Problem Set 9

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GitHub link

`https://github.com/dariolop76/phys-ga2000`

1 Problem 1

1.1 Part a

We numerically solve the differential equation of a simple harmonic oscillator:

$$\frac{d^2x}{dt^2} = -\omega^2 x. \tag{1}$$

First, we turn this second-order equation into the following two coupled first-order equations:

$$\begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = -\omega^2 x \end{cases} \tag{2}$$

We apply the fourth-order Runge-Kutta method to simultaneously solve these two equations. We solve for the case $\omega = 1$, in the range $t = 0$ to $t = 50$, using $x = 1$ and $\frac{dx}{dt} = 0$ as initial conditions. Fig. 1 shows the solution x as a function of time.

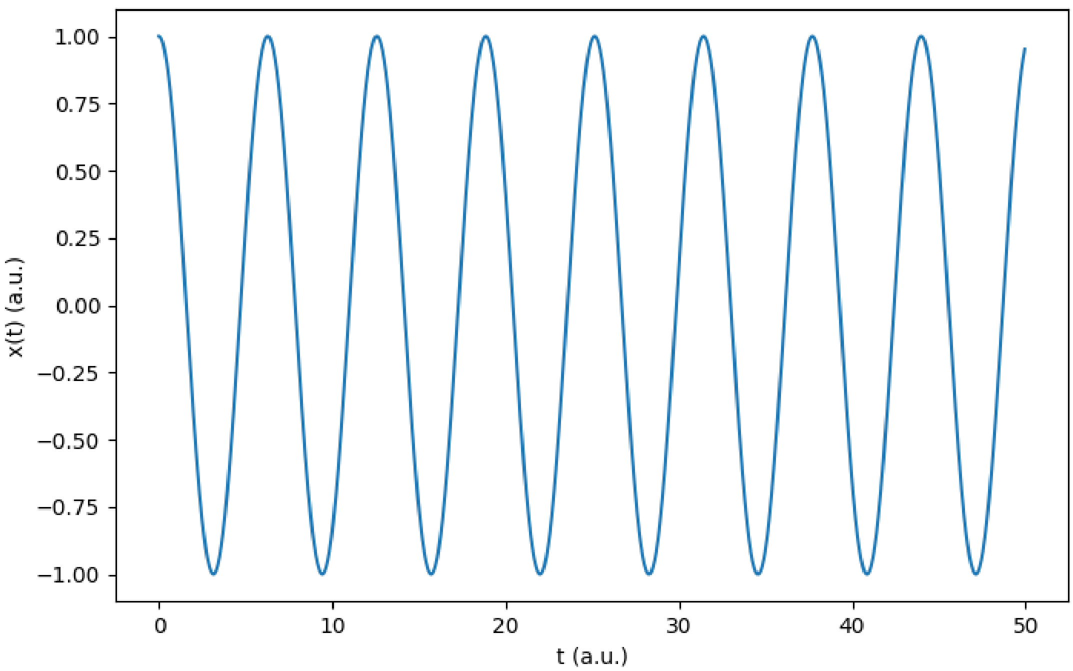

Figure 1: Plot of the position x as a function of time t in the range $t \in [0, 50]$ obtained by using the fourt-order Runge-Kutta method.

1.2 Part b

We increase the amplitude to $x = 2$ and plot the solution. We can see in Figure 2 that the new solution has the same period as the previous one, as it is supposed to be in the case of a harmonic oscillator.

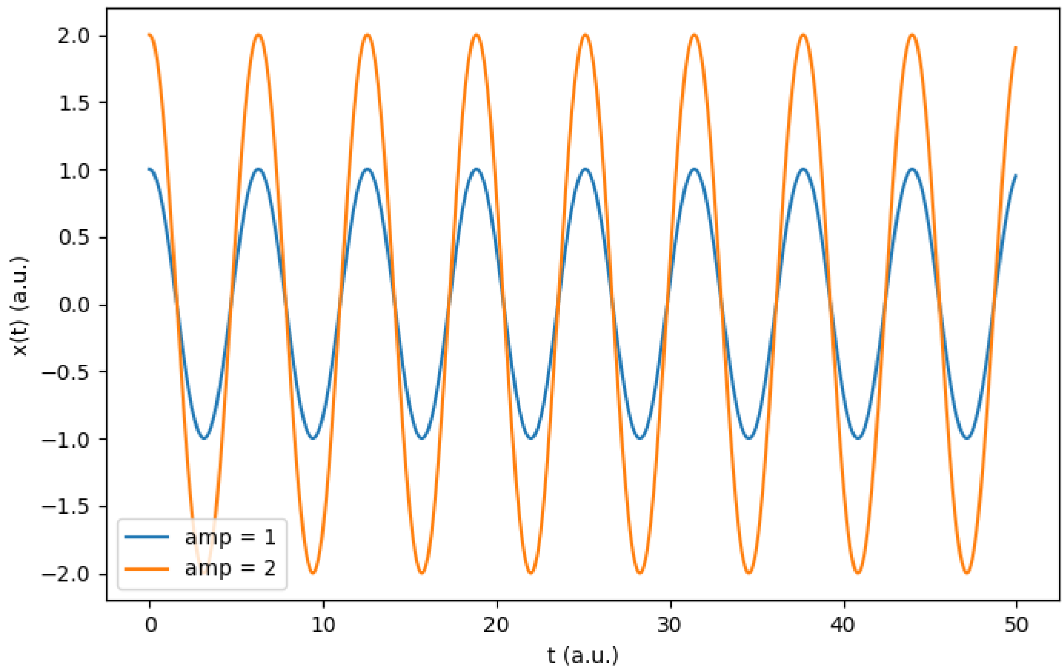

Figure 2: Position as a function of time for amplitudes equal to 1 and 2.

1.3 Part c

We solve for the motion of the anharmonic oscillator, which is described by the equation

$$\frac{d^2x}{dt^2} = -\omega x^3, \tag{3}$$

using the same method as before. Figure 3 shows the solutions for different amplitudes. In this case, the period of oscillation decreases with increasing amplitude.

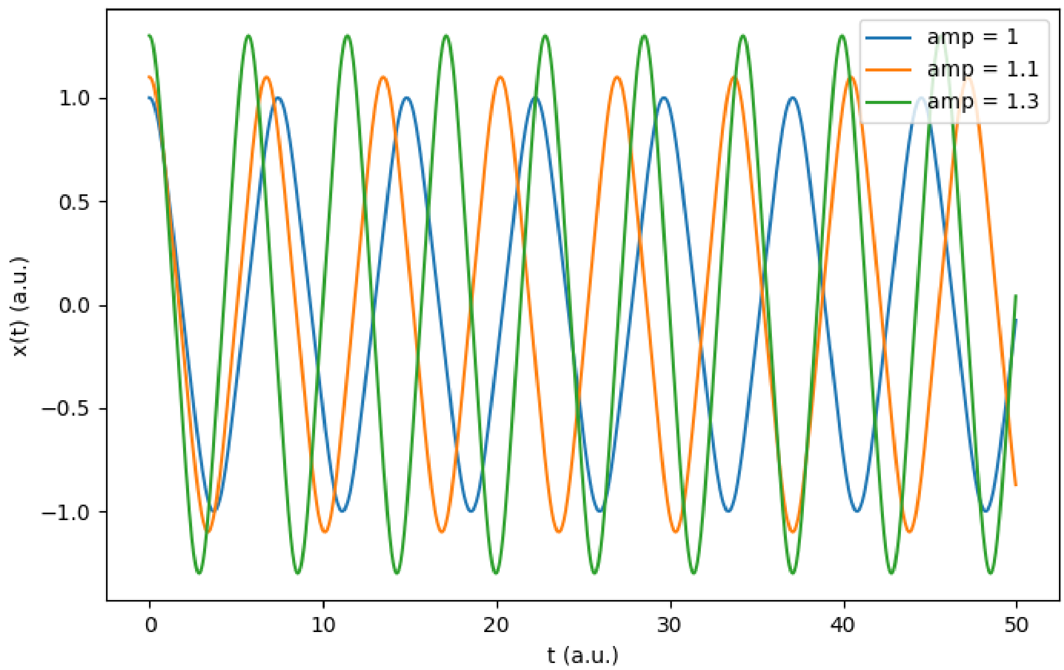

Figure 3: Position as a function of time for amplitudes equal to 1, 1.1 and 1.3 in the case of the anharmonic oscillator.

1.4 Part d

Figure 4 shows the phase space diagrams for the solutions computed in Part c.

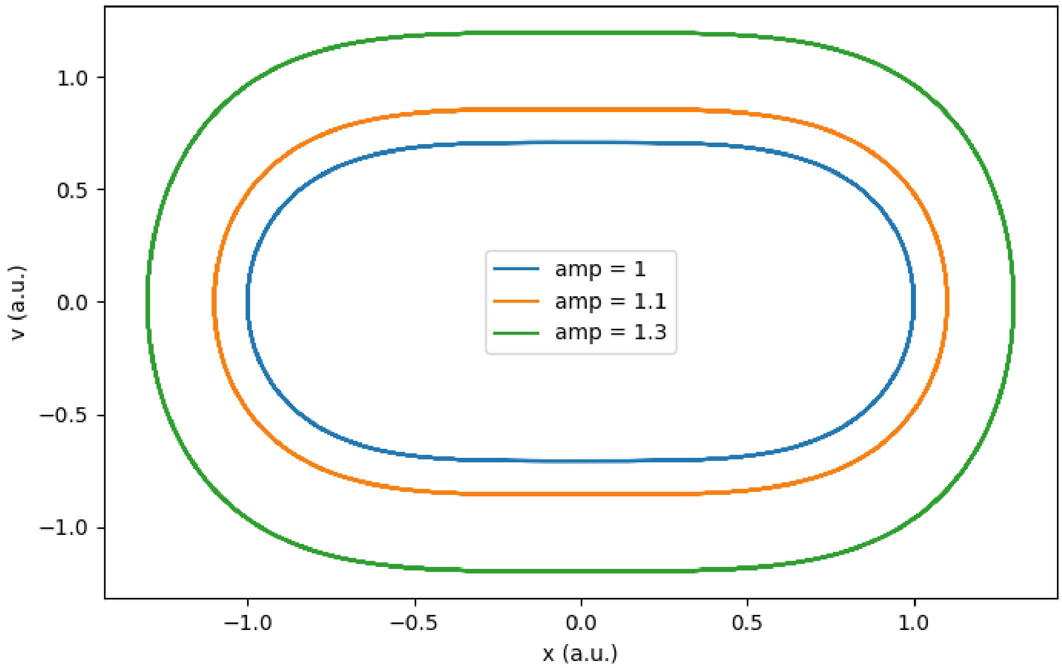

Figure 4: Phase space plots in the case of the anharmonic oscillator for amplitudes equal to 1, 1.1 and 1.3.

1.5 Part e

We solve for the motion of the van der Pol oscillator, described by the equation

$$\frac{dx^2}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + \omega^2 x = 0, \tag{4}$$

using again the fourth-order Runge-Kutta method. We solve this equation in the range $t \in [0, 20]$ and for three values of the parameter μ. Figure 5 shows the phase space plot for these solutions.

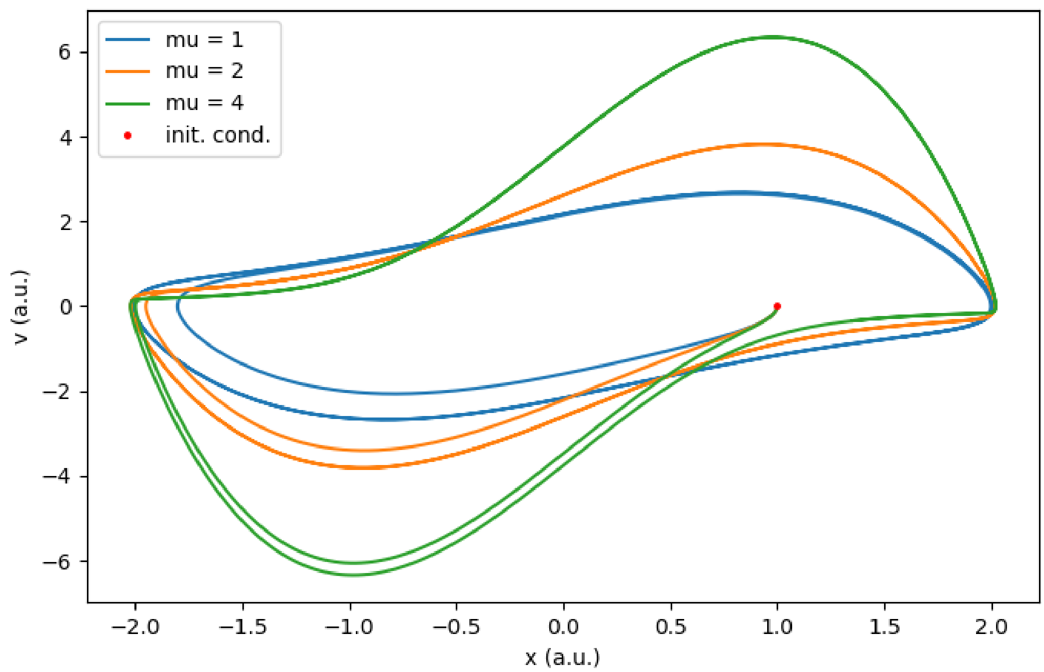

Figure 5: Phase space plots in the case of the van der Pol oscillator, for μ equal to $1, 2$, and 4.

2 Problem 2

2.1 Part a

We consider the problem of a spherical cannonball shot from a cannon standing on level ground. The motion is in the (x,y) plane, whose origin coincides with the starting position of the cannonball. We apply Newton's second law $F = ma$ to obtain two second-order differential equations to be solved for the positions x and y. On the x axis, the cannonball is subject to the x-component of the force due to air resistance, which is given by

$$\vec{F_x} = -\frac{1}{2}\pi R^2 \rho C v^2 \cos\theta \hat{x}, \tag{5}$$

where R is the sphere's radius, ρ is the density of air, v is the magnitude of the velocity, C is the coefficient of drag and θ is the angle between the direction of the velocity and the x axis. The minus sign is due to the fact that the force acts in the opposite direction of the motion, slowing the cannonball down. Since the x-component of the velocity is given by $\dot{x} = v_x = v\cos\theta$ and the magnitude of the velocity is $v = \sqrt{\dot{x}^2 + \dot{y}^2}$, the equation of motion for the position x becomes:

$$m\ddot{x} = F_x = -\frac{1}{2}\pi R^2 \rho C \dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}, \tag{6}$$

where m is the mass of the cannonball. Along the y axis, the force the cannonball is subject to is the sum of the gravitational force and the drag force:

$$\vec{F_y} = \left(-mg - \frac{1}{2}\pi R^2 \rho C v^2 \sin\theta\right)\hat{y}, \tag{7}$$

where g is the gravitational acceleration. So the differential equation for the position y is given by

$$m\ddot{y} = F_y = -mg - \frac{1}{2}\pi R^2 \rho C \dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} \tag{8}$$

We would like to make Equations 6 and 8 dimensionless, so we rescale t, x and y in the following way:

$$t \rightarrow t' = \frac{t}{T}, \tag{9}$$

$$x \rightarrow x' = \frac{x}{T^2 g}, \tag{10}$$

$$y \rightarrow y' = \frac{y}{T^2 g}, \tag{11}$$

$$\tag{12}$$

where T is the typical timescale of the motion. Using

$$\frac{d^2}{dt^2} = \frac{1}{T^2}\frac{d^2}{dt'^2}, \tag{13}$$

Equations 6 and 8 can be rewritten in terms of dimensionless quantities and a dimensionless parameter A as follows:

$$\frac{d^2 x'}{dt'^2} = -\frac{\pi}{2} A \dot{x'}\sqrt{\dot{x'}^2 + \dot{y'}^2} \tag{14}$$

$$\frac{d^2 y'}{dt'^2} = -1 - \frac{\pi}{2} A \dot{y'}\sqrt{\dot{x'}^2 + \dot{y'}^2} \tag{15}$$

where $A \equiv \frac{R^2 \rho C g T^2}{m}$. We apply the fourth-order Runge-Kutta method to numerically solve the system of second-order differential equations 14 and 15, which can be turned into a system of four first-order differential equations:

$$
\begin{cases}
\dfrac{dx'}{dt'} = v'_x \\
\dfrac{dy'}{dt'} = v'_y \\
\dfrac{dv'_x}{dt'} = -\dfrac{\pi}{2} A v'_x \sqrt{v'^2_x + v'^2_y} \\
\dfrac{dv'_y}{dt'} = -1 - \dfrac{\pi}{2} A v'_y \sqrt{v'^2_x + v'^2_y}
\end{cases}
\tag{16}
$$

Figure 7 shows the trajectory of the cannonball, with the parameters and initial conditions provided in the prompt of Exercise 8.7.

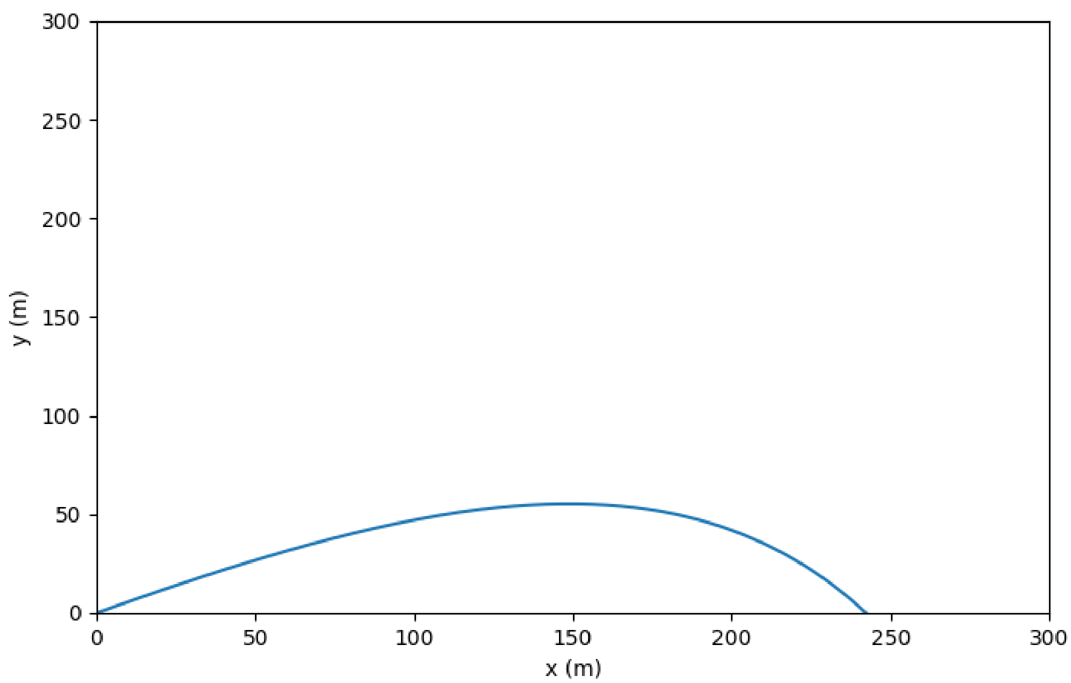

Figure 6: Trajectory of the cannonball subject to gravity and air resistance.

2.2 Part b

We numerically solve for the motion of the cannonball using different values for the mass m. Figure 7 shows the obtained trajectories.

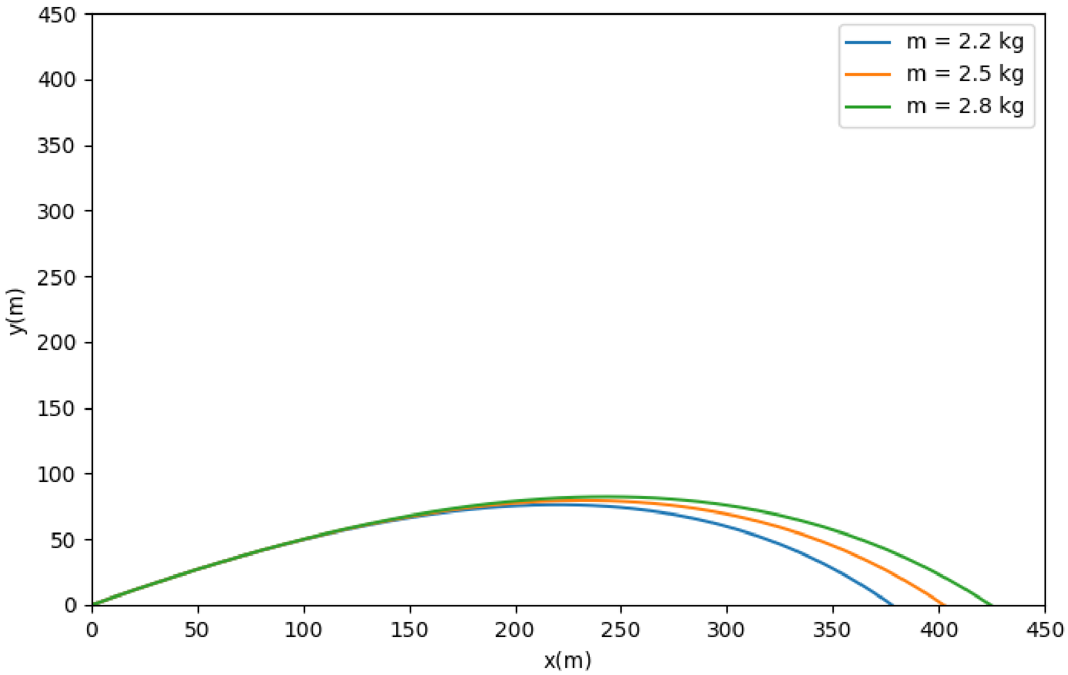

Figure 7: Trajectory of the cannonball for three different values of its mass.

As we can see, the distance traveled by the cannonball increases with its mass. Indeed, the greater the mass, the greater the inertia. As a result, the cannonball will experience slower deceleration due to the drag force and travel farther distances. Figure 8 shows the distance traveled by the cannonball as a function of the mass.

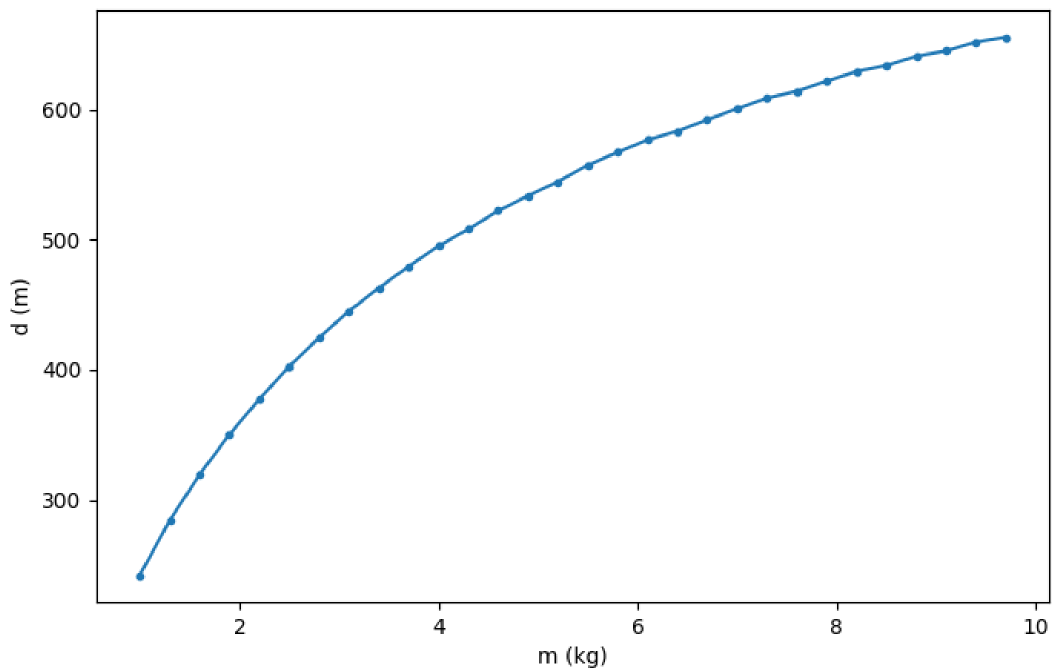

Figure 8: Distance traveled by the cannonball as a function of the mass.