CMU 17-651 | Models of Software Systems | Fall 2018

Homework #4:
Structures, Equational Reasoning, and Induction

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I. Sets, Relations, and Functions

1. Use set comprehension to define the set $SumOfSquares$ containing all the natural numbers that can be expressed as the sum $a^2 + b^2$ where a and b are natural numbers.
   $\mathit{SumOfSquares} == \{x : \mathbb{N} \mid (\exists\, a : \mathbb{N})(\exists\, b : \mathbb{N}) \bullet x = a^2 + b^2\}$

2. Write out in full the powersets of each of the following.
   a. $\mathbb{P}\{7, 1\} = \{\emptyset, \{1\}, \{7\}, \{7, 1\}\}$
   b. $\mathbb{P}\{5\} = \{\emptyset, \{5\}\}$
   c. $\mathbb{P}\emptyset = \{\emptyset\}$
   d. $\mathbb{P}\{\emptyset\} = \{\emptyset, \{\emptyset\}\}$
3. Write out in full the following Cartesian products.
   a. $\{4, 2\} \times \{2, 4\}$
   b. $\{0\} \times \emptyset$
   c. $\{1, 2\} \times \{a\}$
   d. $\{\emptyset\} \times \{a\}$
4. Suppose $R == 2 \mathinner{..} 5$ and $S == 4 \mathinner{..} 6$. Enumerate the elements of the following sets.
   a. $R \cup S$
   b. $R \cap S$
   c. $R \setminus S$
   d. $S \setminus R$
   e. $S \times R$
5. Let S be the set of numbers from 1 to 12 inclusive. Let R be a relation, such that $R : S \leftrightarrow S$ and such that x is related to y exactly when y is greater than the square of x but less than the square of $x + 1$. Provide an axiomatic definition for R.
   (Note: be sure to check your notation and formatting — refer to page 152 in GWC10.)
6. Suppose Let and Num are defined as follows:

$$\begin{aligned} Let &== \{a, b, c, d, e\} \\ Num &== \{1, 2, 3, 4, 5\} \end{aligned}$$

   a. Give an example of each of the following:
      i. A function whose declaration is $Let \rightarrow Num$
      ii. A function whose declaration is $Let \nrightarrow Num$
      iii. A total injection from Let to Num
   b. Is it possible to give an example of a total injection from Let to $\{1, 2, 3, 4\}$? If so, provide one; if not, explain why not.

II. Proof: Equational Reasoning

You may use any of the following theorems in your equational proofs:

$\vdash p \wedge true \Leftrightarrow p$	$\wedge$-True
$\vdash p \wedge false \Leftrightarrow false$	$\wedge$-False
$\vdash p \vee true \Leftrightarrow true$	$\vee$-True
$\vdash p \vee false \Leftrightarrow p$	$\vee$-False
$\vdash p \vee \neg p$	Excluded Middle
$\vdash p \vee q \Leftrightarrow q \vee p$	$\vee$-Commutativity
$\vdash p \wedge q \Leftrightarrow q \wedge p$	$\wedge$-Commutativity
$\vdash (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$\vee$-Associativity
$\vdash (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	$\wedge$-Associativity
$\vdash p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$\vee\wedge$-Distributivity
$\vdash p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$\wedge\vee$-Distributivity
$\vdash p \Rightarrow q \Leftrightarrow \neg p \vee q$	$\Rightarrow$-Alternative
$\vdash p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$	Contrapositives
$\vdash \neg\neg p \Leftrightarrow p$	Double Negation
$\vdash \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	De Morgan
$\vdash \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan
$\vdash x \in \emptyset \Leftrightarrow false$	$\emptyset$ Membership

7. Prove in equational style the following laws for set union:

   a. $S \cup T = T \cup S$

   b. $S \cap \emptyset = \emptyset$

   (HINT: To prove $S = T$ show $\forall x : U \bullet x \in S \Leftrightarrow x \in T$, where U is the type of elements in sets S and T.)

8. Prove the following theorem in equational style:

$$\vdash \neg(\neg p \Rightarrow (q \wedge r)) \Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$$

III. Sequences

9. Define the following sequences by enumeration:

   a. *Threes*: natural numbers smaller than 18 that are divisible by 3.

   b. *Twos*: natural numbers smaller than 20 that are divisible by 2.

10. Given the above, what are each of the following:

    a. $Threes \cup Twos$

    b. $Threes \cap Twos$

    c. $\operatorname{dom} Threes$

    d. $\operatorname{ran} Threes$

    e. $\operatorname{dom} Twos \lhd Threes$

    f. $(5 \mathinner{..} 8) \lhd (Threes \frown Twos)$
    (NOTE: $\frown$ is the concatenation operator).

IV. Proof: Natural Induction

Prove the following claim by induction over the natural numbers:

$$0^2 + 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

V. Proof: Structural Induction

Consider the definition of binary trees in Chapter 7.

(a) Show that

$$\forall\, t : \mathit{TREE} \bullet \mathit{leaves}(t) = \mathit{nodes}(t) + 1$$

(b) Define a *mirror* function that recursively swaps the branches of a tree.

(c) Using the definition of *mirror* show that

$$\forall\, t : \mathit{TREE} \bullet \mathit{size}(\mathit{mirror}(t)) = \mathit{size}(t)$$

(d) Using the definition of *mirror* show that

$$\forall\, t : \mathit{TREE} \bullet \mathit{mirror}(\mathit{mirror}(t)) = t$$

(Hint: use structural induction over trees.)