

# Digital Communications and Laboratory

## Third Homework

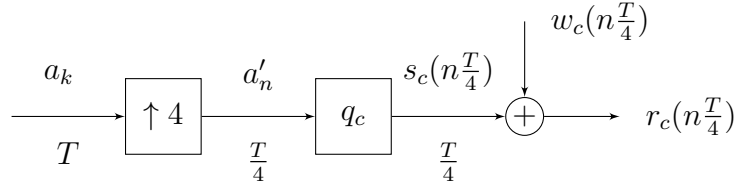
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# PROBLEM

The following system was considered. A stream of QPSK symbols is upsampled with period  $T/4$  and filtered with a filter  $q_c$  which output is  $s_c(n\frac{T}{4}) = \alpha s_c((n-1)\frac{T}{4}) + \beta a'_{n-5}$ . This signal is transmitted through the channel, which introduces the noise component  $w_c(n\frac{T}{4})$  with PSD  $\mathcal{P}_{w_c}(f) = N_0$ . Note that noise components are iid with  $pmd \sim \mathcal{CN}(0, \sigma_{w_c}^2)$ . The SNR at the output of the system is therefore

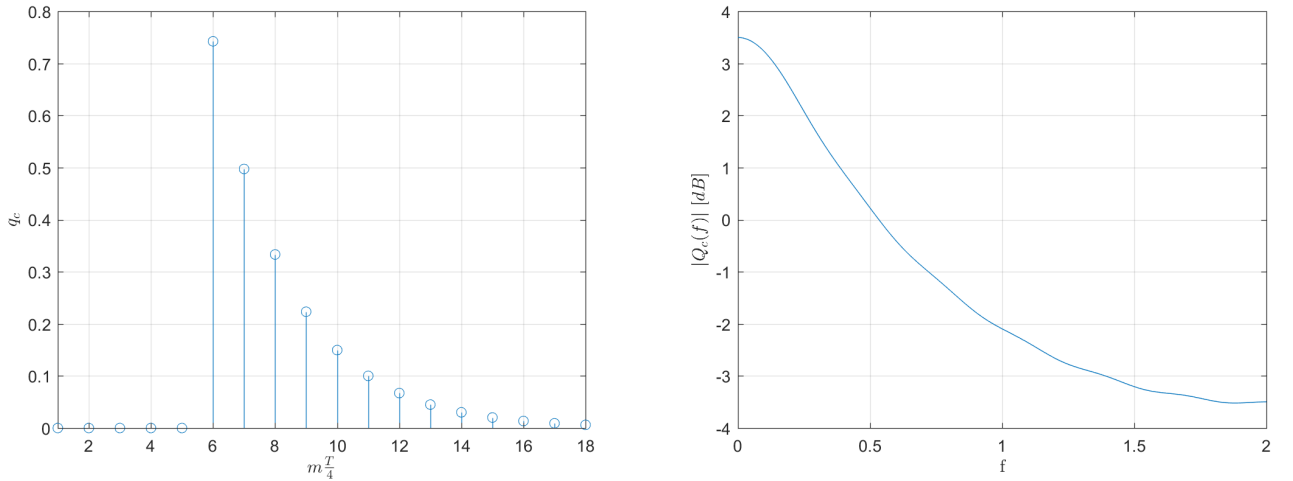
$$\Gamma = \frac{M_{s_c}}{N_0 \frac{1}{T}} = \frac{\sigma_a^2 E_{q_c}}{\sigma_{w_c}^2}$$

with  $\sigma_a^2 = 2$  and  $E_{q_c} = \sum_m |q_c(m\frac{T}{4})|^2$ .



**Figure 1.** Model for the transmission system of Problem 1.

The QPSK symbols are generated with a PN sequence of length  $L = 2^{20} - 1$  in order to provide a stream of bits with spectral characteristics similar to those of a white noise signal. Two consecutive bits are then coupled and mapped into one of the possible constellations symbols, associating the first and second bit to the real and imaginary part respectively. The  $q_c$  filter in linear and frequency domain is given in Figures [2].



**Figure 2.** Impulse response (left) and Frequency response (right) of the filter  $q_c$ .

In the following, 6 different receiver configurations are studied. For each of this, an SNR value of  $\Gamma = 10$  dB was assumed.

## Channel equalization

Receivers (a), (b), (c) and (d) implement a technique called *Decision feedback equalizer* (DFE) which attempts to reduce the intersymbol interference (ISI).

The desired signal at the receiver input is

$$x_k = \sum_{i=-\infty}^{\infty} a_i h_{k-i} + \tilde{w}_k$$

where  $h_n = h_{Tx} * g_C * g_M(t)$ . Assuming  $\{h_n\}$  has finite duration and support  $[-N_1, -N_1 + 1, \dots, N_2 - N_2]$ , we define postcursors the samples with positive index and precursors those with negative index.

Writing  $x_k$  explicitly we can see that it depends both on previous and future symbols.

The DFE is composed by two filters:

1. *Feedforward (FF) filter*  $c$ , with  $M_1$  coefficients:

$$x_{FF,k} = \sum_{i=0}^{M_1-1} c_i x_{k-i} \quad (1)$$

2. *Feedback (FB) filter*  $b$ , with  $M_2$  coefficients:

$$x_{FB,k} = \sum_{i=1}^{M_2} b_i a_{k-i-D} \quad (2)$$

Ideally the task of the FF filter is to obtain an overall impulse response  $\{\psi_n = h * c_n\}$  with small precursors, while the FB filter cancels almost all the ISI.

The output of the DFE can be written as

$$\begin{aligned} y_k &= x_{FF,k} + x_{FB,k} \\ &= \sum_{i=0}^{M_1-1} c_i x_{k-i} + \sum_{j=1}^{M_2} b_j a_{k-D-j} \end{aligned} \quad (3)$$

The Wiener filter theory can be exploited to determine the optimum coefficients, which minimize the cost function

$$J = E [|a_{k-D} - y_k|^2] \quad (4)$$

The optimum FF filter  $c$  is given by

$$\mathbf{c}_{opt} = \mathbf{R}^{-1} \mathbf{p} \quad (5)$$

where the matrices  $\mathbf{R}$  and  $\mathbf{p}$  are computed using

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left( \sum_{j=-N_1}^{N_2} h_j h_{j-(p-q)}^* - \sum_{j=1}^{M_2} h_{j+D-q} h_{j+D-p}^* \right) + r_{\tilde{w}}(p-q) \quad (6)$$

$$[\mathbf{p}]_p = \sigma_a^2 h_{D-p}^*, \quad p, q = 0, 1, \dots, M_1 - 1 \quad (7)$$

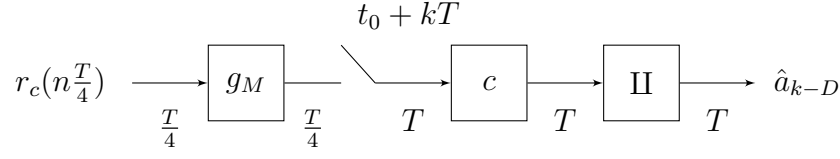
The optimum FB coefficients  $\{b_i\}$  are given by

$$b_i = - \sum_{l=0}^{M_1-1} c_{opt,l} h_{i+D-l} \quad i = 1, 2, \dots, M_2 \quad (8)$$

The minimum-cost function can be expressed in close form as:

$$J_{min} = \sigma_a^2 \left( 1 - \sum_{l=0}^{M_1-1} c_{opt,l} h_{D-l} \right) \quad (9)$$

## Receiver a

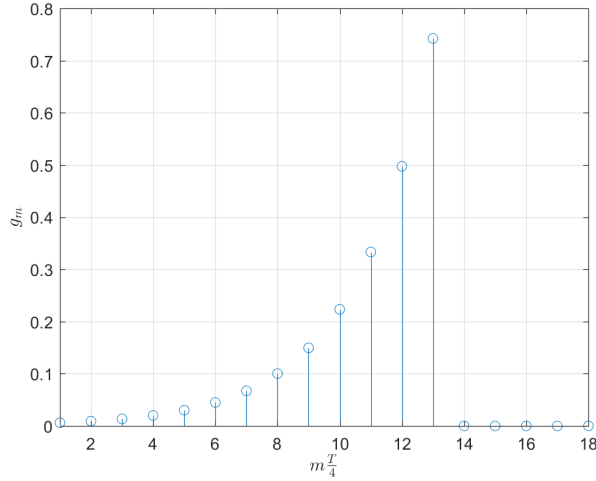


**Figure 3.** Model for the receiver (a).

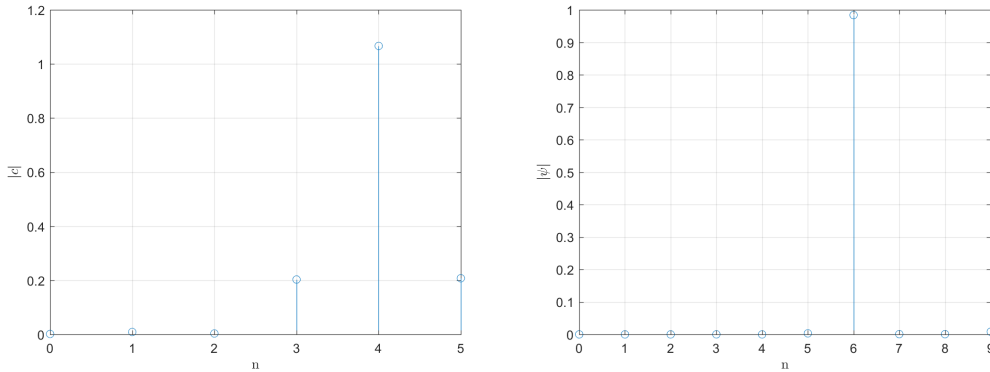
The receiver filter consists of a filter  $g_M$  matched to the transmission filter  $q_c$  followed by a *Linear Equalizer (LE)* filter  $c$ . The matched filter is simply computed as  $g_m(t) = q_c^*(t_0 + t)$ , where  $t_0$  is the timing phase. It is given in Figure [4]. From now on, we may refer to the global impulse response of the system at the input of  $c$  as  $h = g_c * g_M$ . Since it is defined @ $\frac{T}{4}$ , a downsampling of a factor 4 is required between the output of  $h$  and the input of  $c$ .

The filter  $c$  corresponds to the Feedforward (FF) filter  $c$  of Equation (1). Since the LE can be seen as a particular case of a DFE where  $M_2 = 0$ , we developed the general DFE algorithm and set  $\{b_i\} = 0$  in this case.

The equalizing filter  $c$  and the overall impulse response  $\psi_i = h * c_i$  for an SNR of  $\Gamma = 10$  dB are given in Figure [5].



**Figure 4.** Matched filter  $g_m$  of the receiver filter.



**Figure 5.** Coefficients of the Equalizer filter  $c$  (left) and of the overall impulse response  $\psi_i$  (right).

The parameters we obtained are reported in Table [1].

$\bar{\mathbf{t}}_0$	$\mathbf{M}_1$	$\mathbf{M}_2$	$\mathbf{D}$
1	7	0	6

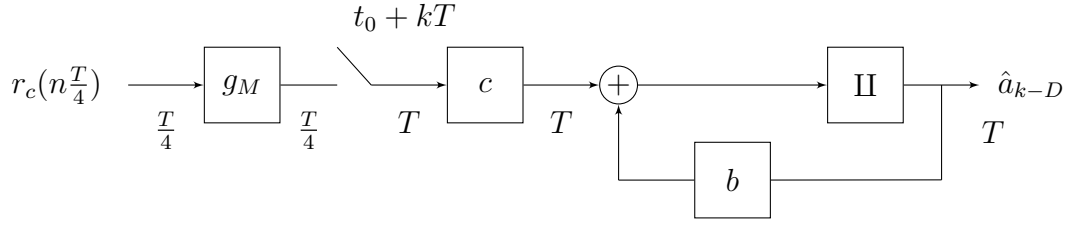
**Table 1.** Parameters of the Receiver (a).

The final signal at the output of the equalizer filter is processed by a threshold detector, which maps each received symbol according to the following rule:

$$\tilde{a}_k \mapsto \hat{a}_k = \text{sgn}(\Re[\tilde{a}_k]) + i \cdot \text{sgn}(\Im[\tilde{a}_k])$$

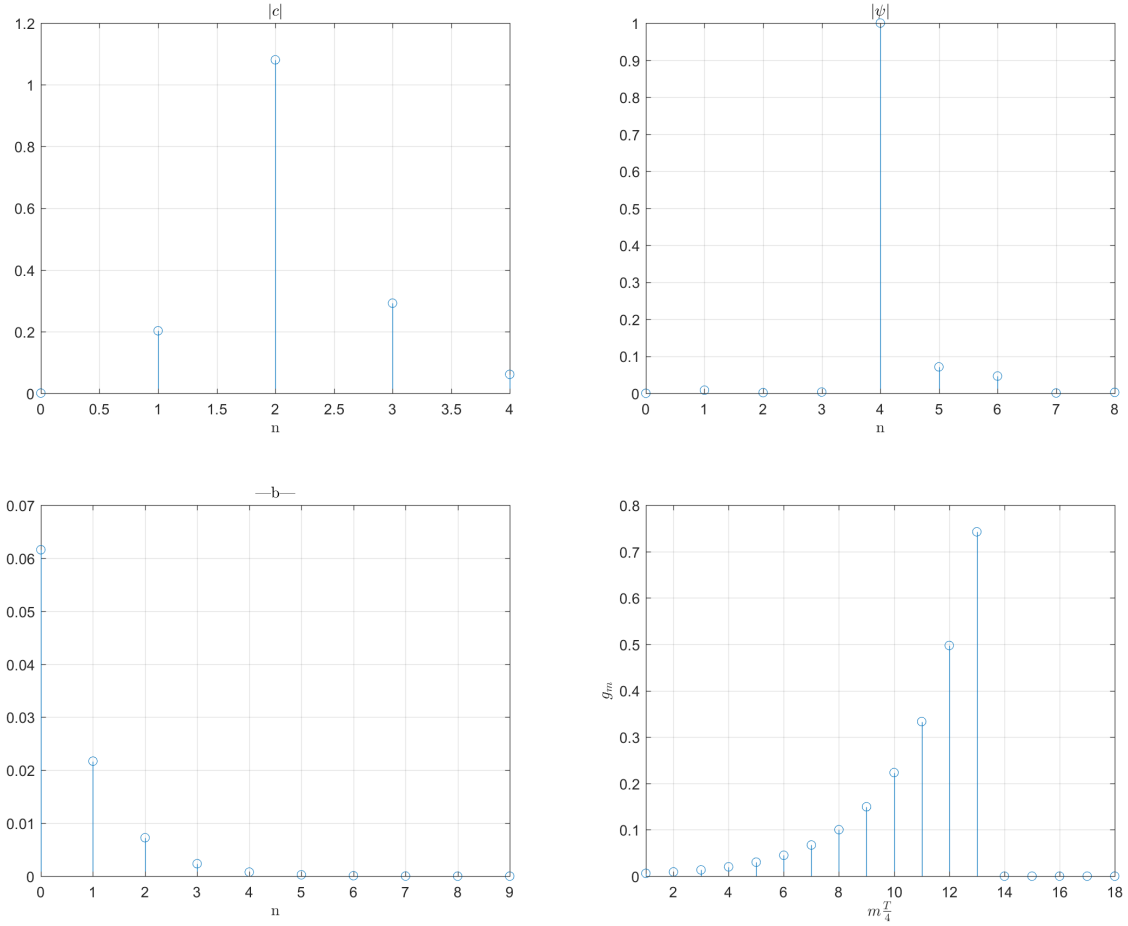
Note that the actual transmitted symbol at time  $k$  is detected at time  $k + D$  because of the delay introduced by the filter  $c$ . This detection configuration will be used also for receivers  $b$ ,  $c$  and  $d$ .

## Receiver b



**Figure 6.** Model for the receiver (b).

The only difference with respect to Receiver (a) is that now the DFE has both the FF filter  $c$  of Equation (1) and the FB filter  $b$  of Equation (2). Using the same algorithm as previous case, we obtained the following filters.

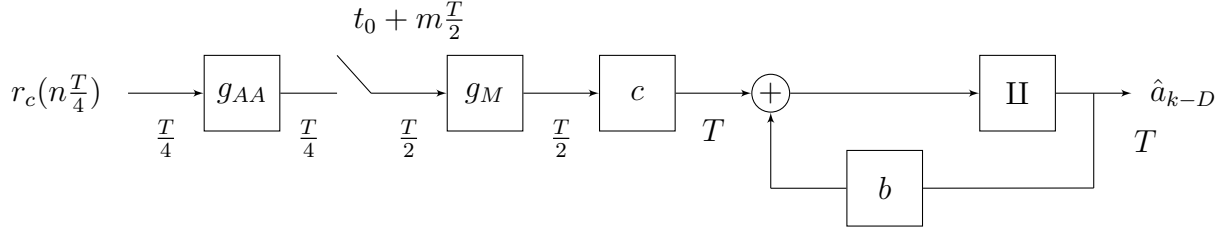


**Figure 7.** Coefficients of the FF filter  $c$ , of the overall impulse response  $\psi_i$ , of the FB filter  $b$  and of the matched filter  $g_m$ . Note that  $g_m$  is the same as for receiver (a).

$\bar{t}_0$	$M_1$	$M_2$	$D$
1	1	1	1

**Table 2.** Parameters of the Receiver (b).

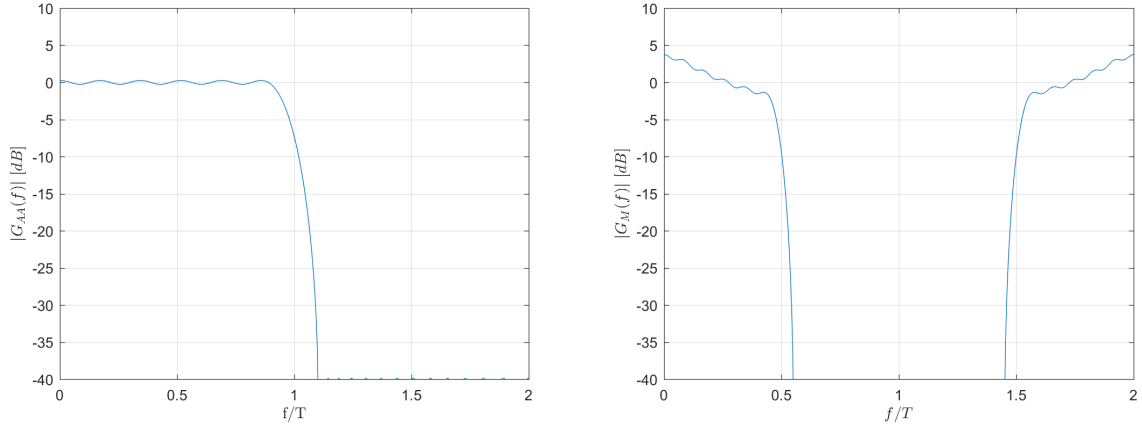
## Receiver c



**Figure 8.** Model for the receiver (c).

In this case the analog filter is replaced by a digital *anti-aliasing* filter  $g_{AA}$ . The signal at the output is downsampled of a factor 2, then a cascade of a matched filter and a DFE filter processes the signal and produces the output symbols  $\hat{a}_{k-D}$ . The filter  $g_m$  is needed in order to allow the equalizing filter defined in equation 5 to act as a match filter. Note also that the filter  $c$  is followed by a downsampling of a factor 2.

The anti-aliasing and the matched filters are given in Figure [9].

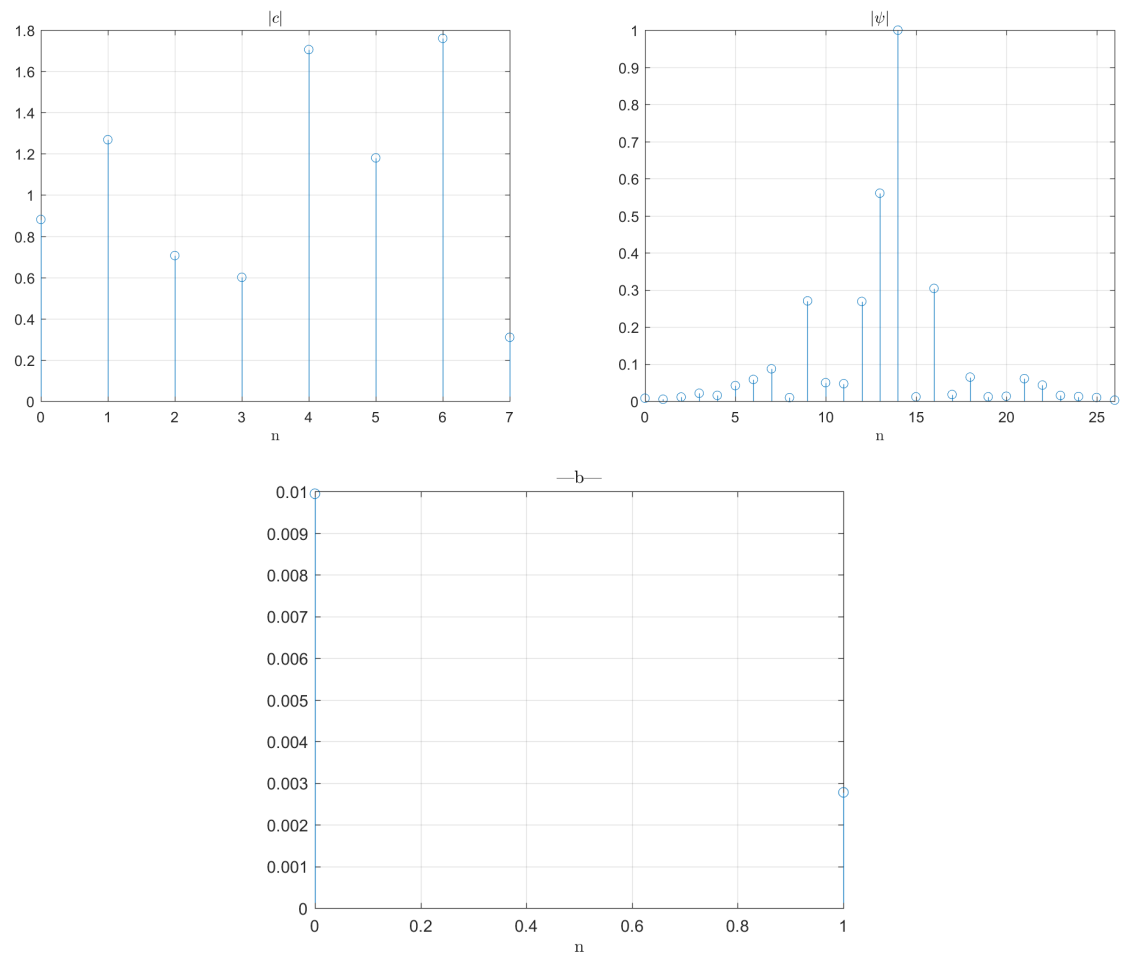


**Figure 9.** Anti-aliasing and matched filters of the Receiver (c).

The remaining filters are given in Figure [10].

$\bar{t}_0$	$M_1$	$M_2$	$D$
1	1	1	1

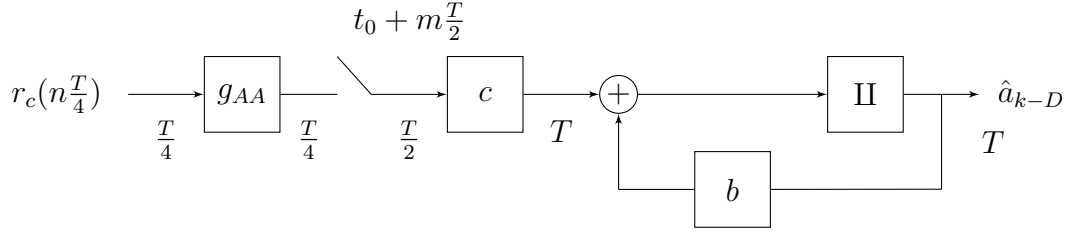
**Table 3.** Parameters of the Receiver (c).



**Figure 10.** Coefficients of the FF filter  $c$ , of the overall impulse response  $\psi_i$  and of the FB filter  $b$ .

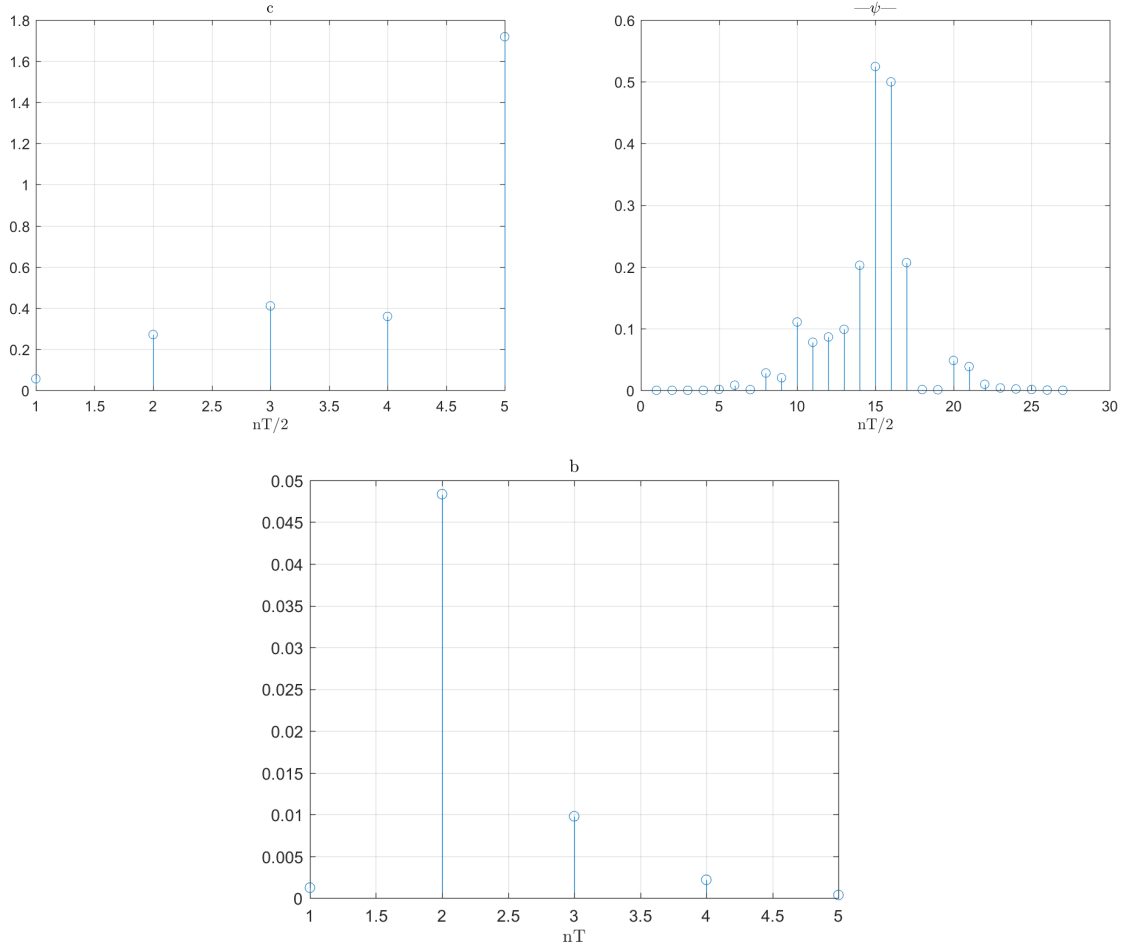


## Receiver d



**Figure 11.** Model for the receiver (d).

The same configuration of receiver (c) is here simulated without the match filter at the input of the DFE. In this case we expect lower performances with respect to the previous configuration, since the equalizing filter is not matched with the impulsive response at its input. The following filters were computed.

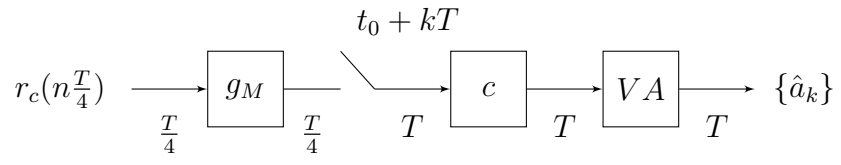


**Figure 12.** Coefficients of the FF filter  $c$ , of the overall impulse response  $\psi_i$  and of the FB filter  $b$ .

$\bar{t}_0$	$M_1$	$M_2$	$D$
1	1	1	1

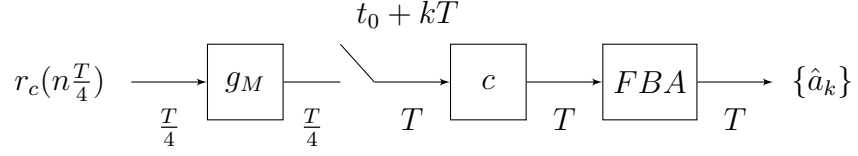
**Table 4.** Parameters of the Receiver (d).

## Receiver e



**Figure 13.** Model for the receiver (e).

## Receiver f



**Figure 14.** Model for the receiver (f).

Still using the channel as given in the previous section, the signal at the output of the  $c$  filter is now processed by a *Forward-Backward algorithm* exploiting the Max-Log-MAP to detect the received symbols  $\hat{a}_k$ . For each instant  $k = 0, 1, \dots, K - 1$ , the algorithm performs the computation of five metrics, respectively:

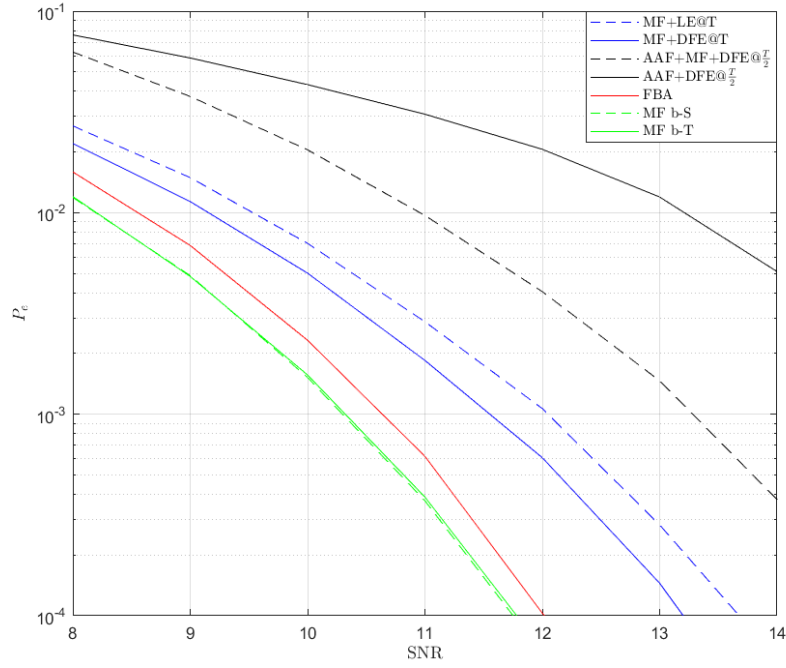
1. The channel transition metric,  $c_k(i, j)$ ,  $i, j = 1, \dots, N_s$ ;
2. The Backward metric,  $\tilde{b}_k(i)$ ,  $i = 1, \dots, N_s$ ;
3. The Forward metric,  $\tilde{f}_k(j)$ ,  $j = 1, \dots, N_s$ ;
4. The State metric,  $\tilde{v}_k(i)$ ,  $i = 1, \dots, N_s$ ;
5. The Log-Likelihood function of the isolated symbol  $\tilde{l}_k(\beta) = \max_{\substack{i \in [1, \dots, N_s], \\ [\sigma_i]_1 = \beta}} = \tilde{v}_k(i)$ .

The detected symbol is then evaluated using the following decision rule:

$$\hat{a}_{k+L_1} = \operatorname{argmax}_{\beta \in \mathcal{A}} \tilde{l}_k(\beta) \quad (10)$$

## BER comparison

For each of the different receiver configurations previously described, the symbol error probability is evaluated by simulations with different values of the SNR  $\Gamma$ .



**Figure 15.** Simulated BER for the different receivers at different SNR values.