My solutions to

Deep Learning: Foundations and Concepts

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14 Sampling

14.1

$$
\begin{aligned}
\mathbb{E}\left[\frac{1}{L}\sum_{l=1}^{L} f\left(Z_l\right)\right] &= \frac{1}{L}\sum_{l=1}^{L}\mathbb{E}\left[f\left(Z_l\right)\right] && \text{linearity of expectation} \\
&= \frac{1}{L}\sum_{l=1}^{L}\mathbb{E}\left[f(Z)\right] && Z_1,\ldots,Z_L \text{ identically distributed} \\
&= \frac{1}{L}\cdot L\cdot\mathbb{E}\left[f(Z)\right] \\
&= \mathbb{E}\left[f(Z)\right]
\end{aligned}
$$

14.2

$$
\begin{aligned}
\mathbb{V}\left[\frac{1}{L}\sum_{l=1}^{L} f(Z_l)\right] &= \mathbb{E}\left[\left(\frac{1}{L}\sum_{l=1}^{L} f(Z_l)\right)^2\right] - \mathbb{E}\left[\frac{1}{L}\sum_{l=1}^{L} f(Z_l)\right]^2 \\
&= \mathbb{E}\left[\left(\frac{1}{L}\sum_{l=1}^{L} f(Z_l)\right)^2\right] - \mathbb{E}\left[f(Z)\right]^2 && \text{i.d. - cf. (14.1)} \\
&= \mathbb{E}\left[\frac{1}{L^2}\sum_{l=1}^{L} f(Z_l)^2 + \frac{1}{L^2}2\sum_{l=1}^{L}\sum_{k<l}^{L} f(Z_l)f(Z_k)\right] - \mathbb{E}\left[f(Z)\right]^2 \\
&= \frac{1}{L^2}\sum_{l=1}^{L}\mathbb{E}\left[f(Z_l)^2\right] + \frac{2}{L^2}\sum_{l=1}^{L}\sum_{k<l}^{L}\mathbb{E}\left[f(Z_l)f(Z_k)\right] - \mathbb{E}\left[f(Z)\right]^2 \\
&\overset{\text{ind.}}{=} \frac{1}{L^2}\sum_{l=1}^{L}\mathbb{E}\left[f(Z_l)^2\right] + \frac{2}{L^2}\sum_{l=1}^{L}\sum_{k<l}^{L}\mathbb{E}\left[f(Z_l)\right]\mathbb{E}\left[f(Z_k)\right] - \mathbb{E}\left[f(Z)\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{i.d.}}{=} \frac{1}{L^2}\sum_{l=1}^{L}\mathbb{E}\left[f(Z)^2\right] + \frac{2}{L^2}\sum_{l=1}^{L}\sum_{k<l}^{L}\mathbb{E}\left[f(Z)\right]^2 - \mathbb{E}\left[f(Z)\right]^2 \\
&= \frac{1}{L^2}L\cdot\mathbb{E}\left[f(Z)^2\right] + \frac{\cancel{2}}{L^2}\frac{L(L-1)}{\cancel{2}}\mathbb{E}\left[f(Z)\right]^2 - \mathbb{E}\left[f(Z)\right]^2 \\
&= \frac{1}{L}\mathbb{E}\left[f(Z)^2\right] + \frac{\cancel{L^2} - L \cancel{- L^2}}{L^2}\mathbb{E}\left[f(Z)\right]^2 \\
&= \frac{1}{L}\left(\mathbb{E}\left[f(Z)^2\right] - \mathbb{E}\left[f(Z)\right]^2\right) \\
&= \frac{1}{L}\mathbb{V}\left[f(Z)\right]
\end{aligned}
$$

14.4

$$
\begin{aligned}
F(y) &:= \int_{-\infty}^{y}\frac{1}{\pi}\frac{1}{1+\hat{y}^2}d\hat{y} \\
&= \frac{1}{\pi}\left[\arctan y\right]_{-\infty}^{y} \\
&= \frac{1}{\pi}\left(\arctan y - \left(-\frac{\pi}{2}\right)\right) \\
&= \frac{\arctan y}{\pi} + \frac{1}{2}
\end{aligned}
$$

with

$$
F^{-1}(z) = \tan\left(\pi\left(z-0.5\right)\right) : (0,1) \longrightarrow \mathbb{R}
$$

So $Y := F^{-1}(Z) = \tan(\pi(Z-0.5)) \sim \text{Cauchy}(0,1)$.