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A new constrained weighted least squares algorithm for TDOA-based localization

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ABSTRACT

The linear least squares (LLS) technique is widely used in time-difference-of-arrival based positioning because of its computational efficiency. Two-step weighted least squares (2WLS) and constrained weighted least squares (CWLS) algorithms are two common LLS schemes where an additional variable is introduced to obtain linear equations. However, they both have the same measurement matrix that becomes ill-conditioned when the sensor geometry is a uniform circular array and the source is close to the array center. In this paper, a new CWLS estimator is proposed to circumvent this problem. The main strategy is to separate the source coordinates and the additional variable to different sides of the linear equations where the latter is first solved via a quadratic equation. In doing so, the matrix to be inverted has a smaller condition number than that of the conventional LLS approach. The performance of the proposed method is analyzed in the presence of zero-mean white Gaussian disturbances. Numerical examples are also included to evaluate its localization accuracy by comparing with the existing 2WLS and CWLS algorithms as well as the Cramér–Rao lower bound.

1. Introduction

Localization of an emitting source using an array of spatially distributed sensors has received considerable interests in recent years [1–3]. Time-difference-of-arrival (TDOA), which is the difference in arrival times between signals received at spatially separated receivers, is one of the commonly used measurements for source localization [4]. Nevertheless, locating the source is not a trivial task because the TDOA measurements are nonlinear functions of the source coordinates. The maximum-likelihood method, which is a nonlinear estimator, requires relatively high computational complexity and the global solution is not guaranteed to be obtained [5,6]. On the other hand, the linear least squares (LLS) [7–9] approach takes advantages of linear operation via the introduction of an additional variable. Without considering the known relationship of this variable with the source position, it is estimated together with the source position by solving a system of linear equations. Two improved versions for [7], namely, two-step weighted least squares (2WLS) [10] and constrained weighted least squares (CWLS) [11] estimators, which exploit the relationship between the additional variable and the source position, can provide estimation performance equal to Cramér–Rao lower bound (CRLB) at sufficiently small noise conditions. However, a matrix ill-conditioned problem occurs when the distances from the source to all receivers are identical or approximately the same. For example, when the source is located at or near the centroid of the receivers whose geometry is close to or equal to a uniform circular array (UCA), the 2WLS [10] performs poorly because the system matrix corresponding

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to the linear equations is singular or ill-conditioned. This will also significantly degrade the performance of [11]. The UCA geometry is the optimal configuration for localization [12,13] when the receivers on the circumference have equal angular spacing and the source is at the center of the circle. Hence an approximation of the UCA geometry occurs often in practice, such as when mobile sensors in a wireless sensor network [12,13] or a team of unmanned aerial vehicles, maneuver to form an UCA around a source [14]. Moreover, circular microphone array is typically used for speaker localization [15] while sonobuoys deployed in a semi-circular pattern have been used in the detection and localization of submarines [16].

The spherical-intersection [8,9] method eliminates the additional variable through subtraction of the localization equations, but is not optimal [10]. This paper gives a new formulation that avoids the possibility of having ill-conditioning problem that occurs in [10] and [11] when the localization geometry is close to a UCA. Instead of eliminating the additional variable, the new method estimates it from a quadratic equation. The result is a closed-form estimator that is optimal at low noise for all localization geometry, even when it is a UCA.

The rest of the paper is organized as follows. In Section 2, the 2WLS and CWLS algorithms are reviewed and then the matrix ill-conditioned problem is noted. Subsequently, a new and optimal CWLS method is proposed to circumvent this issue in Section 3. Analytical study of the devised scheme is also included. Simulation results are presented in Section 4 to evaluate the localization accuracy of the proposed CWLS estimator by comparing with that of [10,11] as well as the CRLB. Finally, conclusions are drawn in Section 5.

2. Problem statement

2.1. Review of existing LLS algorithms [10,11]

Consider an array of $M \geq 4$ receivers in a two-dimensional (2-D) space. Note that extension to three-dimensional space is straightforward. Let $\mathbf{x} = [x\ y]^T$ be the source position to be determined and $\mathbf{x}_i = [x_i\ y_i]^T$, $i = 1, 2, \ldots, M$, be the known coordinates of the ith receiver. Without loss of generality, the first sensor is assigned as the reference. The range difference measurements converted from the TDOAs, denoted by $r_{i,1}$, are modeled as

$$r_{i,1} = d_{i,1} + n_{i,1}, \quad i = 2, 3, \ldots, M \tag{1}$$

where $n_{i,1}$ is the range difference error and $d_{i,1} = d_i - d_1$ with d_i being the distance between the source and the ith receiver, that is

$$d_i = \sqrt{(x-x_i)^2 + (y-y_i)^2} \tag{2}$$

Substituting (2) into (1) yields

$$r_{i,1} + \sqrt{(x-x_1)^2 + (y-y_1)^2} = \sqrt{(x-x_i)^2 + (y-y_i)^2} + n_{i,1}, \quad i = 2, 3, \ldots, M \tag{3}$$

Squaring both sides of (3) and introducing an additional variable

$$R_1 = d_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2} \tag{4}$$

we obtain

$$(x_i-x_1)(x-x_1) + (y_i-y_1)(y-y_1) + r_{i,1}R_1 = 0.5[(x_i-x_1)^2 + (y_i-y_1)^2 - r_{i,1}^2] + m_{i,1}, \quad i = 2, 3, \ldots, M \tag{5}$$

where $m_{i,1} = d_i n_{i,1} + 0.5 n_{i,1}^2$.

In the 2WLS [10] and CWLS [11] methods, (5) is expressed in the following matrix form:

$$\mathbf{A}\boldsymbol{\theta} = \mathbf{b} + \mathbf{m} \tag{6}$$

where

$$\mathbf{A} = \begin{bmatrix} x_2-x_1 & y_2-y_1 & r_{2,1} \\ \vdots & \vdots & \vdots \\ x_M-x_1 & y_M-y_1 & r_{M,1} \end{bmatrix} \tag{7}$$

$$\boldsymbol{\theta} = [x-x_1\ y-y_1\ R_1]^T \tag{8}$$

$$\mathbf{b} = 0.5 \begin{bmatrix} (x_2-x_1)^2 + (y_2-y_1)^2 - r_{2,1}^2 \\ \vdots \\ (x_M-x_1)^2 + (y_M-y_1)^2 - r_{M,1}^2 \end{bmatrix} \tag{9}$$

$$\mathbf{m} = [m_{2,1} \cdots m_{M,1}]^T \tag{10}$$

with ^{T} being the transpose operation. Note that the additional variable R_1 makes the equation in (5) linear.

The basic idea of [10,11] is to determine $\boldsymbol{\theta}$ from

$$\hat{\boldsymbol{\theta}} = \arg\min_{\tilde{\boldsymbol{\theta}}} J(\tilde{\boldsymbol{\theta}}) = \arg\min_{\tilde{\boldsymbol{\theta}}} (\mathbf{A}\tilde{\boldsymbol{\theta}} - \mathbf{b})^T \mathbf{W} (\mathbf{A}\tilde{\boldsymbol{\theta}} - \mathbf{b}) \tag{11}$$

$$\text{s.t.} \quad \tilde{\boldsymbol{\theta}}^T \mathbf{S} \tilde{\boldsymbol{\theta}} = (\tilde{x}-x_1)^2 + (\tilde{y}-y_1)^2 - \tilde{R}_1^2 = 0 \tag{12}$$

where $\tilde{\boldsymbol{\theta}} = [\tilde{x}-x_1,\ \tilde{y}-y_1,\ \tilde{R}_1]^T$ is an optimization variable vector, $\hat{\boldsymbol{\theta}}$ is the estimate of $\boldsymbol{\theta}$, $\mathbf{S} = \mathrm{diag}(1,\ 1,\ -1)$ with diag being the diagonal function, and $\mathbf{W} = (\mathbb{E}\{\mathbf{m}\mathbf{m}^T\})^{-1}$ is the weighting matrix with $\mathbb{E}\{\cdot\}$ being the expectation and $^{-1}$ being the matrix inverse. When the received data at the sensors are processed optimally [17], the range difference errors $\{n_{i,1}\}$ can be modeled as $(n_i - n_1)$ where $\{n_i\}$ are uncorrelated zero-mean Gaussian processes with variances $\{\sigma_i^2\}$. Ignoring the second-order error term, the weighting matrix is approximated as

$$\mathbf{W} \approx (\hat{\mathbf{D}}^T \boldsymbol{\Sigma} \hat{\mathbf{D}})^{-1} \tag{13}$$

where $\hat{\mathbf{D}} = \mathrm{diag}(\hat{d}_2, \hat{d}_3, \ldots, \hat{d}_M)$ and $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_2^2, \sigma_3^2, \ldots, \sigma_M^2) + \sigma_1^2 \mathbf{1}_{M-1}\mathbf{1}_{M-1}^T$ with $\hat{d}_i$ and $\mathbf{1}_M$ being the estimate of d_i and $M \times 1$ vector with all elements 1, respectively.

The unconstrained solution of (11), denoted by $\hat{\boldsymbol{\theta}}_{\mathrm{WLS}}$, corresponds to the first step of the 2WLS method [10], which is given as

$$\hat{\boldsymbol{\theta}}_{\mathrm{WLS}} = (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{b} \tag{14}$$

As $\{\hat{d}_i\}$ are not available, they can be initialized by setting $\mathbf{W} = \mathbf{I}_{M-1}$ where $\mathbf{I}_{M-1}$ is the $(M-1) \times (M-1)$ identity matrix. A second weighted least squares step which utilizes the relationship in (12) is then applied to compute the final position estimate. The interested reader is referred to [10] for details.

On the other hand, [11] solves (11)–(12) by minimizing the Lagrangian:

$$L(\tilde{\boldsymbol{\theta}},\lambda)=(\mathbf{A}\tilde{\boldsymbol{\theta}}-\mathbf{b})^T\mathbf{W}(\mathbf{A}\tilde{\boldsymbol{\theta}}-\mathbf{b})+\lambda\tilde{\boldsymbol{\theta}}^T\mathbf{S}\tilde{\boldsymbol{\theta}} \tag{15}$$

where λ is the Lagrange multiplier. The CWLS solution, denoted by $\hat{\boldsymbol{\theta}}_{\text{CWLS}}$, is

$$\hat{\boldsymbol{\theta}}_{\text{CWLS}}=(\mathbf{A}^T\mathbf{W}\mathbf{A}+\lambda\mathbf{S})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{b} \tag{16}$$

where λ is a root of the 4th-order polynomial [11]:

$$\sum_{i=1}^{3}\frac{u_i w_i}{(\lambda+\zeta_i)^2}=0 \tag{17}$$

with $[u_1\ u_2\ u_3]^T=\mathbf{U}^T\mathbf{S}\mathbf{A}^T\mathbf{W}\mathbf{b}$, $[w_1\ w_2\ w_3]^T=\mathbf{U}^{-1}\mathbf{A}^T\mathbf{W}\mathbf{b}$ and $\mathbf{A}^T\mathbf{W}\mathbf{A}\mathbf{S}=\mathbf{U}\,\text{diag}(\zeta_1,\zeta_2,\zeta_3)\mathbf{U}^{-1}$. In [11], λ is the real-valued root which gives the minimum value of $J(\tilde{\boldsymbol{\theta}})$ in (11).

2.2. Matrix ill-conditioned problem

Consider the example of a UCA geometry where all receivers lie on the circumference of a circle, with the source at the center. Then the source is at equidistance from all receivers and $r_{i,1}=0$ for all i in (7). Now $\mathbf{A}$ is a singular matrix. In practice, the receiver-source geometry may not be exactly UCA, and there are TDOA measurement noises. Nevertheless, a localization geometry that approximates a UCA circle has an ill-conditioned matrix, when the noise terms are sufficiently small.

Now when $\mathbf{A}$ and hence $\mathbf{A}^T\mathbf{W}\mathbf{A}$ is ill-conditioned, $\mathbb{E}\{\lambda\}=0$ for λ in (16), so that (16) is an ill-conditioned solution. To see this, (16) is expressed as

$$[(\overline{\mathbf{A}}+\Delta\mathbf{A})^T\mathbf{W}(\overline{\mathbf{A}}+\Delta\mathbf{A})+\lambda\mathbf{S}](\boldsymbol{\theta}+\Delta\boldsymbol{\theta})=(\overline{\mathbf{A}}+\Delta\mathbf{A})^T\mathbf{W}(\overline{\mathbf{b}}+\Delta\mathbf{b}) \tag{18}$$

where $\mathbf{A}=\overline{\mathbf{A}}+\Delta\mathbf{A}$, $\mathbf{b}=\overline{\mathbf{b}}+\Delta\mathbf{b}$ and $\hat{\boldsymbol{\theta}}_{\text{CWLS}}=\boldsymbol{\theta}+\Delta\boldsymbol{\theta}$ with $\overline{\mathbf{A}}$ being the noise-free form of $\mathbf{A}$. Assuming sufficiently small noise conditions, we ignore the second-order error terms to obtain

$$\lambda\mathbf{S}(\boldsymbol{\theta}+\Delta\boldsymbol{\theta})\approx\overline{\mathbf{A}}^T\mathbf{W}\Delta\mathbf{b}+\Delta\mathbf{A}^T\mathbf{W}\overline{\mathbf{b}}-\overline{\mathbf{A}}^T\mathbf{W}\Delta\mathbf{A}\boldsymbol{\theta}-\Delta\mathbf{A}^T\mathbf{W}\overline{\mathbf{A}}\boldsymbol{\theta}-\overline{\mathbf{A}}^T\mathbf{W}\overline{\mathbf{A}}\Delta\boldsymbol{\theta} \tag{19}$$

As $\mathbb{E}\{\Delta\mathbf{A}\}=\mathbf{0}_{(M-1)\times 3}$, $\mathbb{E}\{\Delta\mathbf{b}\}\approx\mathbf{0}_{(M-1)\times 1}$ where $\mathbf{0}_{m\times n}$ denotes the $m\times n$ zero matrix, and $\mathbb{E}\{\Delta\boldsymbol{\theta}\}=\mathbf{0}_{3\times 1}$ [11], we have

$$\mathbb{E}\{\lambda\}=0 \tag{20}$$

That is to say, the solution of (16) will suffer from the problem of ill-conditioning of $\mathbf{A}^T\mathbf{W}\mathbf{A}$ for sufficiently small $\{\sigma_i^2\}$, i.e., [11] cannot give the correct λ when $\mathbf{A}$ is ill-conditioned. More precisely, the correct value of λ in (17) may not be determined in [11]. These findings have been verified in the simulation results provided in Section 4.

3. Algorithm development and analysis

In this section, we devise a new CWLS algorithm, called separated CWLS (SCWLS) because the additional variable R_1 is separated from the unknowns $(x-x_1)$ and $(y-y_1)$. We then analytically compare its performance with [10,11].

3.1. The SCWLS algorithm

To fix the matrix ill-conditioned problem in (14) or (16), we first express (6) as [8]

$$\mathbf{G}\boldsymbol{\eta}=\mathbf{b}-\mathbf{g}R_1+\mathbf{m} \tag{21}$$

where $\mathbf{G}=[[\mathbf{A}]_{:,1}\ [\mathbf{A}]_{:,2}]$ and $\mathbf{g}=[\mathbf{A}]_{:,3}$ with $[\mathbf{A}]_{:,i}$ being the ith column of the matrix $\mathbf{A}$ and $\boldsymbol{\eta}=[[\boldsymbol{\theta}]_1\ [\boldsymbol{\theta}]_2]^T$ with $[\boldsymbol{\theta}]_i$ being the ith element of the vector $\boldsymbol{\theta}$.

According to (21), the cost function of SCWLS, denoted by $J(\tilde{\boldsymbol{\eta}},\ \tilde{R}_1)$, is

$$J(\tilde{\boldsymbol{\eta}},\ \tilde{R}_1)=(\mathbf{G}\tilde{\boldsymbol{\eta}}-\mathbf{b}+\mathbf{g}\tilde{R}_1)^T\mathbf{W}(\mathbf{G}\tilde{\boldsymbol{\eta}}-\mathbf{b}+\mathbf{g}\tilde{R}_1) \tag{22}$$

where $\tilde{\boldsymbol{\eta}}=[[\tilde{\boldsymbol{\theta}}]_1\ [\tilde{\boldsymbol{\theta}}]_2]^T$ is the optimization variable for $\boldsymbol{\eta}$.

The SCWLS position estimate is the $\tilde{\boldsymbol{\eta}}$ which gives the smallest value of $J(\tilde{\boldsymbol{\eta}},\ \tilde{R}_1)$ with a constraint, that is

$$\hat{\boldsymbol{\eta}}=\arg\min_{\tilde{\boldsymbol{\eta}}} J(\tilde{\boldsymbol{\eta}},\ \tilde{R}_1) \tag{23}$$

$$\text{s.t.}\quad \tilde{\boldsymbol{\eta}}^T\tilde{\boldsymbol{\eta}}=\tilde{R}_1^2 \tag{24}$$

Solving the SCWLS problem in (23) and (24) is equivalent to minimizing the Lagrangian:

$$L(\tilde{\boldsymbol{\eta}},\beta,\tilde{R}_1)=(\mathbf{G}\tilde{\boldsymbol{\eta}}-\mathbf{b}+\mathbf{g}\tilde{R}_1)^T\mathbf{W}(\mathbf{G}\tilde{\boldsymbol{\eta}}-\mathbf{b}+\mathbf{g}\tilde{R}_1)+\beta(\tilde{\boldsymbol{\eta}}^T\tilde{\boldsymbol{\eta}}-\tilde{R}_1^2) \tag{25}$$

where β is the Lagrange multiplier.

The minimum of (25) is found by differentiating $L(\tilde{\boldsymbol{\eta}},\beta,\tilde{R}_1)$ with respect to one of the variables while the remaining two are kept constant, and then equating the resultant expressions to zero. The differentiation of $L(\tilde{\boldsymbol{\eta}},\beta,\tilde{R}_1)$ with respect to $\tilde{\boldsymbol{\eta}}$ is

$$\frac{\partial L(\tilde{\boldsymbol{\eta}},\beta,\tilde{R}_1)}{\partial\tilde{\boldsymbol{\eta}}}=2\mathbf{G}^T\mathbf{W}(\mathbf{G}\tilde{\boldsymbol{\eta}}-\mathbf{b}+\mathbf{g}\tilde{R}_1)+2\beta\tilde{\boldsymbol{\eta}} \tag{26}$$

The position estimate based on (26) is then

$$\hat{\boldsymbol{\eta}}=(\mathbf{G}^T\mathbf{W}\mathbf{G}+\beta\mathbf{I}_2)^{-1}(\mathbf{G}^T\mathbf{W}\mathbf{b}-\mathbf{G}^T\mathbf{W}\mathbf{g}\hat{R}_1) \tag{27}$$

where β as well as $\hat{R}_1$ have yet to be determined.

In addition, differentiating $L(\tilde{\boldsymbol{\eta}},\beta,\tilde{R}_1)$ with respect to $\tilde{R}_1$ while keeping $\tilde{\boldsymbol{\eta}}$ and β constant, and then equating the resultant expression to zero yield

$$\hat{R}_1=\frac{\mathbf{g}^T\mathbf{W}\mathbf{G}\hat{\boldsymbol{\eta}}-\mathbf{g}^T\mathbf{W}\mathbf{b}}{\beta-\mathbf{g}^T\mathbf{W}\mathbf{g}} \tag{28}$$

Substituting (27) into the equality constraint of (24) results in a quadratic equation

$$a\hat{R}_1^2+b\hat{R}_1+c=0 \tag{29}$$

where $a=(\mathbf{G}^T\mathbf{W}\mathbf{g})^T(\mathbf{G}^T\mathbf{W}\mathbf{G}+\beta\mathbf{I}_2)^{-2}\mathbf{G}^T\mathbf{W}\mathbf{g}-1$, $b=-2(\mathbf{G}^T\mathbf{W}\mathbf{b})^T(\mathbf{G}^T\mathbf{W}\mathbf{G}+\beta\mathbf{I}_2)^{-2}\mathbf{G}^T\mathbf{W}\mathbf{g}$, and $c=(\mathbf{G}^T\mathbf{W}\mathbf{b})^T(\mathbf{G}^T\mathbf{W}\mathbf{G}+\beta\mathbf{I}_2)^{-2}\ \mathbf{G}^T\mathbf{W}\mathbf{b}$, and we see that there is no inversion of $\mathbf{A}^T\mathbf{W}\mathbf{A}$.

Since $\hat{\boldsymbol{\eta}}$ is a function of $\hat{R}_1$ and β in (27), and $\hat{R}_1$ is related to β in (29), we have to compute β first. One direction for solving β is to express $\hat{R}_1$ in terms of β only using (27) and (28) and then putting the resultant expression in (29), which will be computationally demanding. On the other hand, comparing (11)–(12) with (23)–(24), it is deduced that the two Lagrange multipliers, λ and β, must be equal because $\mathbf{A}\tilde{\boldsymbol{\theta}}-\mathbf{b}=\mathbf{G}\tilde{\boldsymbol{\eta}}-\mathbf{b}+\mathbf{g}\tilde{R}_1$ and $\tilde{\boldsymbol{\theta}}^T\mathbf{S}\tilde{\boldsymbol{\theta}}=\tilde{\boldsymbol{\eta}}^T\tilde{\boldsymbol{\eta}}-\tilde{R}_1^2$.

From (17), we know that there are at most four possible real values of β and at most two $\hat{R}_1$ with positive values in (29). Basically, we will choose the set of $\{\beta, \hat{R}_1, \hat{\boldsymbol{\eta}}\}$ which minimizes (22) as the final solution. Starting with a potential candidate of β and taking care of the cases of complex-valued and negative roots of (29), the root selection for $\hat{R}_1$ is given as follows:

Let $\Delta=\sqrt{b^2-4ac}$. Since $\hat{R}_1=\sqrt{(x-x_1)^2+(y-y_1)^2}\geq 0$, we have:

(a) If $\Delta<0$ or the roots of (29) are complex:
  (i) If $-b/(2a)\geq 0$, $\hat{R}_1=-b/(2a)$. That is, $\hat{R}_1$ is given by the real part of the root.
  (ii) If $-b/(2a)<0$, $\hat{R}_1=0$.

  That is, we only consider the real part of the roots. Since $\hat{R}_1$ should be non-negative, it will be set to zero if the real component is negative.

(b) If $\Delta\geq 0$, there are two real roots, $(-b+\Delta)/(2a)$ and $(-b-\Delta)/(2a)$, respectively. Only one root is selected as $\hat{R}_1$ by the following rules:
  (i) If one root is positive and the other is negative, the positive one is selected.
  (ii) If both roots are positive, we solve $\hat{\boldsymbol{\eta}}$ using (27) and choose the root which makes the cost function $J(\hat{\boldsymbol{\eta}},\ \hat{R}_1)$ given in (22) minimal.
  (iii) If both roots are negative, then we set $\hat{R}_1=0$ as it must be positive.

Note that (b)(i) and (b)(ii) are the nominal cases while the remaining ones will correspond to abnormal scenarios which occur when the noise is large.

Finally, as the ideal weighting matrix for $\mathbf{W}$ is a function of $\widehat{\mathbf{D}}$, and $\widehat{\mathbf{D}}$ is constructed from $\widehat{\mathbf{x}}=\widehat{\boldsymbol{\eta}}+\mathbf{x}_1$ which is to be determined, an iterative procedure to update the weighting matrix is thus required [10]. To summarize, the estimate of $\boldsymbol{\eta}$ is found by the following procedure:

(1) Set $\mathbf{W}=\mathbf{I}_{M-1}$.
(2) Solve the roots of (17) to obtain the real-valued β, denoted by $\beta^{(l)}$, $l=1,2,\ldots,L$, with $L\leq 4$.
(3) Substitute $\{\beta^{(l)}\}$ into (29) and determine the estimates of R_1, denoted by $\{\hat{R}_1^{(l)}\}$, according to the above-mentioned root selection procedure.
(4) Obtain $\{\widehat{\boldsymbol{\eta}}^{(l)}\}$ from (27) with the use of $\{\beta^{(l)}\}$ and $\{\hat{R}_1^{(l)}\}$.
(5) Choose the set of $\{\beta^{(l)}, \hat{R}_1^{(l)}, \widehat{\boldsymbol{\eta}}^{(l)}\}$ which minimizes (22) as the solution.
(6) Construct $\mathbf{W}$ according to (2) and (13) with the use of $\widehat{\boldsymbol{\eta}}$.
(7) Repeat Steps (2)–(6) until a stopping criterion is satisfied.

Finally, the estimate of $\mathbf{x}$ is computed as $\widehat{\mathbf{x}}=\widehat{\boldsymbol{\eta}}+\mathbf{x}_1$. Note that $\hat{R}_1$ and β are dependent of $\widehat{\boldsymbol{\eta}}$ in (27)–(29) and we determine their values at each iteration by minimizing (22). That is to say, we basically update $\widehat{\boldsymbol{\eta}}$ in an iterative manner. This relaxation procedure is known as iterative quadratic maximum likelihood technique or the Steiglitz–McBride algorithm [18], which has local convergence property with a linear rate of convergence [19].

3.2. Comparison with existing LLS algorithms

Conceptually, the 2WLS, CWLS and SCWLS should provide the same estimation performance because all of them are based on the same cost function of (11) and constraint of (12). But when $\mathbf{A}^T\mathbf{W}\mathbf{A}$ is ill-conditioned, 2WLS and CWLS have larger error than SCWLS.

In solving linear systems based on the l_2 norm, the condition number of the coefficient matrix, which is the ratio of the maximum and minimum singular values of the corresponding matrix, measures the sensitivity of the system to errors in the data [20]. Briefly speaking, the larger the condition number, the more ill-conditioned or sensitive the system is. The following gives a comparison of the condition numbers of $\mathbf{A}^T\mathbf{W}\mathbf{A}$ and $\mathbf{G}^T\mathbf{W}\mathbf{G}$, denoted as $\mathcal{K}(\mathbf{A}^T\mathbf{W}\mathbf{A})$ and $\mathcal{K}(\mathbf{G}^T\mathbf{W}\mathbf{G})$, respectively.

Lemma 1. *Let $\mathbf{P}\in\mathbb{R}^{K\times N}$ where $K\geq N\geq 2$, $\mathbf{Q}$ is the same as $\mathbf{P}$ but with a column being removed and $\mathbf{Q}$ has full rank. Then, the condition number of $\mathbf{P}$ is not less than that of $\mathbf{Q}$.*

Proof. Without loss of generality, we assume that

$$\mathbf{P}=[\mathbf{Q}\ \ \mathbf{q}]. \tag{30}$$

Denote the smallest and largest singular values of $\mathbf{Z}$ by $\sigma_{\min}(\mathbf{Z})$ and $\sigma_{\max}(\mathbf{Z})$, respectively. According to Eckart–Young–Mirsky matrix approximation theorem [21], we have

$$\tilde{\mathbf{X}}=\arg\min_{\operatorname{rank}(\mathbf{X})=N-2}\|\mathbf{Q}-\mathbf{X}\|_2 \quad\text{and}\quad \sigma_{\min}(\mathbf{Q})=\|\mathbf{Q}-\tilde{\mathbf{X}}\|_2, \tag{31}$$

$$\tilde{\mathbf{Y}}=\arg\min_{\operatorname{rank}(\mathbf{Y})=N-1}\|\mathbf{P}-\mathbf{Y}\|_2 \quad\text{and}\quad \sigma_{\min}(\mathbf{P})=\|\mathbf{P}-\tilde{\mathbf{Y}}\|_2 \tag{32}$$

where $\operatorname{rank}(\cdot)$ and $\|\cdot\|_2$ are the matrix rank and 2-norm operators, respectively. To relate $\sigma_{\min}(\mathbf{P})$ and $\sigma_{\min}(\mathbf{Q})$, we construct a rank-$(N-1)$ matrix, namely, $\tilde{\mathbf{P}}=[\tilde{\mathbf{X}}\ \mathbf{q}]$, to obtain

$$\sigma_{\min}(\mathbf{P})\leq\|\mathbf{P}-\tilde{\mathbf{P}}\|_2=\|[\mathbf{Q}\ \ \mathbf{q}]-[\tilde{\mathbf{X}}\ \ \mathbf{q}]\|_2=\|\mathbf{Q}-\tilde{\mathbf{X}}\|_2=\sigma_{\min}(\mathbf{Q}). \tag{33}$$

Next, we consider $\sigma_{\max}(\mathbf{P})$. It is known that [22]

$$(\tilde{\mathbf{u}},\tilde{\mathbf{v}})=\arg\max_{\substack{\mathbf{u}\in\mathbb{R}^K \& \|\mathbf{u}\|_2=1\\ \mathbf{v}\in\mathbb{R}^N \& \|\mathbf{v}\|_2=1}}\mathbf{u}^T\mathbf{P}\mathbf{v} \quad\text{and}\quad \sigma_{\max}(\mathbf{P})=\tilde{\mathbf{u}}^T\mathbf{P}\tilde{\mathbf{v}}, \tag{34}$$

$$(\tilde{\mathbf{s}},\tilde{\mathbf{t}})=\arg\max_{\substack{\mathbf{s}\in\mathbb{R}^K \& \|\mathbf{s}\|_2=1\\ \mathbf{t}\in\mathbb{R}^{N-1} \& \|\mathbf{t}\|_2=1}}\mathbf{s}^T\mathbf{Q} \quad\text{and}\quad \sigma_{\max}(\mathbf{Q})=\tilde{\mathbf{s}}^T\mathbf{Q}\tilde{\mathbf{t}} \tag{35}$$

Then we construct $\mathbf{v}=[\tilde{\mathbf{t}}^T\ \ 0]^T$ and it is seen that

$$\sigma_{\max}(\mathbf{Q})=\tilde{\mathbf{s}}^T\mathbf{Q}\tilde{\mathbf{t}}=\tilde{\mathbf{s}}^T[\mathbf{Q}\ \ \mathbf{q}]\begin{bmatrix}\tilde{\mathbf{t}}\\0\end{bmatrix}=\tilde{\mathbf{s}}^T\mathbf{P}\mathbf{v}\leq\sigma_{\max}(\mathbf{P}) \tag{36}$$

where the last inequality follows from (34). Hence, combining (33) and (36) and using the fact that $\sigma_{\min}(\mathbf{Q})>0$, we have

$$\frac{\sigma_{\max}(\mathbf{P})}{\sigma_{\min}(\mathbf{P})}\geq\frac{\sigma_{\max}(\mathbf{Q})}{\sigma_{\min}(\mathbf{Q})} \tag{37}$$

which indicates that the condition number of any $K\times N$ matrix $\mathbf{P}$ is larger than or equal to that of its $K\times(N-1)$ submatrix $\mathbf{Q}$. □

Corollary 1. $\mathcal{K}(\mathbf{A}^T\mathbf{W}\mathbf{A})\geq\mathcal{K}(\mathbf{G}^T\mathbf{W}\mathbf{G})$

Proof. Let $\mathbf{P}=\mathbf{W}^{0.5}\mathbf{A}$ and $\mathbf{Q}=\mathbf{W}^{0.5}\mathbf{G}$ where $\mathbf{G}$ is $\mathbf{A}$ but without the last column. As $\mathbf{W}^{0.5}\mathbf{A}\in\mathbb{R}^{(M-1)\times 3}$ where $M\geq 4$ and $\mathbf{W}^{0.5}\mathbf{G}$ has full rank, by Lemma 1, we have

$$\frac{\sigma_{\max}(\mathbf{W}^{0.5}\mathbf{A})}{\sigma_{\min}(\mathbf{W}^{0.5}\mathbf{A})}\geq\frac{\sigma_{\max}(\mathbf{W}^{0.5}\mathbf{G})}{\sigma_{\min}(\mathbf{W}^{0.5}\mathbf{G})}. \tag{38}$$

As $\sigma_i(\mathbf{P}^T\mathbf{P})=\sigma_i^2(\mathbf{P})$ where $\sigma_i(\mathbf{P})$ denotes the ith largest singular value of $\mathbf{P}$, we have

$$\frac{\sigma_{\max}(\mathbf{A}^T\mathbf{W}\mathbf{A})}{\sigma_{\min}(\mathbf{A}^T\mathbf{W}\mathbf{A})}=\frac{\sigma_{\max}^2(\mathbf{W}^{0.5}\mathbf{A})}{\sigma_{\min}^2(\mathbf{W}^{0.5}\mathbf{A})}\geq\frac{\sigma_{\max}^2(\mathbf{W}^{0.5}\mathbf{G})}{\sigma_{\min}^2(\mathbf{W}^{0.5}\mathbf{G})}=\frac{\sigma_{\max}(\mathbf{G}^T\mathbf{W}\mathbf{G})}{\sigma_{\min}(\mathbf{G}^T\mathbf{W}\mathbf{G})} \tag{39}$$

where the inequality of (39) follows from (38). □

Since $\mathbb{E}\{\lambda\}=\mathbb{E}\{\beta\}=0$ under sufficiently small noise conditions and $\mathcal{K}(\mathbf{G})\leq\mathcal{K}(\mathbf{A})$, we can conclude that SCWLS is less sensitive to data errors than 2WLS and CWLS, especially when $\mathbf{A}^T\mathbf{W}\mathbf{A}$ is ill-conditioned.

In addition, it has been proved that the performance of [11] achieves the CRLB when noise is sufficiently small. Together with the fact that the cost function and constraint of the SCWLS approach are the same as those of CWLS, it is clear that the mean square position error (MSPE) of $\hat{\mathbf{x}}$ is identical to the CRLB for TDOA-based positioning [10], indicating the optimality of $\hat{\mathbf{x}}$ for small noise conditions.

4. Simulation results

This section contains four simulation examples to assess the relative performance of SCWLS, 2WLS [10] and CWLS [11]. In all three algorithms, we use three iterations in updating the weighting matrix of $\mathbf{W}$ because there is no obvious improvement with more iterations. Note that apart from employing a fixed number of iterations, we can also exploit the difference between successive estimates or an adaptive number of iterations as the stopping criterion, which may lead to higher accuracy and/or smaller computational load.

Each example has the same 4-receiver configuration, at coordinates $[0\ 0]^T$, $[0\ 10]^T$, $[10\ 0]^T$ and $[10\ 10]^T$, i.e., they are on the circumference of a circle centered at $[5\ 5]^T$. The source position is different for each example. This is to demonstrate that 2WLS and CWLS have larger MSPE than SCWLS when the source is at or near the center of the circle. For other geometries, all three methods give MSPE that are close to the CRLB, although SCWLS has the highest threshold (see Example 2). The performance measure is the MSPE as a function of $p=10\log_{10}(\sigma^2)$ dB, where σ^2 is the variance of the noise term n_i in $n_{i,1}=n_i-n_1$ of (1). The $\{n_i\}$ are zero mean Gaussian random variables. The MSPE is computed using 1000 independent trials:

$$\text{MSPE}=10\log_{10}\left(\frac{\sum_{l=1}^{1000}\|\hat{\mathbf{x}}(l)-\mathbf{x}\|_2^2}{1000}\right)\ \text{dB} \tag{40}$$

where $\hat{\mathbf{x}}(l)$ is the estimate of $\mathbf{x}$ at the lth run.

Example 1. The source is at $\mathbf{x}=[5.1\ 4.9]^T$, i.e., it is close to the center $[5\ 5]^T$ of the UCA geometry. Fig. 1 plots the MSPE versus p. As expected, for $p\leq-5$ dB, the MSPE of SCWLS is on the CRLB, while those of 2WLS and CWLS are several dBs above the CRLB. For $p>-5$ dB, both SCWLS and CWLS have MSPE that meet the CRLB, while 2WLS cannot. At higher noise power, the noise terms decrease $\mathcal{K}(\mathbf{A}^T\mathbf{W}\mathbf{A}+\lambda\mathbf{S})$ in (16) and reduce the sensitivity of CWLS to data errors. Hence the MSPE of CWLS is able to meet the CRLB.

Example 2. The source is at $\mathbf{x}=[2\ 8]^T$. In Fig. 2, the MSPEs of 2WLS, CWLS and SCWLS meet the CRLB for $p\leq-24$ dB, -10 dB, and 0 dB, respectively. Above those respective values, called thresholds, the respective estimator has MSPE much higher than the CRLB. This is the so-called threshold phenomenon, which occurs in nonlinear estimation when the noise powers exceed a threshold value [10]. A plausible explanation for SCWLS having a higher threshold than 2WLS and CWLS is that its root selection procedure for R_1 is better in mitigating the noise effects.

Example 3. The source is within the square perimeter of the four receivers but is at a different (random) position for each of the 1000 trials. Again, Fig. 3 shows that SCWLS has smaller MSPE than 2WLS or CWLS. Since the source position is random, it can be near $[5\ 5]^T$ occasionally,

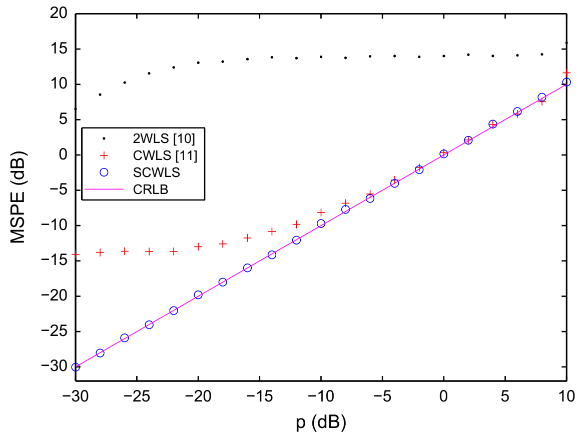

Fig. 1. Mean square position error versus p at $\mathbf{x}=[5.1\ 4.9]^T$.

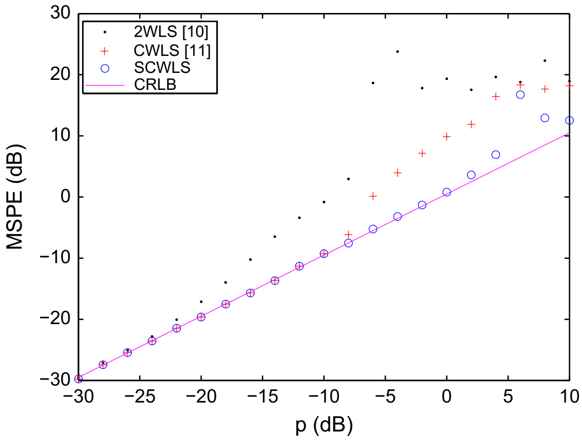

Fig. 2. Mean square position error versus p at $\mathbf{x}=[2\ 8]^T$.

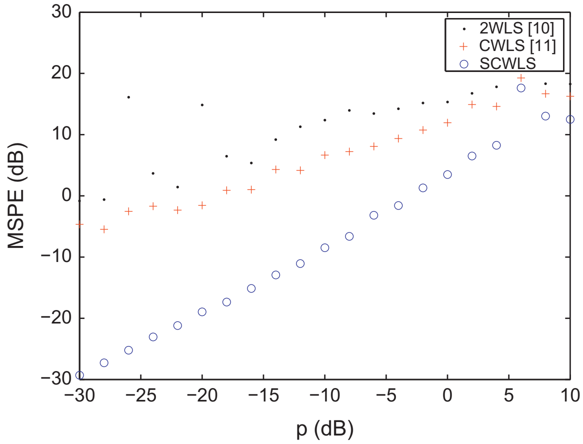

Fig. 3. Mean square position error versus p with random source position inside boundary.

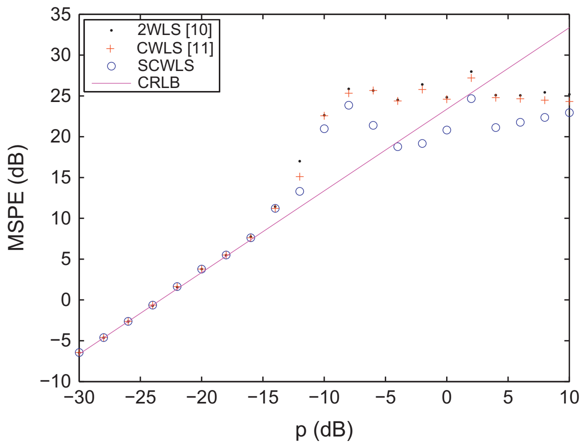

Fig. 4. Mean square position error versus p at $\mathbf{x}=[20\ 8]^T$.

causing ill-conditioning in 2WLS and CWLS, resulting in large MSPE.

Example 4. The source is at $\mathbf{x}=[20\ 8]^T$, which is outside the square bounded by the receivers. Fig. 4 shows that for this less favorable geometry, the threshold value falls to $p\approx-10$ dB. For $p>-5$ dB, the algorithms give biased estimates, and their MSPEs are below the CRLB. It is possible that when an estimate is biased, its mean square error can be smaller than the CRLB [23].

The computation times per trial, using MATLAB in a personal computer, are 2.93×10^{-4} s, 1.03×10^{-3} s and 1.18×10^{-3} s, for 2WLS, CWLS and SCWLS, respectively.

5. Conclusion

Motivated by the matrix ill-conditioned problem in the 2WLS and CWLS methods, a SCWLS approach is proposed in this paper. It is shown that the SCWLS is optimal even when the distances from the source to all receivers are identical or approximately equal, while the other two methods have MSPE above the CRLB. Simulation results also suggest that it is worthy to reformulate the cost function to a more well-conditioned form, which leads to a better threshold performance particularly when the receiver geometry is equal to a UCA and the source is near the array centroid. As a future work, the proposed algorithm will be evaluated using real-world TDOA data.

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