RPY Singularity Inverse Proof

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November 8, 2023

1 Proof

Definition 1.1 (RPY Inverse)**.** A singularity occurs in an inverse problem of a fixed-axis RPY matrix if the pitch angle θ is equal to $\pm\frac{\pi}{2}$. We can still extract the roll and yaw angles as a sum/difference, but we cannot recover the individual angles precisely.

The formula for these combinations is:

$$\begin{cases} \phi - \psi = \operatorname{atan2}\{\texttt{R[1][2]}, \texttt{R[0][2]}\} & \text{if } \theta = \frac{\pi}{2} \\ \phi + \psi = \operatorname{atan2}\{\texttt{-R[1][2]}, \texttt{R[1][1]}\} & \text{if } \theta = -\frac{\pi}{2} \end{cases} \tag{1}$$

Where `R` is the rotation matrix, indexed starting from 0. (i.e.: `R[1][2]` is the first row, third column).

RPY Inverse Proof.

$$R(\psi, \theta, \phi) = \begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi - s\phi c\psi & c\phi s\theta c\psi + s\phi s\psi \\ s\phi c\theta & s\phi s\theta s\psi + c\phi c\psi & s\phi s\theta c\psi - c\phi s\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix} \tag{2}$$

Considering the case where $\theta = \frac{\pi}{2}$:

$$R(\psi, \frac{\pi}{2}, \phi) = \begin{bmatrix} 0 & c\phi s\psi - s\phi c\psi & c\phi c\psi + s\phi s\psi \\ 0 & s\phi s\psi + c\phi c\psi & s\phi c\psi - c\phi s\psi \\ -1 & 0 & 0 \end{bmatrix} \tag{3}$$

Now consider two trigonometric properties:

$$\sin(\phi - \psi) = \underbrace{\sin(\phi)\cos(\psi) - \cos(\phi)\sin(\psi)}_{\texttt{R[1][2]}} \tag{4}$$

$$\cos(\phi - \psi) = \underbrace{\cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)}_{\texttt{R[0][2]}} \tag{5}$$

We can now solve for $\phi - \psi$ using the inverse tangent function:

$$\phi - \psi = \operatorname{atan2}\{\texttt{R[1][2]}, \texttt{R[0][2]}\} \tag{6}$$

Similarly, for $\theta = -\frac{\pi}{2}$:

$$R(\psi, -\frac{\pi}{2}, \phi) = \begin{bmatrix} 0 & -(s\phi c\psi + c\phi s\psi) & s\phi s\psi - c\phi c\psi \\ 0 & c\phi c\psi - s\phi s\psi & -(s\phi c\psi + c\phi s\psi) \\ 1 & 0 & 0 \end{bmatrix} \tag{7}$$

Through the dual of the previous properties:

$$\sin(\phi + \psi) = \underbrace{\sin(\phi)\cos(\psi) + \cos(\phi)\sin(\psi)}_{\texttt{-R[1][2]}} \tag{8}$$

$$\cos(\phi + \psi) = \underbrace{\cos(\phi)\cos(\psi) - \sin(\phi)\sin(\psi)}_{\texttt{R[1][1]}} \tag{9}$$

We can now solve for $\phi + \psi$ using the inverse tangent function:

$$\phi + \psi = \operatorname{atan2}\{\texttt{-R[1][2]}, \texttt{R[1][1]}\} \tag{10}$$

Additionally, one can freely choose an assignment for either ϕ or ψ, and then solve for the other angle. $\square$