Simulation and Analysis of 1D Wave Propagation under Various Physical Models

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September 6th 2025

Course of **Quantum Information and Computing**

Academic Year 2024/2025

Numerical methods for differential equations

Ordinary **D**ifferential **E**quations

- Euler Method
- Crank-Nicolson Method
- Trotter-Suzuki Formula

Euler Method, Crank-Nicolson Method: **R**unge-**K**utta

Partial **D**ifferential **E**quations

- **F**inite **D**ifference **M**ethod
- **F**inite **V**olume **M**ethod
- Galerkin methods
  - Spectral Method
  - **F**inite **E**lement **M**ethod

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Ordinary Differential Equations

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$\underbrace{\text{Euler Method} \quad \text{Crank-Nicolson Method}}_{\text{Runge-Kutta}}$

Partial Differential Equations

- Finite Difference Method
- Finite Volume Method
- Galerkin methods
  - Spectral Method
  - Finite Element Method

Introduction to the problem

Solving a **PDE** means to find a function u such that

$$\mathcal{L}u = f$$

where $\mathcal{L}$ is a differential operator and f is a source term.

The equation holds in a domain Ω and is completed by prescribing **boundary conditions** on $\partial\Omega$.

In most physical applications $\mathcal{L}$ is a second-order operator

- **Poisson** equation: $\mathcal{L} = -\Delta$
- **Heat** equation: $\mathcal{L} = \frac{\partial}{\partial t} - \Delta$
- **Wave** equation: $\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2\Delta$

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Weak formulation

Galerkin methods rely on a **weak formulation**

- Multiply by a test function v and integrate over the entire domain

$$-\int_{\Omega} (\Delta u) v d\Omega = \int_{\Omega} f v d\Omega$$

- Integrate by parts the left hand side

$$-\int_{\Omega} (\Delta u) v d\Omega = \int_{\Omega} \nabla u \cdot \nabla v d\Omega - \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds$$

- Substitute and get the new expression

$$\int_{\Omega} \nabla u \cdot \nabla v d\Omega = \int_{\Omega} f v d\Omega + \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds$$

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About the test function

The test function v is introduced to check whether the PDE is satisfied *on average* throughout the domain.

The problem becomes to find u such that

$$a(u,v) = F(v) \qquad \forall v \in V$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v d\Omega \qquad \text{is a bilinear form}$$

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Benefits of the weak formulation

Strong formulation	Weak formulation
$u \in C^2(\Omega)$	$u, v \in H^1(\Omega)$*
Holds pointwise in Ω	Holds on average on Ω
Derivatives exist classically	Derivatives exist in the distributional sense

In short: weak formulation requires **less regularity**

* $H^1(\Omega)$ is a **Sobolev space** of functions with square-integrable first derivatives:

$$w \in H^1(\Omega) = \left\{ w \in L^2(\Omega) \mid \nabla w \in L^2(\Omega)^d \right\}$$

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On boundary conditions

Another difference lies in the boundary condition prescription.

In strong formulation

→ **Dirichlet**: $u = g \quad \text{on } \partial\Omega$

→ **Neumann**: $\frac{\partial u}{\partial n} = h \quad \text{on } \partial\Omega$

In weak formulation

Dirichlet becomes
essential condition

Set $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega$ to cancel the boundary term, since no information is available about $\frac{\partial u}{\partial n}$

Impose $\boldsymbol{u} = \boldsymbol{g}$ on $\partial\Omega$ to the computed solution

Neumann becomes
natural condition