Homework #3: Proof

Dario A Lencina-Talarico Due: 17 September 2018

INSTRUCTIONS: before completing your homework, make sure you have considered the following:

- You are allowed to use only primitive inference rules.
- Remember to include vertical lines to represent the scope any assumptions.
- Remember that all assumptions in your proofs must be discharged.
- Double-check that your line references are correct when applying inference rules.

I. Proof: Propositional Logic

In this section *use only the primitive inference rules* of propositional calculus.

1. Provide derivations for each of the following, using natural deduction:

a. $p \wedge q, p \Rightarrow s, q \Rightarrow t \vdash s \wedge t$

1.	$p \wedge q$	premise
2.	$p \Rightarrow s$	premise
3.	$q \Rightarrow t$	premise
4.	p	$\wedge$-elim, 1
5.	s	modus ponens, 2,4
6.	q	$\wedge$-elim, 1
7.	t	modus ponens, 3,6
8.	$s \wedge t$	$\wedge$-intro, 5,7

b. $q \Rightarrow \neg p, p \wedge q \vdash r$

1.	$q \Rightarrow \neg p$	premise	
2.	$p \wedge q$	premise	
3.	q	$\wedge - elim, 2$	
4.	$\neg p$	modus ponens, 1	
5.	p	$\wedge - elim, 2$	
6.	$\neg r$	assumption	l
7.	$\neg p$	Copy from 4	l
8.	p	Copy from 5	l
4.	r	$\neg - elim$, 6-8	

c. $p \wedge q \vdash p \Rightarrow q$

1.	$p \wedge q$	assuption	l
2.	p	$\wedge - elim$,1	l
3.	q	$\wedge - elim$,1	l
4.	$p \Rightarrow q$	$\Rightarrow - intro$,1-3,2,3	

d. $\neg\neg q \vdash q \vee r$

1.	$\neg\neg q$	assuption	l
2.	$\neg q$	$\neg - elim, 1$	l
3.	q	$\neg - elim, 2$	l
4.	$q \vee r$	$\vee - intro, 1 - 3$	

e. $p \Rightarrow (q \wedge r), \neg p \Rightarrow r, p \vee \neg p \vdash r$

1.	$\neg p \Rightarrow r$	premise		
2.	$p \vee \neg p$	assumption		l
3.	p	$\vee$-elim, 2		l
4.	$p \Rightarrow (q \wedge r)$	assumption	l	l
5.	$(q \wedge r)$	modus ponens,4,3	l	l
6.	r	$\wedge - elim$, 2-5		

II. Proof: Predicate Logic

In this section *use only the primitive inference rules* of predicate calculus.

2. Show using natural deduction:

a. $\forall x : T \bullet P(x) \wedge Q(x) \dashv\vdash (\forall x : T \bullet P(x)) \wedge (\forall y : T \bullet Q(y))$

$\forall x : T \bullet P(x) \wedge Q(x) \vdash (\forall x : T \bullet P(x)) \wedge (\forall y : T \bullet Q(y))$

1.	$\forall x : T \bullet P(x) \wedge Q(x)$	assumption			l
2.	$x \in T$	assumption		l	l
3.	$P(x) \wedge Q(x)$	$\forall - elim$, 1		l	l
4.	$P(x)$	$\wedge - elim$, 3		l	l
5.	$Q(x)$	$\wedge - elum$, 3		l	l
6.	$y \in T$	assumption	l	l	l
7.	$Q(y)$	transforming x to y, 5,6	l	l	l
8.	$\forall y : T \bullet Q(y)$	$\forall - intro$ 6,7		l	l
9.	$\forall x : T \bullet P(x)$	$\forall - intro$ 2,4		l	l
10.	$\forall x : T \bullet P(x)) \wedge (\forall y : T \bullet Q(y))$	$\wedge - intro$ 8,9			

$(\forall x : T \bullet P(x)) \wedge (\forall y : T \bullet Q(y)) \vdash \forall x : T \bullet P(x) \wedge Q(x)$

1.	$(\forall x : T \bullet P(x)) \wedge (\forall y : T \bullet Q(y))$	assumption			l
2.	$(\forall x : T \bullet P(x)$				l
3.	$(\forall y : T \bullet Q(y))$				l
4.	$a \in T$	assumption		l	l
5.	$b \in T$	assumption	l	l	l
6.	$P(a)$	$\forall - elim, 2, 4$	l	l	l
7.	$Q(b)$	$\forall - elim, 3, 5$	l	l	l
8.	$P(a) \wedge Q(b)$	$\wedge - intro, 6, 7$	l	l	l
9.	$\forall x : T \bullet P(x) \wedge Q(x)$	$\forall - intro, 4 - 8$			

b. $\exists x : T \bullet P(x) \vee Q(x) \dashv\vdash (\exists x : T \bullet P(x)) \vee (\exists x : T \bullet Q(x))$

$\exists x : T \bullet P(x) \vee Q(x) \vdash (\exists x : T \bullet P(x)) \vee (\exists x : T \bullet Q(x))$

1. $\exists x : T \bullet P(x) \vee Q(x)$	assumption		l
2. $a \in T \wedge P(a) \vee Q(a)$	assumption	l	l
3. $P(a) \vee Q(a)$	$\wedge -elim, 2$	l	l
4. $P(a)$	$\vee -elim, 3$	l	l
5. $Q(a)$	$\vee -elim, 3$	l	l
6. $a \in T$	$\wedge -elim, 2$	l	l
6. $\exists x : T \bullet P(x)$	$\exists -intro, 6, 4$	l	l
7. $\exists x : T \bullet Q(x)$	$\exists -intro, 6, 5$	l	l
8. $(\exists x : T \bullet P(x)) \vee (\exists x : T \bullet Q(x))$	$\vee -intro, 6, 7$		

$(\exists x : T \bullet P(x)) \vee (\exists x : T \bullet Q(x)) \vdash \exists x : T \bullet P(x) \vee Q(x)$

1. $(\exists x : T \bullet P(x)) \vee (\exists x : T \bullet Q(x))$	assumption			l
2. $\exists x : T \bullet P(x)$	$\vee -elim, 1$			l
3. $\exists x : T \bullet Q(x)$	$\vee -elim, 1$			l
4. $a \in T \wedge P(a)$	assumption		l	l
5. $P(a)$	$\wedge -elim\ 4$		l	l
6. $b \in T \wedge Q(b)$	assumption	l	l	l
7. $Q(b)$	$\wedge -elim\ 6$	l	l	l
8. $b \in T$	$\wedge -elim\ 6$	l	l	l
9. $P(a) \vee Q(b)$	$\vee -intro, 2-4$	l	l	l
10. $\exists x : T \bullet P(x) \vee Q(x)$	$\exists -intro, 8, 9$		l	l
11. $\exists x : T \bullet P(x) \vee Q(x)$	$\exists -elim, 2, 4, 10$			
12. $\exists x : T \bullet P(x) \vee Q(x)$	$\exists -elim, 3, 6-10$			

(NOTE: $p \dashv\vdash q$ is a shorthand for "$p \vdash q$ and $q \vdash p$." That is, for $p \dashv\vdash q$ you need to show two separate derivations: one for $p \vdash q$ and another for $q \vdash p$.)