CMU 17-651 Models of Software Systems Fall 2016

Homework #4:
Structures, Equational Reasoning, and Induction

Garlan Due: 19 September 2016

I. Sets, Relations, and Functions

1. Use set comprehension to define the set $SumOfSquares$ containing all the natural numbers that can be expressed as the sum $a^2 + b^2$ where a and b are natural numbers.
2. Write out in full the powersets of each of the following.
   a. $\{7, 1\}$
   b. $\{5\}$
   c. $\emptyset$
   d. $\{\emptyset\}$
3. Write out in full the following Cartesian products.
   a. $\{4, 2\} \times \{2, 4\}$
   b. $\{0\} \times \emptyset$
   c. $\{1, 2\} \times \{a\}$
   d. $\{\emptyset\} \times \{a\}$
4. Suppose $R == 2\,..\,5$ and $S == 4\,..\,6$. Enumerate the elements of the following sets.
   a. $R \cup S$
   b. $R \cap S$
   c. $R \setminus S$
   d. $S \setminus R$
   e. $S \times R$
5. Let S be the set of numbers from 1 to 12 inclusive. Let R be a relation, such that $R : S \leftrightarrow S$ and such that x is related to y exactly when y is greater than the square of x but less than the square of $x + 1$. Provide an axiomatic definition for R.
   (Note: be sure to check your notation and formatting — refer to page 152 in GWC10.)
6. Suppose Let and Num are defined as follows:

$$
\begin{aligned}
Let &== \{a, b, c, d, e\} \\
Num &== \{1, 2, 3, 4, 5\}
\end{aligned}
$$

   a. Give an example of each of the following:
      i. A function whose declaration is $Let \rightarrow Num$
      ii. A function whose declaration is $Let \nrightarrow Num$
      iii. A total injection from Let to Num
   b. Is it possible to give an example of a total injection from Let to $\{1, 2, 3, 4\}$? If so, provide one; if not, explain why not.

II. Proof: Equational Reasoning

You may use any of the following theorems in your equational proofs:

$\vdash p \wedge true \;\Leftrightarrow\; p$	$\wedge$-True
$\vdash p \wedge false \;\Leftrightarrow\; false$	$\wedge$-False
$\vdash p \vee true \;\Leftrightarrow\; true$	$\vee$-True
$\vdash p \vee false \;\Leftrightarrow\; p$	$\vee$-False
$\vdash p \vee \neg p$	Excluded Middle
$\vdash p \vee q \;\Leftrightarrow\; q \vee p$	$\vee$-Commutativity
$\vdash p \wedge q \;\Leftrightarrow\; q \wedge p$	$\wedge$-Commutativity
$\vdash (p \vee q) \vee r \;\Leftrightarrow\; p \vee (q \vee r)$	$\vee$-Associativity
$\vdash (p \wedge q) \wedge r \;\Leftrightarrow\; p \wedge (q \wedge r)$	$\wedge$-Associativity
$\vdash p \vee (q \wedge r) \;\Leftrightarrow\; (p \vee q) \wedge (p \vee r)$	$\vee\wedge$-Distributivity
$\vdash p \wedge (q \vee r) \;\Leftrightarrow\; (p \wedge q) \vee (p \wedge r)$	$\wedge\vee$-Distributivity
$\vdash p \Rightarrow q \;\Leftrightarrow\; \neg p \vee q$	$\Rightarrow$-Alternative
$\vdash p \Rightarrow q \;\Leftrightarrow\; \neg q \Rightarrow \neg p$	Contrapositives
$\vdash \neg\neg p \;\Leftrightarrow\; p$	Double Negation
$\vdash \neg(p \wedge q) \;\Leftrightarrow\; \neg p \vee \neg q$	De Morgan
$\vdash \neg(p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q$	De Morgan
$\vdash x \in \emptyset \;\Leftrightarrow\; false$	$\emptyset$ Membership

7. Prove in equational style the following laws for set union:
   a. $S \cup T = T \cup S$
   b. $S \cap \emptyset = \emptyset$

   (Hint: To prove $S = T$ show $\forall\, x : U \bullet x \in S \Leftrightarrow x \in T$, where U is the type of elements in sets S and T.)

8. Prove the following theorem in equational style:

$$\vdash \neg(\neg p \Rightarrow (q \wedge r)) \;\Leftrightarrow\; (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$$

III. Sequences

9. Define the following sequences by enumeration:
   a. *Threes*: natural numbers smaller than 18 that are divisible by 3.
   b. *Twos*: natural numbers smaller than 20 that are divisible by 2.

10. Given the above, what are each of the following:
    a. $Threes \cup Twos$
    b. $Threes \cap Twos$
    c. $\operatorname{dom} Threes$
    d. $\operatorname{ran} Threes$
    e. $\operatorname{dom} Twos \lhd Threes$
    f. $(5 \mathrel{..} 8) \lhd (Threes \frown Twos)$
    (Note: $\frown$ is the concatenation operator).

IV. Proof: Natural Induction

Prove the following claim by induction over the natural numbers:

$$0^2 + 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

V. Proof: Structural Induction

Consider the definition of binary trees in Chapter 7.

(a) Show that

$$\forall\, t : \mathit{TREE} \bullet \mathit{leaves}(t) = \mathit{nodes}(t) + 1$$

(b) Define a *mirror* function that recursively swaps the branches of a tree.

(c) Using the definition of *mirror* show that

$$\forall\, t : \mathit{TREE} \bullet \mathit{size}(\mathit{mirror}(t)) = \mathit{size}(t)$$

(d) Using the definition of *mirror* show that

$$\forall\, t : \mathit{TREE} \bullet \mathit{mirror}(\mathit{mirror}(t)) = t$$

(Hint: use structural induction over trees.)