

Homework #4:

Structures, Equational Reasoning, and Induction

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I. Sets, Relations, and Functions

1. Use set comprehension to define the set *SumOfSquares* containing all the natural numbers that can be expressed as the sum $a^2 + b^2$ where a and b are natural numbers.
 $SumOfSquares == \{x : \mathbb{N} \mid (\exists a : \mathbb{N})(\exists b : \mathbb{N}) \bullet x = a^2 + b^2\}$
2. Write out in full the powersets of each of the following.
 - a. $\mathbb{P}\{7, 1\} = \{\emptyset, \{1\}, \{7\}, \{7, 1\}\}$
 - b. $\mathbb{P}\{5\} = \{\emptyset, \{5\}\}$
 - c. $\mathbb{P}\emptyset = \{\emptyset\}$
 - d. $\mathbb{P}\{\emptyset\} = \{\emptyset, \{\emptyset\}\}$
3. Write out in full the following Cartesian products.
 - a. $\{4, 2\} \times \{2, 4\} = \{(4, 2), (4, 4), (2, 2), (2, 4)\}$
 - b. $\{0\} \times \emptyset = \emptyset$
 - c. $\{1, 2\} \times \{a\} = \{(1, a), (2, a)\}$
 - d. $\{\emptyset\} \times \{a\} = \{\emptyset\}$
4. Suppose $R == 2 \dots 5$ and $S == 4 \dots 6$. Enumerate the elements of the following sets.
 - a. $R \cup S = \{2, 3, 4, 5, 6\}$
 - b. $R \cap S = \{4, 5\}$
 - c. $R \setminus S = \{2, 3\}$
 - d. $S \setminus R = \{6\}$
 - e. $S \times R = \{(4, 2), (4, 3), (4, 4), (4, 5), (5, 2), (5, 3), (5, 4), (5, 5), (6, 2), (6, 3), (6, 4), (6, 5)\}$
5. Let S be the set of numbers from 1 to 12 inclusive. Let R be a relation, such that $R : S \leftrightarrow S$ and such that x is related to y exactly when y is greater than the square of x but less than the square of $x + 1$. Provide an axiomatic definition for R .
 (Note: be sure to check your notation and formatting — refer to page 152 in GWC10.)
6. Suppose *Let* and *Num* are defined as follows:

$$Let == \{a, b, c, d, e\}$$

$$Num == \{1, 2, 3, 4, 5\}$$

- a. Give an example of each of the following:
 - i. A function whose declaration is $Let \rightarrow Num$
 - ii. A function whose declaration is $Let \leftrightarrow Num$
 - iii. A total injection from *Let* to *Num*
- b. Is it possible to give an example of a total injection from *Let* to $\{1, 2, 3, 4\}$? If so, provide one; if not, explain why not.

II. Proof: Equational Reasoning

You may use any of the following theorems in your equational proofs:

$\vdash p \wedge \text{true} \Leftrightarrow p$	\wedge -True
$\vdash p \wedge \text{false} \Leftrightarrow \text{false}$	\wedge -False
$\vdash p \vee \text{true} \Leftrightarrow \text{true}$	\vee -True
$\vdash p \vee \text{false} \Leftrightarrow p$	\vee -False
$\vdash p \vee \neg p$	Excluded Middle
$\vdash p \vee q \Leftrightarrow q \vee p$	\vee -Commutativity
$\vdash p \wedge q \Leftrightarrow q \wedge p$	\wedge -Commutativity
$\vdash (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	\vee -Associativity
$\vdash (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	\wedge -Associativity
$\vdash p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$\vee \wedge$ -Distributivity
$\vdash p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$\wedge \vee$ -Distributivity
$\vdash p \Rightarrow q \Leftrightarrow \neg p \vee q$	\Rightarrow -Alternative
$\vdash p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$	Contrapositives
$\vdash \neg \neg p \Leftrightarrow p$	Double Negation
$\vdash \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	De Morgan
$\vdash \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan
$\vdash x \in \emptyset \Leftrightarrow \text{false}$	\emptyset Membership

7. Prove in equational style the following laws for set union:

- $S \cup T = T \cup S$
- $S \cap \emptyset = \emptyset$

(HINT: To prove $S = T$ show $\forall x : U \bullet x \in S \Leftrightarrow x \in T$, where U is the type of elements in sets S and T .)

8. Prove the following theorem in equational style:

$$\vdash \neg(\neg p \Rightarrow (q \wedge r)) \Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$$

III. Sequences

9. Define the following sequences by enumeration:

- Threes*: natural numbers smaller than 18 that are divisible by 3.
- Twos*: natural numbers smaller than 20 that are divisible by 2.

10. Given the above, what are each of the following:

- $\text{Threes} \cup \text{Twos}$
- $\text{Threes} \cap \text{Twos}$
- $\text{dom } \text{Threes}$
- $\text{ran } \text{Threes}$
- $\text{dom } \text{Twos} \triangleleft \text{Threes}$
- $(5 \dots 8) \triangleleft (\text{Threes} \frown \text{Twos})$
(NOTE: \frown is the concatenation operator).

IV. Proof: Natural Induction

Prove the following claim by induction over the natural numbers:

$$0^2 + 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

V. Proof: Structural Induction

Consider the definition of binary trees in Chapter 7.

(a) Show that

$$\forall t : TREE \bullet leaves(t) = nodes(t) + 1$$

(b) Define a *mirror* function that recursively swaps the branches of a tree.

(c) Using the definition of *mirror* show that

$$\forall t : TREE \bullet size(mirror(t)) = size(t)$$

(d) Using the definition of *mirror* show that

$$\forall t : TREE \bullet mirror(mirror(t)) = t$$

(HINT: use structural induction over trees.)