week 4

Velocity derivative in SPH

← relevant for continuity and energy eqn

$$\nabla_i \vec{v}_i \cong \sum_j \frac{m_j}{\rho_j} \vec{v}_j \underbrace{\nabla_i W(\vec{r}_i - \vec{r}_j ; h)}_{= \nabla_i W_{ij}}$$

← kernel size

Def: $\nabla \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(v_x, v_y, v_z\right) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

$$\nabla(\rho \vec{v}) \overset{\text{chain rule}}{=} \rho \nabla \vec{v} + \vec{v} \nabla \rho \iff \nabla \vec{v} = \frac{1}{\rho}\left[\nabla(\rho \vec{v}) - \vec{v} \nabla \rho\right]$$

$$\nabla_i \vec{v}_i \cong \frac{1}{\rho_i}\left[\underbrace{\sum_j \frac{m_j}{\rho_j} \rho_j \vec{v}_j \nabla_i W_{ij}}_{\nabla_i(\rho_i \vec{v}_i)} - \underbrace{\vec{v}_i \sum_j \frac{m_j}{\rho_j} \rho_j \nabla_i W_{ij}}_{\vec{v}_i \nabla_i \rho}\right]$$

$$= \frac{1}{\rho_i}\left[\sum_j m_j (\vec{v}_j - \vec{v}_i) \nabla_i W_{ij}\right] = -\frac{1}{\rho_i} \sum_j m_j \underbrace{(\vec{v}_i - \vec{v}_j)}_{\vec{v}_{ij}} \nabla_i W_{ij}$$

$$= -\frac{1}{\rho_i} \sum_j m_j \vec{v}_{ij} \nabla_i W_{ij}$$

Continuity equation: $\frac{D\rho_i}{Dt} = -\rho_i \nabla_i \vec{v}_i = + \rho_i \frac{1}{\rho_i} \sum_j m_j (\vec{v}_i - \vec{v}_j) \nabla_i W_{ij}$

+: material interf, free surf

−: tot. mass not conserved

$$\boxed{\frac{D\rho_i}{Dt} = \sum_j m_j \vec{v}_{ij} \nabla_i W_{ij}}$$

continuity equation

$$\boxed{\rho_i = \sum_j m_j W_{ij}}$$

SPH density ("integral" form)

Momentum equation: (Euler eqn)

$$\frac{D\vec{v}_i}{Dt} = -\frac{1}{\rho_i}\nabla_i P_i = -\frac{1}{\rho_i^2}\sum_j m_j \underbrace{(P_j - P_i)}_{\text{not symmetric } i \leftrightarrow j} \nabla_i W_{ij}$$

force $\to 0$ for $P = \text{const}$ but not symmetric under $i \leftrightarrow j$

$\Rightarrow$ linear and angular momentum not conserved!

(Newton: $\vec{F}_{ij} = -F_{ji}$)

Instead:

$$\frac{\nabla P}{\rho} = \nabla\left(\frac{P}{\rho}\right) + \frac{P}{\rho^2}\nabla\rho$$

Chain rule!

$$\Rightarrow \frac{D\vec{v}_i}{Dt} = -\nabla_i\left(\frac{P_i}{\rho_i}\right) - \frac{P_i}{\rho_i^2}\nabla_i\rho_i = -\sum_j \frac{m_j}{\rho_j}\frac{P_j}{\rho_j}\nabla_i W_{ij} - \frac{P_i}{\rho_i^2}\sum_j \frac{m_j}{\rho_j}\rho_j \nabla_i W_{ij}$$

$$\boxed{\frac{D\vec{v}_i}{Dt} = -\sum_j m_j \underbrace{\left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right)}_{\text{symmetric } i \leftrightarrow j} \nabla_i W_{ij}}$$

SPH momentum equation (symmetric)

momentum conserved

Caveat: if particles i and j have very different kernel sizes h_i, h_j then this is not quite true.

3) **Energy equation:**

$$\frac{Du_i}{Dt} = -\frac{P_i}{\rho_i}\nabla_i \vec{v}_i = +\frac{P_i}{\rho_i^2}\sum_j m_j \vec{v}_{ij}\nabla_i W_{ij}$$

Kernel derivative

$$\boxed{\frac{Du_i}{Dt} = -\frac{P_i}{\rho_i^2}\sum_j m_j \vec{v}_{ij}\nabla_i W_{ij}}$$

SPH energy equation (Benz formulation)

alternative derivation:

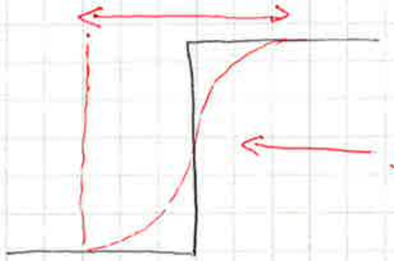
$$L = \int_V \rho\left(\frac{1}{2}\vec{v}^2 - u\right) dV$$

Lagrangian of an ideal fluid

$\hookrightarrow$ unique EOM but some "good" formulations are missing!

Artificial viscosity

Inviscid fluid $\longrightarrow$ what happens in shocks?



smooth discontinuity

shock: need viscosity (kinetic energy $\to$ thermal energy)

AV: add dissipation term in momentum and energy equation to smoothen shock and include shock heating!

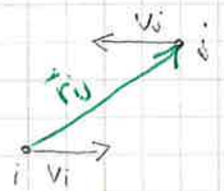
$$\frac{D\vec{v}_i}{Dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{ij} \right) \nabla_i W_{ij}$$

AV

$$\frac{Du_i}{Dt} = + \frac{P_i}{\rho_i^2} \sum_j m_j \vec{v}_{ij} \nabla_i W_{ij} + \underbrace{\frac{1}{2} \sum_j m_j \Pi_{ij} \vec{v}_{ij} \nabla_i W_{ij}}_{\left.\frac{Du_i}{Dt}\right|_{AV}}$$

$$\Pi_{ij} = \begin{cases} \dfrac{-\overset{=1 \text{ or } 0.5}{\alpha} \bar{c}_{ij} \mu_{ij} + \overset{2\alpha}{\beta} \mu_{ij}^2}{\bar{\rho}_{ij}}, & \vec{v}_{ij} \cdot \vec{r}_{ij} < 0 \\ 0, & \text{else} \end{cases}$$

converging flow



where: $\bar{\rho}_{ij} = \frac{1}{2}(\rho_i + \rho_j)$, $\underset{\text{sound speed}}{\bar{c}_{ij}} = \frac{1}{2}(c_i + c_j)$, $\mu_{ij} = \dfrac{\bar{h}_{ij} \vec{v}_{ij} \cdot \vec{r}_{ij}}{|\vec{r}_{ij}| + \eta^2}$

$\bar{h}_{ij} = \frac{1}{2}(h_i + h_j)$

η^2: small, divergence!

Ideal gas: $P = (\gamma - 1)\rho u$; $c_s = \sqrt{\gamma(\gamma-1)u}$ sqrt() expensive; $\dfrac{P}{\rho^2} = \dfrac{c_s^2}{\gamma \rho}$

momentum eqn

Code: Variables for particles: $\vec{r}, \vec{v}, u, c, \rho, h$ ← kernel size from NN search

Calculate: $\vec{a} = \frac{D\vec{v}}{Dt}$ and $\dot{u} = \frac{Du}{Dt}$ Store too!

2 extra temp. var: $\underbrace{\vec{v}_{pred}, \; u_{pred}}_{\text{predicted quantities}}$ ← ODE "Leap frog"

$\vec{a} = \vec{a}(\underbrace{\vec{r}, u}_{\rho, c, P}, \vec{v})$; $\dot{u} = \dot{u}(\vec{r}, u, \vec{v})$ ← AV

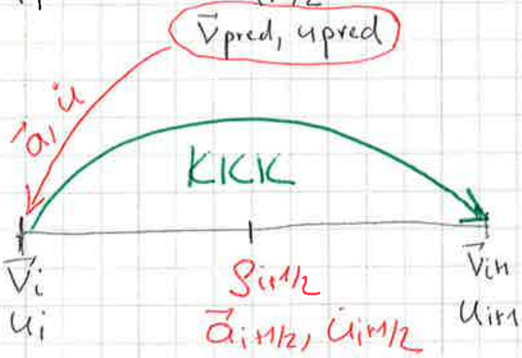
Total: $\vec{r}, \vec{v}, u, c, \rho, h, \vec{a}, \dot{u}, \vec{v}_{pred}, u_{pred}$

"Leap frog": DKD

DRIFT1:

$$\vec{r}_{i+1/2} = \vec{r}_i + \vec{v}_i \tfrac{1}{2}\Delta t$$
$$\vec{v}_{pred} = \vec{v}_i + \vec{a}_i \tfrac{1}{2}\Delta t$$
$$u_{pred} = u_i + \dot{u}_i \tfrac{1}{2}\Delta t$$

KICK:



$$\vec{v}_{i+1} = \vec{v}_i + \vec{a}(\vec{r}_{i+1/2}, \vec{v}_{pred}, u_{pred})\Delta t$$
$$u_{i+1} = u_i + \dot{u}_i(\vec{r}_{i+1/2}, \vec{v}_{pred}, u_{pred})\Delta t$$

DRIFT2:

$$\vec{r}_{i+1} = \vec{r}_{i+1/2} + \vec{v}_{i+1} \tfrac{1}{2}\Delta t$$

Functions: $\underbrace{\text{DRIFT1( )}}_{\text{Calc. } \vec{v}_{pred},\, u_{pred}}$, KICK( ), DRIFT2( ), CALC FORCES( )

Week 4

SPH init:
DRIFT($\Delta t = 0$) ← zero time drift: $\vec{v}_{pred}$, u_{pred}
CALCFORCE()

Evolution:

```
for (step=0; step < NSTEP; step++) {
    DRIFT1 (Δt/2)
    CALCFORCE()
    KICK (Δt)
    DRIFT2 (Δt/2)
}
```

DKD

Set $\vec{v}_{pred}$, u_{pred}

CALCFORCE: TREEBUILD
NN-DENSITY ← $\forall p$: calculate ρ_i
CALCSOUND ← $\forall p$: $c_i = \sqrt{\gamma(\gamma-1) u_{pred}}$
NN-SPHFORCE ← $\forall p$: calculate $\vec{a}, \dot{u}$

use: $\frac{P}{\rho^2} = \frac{c^2}{\gamma \rho}$

DRIFT1(Δt / $\Delta t/2$): $\vec{r}_i \mathrel{+}= \vec{v}_i \Delta t$
$\vec{v}_{pred} = \vec{v}_i + \vec{a} \Delta t$
$u_{pred} = u_i + \dot{u} \Delta t$

KICK(Δt): $\vec{v}_i \mathrel{+}= \vec{a} \Delta t$
$u \mathrel{+}= \dot{u} \Delta t$

DRIFT2(Δt): $r_i \mathrel{+}= v_i \cdot \Delta t$

Symmetric kernel: $\bar{W}_{ij} = \frac{1}{2}\left(W_{ij}(r_{ij}, h_i) + W_{ij}(r_{ij}, h_b)\right)$
↳ symmetric → true conservation of momentum!