Digital Communications and Laboratory
Second Homework

Faccin Dario, Santi Giovanni

Problem 1

The following system was given:

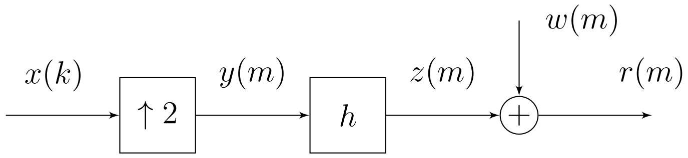

Figure 1. Model for the transmission system of Problem 1.

The parameters are as follow:

- $y(m) = \begin{cases} x(m/2) & \text{if } m \text{ is even} \\ 0 & \text{otherwise} \end{cases}$
- $z(m) = -a_1 z(m-1) - a_2 z(m-2) + y(m)$, $m = 0, 1, \ldots$, with initial values $z(-1) = z(-2) = 0$ and coefficients $a_1 = -0.9635$ and $a_2 = 0.4642$;
- noise samples iid with $w(m) \sim \mathcal{N}(0, \sigma_w^2)$, $\sigma_w^2 = -8$ dB;
- $r(m) = z(m) + w(m)$.

We assumed the receiver to know the input signal $\{x(k)\}$ and a bound on the length of h, respectively $N_h \leq 20$. Note that in this way we approximate the IIR filter with a FIR filter of order N_h. In order to estimate the channel, i.e. the impulse response $\hat{h}_i$, $i = 0, 1, \ldots, N-1$, we exploited the *polyphase decomposition* of scheme in Figure [1] deriving from the first noble identity. This step is necessary, since we know how to estimate the channel only if both input and output of the system have the same period. The scheme is shown in Figure [2].

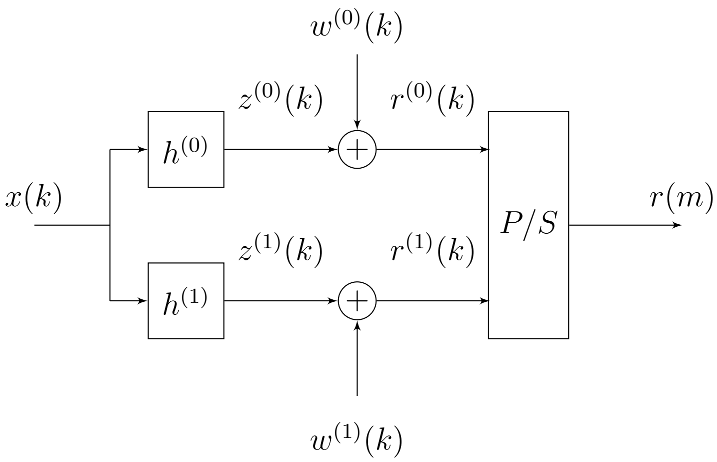

Figure 2. Polyphase decomposition of h.

Since the system is described by a FIR filter, this is a linear channel estimation problem that is solved by taking as input a PN sequence with period L and statistical power $r_x(0)$, $\{p(i)\}$, $i = 0, 1, \ldots, L-1$. In this way, infact, the cross-correlation between the output signal d and the input x is proportional with a factor $r_x(0)$ to the impulse response h_i, respectively:

$$r_{dx}(n) = r_{zx}(n) = r_x * h(n) = r_x(0) \cdot h_n$$

We recall that the autocorrelation of a PN sequence is periodic with period L, thus even the output of the time-invariant filter is periodic with the same period. In the following analysis we explain how to estimate only the first polyphase component of h, h_0, since the other component is estimated using the same procedure. The model we used is given in Figure [3].

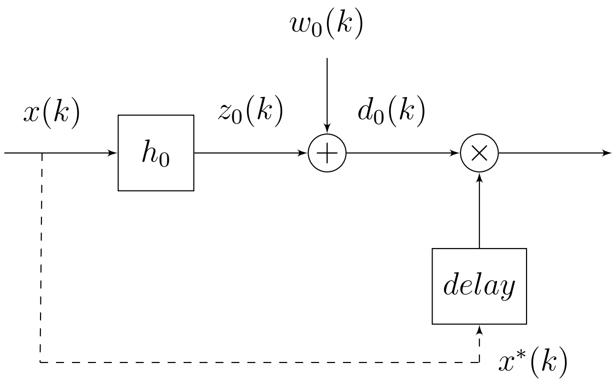

Figure 3. Correlation method to estimate the first polyphase component of the system.

The input signal is a Maximal-length PN sequence of length L repeated once. Since h_0 is defined only on the even values of k, we implicitly consider all involved signal to be defined in the same intervals. Since the output of the FIR filter is affected by a transient, $z(k)$ avoids the first L samples. The signal at the output of the unknown system is $d_0(k) = z_0(k) + w_0(k)$.

Correlation method

According to the model of Figure [3], the correlation method computes the coefficients h_i, $i = 0, 1, \dots, N-1$ exploiting the cross-correlation between the output signal $d_0(k)$ and the input one delayed of m samples, respectively:

$$\hat{r}_{dx,0}(m) = \frac{1}{L} \sum_{k=L-1}^{2L-2} d(k) x^*(k-m) \simeq h_m \qquad m = 0, 1, \dots, N-1 \tag{1}$$

According to Figure [2], once computed both $\hat{h}_0$ and $\hat{h}_1$, $\hat{h}$ is just the P/S representation of the two.
An example of the two estimates with $N_h = 20$ and $L = 127$ is given in Figure [4].

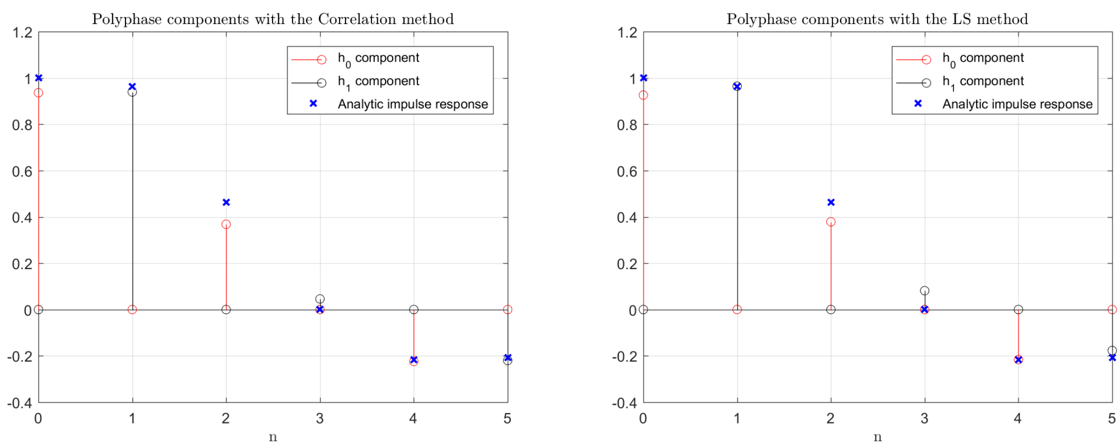

Figure 4. Estimates of the Polyphase components $\hat{h}_0$ and $\hat{h}_1$ computed with the correlation method (left) and with the least-square method (right).

Least-Square method

With reference to the model of Figure [3], the noisy output of the unknown system can be written as:

$$d(k) = \mathbf{h}^T \mathbf{x}(k) + w(k) \qquad k = L-1, ..., 2L-2$$

For a certain estimate of $\hat{\mathbf{h}}$ of the unknown system, the cost function is given by

$$\mathcal{E} = \sum_{k=L-1}^{2L-2} |d(k) - \hat{d(k)}| \tag{2}$$

where $\hat{d}(k) = \mathbf{h}^T \mathbf{x}(k)$. This is minimized by

$$\hat{\mathbf{h}}_{LS} = \underset{\hat{h}}{\operatorname{argmin}} \, \mathcal{E} = (\mathcal{I}^{\mathcal{H}} \mathcal{I})^{-1} \mathcal{I}^{\mathcal{H}} o \tag{3}$$

where $\mathcal{I}$ is the *observation matrix* and o is the *desired sample vector*, respectively:

$$\mathcal{I} = \begin{bmatrix} x(L-1) & \dots & x(0) \\ \vdots & \vdots & \ddots \\ x((L-1)+(L-1)) & \dots & x(L-1) \end{bmatrix}$$

$$o^T = [d(L-1) \dots d((L-1)+(L-1))]$$

Conclusions

To estimate the noise variance $\hat{\sigma}_w^2$, we computed the cost function given in Equation (2) for both methods. Since the error

$$d(k) - \hat{d}(k) = (z(k) - \hat{d}(k)) + w(k)$$

consist of two terms, one due to the estimation error and the other due to the noise of the system, for $\hat{\mathbf{h}} \simeq \mathbf{h}$ the estimate of the variance of the noise w can be assumed equal to

$$\hat{\sigma}_w^2 = \frac{\mathcal{E}}{L}$$

We run the algorithm using $L \in \{31, 63, 127, 255, 511, 1023\}$ and varying N_i, $i = 1, 2, \dots, 20$. Note that, in order to guarantee that the condition $L \gg N$ holds, we a priori excluded the first three possible vales for the length of the PN sequence, i.e. $L = 3, 7, 15$. The results are shown in Figure [5].
At the receiver side, the rationale behind the choice of the L, N parameters that would be the most efficient to estimate the channel impulse response is given by the following facts: first of all, the functional $\frac{\mathcal{E}}{L}$ should be as near as possible to the performance bound σ_w^2, meaning that the error due to the estimate is negligible with respect to the error due to the noise; secondly, the trend of $\frac{\mathcal{E}}{L}$ as N grows should be decreasing until $N = N_h$, and then it should be approximately constant. At the same time, a smaller N should result in a higher computational speed.
For all these reason we chose $L = 63$ and $N = 6$. This is graphically shown in Figure [6], where a zoomed version of the Figure [5] is given. The values of $\{h_i\}$ and $\{\hat{h}_i\}$, $i = 1, \dots, 6$ are reported in Table [1]. The variance of the estimates are respectively $\hat{\sigma}_h^2 = -8.1108$ for the correlation method and $\hat{\sigma}_h^2 = -8.0559$ for the ls method.

N	1	2	3	4	5	6
h	1.0000	0.9635	0.4641	-0.0001	-0.2155	-0.2076
$\hat{h}_{cor}$	0.9360	0.9394	0.3682	0.0459	-0.2240	-0.2200
$\hat{h}_{ls}$	0.9260	0.9645	0.3787	0.0814	-0.2159	-0.1768

Table 1. Coefficients of the impulse responses and estimated variances.

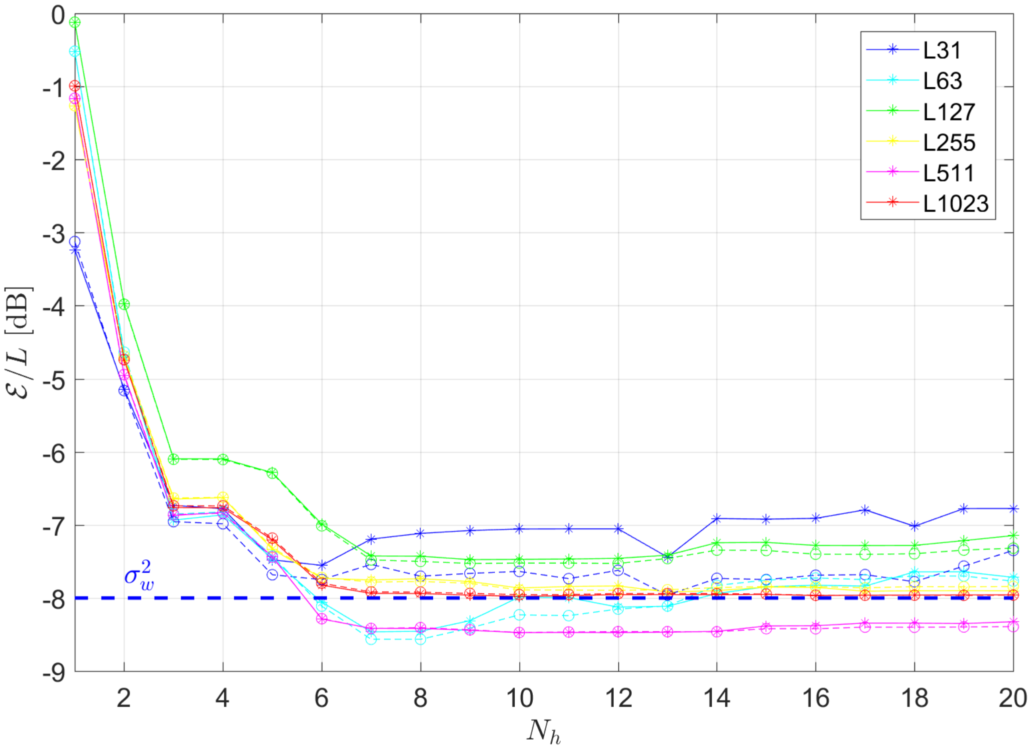

Figure 5. Estimate of noise variance $\hat{\sigma}_w^2$ as a function of N, L using the Correlation Method (dashed lines) and the Least-Square Method (solid lines).

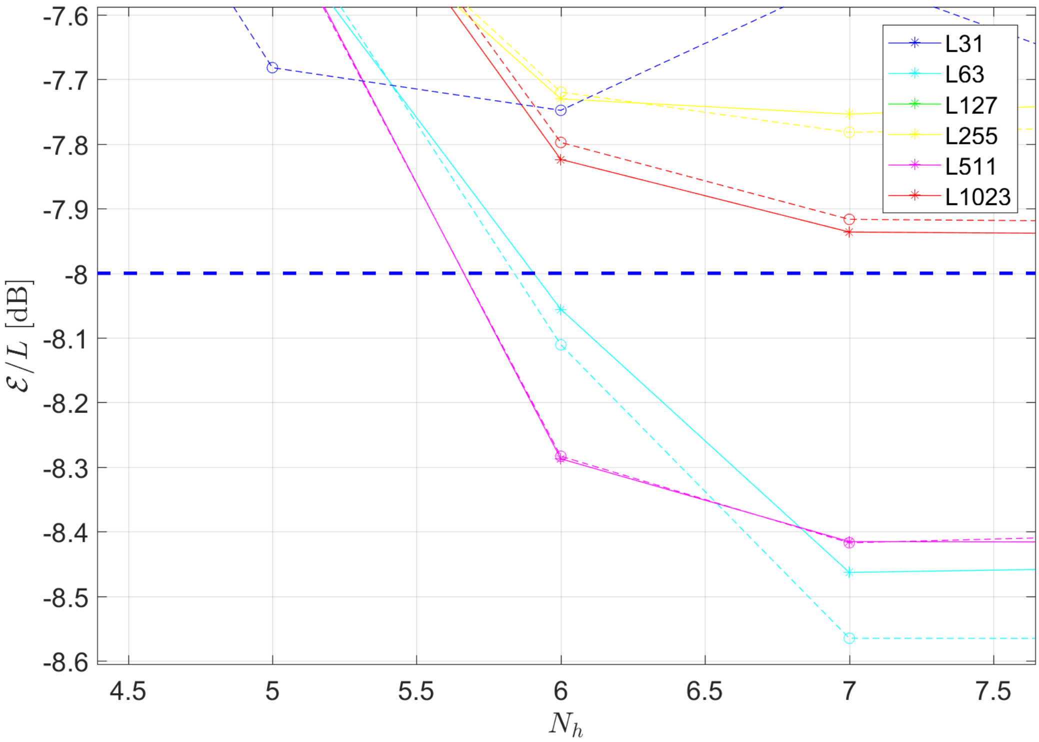

Figure 6. Zoom of Figure [5].

Problem 2

A flat fading channel with only one tap $h_0(nT_c)$ was studied, assuming a *Rice factor* of k=2 dB and normalized M_{h_0}. Moreover, a classical *Doppler Spectrum* with $f_d T_c = 40 \cdot 10^{-5}$ was considered. The schematic model to generate the coefficient h_0 of the channel is given in Figure 7.

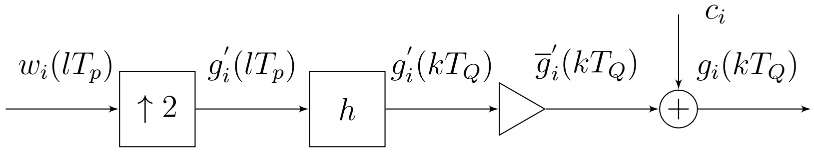

Figure 7. Model to generate the coefficient h_0 of the time-varying channel.

The Doppler Spectrum can be generated using a filter h_{ds} such that $|\mathcal{H}_{ds}(f)|^2 = D(f)$.
In Table [2] are shown the coefficients used for such filter [1]:

$H_{ds}(z) = B(z)/A(z)$	$f_d T_p = 0.1$		
$\{a_n\}$,	$n = 0, \ldots, 11$:		
1.0000	-4.4153	8.6283	-9.4592
6.1051	-1.3542	-3.3622	7.2390
-7.9361	5.1221	-1.8401	2.8706e-1
$\{b_n\}$,	$n = 0, \ldots, 21$:		
1.3651e-4	8.1905e-4	2.0476e-3	2.7302e-3
2.0476e-3	9.0939e-4	6.7852e-4	1.3550e-3
1.8076e-3	1.3550e-3	5.3726e-4	6.1818e-5
-7.1294e-5	-9.5058e-5	-7.1294e-5	-2.5505e-5
1.3321e-5	4.5186e-5	6.0248e-5	4.5186e-5
1.8074e-5	3.0124e-6		

Table 2. Coefficients for the IIR filter.

The graphical representation of the impulse response of the IIR filter and the Doppler Spectrum is shown in Figure [8]. To obtain h_0, following the scheme of Figure [7], the noise component $w \sim \mathcal{CN}(0,1)$ is filtered with the IIR filter previously described. Note that the frequency response of this filter is $\mathcal{H}_{ds}(f) = \sqrt{\mathcal{D}(f)}$ while the PSD of the noise is constant and equal to 1. For this reason, the equivalent impulse response of this part is equal to $\mathcal{D}(f) = 1 \cdot |\mathcal{H}_{ds}|^2$ which is actually the Doppler spectrum.
The output of the filter is affected by a transient, which we avoided by considering only values after $5N_{eq}T_p$, where $N_{eq} = \left\lceil -\frac{1}{\ln(|p|)} \right\rceil$ is the equivalent time constant, and p is the pole with the highest magnitude. Then, after scaling the coefficient such that $M_{h_0}/\sqrt{E_{h_{ds}}} = 1$, the signal is filtered with an interpolation filter of factor $1/T_Q = T_p/T_c = 250$.
The interpolator output signal is multiplied by a constant $\sigma_0 = \sqrt{M_d}$ to impose the desired power delay profile. Finally the signal is added with another constant, C, which includes the deterministic component according to [1], Page 307. The final signal is given in Figure [9].

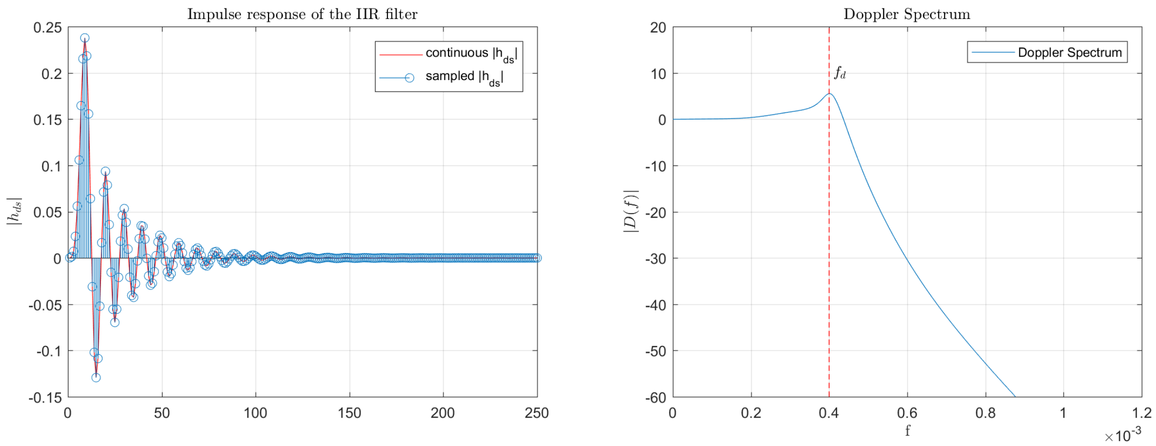

Figure 8. Impulse response of the IIR filter and Doppler Spectrum.

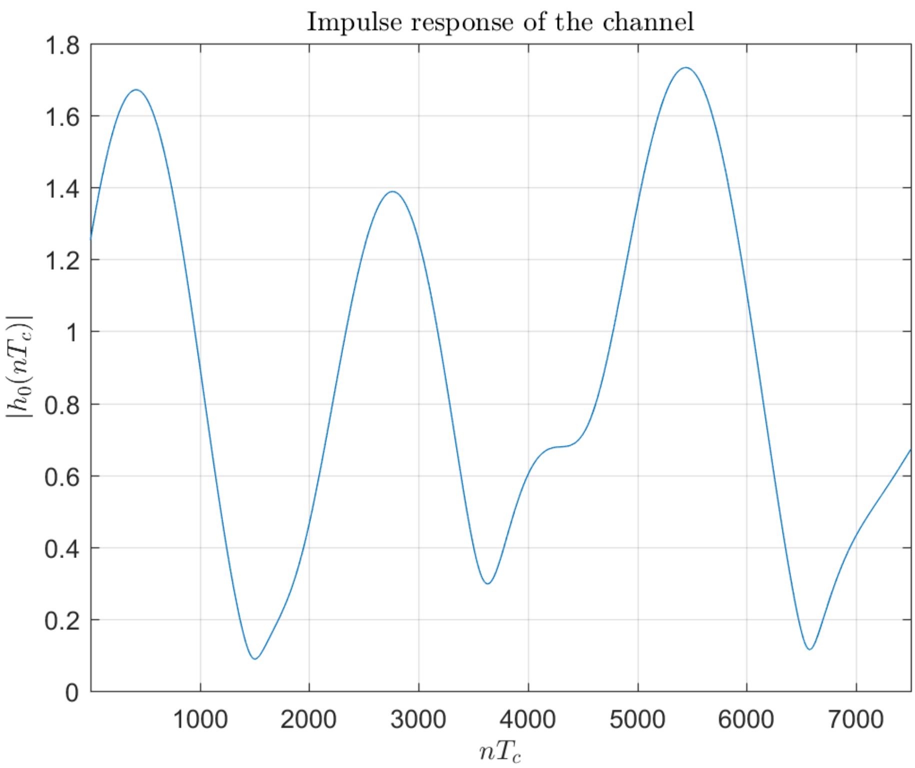

Figure 9. Magnitude of the simulated h_0 for 7500 samples.

PDF of $\frac{|\mathbf{h_0}|}{\sqrt{\mathbf{M}_{|\mathbf{h_0}|}}}$

The signal $h' = \frac{h_0}{\sqrt{M_{|h_0|}}}$ for 80000 samples is now studied. Note that, according to Figure [7], h' contains a deterministic component in addiction to a random component, which is complex Gaussian with zero-mean and variance equal to one. For this reason the *pdf* of $|h'|$ is a Rice distribution given by

$$p_{|h'|} = \begin{cases} 2(1+K)ae^{-K-(1+K)a^2} I_0(2a\sqrt{K(1+K)}) & a \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where I_0 is the *modified Bessel function of the first type and order zero*, respectively

$$I_0 = \frac{1}{2\pi} \int_{-\pi}^{pi} e^{x\cos\alpha} d\alpha$$

The histogram of h' is shown in Figure [10]. Here it is given also the theoretical *pdf* evaluated according to Equation (4).

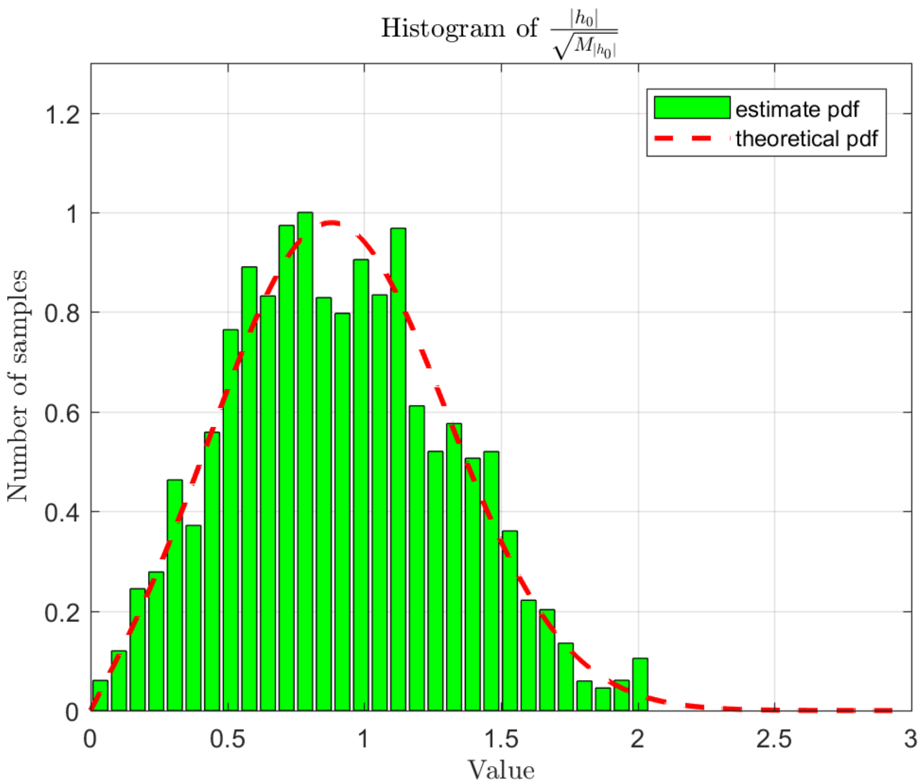

Figure 10. Plot of both estimate and theoretical curve of the pdf of h'.

Spectrum of $\mathbf{h_0}$

In this section the spectrum of h_0 is computed using the Welch Periodogram. This method extracts different subsequences of consecutive D samples which eventually overlap, and for each of these it computes the periodogram $\mathcal{P}^{(s)}_{PER}(f)$. We chose $D = 4000$ and $S = 20000$ in our analysis. The mathematical model is given by

$$\mathcal{P}_{WE}(F) = \frac{1}{N_s} \sum_{x=0}^{N_s - 1} \mathcal{P}^{(s)}_{PER}(f)$$

where $N_s = \lfloor \frac{K-D}{D-S} - 1 \rfloor = 2$ is the total number of subsequences.
In order to compare the estimate with the theoretical case, the ideal PSD is computed. By definition, the *Doppler Spectrum* is the Fourier Transform of the autocorrelation function of the impulse response, in correspondence of the same delay τ, evaluated at two different instants. In out analysis, it is provided as IIR filter resulting from the Jakes model, and the coefficients are reported in Table [8]. Note that this is actually an approximation deriving from an empirical model such Jakes model is. The deterministic component determines a spectral line on the central frequency, as shown in Figure [11].
As a remark, notice that both curves are normalized according to the specifics of our problem.

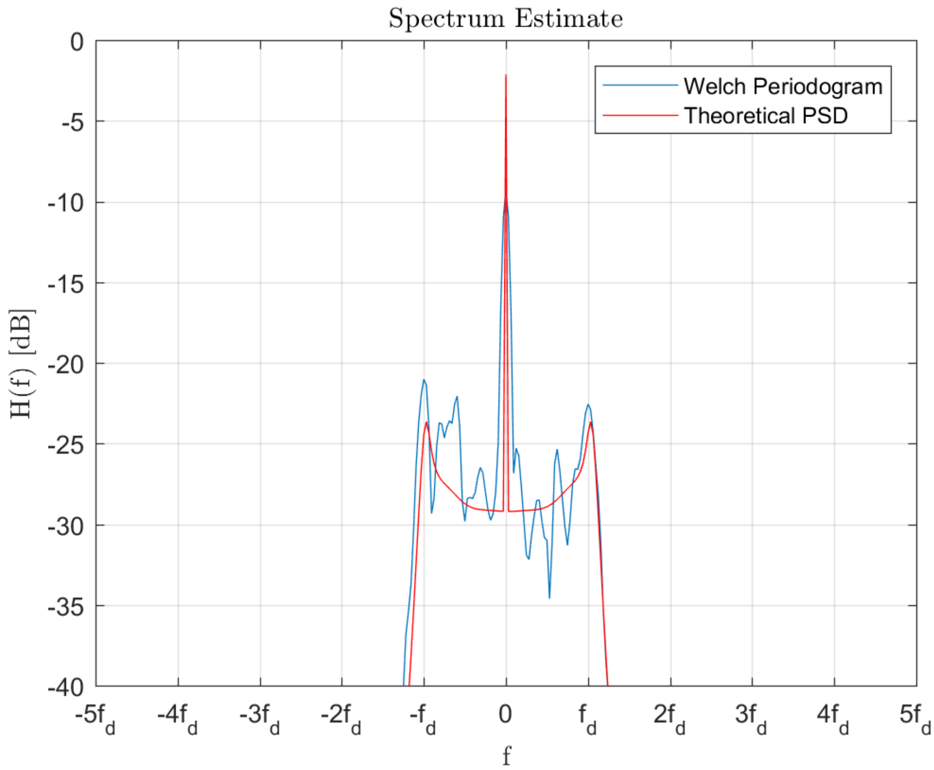

Figure 11. PSD estimate of h_0 with the theoretical curve.

Bibliography

[1] Nevio Benvenuto, Giovanni Cherubini, *Algorithms for Communication Systems and their Applications*. Wiley, 2002.