Homework #3: Proof

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INSTRUCTIONS: before completing your homework, make sure you have considered the following:

- You are allowed to use only primitive inference rules.

- Remember to include vertical lines to represent the scope any assumptions.

- Remember that all assumptions in your proofs must be discharged.

- Double-check that your line references are correct when applying inference rules.

I. Proof: Propositional Logic

In this section *use only the primitive inference rules* of propositional calculus.

1. Provide derivations for each of the following, using natural deduction:

   a. $p \wedge q, p \Rightarrow s, q \Rightarrow t \vdash s \wedge t$
1.	$p \wedge q$	premise
2.	$p \Rightarrow s$	premise
3.	$q \Rightarrow t$	premise
4.	p	$\wedge$-elim, 1
5.	s	modus ponens, 2,4
6.	q	$\wedge$-elim, 1
7.	t	modus ponens, 3,6
8.	$s \wedge t$	$\wedge$-intro, 5,7

   b. $q \Rightarrow \neg p, p \wedge q \vdash r$
1.	$q \Rightarrow \neg p$	premise	
2.	$p \wedge q$	premise	
3.	q	$\wedge - elim, 2$	
4.	$\neg p$	modus ponens, 1	
5.	p	$\wedge - elim, 2$	
6.	$\neg r$	assumption	l
7.	$\neg p$	Copy from 4	l
8.	p	Copy from 5	l
4.	r	$\neg - elim$, 6-8	

   c. $p \wedge q \vdash p \Rightarrow q$
1.	$p \wedge q$	assuption	l
2.	p	$\wedge - elim$,1	l
3.	q	$\wedge - elim$,1	l
4.	$p \Rightarrow q$	$\Rightarrow - intro$,1-3,2,3	

d. $\neg\neg q \vdash q \lor r$

    1. $\neg\neg q$ assuption   1
    2. $\neg q$ $\neg - elim, 1$ 1
    3. q $\neg - elim, 2$ 1
    4. $q \lor r$ $\lor - intro, 1 - 3$

e. $p \Rightarrow (q \land r), \neg p \Rightarrow r, p \lor \neg p \vdash r$

    1. $\neg p \Rightarrow r$ premise
    2. $p \lor \neg p$ assumption   1
    3. p $\lor$-elim, 2   1
    4. $p \Rightarrow (q \land r)$ assumption   1   1
    5. $(q \land r)$ modus ponens,4,3   1   1
    6. r $\land - elim$, 2-5

II. Proof: Predicate Logic

In this section *use only the primitive inference rules* of predicate calculus.

2. Show using natural deduction:

a. $\forall x : T \bullet P(x) \land Q(x) \dashv\vdash (\forall x : T \bullet P(x)) \land (\forall y : T \bullet Q(y))$

    $\forall x : T \bullet P(x) \land Q(x) \vdash (\forall x : T \bullet P(x)) \land (\forall y : T \bullet Q(y))$

    1. $\forall x : T \bullet P(x) \land Q(x)$ assumption   1
    2. $x \in T$ assumption   1   1
    3. $P(x) \land Q(x)$ $\forall - elim$, 1   1   1
    4. $P(x)$ $\land - elim$, 3   1   1
    5. $Q(x)$ $\land - elum$, 3   1   1
    6. $y \in T$ assumption   1   1   1
    7. $Q(y)$ transforming x to y, 5,6   1   1   1
    8. $\forall y : T \bullet Q(y)$ $\forall - intro$ 6,7   1   1
    9. $\forall x : T \bullet P(x)$ $\forall - intro$ 2,4   1   1
    10. $\forall x : T \bullet P(x)) \land (\forall y : T \bullet Q(y))$ $\land - intro$ 8,9

    $(\forall x : T \bullet P(x)) \land (\forall y : T \bullet Q(y)) \vdash \forall x : T \bullet P(x) \land Q(x)$

    1. $(\forall x : T \bullet P(x)) \land (\forall y : T \bullet Q(y))$ assumption   1
    2. $(\forall x : T \bullet P(x)$ 1
    3. $(\forall y : T \bullet Q(y))$ 1
    4. $a \in T$ assumption   1   1
    5. $b \in T$ assumption   1   1   1
    6. $P(a)$ $\forall - elim, 2, 4$ 1   1   1
    7. $Q(b)$ $\forall - elim, 3, 5$ 1   1   1
    8. $P(a) \land Q(b)$ $\land - intro, 6, 7$ 1   1   1
    9. $\forall x : T \bullet P(x) \land Q(x)$ $\forall - intro, 4 - 8$

b. $\exists x : T \bullet P(x) \lor Q(x) \dashv\vdash (\exists x : T \bullet P(x)) \lor (\exists x : T \bullet Q(x))$

    $\exists x : T \bullet P(x) \lor Q(x) \vdash (\exists x : T \bullet P(x)) \lor (\exists x : T \bullet Q(x))$

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1. $\exists\, x : T \bullet P(x) \vee Q(x)$ assumption   1

1. $\exists\, x : T \bullet P(x) \vee Q(x)$	assumption	1		
2. $a \in T \wedge P(a) \vee Q(a)$	assumption	1	1	
3. $P(a) \vee Q(a)$	$\wedge-elim, 2$	1	1	
4. $P(a)$	$\vee-elim, 3$	1	1	
5. $Q(a)$	$\vee-elim, 3$	1	1	
6. $a \in T$	$\wedge-elim, 2$	1	1	
6. $\exists\, x : T \bullet P(x)$	$\exists-intro, 6, 4$	1	1	
7. $\exists\, x : T \bullet Q(x)$	$\exists-intro, 6, 5$	1	1	
8. $(\exists\, x : T \bullet P(x)) \vee (\exists\, x : T \bullet Q(x))$	$\vee-intro, 6, 7$			

$(\exists\, x : T \bullet P(x)) \vee (\exists\, x : T \bullet Q(x)) \vdash \exists\, x : T \bullet P(x) \vee Q(x)$

1. $(\exists\, x : T \bullet P(x)) \vee (\exists\, x : T \bullet Q(x))$	assumption	1		
2. $\exists\, x : T \bullet P(x)$	$\vee-elim, 1$	1		
3. $\exists\, x : T \bullet Q(x)$	$\vee-elim, 1$	1		
4. $a \in T \wedge P(a)$	assumption	1	1	
5. $P(a)$	$\wedge-elim\ 4$	1	1	
6. $b \in T \wedge Q(b)$	assumption	1	1	1
7. $Q(b)$	$\wedge-elim\ 6$	1	1	1
8. $b \in T$	$\wedge-elim\ 6$	1	1	1
9. $P(a) \vee Q(b)$	$\vee-intro, 2-4$	1	1	1
10. $\exists\, x : T \bullet P(x) \vee Q(x)$	$\exists-intro, 8, 9$	1	1	
11. $\exists\, x : T \bullet P(x) \vee Q(x)$	$\exists-elim, 2, 4, 10$			
12. $\exists\, x : T \bullet P(x) \vee Q(x)$	$\exists-elim, 3, 6-10$			

(NOTE: $p \dashv\vdash q$ is a shorthand for "$p \vdash q$ and $q \vdash p$." That is, for $p \dashv\vdash q$ you need to show two separate derivations: one for $p \vdash q$ and another for $q \vdash p$.)