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# Source Localization from Range-Difference Measurements

The problem considered in this lecture note is that of locating a radiating source from range-difference observations. In a constant-velocity propagation medium, such observations can be obtained from the time differences of arrival measured using an array (or network) of passive sensors under the assumption that the speed of propagation is known. This specific source localization problem has received significant attention in the signal processing literature for at least 20 years. Several solutions have been proposed to solve it either approximately or exactly [1]–[7]. However, some of these solutions have not been described clearly in the original papers, and confusions seem to persist in more recent works.

We will clarify and streamline the most successful solutions. Furthermore, we will introduce a new approximate solution. Finally, we will comment briefly on the related problem of source localization from energy or range measurements [5], [8], [9].

## RELEVANCE

Applications of the source localization methods discussed here include surveillance, navigation, wireless communications, teleconferencing, and geophysics. The material we present herein fits well into graduate signal processing courses in electrical engineering, such as courses on signal processing for wireless communications and networks as well as on spectral estimation and array processing.

## PREREQUISITES

The prerequisites for understanding and using this lecture note include basic estimation theory and linear algebra.

## PROBLEM STATEMENT

Consider an array of $N+1$ sensors, and let $\mathbf{a}_n$ denote the coordinates of the $n$th sensor (in practical applications, $\mathbf{a}_n$ is thus a $2\times 1$ or $3\times 1$ vector). We assume that $\mathbf{a}_0 = \mathbf{0}$; therefore, the sensor 0 is chosen as the origin of the coordinate system. Let $\mathbf{x}$ denote the source's coordinate vector. Finally, let $\mathrm{d}_n$ denote the source's range-difference between sensor $n$ and sensor 0, i.e.,

$$d_n = \|\mathbf{a}_n - \mathbf{x}\| - \|\mathbf{x}\|, \quad n = 1, \cdots, N, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean vector norm. It follows from (1) that

$$\|\mathbf{a}_n - \mathbf{x}\|^2 = (d_n + \|\mathbf{x}\|)^2, \tag{2}$$

which yields the following equations in the unknown vector $\mathbf{x}$:

$$d_n\|\mathbf{x}\| + \mathbf{a}_n^T\mathbf{x} = b_n, \quad n = 1, \cdots, N, \tag{3}$$

where $(\cdot)^T$ denotes the transpose and

$$b_n = \frac{\|\mathbf{a}_n\|^2 - d_n^2}{2}. \tag{4}$$

The equations in (3) hold only approximately in practice due to measurement errors and sensor calibration errors. Consequently, a reasonable way to estimate $\mathbf{x}$ based on (3) is via the minimization of the following least-squares (LS) criterion:

$$c_1 = \sum_{n=1}^{N} \left(d_n\|\mathbf{x}\| + \mathbf{a}_n^T\mathbf{x} - b_n\right)^2. \tag{5}$$

The most successful methods for approximately minimizing (5) have been presented in [1], [2] (also see the references therein). The approximate minimization method proposed in [4] was less successful in that its performance was shown to be rather poor [1], [2]. The exact minimization of (5) has been considered in [3]. However, the methods in [1]–[3] have not been described clearly, and confusions about them persist in the literature. In the following sections, we will try to clarify and streamline these methods.

## METHOD 1: THE UNCONSTRAINED LS ESTIMATE

Let

$$\mathbf{y}(\mathbf{x}) = \begin{bmatrix} \|\mathbf{x}\| \\ \mathbf{x} \end{bmatrix}; \quad \mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix};$$

$$\Phi = \begin{bmatrix} d_1 & \mathbf{a}_1^T \\ \vdots & \vdots \\ d_N & \mathbf{a}_N^T \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}. \tag{6}$$

Using this notation, we can rewrite the LS criterion in (5) as:

$$c_1 = \|\Phi\mathbf{y}(\mathbf{x}) - \mathbf{b}\|^2. \tag{7}$$

Let $y_i$ denote the $i$th element of $\mathbf{y}$. The unconstrained LS estimate of $\mathbf{y}$ (and hence of $\mathbf{x}$) is obtained by minimizing (7) with respect to $\mathbf{y}$, ignoring the dependence of $y_1 = \|\mathbf{x}\|$ on the remaining elements of $\mathbf{y}$. As is well known, this unconstrained minimization of (7) gives

$$\tilde{\mathbf{y}} = (\Phi^T\Phi)^{-1}\Phi^T\mathbf{b}. \tag{8}$$

The corresponding unconstrained LS estimate $\tilde{\mathbf{x}}$ *of* $\mathbf{x}$ is given by

$$\tilde{\mathbf{x}} = [\,\mathbf{0} \quad \mathbf{I}\,]\,\tilde{\mathbf{y}}, \tag{9}$$

where $\mathbf{0}$ is a vector made from zeros. Whenever the errors affecting (3) are



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small enough, $\tilde{\mathbf{x}}$ can be shown to be a reasonably accurate approximation to the minimizer of (5).

Next, we consider the so-called spherical-interpolation (SI) method, introduced in [1] and [2], which also estimates $\mathbf{x}$ by approximately minimizing (5). In [1] and [2], the SI method was obtained as follows:

1) (7) was minimized with respect to $\mathbf{x}$ for fixed $y_1$; let us denote the result of this minimization operation by $\tilde{\mathbf{x}}(y_1)$
2) $\tilde{\mathbf{x}}(y_1)$ was inserted in (7) to obtain a function of only $y_1$ that was minimized with respect to $y_1$; let $\tilde{y}_1$ denote the result of this minimization
3) $\mathbf{x}$ was estimated as $\tilde{\mathbf{x}}(\tilde{y}_1)$.

Because (7) has a unique minimum with respect to $\mathbf{y}$, given by (8), the final result of the previous unnecessarily complicated two-stage unconstrained minimization process can be nothing but $\tilde{\mathbf{x}}$ (this simple fact was also noted in [3]). Consequently, the SI method of [1] and [2] is identical to the unconstrained LS method in (8) and (9).

Another method for the approximate minimization of (5) is the so-called subspace minimization (SM) method (see [1], [5], and the references therein). This method first eliminates the term depending on $\|\mathbf{x}\|$ from (3) by premultiplying (3) with the orthogonal projector onto the null space of $\mathbf{d}^T$, i.e.,

$$\mathbf{P}_{\mathbf{d}}^{\perp} = \mathbf{I} - \frac{\mathbf{d}\mathbf{d}^T}{\|\mathbf{d}\|^2}. \tag{10}$$

The result of this multiplication is

$$\mathbf{P}_{\mathbf{d}}^{\perp}(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{0}, \tag{11}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_N^T \end{bmatrix}. \tag{12}$$

Next, $\mathbf{x}$ is estimated by minimizing the LS criterion derived from (11)

$$c_2 = \|\mathbf{P}_{\mathbf{d}}^{\perp}(\mathbf{A}\mathbf{x} - \mathbf{b})\|^2. \tag{13}$$

Now, evidently, (13) can also be obtained by minimizing the original LS criterion (7) with respect to $y_1$ for fixed $\mathbf{x}$. It follows from this simple observation that the minimizer of (13) is $\tilde{\mathbf{x}}$. Consequently, the SM method, like the SI method, is identical to the unconstrained LS method.

To end this section, we note that the unconstrained LS method can be easily extended to the case of unknown speed of propagation, as shown in [6].

## METHOD 2: THE CONSTRAINED LS METHOD

The dependence among the elements of $\mathbf{y}(\mathbf{x})$ can be described as a constraint on $\mathbf{y}$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = 0; \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{I} \end{bmatrix}. \tag{14}$$

The Lagrangian function associated with the constrained LS problem (7), (14) is given by

$$L(\mathbf{y}, \lambda) = \|\Phi\mathbf{y} - \mathbf{b}\|^2 + \lambda\mathbf{y}^T\mathbf{D}\mathbf{y}, \tag{15}$$

where $\lambda$ is the Lagrange multiplier. Equating $\partial L(\mathbf{y}, \lambda)/\partial\mathbf{y}$ to zero gives

$$\hat{\mathbf{y}}(\lambda) = (\Phi^T\Phi + \lambda\mathbf{D})^{-1}\Phi^T\mathbf{b}, \tag{16}$$

where $\lambda$ (which should be such that the matrix $\Phi^T\Phi + \lambda\mathbf{D}$ is positive definite) is obtained by solving the constraint equation

$$\hat{\mathbf{y}}^T(\lambda)\mathbf{D}\hat{\mathbf{y}}(\lambda) = 0. \tag{17}$$

In general (for three-dimensional applications), there is no known way to solve (17) in closed form, and thus iterative methods must be used for finding $\lambda$. Once $\lambda$ was found, the constrained LS estimate of $\mathbf{x}$ is given by

$$\hat{\mathbf{x}} = [\,\mathbf{0} \quad \mathbf{I}\,]\,\hat{\mathbf{y}}(\lambda). \tag{18}$$

The constrained minimization of (7) by means of the Lagrange method, as outlined above, was considered in [3], but the discussion there, like those in [1], [2], and some more recent literature, is misleading. In particular, [3] appears to ignore the estimate in (16), given by the Lagrange approach and instead considers a so-called linear correlation (LC) estimate

$$\hat{\hat{\mathbf{y}}}(\lambda) = [\mathbf{I} + \lambda(\Phi^T\Phi)^{-1}\mathbf{D}]^{-1}\tilde{\mathbf{y}}, \tag{19}$$

where $\tilde{\mathbf{y}}$ is the unconstrained LS estimate given by (8), and $\lambda$ is obtained by solving the equation $\hat{\hat{\mathbf{y}}}^T(\lambda)\mathbf{D}\hat{\hat{\mathbf{y}}}(\lambda) = 0$. Evidently, we have

$$\begin{aligned}\hat{\hat{\mathbf{y}}}(\lambda) &= (\Phi^T\Phi + \lambda\mathbf{D})^{-1}(\Phi^T\Phi)\,\tilde{\mathbf{y}} \\ &= \hat{\mathbf{y}}(\lambda), \end{aligned} \tag{20}$$

and hence the reason for the lengthy calculations and arguments in [3] leading to (19) as well as the motivation for the name of LC given to the so-obtained method are unclear. Since the LC method of [3] is identical to the constrained LS method, we propose to call it accordingly.

## METHOD 3: A NEW APPROXIMATE LS ESTIMATE

As explained in the previous section, the constrained LS estimate requires the use of an iterative solver for the Lagrange multiplier equation in (17). Due to the potential convergence problems of such an iterative procedure, approximate but closed-form estimates might be preferable in applications. In the following, we propose a closed-form approximate LS estimate that can be viewed as an enhanced version of the unconstrained LS estimate in a sense that will become clear shortly.

A simple calculation shows that the LS criterion in (7) can be rewritten as (to within an additive constant)

$$c_3 = [\mathbf{y}(\mathbf{x}) - \tilde{\mathbf{y}}]^T(\Phi^T\Phi)[\mathbf{y}(\mathbf{x}) - \tilde{\mathbf{y}}], \tag{21}$$

where $\tilde{\mathbf{y}}$ is given by (8). We linearize $\mathbf{y}(\mathbf{x}) - \tilde{\mathbf{y}}$ for $\mathbf{x}$ in the vicinity of $\tilde{\mathbf{x}}$, by means of a Taylor series expansion

$$\begin{aligned}\mathbf{y}(\mathbf{x}) - \tilde{\mathbf{y}} &\approx \mathbf{y}(\tilde{\mathbf{x}}) - \tilde{\mathbf{y}} + \left.\frac{\partial\mathbf{y}(\mathbf{x})}{\partial\mathbf{x}^T}\right|_{\mathbf{x}=\tilde{\mathbf{x}}}(\mathbf{x} - \tilde{\mathbf{x}}) \\ &= \delta + \mathbf{G}(\mathbf{x} - \tilde{\mathbf{x}}), \end{aligned} \tag{22}$$

where

$$\delta = \begin{bmatrix} \|\tilde{\mathbf{x}}\| - \tilde{y}_1 \\ \mathbf{0} \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} \frac{\tilde{\mathbf{x}}^T}{\|\tilde{\mathbf{x}}\|} \\ \mathbf{I} \end{bmatrix}. \tag{23}$$

Using (22) in (21), we get a quadratic approximation of (7)

$$c_4 = [\delta + \mathbf{G}(\mathbf{x} - \tilde{\mathbf{x}})]^T (\Phi^T \Phi) \times [\delta + \mathbf{G}(\mathbf{x} - \tilde{\mathbf{x}})]. \quad (24)$$

We propose to estimate $\mathbf{x}$ as the minimizer of (24), which gives

$$\hat{\mathbf{x}} = \tilde{\mathbf{x}} - (\mathbf{G}^T \Phi^T \Phi \mathbf{G})^{-1} \times (\mathbf{G}^T \Phi^T \Phi \delta). \quad (25)$$

The above estimate can also be interpreted as the result of one iteration with a Gauss-Newton algorithm applied to the LS criterion in (7) and initialized at $\tilde{\mathbf{x}}$. It is well known that the estimate in (25) approaches the constrained LS estimate, as the estimation error in $\tilde{\mathbf{x}}$ approaches zero.

## EXTENSIONS

Note that the source localization problem from range-difference measurements discussed earlier is related to the problem of source localization from energy or range measurements (see, e.g., [5], [8], and [9]). The energy measurement based source localization approach, advocated in [5] (and some of its references), is based on the fact that the energy of the signal received by the $n$th sensor over a (relatively small) time interval is inversely proportional to $\|\mathbf{x} - \mathbf{a}_n\|^2$ (for $n = 1, 2, \ldots, N$; we assume now that there are $N$ sensors, for notational convenience). Using this fact and some simple manipulations (see, e.g., [5] for details), it is possible to obtain an equation in the unknown vector $\mathbf{x}$ that is somewhat similar to (2), namely:

$$\|\mathbf{x} - \gamma_n\|^2 = \rho_n, \quad n = 1, \ldots, N, \quad (26)$$

where $\gamma_n$ and $\rho_n$ are functions of the energy measurements (as well as the sensor locations, etc., see [5]). From (26), we have that

$$\|\mathbf{x}\|^2 + \mathbf{a}_n^T \mathbf{x} = b_n, \quad n = 1, \ldots, N, \quad (27)$$

where $\mathbf{a}_n = -2\gamma_n$ and $b_n = \rho_n - \|\gamma_n\|^2$. Here $\mathbf{a}_n$ is no longer the coordinate vector of the $n$th sensor; however, by a slight abuse of notation, we have used $\mathbf{a}_n$ in (27) to stress the mathematical analogy with (3).

The same data model is also applicable to the problem of source localization from range measurements (see, e.g., [8], [9], and the references therein). Let $r_n$ denote the range between the source and the $n$th sensor:

$$r_n = \|\mathbf{a}_n - \mathbf{x}\|, \quad n = 1, \ldots, N, \quad (28)$$

where $\mathbf{a}_n$ and $\mathbf{x}$ are the same as those in (1). It follows from (28) that

$$\|\mathbf{x}\|^2 + \breve{\mathbf{a}}_n^T \mathbf{x} = \breve{b}_n, \quad n = 1, \ldots, N, \quad (29)$$

where $\breve{\mathbf{a}}_n = -2\mathbf{a}_n$ and $\breve{b}_n = r_n^2 - \|\mathbf{a}_n\|^2$. Note that (29) has the same form as (27).

The LS problem, derived from the model equations in (27), has the previous form in (7) with the only difference that now

$$\mathbf{y}(\mathbf{x}) = \begin{bmatrix} \|\mathbf{x}\|^2 \\ \mathbf{x} \end{bmatrix} \quad (30)$$

(and $d_n \equiv 1$). Due to the similarity between these two LS problems, all three estimation methods discussed previously, viz. the unconstrained LS, the constrained LS, and the approximate LS methods, can be used to estimate $\mathbf{x}$ in the present case, after some minor modifications. We omit the details of these modifications in the interest of brevity.

## CONCLUSIONS: WHAT WE HAVE LEARNED

The problem of source localization from range-difference measurements, which have been obtained using an array of passive sensors, is important to many applications and consequently it has received considerable attention in the literature. In this lecture note, we commented on, clarified, and streamlined some approximate and exact solutions to this problem. We also introduced a new closed-form approximate solution that can be viewed as an enhanced version of the unconstrained LS estimate. Finally, we commented briefly on the related problem of source localization from energy or range measurements. In nonpathological simulated examples, all these approaches provide location estimates that typically agree to within a few percent. On the other hand, in practical situations there will likely be wide variations in the quality of these estimates, depending on modelling errors and other application properties. See, for example, the numerical results in the cited references herein.


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*(continued on page 69)*

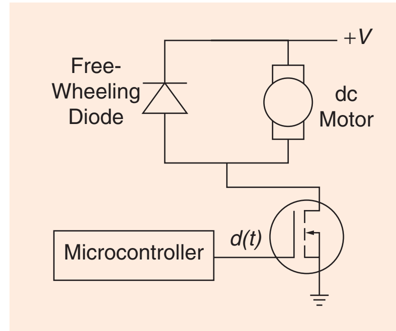
When $x(n) = A = 128$, the output of the digital portion, $y(n)$, oscillates between the values $A/2$ (64) and $3A/2$ (192) and has an average value of 128, as it should. In this situation, a low-cost analog low-pass filter will work very well indeed.

Figure 4 shows $z(n)$, $y(n)$, and $d(t)$ when $A = 128$ and $x(n) = 68$. Again $z(n)$ remains between zero and 255 and the average value of $y(n)$ is equal to the value of the $x(n)$ input.

Additionally, in each figure, the period of the output is significantly less than the maximum period that the theory permits—and this means that our low-pass filtering will be more effective than we predicted.

If you are interested in controlling the speed of a dc motor control, you can take care of the problem of the not-very-well-defined logic levels and the analog low-pass filtering in a particularly simple fashion. DC motors are themselves low-pass filters, so no low-pass filtering is required. Additionally, the dc motor requires an input that is capable of driving the motor and not the signal-level outputs of a microcontroller. One way to produce such a final output signal is to use the logic-level outputs of the microcontroller to control some form of power amplifier—and if done properly, the power amplifier can take care of the level shifting too.

If you are interested in using the motor to turn in one direction only, then connecting a MOSFET with a freewheeling diode to the output of the microcontroller, as shown in Figure 5, will do the job. As the MOSFET's resistance when turned on is very close to zero, the voltage driving the dc motor will be almost precisely $+V$. If you are interested in bidirectional control, then an H-bridge dc motor motion control chip (such as the National Semiconductor LM18201, or the Texas Instruments L293) is called for.




**[FIG5] Unidirectional speed control of a dc motor.**

## CONCLUSION

We have presented the properties of a simple $\Sigma\Delta$ DAC. We have shown that when one uses fixed-point arithmetic, a constant input leads to a limit cycle whose average value is precisely the value of the constant. We have also shown that it is possible to actually implement the digital portion of DACs of this type in a simple fashion using microprocessors from the simple and popular 8051 family. Finally, we considered some implementation issues.

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