

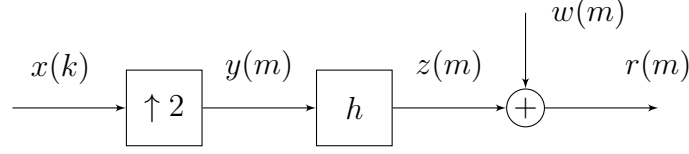
# Digital Communications and Laboratory

## Second Homework

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# Problem 1

The following system was given:

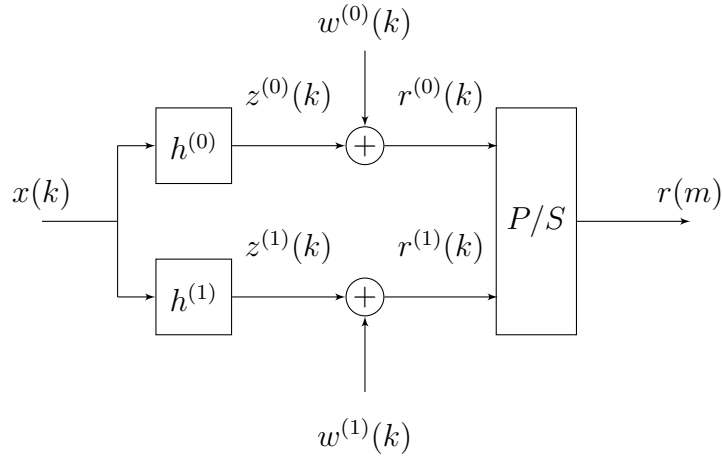


**Figure 1.** Model for the transmission system of Problem 1.

The parameters are as follow:

- $y(m) = \begin{cases} x(m/2) & \text{if } m \text{ is even} \\ 0 & \text{otherwise} \end{cases}$
- $z(m) = -a_1 z(m-1) - a_2 z(m-2) + y(m)$ ,  $m = 0, 1, \dots$ , with initial values  $z(-1) = z(-2) = 0$  and coefficients  $a_1 = -0.9635$  and  $a_2 = 0.4642$ ;
- noise samples iid with  $w(m) \sim \mathcal{N}(0, \sigma_w^2)$ ,  $\sigma_w^2 = -8$  dB;
- $r(m) = z(m) + w(m)$ .

We assumed the receiver to know the input signal  $\{x(k)\}$  and a bound on the length of  $h$ , respectively  $N_h \leq 20$ . In order to estimate the channel, i.e. the impulse response  $\hat{h}_i$ ,  $i = 0, 1, \dots, N-1$ , we exploited the *polyphase decomposition* of scheme in Figure [1] deriving from the first noble identity. This is shown in Figure [2].



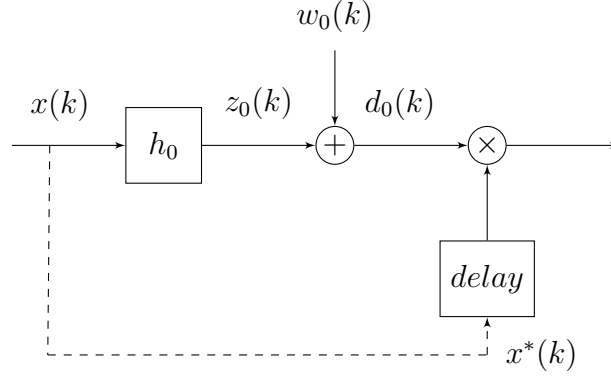
**Figure 2.** Polyphase decomposition of  $h$ .

Since the system is described by a FIR filter, this is a linear channel estimation problem that is solved by taking as input a PN sequence with period  $L$  and statistical power  $r_x(0)$ ,  $\{p(i)\}$ ,  $i = 0, 1, \dots, L-1$ . In this way, infact, the cross-correlation between the output signal  $d$  and the input  $x$  is proportional with a factor  $r_x(0)$  to the impulse response  $h_i$ , respectively:

$$r_{dx}(n) = r_{zx}(n) = r_x * h(n) = r_x(0) \cdot h_n$$

We recall that the autocorrelation of a PN sequence is periodic with period  $L$ , thus even the output of the time-invariant filter is periodic with the same period. In the following analysis we

explain how to estimate only the first polyphase component of  $h$ ,  $h_0$ , since the other component is estimated using the same procedure. The model we used is given in Figure [3]



**Figure 3.** Correlation method to estimate the first polyphase component of the system

The input signal is a Maximal-length PN sequence of length  $L$  repeated once. Since  $h_0$  is defined only on the even values of  $k$ , we implicitly consider all involved signal to be defined in the same intervals. The output of the FIR filter is affected by a transient, then  $z(k)$  avoids the first  $L$  samples. The signal at the output of the unknown system is  $d_0(k) = z_0(k) + w_0(k)$ .

## Correlation method

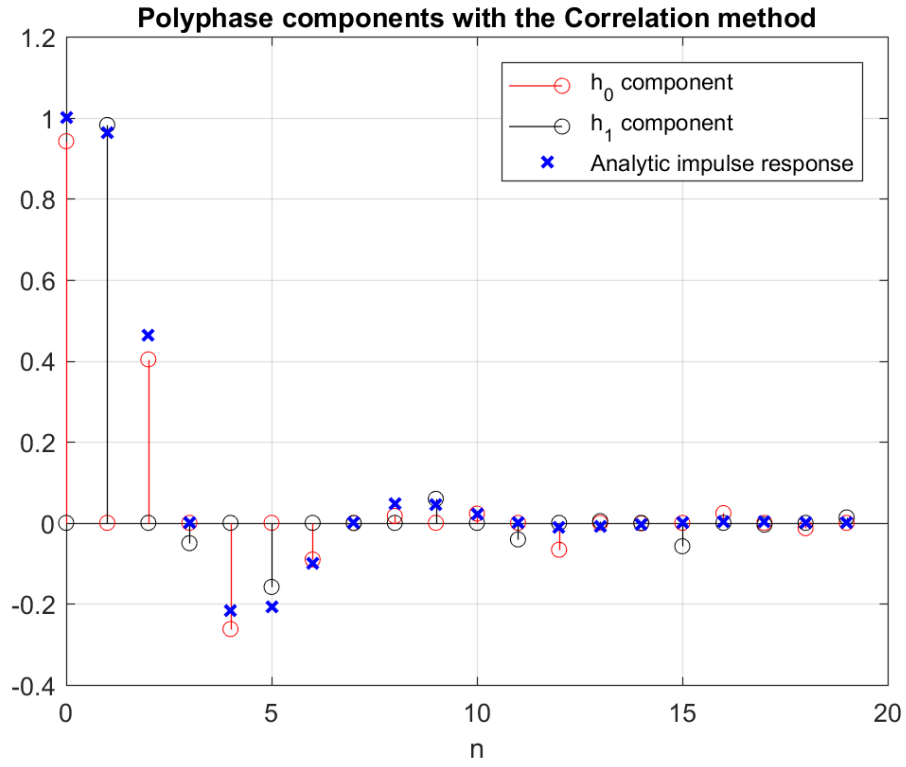
According to the model of Figure [3], the correlation method computes the coefficients  $h_i$ ,  $i = 0, 1, \dots, N - 1$  exploiting the cross-correlation between the output signal  $d_0(k)$  and the input one delayed of  $m$  samples, respectively:

$$\hat{r}_{dx,0}(m) = \frac{1}{L} \sum_{k=L-1}^{2L-2} d(k)x^*(k-m) \simeq h_m \quad m = 0, 1, \dots, N - 1 \quad (1)$$

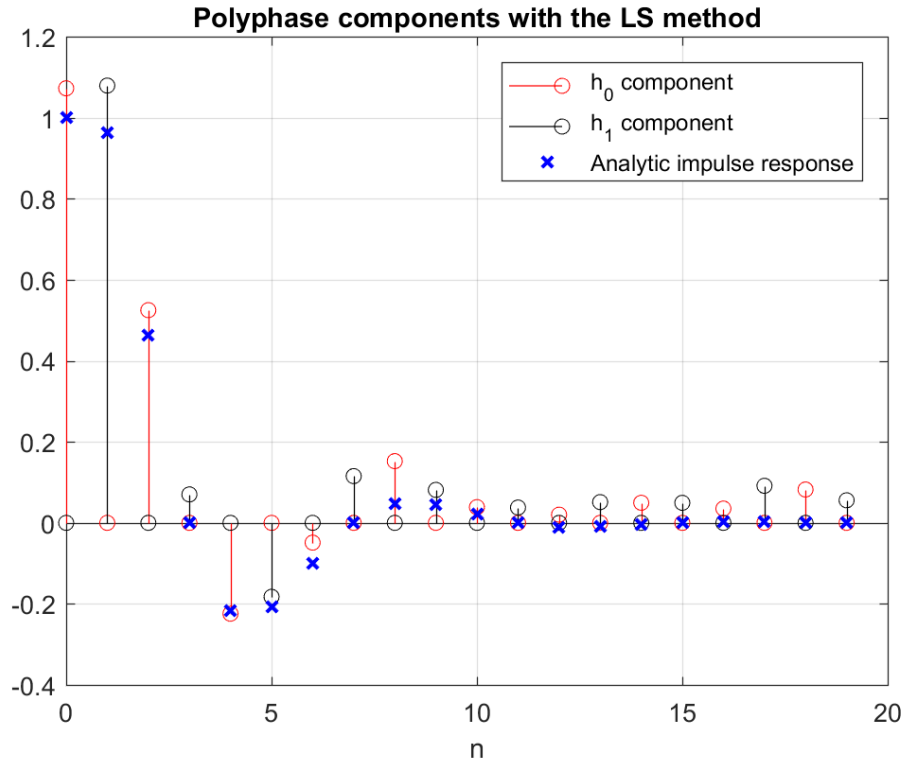
According to Figure [2], once computed both  $\hat{h}_0$  and  $\hat{h}_0$ ,  $\hat{h}$  is just the PS representation of the two. The variance of the estimate is:

$$\hat{\sigma}_w^2 = \text{var}[\hat{r}_{dx}(m)] \simeq \frac{\sigma_w^2}{L}$$

An example of the two estimates with  $N_h = 20$  and  $L = 127$  is given in the following Figures.



**Figure 4.** Estimates of the Polyphase components  $\hat{h}_0$  and  $\hat{h}_1$  computed with the correlation method.



**Figure 5.** Estimates of the Polyphase components  $\hat{h}_0$  and  $\hat{h}_1$  computed with the least-square method.

## Least-Square method

With reference to the model of Figure [3], the noisy output of the unknown system can be written as:

$$d(k) = \mathbf{h}^T \mathbf{x}(k) \quad k = L - 1, \dots, 2L - 2$$

For a certain estimate of  $\hat{\mathbf{h}}$  of the unknown system, the cost function is given by

$$\mathcal{E} = \sum_{k=L-1}^{2L-2} |d(k) - d(\hat{k})| \quad (2)$$

where  $d(\hat{k}) = \mathbf{h}^T \mathbf{x}(k)$ . This is minimized by

$$\hat{\mathbf{h}}_{LS} = \underset{\hat{\mathbf{h}}}{\operatorname{argmin}} \mathcal{E} = (\mathcal{I}^H \mathcal{I})^{-1} \mathcal{I}^H o \quad (3)$$

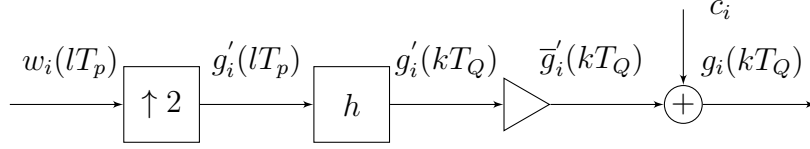
where  $\mathcal{I}$  is the *observation matrix* and  $o$  is the *desired sample vector*, respectively:

$$\mathbf{I} = \begin{bmatrix} x(L-1) & \dots & x(0) & \\ \vdots & \vdots & \ddots & \vdots \\ x((L-1) + (L-1)) & \dots & x(L-1) & \end{bmatrix}$$

$$o^T = [d(L-1) \dots d((L-1) + (L-1))]$$

## Problem 2

A flat fading channel with only one tap  $h_0(nT_c)$  was studied, assuming a *Rice factor* of  $k=2$  dB and normalized  $M_{h_0}$ . Moreover, a classical *Doppler Spectrum* with  $f_d T_c = 40 \cdot 10^{-5}$  was considered. The schematic model to generate the coefficient  $h_0$  of the channel is given in Figure 6.



**Figure 6.** Model to generate the coefficient  $h_0$  of the time-varying channel.

The Doppler Spectrum can be generated using a filter  $h_{ds}$  such that  $|\mathcal{H}_{ds}(f)|^2 = D(f)$ . In Table 1 are shown the coefficients used for such filter [1]:

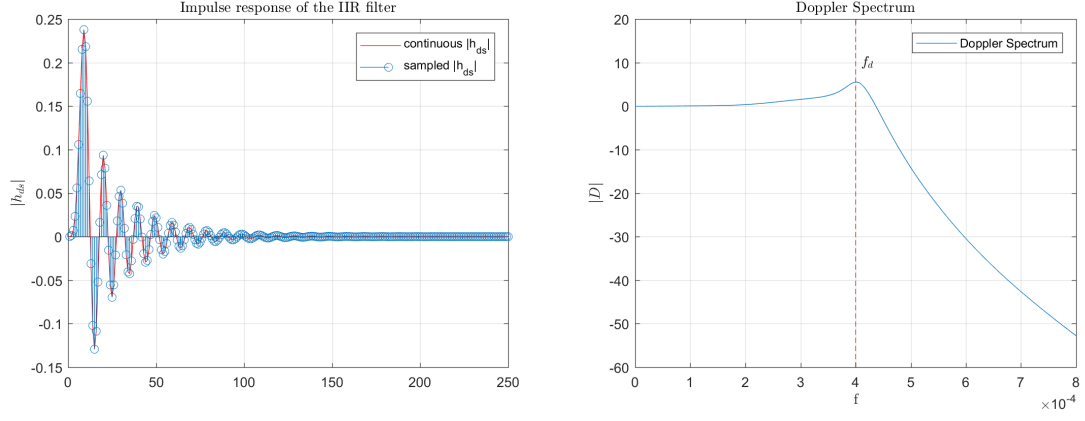
$H_{ds}(z) = B(z)/A(z) \quad f_d T_p = 0.1$			
$\{a_n\}$ ,	$n = 0, \dots, 11:$		
1	-4.4153	8.6283	-9.4592
6.1051	-1.3542	-3.3622	7.2390
-7.9361	5.1221	-1.8401	2.8706e-1
$\{b_n\}$ ,	$n = 0, \dots, 21:$		
1.3651e-4	8.1905e-4	2.0476e-3	2.7302e-3
2.0476e-3	9.0939e-4	6.7852e-4	1.3550e-3
1.8076e-3	1.3550e-3	5.3726e-4	6.1818e-5
-7.1294e-5	-9.5058e-5	-7.1294e-5	-2.5505e-5
1.3321e-5	4.5186e-5	6.0248e-5	4.5186e-5
1.8074e-5	3.0124e-6		

**Table 1.** Coefficients for the IIR filter

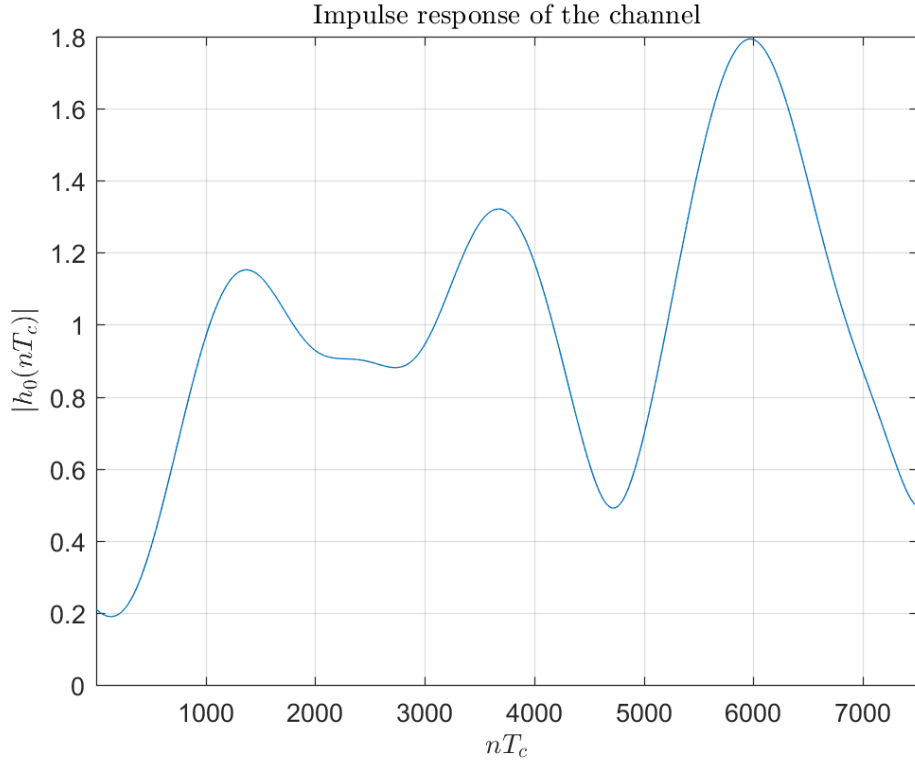
The graphical representation of the impulse response of the IIR filter and the Doppler Spectrum is shown in Fig. 7. To obtain  $h_0$ , following the scheme of Fig. 6, the noise component  $w \sim \mathcal{CN}(0, 1)$  is filtered with the IIR filter previously described. Note that the frequency response of this filter is  $\mathcal{H}_{ds}(f) = \sqrt{\mathcal{D}(f)}$  while the PSD of the noise is constant and equal to 1. For this reason, the equivalent impulse response of this part is equal to  $\mathcal{D}(f) = 1 \cdot |\mathcal{H}_{ds}|^2$  which is actually the Doppler spectrum.

The output of the filter is affected by a transient, which we avoided by considering only values after  $5N_{eq}T_p$ , where  $N_{eq} = \left\lceil -\frac{1}{\ln(|p|)} \right\rceil$  is the equivalent time constant, and  $p$  is the pole with the highest magnitude. Then, after scaling the coefficient such that  $M_{h_0}/\sqrt{E_{h_{ds}}} = 1$ , the signal is filtered with an interpolation filter of factor  $1/T_Q = T_p/T_c = 250$ .

The interpolator output signal is multiplied by a constant  $\sigma_0 = \sqrt{M_d}$  to impose the desired power delay profile, and finally added up with another constant,  $C$ , which included the deterministic component according to [1], Page 307. The final signal is given in Fig. 8.



**Figure 7.** Impulse response of the IIR filter and Doppler Spectrum



**Figure 8.** Magnitude of the simulated  $h_0$  for 7500 samples.

## PDF of $\frac{|h_0|}{\sqrt{M|h_0|}}$

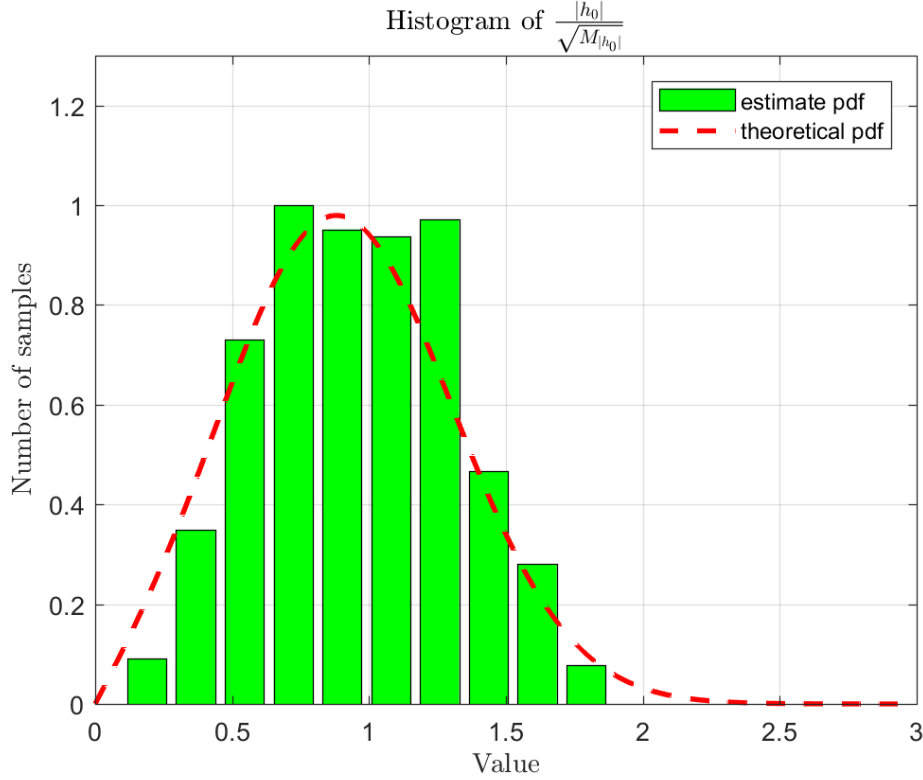
The signal  $h' = \frac{h_0}{\sqrt{M|h_0|}}$  for 80000 samples is now studied. Note that, according to Fig. [6],  $h'$  contains a deterministic component in addition to a random component, which is complex gaussian with zero-mean and variance equal to one. For this reason the *pdf* of  $|h'|$  is a Rice distribution given by

$$p_{|h'|} = \begin{cases} 2(1+K)ae^{-K-(1+K)a^2}I_0(2a\sqrt{K(1+K)}) & a \geq 0 \\ 0 & otherwise \end{cases} \quad (4)$$

where  $I_0$  is the *modified Bessel function of the first type and order zero*, respectively

$$I_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos \alpha} d\alpha$$

The histogram of  $h'$  is shown in Figure [9]. Here it is given also the theoretical *pdf* evaluated according to equation 4.



**Figure 9.** Plot of both estimate and theoretical curve of the pdf of  $h'$  .



## Spectrum of $h_0$

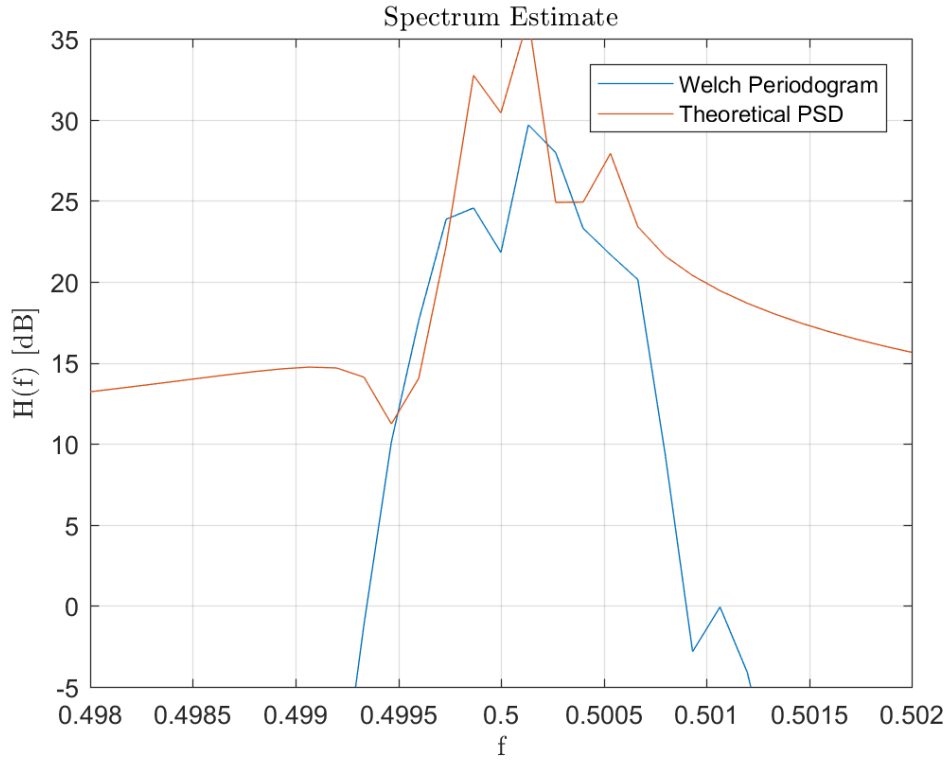
In this section the spectrum of  $h_0$  is computed using the Welch Periodogram. This method extracts different subsequences of consecutive  $D$  samples which eventually overlap, and for each of these it computes the periodogram  $\mathcal{P}_{PER}^{(s)}(f)$ . The mathematical model is given by

$$\mathcal{P}_{WE}(F) = \frac{1}{N_s} \sum_{x=0}^{N_s-1} \mathcal{P}_{PER}^{(s)}(f)$$

where  $N_s = \lfloor \frac{K-D}{D-S} - 1 \rfloor$  is the total number of subsequences.

In order to compare the estimate with the theoretical case, the ideal PSD is computed. By definition, the *Doppler Spectrum* is the Fourier Transform of the autocorrelation function of the impulse response, in correspondence of the same delay  $\tau$ , evaluated at two different instants. In our analysis, it is provided as IIR filter resulting from the Jakes model, and the coefficients are reported in table [7]

The result is given in Figure [10].



**Figure 10.** PSD estimate of  $h_0$  with the theoretical curve.

# Bibliography

- [1] Nevio Benvenuto, Giovanni Cherubini, *Algorithms for Communication Systems and their Applications*. Wiley, 2002.