Supplemental material to The Crane Operator's Tricks and other Shenanigans with a Pendulum

1 Damping the torsion balance

The differential equation of the torsion balance subjected to an external torque $n(t)$ is given by

$$\ddot{\varphi}_t + \omega_0^2 \varphi_t = \frac{n(t)}{I}, \tag{1}$$

where $\omega_0^2 = \kappa/I$, I is the moment of inertia of the pendulum, κ is the torsional spring constant of the pendulum restoring force, and ϕ is the angular deflection. In the Laplace domain, it is

$$\frac{\Phi_t(s)}{N(s)} = \frac{1}{I}\frac{1}{s^2 + \omega_0^2}. \tag{2}$$

The torque acting on the pendulum is proportional to $\sin\big((\varphi_s - \varphi_t)/\varphi_{\text{norm}}\big)$.

The reason for this is that the torque does not change in a linear fashion, but rather sinusoidal. To calculate the response we need three functions that will be combined with different time shifts. The three functions are

$$\begin{aligned} f_1(t) &= u(t), & (3)\\ f_2(t) &= u(t)\cos\left(\frac{t}{\tau}\frac{\pi}{2}\right), \text{ and} & (4)\\ f_3(t) &= u(t)\sin\left(\frac{t}{\tau}\frac{\pi}{2}\right), & (5) \end{aligned}$$

where $u(t)$ denotes the Heaviside step function and τ is the duration of one move.

The external torque in the time domain with a total amplitude of n_a and the moves starting at t_1 and t_3 is

$$n(t) = \frac{n_a}{2}\Big(f_1(t-t_1) - f_2(t-t_1) - f_3(t-t_2) + f_3(t-t_3) - f_2(t-t_4) + f_1(t-t_4)\Big). \tag{6}$$

The first move starts at t_1 and is completed at $t_2 = t_1 + \delta t$. The second move starts at t_3 and ends at $t_4 = t_3 + \delta t$. Consistent with the main text, the duration of the move is abbreviated by δt.

Figure 1 shows the torque for $n_a = 1 \times 10^{-8}\,\mathrm{N\,m}$, $t_1 = 20\,\mathrm{s}$, $t_3 = 80\,\mathrm{s}$, and $\delta t = 28\,\mathrm{s}$.

In the s domain using the abbreviation $\nu = \pi/(2\tau)$, the torque is given by

$$N(s) = \frac{n_A}{2}\left(e^{-st_1}\frac{1}{s} - e^{-st_1}\frac{s}{s^2+\nu^2} - e^{-st_2}\frac{\nu}{s^2+\nu^2} + e^{-st_3}\frac{\nu}{s^2+\nu^2} - e^{-st_4}\frac{s}{s^2+\nu^2} + e^{-st_4}\frac{1}{s}\right) \tag{7}$$

Including the unit pulse that makes the pendulum swing at $t = 0$ through the equilibrium position with $\dot{\phi}(0) = v_0$ and multiplying with

$$\frac{\Phi_t(s)}{N(s)} = \frac{1}{I}\frac{1}{s^2 + \omega_o^2}. \tag{8}$$

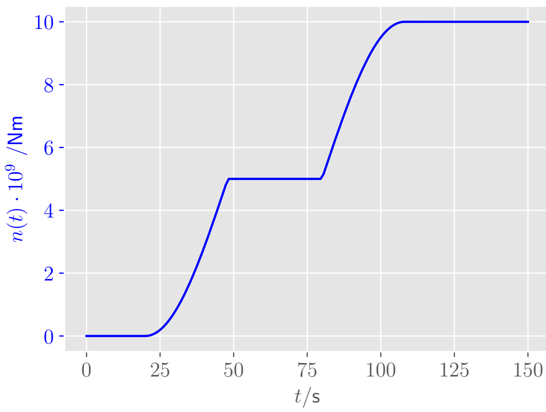

Figure 1: The torque as a function of time, according to equation 6.

yields,

$$\begin{aligned}\Phi_t(s) = {} & \frac{v_0}{s^2+\omega_o^2} + \frac{n_A}{2I}\Bigg(e^{-st_1}\left(\frac{1}{\omega_0^2 s} - \frac{s}{\omega_o^2\left(\omega_o^2+s^2\right)}\right) + e^{-st_4}\left(\frac{1}{\omega_0^2 s} - \frac{s}{\omega_o^2\left(\omega_o^2+s^2\right)}\right) \\ & - e^{-st_1}\left(\frac{s}{(\nu^2+\omega_o^2)(\omega_o^2+s^2)} - \frac{s}{(\nu^2+\omega_o^2)(\nu^2+s^2)}\right) - e^{-st_4}\left(\frac{s}{(\nu^2+\omega_o^2)(\omega_o^2+s^2)} - \frac{s}{(\nu^2+\omega_o^2)(\nu^2+s^2)}\right) \\ & - e^{-st_2}\left(\frac{\nu}{(\nu^2-\omega_o^2)(\omega_o^2+s^2)} - \frac{\nu}{(\nu^2-\omega_o^2)(\nu^2+s^2)}\right) + e^{-st_3}\left(\frac{\nu}{(\nu^2-\omega_o^2)(\omega_o^2+s^2)} - \frac{\nu}{(\nu^2-\omega_o^2)(\nu^2+s^2)}\right)\Bigg)\end{aligned} \tag{9}$$

Transforming back to the time domain yields for $t > t_4$,

$$\begin{aligned}\varphi_t(t) = {} & v_0 \sin\left(\omega_o t\right) + \frac{n_A}{2I}\Bigg(\frac{2}{\omega_o^2} - \frac{1}{\omega_o^2}\cos\left(\omega_o(t-t_1)\right) - \frac{1}{\omega_o^2}\cos\left(\omega_o\left(t-t_4\right)\right) \\ & - \frac{1}{\nu^2+\omega_o^2}\cos\left(\omega_o\left(t-t_1\right)\right) + \frac{1}{\nu^2+\omega_o^2}\cos\left(\nu\left(t-t_1\right)\right) - \frac{1}{\nu^2+\omega_o^2}\cos\left(\omega_o\left(t-t_4\right)\right) + \frac{1}{\nu^2+\omega_o^2}\cos\left(\nu\left(t-t_4\right)\right) \\ & - \frac{\nu}{\omega_o}\frac{1}{\nu^2-\omega_o^2}\sin\left(\omega_o\left(t-t_2\right)\right) + \frac{1}{\nu^2-\omega_o^2}\sin\left(\nu\left(t-t_2\right)\right) + \frac{\nu}{\omega_o}\frac{1}{\nu^2-\omega_o^2}\sin\left(\omega_o\left(t-t_3\right)\right) - \frac{1}{\nu^2-\omega_o^2}\sin\left(\nu\left(t-t_3\right)\right)\Bigg)\end{aligned} \tag{10}$$

From this equation the time derivative is calculated. The trigonometric functions are expanded and sorted into the sine and cosine terms. We obtain,

$$\dot{\varphi}_t(t) = C\cos\left(\omega_o t\right) + S\sin\left(\omega_o t\right) \quad \text{with} \tag{11}$$

$$C = v_0 - \frac{n_a\pi\omega_o\Big(-2\omega_o\tau\cos\left(\omega_o t_3\right) + 2\omega_o\tau\cos\big(\omega_o\left(t_1+\delta t\right)\big) + \pi\sin\left(\omega_o t_1\right) + \pi\sin\big(\omega_o\left(t_3+\delta t\right)\big)\Big)}{2\kappa(\pi^2-4\omega_o^2\delta t^2)} \quad \text{and} \tag{12}$$

$$S = \frac{n_a\pi\omega_o\Big(\pi\cos\left(\omega_o t_1\right) + \pi\cos\big(\omega_o\left(t_3+\delta t\right)\big) + 2\omega_o\delta t\sin\left(\omega_o t_3\right) - 2\omega_o\delta t\sin\big(\omega_o\left(t_1+\delta t\right)\big)\Big)}{2\kappa(\pi^2-4\omega_o^2\delta t^2)}. \tag{13}$$

To damp the pendulum to a desired velocity amplitude v_{des}, we minimize the term

$$\left|\sqrt{\big(C(t_1,t_3)\big)^2 + \big(S(t_1,t_3)\big)^2} - v_{\text{des}}\right| \tag{14}$$

numerically.