

Simulation and Analysis of 1D Wave Propagation under Various Physical Models

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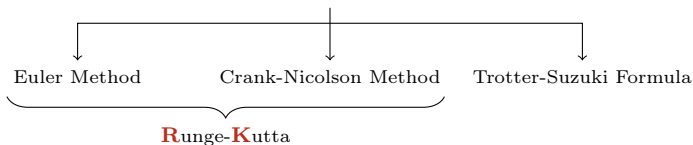
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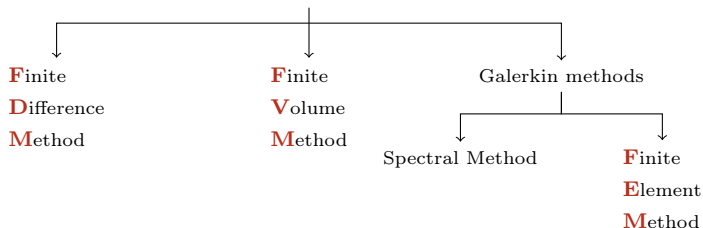
Course of **Quantum Information and Computing**
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Numerical methods for differential equations

Ordinary Differential Equations

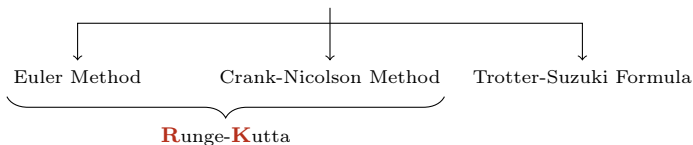


Partial Differential Equations

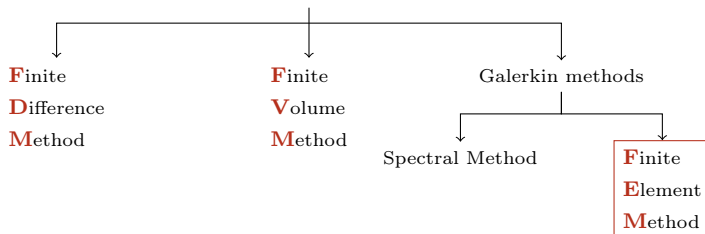


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Partial Differential Equations



Introduction to the problem

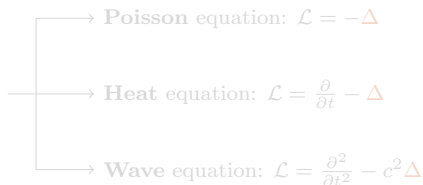
Solving a **PDE** means to find a function u such that

$$\mathcal{L}u = f$$

where \mathcal{L} is a differential operator and f is a source term.

The equation holds in a domain Ω and is completed by prescribing **boundary conditions** on $\partial\Omega$.

In most physical
applications \mathcal{L} is a
second-order operator



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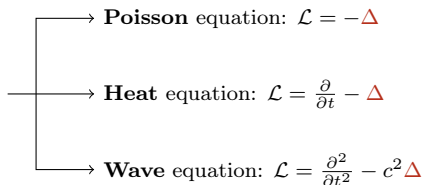
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Weak formulation

Galerkin methods rely on a **weak formulation**

- Multiply by a **test function** v and integrate over the entire domain

$$-\int_{\Omega} (\Delta u) v d\Omega = \int_{\Omega} f v d\Omega$$

- Integrate by parts the left hand side

$$-\int_{\Omega} (\Delta u) v d\Omega = \int_{\Omega} \nabla u \cdot \nabla v d\Omega - \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds$$

- Substitute and get the new expression

$$\int_{\Omega} \nabla u \cdot \nabla v d\Omega = \int_{\Omega} f v d\Omega + \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds$$

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About the test function

The test function v is introduced to check whether the PDE is satisfied on average throughout the domain.

The problem becomes to find u such that

$$a(u, v) = F(v) \quad \forall v \in V$$

where

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v d\Omega \quad \text{is a bilinear form}$$

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Benefits of the weak formulation

Strong formulation

$$u \in C^2(\Omega)$$

Holds pointwise in Ω

Derivatives exist classically

Weak formulation

$$u, v \in H^1(\Omega)^*$$

Holds on average on Ω

Derivatives exist in the
distributional sense

In short: weak formulation requires less regularity

* $H^1(\Omega)$ is a **Sobolev space** of functions with square-integrable first derivatives:

$$w \in H^1(\Omega) = \left\{ w \in L^2(\Omega) \mid \nabla w \in L^2(\Omega)^d \right\}$$

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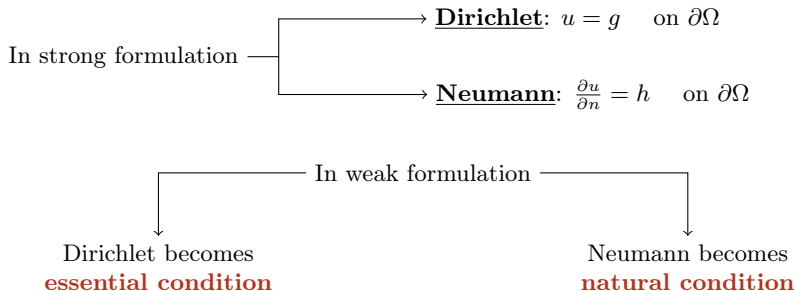
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On boundary conditions

Another difference lies in the boundary condition prescription.



Set $\mathbf{v} = \mathbf{0}$ on $\partial\Omega$ to cancel the boundary term, since no information is available about $\frac{\partial u}{\partial n}$

Impose $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ to the computed solution