- α_i angle between z_{i-1} and z_i measured about x_i.

6.3 Transformation matrix from DH parameters

$${}^{i-1}A_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{40}$$

6.4 DH parameters from transformation matrix

First we have to check that the first three by three submatrix is a rotation matrix (see section 3.1).

Then we can find the parameters:

$$\theta_i = \text{atan2}\left(R_{12}, R_{11}\right) \tag{41}$$
$$\alpha_i = \text{atan2}\left(R_{32}, R_{33}\right) \tag{42}$$
$$d_i = R_{34} \tag{43}$$
$$a_i = R_{14}\cos(\theta_i) + R_{24}\sin(\theta_i) \tag{44}$$
$$\tag{45}$$

7 Workspace

7.1 2-DOF robot

The primary workspace is defined by two concentric circles of radius r_1 and r_2 where:

$$r_1 = |l_1 - l_2| \tag{46}$$

$$r_2 = l_1 + l_2 \tag{47}$$

7.2 3-DOF robot

The primary workspace is defined by two concentric spheres of radius r_{in} and r_{out} where:

$$r_{out} = l_{min} + l_{med} + l_{max} \tag{48}$$

$$r_{in} = \max(0, l_{max} - l_{med} - l_{min}) \tag{49}$$

where:

- l_{min} is the length of the shortest link
- l_{med} is the length of the medium link
- l_{max} is the length of the longest link

8 Inverse Kinematic

8.1 Trigonometry

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \tag{50}$$
$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) \tag{51}$$

8.2 algebraic transformation

if we have a system of the form:

$$a\cos(\theta) + b\sin(\theta) = c \tag{52}$$

we can transform it in a system of the form:

$$u_{12} = \frac{a \pm \sqrt{a^2 + b^2 - c^2}}{b + c} \tag{53}$$

$$\theta_{12} = \text{atan2}(u_{12}) \tag{54}$$

Note: we have to check that $a^2 + b^2 - c^2 \geq 0$

8.3 algebraic solution

Rewrite a system of equations in the form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{55}$$

and obtain the solution:

$$det = (a_{11}a_{22} - a_{12}a_{21}) \tag{56}$$

$$c_1 = \frac{a_{11}b_1 + a_{21}b_2}{det} \tag{57}$$

$$s_1 = \frac{a_{12}b_1 + a_{22}b_2}{det} \tag{58}$$

Note: we have to check that $det \neq 0$