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CINVESTAV - ORACLE

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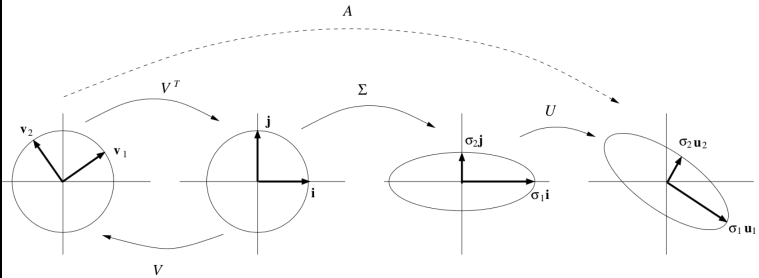
What does $A = U\Sigma V^T$ mean?

Geometric interpretation

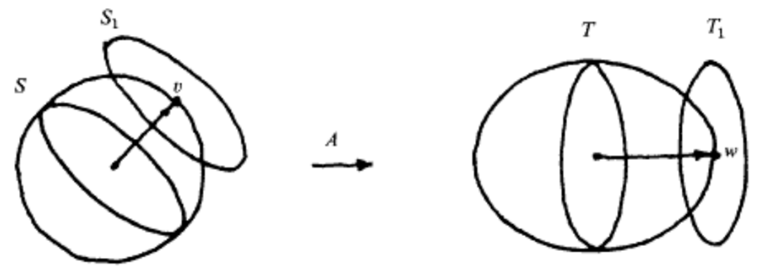


Why SVD is possible? Geometrical insight

Is all about orthogonality

$A = U\Sigma V^T \implies A\mathbf{v_i} = \sigma_i \mathbf{u_i} \ni \ i = 1 \ldots \mathrm{rank}(A)$

What is special about v?



Got curious?

Check the report, it has the full proof.

SVD as an Eigenproblem

Algebraic Proof

Almost a whole chapter of the report, check it out!

The eigenproblem, solved for symmetric matrices

$B^{n \times n} = Q \Lambda Q^T$

$A^T A$ is symmetric!

$A = U \Sigma V^T \iff A^T A = V \Sigma^2 V^T$

$A A^T$ is symmetric too! (used in GenSim)

$A = U \Sigma V^T \iff A A^T = U \Sigma^2 U^T$

Is SVD problem well conditioned?

A problem is well conditioned if ...

$x \simeq (x + \delta) \implies f(x) \simeq f(x + \delta)$

Weyl's Theorem

$|\tilde{\sigma}_i - \sigma_i| \leq \|E\|_2 \quad \ni \quad \tilde{A} = A + E$

Wedin's Theorem

The singular vectors subproblem is not well conditioned in theory, but it has bounds.

In practice we are fine

We do not need singular vectors by themselves ($\mathbf{v} = \Sigma^{-1}\, U^T \mathbf{x}$).

Algorithm 1: The Single-Vector Lanczos Algorithm

Input : A matrix $A^{m \times n}$ and a truncation factor k
Output: The k singular values and its associated right singular vectors of A. Both are numeric approximations.

Use Lanczos Tridiagonalization Step to generate a family of c symmetric tridiagonal matrices $\{T_j\}$ for $A^T A$ $(c > k)$

Compute the eigenvalues and eigenvectors of T_k using the (implicit) QL Method.

For each computed λ_i of T_k, calculate the associated unit eigenvector $\mathbf{z_i}$ such that $T_k \mathbf{z_i} = \lambda_i \mathbf{z_i}$.

For each calculated eigenvector $\mathbf{z_i}$ of T_k, compute the Ritz vectors $v_i = Q_c \mathbf{z_i}$ as an approximation to the i-th eigenvector of $A^T A$. Note that the matrix Q_c is a side product of the first step.

return $(\{\lambda_1, \lambda_2, \cdots, \lambda_k\}, \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\})$

Algorithm 2: Lanczos Tridiagonalization Step (sparse,2)

Input : A unit vector $\mathbf{q_1} \in \mathbb{R}^n$ and a symmetric matrix $A^{n \times n}$

Output: The sequences $\{\alpha_i\}$, $\{\beta_i\}$ and matrix $Q = [\mathbf{q_1}|\mathbf{q_2}|\cdots]$

$k \leftarrow 0, \beta_0 \leftarrow 1, \mathbf{q_0} \leftarrow 0, r_0 \leftarrow \mathbf{q_1}$

while $k = 0 \vee \beta_k \neq 0$ **do**

$\quad \mathbf{q_{k+1}} \leftarrow \dfrac{\mathbf{r_k}}{B_k}$

$\quad k \leftarrow k + 1$

$\quad \alpha_k \leftarrow \mathbf{q_k}^T A\mathbf{q_k}$

$\quad \mathbf{r_k} \leftarrow A\mathbf{q_k} - \alpha_k q_k - \beta_{k-1}\mathbf{q_{k-1}}$

$\quad \beta_k \leftarrow \|\mathbf{r_k}\|_2$

return $(\{\alpha_i\}, \{\beta_i\}, Q = [\mathbf{q_1}|\mathbf{q_2}|\cdots])$

The T_k from Lanczos Tridiagonalization Step

$$T_k = \begin{bmatrix} \alpha_1 & \beta_1 & \cdots & 0 \\ \beta_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \beta_{k-1} \\ 0 & \cdots & \beta_{k-1} & \alpha_k \end{bmatrix}$$

Lanczos Algorithm Reality Check

Is it numerically stable?

No, the tridiagonalization step needs to be rewritten (thanks Paige!).

Is orthogonality really guaranteed?

No, it is lost precisely with the converved singular vectors (thanks again Peige!). Selective re-orthogonalization is required (thanks Parlett and Simon!).

What to do then?

Berry's implementation, LASVD/LAS2, consider all aspects above; including performance (parallelization).

LASVD/LAS2 Profiling by Berry

Routine	Library	Description
SPMXV	BLAS level 2	Sparse matrix-vector mult.
IMTQL2 / TRED2	EISPACK	Implicit QL Algorithm.
DAXPY	BLAS level 1	$\mathbf{x} \leftarrow \gamma\mathbf{x} + \mathbf{y}$
DAXPY	BLAS level 1	$\mathbf{x} \leftarrow \mathbf{y}$
DDOT	BLAS level 1	$\mathbf{x} \cdot \mathbf{y}$

	Alliant FX/80		Cray-2S/4-128	
Routine	Speedup	%CPU Time	Speedup	%CPU Time
SPMXV	3	27%	-	72%
IMTQL2	4.3	14%	-	12%
DAXPY	5	17%	-	-
DCOPY	3.6	20%	-	-
DDOT	7.7	2%	-	-

SVDPACK $\Longrightarrow$ SVDLIBC: lost parallelism

- Berry's SVDPACK LAS2: SPMXV (BLAS?), IMTQL2 (EISPACK)
- Berry ports to SVDPACKC: opa/opb (user), IMTQL2 (serial!)
- Rohde new skin of SVDLIBC: opa/opb (serial!), IMTQL2 (serial!).
- Radim's python wrapper SPARSESVD : same all
- Radim's GenSim uses a serial LAS2 routine!