Localization Algorithms in Passive Sensor Networks

Darya Ismailova

Department of Electrical and Computer Engineering
University of Victoria, Victoria, BC, Canada

December 16, 2016

Outline

1. Motivation
2. Basic Localization Systems and Methods
3. Iterative Re-Weighting Least-Squares Methods for Source Localization
4. Penalty Convex-Concave Procedure for Range-based Localization
5. Conclusions and Future Work

Geolocation and Why it Matters

- Navigation: outdoor; indoor
- Surveillance
- Localization of emergency callers
- Emergency and rescue operations / first responders
- Self-organizing networks
- Asset monitoring and tracking
- Other commercial location-based servises
- ...

Basic Localization Systems and Methods

- Ranging methods
  - range measurements (Time Of Arrival)
  - range-difference measurements (Time-Difference of Arrival)
  - received signal strength
- Angle Of Arrival Techniques
- Survey-Based Systems (fingerprinting)
  - memoryles systems (SVM, NN)
  - memory systems (Bayesian inference, grid-based Markov)
  - channel impulse response fingerprinting
  - non-RF features

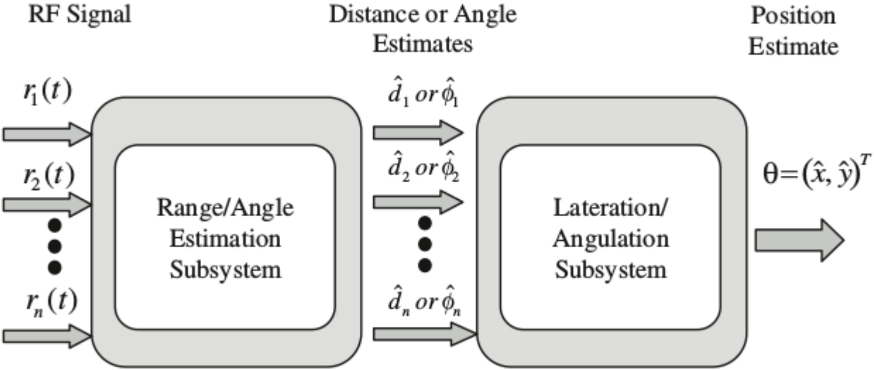

Figure: Classical geolocation system. Range or angle information is extracted from received RF signals. Location is then estimated by lateration/angulation techniques [GeoLoc].

Time Of Arrival Localization (TOA)

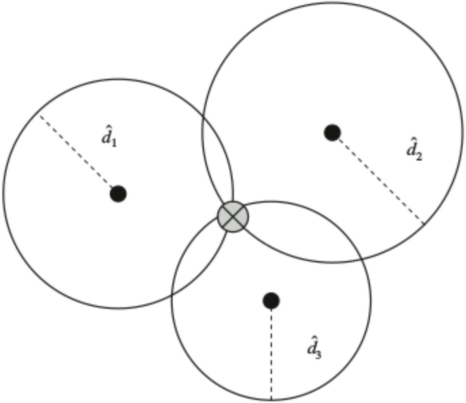

Figure: TOA-based trilateration. Range measurements to at least three BS make up a set of nonlinear equations that can be solved to estimate the position of a signal source [GeoLoc].

Time Of Arrival Localization (TOA)

The nonlinear least squares (NLLS) source location extimate $\hat{\boldsymbol{x}}$ is found by

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \left\{ \sum_{i=1}^{m} \beta_i \left(d_n^{(i)} - \|\boldsymbol{x} - \boldsymbol{a}_i\| \right)^2 \right\}$$

where

$\boldsymbol{a}_i$ - a vector of known coordinates of reference points (sensors)

$d_n^{(i)}$ - a noisy range measurement associated with it

β_i - a weight used to emphasize the degree of confidence in the measurement

m - the number of sensors.

Time-Difference Of Arrival Localization (TDOA)

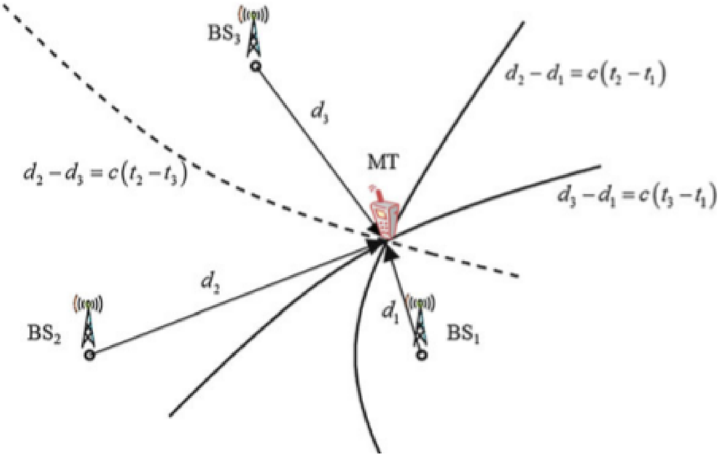

Figure: Example of observed time-difference of arrival (O-TDOA) method [GeoLoc].

Time-Difference Of Arrival Localization (TDOA)

Given the range-difference measurements

$$d_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x} - \boldsymbol{a}_0\| = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x}\|, \text{ for } i = 1, 2, \ldots, m$$

The standard NLLS location estimate $\hat{\boldsymbol{x}}$ is found by

$$\text{minimize} \sum_{i=1}^{m} \left(\|\hat{\boldsymbol{x}} - \boldsymbol{a}_i\| - \|\hat{\boldsymbol{x}}\| - d_i\right)^2$$

with

$\boldsymbol{a}_i$ - a vector of known coordinates of reference points (sensors)

$d_n^{(i)}$ - a noisy range measurement associated with it

m - the number of sensors.

Methods Based on Received Signal Strength (RSS-based)

The relationship between the RSS reading and the distance can be approximated by

$$P_x(d) = P_0(d_0) - 10 n_p \log_{10}\left(\frac{d_i}{d_0}\right) + X_\sigma$$

where
$P_0(d_0)$ - a reference power in dB milliwatts at a reference distance d_0 away from the transmitter
n_p - the pathloss exponent
X_σ - the log-normal shadow fading component with variance σ^2
d_i - the distance between the mobile devices and the ith base station.
σ and n_p are environment dependent

Why Least Squares

- Least squares (LS) algorithms for range-based localization:
  - geometrically meaningful
  - provide low complexity solutions with competitive accuracy

- However:
  - the error measure is non-convex
  - excludes many *local* methods, that are iterative

- Solutions obtained using *global* localization techniques such as semidefinite programming (SDP) are not optimal in LS sense.

Iterative Re-Weighting Least-Squares Methods for Source Localization

- Methods developed by A. Beck, P. Stoica, J. Li [BSL2008] for *squared* range LS (SR-LS) and *squared* range difference LS (SDR-LS) problems allow to obtain exact and *global* solutions.
- The results produced are merely approximations of the original LS problems because SR-LS and SRD-LS are no longer an ML solutions.
- Proposed iterative procedure where the SR-LS (or SRD-LS) algorithm is applied to a *weighted* sum of squared terms and special weights construction allow to obtain a solution which is conciderably closer to the original range-based (or range-difference-based) LS solution.

Source Localization From Range Measurements

Measurement Model

- Throughout it is assumed that *range measurements* obey the model

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m.$$

  where $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ - given array of m sensors;
  $\boldsymbol{a}_i \in R^n$ contains n coordinates of the ith sensor in space R^n;
  r_i - received noisy distance reading from the ith sensor;
  ε_i - unknown noise associated with measurement from the ith sensor.

- The problem can be stated as to estimate the exact source location $\boldsymbol{x} \in R^n$ from noisy range measurements $\boldsymbol{r} = [r_1 \ r_2 \ldots r_m]^T$.

LS Formulations

- The range-based least squares (R-LS) estimate refers to the solution of the problem

$$\underset{\boldsymbol{x}}{\text{minimize}}\, f(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \tag{R}$$

- If $\varepsilon \sim N(0, \boldsymbol{\Sigma})$ and $\boldsymbol{\Sigma} \propto \boldsymbol{I}$, then the R-LS solution of problem (R) is identical to the ML location estimator.

- Unfortunately, the objective in (R) is highly non-convex, posessing many local minimizers even for small-scale systems.

Source Localization From Range Measurements

LS Formulations

- Alternatively, location estimate can be obtained by solving the *squared range based LS* (SR-LS) problem [BSL2008]

$$\underset{\boldsymbol{x}}{\text{minimize}} \sum_{i=1}^{m} (\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2)^2 \tag{SR}$$

- The SR-LS estimate is no longer an ML solution, hence, only an approximation of the original R-LS problem.
- To reduce the gap between the two solutions we propose a weighted SR-LS (WSR-LS) problem:

$$\underset{\boldsymbol{x}}{\text{minimize}} \sum_{i=1}^{m} w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2\right)^2 \tag{WSR}$$

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- WSR-LS with properly chosen weights facilitates an excellent approximation of the R-LS estime.

- The main idea is to use the weigths $w_i, i = 1, \ldots, m$ to tune the objective in (WSR) toward the objective in (R). We compare the ith term of the objective in (WSR) with its counterpart in (R) as:

$$\underbrace{w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2 \right)^2}_{\text{in (WSR)}} \leftrightarrow \underbrace{\left(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i \right)^2}_{\text{in (R)}}$$

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- By writing the ith term in (WSR) as

$$w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2 \right)^2 = w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\| + r_i \right)^2 \underbrace{\left(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i \right)^2}_{\text{same as in (R)}}$$

we note that the objective in (WSR) would be the same as in (R) if the weight w_i was assigned to $1/\left(\|\boldsymbol{x} - \boldsymbol{a}_i\| + r_i \right)^2$.

- Evidently, such weight assignments cannot be realized.

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- In the proposed iterative procedure we solve a weighted SR-LS sub-problem,where at each iteration the weights are fixed:

$$\underset{x}{\text{minimize}} \sum_{i=1}^{m} w_i^{(k)} \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2 \right)^2 \qquad \text{(IRWSR)}$$

- for $k = 1$ all weights $\{w_i^{(1)}, i = 1, \ldots, m\}$ are set to unity;
- for $k \geq 2$ the weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ are assigned using the previous iterate $\boldsymbol{x}_{k-1}$ as

$$w_i^{(k)} = \frac{1}{\left(\|\boldsymbol{x}_{k-1} - \boldsymbol{a}_i\| + r_i\right)^2}.$$

Source Localization From Range-Difference Measurements

Problem Statement

- It is assumed that the range-difference measurements obey the model:

$$d_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x} - \boldsymbol{a}_0\| = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x}\|, \quad i = 1, \ldots, m$$

  where $\boldsymbol{a}_0$ - reference sensor placed at the origin.

- The standard range-difference LS (RD-LS) problem is formulated as

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}}\, F(\boldsymbol{x}) = \sum_{i=1}^{m} (d_i + \|\boldsymbol{x}\| - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \qquad \text{(RD)}$$

Source Localization From Range-Difference Measurements

SRD-LS and WSRD-LS formulations

- An approximation of the RD-LS solution can be obtained by solving the *squared range difference based LS* (SRD-LS) problem.
- By re-writing the measurements model as $d_i + \|\boldsymbol{x}\| = \|\boldsymbol{x} - \boldsymbol{a}_i\|$ and squaring both sides, we obtain

$$-2d_i\|\boldsymbol{x}\| - 2\boldsymbol{a}_i^T\boldsymbol{x} = g_i, \quad i = 1, \ldots, m$$

where $g_i = d_i^2 - \|\boldsymbol{a}_i\|^2$. The SRD-LS solution can be obtained by minimizing following criterion:

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}} \sum_{i=1}^{m} \left(-2\boldsymbol{a}_i^T\boldsymbol{x} - 2d_i\|\boldsymbol{x}\| - g_i\right)^2$$

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- We now present a method for improved solutions over SRD-LS solutions.

- We consider the weighted SRD-LS problem

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}} \sum_{i=1}^{m} w_i \left(-2\boldsymbol{a}_i^T \boldsymbol{x} - 2d_i \|\boldsymbol{x}\| - g_i \right)^2 \tag{WSRD}$$

where weights w_i for $i = 1, \ldots, m$ are *fixed* nonnegative constants.

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- The ith term of the objective function in (WSRD) can be written as:

$$
\begin{aligned}
& w_i \left(-2d_i \|\boldsymbol{x}\| - 2\boldsymbol{a}_i^T \boldsymbol{x} - g_i \right)^2 \\
= & w_i \left(d_i + \|\boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{a}_i\| \right) \underbrace{\left(d_i + \|\boldsymbol{x}\| - \|\boldsymbol{x} - \boldsymbol{a}_i\| \right)}_{\text{same as in RD}}
\end{aligned}
$$

- If weights w_i were set to $1/\left(d_i + \|\boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{a}_i\|\right)^2$ the objective in (WSRD) would be the same as in (RD).

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- We employ an iterative procedure where the weights in the kth iteration are assigned to

$$w_i^{(k)} = \frac{1}{(d_i + \|\boldsymbol{x}_{k-1}\| + \|\boldsymbol{x}_{k-1} - \boldsymbol{a}_i\|)^2}, i = 1, \ldots, m$$

with $\{w_i^{(1)} = 1, i = 1, \ldots, m\}$.

- We will refer to the derived problem as the iterative re-weighted SRD-LS (WSRD-LS) problem and the solution obtained as IRWSRD-LS solution.

Performance Evaluation for SR-LS and IRWSR-LS

- We can see that IRWSR-LS solutions offer considerable improvement over SR-LS solutions.

Table: Averaged MSE for SR-LS and IRWSR-LS methods by noise level

σ	**SR - LS**	**IRWSR-LS**	**Improvement (%)**
1e-03	2.03251062e-06	1.19962894e-06	41
1e-02	1.83717590e-04	1.24797437e-04	32
1e-01	1.83611315e-02	1.22233840e-02	33

Performance Evaluation for SRD-LS and IRWSRD-LS

Table: Averaged MSE for SRD-LS and IRWSRD-LS methods by noise level

σ	**SRD - LS**	**IRWSRD-LS**	**Improvement (%)**
1e-04	1.38301598e-08	8.22705918e-09	40
1e-03	1.60398717e-06	1.03880406e-06	35
1e-02	1.11632818e-04	6.67785604e-05	40
1e-01	1.20947651e-02	7.20891487e-03	40
1e+0	1.57050323e+00	9.70756420e-01	40

Range-Based Localization Revisited

Measurement Model

- The *range measurements* model is assumed to be given by

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m.$$

$\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ - given array of m sensors;
r_i - received noisy distance reading from sensor i;
ε_i - unknown noise associated with measurement from the ith sensor.

- The range-based least squares (R-LS) estimate refers to the solution of

$$\underset{\boldsymbol{x}}{\text{minimize}}\, F(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2$$

Penalty Convex-Concave Procedure for Source Localization

- We frame the localization problem as difference-of-convex-functions (DC) program.
- Proposed formulation:
  - based on a penalty convex-concave procedure (PCCP)
  - accepts infeasible initial points
  - additional constraints that enforce the algorithms iteration path towards the LS solution
  - strategies to secure good initial points

Basic Convex-Concave Procedure (CCP)

- The CCP finds local optima of *nonconvex* problems of the form

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) - g(\boldsymbol{x}) \\ \text{subject to:} \quad & f_i(\boldsymbol{x}) \leq g_i(\boldsymbol{x}) \quad \text{for: } i = 1, 2, \ldots, m \end{aligned}$$

  where $f(\boldsymbol{x}), g(\boldsymbol{x}), f_i(\boldsymbol{x}), g_i(\boldsymbol{x})$ for $i = 1, 2 \ldots, m$ are convex.

- It is a descent algorithm that requires a *feasible* initial point $\boldsymbol{x}_0$, i.e. $f_i(\boldsymbol{x}) - g_i(\boldsymbol{x}) \leq 0$ for $i = 1, 2 \ldots, m$.

Basic Convex-Concave Procedure (CCP)

- The basic CCP algorithm is an iterative procedure including two key steps (in the k-th iteration):

1. Convexify: form $\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) = g(\boldsymbol{x}_k) + \triangledown g(\boldsymbol{x}_k)^T(\boldsymbol{x} - \boldsymbol{x}_k)$

   and $\hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = g_i(\boldsymbol{x}_k) + \triangledown g_i(\boldsymbol{x}_k)^T(\boldsymbol{x} - \boldsymbol{x}_k)$
   for $i = 1, 2 \ldots, m$

2. Solve the convex problem:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) - \hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) \\ \text{subject to:} \quad & f_i(\boldsymbol{x}) - \hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) \le 0 \\ & \text{for: } i = 1, 2, \ldots, m \end{aligned}$$

An example of the basic CCP procedure.

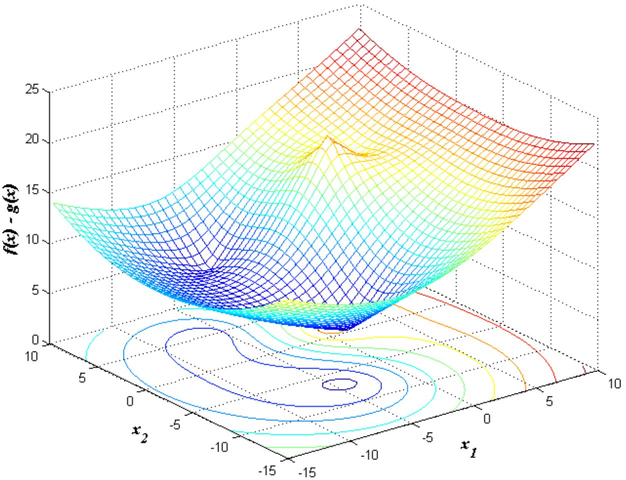

Figure: A nonconvex function in the form of the difference of two convex functions and its contour plot.

An example of the basic CCP procedure.

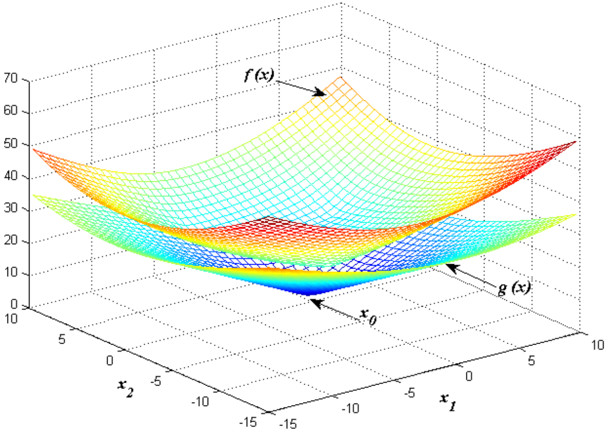

Figure: Separation of the nonconvex function into two convex functions.

An example of the basic CCP procedure.

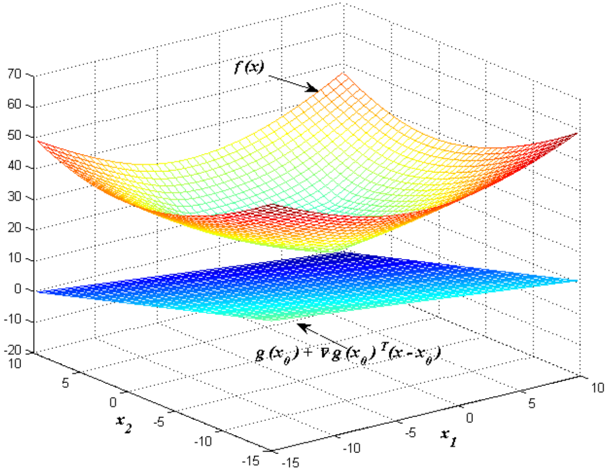

Figure: First order approximation of $g(\boldsymbol{x})$.

An example of the basic CCP procedure.

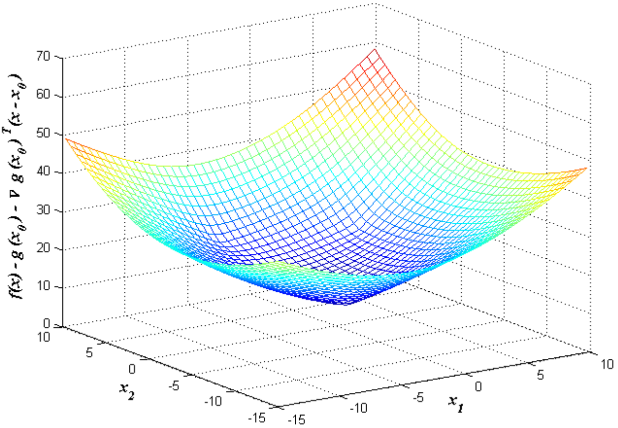

Figure: A convex approximation of the original nonconvex function at $\boldsymbol{x}_0 = (0, 0)$.

Range-Based Localization Revisited

- The range-based least squares (R-LS) estimate

$$\underset{\boldsymbol{x}}{\text{minimize}}\, F(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \tag{R}$$

Problem Reformulation

- We begin by re-writing the objective $F(\boldsymbol{x})$ up to a constant as:

$$\sum_{i=1}^{m}(r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\sum_{i=1}^{m}\boldsymbol{a}_i - 2\sum_{i=1}^{m} r_i\|\boldsymbol{x} - \boldsymbol{a}_i\|$$

  which allows to formulate it in a basic CCP form $F(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\boldsymbol{x})$ with

$$f(\boldsymbol{x}) = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\sum_{i=1}^{m}\boldsymbol{a}_i \quad \text{- convex}$$

$$g(\boldsymbol{x}) = 2\sum_{i=1}^{m} r_i\|\boldsymbol{x} - \boldsymbol{a}_i\| \quad \text{- convex.}$$

Problem Reformulation

- Since $g(\boldsymbol{x})$ is not differentiable at the point where $\boldsymbol{x} = \boldsymbol{a}_i$ for some $1 \leq i \leq m$, we replace $\bigtriangledown g(\boldsymbol{x}_k)$ by a subgradient of $g(\boldsymbol{x})$ at $\boldsymbol{x}_k$ as

$$\partial g(\boldsymbol{x}_k) = 2 \sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|$$

where

$$\partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\| = \begin{cases} \dfrac{\boldsymbol{x}_k - \boldsymbol{a}_i}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|}, & \text{if } \boldsymbol{x}_k \neq \boldsymbol{a}_i \\ \boldsymbol{0}, & \text{otherwise} \end{cases}$$

Problem Reformulation

- Up to a multiplicative factor $1/m$ and an additive constant term the objective in (R) can be written as

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \hat{F}(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{x} - 2\boldsymbol{x}^T \boldsymbol{v}_k$$

where

$$\boldsymbol{v}_k = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|, \quad \bar{\boldsymbol{a}} = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{a}_i$$

- Given $\boldsymbol{x}_k$ (in the k-th iteration) the solution of the quadratic problem can be obtained as

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|$$

Imposing Error Bounds

- The algorithm can be enhanced by imposing a bound on each squared measurement error

$$(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i)^2 \leq \delta_i^2$$

which leads to

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \leq 0 & \qquad (C1) \\ r_i - \delta_i \leq \|\boldsymbol{x} - \boldsymbol{a}_i\|, \quad \text{for } 1 \leq i \leq m. & \qquad (C2) \end{aligned}$$

Both sets of constraints can be written in a form $f_i(\boldsymbol{x}) \leq g_i(\boldsymbol{x})$.

- Constraints in (C1) are convex, with $f_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i$, and $g_i(\boldsymbol{x}) = 0$.

Imposing Error Bounds

- In case of (C2): define $f_i(\boldsymbol{x}) = r_i - \delta_i$ and $g_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\|$. Replace $g_i(\boldsymbol{x})$ with its approximation

$$\hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k)$$

  This allows to convexify constraints $r_i - \delta_i \leq \|\boldsymbol{x} - \boldsymbol{a}_i\|$ as

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \leq 0$$

- Summarizing, the problem in the k-th iteration can be stated as

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k \\ \text{subject to:} \quad & \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \leq 0 \end{aligned}$$

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \leq 0$$

Penalty CCP (PCCP)

- Technical problem: the formulation requires a feasible initial point $\boldsymbol{x}_0$.
- Solution approach: allow *infeasible* initinial points by introducing slack variables $s_i \geq 0, \hat{s}_i \geq 0$, $1 \leq i \leq m$ into constraints (C1) and (C2) and penalizing the sum of violations.
- This leads to a *penalty* CCP:

$$
\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{s},\hat{\boldsymbol{s}}}{\text{minimize}} \quad & \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k + \tau_k \sum_{i=1}^{m}(s_i + \hat{s}_i) \\
\text{subject to:} \quad & \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \leq s_i \\
-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - & \frac{(\boldsymbol{x}_k - \boldsymbol{a}_i)^T}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|}(\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \leq \hat{s}_i \\
& s_i \geq 0, \hat{s}_i \geq 0, \text{ for: } i = 1, 2, \ldots, m
\end{aligned}
$$

where $0 \leq \tau_k \leq \tau_{max}$.

The Algorithm: Input parameters

Bound δ_i on the measurement error

- Lower δ_i leads to a "tighter" solution.
- Larger δ_i makes the algorithm less sensitive to outliers.
- If ε obeys a Gaussian distribution with zero mean and $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, then $\delta_i = \gamma\sigma_i$, where γ determines the width of confidence interval.
- For example, for $\gamma = 3$ we have the probability $Pr\{|\varepsilon_i| \leq 3\sigma_i\} \approx 0.99$.

The Algorithm: Input parameters

Initial point x_0

Techniques to select a good initial point:

- select the initial point uniformly randomly over the same region as the unknown source;
- set the initial point to the origin;
- run the algorithm from a set of candidate initial points and identify the solution as the one with lowest LS error;
- apply a *global* localization algorithm to generate an approximate LS solution, then take it as the initial point.

Numerical Results

System setup

- Sensors: $\{\boldsymbol{a}_i, i = 1, 2, \dots, 5\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$
- Source: $\boldsymbol{x}_s$, located randomly in $\{\boldsymbol{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$
- Noise: $\{\varepsilon_i, i = 1, \dots, m\}$ was modelled as i.i.d random variables with zero mean and variance σ^2, $\sigma \in \{10^{-3}, 10^{-2}, 10^{-1}, 1\}$
- $\gamma = 3$, $K_{max} = 20$

Numerical Results

Table: Averaged MSE for SR-LS and PCCP methods

σ	MLE	SR - LS	PCCP	R.I.
1e-03	6.0159e-01	1.3394e-06	**9.5243e-07**	29%
1e-02	3.5077e-01	1.4516e-04	**9.5831e-05**	34%
1e-01	3.7866e-01	1.2058e-02	**8.7107e-03**	28%
1e+0	1.4470e+00	1.3662e+00	**1.2346e+00**	10%

Conclusions

- New iterative methods for locating a radiating source based on noisy range and range-difference measurements.
- The iterative re-weighting methods are developed by transforming the SR-LS and SRD-LS algorithms [BSL2008] into an iterative procedure so that a weighted SR-LS (SRD-LS) objective assymptotically approaches the original R-LS objective.
- Convex minimization method based on PCCP that can be efficiently solved with an infeasible initial point
- Proposed algorithms are found to outperform the existing methods.

Future Work

- Study and mitigation of the influence of sensor geometry on the accuracy of the developed methods (for example, geometric dilusion of precision).
- Multiple source localization in wireless sensor networks.

Q & A

Appendix

Source Localization From Range Measurements

Weighted Squared Range Least Squares Formulation

- Following [BSL2008], we convert (WSR) into a GTRS as

$$\underset{\boldsymbol{y} \in R^{n+1}}{\text{minimize}} \|\boldsymbol{A}_w \boldsymbol{y} - \boldsymbol{b}_w\|^2 \tag{1a}$$

$$\text{subject to:} \quad \boldsymbol{y}^T \boldsymbol{D} \boldsymbol{y} + 2\boldsymbol{f}^T \boldsymbol{y} = 0 \tag{1b}$$

where $\boldsymbol{y} = [\boldsymbol{x}^T \ \alpha]^T$, $\alpha = \|\boldsymbol{x}\|$, $\boldsymbol{A}_w = \boldsymbol{\Gamma A}$ and $\boldsymbol{b}_w = \boldsymbol{\Gamma b}$ with fixed $\boldsymbol{\Gamma} = \text{diag}\left(\sqrt{w_1}, \ldots, \sqrt{w_m}\right)$, and

$$\boldsymbol{A} = \begin{pmatrix} -2\boldsymbol{a}_1^T & 1 \\ \vdots & \vdots \\ -2\boldsymbol{a}_m^T & 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} r_1^T - \|\boldsymbol{a}_1\|^T \\ \vdots \\ r_m^T - \|\boldsymbol{a}_m\|^T \end{pmatrix} \tag{2}$$

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{I}_{n\times n} & \boldsymbol{0}_{n\times 1} \\ \boldsymbol{0}_{1\times n} & 0 \end{pmatrix}, \boldsymbol{f} = \begin{pmatrix} \boldsymbol{0} \\ -0.5 \end{pmatrix} \tag{3}$$

Source Localization From Range Measurements

The Algorithm

1. Input data: Sensor locations $\{\boldsymbol{a}_i, i = 1, \ldots, m\}$, range measurements $\{r_i, i = 1, \ldots, m\}$, maximum number of iterations k_{max} and convergence tolerance ζ.
2. Generate data set $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{D}, \boldsymbol{f}$ using (2) and (3). Set $k = 1, w_i^{(1)} = 1$ for $i = 1, \ldots, m$.
3. Set $\boldsymbol{\Gamma}_k = \text{diag}\left(\sqrt{w_1^{(k)}}, \ldots, \sqrt{w_m^{(k)}}\right)$, $\boldsymbol{A}_w = \boldsymbol{\Gamma}_k \boldsymbol{A}$ and $\boldsymbol{b}_w = \boldsymbol{\Gamma}_k \boldsymbol{b}$.
4. Solve the WSR-LS problem (IRWSR) via (1) to obtain its global solution $\boldsymbol{x}_k$.
5. If $k = k_{max}$ or $\|\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\| < \zeta$, terminate and output $\boldsymbol{x}_k$ as the solution; otherwise, set $k = k + 1$, update weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ and repeat from Step 3).

Source Localization From Range-Difference Measurements

Weighted Squared Range-Difference Least Squares Formulation

- By introducing new variable $\boldsymbol{y} = [\boldsymbol{x}^T \ \|\boldsymbol{x}\|]^T$ and noticing nonnegativity of the component y_{n+1} problem (WSRD) is converted to

$$\underset{\boldsymbol{y} \in R^{n+1}}{\text{minimize}} \|\boldsymbol{B}_w \boldsymbol{y} - \boldsymbol{g}_w\| \tag{4a}$$

$$\text{subject to: } \boldsymbol{y}^T \boldsymbol{C} \boldsymbol{y} = 0 \tag{4b}$$

$$y_{n+1} \geq 0 \tag{4c}$$

- where $\boldsymbol{B}_w = \boldsymbol{\Gamma} \boldsymbol{B}$, $\boldsymbol{g}_w = \boldsymbol{\Gamma} \boldsymbol{g}$, $\boldsymbol{\Gamma} = \text{diag}\{\sqrt{w_1}, \ldots, \sqrt{w_m}\}$, $\boldsymbol{g} = [g_1 \ldots g_m]^T$ and

$$\boldsymbol{B} = \begin{pmatrix} -2\boldsymbol{a}_1^T & -2d_1 \\ \vdots & \vdots \\ -2\boldsymbol{a}_m^T & -2d_m \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{0}_{n \times 1} \\ \boldsymbol{0}_{1 \times n} & -1 \end{pmatrix} \tag{5}$$

The Algorithm

1. Input data: Sensor locations $\{\boldsymbol{a}_i, i = 0, 1, \ldots, m\}$ with $\boldsymbol{a}_0 = \mathbf{0}$, range-difference measurements $\{d_i, i = 1, \ldots, m\}$, maximum number of iterations k_{max} and convergence tolerance ξ.
2. Generate data set $\{\boldsymbol{B}, \boldsymbol{g}, \boldsymbol{C}\}$ using (5). Set $k = 1$, $w_i^{(1)} = 1$ for $i = 1, \ldots, m$.
3. Set $\boldsymbol{\Gamma}_k = \text{diag}\left(\sqrt{w_1^{(k)}}, \ldots, \sqrt{w_m^{(k)}}\right)$, $\boldsymbol{B}_w = \boldsymbol{\Gamma}_k \boldsymbol{B}$ and $\boldsymbol{g}_w = \boldsymbol{\Gamma}_k \boldsymbol{g}$.
4. Solve WSRD-LS problem (4) to obtain its global solution $\boldsymbol{x}_k$.
5. If $k = k_{max}$ or $\|\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\| < \xi$, terminate and output $\boldsymbol{x}_k$ as the solution; otherwise, set $k = k + 1$, update weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ and repeat from Step 3).

References I

[SL2006] P. Stoica and J. Li, "Source localization from range-difference measurements," *IEEE Signal Process. Mag.*, vol. 23, pp. 63–65,69, Nov. 2006.

[BSL2008] A. Beck, P. Stoica and J. Li, "Exact and approximate solutions of source localization problems," *IEEE Trans. Signal Processing*, vol. 56, no. 5, pp. 1770–1777, May 2008.

[GeoLoc] C. Gentile, N. Alsindi, R. Raulefs, and C. Teolis, *Geolocation Techniques: Principles and Applications*, Springer, New-York, 2013.