A Source Representation Space Approach To Eigenstructure Based Broadband Source Location Estimation

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ABSTRACT

For broadband Source Location Estimation (SLE), two new broadband spatial spectral estimation algorithms are described which utilize the eigenstructure of a broadband spatial/temporal data covariance matrix. The algorithms are formulated using a broadband source model, termed the *broadband source representation space.* The model, a subspace of the source observation space, is developed and employed to establish the plausibility of broadband matrix eigenstructure methods. For one of the proposed algorithms, broadband source representation space basis vectors are used directly. Simulations are presented illustrating performance and extensions are noted.

INTRODUCTION

In sensor array processing applications, such as spread spectrum communications and sonar, there is growing interest in the analysis of broadband sources and data. One important function is Source Location Estimation (SLE), where received broadband array data is evaluated to extract estimates of locations of multiple sources. Two basic approaches to SLE are: direct estimation of locations; and source detection and location estimation via an estimated spatial spectrum. As an example of the former approach, Wax and Kailath [1] described a maximum likelihood broadband source location estimator for gaussian data. This algorithm requires: 1) knowledge of the data density function conditioned on the source locations; 2) knowledge of the number of impinging sources; and 3) a computationally expensive search of the location parameter space. Alternatively, methods based on spatial spectral estimates require only a partial additive noise statistical characterization and typically much less computation. In this paper, broadband spatial spectral estimation is addressed.

Until recently, established approaches to high resolution broadband spatial spectral estimation (e.g. Bienvenu [2] and Wax, Shan and Kailath [3]) involved: 1) the decomposition of the array snapshot vectors into a discrete set of narrowband frequency components; 2) the independent processing, across the sensors, of the individual data narrowband components to obtain narrowband null spectra; and 3) superposition of the narrowband null spectra to form a single broadband spatial spectrum. For each processing frequency, a narrowband spatial covariance matrix must be estimated and compared against narrowband source representation vectors to generate the narrowband null spectrum.

Narrowband decomposition approaches utilize previously developed narrowband data processing techniques which, in turn, employ a simple source model. For K array elements, a source impinging form location θ, and a processing angular frequency ω, this model is the K-dimensional vector $\alpha_\theta(\omega)$ of the relative amplitudes and phases of the samples of a source at the outputs of the array elements. For each processing frequency, this model of a source is *rank 1*. Since broadband sources, observed directly at the array output, are not rank 1, direct broadband data processing has not been suggested.

Any spectral estimation implementation processes a finite data sampling and finite observed source energy. Wang and Kaveh [4] have illustrated that, with a narrowband decomposition approach, observed broadband source energy is utilized inefficiently since it is essentially distributed over a set of narrowband spatial spectral estimation problems. To show this, consider the problem of estimating eigenvectors for a broadband source with flat spectrum over the processing bandwidth. In high resolution spectra estimation, performance is critically dependent on the accuracy to which source eigenvectors can be estimated from a spatial covariance matrix. Define $M \cdot SNR$ as the source observed energy to noise power ratio, where M is the number of data samples and SNR is the signal-to-noise power ratio. This is a fundamental measure of source detection or parameter estimation capability (note, for example, Cramer Rao Bounds on estimation variance). For eigenvector estimation, SNR is the ratio of source to noise eigenvalues.

Consider N temporal samples of a single broadband source with power σ_s^2 observed with a K element array. At each sensor, assume spatially uncorrelated noise with variance σ_n^2. In a narrowband decomposition approach, where L narrowband components are generated and processed separately, each source will contribute approximately one eigenvector to an estimated narrowband spatial covariance matrix. For a narrowband source eigenvector,

$$(M \cdot SNR)_{nb} = \left(\frac{N}{L}\right)\left(\frac{K\sigma_s^2}{\sigma_n^2}\right) \quad , \tag{1}$$

where nb denotes narrowband. The number of independent data samples is reduced by L in each bin while the SNR is unimproved. For the broadband data spatial covariance matrix, let p_i be the fraction of source power associated with the i^{th} source eigenvector. For the i^{th} eigenvector,

$$(M \cdot SNR)_i = N \cdot p_i \left(\frac{K\sigma_s^2}{\sigma_n^2}\right) \quad . \tag{2}$$

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In narrowband processing, L must be large for the rank 1 assumption to hold. For comparison, in direct broadband processing of the $K \times K$-dimensional broadband matrix for a source from array broadside $p_1 = 1.$. So $p_i >> 1/L$, $(M \cdot SNR)_i >> (M \cdot SNR)_{nb}$, and more accurate source broadband eigenvector estimation can be realized.

Generally, however, broadband source observations are not rank 1. Covariance matrix transformations, described in [4], have been employed to focus sources from pre-selected locations onto single dimensions, thereby improving SLE resolution. However, biased SLE can result. In this paper, a low rank property of broadband sources is identified. Improved SLE through direct broadband data processing can therefore be realized. Below, eigenvector methods of direct broadband spatial spectral estimation are discussed. See [5] for a description of broadband power spatial spectral estimators analogous to Capon's "maximum likelihood method". Unbiased focusing is a future research topic.

Below, eigenvector methods of spatial spectral estimation are first reviewed. A low rank source representation, termed the *source representation space*, is then defined. This representation is employed to develop two estimators which derive broadband spatial spectra from the eigenstructure of a broadband data spatial/temporal covariance matrix. Simulations are presented to illustrate performance of new algorithms.

BACKGROUND

Narrowband SLE

Existing narrowband algorithms are extensions of Schmidt's *M*ultiple *S*ignal *C*lassification (MUSIC) algorithm [6]. This algorithm is now summarized. For processing angular frequency ω_0, assume $D < K$ complex narrowband sources are impinging on an array of K elements. Each element of the K-dimensional array data vector $\chi(t)$ is assumed to be of the form,

$$x_i(t) = \sum_{k=1}^{D} s_k(t)\, a_{i,k}(\omega_0)\, e^{-j\omega_0 \tau_{i,\theta_k}} + n_i(t) \quad , \qquad (3)$$

where $s_k(t)$ is the k^{th} source as observed at the array coordinate origin, $a_{i,k}(\omega_0)\, e^{-j\omega_0 \tau_{i,\theta_k}}$ is the propagation/element response, and $n_i(t)$ is additive noise which is uncorrelated with the sources. The data vector can be written as $\chi(t) = A\ \sigma(t) + \eta(t)$ where $\sigma(t) = [\, s_1(t), \cdots, s_D(t)\,]^T$, $\eta(t) = [\, n_1(t), \cdots, n_K(t)\,]^T$, and

$$A = \left[\, \alpha_{\theta_1}(\omega_0), \cdots, \alpha_{\theta_D}(\omega_0) \,\right] . \qquad (4)$$

The Hermitian positive definite spatial covariance matrix is,

$$R_{\chi\chi} = A\ R_{\sigma\sigma} A^{\dagger} + R_{nn} \quad , \qquad (5)$$

where $R_{\sigma\sigma} = E[\, \sigma(t)\, \sigma^{\dagger}(t)\,]$, and R_{nn} is the noise spatial covariance matrix. $R_s = A\ R'_{\sigma\sigma} A^{\dagger}$ is a composite source sample covariance matrix. For an eigenstructure approach, $R_{nn} = \sigma_n^2 I_K$ is required. Therefore, the additive noise must be either spatially white or pre-whitened.

The source cross covariance matrix $R_{\sigma\sigma}$ is a $D \times D$ positive non-negative matrix. If the narrowband sources are not coherent (i.e. if sources observed at the array origin are not related simply by an amplitude scale and a time shift, propagation is not multipath), $R_{\sigma\sigma}$ is positive definite and has rank D. Assume non-coherent sources. Let $\lambda_i\, ; i = 1,2, \cdots, K$ and $\epsilon_i\, ; i = 1,2, \cdots, K$ denote, respectively, the ordered eigenvalues (most significant first) and corresponding eigenvectors of $R_{\chi\chi}$. Schmidt has shown that the smallest $(K - D)$ eigenvalues are $\lambda_i = \sigma_n^2$. These are termed noise-only eigenvalues. The noise-only eigenvectors, those associated with the noise-only eigenvalues, have the property that,

$$A^{\dagger}\, \epsilon_i = 0. \quad i = D+1, \cdots, K \ . \qquad (6)$$

The spans of the signal and of the noise-only eigenvectors are the signal and noise-only subspaces, respectively. Eq(6) describes the property of the eigenvectors of the data spatial covariance matrix which is utilized in generating the narrowband spatial spectra. Each narrowband source location vector corresponding to an impinging source is orthogonal to the noise-only subspace, and therefore in the span of the signal subspace. For an unambiguous array configuration, only those location vectors, out of all possible location vectors, will be in the signal subspace. The spatial spectra is equivalently estimated by *projecting* proposed source location vectors onto either the signal or noise-only subspace. The noise-only subspace estimator is,

$$P_{music}(\theta) = \left[\sum_{i=D+1}^{K} |\, \alpha_\theta^{\dagger}(\omega_0)\, \epsilon_i \,|^2 \right]^{-1} . \qquad (7)$$

Broadband Source SLE

There are three eigenstructure based broadband source algorithms that represent existing approaches. These are narrowband decomposition algorithms proposed by Bienvenu [2] and by Wax, Shan and Kailath [3], and Wang and Kaveh's focusing algorithm [4]. The former two algorithms are described below. Focusing is not addressed in this paper.

Consider the KL-dimensional observation $\mathbf{x}(t) = [\, \chi^T(t), \cdots, \chi^T(t-L+1)\,]^T$. A KL-dimensional location vector is $\mathbf{a}_\theta(\omega) = \mathbf{e}_L(\omega) \otimes \alpha_\theta(\omega)$ where $\mathbf{e}_L(\omega)$ is the L-dimensional Fourier vector for angular frequency ω. Bienvenu notes that, as $L \to \infty$, the location vectors $\{\, \mathbf{a}_\theta(\omega_k);\, \omega_k = 2\pi k/L\, ;\, k = 0,1, \cdots, L-1 \,\}$ form a basis for KL-dimensional sample vectors $\mathbf{s}_\theta(t)$ of a source from θ.

For large L, the KL-dimensional stacked snapshot vector can be written as $\mathbf{x}(t) = \mathbf{A}\, \sigma(t) + \mathbf{n}(t)$, where $\sigma(t)$ and $\mathbf{n}(t)$ are decomposed source and noise vectors, and

$$\mathbf{A} = \Big[\mathbf{a}_{\theta_1}(\omega_0), \cdots, \mathbf{a}_{\theta_1}(\omega_{L-1}) \ |\ , \cdots, \qquad (8)$$
$$|\ \mathbf{a}_{\theta_D}(\omega_0), \cdots, \mathbf{a}_{\theta_D}(\omega_{L-1}) \Big] .$$

For spatially and temporally white noise, for non-coherent sources, and for $D < K$, the KL-dimensional data spatial/temporal covariance matrix is characterized by a signal and noise-only subspace. Bienvenu has suggested the noise-only subspace spatial spectra estimator,

$$P_{bien}(\theta) = \left(\frac{1}{L} \sum_{k=0}^{L-1} \left[\sum_{i=DL+1}^{KL} |\, \mathbf{a}_\theta^{\dagger}(2\pi k/L)\, \mathbf{e}_i \,|^2 \right] \right)^{-1} , \qquad (9)$$

where the $\mathbf{e}_i$ are the ordered eigenvectors of the data spatial/temporal covariance matrix.

Wax, et.al. [3] have suggested a narrowband decomposition approach to eigenvector broadband spatial spectra estimation. In their algorithm, the K-dimensional snapshot vector $\chi(t)$ is decomposed into L narrowband vectors $\varsigma_k(t)$ for $\omega_k = 2\pi k/L\, ;\, k = 0,1, \cdots, L-1$. A null spectrum is than computed from the eigenstructures of the estimated narrowband covariance matrices

$R_{\varsigma_k,\varsigma_k} = E\{\varsigma_k(t)\varsigma_k^\dagger(t)\}$; $k=0,1,\cdots,L-1$.

For each estimated $R_{\varsigma_k,\varsigma_k}$, denote the eigenvalues and eigenvectors as $\{\lambda_i(k);\ i=1,2,\cdots,K\}$ and $\{\epsilon_i(k);\ i=1,2,\cdots,K\}$. For D full bandwidth non-coherent broadband sources, each narrowband covariance matrix is characterized by a D-dimensional signal subspace and a $(K-D)$-dimensional noise-only subspace. They proposed a class of broadband spatial spectra estimators which are based on a varying rule for combining narrowband null spectra. For example, the arithmetic mean of the narrowband projections forms the noise-only subspace estimator,

$$P_{wax,a}(\theta) = \left(\frac{1}{L}\sum_{k=0}^{L-1} \left[\sum_{i=D+1}^{K} |\, \alpha_\theta^\dagger(\omega_k)\,\epsilon_i(k)\,|^2 \right] \right)^{-1} , \quad (10)$$

where the subscript wax,a suggests Wax's arithmetic mean estimator. Bienvenu has shown that Eqs (9) and (10) are asymptotically (with L) equivalent. Alternatively, the geometric mean of the projections can be used to form,

$$P_{wax,g}(\theta) = \left(\prod_{k=0}^{L-1} \left[\sum_{i=D+1}^{K} |\, \alpha_\theta^\dagger(\omega_k)\,\epsilon_i(k)\,|^2 \right]^{1/L} \right)^{-1} . \quad (11)$$

THE SOURCE REPRESENTATION SPACE

The source representation space, as defined in [5], is a model of sources, as observed with an array processing structure, which accounts for variation over ranges of source propagation parameters (e.g. source location and temporal spectral content) and observation parameters (e.g. sensor location and response). Although this general model can be useful in the spatial spectral estimation problem, the *broadband source representation space*, which is a more specific model of a single broadband source, is employed in this paper. The algorithms described below can be modified for the general representation.

The broadband source representation space provides: 1) a basis vector description of a broadband source which is used directly in the basis vector algorithm described below; and 2) an indication of dimension or rank of a broadband source which is used the establish the plausibility of a broadband covariance matrix eigenvector approach.

Consider a single broadband source radiating from location θ, propagating in an environment with known deterministic propagation characteristics, and observed with an array of known geometry and element transfer functions. Let $s_\theta(t)$ be the source random process as observed at the array reference. Denote the *source sample vector*, the KL-dimensional observation of $s_\theta(t)$, as $\mathbf{s}_\theta(t)$. For source representation, let $S_\theta(\omega)$ be a temporal spectral weighting function for angular frequency band B_θ. The *source sample covariance matrix* is defined as,

$$\mathbf{R}_\theta = E\{\mathbf{s}_\theta(t)\,\mathbf{s}_\theta^\dagger(t)\} = \int_{B_\theta} S_\theta(\omega)\,\mathbf{a}_\theta(\omega)\,\mathbf{a}_\theta^\dagger(\omega)\ d\omega \ . \quad (12)$$

Note that the range (column space) of $\mathbf{R}_\theta$ is the span of $\{\mathbf{a}_\theta(\omega);\ \omega\in B_\theta\}$. If an orthonormal basis for the range of $\mathbf{R}_\theta$ were derived, $S_\theta(\omega)$ would control the average energy of $\mathbf{s}_\theta(t)$ projected onto the basis vectors.

We are interested in an determining the orthonormal ordered basis $\{\mathbf{v}_{1,\theta},\cdots,\mathbf{v}_{KL,\theta}\}$ for source sample vectors $\mathbf{s}_\theta(t)$ which is most most efficient in the 2^{nd} order statistical sense. This is the Discrete KarhunenLoève Expansion problem described by Ahmed and Rao [7, pp200-203]. The desired basis is the solution to,

$$\mathbf{R}_\theta\,\mathbf{v}_j = \lambda_j\ \mathbf{v}_j \qquad j=1,2,\cdots,KL \ , \quad (13)$$

where the $\{\lambda_j;\ \lambda_j \ge \lambda_{j+1}\}$ and the $\{\mathbf{v}_{j,\theta}\}$ are the ordered eigenvalues and eigenvectors of $\mathbf{R}_\theta$.

For a given source location θ and frequency band B_θ, a Hermitian, positive semi-definite source sample covariance matrix $\mathbf{R}_\theta$ is generated and its eigenstructure $\{\mathbf{v}_j;\ \lambda_j;\ j=1,2,\cdots,KL;\ \lambda_j \ge \lambda_{j+1}\}$ is computed. A *broadband source representation space* is defined as the span of the eigenvectors associated with the D_θ "significant" eigenvalues. This is the D_θ-dimensional subspace of the KL-dimensional observation space which contains the most, and for large enough D_θ essentially all, source energy. The term significant is used subjectively here. For a single broadband source, propagating in a pure delay environment and sampled isotropicly with an arbitrary configured array, Ref [8] provides an accurate indication of the dimension of the source representation space. For this case, the number of significant eigenvalues of $\mathbf{R}_\theta$ is equal to the *observed source time-bandwidth product*, which is product of the source bandwidth (as limited by the array) and the temporal duration of the source in the array/delay-line structure. This product is essentially the source rank since, beyond this number, the eigenvalues decrease very rapidly.

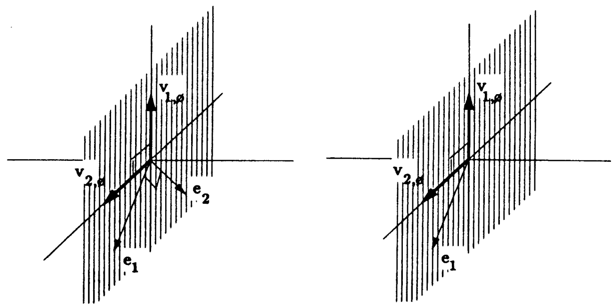
BROADBAND ALGORITHMS

Consider a single broadband source impinging form θ and observed with a KL-dimensional array/delay-line structure. For a selected processing frequency band B_θ, let $\{\mathbf{v}_{i,\theta};\ i=1,2,\cdots,D_\theta\}$ form a basis for the broadband source representation space. No robustness to perturbation of observation parameters is incorporated. Assume that this source representation space is accurate so that it essentially spans the actual source location vectors $\{\mathbf{a}_\theta(\omega);\ \omega\in B_\theta\}$. Now consider the signal subspace resulting from one source which is impinging from θ. For a *full bandwidth source*, one with energy over the entire processing band B_θ, the signal subspace and the broadband source representation space are essentially equivalent. That is, $D=D_\theta$ the signal and subspace eigenvectors $\{\mathbf{e}_i:\ i=1,2,\cdots,D\}$ span the same space as $\{\mathbf{v}_{j,\theta};\ j=1,2,\cdots,D_\theta\}$. Figure (1-a) illustrates this relationship for 2-dimensional source representation and signal spaces. Since, for the impinging source location, each source representation space basis vector is in the signal subspace, methods based on basis vector projections can correctly indicate source location.

Now consider the *partial bandwidth source* case where the bandwidth of the impinging source is a subset of the processing bandwidth B_θ. Now, $D < D_\theta$, and the signal subspace is only a portion of the broadband source representation space. This is demonstrated in Figure (1-b) where the source representation space is rank 2 and the signal subspace is rank 1. Note that neither of the source representation space basis vectors are in the signal subspace. This example illustrates why eigenvector methods based on source representation space basis vector projections will fail in the partial bandwidth source case.

Basis vector projection is not enough for partial rank sources. An *exhaustive* search of unit length vectors in the source representation space is required. Dense location vector projection is suggested when partial dimensional sources are

Fig(1): For a 3-D Observation Space, a 2-D Source Representation Space (for a Source From θ) and the Signal Subspace due to a source from θ: a) Full Bandwidth Source; b) Partial Bandwidth Source.



to be processed.

Whether basis vectors or location vectors are projected, broadband spatial spectra estimation characteristics critically depends on how, for a given source location, projections (i.e. individual null spectra) are combined. Bienvenu proposed the inverse of the sum of the projections of the source representation vectors, i.e. Eq(9). Wax, et.al. suggested the inverse of combined sums and products of individual null spectra. Eqs(10,11) are two examples. Both of these estimators are sensitive the source spectral density. The nature of this sensitivity is discussed by Buckley and Griffiths [8]. In [8], a third generic form,

$$P(\theta) = \sum_{i=1}^{D} \left\{ \frac{1}{(\ proj.\ )_i} \right\} \ , \qquad (14)$$

is suggested as being more robust to source temporal spactral characteristics than Eq(10,11). Extensive simulations, based on exact data statistics, have verified this claim.

Plausibility of Broadband Data Eigenstructure Approaches

Theoretically, covariance matrix eigenstructure based spectra estimation approaches require the existence of a noise-only subspace (i.e. the composite source sample covariance matrix $\mathbf{R}_s$ is rank deficient). In narrowband decomposition approaches to broadband source SLE, this rank deficiency is only approximated in limited temporal sampling applications. The practical requirement on $\mathbf{R}_s$, from which all application of non-monochromatic source eigenvector based algorithms are predicated, is that it be ill-conditioned. Bias resulting from the absence of a strict noise-only subspace is then negligible.

The broadband source representation space model illustrates that a single source KL-dimensional sample covariance matrix is essentially rank deficient if L is selected large enough. For pure delay propagation and an isotropic, filled array (i.e. an array in adherence with Nyquist spatial sampling, see [8]), $L \geq 2$ is sufficient. It follows that, with proper selection of L, a composite broadband sample covariance matrix due to impinging sources will be essentially rank deficient and a noise-only subspace essentially exists.

The Basis Vector Algorithm

For sources which fill their employed source representation space, the basis vector estimator,

$$P_{bv}(\theta) = \sum_{j=1}^{D_\theta} \left(\sum_{i=D_t+1}^{KL} |\ \mathbf{v}_{j,\theta}^{\dagger}\, \mathbf{e}_i\ |^2 \right)^{-1} \qquad (15)$$

is suggested, where D_t is the *dimension* of the composite broadband source sample covariance matrix, $\{\ \mathbf{e}_i\ ;\ i = D_t+1, \cdots, KL\ \}$ are the data covariance matrix noise-only eigenvectors, and $\{\ \mathbf{v}_{j,\theta};\ j=1,2, \cdots, D_\theta\ \}$ are the basis vectors for the θ location representation. An equivalent signal subspace algorithm exists.

The Location Vector Algorithm

For partial rank sources, an alternative algorithm is required. At the cost of increased computation, projections of densely spaced source location vectors in the source representation space can be used. The proposed noise-subspace location vector estimator is,

$$P_{lv}(\theta) = \sum_{k=1}^{N_\theta} \left(\sum_{i=D_t+1}^{KL} |\ \mathbf{a}_\theta^{\dagger}(\omega_k)\, \mathbf{e}_i\ |^2 \right)^{-1} \ , \qquad (16)$$

where the ω_k are equally spaced over B_θ and N_θ is the number of location vectors employed to represent location θ..

SIMULATIONS

Simulation results are shown if Figs(2-5). All pertinent parameters accompany the figures. For each figure, a linear equi-spaced filled array of isotropic elements were simulated. Propagation was pure delay. All frequencies are normalized to a 1 Hz. sampling rate. All plots are broadband spatial spectra vs. arrival angle of far field sources, measured with respect to array broadside.

Fig(2) illustrates the basis vector algorithm performance (and sensitivity to source spectral content) for exact data covariances. Fig(3) illustrates the effectiveness of the location vector algorithm for partial bandwidth sources. With Figs(4,5), the location vector algorithm is compared to a narrowband decomposition approach. For combining the narrowband null spectra, Eq(14) was used. Broadband spectra generated with Eqs(10,11) were no better. The two sources are resolved only for the direct broadband processing algorithm.

In the comparative simulation, the advantage in direct broadband processing is attributed to the more concentrated observed source energy. The advantage, while modest, is definite. Note that, compared to the 0.° source, the 3.° source is not as distinguished. The 3.° source observed energy is less concentrated. With source focusing, resolution should be further improved.

SUMMARY

Direct broadband data processing has been motivated as providing increased resolution compared to a narrowband decomposition approach. Two new algorithms based on the eigenvectors of a broadband spatial/temporal covariance matrix were proposed. Simulations show a modest improvement in broadband spatial spectral estimation compared to narrowband decomposition algorithm. Focusing is required to significantly increase resolution. Methods similar to Wang and Kaveh's , but with non-steer location source representation (and therefore no SLE bias) are currently under investigation.

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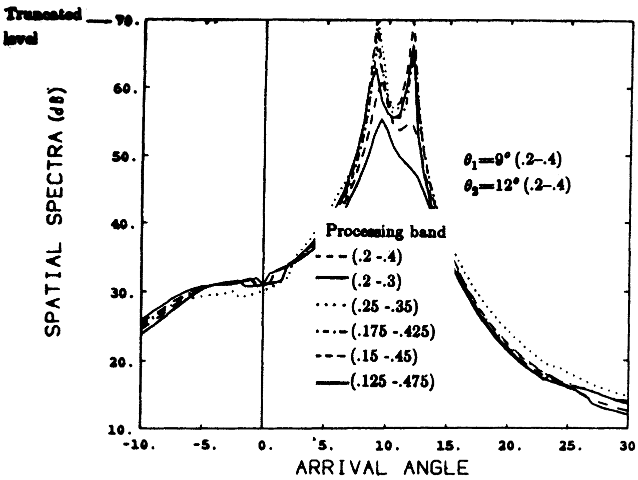
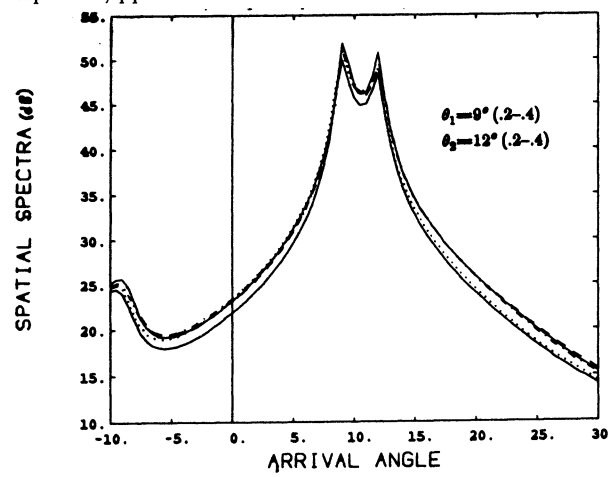
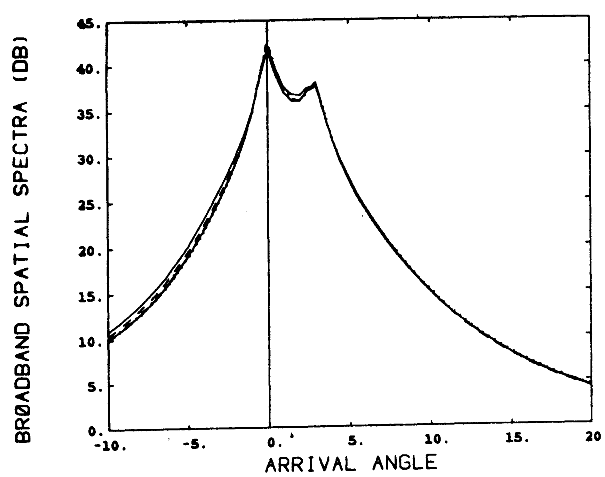


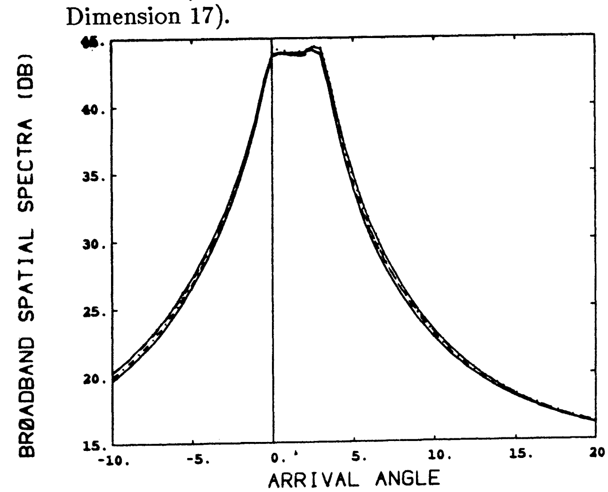
Fig(2): Basis Vector Algorithm Broadband Spatial Spectra - from a Computed Broadband Covariance Matrix - for Varying Source Representation Space Bandwidths (K =10, L =3, and Signal Subspace Dimension 9).



Fig(3): Location Vector Algorithm Broadband Spatial Spectra - from Simulated Data Broadband Covariance Matrices, 4096 Array Data Vectors/Curve - for 20 Location Vectors over .1 - .5 Hz. (K =10, L =3, and Signal Subspace Dimension 17).



Fig(4): Location Vector Algorithm Broadband Spatial Spectra - from Simulated Data Broadband Covariance Matrices, 4096 Array Data Vectors/Curve - for 10 Location Vectors over .1 - .5 Hz. (SNR=10dB, Source BW (.1 - .5 Hz.), θ_1=0.^{o} , θ_2=3.^{o} , K =16, L =3, and Signal Subspace Dimension 9).



Fig(5): For the Data used for Fig. (4), Wax, et. al. Algorithm Eq(11) (L =64 (only freq. bins 7 thru 33 processed) K =16, for each freq., 2-D Signal Subspace).