Chapter 1

Penalty Convex-Concave Procedure for Source Localization Problem

This chapter is focused on the least squares formulation for the problem of localizing a single radiating source based on range measurements. We exploit special structure of the cost function of an unconstrained LS formulation and show that it is well suited for being investigated in a setting known as difference-of-convex-functions (DC) programming. Further, we present an algorithm for solving the LS problem at hand based on a penalty convex-concave procedure (PCCP) [30] that accommodates infeasible initial points. We also provide algorithmic details that are tailored to the localization problem at hand, these include additional constraints that enforce the algorithms iteration path towards the LS solution and strategies to secure good initial points for the algorithm. Numerical results are presented to demonstrate that the proposed algorithm offers substantial performance improvement relative to some best known results from the literature.

1.1 Notes TODO

Min number of iterations: 3 - proof?

stop when $\triangle F_{objectivefunstion}$ is increasing?

set Max number of iterations?

study the geometry of not-so-well results

add cases with 5 - 15 sensors

use SMACOF paper as an example of the divergence proof

1.2 Problem Statement and Review of Related Work

Locating a radiating source from range measurements in a passive sensor network has recently attracted an increasing amount of research interest as it finds applications in a wide range of network-based wireless systems. Least squares (LS) based algorithms for source localization problems constitute an important class of solution techniques as they are geometrically meaningful and often provide low complexity solution procedures with competitive estimation accuracy [13]-[15]. On the other hand, the error measure in an LS formulation for the localization problem of interest is shown to be highly non-convex, possessing multiple local solutions with degraded performance. This non-convexity excludes many local methods that are iterative, hence extremely sensitive to where the iteration begins. Several non-iterative *global* localization techniques are available from the literature. A global solution may be obtained by relaxing the LS model at hand to a semidefinite programming (SDP) problem which is known to be convex [32]. In doing so, however, the convexification based solution is no longer optimal in LS sense. Another representative in this class is the method proposed in [15], where localization problems for range measurements are addressed by developing solution methods for *squared* range LS (SR-LS) problems. Although these methods are efficient in terms of complexity, they remain to be suboptimal in the maximum likelihood (ML) sense because the solutions produced are merely approximations of the ML estimate.

same: *The source localization problem considered here involves a given array of m sensors specified by $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ where $\boldsymbol{a}_i \in R^n$ contains n coordinates of the ith sensor in space R^n. Each sensor measures its distance to a radiating source $\boldsymbol{x} \in R^n$. Throughout it is assumed that only noisy copies of the distance data are available, hence the range measurements obey the model*

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m. \tag{1.1}$$

where ε_i denotes the unknown noise that has occurred when the ith sensor measures its distance to source $\boldsymbol{x}$. Let $\boldsymbol{r} = [r_1\ r_2 \ldots r_m]^T$ and $\boldsymbol{\varepsilon} = [\varepsilon_1\ \varepsilon_2 \ldots \varepsilon_m]^T$, the source

localization problem can be stated as to estimate the exact source location $\boldsymbol{x}$ *from the noisy range measurements* $\boldsymbol{r}$. **In the rest of this section: shorten the review of the problem. add more lit review on methods** For the localization problem at hand, the range-based least squares (R-LS) estimate refers to the solution of the problem

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad F(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \tag{1.2}$$

Formulation (1.2) is connected to the maximum-likelihood (ML) location estimation that determines $\boldsymbol{x}$ by examining the probabilistic model of the error vector $\boldsymbol{\varepsilon}$. If $\boldsymbol{\varepsilon}$ obeys a Gaussian distribution with zero mean and covariance $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, then the maximum likelihood (ML) location estimator in this case is known to be

$$\boldsymbol{x}_{ML} = \arg\min_{\boldsymbol{x} \in R^n} (\boldsymbol{r} - \boldsymbol{g})^T \Sigma^{-1} (\boldsymbol{r} - \boldsymbol{g}) \tag{1.3}$$

where $\boldsymbol{g} = [g_1 \ g_2 \ldots \ g_m]^T$ with $g_i = \|\boldsymbol{x} - \boldsymbol{a}_i\|$. It follows immediately that the ML solution in (1.3) is identical to the R-LS solution of problem (1.2) when covariance $\boldsymbol{\Sigma}$ is proportional to the identity matrix, i.e., $\sigma_1^2 = \ldots = \sigma_m^2 = 1$. In the literature this is known as the equal noise power case. For notation simplicity this paper focuses on the equal noise power case, however the method developed below is also applicable to the unequal noise power case by working on a weighted version of the objective in (1.2) with $\{\sigma_i^{-2}, i = 1, \ldots, m\}$ as the weights.

There are many methods for continuous unconstrained optimization [33], however most of them are *local* methods in the sense they are sensitive to the choice of initial point, and give no guarantee to yield global solutions when applied to non-convex objective functions. Unfortunately, the objective function in (1.2) is highly non-convex, possessing many local minimizers even for small-scale systems. In this paper we present an different approach to solve the positioning problem, which employs a successive convex-concave procedure.

Advantages of convex-concave procedure from citeLBoyd

1.3 Fitting the Localization Problem to the CCP Framework

1.3.1 Basic Convex-Concave Procedure

describe what a DC problem is The CCP refers to an effective heuristic method to deal with a class of *nonconvex* problems of the form

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) - g(\boldsymbol{x}) && \text{(1.4a)} \\ \text{subject to:} \quad & f_i(\boldsymbol{x}) \leq g_i(\boldsymbol{x}) \quad \text{for: } i = 1, 2, \ldots, m && \text{(1.4b)} \end{aligned}$$

where $f(\boldsymbol{x}), g(\boldsymbol{x}), f_i(\boldsymbol{x}), g_i(\boldsymbol{x})$ for $i = 1, 2, \ldots, m$ are convex. The basic CCP algorithm is an iterative procedure including two key steps (in the k-th iteration where iterate $\boldsymbol{x}_k$ is known):

(i) Convexification of the objective function and constraints by replacing $g(\boldsymbol{x})$ and $g_i(\boldsymbol{x})$, respectively, with their affine approximations

$$\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) = g(\boldsymbol{x}_k) + \bigtriangledown g(\boldsymbol{x}_k)^T (\boldsymbol{x} - \boldsymbol{x}_k) \tag{1.5a}$$

and

$$\begin{aligned} \hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = g_i(\boldsymbol{x}_k) + \bigtriangledown g_i(\boldsymbol{x}_k)^T (\boldsymbol{x} - \boldsymbol{x}_k) \\ \text{for: } i = 1, 2, \ldots, m \end{aligned} \tag{1.5b}$$

(ii) Solving the convex problem

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) - \hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) && \text{(1.6a)} \\ \text{subject to:} \quad & f_i(\boldsymbol{x}) - \hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) \leq 0 && \text{(1.6b)} \\ & \text{for: } i = 1, 2, \ldots, m \end{aligned}$$

Because of the convexity of all the functions involved, it can be shown that the basic CCP is a descent algorithm and the iterates $\boldsymbol{x}_k$ converge to the critical point of the original problem (4) [30]. The basic CCP requires a *feasible* initial point $\boldsymbol{x}_0$ (in the sense that $\boldsymbol{x}_0$ satisfies (6b) for $i = 1, 2, \ldots, m$) to start the procedure. By introducing additional slack variables, a penalty CCP has been adopted to accept infeasible initial points.

1.3.2 Problem Reformulation

We begin by re-writing the objective function in (2) up to a constant as:

$$\begin{aligned} F(\boldsymbol{x}) = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\sum_{i=1}^{m}\boldsymbol{a}_i \\ - 2\sum_{i=1}^{m} r_i\|\boldsymbol{x}-\boldsymbol{a}_i\| \end{aligned} \tag{1.7}$$

The objective in (7) is not convex. This is because, for points $\boldsymbol{x}$ that are not coincided with $\boldsymbol{a}_i$ for $1 \leq i \leq m$, the Hessian of $F(\boldsymbol{x})$ is given by

$$\begin{aligned} \bigtriangledown^2 F(\boldsymbol{x}) = 2m\boldsymbol{I} + 2\sum_{i=1}^{m}\frac{r_i}{\|\boldsymbol{x}-\boldsymbol{a}_i\|^3}\cdot \\ \cdot\Big((\boldsymbol{x}-\boldsymbol{a}_i)(\boldsymbol{x}-\boldsymbol{a}_i)^T - \|\boldsymbol{x}-\boldsymbol{a}_i\|^2\boldsymbol{I}\Big) \end{aligned}$$

which is not always positive semidefinite. On the other hand, by defining

$$\begin{aligned} f(\boldsymbol{x}) &= m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\sum_{i=1}^{m}\boldsymbol{a}_i \\ g(\boldsymbol{x}) &= 2\sum_{i=1}^{m} r_i\|\boldsymbol{x}-\boldsymbol{a}_i\| \end{aligned} \tag{1.8}$$

the objective in (7) can be expressed as $F(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\boldsymbol{x})$ with both $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ convex, hence it fits naturally into (4a). Note that $g(\boldsymbol{x})$ in (8) is not differentiable at the point where $\boldsymbol{x} = \boldsymbol{a}_i$ for some $1 \leq i \leq m$, thus we replace the term $\bigtriangledown g(\boldsymbol{x}_k)$ in (5a) by a subgradient [34] of $g(\boldsymbol{x})$ at $\boldsymbol{x}_k$, denoted by $\partial g(\boldsymbol{x}_k)$ as

$$\partial g(\boldsymbol{x}_k) = 2\sum_{i=1}^{m} r_i\partial\|\boldsymbol{x}_k-\boldsymbol{a}_i\|$$

where

$$\partial\|\boldsymbol{x}_k-\boldsymbol{a}_i\| = \begin{cases} \dfrac{\boldsymbol{x}_k-\boldsymbol{a}_i}{\|\boldsymbol{x}_k-\boldsymbol{a}_i\|}, & \text{if } \boldsymbol{x}_k \neq \boldsymbol{a}_i \\ \boldsymbol{0}, & \text{otherwise} \end{cases}$$

Hence $\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k)$ in (5a) is given by

$$\begin{aligned}\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) &= 2\sum_{i=1}^{m} r_i \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + 2\left(\boldsymbol{x} - \boldsymbol{x}_k\right)^T \sum_{i=1}^{m} r_i \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\| \\ &= 2\boldsymbol{x}^T \sum_{i=1}^{m} r_i \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\| + c\end{aligned}$$

where c is a constant given by

$$c = -2\sum_{i=1}^{m} r_i \boldsymbol{a}_i^T \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|.$$

It follows that up to a multiplicative factor $1/m$ and an additive constant term the convex objective function in (6a) can be written as

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \hat{F}(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k \tag{1.9}$$

where

$$\boldsymbol{v}_k = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|, \quad \bar{\boldsymbol{a}} = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{a}_i \tag{1.10}$$

It is rather straightforward to see that given $\boldsymbol{x}_k$ (in the k-th iteration) the solution of the quadratic problem (9) can be obtained as

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\| \tag{1.11}$$

1.3.3 Imposing Error Bounds and Penalty Terms

The algorithm being developed can be enhanced by imposing a bound on each squared measurement error, namely

$$\left(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i\right)^2 \leq \delta_i^2 \tag{1.12}$$

which leads to

$$\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \leq 0 \tag{1.13a}$$

$$r_i - \delta_i \leq \|\boldsymbol{x} - \boldsymbol{a}_i\| \tag{1.13b}$$

for $1 \leq i \leq m$. The constraints in (13a) are convex and fit into those in (6b) with $f_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i$ and $g_i(\boldsymbol{x}) = 0$, while those in (13b) are in the form of (4b) with $f_i(\boldsymbol{x}) = r_i - \delta_i$ and $g_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\|$. Following CCP (see (5b)), $g_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\|$ is linearized around iterate $\boldsymbol{x}_k$ to

$$\hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k)$$

and (13b) is convexified as

$$r_i - \delta_i \leq \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k)$$

which now fits into (6b), or equivalently

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \leq 0 \tag{1.14}$$

We remark that constraint (14) is not only convex but also tighter than (13b). As a matter of fact, the convexity of the norm $\|\boldsymbol{x} - \boldsymbol{a}_i\|$ implies that it obeys the property

$$\|\boldsymbol{x} - \boldsymbol{a}_i\| \geq \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k)$$

Therefore, a point $\boldsymbol{x}$ satisfying (14) automatically satisfies (13b). Summarizing, the convexified problem in the k-th iteration can be stated as

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k && \text{(1.15a)} \\ \text{subject to:} \quad & \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \leq 0 && \text{(1.15b)} \end{aligned}$$

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \leq 0 \tag{1.15c}$$

A technical problem making the formulation in (15) difficult to implement is that it requires a feasible initial point $\boldsymbol{x}_0$. The problem can be overcome by introducing nonnegative slack variables $s_i \geq 0, \hat{s}_i \geq 0$, for $i = 1, \ldots, m$ into the constraints in (15b) and (15c) to replace their right-hand sides (which are zeros) by relaxed upper bounds (as these new bounds themselves are nonnegative variables). This leads to a

penalty CCP (PCCP) based formulation as follows:

$$\underset{\boldsymbol{x},\boldsymbol{s},\hat{\boldsymbol{s}}}{\text{minimize}} \quad \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k + \tau_k \sum_{i=1}^{m}(s_i + \hat{s}_i) \tag{1.16a}$$

$$\text{subject to:} \quad \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \leq s_i \tag{1.16b}$$

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \frac{(\boldsymbol{x}_k - \boldsymbol{a}_i)^T}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|}(\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \leq \hat{s}_i \tag{1.16c}$$

$$s_i \geq 0, \hat{s}_i \geq 0, \text{ for: } i = 1, 2, \ldots, m \tag{1.16d}$$

where the weight $\tau_k \geq 0$ increases as iterations proceed until it reaches an upper limit τ_{max}. By using a monotonically increasing τ_k for the penalty term in (16a), the algorithm reduces the slack variables s_i and $\hat{s}_i$ very quickly. As a result, new iterates quickly become feasible as s_i and $\hat{s}_i$ vanish. The upper limit τ_{max} is imposed to avoid numerical difficulties that may occur if τ_k becomes too large and to ensure convergence if a feasible region is not found [9]. Consequently, while formulation (16) accepts *infeasible* initial points, the iterates obtained by solving (16) are practically identical to those obtained by solving (15).

[30] PCCP is not descent algorithm, but it will converge.

1.3.4 The Algorithm

The input parameters for the algorithm include the bound δ_i on the measurement error. Setting δ_i to a lower value leads to a "tighter" solution. On the other hand, a larger δ_i would make the algorithm less sensitive to outliers. If measurement noise $\boldsymbol{\varepsilon}$ obeys a Gaussian distribution with zero mean and known covariance $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, then δ_i can be expressed as $\delta_i = \gamma\sigma_i$, where γ is a parameter that determines the width of confidence interval. For example, for $\gamma = 3$ we have the probability $Pr\{|\varepsilon_i| \leq 3\sigma_i\} \approx 0.99$. Other input parameters are initial point $\boldsymbol{x}_0$, maximum number of iterations K_{max}, initial weight τ_0, and upper limit of weight τ_{max} (to avoid numerical problems that may occur if τ_i becomes too large).

As mentioned in Sec. 2, the original LS objective is highly non-convex with many local minimums even for small-scale systems. Consequently, it is of critical importance to select a good initial point for the proposed PCCP-based algorithm because PCCP is essentially a local procedure. Several techniques are available, these include: (i) Select the initial point uniformly randomly over the same region as the unknown

radiating source; (ii) Set the initial point to the origin; (iii) Run the algorithm from a set of candidate initial points and identify the solution as the one with lowest LS error. Typically, comparing the results from n distinct initial points shall suffice. For the planar case ($n = 2$), for example, it is sufficient to compare the two intersection points of the two circles that are associated with the two smallest distance readings as the target is very likely to be in the vicinity of these sensors; and (iv) Apply a global localization algorithm such as those in [15] to generate an approximate LS solution, then take it as the initial point to run the proposed algorithm. The algorithm can be now outlined as follows.

PCCP-based LS Algorithm for Source Localization

Step 1: Input sensor locations $\{\boldsymbol{a}_i, i = 1, \ldots, m\}$, range measurements $\{r_i, i = 1, \ldots, m\}$, $\boldsymbol{x}_0, K_{max}, \tau_0, \tau_{max}, \mu > 0, \gamma, \sigma$, and set $k = 0$.

Step 2: Form $\boldsymbol{v}_k$ as in (10) and solve (16). Denote the solution as $(\boldsymbol{s}^*, \hat{\boldsymbol{s}}^*, \boldsymbol{x}^*)$.

Step 3: Update $\tau_{k+1} = \min(\mu\tau_k, \tau_{max})$, set $k = k + 1$.

Step 4: If $k = K_{max}$, terminate and output $\boldsymbol{x}^*$ as the solution; otherwise, set $\boldsymbol{x}_k = \boldsymbol{x}^*$ and repeat from Step 2.

1.4 Numerical Results

For illustration purposes, the proposed algorithm was applied to a network with five sensors, and its performance was evaluated and compared with existing state-of-the-art methods by Monte Carlo simulations with a set-up similar to that of [**?**]. SR-LS solutions were used as performance benchmarks for the PCCP-based LS Algorithm. The system consisted of 5 sensors $\{\boldsymbol{a}_i, i = 1, 2, \ldots, 5\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$, and a radiating source $\boldsymbol{x}_s$, located randomly in the region $\{\boldsymbol{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$. The coordinates of the source and sensors were generated for each dimension following a uniform distribution. Measurement noise $\{\varepsilon_i, i = 1, \ldots, m\}$ was modelled as independent and identically distributed (i.i.d) random variables with zero mean and variance σ^2, with σ being one of four possible levels $\{10^{-3}, 10^{-2}, 10^{-1}, 1\}$. The range measurements $\{r_i, i = 1, 2, \ldots, 5\}$ were calculated using (1). Accuracy of source location estimation was evaluated in terms of average of the squared position error error in the form $\|\boldsymbol{x}^* - \boldsymbol{x}_s\|^2$, where $\boldsymbol{x}_s$ denotes the exact source location and $\boldsymbol{x}^*$ is its estimation obtained by SR-LS and PCCP methods, respectively. In our simulations parameter γ was set to 3 and the number

Table 1.1: Averaged MSE for SR-LS and PCCP methods

σ	MLE	SR - LS	PCCP	R.I.
1e-03	6.0159e-01	1.3394e-06	**9.5243e-07**	29%
1e-02	3.5077e-01	1.4516e-04	**9.5831e-05**	34%
1e-01	3.7866e-01	1.2058e-02	**8.7107e-03**	28%
1e+0	1.4470e+00	1.3662e+00	**1.2346e+00**	10%

of iterations was set to 20. The proposed method was implemented by using CVX [36] and implementation of SR-LS followed [**?**]. The PCCP algorithm was initialized with intersection points of the two circles that are associated with the two smallest distance readings. A candidate solution point with lowest LS error in (2) was chosen as a PCCP solution. In cases when the circles did not intersect due to high noise level, the initial point was set as a midpoint between the centers of the two circles.

Table 1.1 provides comparisons of the PCCP with SR-LS and MLE, where each entry is averaged squared error over 1,000 Monte Carlo runs of the method. The MLE was implemented using Matlab function *lsqnonlin* [37], initialized with the same point as PCCP. It is observed that, comparing with SR-LS, the estimates produced by the proposed algorithm are found to be closer to the true source locations in MSE sense. The last column of the table represents relative improvement of the proposed method over SR-LS solutions in percentage.

Chapter 2

Least Squares Localization by Sequential Convex Relaxation

Notes for further development.

2-step approach:

1) identify outliers to reduce the error-prone data points

2) apply the algorithm to the redused data set

Bibliography

[1] J. O. Smith and J. S. Abel, "Closed-form least-squares source location estimation from range-difference measurements," *IEEE Trans. on Acoustic, Speech Signal Proc.*, vol. 12, pp. 1661–1669, Dec. 1987.

[2] H. Schau and A. Robinson, "Passive source localization employing intersecting spherical surfaces from time-of-arrival differences," *IEEE Trans. on Acoustic, Speech Signal Proc.*, vol. ASSP–35, pp. 1223–1225, Aug. 1987.

[3] K. Yao, R. Hudson, C. Reed, D. Chen, and F. Lorenzelli, "Blind beamforming on a randomly distributed sensor array system," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1555-1567, Oct. 1998.

[4] M. A. Sprito, "On the accuracy of cellular mobile station location estimation," *IEEE Trans. on Veh. Technol.*, vol. 50, pp. 674-685, May 2001.

[5] Y. Huang, J. Benesty, G. W. Elko, and R. M. Mersereau, "Realtime passive source localization: A practical linear correction least-squares approach," *IEEE Trans. on Speech Audio Proc.*, vol. 9, no. 8, pp. 943-956, Nov. 2002.

[6] K. W. Cheung, H. C. So, W. K. Ma, and Y. T. Chan, "Least squares algorithms for time-of-arrival-based mobile location," *IEEE Trans. on Signal Proc.*, vol. 52, no. 4, pp. 1121–1228, Apr. 2004.

[7] D. Li and H. Hu, "Energy-Based Collaborative Source Localization Using Acoustic Microsensor Array," in *EURASIP Journal on Applied Signal Proc.*, vol. 4, 321–337, 2003.

[8] X. Sheng and Y.-H. Hu, "ML Multiple-source localization using acoustic energy measurements with wireless sensor networks," *IEEE Trans. on Signal Proc.*, vol. 53, no.1, pp. 44–53, Jan. 2005.

[9] Z. M. Saric, D. D. Kukolj, and N. D. Teslic, "Acoustic source localization in wireless sensor network", *Circuits Syst Signal Proc.*, vol. 29, pp. 837–856, 2010.

[10] L.Lu, H.-C. Wu, K.Yan, and S.Iyengar, "Robust expectation maximization algorithm for multiple wideband acoustic source localization in the presence of nonuniform noise variances," *IEEE Sensors J.*, vol. 11, no. 3, pp. 536–544, Mar. 2011.

[11] K.W. Cheung, W.K. Ma, and H.C. So, "Accurate approximation algorithm for TOA-based maximum-likelihood mobile location using semidefinite programming," in *Proc. ICASSP*, vol. 2, pp. 145–148, 2004.

[12] A. H. Sayed, A. Tarighat, and N. Khajehnouri, "Network-based wireless location," *IEEE Signal Pro. Mag.*, vol. 22, no. 4, pp. 24–40, July 2005.

[13] Y. T. Chan, H. Y. C. Hang, and P. C. Ching, "Exact and approximate maximum likelihood localization algorithms," *IEEE Trans. on Veh. Technol.*, vol. 55, no. 1, pp. 10–16, Jan. 2006.

[14] P. Stoica and J. Li, "Source localization from range-difference measurements," *IEEE Signal Proc. Mag.*, vol. 23, pp. 63–65,69, Nov. 2006.

[15] A. Beck, P. Stoica and J. Li, "Exact and approximate solutions of source localization problems," *IEEE Trans. on Signal Proc.*, vol. 56, no. 5, pp. 1770–1777, May 2008.

[16] A. Beck, M. Teboulle, and Z. Chikishev, "Iterative minimization schemes for solving the single source localization problem," *SIAM J. Optim.*, vol. 19, no. 3, pp. 1397–1416, Nov. 2008.

[17] I. Daubechies, R. DeVore, M. Fornasier, and C. S. Güntürk, "Iteratively reweighted least squares minimization for sparse recovery," *Comm. Pure Appl. Math.*, vol. 63, pp. 1–38, 2010.

[18] A. Beck, and D. Pan, "On the solution of the GPS localization and circle fitting problems," *SIAM J. Optim.*, vol. 22, no. 1, pp. 108–134, Jan. 2012.

[19] A. Beck, "On the convergence of alternating minimization for convex programming with applications to iteratively re-weighted least squares and decomposition schemes," *SIAM J. Optim.*, vol. 25, no. 1, pp. 185-209, Jan. 2015.

[20] J.J. More, "Generalizations of the trust region subproblem," *Optim. Methods Softw.*, vol. 2, pp. 189–209, 1993.

[21] C. Fortin and H. Wolkowicz, "The trust region subproblem and semidefinite programming," *Optim. Methods Softw.*, vol. 19, no.1, pp. 41–67, 2004.

[22] D.P. O'Leary, "Robust regression computation using iteratively reweighted least squares," *SIAM J. Matrix Anal. Appl.*, vol. 11, no. 3, pp. 466–480, 1990.

[23] N. Bissantz, L. Dümbgen, A. Munk, and B. Stratmann, "Convergence analysis of generalized iteratively reweighted least squares algorithms on convex function spaces," *SIAM J. Optim.*, vol. 19, no. 4, pp 1828–1845, 2009.

[24] K. W. Cheung and H. C. So, "A multidimensional scaling framework for mobile location using time-of-arrival measurements," *IEEE Trans. on Signal Proc.*, vol. 53, no. 2, pp. 460–470, Feb. 2005.

[25] H. Liu, H. Darabi, P. B, and J. Liu, "Survey of Wireless Indoor Positioning Techniques and Systems," *IEEE Trans. on Systems, Man and Cybernetics. Part C: Applications and Revies*, vol. 37, no. 6, pp. 1067–1080, Nov. 2007.

[26] Y. Liu, Y.H. Hu, and Q. Pan, "Distributed, Robust Acoustic Source Localization in a Wireless Sensor Network," *IEEE Trans. on Signal Proc.*, vol. 60, no. 8, pp. 4350–4359, Aug. 2012.

[27] W. Kim, J. Lee, and G. Jee, "The interior-point method for an optimal treatment of bias in trilateration location," *IEEE Trans. Veh. Technol.*, vol. 55, no. 4, pp. 1291–1301, Jul. 2006.

[28] T. Qiao, S. Redfield, A. Abbasi, Z. Su, and H. Liu, "Robust coarse position estimation for TDOA localization," *IEEE Wireless Commun. Lett.*, vol. 2, no. 6, pp. 623–626, Dec. 2013.

[29] A. L. Yuille and A. Rangarajan, "The concave-convex procedure,' *Neural Computation*, vol. 15, no. 4, pp. 915–936, 2003.

[30] T. Lipp and S. Boyd, "Variations and extensions of the convex-concave procedure," *Research Report*, Stanford University, Aug. 2014.

[31] G. R. Lanckreit and B. K. Sriperumbudur, "On the convergence of the concave-convex procedure," in *Advances in Neural Information Proc. Systems*, pp. 1759–1767, 2009.

[32] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM Rev.*, vol. 38, no. 1, pp. 40–95, Mar. 1996.

[33] A. Antoniou and W.-S. Lu, *Practical Optimization: Algorithms and Engineering Applications*, Springer, New-York, 2007.

[34] Y. Nesterov, *Introductory Lectures on Convex Optimization*, Kluwer Academic Publishers, Boston, 2004.

[35] C. Gentile, N. Alsindi, R. Raulefs, and C. Teolis, *Geolocation Techniques: Principles and Applications*, Springer, New-York, 2013.

[36] CVX Research, http://cvx.com/cvx, August 2012.

[37] The Mathworks Inc., http://mathworks.com, 2015.