Numerical Methods

Computational Basics

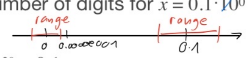
- 32 or 64 bits
- computations with limited accurancy can lead to results with even lower accuaracy

important!

  - Scientific notation, omitting leading/trailing zeros
    - $7000000000 = 0.7 \cdot 10^{10} = 0.7 \cdot E10$ or $0.000000001 = 0.1 \cdot 10^{-8} = 0.1 \cdot E\text{-}8$
  - IEEE single precision (32bit) float numbers are represented as
    $\pm 0.x_1 x_2 x_3 \ldots x_{23} \pm e_1 \ldots e_7$
    - Sign(s), significand, exponent in base 2 (binary digits), range $10^{\pm 308}$
  - IEEE double precision (64bit) → Today normal, very precise
    $\pm 0.x_1 x_2 x_3 \ldots x_{52} \pm e_1 \ldots e_{10}$
  - A large set of 2^{64} float numbers are representable in binary form, but there are infinitely many real numbers!

- never use comarisons like x == y for floating point numbers

Arithmetic evaluation with finite number of digits for $x = 0.1 \cdot 10^0 = 0.1$ and $y = 0.1 \cdot 10^{-20}$, compute x + y:



$0.1 \cdot 10^0 + 0.1 \cdot 10^{-20} = 0.1$

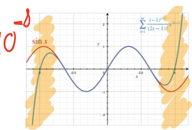
0.1 + 0.0000000000000000001

- Because x and y have different range of significant bits.

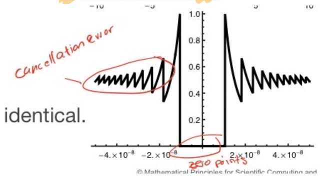
bits that you cannot represent anymore

$0.1 + 0.7 \neq 0.8$ (because of Trucation Error)

$|0.1 + 0.7 - 0.8| < 10^{-8}$



- Truncation Error = binary digits cannot represent decimal loats exactly
- Cancellation Error = when a Term behaves almost identical to another one



Linear Equations

- A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

- A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions.
- A system of linear equation is said to be **inconsistent** if it has no solution.

- two lin. sys are equivalent if they have the same solution set

- augmented matrix:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

lin. equation system

use Eleminationsverfahren and bring it to 1's „Führenden Elementen" -> siehe Mathe zusammenfassung

Reduced echelon form with leading entries 1, and 0s above and below each leading 1

backward phase

pivot position (5) → basic variables

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

forward phase is to put in reduction algorithm

free variables

pivot colomn (5)

Theorem 2: Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—i.e., if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \dots \ 0 \quad b] \text{ with } b \text{ nonzero.}$$

If a linear system is consistent, then the solution set contains either
(i) a unique solution, when there are no free variables, or
(ii) infinitely many solutions, when there is at least one free variable.

partial pivoting = wisely choosen steps of the gaussian elimination

- In general, the forward phase of row reduction takes much longer than the backward phase.
  - An algorithm for solving a system is usually measured in flops (or floating point operations). A flop is one arithmetic operation (+;-;*;/) on two real floating point numbers.
- For an $n\times(n+1)$ matrix, the reduction to echelon form can take $2n^3/3 + n^2/2 - 7n/6$ flops.
  - Which is approximately $2n^3/3$ flops or $O(n^3)$ when n is moderately large—say, $n \geq 30$.
- In contrast, further reduction to reduced echelon form needs at most n^2 flops.

When the pivot element is in the last colone it means it is inconsistent since $0 \neq$ pivot element → no solution

Matrix Vector Equation Ax=b

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

$$x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}2\\-5\end{bmatrix} + x_3\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}.$$ matrix times a vector = linear equation

$$\underbrace{\begin{bmatrix}1 & 2 & -1\\0 & -5 & 3\end{bmatrix}}_{\text{coefficient matrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}.$$ form $Ax=b$ (matrix equation)

Theorem 4: Let A be an $m\times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x}=\mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row. not included in column b! pivot coefficient matrix

- Applies to a **coefficient matrix**, not an augmented matrix. If an augmented matrix $[\,A \ \ \mathbf{b}\,]$ has a pivot position in every row, then the equation $A\mathbf{x}=\mathbf{b}$ may or may not be consistent.

ex:

$$= \begin{bmatrix}2x_1\\-x_1\\6x_1\end{bmatrix} + \begin{bmatrix}3x_2\\5x_2\\-2x_2\end{bmatrix} + \begin{bmatrix}4x_3\\-3x_3\\8x_3\end{bmatrix} = \begin{bmatrix}2x_1+3x_2+4x_3\\-x_1+5x_2-3x_3\\6x_1-2x_2+8x_3\end{bmatrix}$$

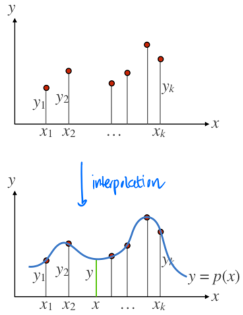
sum of product = a dot product

dot product between row i of A and x

Theorem 5: If A is an $m\times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then

a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ (we can't change order)
b. $A(c\mathbf{u}) = c(A\mathbf{u})$. (we can change order)

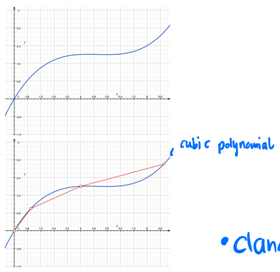
- polynomial interpolation



- Assume, for $\boxed{k=3}$, measurements y_0, y_1, y_2, y_3 are given for some time steps t_0, t_1, t_2, t_3, thus $y_i = p(t_i)$ for $i = 0\ldots3$.
- Find a cubic polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ that interpolates the given values y_0, y_1, y_2, y_3.
  - Four unknown variables a_0, a_1, a_2, a_3, which require four equations.
- Given our data y_0, y_1, y_2, y_3 and t_0, t_1, t_2, t_3 we can write down four equations

$$y_0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3$$
$$y_1 = a_0 + a_1 t_1 + a_2 t_1^2 + a_3 t_1^3$$
$$y_2 = a_0 + a_1 t_2 + a_2 t_2^2 + a_3 t_2^3$$
$$y_3 = a_0 + a_1 t_3 + a_2 t_3^2 + a_3 t_3^3$$

  - So we have four equations and four unknown variables.



- As matrix notation $T\mathbf{a}=\mathbf{y}$ of the given data we get

unknown variable — measured output variables

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$t=0$ — time step

  - This coefficient matrix T is also called a **Vandermonde matrix.**

- danger of interpolation: - for higher order polynomials the result tend to suffer from overfitting with oscillations
  - depending on the basics function I'm using the curve will look a bit different, but it will always interpolate correctly. (have to be independent of each other!)
  - i.e: $a_1 \cdot x_0 + a_2 \cdot x_1 + a_3 \sin(x) + a_4 \cos(x)$ (weights) basis functions (independent of each other)

Solution sets of linear systems

- **homogenous**: $Ax = 0$ (a non-trivial solution iff there is at least one free variable)!
  - ↑ trivial solution
  - ↳ many solutions

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced echelon form

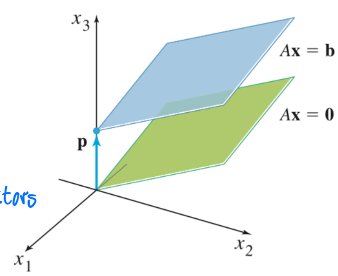
$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \quad - \tfrac{4}{3}x_3 &= 0 \\ x_2 \quad &= 0 \\ 0 &= 0 \end{aligned}$$

x_3 is free

Since x_3 is a free variable, $A\mathbf{x}=\mathbf{0}$ has nontrivial solutions (one for each choice of x_3.)

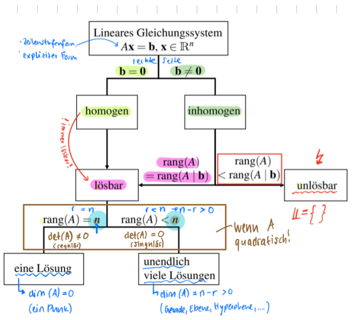
- **parametric vector form**: $x = p + x_3 v \mid s, t \in \mathbb{R}$ (free variable; constant vector)
  - ↳ if trivial solution: a dot
  - ↳ if non-trivial solution: a line in $\mathbb{R}^n$ through the origin

Theorem 6: Suppose the equation $A\mathbf{x}=\mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x}=\mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x}=\mathbf{0}$.



- **nonhomogeneous**: $Ax = b \mid b \neq 0$ (not zero-vector)
  - ↳ many solution (line) → vector plus an arbitrary linear combination of vectors
    - translated by p to v+p (not through the origin)

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.



- Network flow
  - A *network* consists of a set of points called *nodes*, with lines called *branches* connecting some or all of the nodes. The direction of flow in each branch is indicated and the flow amount is shown.

For example see lecture slides

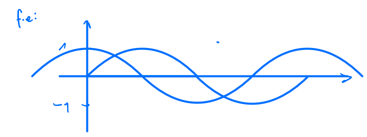
The Matrix of Lin. Transformation

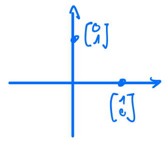
$T: \underbrace{\mathbb{R}^n}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}^m}_{\text{codomain}}$ $T(x)$ = image   set of all images $T(x)$ = range

$T(\alpha): \begin{pmatrix} x \\ y \end{pmatrix} \mapsto$ rotation by "α"

$A = \begin{bmatrix} ? \end{bmatrix}$ $T_\alpha\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$

$T_\alpha\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}$



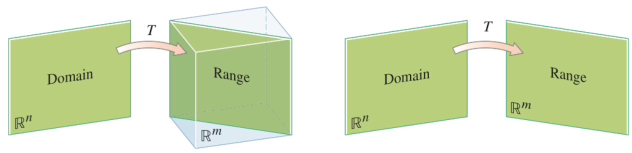


$A = \begin{bmatrix} T_\alpha[e_1] & T_\alpha[e_2] \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{\text{= Rotation matrix}}$ rotation by "α"

- A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
- The set $\mathbb{R}^n$ is called **domain** (input) of T, and the set $\mathbb{R}^m$ is called the **codomain** (output) of T.
  - The notation $T : \underbrace{\mathbb{R}^n}_{\text{domain}} \rightarrow \underbrace{\mathbb{R}^m}_{\text{codomain}}$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$
  - Under the action of T
- The set of all images $T(\mathbf{x})$ is called the **range** of T.

- **Definition**: A transformation (or mapping) T is linear if:
  - i. $\boxed{T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})}$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T;
  - ii. $\boxed{T(c\mathbf{u}) = cT(\mathbf{u})}$ for all scalars c and all $\mathbf{u}$ in the domain of T.
- Linear transformations *preserve the operations of vector addition and scalar multiplication.*
  - $T(0 \cdot v) = 0 \cdot T(v) = 0$
  - iii. If T is a linear transformation, then $\boxed{T(\mathbf{0}) = \mathbf{0}}$, and
  - iv. $\boxed{T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})}$ for all vectors $\mathbf{u}$, $\mathbf{v}$ in the domain of T and all scalars c, d.
- If a transformation satisfies (iv) for all $\mathbf{u}$, $\mathbf{v}$ and c, d, it must be linear.

Definition: A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$

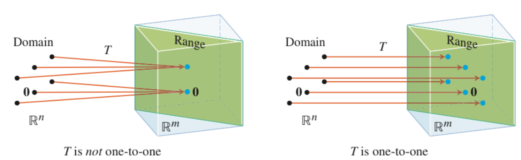


T is *not* onto $\mathbb{R}^m$ T is onto $\mathbb{R}^m$ = range of T is all of the codomain

FIGURE 3 Is the range of T all of $\mathbb{R}^m$? only a subspace of codomain

Surjektiv

- **onto** = Equivalently, T is onto $\mathbb{R}^m$ when the range of T is all of the codomain $\mathbb{R}^m$.
- **not onto** = The mapping T is not onto when there is some $\mathbf{b}$ in $\mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution.
- **one-to-one** = bijektiv
- **Definition**: A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$



T is *not* one-to-one T is one-to-one

FIGURE 4 Is every $\mathbf{b}$ the image of at most one vector?

When an equation is consistent (= each row has a pivot element) means she is also onto the domain!

When an equation has a free variable, the image is not one-to-one

Theorem 4: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x}=\mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row. = consistent

- Applies to a **coefficient matrix**, not an augmented matrix. If an augmented matrix $[\,A\ \ \mathbf{b}\,]$ has a pivot position in every row, then the equation $A\mathbf{x}=\mathbf{b}$ may or may not be consistent.

- what does lin. rotation mean? even if $\cos\alpha$, $\sin\alpha$ aren't lin. functions, but in higher space we can get a lin. Transformation by a transformation matrix

Theorem 11: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x}=\mathbf{0}$.

- linear T means f.e: $T(0) = 0$ → one-to-one (trivial solution)
- if not one-to-one: $T(u) = b$ and $T(v) = b$ → $T(u-v) = b-b = 0$ but $u \neq v$ therefore more than one solution

Theorem 12: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$. → consistent (pivot = m)
b. T is one-to-one if and only if the columns of A are linearly independent.
→ only the trivial solution 0!

$$\begin{bmatrix} A \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} = x_1 \cdot \alpha_1 + x_2 \cdot \alpha_2 + \dots$$

matrix transformation

image

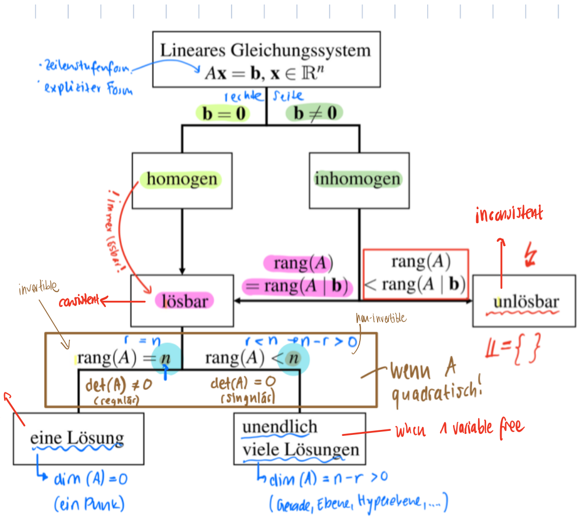
$$Ax = T(x)$$

- recurrence relation or linear difference equation - if $x_1 = Ax_0$, $x_2 = Ax_1$ then $\boxed{x_{k+1} = Ax_k}$

recurrence relation (lin. transformation on every step)

2. Matrix Algebra

Invertible Matrices



- If $n \times n$ matrix A is invertible, its unique inverse is denoted by A^{-1}, so that $A^{-1}A = I$ and $AA^{-1} = I$.
- If $n \times n$ matrix A is invertible, then for each $\mathbf{b}$ in $\mathbb{R}^n$ the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ (Theorem 5 in 2.2)
- If $n \times n$ matrices A and B are invertible, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$ (Theorem 6 in 2.2)
  - Also if A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$

Theorem 4: Let A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\underbrace{ad-bc \neq 0}_{\det(A) \neq 0}$ then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Äquivalente Aussagen für eine $\boxed{n \times n}$ Matrix A

det A != 0
⇔ A ist regulär.
⇔ r(A)=n (A hat den Rang n).
⇔ A^{-1} (die Inverse von A) existiert.
⇔ Die n Spaltenvektoren von A sind linear unabhängig. (analog: Zeilenvektoren)
⇔ A$\underline{x}$ = 0 besitzt nur die triviale Lösung ($\underline{x}$ = 0).
⇔ A$\underline{x}$ = b besitzt eine eindeutige Lösung für jede beliebige rechte Seite b.
⇔ Jeder Vektor b ∈ $\mathbb{R}^n$ lässt sich als Linearkombination der Spaltenvektoren der Matrix A darstellen (r(A | b) = n).
⇔ Die n Spaltenvektoren von A bilden eine Basis und ein Erzeugendensystem des $\mathbb{R}^n$. (analog: Zeilenvektoren)

* $[A\ I] = A^{-1}$

$AA^{-1} = I$

f.e $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

---

det A = 0
⇔ A ist singulär (nicht regulär).
⇔ r(A) < n
⇔ A^{-1} existiert nicht.
⇔ Die n Spaltenvektoren von A sind linear abhängig. (analog: Zeilenvektoren)
⇔ A$\underline{x}$ = 0 besitzt unendlich viele Lösungen.
⇔ A$\underline{x}$ = $\underline{b}$ besitzt für eine beliebige rechte Seite b entweder keine, oder unendlich viele Lösungen.
⇔ Nicht jeder Vektor b ∈ Rn lässt sich als Linearkombination der Spaltenvektoren der Matrix A darstellen (r(A | b) $\leq$ n).
⇔ Die n Spaltenvektoren von A bilden weder eine Basis, noch ein Erzeugendensystem des $\mathbb{R}^n$. (analog: Zeilenvektoren)

* $[A\ I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$ reduced echelon form $\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$

A^{-1}

Invertible matrix theorem

→ if one statement is not true, all of them are not true

Theorem 8: Let A be an $n{\times}n$ matrix. Then the following statements are equivalent, thus being either all true or all false.

only quadratic matrix

⇒ A has a pivot in every row and column

a. The matrix A is an invertible matrix. (nonsingular)

b. The matrix A is row equivalent to the $n{\times}n$ identity matrix. → pivot position in every row

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. → no free variables

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. = bijectiv

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. = surjectiv

j. There is an $n{\times}n$ matrix C such that $\boxed{CA = I}$.

k. There is an $n{\times}n$ matrix D such that $\boxed{AD = I}$.

l. A^T is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.
e. The columns of A form a linearly independent set.
h. The columns of A span $\mathbb{R}^n$.

g. $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
n. $\text{Col } A = \mathbb{R}^n$
o. $\text{rank } A = n$

p. $\dim \text{Nul } A = 0$
q. $\text{Nul } A = \{\mathbf{0}\}$
d. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Sei $f: \mathbb{R}^n \to \mathbb{R}^m$ mit $y = f(x) = Ax$ gilt:

- **surjektiv**, wenn Rang $= m$; d.h. ich kann ganzen Raum erzeugen ⇔ $\text{lin}\{a^1, \dots a^m\} = \mathbb{R}^m$
  "onto"
  $\boxed{\mathbb{R}^n \to \mathbb{R}^m \;;\; \text{rang}(A) = m}$ (⇒ wenn jeder x in Z erreicht wird)

- **injektiv**, wenn Rang $= n$; wenn lin. unabhängig.
  $\boxed{\mathbb{R}^n \to \mathbb{R}^m \;;\; \text{rang}(A) = n}$ (⇒ wenn jedes x in Z höchstens einmal erreicht wird)

- **bijektiv**, wenn Abbildungsmatrix eine quadratische Matrix mit vollem Rang!
  "one-to-one" impliziert
  $\boxed{\mathbb{R}^n \to \mathbb{R}^m \;;\; n = m}$ (⇒ wenn jedes x in Z genau einmal erreicht wird)

  ↳ **regulär**, wenn Abbildungsmatrix voller Rang hat:
  "nonsingular"
  $\boxed{\text{rang}(A) = n = m}$

⇒ es existiert eine Inverse Abbildung f^{-1}: $f^{-1}: \mathbb{R}^m \to \mathbb{R}^n$

$\boxed{f^{-1}(x) = A^{-1} \cdot x}$ $\forall x \in \mathbb{R}^n: f^{-1}(f(x)) = x = f(f^{-1}(x)) \Leftrightarrow A^{-1} \cdot A = A \cdot A^{-1} = I$
→ durch simultanes oder parametrisches Lösen

- **singulär**, wenn Abbildungsmatrix kein voller Rang hat: singular = noninvertible
  "singular"
  $\boxed{\text{rang}(A) < \min\{m,n\}}$ → wenn nicht quadratisch!

RECHENREGEL INVERSER MATRIZEN

Seien A & B reguläre Matrizen der Ordnung n, dann gilt:
→ es existiert eine Inverse

$$\boxed{\begin{aligned} &\bullet\ A^{-1} \cdot A = A \cdot A^{-1} = I \\ &\bullet\ (A^{-1})^{-1} = A \\ &\bullet\ (\alpha A)^{-1} = \frac{1}{\alpha} \cdot A^{-1} \quad \forall \alpha \neq 0 \\ &\bullet\ (A^T)^{-1} = (A^{-1})^T \end{aligned}}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $(AB)^{-1} \neq A^{-1}B^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

$AB = BA$ if A has an inverse → $A^{-1}B = BA^{-1}$

Invertible lin. transformation

$T: \mathbb{R}^n \to \mathbb{R}^n$ (invertible) $\exists$ unique function $S(x) = A^{-1}x$

$\to T(x) = T(S(b)) = b$, each b is in the range of T

$\to S(T(x)) = S(Ax) = A^{-1}(Ax) = x$

- **nearly singular matrices**: f.e with floating numbers only nearly singular $\to$ ill-conditioned matrix
  - ↳ can only produce $< n$ pivots
  - ↳ condition number = how close to being singular a matrix is! (as higher it is that far away from being singular)

Matrix Factorizations

A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.

- **LU factorization** = solving a sequence of equations, with same coefficient matrix. f.e: $A\mathbf{x}_1 = \mathbf{b}_1,\ A\mathbf{x}_2 = \mathbf{b}_2,\ \dots A\mathbf{x}_p = \mathbf{b}_p$

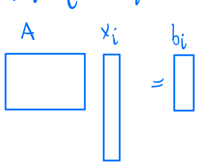
- At first, assume that A is an $m{\times}n$ matrix that can be row reduced to echelon form, *without row interchanges*. !
- Then A can be written in the form $\boxed{A = LU}$, were L is an $m{\times}m$ lower triangular matrix with 1's on the diagonal and U is an $m{\times}n$ echelon form of A. (square matrix)
- Such a factorization is called an **LU factorization** of A. The matrix L is invertible and is called a unit lower triangular matrix.

columns →; echelon form (nonpivot elements)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = LU$$

↑ row ; has to be inverse bc. n pivot elements and transposed in echelon form

$Ax_i = b_i$ $\quad A = LU \to Ax = b \to L(\underbrace{Ux}_{y}) = b \to$ $Ly = b$; $Ux = y$



$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

'e $A\mathbf{x}=\mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$

$$[L\ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [I\ \mathbf{y}]$$

$$[U\ \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix},\ \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

x is what we wanted to find through factorization

• elementary matrix E = single elementary row operation to an identity matrix I.

↳ has either n or $n+1$ nonzero elements

The inverse of E is the elementary matrix of the same type that transforms E back into I.

- Add 3 times row 2 to row 3: - one row operation on a identity matrix
  - Hence $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ = unit lower triangular elementary matrice
- Now apply E to some given matrix A:
  - Given $A = \begin{bmatrix} 2 & -3 & 3 & 1 \\ 1 & 1 & -4 & 2 \\ 4 & 2 & 1 & 3 \\ 5 & -1 & 5 & 1 \end{bmatrix}$,

  applying E will result in $EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 3 & 1 \\ 1 & 1 & -4 & 2 \\ 4 & 2 & 1 & 3 \\ 5 & -1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 3 & 1 \\ 1 & 1 & -4 & 2 \\ 7 & 5 & -11 & 9 \\ 5 & -1 & 5 & 1 \end{bmatrix}$ #3

$E_p \cdots E_1 A = U \quad | (E_p \ldots E_1)^{-1}$

$A = (E_p \cdots E_1)^{-1} U = LU \quad | \; U^{-1}$

$L = (E_p \cdots E_1)^{-1}$ → thus L is unit lower triangular

$E_p \cdots E_1 L = (E_p \cdots E_1)(E_p \cdots E_1)^{-1} = I$

not always possible

Algorithm for an LU Factorization

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the *same sequence of row operations* reduces L to I.

→ see chap_2.2, p.15

①

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

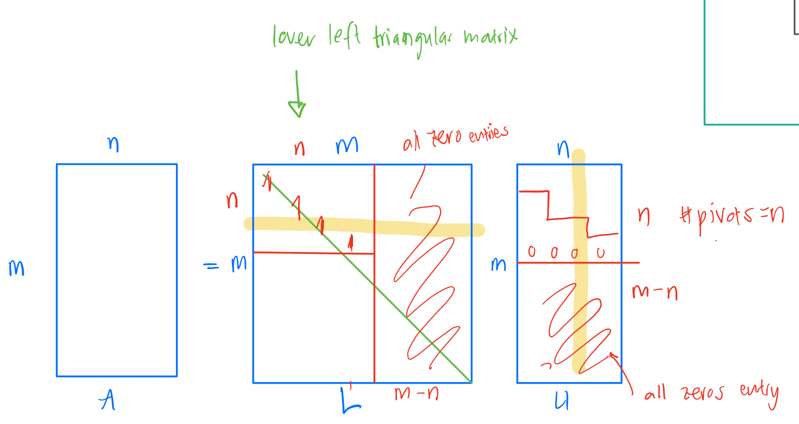
②

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}$$

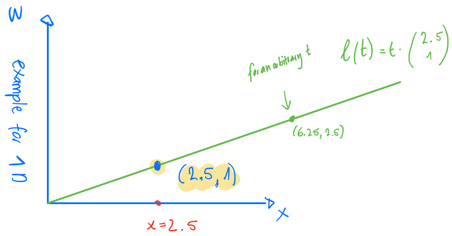
$$\div 2 \quad \div 3 \quad \div 2 \quad \div 5$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Application to computer graphics



Homogeneous coordinates



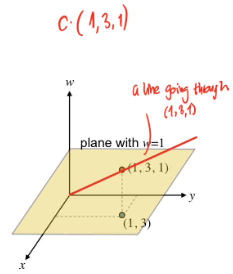
- We say that (x, y) has *homogeneous coordinates* $(x, y, 1)$.

example of 3D

- We say that $(x, y, z, 1)$ are homogeneous coordinates for the point (x, y, z) in $\mathbb{R}^3$.
- In general, (X, Y, Z, H) are **homogeneous coordinates** for (x, y, z) if $\boxed{x = X/H,}$ $\boxed{y = Y/H,}$ $\boxed{z = Z/H}$ $H/H = 1$ and $\boxed{H \neq 0}$.
- Each nonzero scalar multiple of $(x, y, z, 1)$ gives a set of homogeneous coordinates for the same point in $\mathbb{R}^3$.
  - For instance, $(10, 6, 14, 2)$ and $(15, 9, 21, 3)$ are all equivalent homogeneous coordinates for $(5, 3, 7)$.

$c \cdot (x, y, z, 1)$ = hyper line in $\mathbb{R}^4$ going through (x, y, z)

$c \cdot (1, 3, 1)$



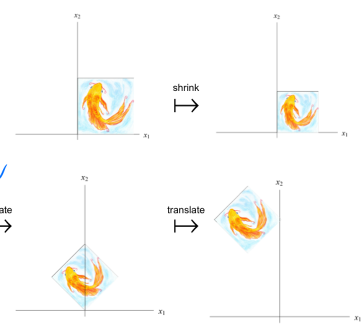
Numerical Methods

$t = 2.5$ $\quad l(t) = \begin{pmatrix} 6.25 \\ 2.5 \end{pmatrix}$ $\qquad t = 3$ $\quad l(t) = \begin{pmatrix} 7.5 \\ 2.5 \end{pmatrix} \longmapsto \begin{pmatrix} 7.5/3 \\ 2.5/3 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 1 \end{pmatrix}$

Composite transformation

$$\underbrace{\overset{\text{translate}}{\begin{bmatrix} 1 & 0 & -.5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\text{rotate}}{\begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\text{shrink}}{\begin{bmatrix} .75 & 0 & 0 \\ 0 & .75 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$

$$= \begin{bmatrix} .75\cos(\pi/4) & -.75\sin(\pi/4) & -.5 \\ .75\sin(\pi/4) & .75\cos(\pi/4) & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

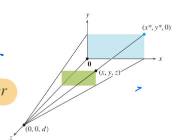


Numerical Methods

9

Perspective projection

- A *perspective projection* maps each point (x, y, z) onto an image point $(x^*, y^*, 0)$.
  - Such that the two points and the eye position, *center of projection*, form a straight line.



x^* of x onto the image plane at $\boxed{z=0}$

$$x^* = \frac{dx}{d-z} = \frac{x}{1 - z/d}. \qquad y^* = y/(1 - z/d).$$

Projection matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix}.$$

Subspaces of R^n

ex. with given P

data to be projected

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 - z/d \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{x}{1-z/d} \\ \frac{y}{1-z/d} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \\ 0 \\ 1 \end{bmatrix}$$

projected point on the image plane (homogenous)

(non-homogenous)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix}}_{\text{projection matrix}} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\text{origin point}} = \underbrace{\begin{bmatrix} x^* \\ y^* \\ 0 \\ w^* \end{bmatrix}}_{\text{projection point}}$$

equal to $A x = b$ (given A, given b, x ?)

y will be solvable with a free variable → consistent with infinite many solutions, mean data point could be anywhere of the line!

Dimension and Rank

$b \in \mathbb{R}^n$ $\quad H = \{a, b, c \ldots\}$

Definition: A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a) The zero vector is in H.

b) For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

c) For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Example 1: If $\mathbf{w}_1$ and $\mathbf{w}_2$ are in $\mathbb{R}^n$ and $H = \mathrm{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$, then H is a subspace of $\mathbb{R}^n$

a) Note that the zero vector is in H because $0 = 0\mathbf{w}_1 + 0\mathbf{w}_2$.

b) Take two arbitrary vectors in H, say, $\mathbf{u} = s_1\mathbf{w}_1 + s_2\mathbf{w}_2$ and $\mathbf{v} = t_1\mathbf{w}_1 + t_1\mathbf{w}_2$.
Then $\mathbf{u}+\mathbf{v} = (s_1+t_1)\mathbf{w}_1 + (s_2+t_2)\mathbf{w}_2$, which is in H.

c) Also, for any scalar c, the vector $c\mathbf{u}$ is in H, because
$c\mathbf{u} = c(s_1\mathbf{w}_1 + s_2\mathbf{w}_2) = cs_1\mathbf{w}_1 + cs_2\mathbf{w}_2$.

Column space & null space of matrix

$A = {}_m\big[\ \big]^n$

$\boxed{\text{Col } A}$ = set of all lin combinations of columns A : i.e. $A = [a_1, \ldots, a_n]$ $\quad$ Col A = Span$\{a_1, \ldots, a_n\}$ → subspace of $\mathbb{R}^m$

- **Example 4**: Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$.

  Determine whether $\mathbf{b}$ is in the column space of A.
- **Solution**: Row reducing the augmented matrix $[A \ \ \mathbf{b}]$,

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free variable
→ many solutions

- We can conclude that $A\mathbf{x} = \mathbf{b}$ is consistent and $\mathbf{b}$ is in Col A.
  - $\mathbf{b}$ is thus a linear combination of the columns of A

$\boxed{\text{Nul } A}$ = set of all solutions of a homogenous equation $Ax = 0$ → subspace of $\mathbb{R}^n$

Proof:

a) The zero vector is in Nul A (because $A\mathbf{0} = \mathbf{0}$).

b) Take any $\mathbf{u}$ and $\mathbf{v}$ in Nul A, then
$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\mathbf{u}+\mathbf{v}$ is in Nul A.

c) For any scalar c, $A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{0} = \mathbf{0}$, so $c\mathbf{u}$ is in Nul A.

The Invertible Matrix Theorem (extended):
Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix

m. The columns of A form a basis of $\mathbb{R}^n$.
n. Col $A = \mathbb{R}^n$
o. rank $A = n$
p. dim Nul $A = 0$
q. Nul $A = \{\mathbf{0}\}$

$\boxed{\text{basis}}$ = lin. independent set in H that spans H

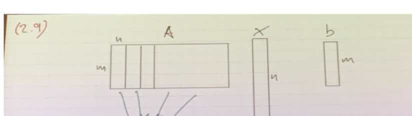
Theorem 13: The pivot columns of a matrix A form a basis for the column space of A.

- **Example**:

$$A = \begin{bmatrix} 2 & 4 & 2 & 6 \\ 1 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

- Basis for Col $A = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 2 \\ 1 \end{bmatrix} \right\}$.

dim = # vectors in a basis
rank = # of pivot columns



- Col A = Span$\{\ \ \}$ $\subseteq \mathbb{R}^m$ (subspace of m)
- Nul A = $\{x : Ax = 0\} \subseteq \mathbb{R}^n$ (subspace of n)

$Ax = b$ consistent $\Leftrightarrow b \in$ Col A

pivot columns of A → basis of Col A

rank A = dim Col A = # of pivot columns of A

rank A + dim Nul$A$ = n $\quad$ (# free variables = dim Nullspace)

Theorem 14: If a matrix A has n columns, then
rank A + dim Nul $A = n$.

Theorem 15: Let H be a p-dimensional subspace of $\mathbb{R}^n$.

a. Any linearly independent set of exactly p elements in H is automatically a basis for H.

b. Also, any set of p elements of H that spans H is automatically a basis for H.

Polynomial interpolation

- Given the output values $y_0, y_1, \dots, y_n$ of some unknown function f for specific input values $x_0, x_1, \dots, x_n$, thus $y_i = f(x_i)$ for $i = 0\dots n$.
- Find a function $p(x)$ that is close to f, with $p(x_i) = f(x_i)$ for $i = 0\dots n$.
  - Using a polynomial function $p(x) = a_0 + a_1 x_i + a_2 x_i^2 + a_3 x_i^3 + \dots + a_n x_i^n$.
- Cubic polynomial interpolation $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ as matrix notation $\underline{A\mathbf{a}=\mathbf{y}}$ (find... $Ax=b$) → solvable, when: x_i all different

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\Rightarrow$ rank$(A) = n$ (fullrang) $\Rightarrow$ invertible $\Rightarrow$ solvable

3. Determinants

2x2: $\boxed{\det A = a_{11}a_{22} - a_{12}a_{21}.}$

$n \geq 2$:

$$\det A = a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \cdot \det A_{1n}$$

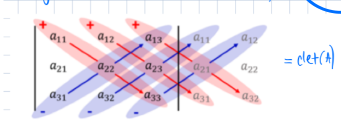
$$= \boxed{\sum_{j=1}^{n} (-1)^{1+j} a_{1j} \cdot \det A_{1j}}$$ ✓ Entwicklungssatz

3x3:

$$\det A = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Regel von Sarrus - nur für 3x3 Matrix



> **Theorem 2:** If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

> **Theorem 3:** Let A be a square matrix.
> a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
> b. If two rows of A are interchanged to produce B, then $\det B$ = -$\det A$.
> c. If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

times of interchanges

$$\det A = (-1)^r \det U$$

echelon form

det(A) ≠ 0

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} \quad \det U \neq 0$$ pivot

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \det U = 0$$

→ when free variable → # pivot < n → det(A) = 0 ϟ

- If A is invertible, all diagonal element are pivots.

$$\det A = \begin{cases} (-1)^r \cdot (\text{product of pivots of } U) & A \text{ invertible} \\ 0 & A \text{ not invertible} \end{cases}$$

If A is a 2x2 matrix with a zero determinant, then one column of A is a multiple of the other.

<u>4. Vector Space</u>

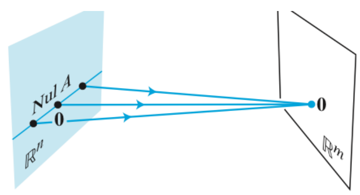
- **Definition**: A **subspace** of a vector space V is a subset H of V that has three properties:
  a) The zero vector of V is in H.
  b) H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
  c) H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.
  - Every subspace is a vector space.
  - Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

zero vector

- $\{0\}$ = zero subspace

spanning set for the nullspace $Ax=0$

- $\boxed{\text{Nul } A}$ is the set of all $\mathbf{x}$ in $\mathbb{R}^n$ that are mapped to the zero vector in $\mathbb{R}^m$ via the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$



→ subspace of $\mathbb{R}^m$

↳ row reduce $[A\ 0]$ to reduced echelon form (will get free variable) → parametric vector form = explicit representation

f.e: $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, free variable

$$\begin{aligned} x_1 - 2x_2 \qquad\quad -x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

explicit representation of nullspace

explicit →

the solutions in parametric vector form are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{w}}$$

$\{u, v, w\}$ is spanning set for Nul A

↓

vectors = # free variables!

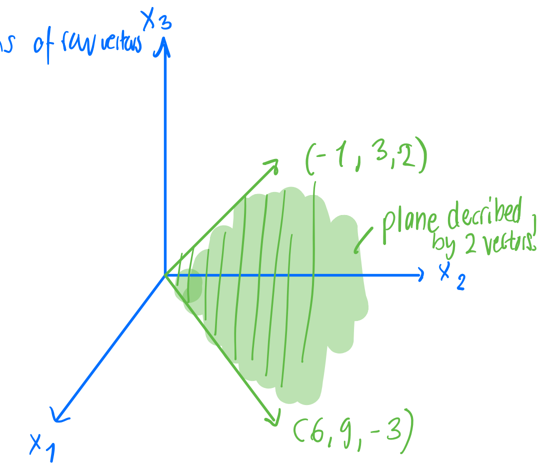
- $\boxed{\text{Col } A}$ ← $\{b : b = Ax \text{ for some } x \text{ in } \mathbb{R}^n\}$

> **Theorem 3**: The column space Col A of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

- The column space of $m \times n$ matrix A is all of $\mathbb{R}^m$ iff equation $Ax=b$ has solution for every b in $\mathbb{R}^m$ → consistent

⇒ $x \mapsto Ax$ is a mapping from $\mathbb{R}^n$ onto $\mathbb{R}^m$ (surjectiv)

$\mathrm{Col}\,A^T$
$\|$
• $\boxed{\text{Row } A}$ = rowspace A → set of all lin combinations of row vectors
→ subspace of $\mathbb{R}^n$



ex: $W = \left\{ x = \begin{bmatrix} 6a & -b \\ 9a & +3b \\ -3b & +2b \end{bmatrix} ;\ a, b \in \mathbb{R} \right\}$

$W = \left\{ x = a \begin{bmatrix} 6 \\ 9 \\ -3 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} ;\ a, b \in \mathbb{R} \right\}$ (changed in vector representation (explicitly))

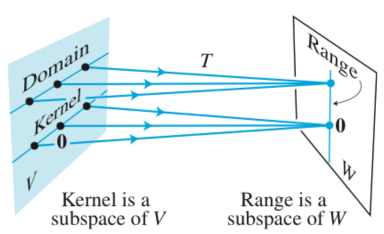
$W = \mathrm{Span}\left\{ \begin{pmatrix} 6 \\ 9 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \right\} = \mathrm{Col}\,A$

(coefficient matrix
$A = \begin{bmatrix} 6 & -1 \\ 9 & 3 \\ -3 & 2 \end{bmatrix}$

- Subspaces of vector spaces other than $\mathbb{R}^n$ are often described in terms of a linear transformation instead of a matrix.

  i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in V, and
  ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all u in V and all scalars c.

• $\boxed{\text{Kernel}}$ (= Nullspace) → of T is set of all x in V that $T(x) = 0$



- T maps the kernel elements to the zero vector in W.
- The kernel of T is a subspace of V.
- The kernel of T is Nul A of its standard form $A\mathbf{x}$.

• $\boxed{\text{range}}$ of T is set of all vectors in W of the form $T(x)$ for some x in V.

- The range of T is a subspace of W.
- The range of T is Col A of its standard form $A\mathbf{x}$.

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	1. Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; *i.e.*, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	2. Col A is explicitly defined; *i.e.*, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[A\ \ \mathbf{0}]$ are required.	3. It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	4. There is a direct relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $\boxed{A\mathbf{v} = \mathbf{0}}$.	5. A typical vector $\mathbf{v}$ in Col A has the property that the equation $\boxed{A\mathbf{x} = \mathbf{v}}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[A\ \ \mathbf{v}]$ are required.
zero vector 7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	subspace 7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. (bijektiv)	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

$Ax=b$ has to be consistent!
$\rightarrow x \mapsto Ax$ mapping $\mathbb{R}^n$ onto $\mathbb{R}^m$

Let V be the vector space of all real-valued functions f defined on an interval $[a, b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a, b]$.

Let W be the vector space $C[a, b]$ of all continuous functions on $[a, b]$, and let $D : V \to W$ be the transformation (i.e. differentiation) that changes f in V into its derivative f'.

Coordinate Systems

derivation-rule: → derivitives are linear transformation

$D(f+g) = D(f) + D(g)$

$D(c \cdot f) = c \cdot D(f)$

Kernel of derivitives are the constant functions

• linearly independence

- **Definition:** Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a basis for H if
  (i) $\mathcal{B}$ is a linearly independent set, and
  (ii) $H = \text{Span } \mathcal{B}$ – basis = dimension of H! in subspace of dim V!

• spanning set

Theorem 5: Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V, $S \subseteq V$, and let $H = \text{Span } \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$

a. If one of the vectors in S – say, $\mathbf{v}_k$ – is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

• basis

Theorem 6: The pivot columns of a matrix A form a basis for Col A.

lin. independent

• row equivalent

RowA = RowB

Theorem 7: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

The Unique Presentation Theorem

Theorem 8: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\boxed{\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n.}$$

- Suppose $\mathbf{x}$ also has the representation $\mathbf{x} = d_1\mathbf{b}_1 + \dots + d_n\mathbf{b}_n$ for some different scalars $d_1, \dots, d_n$.
- Then, subtracting, we have different scalars

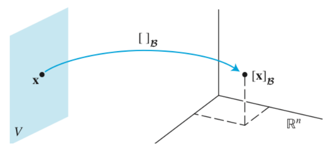
$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \dots + (c_n - d_n)\mathbf{b}_n$$

- Since $\mathcal{B}$, basis, is linearly independent, the weights must all be zero. That is, $\boxed{c_j = d_j}$ for $1 \le j \le n$.
- Therefore, the weights c_j and d_j are not different and thus unique. ↯

Coordinate Mapping $x \mapsto [x]_b$ (means x relative to the basis b)

↳ P_b^{-1} is invertible ⇒ coordinate mapping is a one-to-one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$

ex: $b_1 = \begin{bmatrix}2\\1\end{bmatrix}$ $b_2 = \begin{bmatrix}-1\\1\end{bmatrix}$ $x = \begin{bmatrix}4\\5\end{bmatrix}$ $b = \{b_1, b_2\}$



$$c_1 \cdot \begin{bmatrix}2\\1\end{bmatrix} + c_2 \cdot \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

$$\underset{A}{\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}} \underset{x}{\begin{bmatrix}c_1\\c_2\end{bmatrix}} = \underset{b}{\begin{bmatrix}4\\5\end{bmatrix}} \qquad [x]_b = \begin{bmatrix}c_1\\c_2\end{bmatrix}$$

change-of-coordinate matrix = matrix changes $[x]_b$ into standard representation x

↳ preserves addition and scalar multiplication ⇒ means it is a linear transformation

$$[\mathbf{u} + \mathbf{w}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{w}]_{\mathcal{B}} \qquad [r\mathbf{u}]_{\mathcal{B}} = r[\mathbf{u}]_{\mathcal{B}}$$

• isomorphism: One-to-one lin. transformation from V ont W

example 1 let B be the standard basis of the space $\mathbb{P}_3$ of polynomials; that is, monomial let $B = \{1, t, t^2, t^3\}$. A element p of $\mathbb{P}_3$ has the form $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$$[p]_B = \begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}$$

example 2: Use coordinate vectors to verify that the polynomials $1+2t^2$, $4+t+5t^2$, $3+2t$ are linearly dependent on $\mathbb{P}_2$ $\rightarrow$ $B \overset{\text{span}}{=} \{1, t, t^2\}$

2 pivot element

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}4\\1\\5\end{bmatrix}, \begin{bmatrix}3\\2\\0\end{bmatrix} \rightarrow \begin{bmatrix}1&4&3&0\\0&1&2&0\\2&5&0&0\end{bmatrix} \sim \begin{bmatrix}1&4&3&0\\0&1&2&0\\0&0&0&0\end{bmatrix} \text{— 1 free variable}$$

$-5\times$ $\quad$ $2\times$ $\quad$ $+$

homogenous equation (augmented matrix) $\quad$ (reduced echelon form)

pivot < n $\rightarrow$ linearly dependent! $-5\cdot\begin{bmatrix}1\\0\\2\end{bmatrix}+2\cdot\begin{bmatrix}4\\1\\5\end{bmatrix}=\begin{bmatrix}3\\2\\0\end{bmatrix}$

$p_1(t) = 1+2t^2 \quad p_2(t) = 4+t+5t^2 \quad p_3(t) = 3+2t$

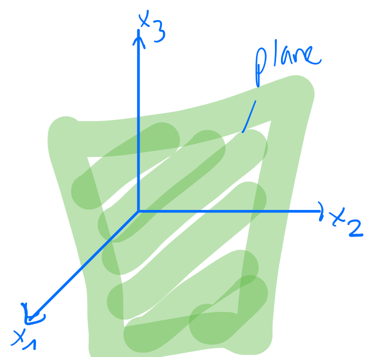
$\text{Span}\{p_1, p_2, p_3\} = H$ (vector space)

$\mathcal{B}_H = \{p_1, p_2\}$ since lin. dependent they can form p_3

$\dim(H) = 2$ (# pivots)

$p(t) = a_0 + a_1 t + a_2 t^2 \in \mathbb{P}_2$

$\mathcal{B}_{\mathbb{P}_2} = \{1, t, t^2\} \rightarrow [p_1]_{\mathcal{B}_{\mathbb{P}_2}} = \begin{bmatrix}1\\0\\2\end{bmatrix}$



The Dimension of a Vector Space

Theorem 10: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ then any set in V containing more than n vectors must be linearly dependent.

Theorem 11: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

- finite-dimensional = vector space spanned by finit set.
- dimension ($\dim V$) = # vectors in a basis for V. (hence they are a basis, they are all lin. independ)

- The subspaces of $\mathbb{R}^3$ can be classified by dimension.
  - 0-dimensional subspace
    - zero vector, origin point
  - 1-dimensional subspace
    - single vector, line through origin
  - 2-dimensional subspace
    - spanned by two vectors, plane through origin
  - 3-dimensional subspace
    - all of $\mathbb{R}^3$, entire volume

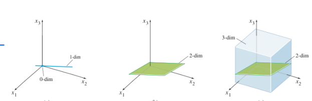


FIGURE 1 Sample subspaces of $\mathbb{R}^3$.