Digital Communications and Laboratory
Third Homework

Faccin Dario, Santi Giovanni

PROBLEM

The following system was considered. A stream of QPSK symbols is upsampled with period T/4 and filtered with a filter q_c which output is $s_c\left(n\frac{T}{4}\right) = \alpha s_c\left((n-1)\frac{T}{4}\right) + \beta a'_{n-5}$. This signal is transmitted through the channel, which introduces the noise component $w_c\left(n\frac{T}{4}\right)$ with PSD $\mathcal{P}_{w_c}(f) = N_0$. Note that noise components are iid with $pmd \sim \mathcal{CN}(0, \sigma^2_{w_c})$. The SNR at the output of the system is therefore

$$\Gamma = \frac{M_{s_c}}{N_0 \frac{1}{T}} = \frac{\sigma_a^2 E_{q_c}}{\sigma^2_{w_c}}$$

with $\sigma_a^2 = 2$ and $E_{q_c} = \sum_m |q_c\left(m\frac{T}{4}\right)|^2$.

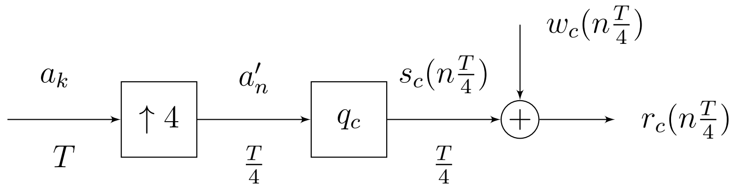

Figure 1. Model for the transmission system of Problem 1.

The QPSK symbols are generated with a PN sequence of length $L = 2^{20} - 1$ in order to provide a stream of bits with spectral characteristics similar to those of a white noise signal. Two consecutive bits are then coupled and mapped into one of the possible constellations symbols, associating the first and second bit to the real and imaginary part respectively.
The q_c filter in linear and frequency domain is given in Figures [2].

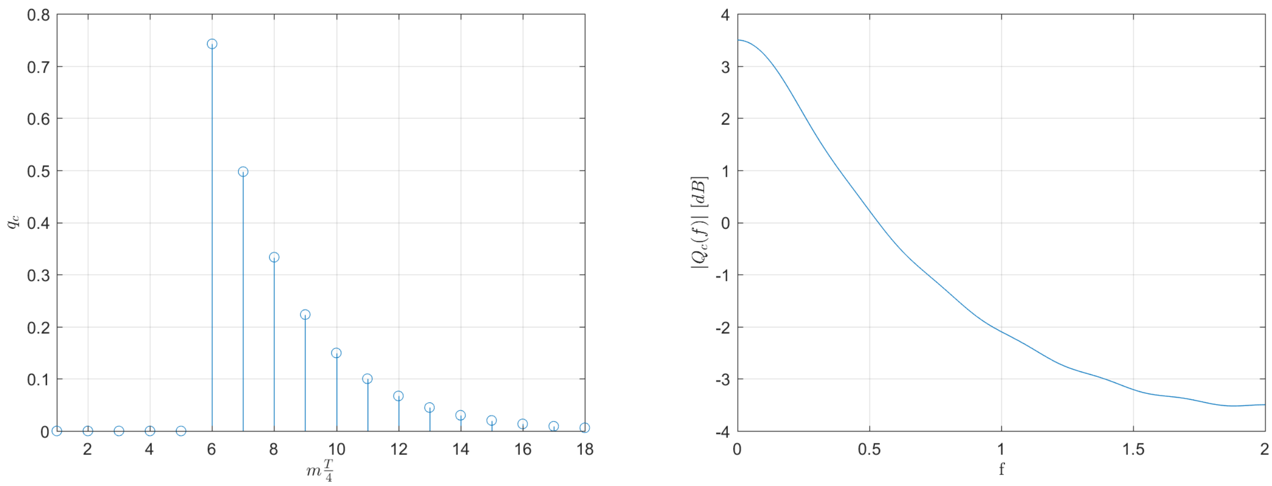

Figure 2. Impulse response (left) and Frequency response (right) of the filter q_c.

In the following, 6 different receiver configurations are studied. For each of this, an SNR value of $\Gamma = 10\ dB$ was assumed.

Channel equalization

Receivers (a), (b), (c) and (d) implement a technique called *Decision feedback equalizer* (DFE) which attempts to reduce the intersymbol interference (ISI).
The desired signal at the receiver input is

$$x_k = \sum_{i=-\infty}^{\infty} a_i h_{k-i} + \tilde{w}_k$$

where $h_n = h_{Tx} * g_C * g_M(t)$. Assuming $\{h_n\}$ has finite duration and support $[-N_1, -N_1 + 1, \dots, N_2 - N_2]$, we define postcursors the samples with positive index and precursors those with negative index.
Writing x_k explicitly we can see that it depends both on previous and future symbols.
The DFE is composed by two filters:

1. *Feedforward (FF) filter* c, with M_1 coefficients:

$$x_{FF,k} = \sum_{i=0}^{M_1-1} c_i x_{k-i} \tag{1}$$

2. *Feedback (FB) filter* b, with M_2 coefficients:

$$x_{FB,k} = \sum_{i=1}^{M_2} b_i a_{k-i-D} \tag{2}$$

Ideally the task of the FF filter is to obtain an overall impulse response $\{\psi_n = h * c_n\}$ with small precursors, while the FB filter cancels almost all the ISI.
The output of the DFE can be written as

$$\begin{aligned} y_k &= x_{FF,k} + x_{FB,k} \\ &= \sum_{i=0}^{M_1-1} c_i x_{k-i} + \sum_{j=1}^{M_2} b_j a_{k-D-j} \end{aligned} \tag{3}$$

The Wiener filter theory can be exploited to determine the optimum coefficients, which minimize the cost function

$$J = E\left[|a_{k-D} - y_k|^2\right] \tag{4}$$

The optimum FF filter c is given by

$$\mathbf{c}_{opt} = \mathbf{R}^{-1}\mathbf{p} \tag{5}$$

where the matrices $\mathbf{R}$ and $\mathbf{p}$ are computed using

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left(\sum_{j=-N_1}^{N_2} h_j h^*_{j-(p-q)} - \sum_{j=1}^{M_2} h_{j+D-q} h^*_{j+d-p} \right) + r_{\tilde{w}}(p-q) \tag{6}$$

$$[\mathbf{p}]_p = \sigma_a^2 h^*_{D-p}, \qquad\qquad p,q = 0,1,\dots,M_1-1 \tag{7}$$

The optimum FB coefficients $\{b_i\}$ are given by

$$b_i = -\sum_{l=0}^{M_1-1} c_{opt,l} h_{i+D-l} \qquad\qquad i = 1,2,\dots,M_2 \tag{8}$$

The minimum-cost function can be expressed in close form as:

$$J_{min} = \sigma_a^2 \left(1 - \sum_{l=0}^{M_1-1} c_{opt,l} h_{D-l} \right) \tag{9}$$

Receiver a

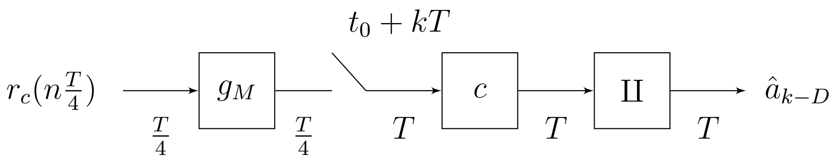

Figure 3. Model for the receiver (a).

The receiver filter consists of a filter g_M matched to the transmission filter q_c followed by a *Linear Equalizer (LE) filter c*. The matched filter is simply computed as $g_m(t) = q_c^*(t_0 + t)$, where t_0 is the timing phase. It is given in Figure [4]. From now on, we may refer to the global impulse response of the system at the input of c as $h = g_c * g_M$. Since it is defined @$\frac{T}{4}$, a downsampling of a factor 4 is required between the output of h and the input of c.
The filter c corresponds to the Feedforward (FF) filter c of Equation (1). Since the LE can be seen as a particular case of a DFE where $M_2 = 0$, we developed the general DFE algorithm and set $\{b_i\} = 0$ in this case.
The equalizing filter c and the overall impulse response $\psi_i = h * c_i$ for an SNR of $\Gamma = 10$ dB are given in Figure [5].

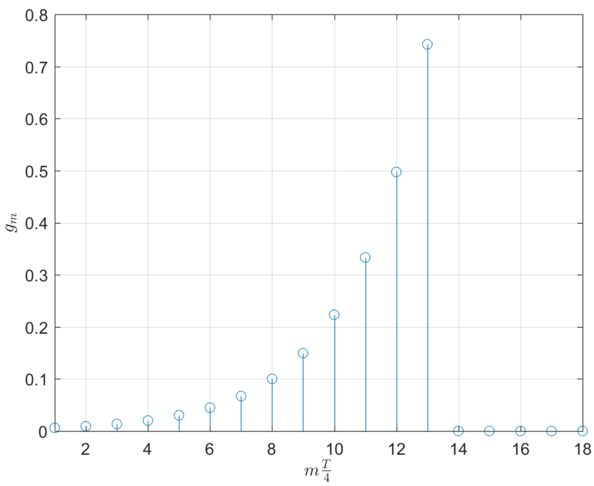

Figure 4. Matched filter g_m of the receiver filter.

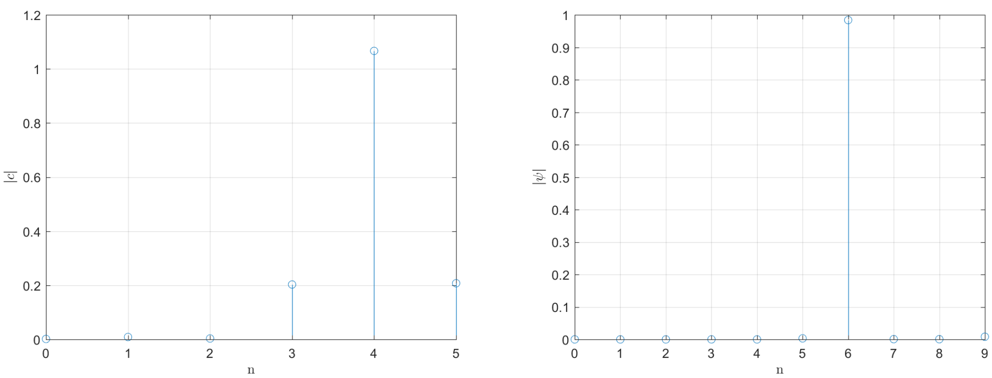

Figure 5. Coefficients of the Equalizer filter c (left) and of the overall impulse response ψ_i (right).

The parameters we obtained are reported in Table [1].

$\bar{\mathbf{t}}_\mathbf{0}$	$\mathbf{M_1}$	$\mathbf{M_2}$	$\mathbf{D}$
1	7	0	6

Table 1. Parameters of the Receiver (a).

The final signal at the output of the equalizer filter is processed by a threshold detector, which maps each received symbol according to the following rule:

$$\tilde{a}_k \mapsto \hat{a}_k = \mathrm{sgn}(\Re[\tilde{a}_k]) + i \cdot \mathrm{sgn}(\Im[\tilde{a}_k])$$

Note that the actual transmitted symbol at time k is detected at time $k + D$ because of the delay introduced by the filter c. This detection configuration will be used also for receivers b, c and d.

Receiver b

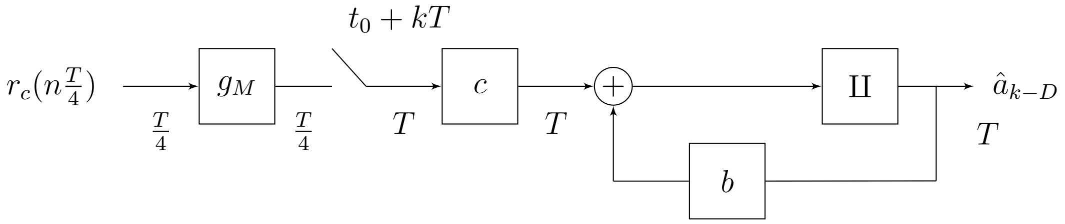

Figure 6. Model for the receiver (b).

The only difference with respect to Receiver (a) is that now the DFE has both the FF filter c of Equation (1) and the FB filter b of Equation (2). Using the same algorithm as previous case, we obtained the following filters.

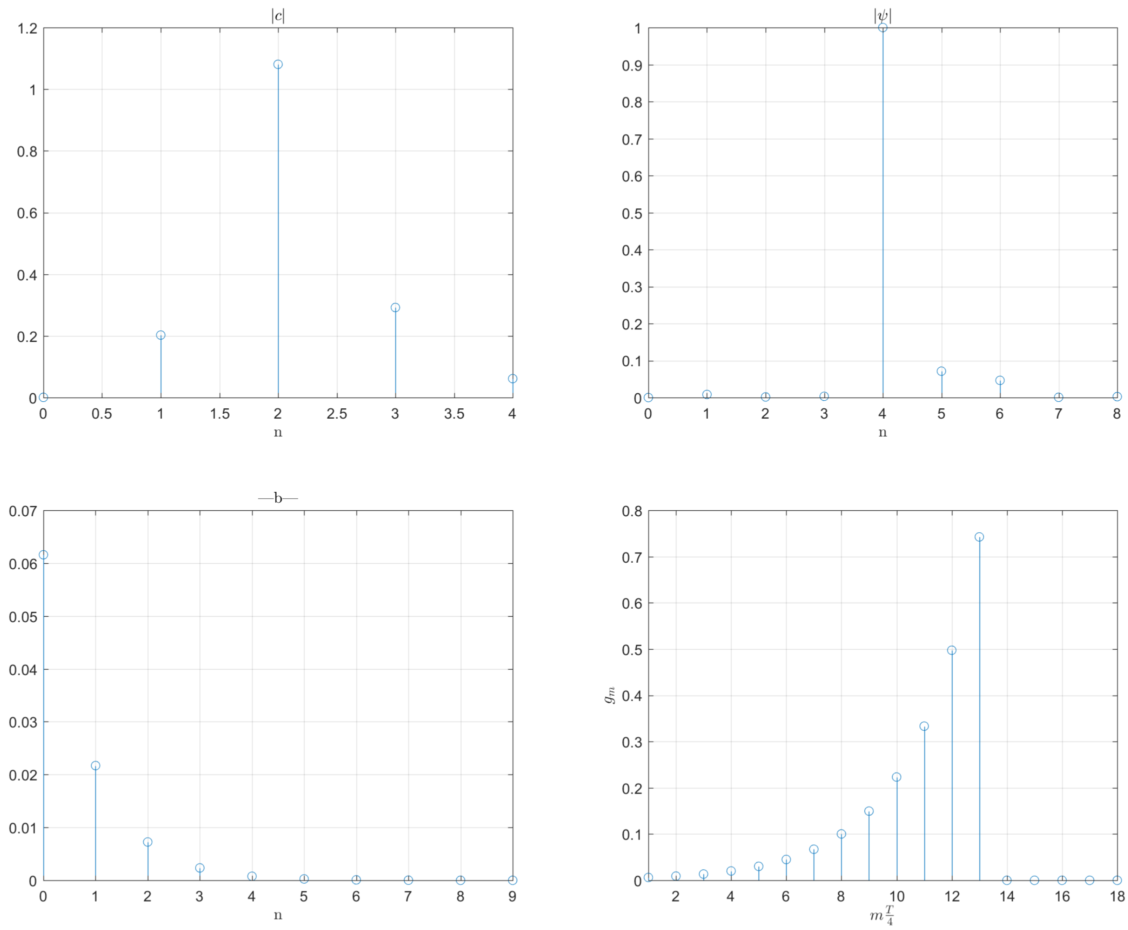

Figure 7. Coefficients of the FF filter c, of the overall impulse response ψ_i, of the FB filter b and of the matched filter g_m. Note that g_m is the same as for receiver (a).

$\bar{\mathbf{t}}_0$	$\mathbf{M_1}$	$\mathbf{M_2}$	**D**
1	1	1	1

Table 2. Parameters of the Receiver (b).

Receiver c

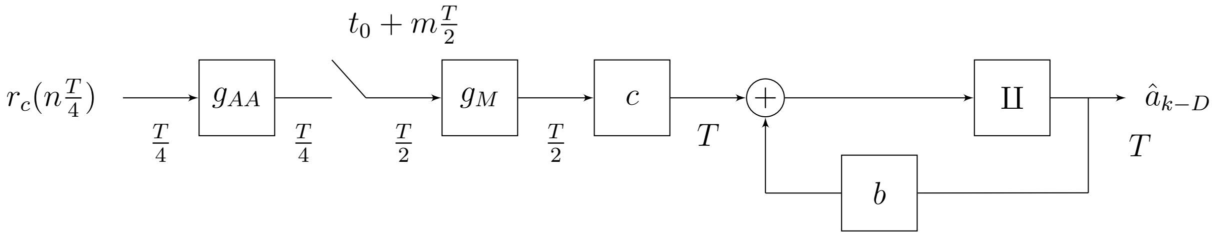

Figure 8. Model for the receiver (c).

In this case the analog filter is replaced by a digital *anti-aliasing* filter g_{AA}. The signal at the output is downsampled of a factor 2, then a cascade of a matched filter and a DFE filter processes the signal and produces the output symbols $\hat{a}_{k-D}$. The filter g_m is needed in order to allow the equalizing filter defined in equation 5 to act as a match filter. Note also that the filter c is now works at T/2, such that the new formulas to compute $\mathbf{c}_{opt}$ are now:

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left(\sum_{n=-\infty}^{+\infty} h_{2n-q} h_{2n-p}^* - \sum_{j=1}^{M_2} h_{j+D-q} h_{2(j+D)-p}^* \right) + r_{\tilde{w}}(p-q) \tag{10}$$

$$[\mathbf{p}]_p = \sigma_a^2 h_{2D-p}^*, \qquad p, q = 0, 1, \ldots, M_1 - 1 \tag{11}$$

The anti-aliasing and the matched filters are given in Figure [9].

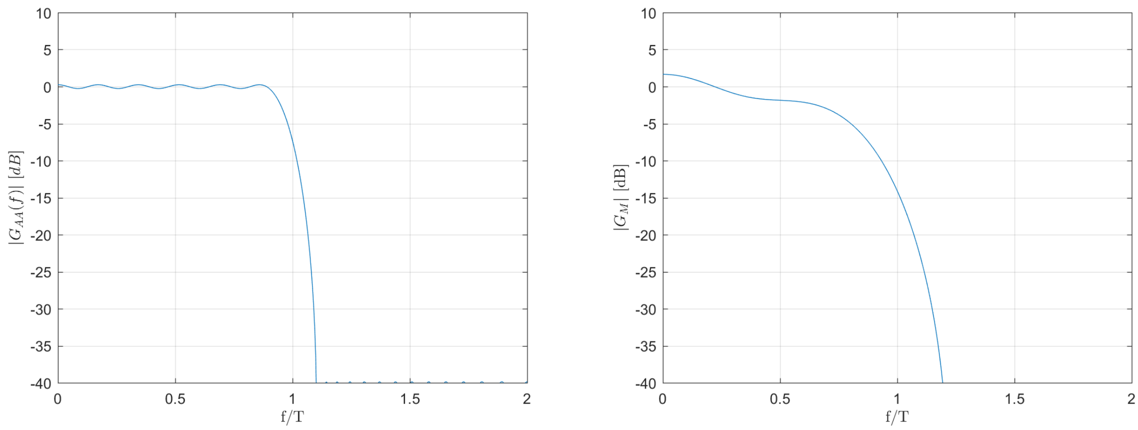

Figure 9. Anti-aliasing and matched filters of the Receiver (c).

The remaining filters are given in Figure [10].

$\bar{t}_0$	M_1	M_2	D
1	1	1	1

Table 3. Parameters of the Receiver (c).

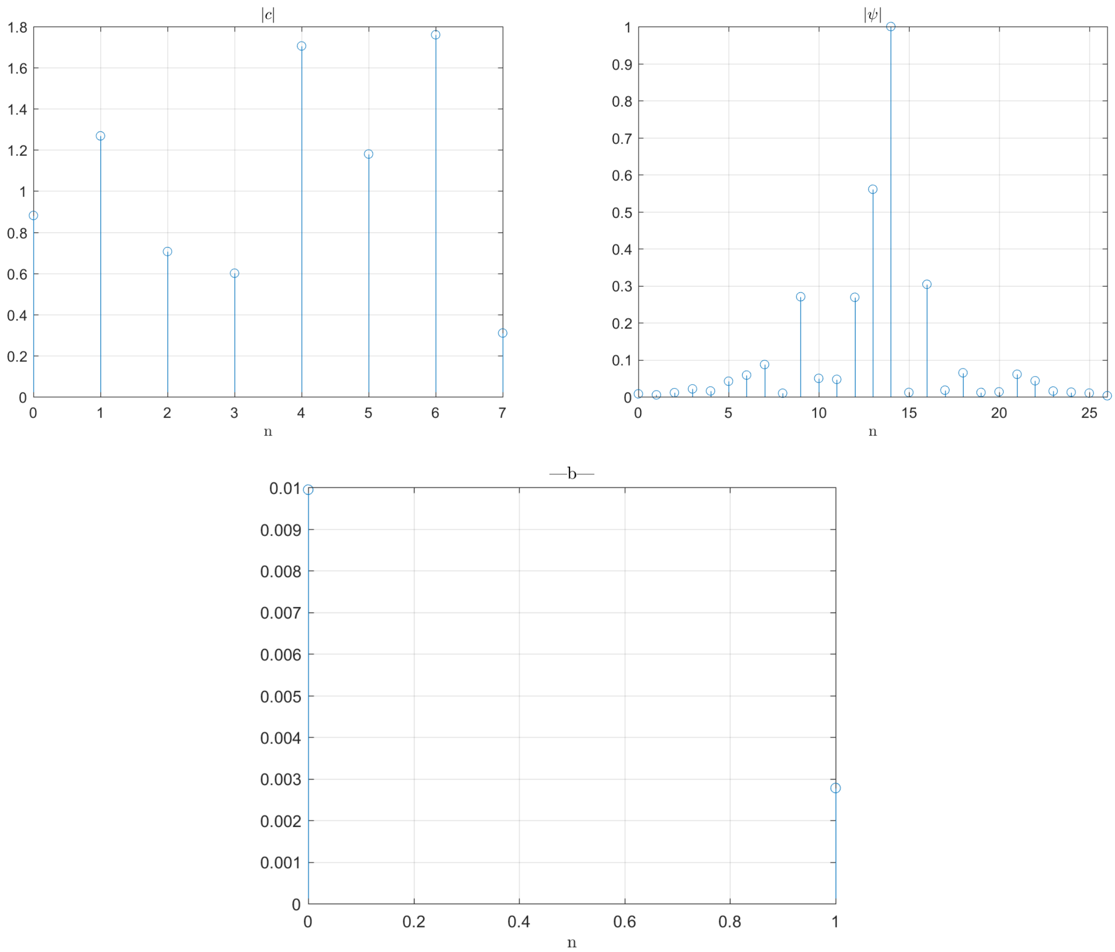

Figure 10. Coefficients of the FF filter c, of the overall impulse response ψ_i and of the FB filter b.

Receiver d

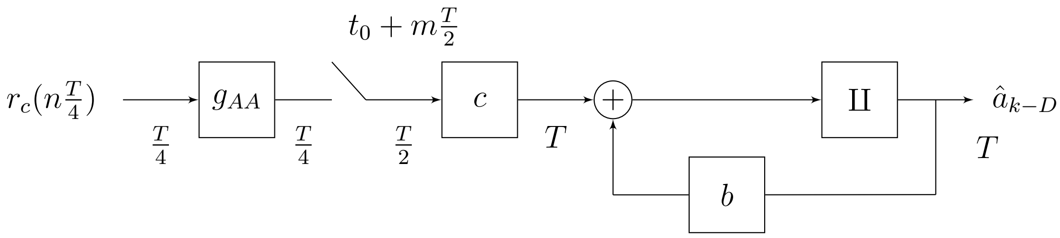

Figure 11. Model for the receiver (d).

The same configuration of receiver (c) is here simulated without the match filter at the input of the DFE. In this case we expect lower performances with respect to the previous configuration, since the equalizing filter is not matched with the impulsive response at its input.
The following filters were computed.

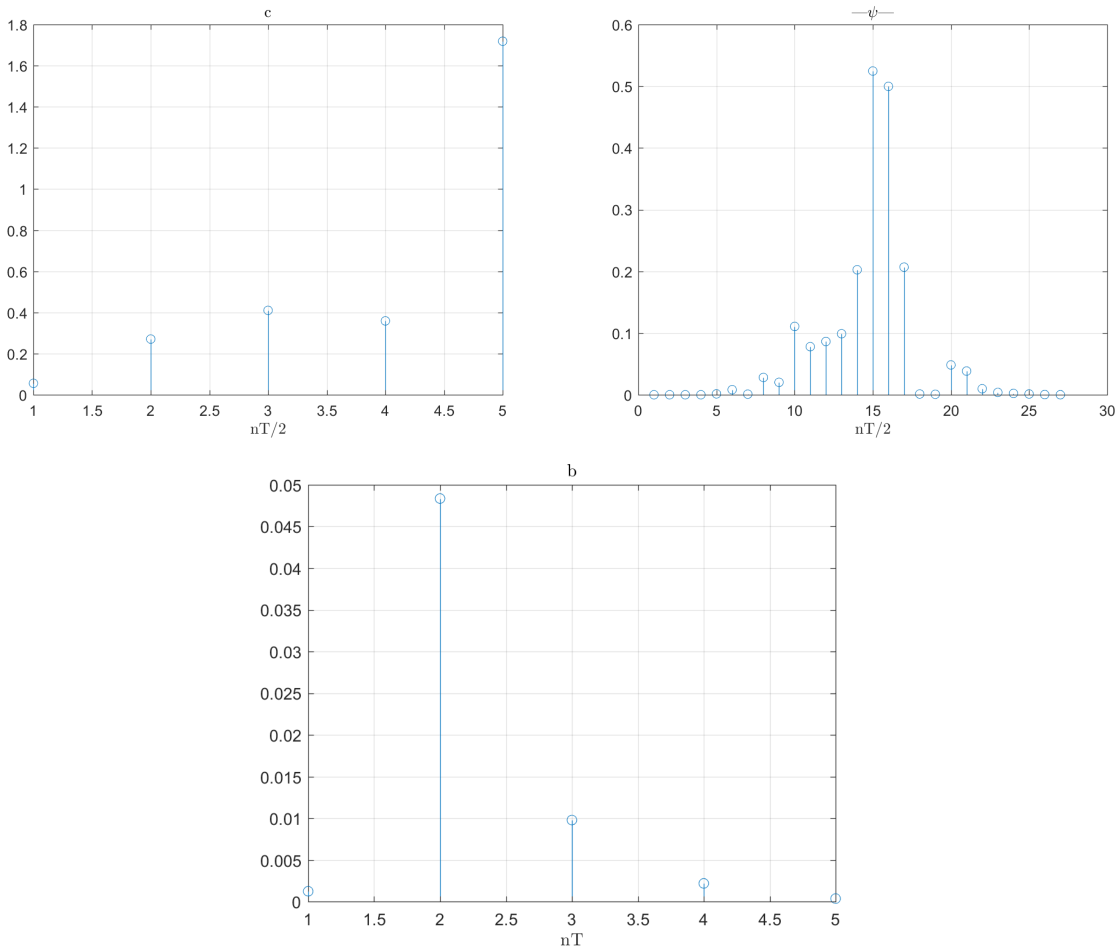

Figure 12. Coefficients of the FF filter c, of the overall impulse response ψ_i and of the FB filter b.

$\bar{t}_0$	M_1	M_2	D
1	1	1	1

Table 4. Parameters of the Receiver (d).

Receiver e

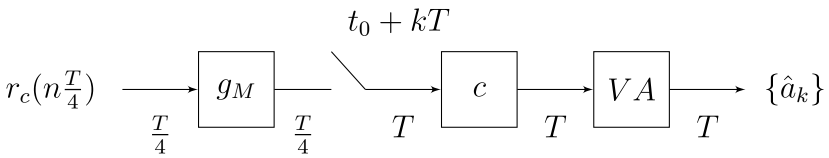

Figure 13. Model for the receiver (e).

Here the receiver uses the channel as given by ψ_D,ψ_{D+1},. . . ,ψ_{D+M_2} of Receiver (b), and performs the maximum likelihood detection through the Viterbi algorithm.
The state of the *finite state machine* associated with the algorithm at time $k = 0, \ldots, K-1$ is

$$\mathbf{s}_k = a_{k+L_1}, a_{k+L_1-1}, \ldots, a_k, \ldots, a_{k-L_2+1} \tag{12}$$

where K is the length of the input sequence, therefore the total number of states is $N_s = M^{L_1+L_2}$.The set of the states $\mathcal{S}$ is the set of the possible values of $\mathbf{s}_k$, such that $\mathbf{s}_k \in \{\sigma_1, \sigma_2, \ldots, \sigma_{N_s}\}$.
With each state σ_j, at instant k the algorithm computes the following quantities:

1. The path metric, $\Gamma(\mathbf{s}_k = \sigma_j)$;

2. The survivor sequence, $\mathcal{L}(\mathbf{s}_k = \sigma_j)$.

These quantities are computed recursively. Starting from $k = 0$, the procedure is repeated until $k = K-1$. The optimum sequence of states is given by the survivor sequence $\mathcal{L}(\mathbf{s}_{K-1} = \sigma_{j,opt})$ associated with $\mathbf{s}_{K-1} = \sigma_{j,opt}$ having minimum cost.

Receiver f

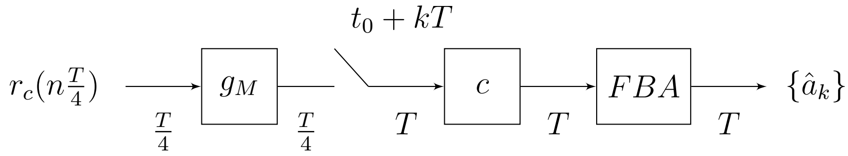

Figure 14. Model for the receiver (f).

Still using the channel as given in the previous section, the signal at the output of the c filter is now processed by a *Forward-Backward algorithm* exploiting the Max-Log-MAP to detect the received symbols $\hat{a}_k$. For each instant $k = 0, 1, \dots, K-1$, the algorithm performs the computation of five metrics, respectively:

1. The channel transition metric, $c_k(i,j)$, $i,j = 1,\dots,N_s$;
2. The Backward metric, $\tilde{b}_k(i)$, $i = 1,\dots,N_s$;
3. The Forward metric, $\tilde{f}_k(j)$, $j = 1,\dots,N_s$;
4. The State metric, $\tilde{v}_k(i)$, $i = 1,\dots,N_s$;
5. The Log-Likelihood function of the isolated symbol $\tilde{l}_k(\beta) = \max\limits_{\substack{i\in[1,\dots,N_s],\\ [\sigma_i]_1=\beta}} = \tilde{v}_k(i)$.

The actual state metric in linear domain is defined as $V_k(i) = P[\mathbf{s}_k = \sigma_i | \mathbf{z}_0^{K-1} = \rho_0^{K-1}]$ and expresses the probability of being in state σ_i at instant k, given the whole observation ρ_0^{K-1}. In the LOG-MAP criterion, the logarithmic variables are indicated with the corresponding lower-case vector, such that $v_k(i) = \ln V_k(i)$. The *log-likelihood* function is then computed as

$$l_k(\beta) = \ln L_k(\beta) = \ln\left(\sum_{\substack{i=1\\ [\sigma_i]_1=\beta}}^{N_s} e^{v_k(i)}\right) \simeq \max_{\substack{i\in[1,\dots,N_s],\\ [\sigma_i]_1=\beta}} v_k(i) \tag{13}$$

from which if follows the approximated likelihood function

$$\tilde{l}_k(\beta) = \max_{\substack{i\in[1,\dots,N_s],\\ [\sigma_i]_1=\beta}} v_k(i) \tag{14}$$

The detected symbol is then evaluated using the MAX-LOG-MAP criterion:

$$\hat{a}_{k+L_1} = \operatorname*{argmax}_{\beta\in\mathcal{A}} \tilde{l}_k(\beta) \tag{15}$$

BER comparison

For each of the different receiver configurations previously described, the symbol error probability is evaluated by simulations with different values of the SNR Γ.

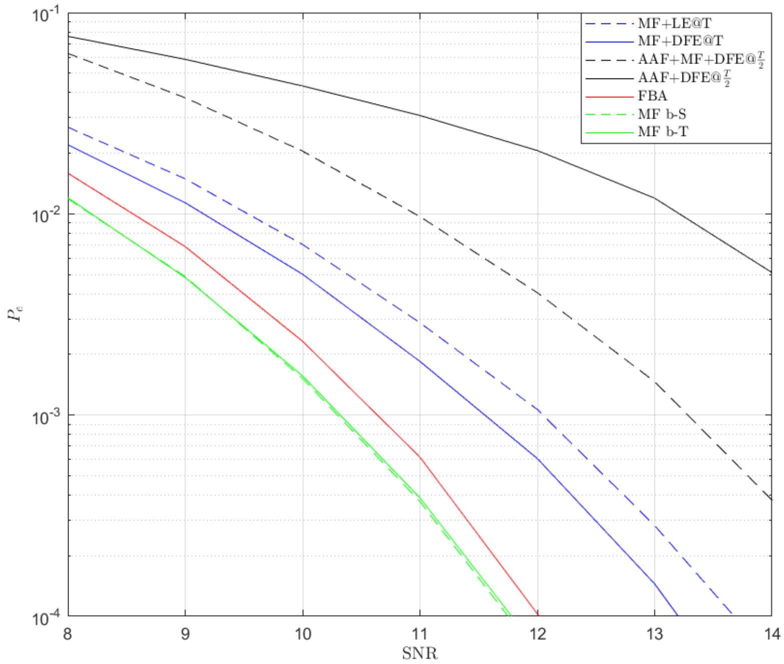

Figure 15. Simulated BER for the different receivers at different SNR values.