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## Homework #4:

### Structures, Equational Reasoning, and Induction

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#### I. Sets, Relations, and Functions

1. Use set comprehension to define the set *SumOfSquares* containing all the natural numbers that can be expressed as the sum  $a^2 + b^2$  where  $a$  and  $b$  are natural numbers.
2. Write out in full the powersets of each of the following.
  - a.  $\{7, 1\}$
  - b.  $\{5\}$
  - c.  $\emptyset$
  - d.  $\{\emptyset\}$
3. Write out in full the following Cartesian products.
  - a.  $\{4, 2\} \times \{2, 4\}$
  - b.  $\{0\} \times \emptyset$
  - c.  $\{1, 2\} \times \{a\}$
  - d.  $\{\emptyset\} \times \{a\}$
4. Suppose  $R == 2 \dots 5$  and  $S == 4 \dots 6$ . Enumerate the elements of the following sets.
  - a.  $R \cup S$
  - b.  $R \cap S$
  - c.  $R \setminus S$
  - d.  $S \setminus R$
  - e.  $S \times R$
5. Let  $S$  be the set of numbers from 1 to 12 inclusive. Let  $R$  be a relation, such that  $R : S \leftrightarrow S$  and such that  $x$  is related to  $y$  exactly when  $y$  is greater than the square of  $x$  but less than the square of  $x + 1$ . Provide an axiomatic definition for  $R$ .  
(Note: be sure to check your notation and formatting — refer to page 152 in GWC10.)
6. Suppose *Let* and *Num* are defined as follows:

$$\begin{aligned} \textit{Let} &== \{a, b, c, d, e\} \\ \textit{Num} &== \{1, 2, 3, 4, 5\} \end{aligned}$$

- a. Give an example of each of the following:
  - i. A function whose declaration is  $\textit{Let} \rightarrow \textit{Num}$
  - ii. A function whose declaration is  $\textit{Let} \leftrightarrow \textit{Num}$
  - iii. A total injection from *Let* to *Num*
- b. Is it possible to give an example of a total injection from *Let* to  $\{1, 2, 3, 4\}$ ? If so, provide one; if not, explain why not.

## II. Proof: Equational Reasoning

You may use any of the following theorems in your equational proofs:

$\vdash p \wedge \text{true} \Leftrightarrow p$	$\wedge$ -True
$\vdash p \wedge \text{false} \Leftrightarrow \text{false}$	$\wedge$ -False
$\vdash p \vee \text{true} \Leftrightarrow \text{true}$	$\vee$ -True
$\vdash p \vee \text{false} \Leftrightarrow p$	$\vee$ -False
$\vdash p \vee \neg p$	Excluded Middle
$\vdash p \vee q \Leftrightarrow q \vee p$	$\vee$ -Commutativity
$\vdash p \wedge q \Leftrightarrow q \wedge p$	$\wedge$ -Commutativity
$\vdash (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$\vee$ -Associativity
$\vdash (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	$\wedge$ -Associativity
$\vdash p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$\vee \wedge$ -Distributivity
$\vdash p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$\wedge \vee$ -Distributivity
$\vdash p \Rightarrow q \Leftrightarrow \neg p \vee q$	$\Rightarrow$ -Alternative
$\vdash p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$	Contrapositives
$\vdash \neg \neg p \Leftrightarrow p$	Double Negation
$\vdash \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	De Morgan
$\vdash \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan
$\vdash x \in \emptyset \Leftrightarrow \text{false}$	$\emptyset$ Membership

7. Prove in equational style the following laws for set union:

- $S \cup T = T \cup S$
- $S \cap \emptyset = \emptyset$

(HINT: To prove  $S = T$  show  $\forall x : U \bullet x \in S \Leftrightarrow x \in T$ , where  $U$  is the type of elements in sets  $S$  and  $T$ .)

8. Prove the following theorem in equational style:

$$\vdash \neg(\neg p \Rightarrow (q \wedge r)) \Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$$

## III. Sequences

9. Define the following sequences by enumeration:

- Threes*: natural numbers smaller than 18 that are divisible by 3.
- Twos*: natural numbers smaller than 20 that are divisible by 2.

10. Given the above, what are each of the following:

- $\text{Threes} \cup \text{Twos}$
- $\text{Threes} \cap \text{Twos}$
- $\text{dom } \text{Threes}$
- $\text{ran } \text{Threes}$
- $\text{dom } \text{Twos} \triangleleft \text{Threes}$
- $(5 \dots 8) \triangleleft (\text{Threes} \frown \text{Twos})$   
(NOTE:  $\frown$  is the concatenation operator).

## IV. Proof: Natural Induction

Prove the following claim by induction over the natural numbers:

$$0^2 + 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

## V. Proof: Structural Induction

Consider the definition of binary trees in Chapter 7.

(a) Show that

$$\forall t : TREE \bullet leaves(t) = nodes(t) + 1$$

(b) Define a *mirror* function that recursively swaps the branches of a tree.

(c) Using the definition of *mirror* show that

$$\forall t : TREE \bullet size(mirror(t)) = size(t)$$

(d) Using the definition of *mirror* show that

$$\forall t : TREE \bullet mirror(mirror(t)) = t$$

(HINT: use structural induction over trees.)