

My solutions to  
Deep Learning: Foundations and Concepts

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## 2 Probabilities

### 2.1

$$\begin{aligned} p(C = 1|T = 1) &= \frac{p(T = 1|C = 1)p(C = 1)}{p(T = 1)} \\ &= \frac{p(T = 1|C = 1)p(C = 1)}{p(T = 1, C = 0) + p(T = 1, C = 1)} \\ &= \frac{p(T = 1|C = 1)p(C = 1)}{p(T = 1|C = 0)p(C = 0) + p(T = 1|C = 1)p(C = 1)} \\ &= \frac{0.9 \cdot 0.001}{0.03 \cdot (1 - 0.001) + 0.9 \cdot 0.001} \\ &\approx 0.029 \end{aligned}$$

### 2.2

Let  $Y$  denote the yellow die,  $B$  the blue die,  $G$  the green die and  $R$  the red die. We consider throws of pairs of independent dice, i.e.  $p(D_1, D_2) = p(D_1)p(D_2)$ . Each die takes on a unique value in a given throw, such that e.g.  $(G = 5) := (G = 5, (B = 0 \text{ or } B = 4))$  and  $(G = 1, B = 0)$  are mutually exclusive events, hence  $p(G = 5 \text{ or } (G = 1, B = 0)) = P(G = 5) + P(G = 1, B = 0)$ .

$$\begin{aligned} p(B > Y) &= p(B = 4, Y = 3) \\ &= p(B = 4)p(Y = 3) \\ &= \frac{4}{6} \cdot \frac{6}{6} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned}
p(G > B) &= p(G = 5 \text{ or } (G = 1, B = 0)) \\
&= p(G = 5) + p(G = 1)p(B = 0) \\
&= \frac{3}{6} + \frac{3}{6} \cdot \frac{2}{6} \\
&= \frac{2}{3} \\
p(R > G) &= p(R = 6 \text{ or } (R = 2, G = 1)) \\
&= p(R = 6) + p(R = 2)p(G = 1) \\
&= \frac{2}{6} + \frac{4}{6} \cdot \frac{3}{6} \\
&= \frac{2}{3} \\
p(Y > R) &= p(Y = 3, R = 2) \\
&= p(Y = 3)p(R = 2) \\
&= \frac{6}{6} \cdot \frac{4}{6} \\
&= \frac{2}{3}
\end{aligned}$$

### 2.3

$$\begin{aligned}
\int_{\mathbb{R}} p_U(u)p_V(y-u)du &= \frac{d}{dy} \int_{-\infty}^y \left( \int_{\mathbb{R}} p_U(u)p_V(\tilde{y}-u)du \right) d\tilde{y} \\
&= \frac{d}{dy} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathbb{1}_{\tilde{y} \leq y}(u, \tilde{y}) p_U(u)p_V(\tilde{y}-u)du \right) d\tilde{y} \\
&= \frac{d}{dy} \int_{\mathbb{R}^2} \mathbb{1}_{\tilde{y} \leq y}(u, \tilde{y}) p_U(u)p_V(\tilde{y}-u)d(u, \tilde{y}) \quad \text{Fubini} \\
&\stackrel{(\star)}{=} \frac{d}{dy} \int_{\mathbb{R}^2} \mathbb{1}_{u+v \leq y}(u, v) p_U(u)p_V(v) \underbrace{\left| \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right|}_{=1} d(u, v) \\
&= \frac{d}{dy} \int_{\mathbb{R}^2} \mathbb{1}_{u+v \leq y}(u, v) p_{U,V}(u, v)d(u, v) \quad U, V \text{ ind.} \\
&= \frac{d}{dy} P(U + V \leq y) \\
&= \frac{d}{dy} P(Y \leq y) \\
&= p_Y(y)
\end{aligned}$$

( $\star$ ) Transformation  $\tilde{y}(u, v) := (u, u + v)$ .

## 2.4

$$\begin{aligned}\int_c^d p(x)dx &= \int_c^d \frac{1}{d-c}dx \\&= \frac{1}{d-c} \int_c^d dx \\&= \frac{1}{d-c} [x]_c^d \\&= \frac{1}{d-c}(d-c) \\&= 1\end{aligned}$$

$$\begin{aligned}\mathbb{E}_p[X] &= \int_c^d xp(x)dx \\&= \int_c^d x \frac{1}{d-c}dx \\&= \frac{1}{d-c} \int_c^d xdx \\&= \frac{1}{d-c} \left[ \frac{1}{2}x^2 \right]_c^d \\&= \frac{1}{2(d-c)}(d^2 - c^2) \\&= \frac{1}{2(d-c)}(d-c)(d+c) \\&= \frac{d+c}{2}\end{aligned}$$

$$\begin{aligned}\mathbb{E}_p[X^2] &= \int_c^d x^2p(x)dx \\&= \int_c^d x^2 \frac{1}{d-c}dx \\&= \frac{1}{d-c} \int_c^d x^2dx \\&= \frac{1}{d-c} \left[ \frac{1}{3}x^3 \right]_c^d \\&= \frac{1}{3(d-c)}(d^3 - c^3) \\&= \frac{1}{3(d-c)}(d-c)(d^2 + c^2 + cd)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3}(d^2 + c^2 + cd) \\
\mathbb{E}_p[X]^2 &= \left(\frac{d+c}{2}\right)^2 \\
&= \frac{d^2 + 2cd + c^2}{4} \\
\mathbb{V}_p[X] &= \mathbb{E}_p[X^2] - \mathbb{E}_p[X]^2 \\
&= \frac{1}{3}(d^2 + c^2 + cd) - \frac{(d+c)^2}{2^2} \\
&= \frac{1}{3}(d^2 + c^2 + cd) - \frac{d^2 + 2cd + c^2}{4} \\
&= \frac{1}{12}(4d^2 + 4c^2 + 4cd - 3d^2 - 6cd - 3c^2) \\
&= \frac{1}{12}(d^2 - 2cd + c^2) \\
&= \frac{1}{12}(d-c)^2
\end{aligned}$$

## 2.5

### Exponential distribution

$$\begin{aligned}
\int p(x|\lambda)dx &= \int_0^\infty \lambda e^{-\lambda x} dx \\
&= \lambda \int_0^\infty e^{-\lambda x} dx \\
&= \lambda \left[ -\frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty \\
&= \lambda \left[ 0 - \left( -\frac{1}{\lambda} e^{-\lambda \cdot 0} \right) \right] \\
&= \lambda \cdot \frac{1}{\lambda} \\
&= 1
\end{aligned}$$

### Laplace distribution

$$\begin{aligned}
\int p(x|\mu, \gamma) &= \int_{-\infty}^\infty \frac{1}{2\gamma} e^{-\frac{|x-\mu|}{\gamma}} dx \\
&= \frac{1}{2\gamma} \int_{-\infty}^\infty e^{-\frac{|x-\mu|}{\gamma}} dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\gamma} \left( \int_{-\infty}^{\mu} e^{-\frac{|x-\mu|}{\gamma}} dx + \int_{\mu}^{\infty} \frac{1}{\gamma} e^{-\frac{|x-\mu|}{\gamma}} dx \right) \\
&= \frac{1}{2\gamma} \left( \int_{-\infty}^{\mu} e^{-\frac{\mu-x}{\gamma}} dx + \int_{\mu}^{\infty} e^{-\frac{x-\mu}{\gamma}} dx \right) \\
&= \frac{1}{2\gamma} \left( \int_{-\infty}^{\mu} e^{\frac{x-\mu}{\gamma}} dx + \int_{\mu}^{\infty} e^{\frac{\mu-x}{\gamma}} dx \right) \\
&= \frac{1}{2\gamma} \left( \left[ \gamma e^{\frac{x-\mu}{\gamma}} \right]_{-\infty}^{\mu} + \left[ -\gamma e^{\frac{\mu-x}{\gamma}} \right]_{\mu}^{\infty} \right) \\
&= \frac{1}{2\gamma} (\gamma [e^0 - 0] - \gamma [0 - (e^0)]) \\
&= \frac{\gamma}{2\gamma} (1 - (-1)) \\
&= \frac{1}{2} \cdot 2 \\
&= 1
\end{aligned}$$

## 2.6

$$\begin{aligned}
\int p(x|\mathcal{D}) &= \int_{-\infty}^{\infty} \frac{1}{N} \sum_{n=1}^N \delta(x - x_n) dx \\
&= \frac{1}{N} \sum_{n=1}^N \int_{-\infty}^{\infty} \delta(x - x_n) dx \quad \text{finite sum} \\
&= \frac{1}{N} \sum_{n=1}^N 1 \quad \text{by def. of } \delta \\
&= \frac{1}{N} \cdot N \\
&= 1
\end{aligned}$$

## 2.8

$$\begin{aligned}
\mathbb{V}[f] &= \mathbb{E} [(f(X) - \mathbb{E}[f(X)])^2] \\
&= \mathbb{E} [f(X)^2 - 2f(X)\mathbb{E}[f(X)] + \mathbb{E}[f(X)]^2] \\
&= \mathbb{E} [f(X)^2] - 2\mathbb{E}[f(X)]\mathbb{E}[f(X)] + \mathbb{E}[f(X)]^2 \quad \text{linearity of } \mathbb{E} \\
&= \mathbb{E} [f(X)^2] - 2\mathbb{E}[f(X)]^2 + \mathbb{E}[f(X)]^2 \\
&= \mathbb{E} [f(X)^2] - \mathbb{E}[f(X)]^2
\end{aligned}$$

## 2.9

For independent random variables it holds that

$$\begin{aligned}
 \mathbb{E}[XY] &:= \int xyp(x, y)d(x, y) \\
 &= \int \int yxp(x)p(y)dx dy \quad \text{ind.} \\
 &= \int y \left( \int xp(x)dx \right) p(y)dy \\
 &= \int xp(x)dx \int yp(y)dy \\
 &= \mathbb{E}[X]\mathbb{E}[Y]
 \end{aligned}$$

and consequently

$$\begin{aligned}
 \text{cov}[X, Y] &:= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
 &= \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] \\
 &= 0
 \end{aligned}$$

## 2.10

$$\begin{aligned}
 \mathbb{E}[X + Z] &= \int (x + z)p(x, z)d(x, z) \\
 &= \int \int (x + z)p(x)p(z)dx dz \quad \text{ind.} \\
 &= \int \int xp(x)p(z)dx dz + \int \int zp(x)p(z)dx dz \\
 &= \int \underbrace{\left( \int xp(x)dx \right)}_{=\mathbb{E}[X]} p(z)dz + \int z \underbrace{\left( \int p(x)dx \right)}_{=1} p(z)dz \\
 &= \mathbb{E}[X] \underbrace{\int p(z)dz}_{=1} + \int zp(z)dz \\
 &= \mathbb{E}[X] + \mathbb{E}[Z]
 \end{aligned}$$

$$\begin{aligned}
 \mathbb{V}[X + Z] &:= \mathbb{E}[(X + Z)^2] - \mathbb{E}[X + Z]^2 \\
 &= \mathbb{E}[X^2 + 2XZ + Z^2] - \mathbb{E}[X + Z]^2 \quad \text{linearity (cf. above)}
 \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}[X^2] + 2\mathbb{E}[XZ] + \mathbb{E}[Z^2] - (\mathbb{E}[X] + \mathbb{E}[Z])^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XZ] + \mathbb{E}[Z^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X]\mathbb{E}[Z] - \mathbb{E}[Z]^2 \quad \text{ind.} \\
&= \mathbb{E}[X^2] + \cancel{2\mathbb{E}[X]\mathbb{E}[Z]} + \mathbb{E}[Z^2] - \mathbb{E}[X]^2 - \cancel{2\mathbb{E}[X]\mathbb{E}[Z]} - \mathbb{E}[Z]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 \\
&= \mathbb{V}[X] + \mathbb{V}[Z]
\end{aligned}$$

## 2.11

### Expectation

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X|Y]] &= \int_{\mathbb{R}} \int_{\mathbb{R}} xp(x|y) dx p(y) dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} x \frac{p(x, y)}{p(y)} dx p(y) dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} xp(x, y) dx dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} xp(x, y) dy dx \quad \text{Fubini } (\star) \\
&= \int_{\mathbb{R}} x \underbrace{\int_{\mathbb{R}} p(x, y) dy}_{=p(x)} dx \\
&= \mathbb{E}[X]
\end{aligned}$$

( $\star$ ) Assuming  $X$  integrable:  $\infty > \mathbb{E}[|X|] = \int_{\mathbb{R}} |x|p(x)dx = \int_{\mathbb{R}} |x| \int_{\mathbb{R}} p(x, y) dy dx = \int_{\mathbb{R}} \int_{\mathbb{R}} |x|p(x, y) dy dx \stackrel{p \geq 0}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} |xp(x, y)| dy dx$ , i.e.  $xp(x, y)$  integrable.

### Variance

$$\begin{aligned}
\mathbb{E}[\mathbb{V}[X|Y]] + \mathbb{V}[\mathbb{E}[X|Y]] &= \mathbb{E}[\mathbb{E}[X^2|Y] - \mathbb{E}[X|Y]^2] \\
&\quad + \mathbb{E}[\mathbb{E}[X|Y]^2] - \mathbb{E}[\mathbb{E}[X|Y]]^2 \\
&= \mathbb{E}[\mathbb{E}[X^2|Y]] - \cancel{\mathbb{E}[\mathbb{E}[X|Y]^2]} \\
&\quad + \cancel{\mathbb{E}[\mathbb{E}[X|Y]^2]} - \mathbb{E}[\mathbb{E}[X|Y]]^2 \quad \mathbb{E} \text{ linear} \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \quad (\dagger) \\
&= \mathbb{V}[X]
\end{aligned}$$

( $\dagger$ ) Assuming  $X^2$  integrable (and consequently  $X$  integrable e.g. via Cauchy-Schwarz), use result for expectation from above.

## 2.14

$$\begin{aligned}
\frac{\partial}{\partial x} p(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
&= p(x) \left( -\frac{2(x-\mu)}{2\sigma^2} \right) \\
&= -\sigma^{-2} p(x) (x - \mu)
\end{aligned}$$

By definition of univariate Gaussian  $\sigma^2 > 0$ , and consequently  $\sigma^{-2} > 0$ . Since also  $p(x) > 0$  for all  $x$ ,  $\frac{\partial}{\partial x} p(x) = 0$  only if  $x = \mu$ .

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} p(x) &= -\sigma^{-2} (-\sigma^{-2} p(x) (x - \mu)^2 + p(x)) \\
&= \sigma^{-2} p(x) (\sigma^{-2} (x - \mu)^2 - 1)
\end{aligned}$$

$\implies \frac{\partial^2}{\partial x^2} p(\mu) = -\sigma^2 p(\mu) < 0$ , so  $\mu$  maximum of  $p$  which means  $\mu$  mode of univariate Gaussian.

## 2.15

### Mean

$$\begin{aligned}
0 &\stackrel{!}{=} \frac{\partial}{\partial \mu} \ln p(x|\mu, \sigma^2) \\
&= \frac{\partial}{\partial \mu} \left( -\frac{1}{2\sigma^2} \sum_{n=1}^N (x_n - \mu)^2 - \frac{N}{2} \ln \sigma^2 - \frac{N}{2} \ln(2\pi) \right) \\
&= -\frac{1}{2\sigma^2} \sum_{n=1}^N 2(x_n - \mu)(-1) \\
&= \frac{1}{\sigma^2} \sum_{n=1}^N (x_n - \mu) \\
&= \frac{1}{\sigma^2} \sum_{n=1}^N x_n - \frac{N\mu}{\sigma^2} \\
\stackrel{N \geq 0}{\implies} \mu &= \frac{1}{N} \sum_{n=1}^N x_n := \tilde{\mu}
\end{aligned}$$



$$\begin{aligned}
\frac{\partial^2}{\partial \mu^2} \ln p(x|\mu, \sigma^2) &= \frac{\partial}{\partial \mu} \left( \frac{1}{\sigma^2} \sum_{n=1}^N x_n - \frac{N\mu}{\sigma^2} \right) \\
&= -\frac{N}{\sigma^2} \\
\Rightarrow \quad \frac{\partial^2 \ln p}{\partial \mu^2}(\tilde{\mu}) &= -\frac{N}{\sigma^2} \stackrel{N>0}{<} 0 \Rightarrow \tilde{\mu} \text{ maximum}
\end{aligned}$$

**Variance**

$$\begin{aligned}
0 &\stackrel{!}{=} \frac{\partial}{\partial \sigma^2} \ln p(x|\mu, \sigma^2) \\
&= \frac{\partial}{\partial \sigma^2} \left( -\frac{1}{2\sigma^2} \sum_{n=1}^N (x_n - \mu)^2 - \frac{N}{2} \ln \sigma^2 - \frac{N}{2} \ln(2\pi) \right) \\
&= -\frac{1}{2\sigma^4} (-1) \sum_{n=1}^N (x_n - \mu)^2 - \frac{N}{2} \frac{1}{\sigma^2} \\
\Rightarrow \quad 0 &= \sum_{n=1}^N (x_n - \mu)^2 - N\sigma^2 \\
\stackrel{N>0}{\Rightarrow} \quad \sigma^2 &= \frac{1}{N} \sum_{n=1}^N (x_n - \mu)^2 := \tilde{\sigma}^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial \sigma^4} \ln p(x|\mu, \sigma^2) &= \frac{\partial}{\partial \sigma^2} \left( \frac{1}{2\sigma^4} \sum_{n=1}^N (x_n - \mu)^2 - \frac{N}{2\sigma^2} \right) \\
&= \frac{1}{2\sigma^6} (-2) \sum_{n=1}^N (x_n - \mu)^2 - \frac{N}{2\sigma^4} (-1) \\
&= -\frac{1}{\sigma^6} \sum_{n=1}^N (x_n - \mu)^2 + \frac{N}{2\sigma^4} \\
&= -\frac{1}{\sigma^4} \left( \frac{1}{\sigma^2} \sum_{n=1}^N (x_n - \mu)^2 - \frac{N}{2} \right) \\
\Rightarrow \quad \frac{\partial^2 \ln p}{\partial \sigma^4}(\tilde{\sigma}^2) &= -\frac{1}{\tilde{\sigma}^4} \left( \frac{1}{\tilde{\sigma}^2} N\tilde{\sigma}^2 - \frac{N}{2} \right) = -\frac{N}{2\tilde{\sigma}^4} \stackrel{N>0}{<} 0 \Rightarrow \tilde{\sigma}^2 \text{ max.}
\end{aligned}$$

## 2.16

If  $n \neq m$ , then by assumption  $X_n \perp X_m$ . Hence  $\mathbb{E}[X_n X_m] = \mathbb{E}[X_n] \mathbb{E}[X_m] \stackrel{(2.52)}{=} \mu \cdot \mu = \mu^2$ . If  $n = m$ , then  $\mathbb{E}[X_n X_m] = \mathbb{E}[X_m^2] \stackrel{(2.53)}{=} \mu^2 + \sigma^2$ . Taken together  $\mathbb{E}[X_n X_m] = \mu^2 + \delta_{nm} \sigma^2$ .

### Mean

$$\begin{aligned}
 \mathbb{E}[\mu_{ML}] &= \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^N X_n\right] & (2.57) \\
 &= \frac{1}{N} \sum_{n=1}^N \mathbb{E}[X_n] & \text{linearity of expectation} \\
 &= \frac{1}{N} \sum_{n=1}^N \mu & \text{by assumption} \\
 &= \frac{1}{N} (N\mu) \\
 &= \mu
 \end{aligned}$$

### Variance

$$\begin{aligned}
 \mathbb{E}[\sigma_{ML}^2] &= \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^N (X_n - \mu_{ML})^2\right] & (2.58) \\
 &= \frac{1}{N} \sum_{n=1}^N \mathbb{E}[(X_n - \mu_{ML})^2] \\
 &= \frac{1}{N} \sum_{n=1}^N \mathbb{E}\left[\left(X_n - \frac{1}{N} \sum_{m=1}^N X_m\right)^2\right] & (2.57) \\
 &= \frac{1}{N} \sum_{n=1}^N \mathbb{E}\left[X_n^2 - \frac{2}{N} \sum_{m=1}^N X_n X_m + \frac{1}{N^2} \sum_{l=1}^N \sum_{k=1}^N X_k X_l\right] \\
 &= \frac{1}{N} \sum_{n=1}^N \left(\mathbb{E}[X_n^2] - \frac{2}{N} \sum_{m=1}^N \mathbb{E}[X_n X_m] + \frac{1}{N^2} \sum_{l=1}^N \sum_{k=1}^N \mathbb{E}[X_k X_l]\right) \\
 &= \frac{1}{N} \sum_{n=1}^N \left(\mu^2 + \sigma^2 - \frac{2}{N} (1 \cdot (\mu^2 + \sigma^2) + (N-1) \cdot \mu^2) \right. \\
 &\quad \left. + \frac{1}{N^2} (N \cdot (\mu^2 + \sigma^2) + (N^2 - N) \cdot \mu^2) \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N} N \left( \frac{1}{N} (N\mu^2 + N\sigma^2) - \frac{1}{N} (2\sigma^2 + 2N\mu^2) + \frac{1}{N} (\sigma^2 + N\mu^2) \right) \\
&= \left( \frac{N-1}{N} \right) \sigma^2
\end{aligned}$$

## 2.17

$$\begin{aligned}
\mathbb{E} [\hat{\sigma}^2] &= \mathbb{E} \left[ \frac{1}{N} \sum_{n=1}^N (X_n - \mu)^2 \right] \\
&= \frac{1}{N} \sum_{n=1}^N \mathbb{E} [(X_n - \mu)^2] \\
&= \frac{1}{N} \sum_{n=1}^N \mathbb{E} [X_n^2 - 2\mu X_n + \mu^2] \\
&= \frac{1}{N} \sum_{n=1}^N (\mathbb{E} [X_n^2] - 2\mu \mathbb{E} [X_n] + \mu^2) \\
&= \frac{1}{N} \sum_{n=1}^N (\mu^2 + \sigma^2 - 2\mu \cdot \mu + \mu^2) \\
&= \frac{1}{N} N \sigma^2 \\
&= \sigma^2
\end{aligned}$$

## 2.18

Analogous to variance in (2.15).

## 2.20

$$J_g = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 + 5 \operatorname{sech}^2(5x_1) & 0 \\ x_1^2 & 1 + 5 \operatorname{sech}^2(5x_2) \end{pmatrix}$$

## 2.22

$$f(p) := - \sum_{i=1}^M p_i \ln p_i$$

$$\begin{aligned}
g(p) &:= \sum_{i=1}^M p_i - 1 \\
L(p, \lambda) &:= f(p) + \lambda g(p) \\
&= - \sum_{i=1}^M p_i \ln p_i + \lambda \left( \sum_{i=1}^M p_i - 1 \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial p_i} &= -1 \cdot \ln p_i - p_i \cdot \frac{1}{p_i} + \lambda \cdot (1 - 0) \\
&= -\ln p_i - 1 + \lambda
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \lambda} &= - \left( \sum_{i=1}^M p_i - 1 \right) \\
&= 1 - \sum_{i=1}^M p_i
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial L}{\partial p_i} \stackrel{!}{=} 0 \\
\Leftrightarrow \quad &-\ln p_i - 1 + \lambda = 0 \\
&\Leftrightarrow \quad -\ln p_i = 1 - \lambda \\
&\Leftrightarrow \quad \ln p_i = \lambda - 1 \\
&\Leftrightarrow \quad p_i = e^{\lambda-1} \quad (1)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial L}{\partial \lambda} \stackrel{!}{=} 0 \\
\Leftrightarrow \quad &1 - \sum_{i=1}^M p_i = 0 \\
\stackrel{(1)}{\Rightarrow} \quad &1 - \sum_{i=1}^M e^{\lambda-1} = 0 \\
\Leftrightarrow \quad &1 - M \cdot e^{\lambda-1} = 0 \\
&\Leftrightarrow \quad 1 = M \cdot e^{\lambda-1} \\
&\Leftrightarrow \quad \frac{1}{M} = e^{\lambda-1}
\end{aligned}$$

Hence for all  $i \in \{1, \dots, M\}$ :

$$p_i = \frac{1}{M}$$

and consequently

$$\begin{aligned} H[p] &= - \sum_{i=1}^M p_i \ln p_i \\ &= - \sum_{i=1}^M \frac{1}{M} \ln \frac{1}{M} \\ &= -M \cdot \frac{1}{M} (-\ln M) \\ &= \ln M \end{aligned}$$