Hydro project

Computational Astrophysics

ADVICE: using a low level programming language (such as for example C/C++) can significantly improve the performances of the code you are going to build up, especially in the second task, where time integration is involved. Higher level languages as Python can be used but be aware it will typically take way longer to complete a simulation, modulo optimization tricks (NumPy, ...).

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1 Task1

1.1 Step 1: Advection with first order differencing

The goal is to solve the 1D advection equation

$$\frac{\partial f}{\partial t} + v_0 \frac{\partial f}{\partial x} = 0$$

where v_0 is constant and uniform.

- Define the domain $0 < x < 1$, with periodic conditions (and ghost cells if necessary). Use $N = 100$ points to begin with.
- The advection velocity is $v_0 = 1$. This defines the timestep by the CFL condition as for example $\Delta t = \frac{\Delta x}{2v_0}$. Because $v_0 > 0$, be careful to define the gradient $\partial f/\partial x$ in a upstream way using finite difference.
- Perform the time integration with a first order explicit scheme for the spacial derivative. Start with a first order Euler time integration, but feel free to test also higher order schemes (RK2,

RK4). Test different initial profiles of f. For example a step function $f(x, t = 0) = 1$ if $x < 0.4$, 2 if $0.4 < x < 0.6$, 1 if $x > 0.6$, or a gaussian $f(x, t = 0) = 1 + \exp\left(-\frac{(x-0.5)^2}{\sigma^2}\right)$, with $\sigma = 0.1$ for instance.

- Explore the evolution over different timescales (number of times it crosses the box), under different resolutions ($N = 100$, 200, 500,...). Discuss the diffusion, stability and accuracy of the scheme with varying the timestep and the space resolution.

1.2 Step 2: Advection with second order finite volume (MUSCL)

Same profiles and set-up as Step 1.

- Use the 2^{nd} order MUSCL scheme: $f(x, t + \Delta t) = f(x, t) - v_0[f_R(x + \Delta x/2, t + \Delta t/2) - f_R(x - \Delta x/2, t+\Delta t/2)]\Delta t/\Delta x$, where f_R is the solution of the Riemann problem at the cell interface. To estimate $f(x \pm \Delta x/2, t + \Delta t/2)$, reconstruct $f(x, t)$ on each cell by a linear function. The slope is given by a slope limiter (Minmod, Van Leer). Then move this linear function to the right ($v_0 > 0$) by a distance $v_0\Delta t/2$. The result provides estimates of the states at $f_R(x + \Delta x/2, t + \Delta t/2)$ and $f_R(x - \Delta x/2, t + \Delta t/2)$.

- At each cell interface one has two estimates of f at $t + \Delta t/2$: one from the cell on the left, one from the cell on the right. The solution of the flux, f_R, is the value from the cell upstream (left if $v_0 > 0$)

- Explore the evolution over different timescales (number of times it crosses the box), under different resolutions ($N = 100$, 200, 500,...). Discuss the diffusion, stability and accuracy of the scheme with varying the timestep, the space resolution, and the slope limiter. Compare with the results in Step 1.

2 Task 2

The goal is to solve the 1D diffusion equation

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

where D is constant and uniform. Use the same set-ups and initial profiles as before.

- Discretize the second derivative with finite differences $\frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1}-2f_i+f_{i-1}}{\Delta x^2}$. Integrate in time with first oder Euler or second order RK. Test different values of D: from 10^{-3} to 1. Test different time steps: a typical value is $\Delta t = \frac{\Delta x^2}{2D}$. Change the resolution N as well.

- Compare with the numerical diffusion you obtain in the first order advection setups. Can you estimate the equivalent diffusion coefficient D for a couple of different values for resolution and timestep[1]?

[1]IMPORTANT. In the lecture you have seen that the equivalent diffusion coefficient should be $D = v\Delta x/2$. This

3 Task 3

The goal is to solve the advection-diffusion equation

$$\frac{\partial f}{\partial t} = -v_0 \frac{\partial f}{\partial x} + D \frac{\partial^2 f}{\partial x^2}$$

This is a heat conducting fluid where f represents the temperature. Use the same set-ups and profiles as before.

- Combine the two schemes (for advection, and for diffusion) using the operator splitting technique. First, solve $\partial f/\partial t = D\, \partial^2 f/\partial x^2$, over a duration $\Delta t/2$. Then, solve $\partial f/\partial t = -v_0\, \partial f/\partial x$ over a duration Δt implementing the MUSCL scheme again. Finally, solve again $\partial f/\partial t = D\, \partial^2 f/\partial x^2$, over a duration $\Delta t/2$.

- Explore the effect of resolution, diffusion coefficient, timestep. How do you estimate a good timestep? With the advection CFL rule or based on the diffusion coefficient?.

4 Task 4 (optional)

The goal is to solve 2D advection

$$\frac{\partial f}{\partial t} = -\mathbf{v_0} \cdot \nabla f = -v_x \frac{\partial f}{\partial x} - v_y \frac{\partial f}{\partial y}$$

where $v_x^2 + v_y^2 = v_0^2$.

- Construct a 2D periodic grid in the domain $0 < x < 1$ and $0 < y < 1$, with N_x and N_y cells. By simplicity, assume $N_y = N_x$.

- To solve the 2D advection, use the operator splitting method, as described in Task 3, now using the x-advection for half a timestep, then y-advection for a full timestep, then x-advection for half a timestep. (you can also switch the direction y,x,y)

- Test different advection velocities: for example $v_x = 1$, $v_y = 0$, or $v_x = 0$, $v_y = 1$, or also $v_x = \sqrt{2}/2$, $v_y = \sqrt{2}/2$.

- As CFL condition, start with using $\Delta t = \frac{1}{2} \frac{\Delta x \Delta y}{\sqrt{(\Delta y v_x)^2 + (\Delta x v_y)^2}}$ and then experiment.

- Use 2D version of the initial conditions (step and gaussian). In addition, you can also test other initial profiles (be careful with the periodicity)

- Explore the evolution over different timescales (number of times it crosses the box), under different resolutions ($N_x = N_y = 100, 200, 500, \ldots$), using different slope limiters (Minmod, van Leer, ...). Discuss the diffusion, stability and accuracy of the scheme.

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is true only if in the advection scheme, your time integration technique is at least second order (RK2 for example). If you use Euler, as in the simple first order advection scheme, then this is not true anymore and you would have $D = v\Delta x/2 \cdot (1 - v\Delta t/\Delta x)$. Can you explain why?