# Simulation and Analysis of 1D Wave Propagation under Various Physical Models

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# Numerical methods for differential equations

## Ordinary Differential Equations

- Euler Method
- Crank-Nicolson Method
- Trotter-Suzuki Formula

$\underbrace{\text{Euler Method} \quad \text{Crank-Nicolson Method}}_{\text{Runge-Kutta}}$

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## Partial Differential Equations

- Finite Difference Method
- Finite Volume Method
- Galerkin methods
  - Spectral Method
  - Finite Element Method

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**P**artial **D**ifferential **E**quations

- **F**inite **D**ifference **M**ethod
- **F**inite **V**olume **M**ethod
- Galerkin methods
  - Spectral Method
  - **F**inite **E**lement **M**ethod

# Introduction to the problem

Solving a **PDE** means to find a function $u$ such that

$$\mathcal{L}u = f$$

where $\mathcal{L}$ is a differential operator and $f$ is a source term.

The equation holds in a domain $\Omega$ and is completed by prescribing **boundary conditions** on $\partial\Omega$.

In most physical applications $\mathcal{L}$ is a second-order operator

- → **Poisson** equation: $\mathcal{L} = -\Delta$
- → **Heat** equation: $\mathcal{L} = \frac{\partial}{\partial t} - \Delta$
- → **Wave** equation: $\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2\Delta$

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# Weak formulation

Galerkin methods rely on a **weak formulation**

- Multiply by a test function $v$ and integrate over the entire domain

$$-\int_{\Omega} (\Delta u) v d\Omega = \int_{\Omega} f v d\Omega$$

- Integrate by parts the left hand side

$$-\int_{\Omega} (\Delta u) v d\Omega = \int_{\Omega} \nabla u \cdot \nabla v d\Omega - \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds$$

- Substitute and get the new expression

$$\int_{\Omega} \nabla u \cdot \nabla v d\Omega = \int_{\Omega} f v d\Omega + \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds$$

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# About the test function

The test function $v$ is introduced to check whether the PDE is satisfied *on average* throughout the domain.

The problem becomes to find $u$ such that

$$a(u,v) = F(v) \qquad \forall v \in V$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v d\Omega \qquad \text{is a bilinear form}$$

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# Benefits of the weak formulation

| Strong formulation | Weak formulation |
|---|---|
| $u \in C^2(\Omega)$ | $u, v \in H^1(\Omega)$* |
| Holds pointwise in $\Omega$ | Holds on average on $\Omega$ |
| Derivatives exist classically | Derivatives exist in the distributional sense |

In short: weak formulation requires **less regularity**

* $H^1(\Omega)$ is a **Sobolev space** of functions with square-integrable first derivatives:

$$w \in H^1(\Omega) = \left\{ w \in L^2(\Omega) \mid \nabla w \in L^2(\Omega)^d \right\}$$

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# On boundary conditions

Another difference lies in the boundary condition prescription.

In strong formulation

- **Dirichlet**: $u = g \quad \text{on } \partial\Omega$
- **Neumann**: $\frac{\partial u}{\partial n} = h \quad \text{on } \partial\Omega$

In weak formulation

| Dirichlet becomes **essential condition** | Neumann becomes **natural condition** |
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| $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega \Rightarrow$ cancels boundary term (no information available on $\frac{\partial u}{\partial n}$) | $\boldsymbol{v}$ **free** on $\partial\Omega$ |
| $\boldsymbol{u} = \boldsymbol{g}$ enforced on $\partial\Omega$ (final solution) | $\frac{\partial u}{\partial n} = \boldsymbol{h}$ naturally enters weak form |

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# Shape functions

Galerkin methods allow to find an approximate solution

$$u_h \in V_h \subset H^1(\Omega) \qquad \text{where } V_h \text{ is a } \textbf{finite-dimensional} \text{ space}$$

In this framework, the goal is to find $u_h$ such that

$$\boxed{a\left(u_h, v_h\right) = F\left(v_h\right) \qquad \forall v_h \in V_h}$$

A **basis of function** $\{\phi_i\}$ is chosen to express $u_h$ and to use it as test:

$$u_h = \sum_{j=1}^{N} u_j \phi_j \implies a\left(\sum_{j=1}^{N} u_j \phi_j, \phi_i\right) = F\left(\phi_i\right) \qquad \forall i = 1, \ldots, N$$

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# Mesh discretization

**FEM** approach consists in the subdivision of the domain in a so-called **mesh**

This choice brings several advantages:

- Good approximation of **complex geometries**
- Better capture of **local effects**
- Possibility of **adaptive refinement**
- Natural construction of a **global solution**

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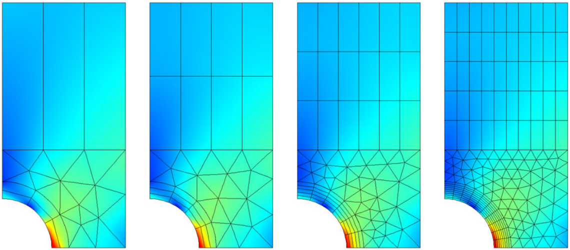

# Elements

Discretization in different dimensions

1D

2D

3D

# Title

In Finite Element Method, $V_h$ is generated by **local basis functions** with compact support. Usually, one discretizes the domain using a so-called mesh