

Homework #4:

Structures, Equational Reasoning, and Induction

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Due: 24 September 2018

I. Sets, Relations, and Functions

1. Use set comprehension to define the set *SumOfSquares* containing all the natural numbers that can be expressed as the sum $a^2 + b^2$ where a and b are natural numbers.

Redo note: Adjusted parenthesis

$$\text{SumOfSquares} == \{x : \mathbb{N} \mid \exists a : \mathbb{N} \bullet \exists b : \mathbb{N} \bullet x = a^2 + b^2\}$$

2. Write out in full the powersets of each of the following.

a. $\mathbb{P}\{7, 1\} = \{\emptyset, \{1\}, \{7\}, \{7, 1\}\}$

b. $\mathbb{P}\{5\} = \{\emptyset, \{5\}\}$

c. $\mathbb{P}\emptyset = \{\emptyset\}$

d. $\mathbb{P}\{\emptyset\} = \{\emptyset, \{\emptyset\}\}$

3. Write out in full the following Cartesian products.

a. $\{4, 2\} \times \{2, 4\} = \{(4, 2), (4, 4), (2, 2), (2, 4)\}$

b. $\{0\} \times \emptyset = \emptyset$

c. $\{1, 2\} \times \{a\} = \{(1, a), (2, a)\}$

d. $\{\emptyset\} \times \{a\} = \{(\emptyset, a)\}$

Redo note: Defined pair

4. Suppose $R == 2..5$ and $S == 4..6$. Enumerate the elements of the following sets.

a. $R \cup S = \{2, 3, 4, 5, 6\}$

b. $R \cap S = \{4, 5\}$

c. $R \setminus S = \{2, 3\}$

d. $S \setminus R = \{6\}$

e. $S \times R = \{(4, 2), (4, 3), (4, 4), (4, 5), (5, 2), (5, 3), (5, 4), (5, 5), (6, 2), (6, 3), (6, 4), (6, 5)\}$

5. Let S be the set of numbers from 1 to 12 inclusive. Let R be a relation, such that $R : S \leftrightarrow S$ and such that x is related to y exactly when y is greater than the square of x but less than the square of $x + 1$. Provide an axiomatic definition for R .

(Note: be sure to check your notation and formatting — refer to page 152 in GWC10.)

6. Suppose *Let* and *Num* are defined as follows:

$$\text{Let} == \{a, b, c, d, e\}$$

$$\text{Num} == \{1, 2, 3, 4, 5\}$$

- a. Give an example of each of the following:

- i. A function whose declaration is $Let \rightarrow Num$
 $f1 == \{a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4, e \mapsto 5\}$
 - ii. A function whose declaration is $Let \rightarrow Num$
 $f2 == \{a \mapsto 1, b \mapsto 2, c \mapsto 3\}$
 - iii. A total injection from Let to Num
 $f3 == \{a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4, e \mapsto 5\}$
- b. Is it possible to give an example of a total injection from Let to $\{1, 2, 3, 4\}$? If so, provide one; if not, explain why not.
- No because the number of values in the source is larger than the target and we can not map more than one source element to the same target because that will violate the injection rule.

II. Proof: Equational Reasoning

You may use any of the following theorems in your equational proofs:

$\vdash p \wedge \text{true} \Leftrightarrow p$	\wedge -True
$\vdash p \wedge \text{false} \Leftrightarrow \text{false}$	\wedge -False
$\vdash p \vee \text{true} \Leftrightarrow \text{true}$	\vee -True
$\vdash p \vee \text{false} \Leftrightarrow p$	\vee -False
$\vdash p \vee \neg p$	Excluded Middle
$\vdash p \vee q \Leftrightarrow q \vee p$	\vee -Commutativity
$\vdash p \wedge q \Leftrightarrow q \wedge p$	\wedge -Commutativity
$\vdash (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	\vee -Associativity
$\vdash (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	\wedge -Associativity
$\vdash p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$\vee \wedge$ -Distributivity
$\vdash p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$\wedge \vee$ -Distributivity
$\vdash p \Rightarrow q \Leftrightarrow \neg p \vee q$	\Rightarrow -Alternative
$\vdash p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$	Contrapositives
$\vdash \neg \neg p \Leftrightarrow p$	Double Negation
$\vdash \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	De Morgan
$\vdash \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan
$\vdash x \in \emptyset \Leftrightarrow \text{false}$	\emptyset Membership

7. Prove in equational style the following laws for set union:

a. $S \cup T = T \cup S$

Redo note: Adjusted proof

$$\begin{aligned}
 & x \in S \cup T \\
 & \Leftrightarrow \quad \text{[Definition of } \cup \text{]} \\
 & x \in S \vee x \in T \\
 & \Leftrightarrow \quad \text{[} \vee \text{ commutative]} \\
 & x \in T \vee x \in S \\
 & \Leftrightarrow \quad \text{[Definition of } \cup \text{]} \\
 & x \in T \cup S \\
 & S \cup T = T \cup S \text{ QED}
 \end{aligned}$$

b. $S \cap \emptyset = \emptyset$

Redo note: Adjusted proof

$$\begin{aligned}
 & x \in S \cap \emptyset \\
 & \Leftrightarrow \quad \text{[Definition of } \cap \text{]} \\
 & x \in S \wedge x \in \emptyset \\
 & \Leftrightarrow \quad \text{[emptyset Membership]} \\
 & x \in S \wedge \text{false} \\
 & \Leftrightarrow \quad \text{[arithmetic]} \\
 & \emptyset \\
 & S \cap \emptyset = \emptyset \text{ QED}
 \end{aligned}$$

(HINT: To prove $S = T$ show $\forall x : U \bullet x \in S \Leftrightarrow x \in T$, where U is the type of elements in sets S and T.)

8. Prove the following theorem in equational style:

$$\vdash \neg(\neg p \Rightarrow (q \wedge r)) \Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$$

III. Sequences

9. Define the following sequences by enumeration:

a. *Threes*: natural numbers smaller than 18 that are divisible by 3.

Using GWC10 notation for seq enums definition:

Threes == < 3, 6, 9, 12, 15 >

b. *Twos*: natural numbers smaller than 20 that are divisible by 2.

Using GWC10 notation for seq enums definition:

Twos == < 2, 4, 6, 8, 10, 12, 14, 16, 18 >

10. Given the above, what are each of the following:

a. *Threes* \cup *Twos*

Redo note: Transformed to sets and applied \cup .

TreesAsSetOfPairs = $\{(1, 3), (2, 6), (3, 9), (4, 12), (5, 15)\}$

TwosAsSetOfPairs = $\{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10), (6, 12), (7, 14), (8, 16), (9, 18)\}$

Threes \cup *Twos* = $\{(1, 3), (2, 6), (3, 9), (4, 12), (5, 15), (1, 2), (2, 4), (3, 6), (4, 8), (5, 10), (6, 12), (7, 14), (8, 16), (9, 18)\}$

b. *Threes* \cap *Twos*

Redo note: Transformed to sets and applied \cap .

Threes \cap *Twos* = $\{\emptyset\}$

c. $\text{dom } \textit{Threes}$ == $\{1, 2, 3, 4, 5\}$

To get the domain of a sequence, we just need to get the indexes from 1 up to the cardinality of the sequence.

d. $\text{ran } \textit{Threes}$ == $\{3, 6, 9, 12, 15\}$

e. $\text{dom } \textit{Twos} \triangleleft \textit{Threes}$

\triangleleft is the domain restrictor operator, informally, the restricted relation contains those elements from *Three* whose first component appears in *Two*.

domTwos == $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Redo note: Corrected set notation.

$\text{dom } \textit{Twos} \triangleleft \textit{Threes}$ == < 3, 6, 9, 12, 15 >

f. $(5 \dots 8) \triangleleft (\textit{Threes} \frown \textit{Twos})$

Threes \frown *Twos* == < 3, 6, 9, 12, 15, 2, 4, 6, 8, 10, 12, 14, 16, 18 >

\triangleleft is asking us to filter elements from $5 \dots 8$

Redo note: Corrected notation to set and swapped 5 to 6.

$(5 \dots 8) \triangleleft (\textit{Threes} \frown \textit{Twos})$ == $\{15, 2, 4, 6\}$ (NOTE: \frown is the concatenation operator).

IV. Proof: Natural Induction

Prove the following claim by induction over the natural numbers:

$$0^2 + 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Base case:

$$P(0) = \frac{0}{6} = 0$$

$$0^2 = 0$$

Inductive step We assume that $k \in \mathbb{N}$ and $P(k)$ holds. We then show that $P(k+1)$ holds, that is $0^2 + 1^2 + 2^2 + \dots + k + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{2k^3+9k^2+13K+6}{6}$

$$\begin{aligned} & 0^2 + 1^2 + 2^2 + \dots + k + (k+1)^2 \\ &= \text{[substitution, induction hypothesis]} \\ & \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \text{[arithmetic]} \\ & \frac{(k^2+k)(2k+1)}{6} + (k+1)^2 \\ &= \text{[arithmetic]} \\ & \frac{2k^3+9k^2+13K+6}{6} \end{aligned}$$

V. Proof: Structural Induction

Consider the definition of binary trees in Chapter 7.

(a) Show that

$$\forall t : TREE \bullet leaves(t) = nodes(t) + 1$$

(b) Define a *mirror* function that recursively swaps the branches of a tree.

The idea of this function is that it recursively calls *mirror* swapping *t1* and *t2* every time that it finds a node. If it finds a leaf it just returns.

$$mirror : TREE \rightarrow TREE$$

Redo: fixed equation, the key aspect is changing the order of *t1*, *t2* and calling *mirror*(*node*) recursively.

$$\begin{aligned} & \forall t1, t2 : TREE \bullet \\ & mirror(leaf) = leaf \wedge \\ & mirror(node(t1, t2)) = node(mirror(t2), mirror(t1)) \end{aligned}$$

(c) Using the definition of *mirror* show that

$$\forall t : TREE \bullet size(mirror(t)) = size(t)$$

Using the definition of *size*(*t*) provided in GWC10:

$$size : TREE \rightarrow \mathbb{N}$$

$$\begin{aligned} & \forall t1, t2 : TREE \bullet \\ & size(leaf) = 1 \wedge \\ & size(node(t1, t2)) = 1 + size(t1) + size(t2) \end{aligned}$$

Proof:

Base Case: Show the property holds for *leaf*, that is, *size*(*leaf*) = *size*(*mirror*(*leaf*))

$$\begin{aligned} & size(leaf) \\ &= \text{[definition of size]} \\ & 1 \\ &= \text{[definition size(mirror(leaf))]} \\ & size(mirror(leaf)) \end{aligned}$$

Induction case: Assume that the property holds for trees t_1 and t_2 , that is $\text{size}(t_1) = \text{size}(\text{mirror}(t_2))$, and $\text{size}(t_2) = \text{size}(\text{mirror}(t_1))$. Show that it holds for $\text{node}(t_1, t_2)$

$$\begin{aligned}
 & \text{size}(\text{node}(t_1, t_2)) \\
 &= && \text{[definition of size]} \\
 & 1 + \text{size}(t_1) + \text{size}(t_2) \\
 &= && \text{[induction hypothesis]} \\
 & 1 + \text{size}(\text{mirror}(t_2)) + \text{size}(\text{mirror}(t_1)) \\
 &= && \text{[definition of mirror]} \\
 & 1 + \text{size}(\text{node}(\text{mirror}(t_2), \text{mirror}(t_1))) + \text{size}(\text{node}(\text{mirror}(t_1), \text{mirror}(t_2))) \\
 & \text{not sure what to do after this...}
 \end{aligned}$$

(d) Using the definition of *mirror* show that

$$\forall t : \text{TREE} \bullet \text{mirror}(\text{mirror}(t)) = t$$

(HINT: use structural induction over trees.)

Proof:

Base Case: Show the property holds for leaf, that is, $\text{leaf} = \text{mirror}(\text{leaf})$

$$\begin{aligned}
 & \text{leaf} \\
 &= && \text{[definition mirror(leaf)]} \\
 & \text{mirror}(\text{leaf})
 \end{aligned}$$

Induction case: Assume that the property holds for trees t_1 and t_2 , that is $\text{node}(t_1, t_2) = \text{node}(\text{mirror}(t_2), \text{mirror}(t_1))$

$$\begin{aligned}
 & \text{node}(t_1, t_2) \\
 &= && \text{[definition of mirror]} \\
 & \text{mirror}(\text{node}(t_1, t_2)) \\
 &= && \text{[applying mirror function to node]} \text{ node}(\text{mirror}(t_2), \text{mirror}(t_1)) \\
 & \text{not sure what to do after this...}
 \end{aligned}$$