

Localization Algorithms in Passive Sensor Networks

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- 1 Motivation
- 2 Basic Localization Systems and Methods
- 3 Iterative Re-Weighting Least-Squares Methods for Source Localization
- 4 Penalty Convex-Concave Procedure for Range-based Localization
- 5 Least Squares Localization by Sequential Convex Relaxation
- 6 Conclusions and Future Work

Introduction

- Navigation: outdoor; indoor
- Surveillance
- Localization of emergency callers
- Emergency and rescue operations / first responders
- Self-organizing networks
- Asset monitoring and tracking
- Other commercial location-based services
- ...

- Ranging methods
 - range measurements (Time Of Arrival)
 - range-difference measurements (Time-Difference of Arrival)
 - received signal strength
- Angle Of Arrival Techniques
- Survey-Based Systems (fingerprinting)
 - memoryless systems (SVM, NN)
 - memory systems (Bayesian inference, grid-based Markov)
 - channel impulse response fingerprinting non-RF features

Basic Localization Systems and Methods

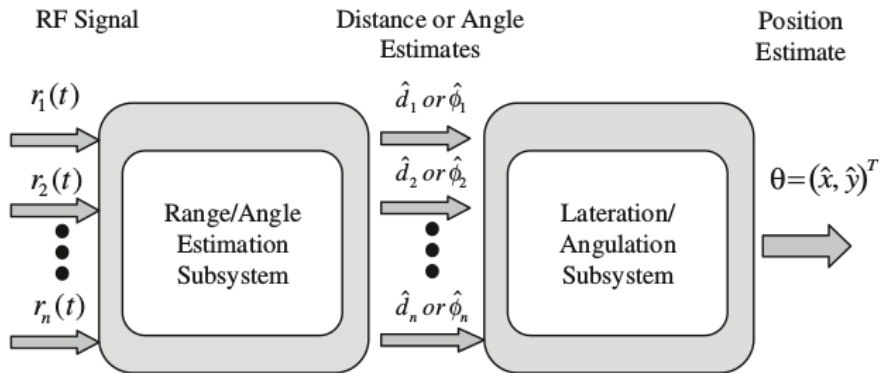


Figure: Classical geolocation system. Range or angle information is extracted from received RF signals. Location is then estimated by lateration/angulation techniques [GeoLoc].

Time Of Arrival Localization (TOA)

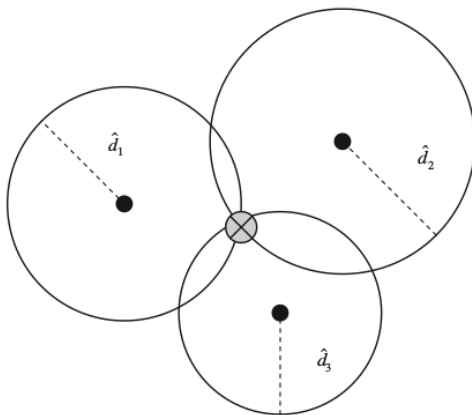


Figure: TOA-based trilateration. Range measurements to at least three BS make up a set of nonlinear equations that can be solved to estimate the position of a signal source [GeoLoc].

Time-Difference Of Arrival Localization (TDOA)

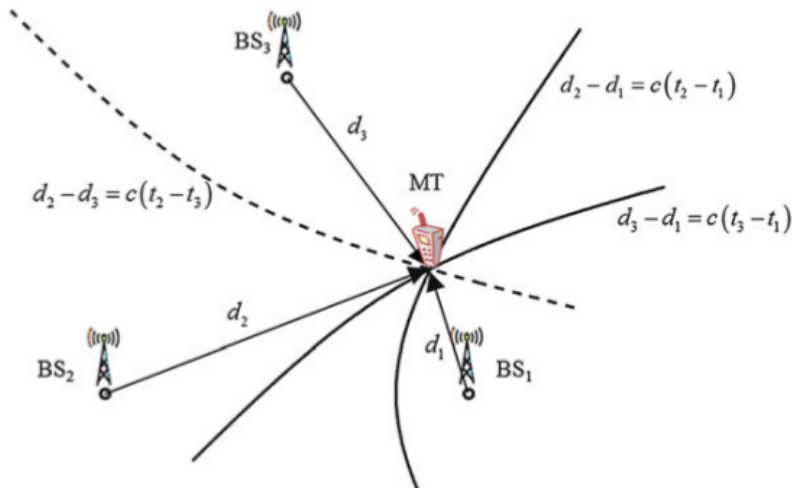


Figure: Example of observed time-difference of arrival (O-TDOA) method [GeoLoc].

Why Least Squares

- Least squares (LS) algorithms for range-based localization:
 - geometrically meaningful
 - provide low complexity solutions with competitive accuracy
- However:
 - the error measure is non-convex
 - excludes many local methods, that are iterative
- Solutions obtained using global localization techniques such as semidefinite programming (SDP) are not optimal in LS sense.

Iterative Re-Weighting Least-Squares Methods for Source Localization

Iterative Re-Weighting Least-Squares Methods for Source Localization

- Methods developed by A. Beck, P. Stoica, J. Li [BSL2008] for *squared* range LS (SR-LS) and *squared* range difference LS (SDR-LS) problems allow us to obtain exact and *global* solutions.
- The results produced are merely approximations of the original LS problems because SR-LS and SDR-LS are no longer ML solutions.
- Proposed iterative procedure where the SR-LS (or SDR-LS) algorithm is applied to a *weighted* sum of squared terms and special weights construction allow to obtain a solution which is considerably closer to the original range-based (or range-difference-based) LS solution.

Source Localization From Range Measurements

Measurement Model

- Throughout it is assumed that *range measurements* obey the model

$$r_i = \|\mathbf{x} - \mathbf{a}_i\| + \varepsilon_i, \quad i = 1, \dots, m.$$

where $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ - given array of m sensors;

$\mathbf{a}_i \in R^n$ contains n coordinates of the i th sensor in space R^n ;

r_i - received noisy distance reading from the i th sensor;

ε_i - unknown noise associated with measurement from the i th sensor.

- The problem can be stated as to estimate the exact source location $\mathbf{x} \in R^n$ from noisy range measurements $\mathbf{r} = [r_1 \ r_2 \ \dots \ r_m]^T$.

Source Localization From Range Measurements

LS Formulations

- The range-based least squares (R-LS) estimate refers to the solution of the problem

$$\underset{\mathbf{x}}{\text{minimize}} f(\mathbf{x}) = \sum_{i=1}^m (r_i - \|\mathbf{x} - \mathbf{a}_i\|)^2 \quad (\text{R})$$

- If $\varepsilon \sim N(0, \mathbf{\Sigma})$ and $\mathbf{\Sigma} \propto \mathbf{I}$, then the R-LS solution of problem (R) is identical to the ML location estimator.
- The objective in (R) is highly non-convex with many local minimizers even for small-scale systems.

Source Localization From Range Measurements

LS Formulations

- Alternatively, location estimate can be obtained by solving the *squared range based LS* (SR-LS) problem [BSL2008]

$$\underset{\mathbf{x}}{\text{minimize}} \sum_{i=1}^m (\|\mathbf{x} - \mathbf{a}_i\|^2 - r_i^2)^2 \quad (\text{SR})$$

- The SR-LS estimate is no longer an ML solution.
- To reduce the gap between the two solutions we propose a weighted SR-LS (WSR-LS) problem:

$$\underset{\mathbf{x}}{\text{minimize}} \sum_{i=1}^m w_i (\|\mathbf{x} - \mathbf{a}_i\|^2 - r_i^2)^2 \quad (\text{WSR})$$

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- The main idea is to use the weights $w_i, i = 1, \dots, m$ to tune the objective in (WSR) toward the objective in (R).

$$\underbrace{w_i (\|\mathbf{x} - \mathbf{a}_i\|^2 - r_i^2)^2}_{\text{in (WSR)}} \leftrightarrow \underbrace{(\|\mathbf{x} - \mathbf{a}_i\| - r_i)^2}_{\text{in (R)}}$$

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- By writing the i th term in (WSR) as

$$w_i (\|\mathbf{x} - \mathbf{a}_i\|^2 - r_i^2)^2 = w_i (\|\mathbf{x} - \mathbf{a}_i\| + r_i)^2 \underbrace{(\|\mathbf{x} - \mathbf{a}_i\| - r_i)^2}_{\text{same as in (R)}}$$

we note that the objective in (WSR) would be the same as in (R) if the weight w_i was assigned to $1/(\|\mathbf{x} - \mathbf{a}_i\| + r_i)^2$.

- Evidently, such weight assignments cannot be realized.

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- We solve a weighted SR-LS sub-problem, where at each iteration the weights are fixed:

$$\underset{\mathbf{x}}{\text{minimize}} \sum_{i=1}^m w_i^{(k)} (\|\mathbf{x} - \mathbf{a}_i\|^2 - r_i^2)^2 \quad (\text{IRWSR})$$

- for $k = 1$ all weights $\{w_i^{(1)}, i = 1, \dots, m\}$ are set to unity;
- for $k \geq 2$ the weights $\{w_i^{(k)}, i = 1, \dots, m\}$ are assigned using the previous iterate \mathbf{x}_{k-1} as

$$w_i^{(k)} = \frac{1}{(\|\mathbf{x}_{k-1} - \mathbf{a}_i\| + r_i)^2}.$$

Source Localization From Range-Difference Measurements

Problem Statement

- It is assumed that the range-difference measurements obey the model:

$$d_i = \|\mathbf{x} - \mathbf{a}_i\| - \|\mathbf{x} - \mathbf{a}_0\| = \|\mathbf{x} - \mathbf{a}_i\| - \|\mathbf{x}\|, \quad i = 1, \dots, m$$

where \mathbf{a}_0 - reference sensor placed at the origin.

- The standard range-difference LS (RD-LS) problem is formulated as

$$\underset{\mathbf{x} \in R^n}{\text{minimize}} F(\mathbf{x}) = \sum_{i=1}^m (d_i + \|\mathbf{x}\| - \|\mathbf{x} - \mathbf{a}_i\|)^2 \quad (\text{RD})$$

Source Localization From Range-Difference Measurements

SRD-LS and WSRD-LS formulations

- An approximation of the RD-LS solution can be obtained by solving the *squared range difference based LS* (SRD-LS) problem.
- We re-write the measurements model as $d_i + \|\mathbf{x}\| = \|\mathbf{x} - \mathbf{a}_i\|$ and square both sides to obtain

$$-2d_i\|\mathbf{x}\| - 2\mathbf{a}_i^T \mathbf{x} = g_i, \quad i = 1, \dots, m$$

where $g_i = d_i^2 - \|\mathbf{a}_i\|^2$. The SRD-LS solution can be found as

$$\underset{\mathbf{x} \in \mathbb{R}^n}{\text{minimize}} \sum_{i=1}^m \left(-2\mathbf{a}_i^T \mathbf{x} - 2d_i\|\mathbf{x}\| - g_i \right)^2$$

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- We consider the weighted SRD-LS problem

$$\underset{\mathbf{x} \in \mathbb{R}^n}{\text{minimize}} \sum_{i=1}^m w_i \left(-2\mathbf{a}_i^T \mathbf{x} - 2d_i \|\mathbf{x}\| - g_i \right)^2 \quad (\text{WSRD})$$

where weights w_i for $i = 1, \dots, m$ are *fixed* nonnegative constants.

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- The i th term of the objective function in (WSRD) can be written as:

$$\begin{aligned} & w_i \left(-2d_i \|\mathbf{x}\| - 2\mathbf{a}_i^T \mathbf{x} - g_i \right)^2 \\ &= w_i (d_i + \|\mathbf{x}\| + \|\mathbf{x} - \mathbf{a}_i\|) \underbrace{(d_i + \|\mathbf{x}\| - \|\mathbf{x} - \mathbf{a}_i\|)}_{\text{same as in RD}} \end{aligned}$$

- If weights w_i were set to $1 / (d_i + \|\mathbf{x}\| + \|\mathbf{x} - \mathbf{a}_i\|)^2$ the objective in (WSRD) would be the same as in (RD).

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- We employ an iterative procedure where the weights in the k th iteration are assigned to

$$w_i^{(k)} = \frac{1}{(d_i + \|\mathbf{x}_{k-1}\| + \|\mathbf{x}_{k-1} - \mathbf{a}_i\|)^2}, i = 1, \dots, m$$

with $\{w_i^{(1)} = 1, i = 1, \dots, m\}$.

- We will refer to the derived problem as the iterative re-weighted SRD-LS (WSRD-LS) problem and the solution obtained as IRWSRD-LS solution.

Performance Evaluation for SR-LS and IRWSR-LS

Table: Averaged MSE for SR-LS and IRWSR-LS methods by noise level

σ	SR - LS	IRWSR-LS	Improvement (%)
1e-03	2.03251062e-06	1.19962894e-06	41
1e-02	1.83717590e-04	1.24797437e-04	32
1e-01	1.83611315e-02	1.22233840e-02	33

Performance Evaluation for SRD-LS and IRWSRD-LS

Table: Averaged MSE for SRD-LS and IRWSRD-LS methods by noise level

σ	SRD - LS	IRWSRD-LS	Improvement (%)
1e-04	1.38301598e-08	8.22705918e-09	40
1e-03	1.60398717e-06	1.03880406e-06	35
1e-02	1.11632818e-04	6.67785604e-05	40
1e-01	1.20947651e-02	7.20891487e-03	40
1e+0	1.57050323e+00	9.70756420e-01	40

Penalty Convex-Concave Procedure for Source Localization

Penalty Convex-Concave Procedure for Source Localization

- We frame the localization problem as difference-of-convex-functions (DC) program.
- Proposed formulation:
 - based on a penalty convex-concave procedure (PCCP)
 - accepts infeasible initial points
 - additional constraints that enforce the algorithm's iteration path towards the LS solution
 - strategies to secure good initial points

Measurement Model

- The *range measurements* model is assumed to be given by

$$r_i = \|\mathbf{x} - \mathbf{a}_i\| + \varepsilon_i, \quad i = 1, \dots, m.$$

$\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ - given array of m sensors;

r_i - received noisy distance reading from sensor i ;

ε_i - unknown noise associated with measurement from the i th sensor.

- The range-based least squares estimate refers to the solution of

$$\underset{\mathbf{x}}{\text{minimize}} F(\mathbf{x}) = \sum_{i=1}^m (r_i - \|\mathbf{x} - \mathbf{a}_i\|)^2 \quad (\text{R})$$

Basic Convex-Concave Procedure (CCP)

- The CCP finds local optima of *nonconvex* problems of the form

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & f(\mathbf{x}) - g(\mathbf{x}) \\ \text{subject to:} & f_i(\mathbf{x}) \leq g_i(\mathbf{x}) \quad \text{for: } i = 1, 2, \dots, m\end{array}$$

where $f(\mathbf{x}), g(\mathbf{x}), f_i(\mathbf{x}), g_i(\mathbf{x})$ for $i = 1, 2, \dots, m$ are convex.

- It is a descent algorithm that requires a *feasible* initial point \mathbf{x}_0 , i.e. $f_i(\mathbf{x}) - g_i(\mathbf{x}) \leq 0$ for $i = 1, 2, \dots, m$.

Basic Convex-Concave Procedure (CCP)

- The basic CCP algorithm is an iterative procedure including two key steps (in the k -th iteration):

① Convexify: form $\hat{g}(\mathbf{x}, \mathbf{x}_k) = g(\mathbf{x}_k) + \nabla g(\mathbf{x}_k)^T(\mathbf{x} - \mathbf{x}_k)$

$$\text{and } \hat{g}_i(\mathbf{x}, \mathbf{x}_k) = g_i(\mathbf{x}_k) + \nabla g_i(\mathbf{x}_k)^T(\mathbf{x} - \mathbf{x}_k) \\ \text{for } i = 1, 2, \dots, m$$

② Solve the convex problem:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && f(\mathbf{x}) - \hat{g}(\mathbf{x}, \mathbf{x}_k) \\ & \text{subject to:} && f_i(\mathbf{x}) - \hat{g}_i(\mathbf{x}, \mathbf{x}_k) \leq 0 \\ & && \text{for: } i = 1, 2, \dots, m \end{aligned}$$

An example of the basic CCP procedure

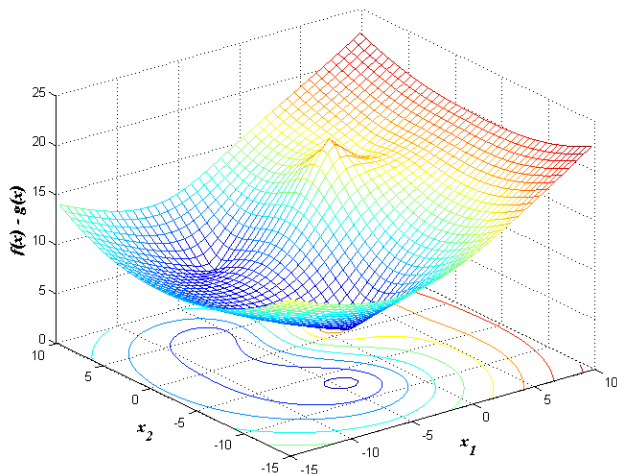


Figure: A nonconvex function in the form of the difference of two convex functions and its contour plot.

An example of the basic CCP procedure

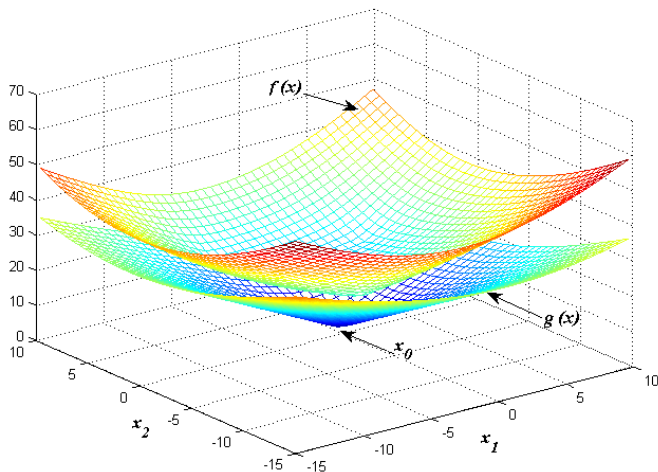


Figure: Separation of the nonconvex function into two convex functions.

An example of the basic CCP procedure

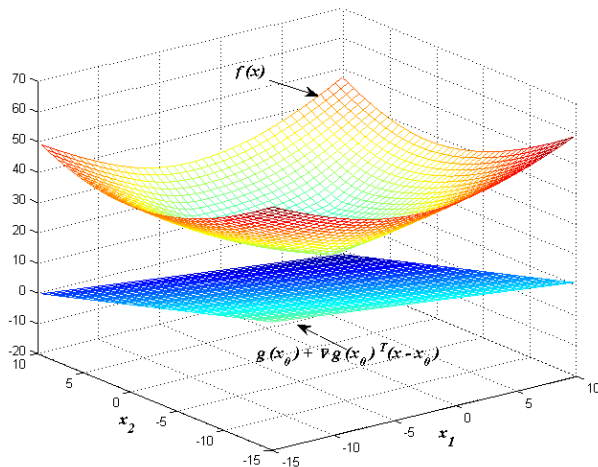


Figure: First order approximation of $g(x)$.

An example of the basic CCP procedure

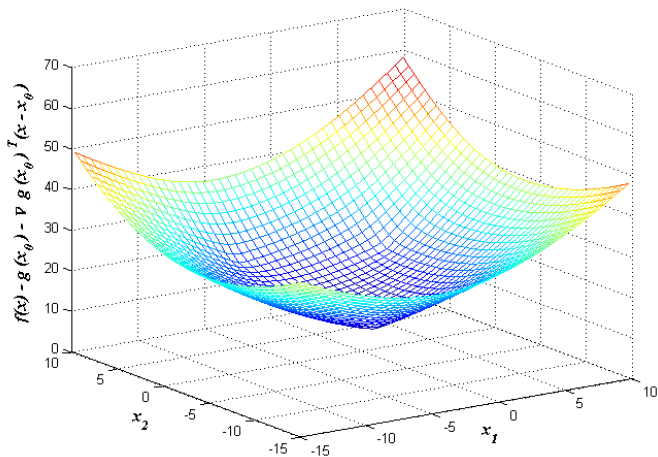


Figure: A convex approximation of the original nonconvex function at $x_0 = (0, 0)$.

Range-Based Localization Revisited

- The range-based least squares (R-LS) estimate:

$$\underset{\mathbf{x}}{\text{minimize}} F(\mathbf{x}) = \sum_{i=1}^m (r_i - \|\mathbf{x} - \mathbf{a}_i\|)^2 \quad (\text{R})$$

Problem Reformulation

- We begin by re-writing the objective $F(\mathbf{x})$ up to a constant as:

$$\sum_{i=1}^m (r_i - \|\mathbf{x} - \mathbf{a}_i\|)^2 = m\mathbf{x}^T \mathbf{x} - 2\mathbf{x}^T \sum_{i=1}^m \mathbf{a}_i - 2 \sum_{i=1}^m r_i \|\mathbf{x} - \mathbf{a}_i\|$$

which allows to formulate it in a basic CCP form $F(\mathbf{x}) = f(\mathbf{x}) - g(\mathbf{x})$ with

$$f(\mathbf{x}) = m\mathbf{x}^T \mathbf{x} - 2\mathbf{x}^T \sum_{i=1}^m \mathbf{a}_i \quad - \text{convex}$$

$$g(\mathbf{x}) = 2 \sum_{i=1}^m r_i \|\mathbf{x} - \mathbf{a}_i\| \quad - \text{convex.}$$

Problem Reformulation

- Since $g(\mathbf{x})$ is not differentiable at the point where $\mathbf{x} = \mathbf{a}_i$ for some $1 \leq i \leq m$, we replace $\nabla g(\mathbf{x}_k)$ by a subgradient of $g(\mathbf{x})$ at \mathbf{x}_k as

$$\partial g(\mathbf{x}_k) = 2 \sum_{i=1}^m r_i \partial \|\mathbf{x}_k - \mathbf{a}_i\|$$

where

$$\partial \|\mathbf{x}_k - \mathbf{a}_i\| = \begin{cases} \frac{\mathbf{x}_k - \mathbf{a}_i}{\|\mathbf{x}_k - \mathbf{a}_i\|}, & \text{if } \mathbf{x}_k \neq \mathbf{a}_i \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

Problem Reformulation

- Up to a multiplicative factor $1/m$ and an additive constant term the objective in (R) can be written as

$$\underset{\mathbf{x}}{\text{minimize}} \quad \hat{F}(\mathbf{x}) = \mathbf{x}^T \mathbf{x} - 2\mathbf{x}^T \mathbf{v}_k$$

where

$$\mathbf{v}_k = \bar{\mathbf{a}} + \frac{1}{m} \sum_{i=1}^m r_i \partial \|\mathbf{x}_k - \mathbf{a}_i\|, \quad \bar{\mathbf{a}} = \frac{1}{m} \sum_{i=1}^m \mathbf{a}_i$$

- Given \mathbf{x}_k (in the k -th iteration) the solution of the quadratic problem can be obtained as

$$\mathbf{x}_{k+1} = \bar{\mathbf{a}} + \frac{1}{m} \sum_{i=1}^m r_i \partial \|\mathbf{x}_k - \mathbf{a}_i\|$$

Imposing Error Bounds

- The algorithm can be enhanced by imposing a bound on each squared measurement error

$$(\|\mathbf{x} - \mathbf{a}_i\| - r_i)^2 \leq \delta_i^2$$

which leads to

$$\|\mathbf{x} - \mathbf{a}_i\| - r_i - \delta_i \leq 0 \quad (C1)$$

$$r_i - \delta_i \leq \|\mathbf{x} - \mathbf{a}_i\|, \quad \text{for } 1 \leq i \leq m. \quad (C2)$$

Both sets of constraints can be written in a form $f_i(\mathbf{x}) \leq g_i(\mathbf{x})$.

- Constraints in (C1) are convex, with $f_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{a}_i\| - r_i - \delta_i$, and $g_i(\mathbf{x}) = 0$.

Imposing Error Bounds

- In case of (C2): define $f_i(\mathbf{x}) = r_i - \delta_i$ and $g_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{a}_i\|$.
Replace $g_i(\mathbf{x})$ with its approximation

$$\hat{g}_i(\mathbf{x}, \mathbf{x}_k) = \|\mathbf{x}_k - \mathbf{a}_i\| + \partial\|\mathbf{x}_k - \mathbf{a}_i\|^T (\mathbf{x} - \mathbf{x}_k)$$

This allows to convexify constraints $r_i - \delta_i \leq \|\mathbf{x} - \mathbf{a}_i\|$ as

$$-\|\mathbf{x}_k - \mathbf{a}_i\| - \partial\|\mathbf{x}_k - \mathbf{a}_i\|^T (\mathbf{x} - \mathbf{x}_k) + r_i - \delta_i \leq 0$$

- Summarizing, the problem in the k -th iteration can be stated as

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^T \mathbf{x} - 2\mathbf{x}^T \mathbf{v}_k \\ & \text{subject to:} && \|\mathbf{x} - \mathbf{a}_i\| - r_i - \delta_i \leq 0 \end{aligned}$$

$$-\|\mathbf{x}_k - \mathbf{a}_i\| - \partial\|\mathbf{x}_k - \mathbf{a}_i\|^T (\mathbf{x} - \mathbf{x}_k) + r_i - \delta_i \leq 0$$

Penalty CCP (PCCP)

- Technical problem: the formulation requires a feasible initial point \mathbf{x}_0 .
- Solution approach: allow *infeasible* initial points by introducing slack variables $s_i \geq 0, \hat{s}_i \geq 0, 1 \leq i \leq m$ into constraints (C1) and (C2) and penalizing the sum of violations.
- This leads to a *penalty* CCP:

$$\begin{aligned} & \underset{\mathbf{x}, \mathbf{s}, \hat{\mathbf{s}}}{\text{minimize}} && \mathbf{x}^T \mathbf{x} - 2\mathbf{x}^T \mathbf{v}_k + \tau_k \sum_{i=1}^m (s_i + \hat{s}_i) \\ & \text{subject to:} && \|\mathbf{x} - \mathbf{a}_i\| - r_i - \delta_i \leq s_i \\ & && -\|\mathbf{x}_k - \mathbf{a}_i\| - \frac{(\mathbf{x}_k - \mathbf{a}_i)^T}{\|\mathbf{x}_k - \mathbf{a}_i\|} (\mathbf{x} - \mathbf{x}_k) + r_i - \delta_i \leq \hat{s}_i \\ & && s_i \geq 0, \hat{s}_i \geq 0, \text{ for: } i = 1, 2, \dots, m \end{aligned}$$

where $0 \leq \tau_k \leq \tau_{\max}$.

The Algorithm: Input parameters

Bound δ_i on the measurement error

- Lower δ_i leads to a “tighter” solution.
- Larger δ_i makes the algorithm less sensitive to outliers.
- If ε obeys a Gaussian distribution with zero mean and $\mathbf{\Sigma} = \text{diag}(\sigma_1^2, \dots, \sigma_m^2)$, then $\delta_i = \gamma\sigma_i$, where γ determines the width of confidence interval.
- For example, for $\gamma = 3$ we have the probability $Pr\{|\varepsilon_i| \leq 3\sigma_i\} \approx 0.99$.

The Algorithm: Input parameters

Initial point x_0

Techniques to select a good initial point:

- select the initial point uniformly randomly over the same region as the unknown source;
- set the initial point to the origin;
- run the algorithm from a set of candidate initial points and identify the solution as the one with lowest LS error;
- apply a *global* localization algorithm to generate an approximate LS solution, then take it as the initial point.

System setup

- Sensors: $\{\mathbf{a}_i, i = 1, 2, \dots, 5\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$
- Source: \mathbf{x}_s , located randomly in $\{\mathbf{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$
- Noise: $\{\varepsilon_i, i = 1, \dots, m\}$ was modelled as i.i.d random variables with zero mean and variance σ^2 , $\sigma \in \{10^{-3}, 10^{-2}, 10^{-1}, 1\}$
- $\gamma = 3$, $K_{max} = 20$

Numerical Results

Table: Averaged MSE for SR-LS and PCCP methods

σ	MLE	SR - LS	PCCP	R.I.
1e-03	6.0159e-01	1.3394e-06	9.5243e-07	29%
1e-02	3.5077e-01	1.4516e-04	9.5831e-05	34%
1e-01	3.7866e-01	1.2058e-02	8.7107e-03	28%
1e+0	1.4470e+00	1.3662e+00	1.2346e+00	10%

Least Squares Localization by Sequential Convex Relaxation

Range-Difference Localization

Problem Statement

- Assumed measurement model:

$$d_i = \|\mathbf{x} - \mathbf{a}_i\| - \|\mathbf{x} - \mathbf{a}_0\| + \varepsilon_i = \|\mathbf{x} - \mathbf{a}_i\| - \|\mathbf{x}\| + \varepsilon_i, \quad i = 1, \dots, m$$

where \mathbf{a}_0 - reference sensor placed at the origin.

- The standard range-difference LS (RD-LS) problem

$$\underset{\mathbf{x} \in \mathbb{R}^n}{\text{minimize}} F(\mathbf{x}) = \sum_{i=1}^m (\|\mathbf{x} - \mathbf{a}_i\| - \|\mathbf{x}\| - d_i)^2 \quad (\text{RD})$$

Sequential Convex Relaxation

- Re-write the unconstrained problem (RD) as a constrained problem

$$\begin{aligned} & \underset{\mathbf{x}, y, \mathbf{z}}{\text{minimize}} && \sum_{i=1}^m (z_i - y - d_i)^2 \\ & \text{subject to:} && \|\mathbf{x} - \mathbf{a}_i\| = z_i, \quad i = 1, 2, \dots, m \\ & && \|\mathbf{x}\| = y \end{aligned}$$

- Assume the k th iterate known $\{\mathbf{x}_k, y_k, \mathbf{z}_k\}$. Let the next iterate be $\{\mathbf{x}_k + \boldsymbol{\delta}_x, y_k + \delta_y, \mathbf{z}_k + \boldsymbol{\delta}_z\}$, i.e. constraints become

$$\begin{aligned} \|\mathbf{x}_k + \boldsymbol{\delta}_x - \mathbf{a}_i\| &\approx z_i^k + \delta_{z_i}, \quad i = 1, 2, \dots, m \\ \|\mathbf{x}_k + \boldsymbol{\delta}_x\| &\approx y_k + \delta_y \end{aligned}$$

Sequential Convex Relaxation

- Replace constraints by their affine approximations

$$\begin{aligned}\|\mathbf{x}_k - \mathbf{a}_i\| + \partial_x^T \|\mathbf{x}_k - \mathbf{a}_i\| \delta_x &\approx z_i^k + \delta_{z_i}, \quad i = 1, 2, \dots, m \\ \|\mathbf{x}_k\| + \partial_x^T \|\mathbf{x}_k\| \delta_x &\approx y_k + \delta_y\end{aligned}$$

- The objective can be written as

$$\begin{aligned}F(\mathbf{x}_{k+1}) &= \sum_{i=1}^m \left(z_i^{(k)} + \delta_{z_i} - (y_k + \delta_y) - d_i \right)^2 \\ &= \sum_{i=1}^m \left(-\delta_y + \delta_{z_i} - \tilde{d}_i^{(k)} \right)^2\end{aligned}$$

$$\text{where } \tilde{d}_i^{(k)} = d_i - y_k - z_i^{(k)}$$

Sequential Convex Relaxation

- In k th iteration we solve the problem

$$\begin{aligned} \underset{\tilde{\boldsymbol{\delta}}}{\text{minimize}} \quad & f(\tilde{\boldsymbol{\delta}}) = \sum_{i=1}^m \left(-\delta_y + \delta_{z_i} - d_i^{(k)} \right)^2 \\ \text{subject to:} \quad & \|\mathbf{x}_k - \mathbf{a}_i\| + \frac{(\mathbf{x}_k - \mathbf{a}_i)^T \boldsymbol{\delta}_x}{\|\mathbf{x}_k - \mathbf{a}_i\|} = z_i^{(k)} + \delta_{z_i}, \\ & i = 1, 2, \dots, m \\ & \|\mathbf{x}_k\| + \frac{\mathbf{x}_k^T \boldsymbol{\delta}_x}{\|\mathbf{x}_k\|} = y_k + \delta_y \\ & \begin{bmatrix} -\beta \mathbf{1}_2 \\ -\beta \\ -\beta \mathbf{1}_m \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{\delta}_x \\ \delta_y \\ \boldsymbol{\delta}_z \end{bmatrix} \leq \begin{bmatrix} \beta \mathbf{1}_2 \\ \beta \\ \beta \mathbf{1}_m \end{bmatrix} \end{aligned}$$

Sequential Convex Relaxation

- Express the problem in a standard form as

$$\begin{array}{ll}\underset{\boldsymbol{\delta}}{\text{minimize}} & f(\tilde{\boldsymbol{\delta}}) \\ \text{subject to} & \mathbf{A}_k \tilde{\boldsymbol{\delta}} = \mathbf{b}_k \\ & \mathbf{C} \tilde{\boldsymbol{\delta}} \leq \mathbf{q}\end{array}$$

- Relax the constraints in order for the problem to be solvable

$$\begin{array}{ll}\underset{\boldsymbol{\delta}}{\text{minimize}} & f(\tilde{\boldsymbol{\delta}}) + \tau \sum_{i=1}^{m+1} (u_i + v_i) + \tau w \\ \text{subject to} & \mathbf{A}_k \tilde{\boldsymbol{\delta}} - \mathbf{b}_k = \mathbf{u} - \mathbf{v} \\ & \mathbf{C} \tilde{\boldsymbol{\delta}} - \mathbf{q} \leq w \mathbf{e} \\ & \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}, w \geq 0\end{array}$$

The Algorithm: Input parameters

- Bound β on the increment vector $\tilde{\delta} = (\delta_x, \delta_y, \delta_z)$.
- The initial point \mathbf{x}_0 .
- Initial weight for penalty terms τ_0 .
- Upper limit of the weight τ_{max} .
- Convergence tolerance ϵ .

System setup

- Sensors: $\{\mathbf{a}_i, i = 1, 2, \dots, 11\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$, $\mathbf{a}_0 = \mathbf{0}$ placed at the origin.
- Source: \mathbf{x}_s , located randomly in $\{\mathbf{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$
- Noise: $\{\varepsilon_i, i = 1, \dots, m\}$ was modelled as i.i.d random variables with zero mean and variance σ^2 , $\sigma \in \{10^{-3}, 10^{-2}, 10^{-1}, 1\}$.
- $\beta = 3$; penalty terms $\tau_0 = 10, \tau_{max} = 10000$.
- Convergence tolerance $\epsilon = 10^{-6}$.

Numerical Results

Table: MSE of position estimation for SRD-LS and SCR-RDLS methods

σ	SRD - LS	SCR-RDLS	(R.I.,%)
1e-03	1.2655e-06	8.4626e-07	33
1e-02	1.4492e-04	6.8385e-05	52
1e-01	1.3329e-02	7.1676e-03	46
1e+0	1.6077e+00	9.5371e-01	40

Range-Based Localization

Measurement Model

- The *range measurements* model

$$r_i = \|\mathbf{x} - \mathbf{a}_i\| + \varepsilon_i, \quad i = 1, \dots, m.$$

$\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ - given array of m sensors;

r_i - received noisy distance reading from sensor i ;

ε_i - unknown noise associated with measurement from the i th sensor.

- The range-based LS estimate refers to the solution of

$$\underset{\mathbf{x}}{\text{minimize}} F(\mathbf{x}) = \sum_{i=1}^m (r_i - \|\mathbf{x} - \mathbf{a}_i\|)^2 \quad (\text{R})$$

Sequential Relaxation

- Equivalent constrained problem

$$\begin{aligned} & \underset{\mathbf{x}, \mathbf{z}}{\text{minimize}} && \sum_i^m (z_i - r_i)^2 \\ & \text{subject to:} && \|\mathbf{x} - \mathbf{a}_i\| = z_i, \quad i = 1, 2, \dots, m \\ & && \mathbf{z} \geq \mathbf{0} \end{aligned}$$

- Relax the constraints

$$\begin{aligned} & \underset{\mathbf{x}, \mathbf{z}}{\text{minimize}} && \sum_i^m (z_i - r_i)^2 \\ & \text{subject to:} && \|\mathbf{x} - \mathbf{a}_i\| \leq (1 + \gamma)z_i \\ & && \|\mathbf{x} - \mathbf{a}_i\| \geq (1 - \gamma)z_i, \quad i = 1, 2, \dots, m \end{aligned}$$

- $\gamma > 0$ is sequentially and monotonically decreasing, $\gamma_0 \in (0, 0.5)$

Sequential Relaxation

- Some constraints are non-convex

$$\begin{aligned} \|\mathbf{x} - \mathbf{a}_i\| &\leq (1 + \gamma)z_i && (\text{convex}) \\ \|\mathbf{x} - \mathbf{a}_i\| &\geq (1 - \gamma)z_i \iff \underbrace{-\|\mathbf{x} - \mathbf{a}_i\|}_{\text{nonconvex}} \leq -(1 - \gamma)z_i && (\text{nonconvex}) \end{aligned}$$

$$i = 1, 2, \dots, m.$$

- Replace non-convex constraints with affine approximation (\mathbf{x}_k is *known*)

$$-\|\mathbf{x}_k - \mathbf{a}_i\| - \partial\|\mathbf{x}_k - \mathbf{a}_i\|^T(\mathbf{x} - \mathbf{x}_k) \leq -(1 - \gamma)z_i$$

Sequential Relaxation

- In the k th iteration (\mathbf{x}_k is known), solve an SOCP problem

$$\underset{\mathbf{x}, \mathbf{z}}{\text{minimize}} \quad \sum_i^m (z_i - r_i)^2$$

$$\text{subject to:} \quad \|\mathbf{x} - \mathbf{a}_i\| \leq (1 + \gamma)z_i$$

$$-\|\mathbf{x}_k - \mathbf{a}_i\| - \partial\|\mathbf{x}_k - \mathbf{a}_i\|^T(\mathbf{x} - \mathbf{x}_k) \leq -(1 - \gamma)z_i$$

$i = 1, 2, \dots, m$. Update γ

$$\gamma_{k+1} = \gamma_0 - k \frac{\gamma_0}{K_{\max} - 1}$$

linearly

$$\gamma_{k+1} = \gamma_0 \frac{(K_{\max} - 1 - k)^2}{(K_{\max} - 1)^2}$$

quadratically

System setup

- Sensors: $\{\mathbf{a}_i, i = 1, 2, \dots, 5\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$.
- Source: \mathbf{x}_s , located randomly in $\{\mathbf{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$
- Noise: $\{\varepsilon_i, i = 1, \dots, m\}$ was modelled as i.i.d random variables with zero mean and variance σ^2 , $\sigma \in \{10^{-3}, 10^{-2}, 10^{-1}, 1\}$.
- Initial relaxation parameter $\gamma_0 = 0.2$
- Number of iterations $K_{max} = 9$.

Table: MSE of position estimation for SR-LS and SCR-RLS methods

σ	SR - LS	SCR-RLS	(R.I.,%)
1e-02	2.5360e-04	2.0596e-04	18
1e-01	1.8696e-02	1.4802e-02	21
1e+0	1.4440e+00	9.6327e-01	33

- New iterative methods for locating a radiating source based on noisy range and range-difference measurements.
- The iterative re-weighting methods are developed by transforming the SR-LS and SRD-LS algorithms [BSL2008] into an iterative procedure so that a weighted SR-LS (SRD-LS) objective asymptotically approaches the original R-LS objective.
- Convex minimization method based on PCCP that can be efficiently solved with an infeasible initial point.
- Proposed algorithms are found to outperform the existing methods.

- Study and mitigation of the influence of sensor geometry on the accuracy of the developed methods (for example, geometric dilution of precision).
- Multiple source localization in wireless sensor networks.

Q & A

Appendix

Performance Comparison of Range-Based Algorithms

Table: CPU time usage

Method	Absolute time, msec.
IRWSR-LS	6.0777e+00
PCCP	5.3790e+03
SCR-RLS	3.7005e+03

Performance Comparison of Range-Based Algorithms

Table: CPU time usage

Method	Relative time
IRWSR-LS	1
PCCP	885
SCR-RLS	609

Performance Comparison of Range-Difference-Based Algorithms

Table: CPU time usage

Method	Absolute time, msec.
IRWSRD-LS	7.4039e+00
SCR-RDLS	3.6347e+03

Performance Comparison of Range-Difference-Based Algorithms

Table: CPU time usage

Method	Relative time
IRWSRD-LS	1
SCR-RDLS	490

Nonconvexity of the R-LS objective

Given the objective

$$F(\mathbf{x}) = \sum_{i=1}^m (r_i - \|\mathbf{x} - \mathbf{a}_i\|)^2$$

its Hessian for points \mathbf{x} that are not coincided with \mathbf{a}_i for $1 \leq i \leq m$, is given by

$$\begin{aligned} \nabla^2 F(\mathbf{x}) = & 2m\mathbf{I} + 2 \sum_{i=1}^m \frac{r_i}{\|\mathbf{x} - \mathbf{a}_i\|^3} \cdot \left((\mathbf{x} - \mathbf{a}_i)(\mathbf{x} - \mathbf{a}_i)^T - \|\mathbf{x} - \mathbf{a}_i\|^2 \mathbf{I} \right) \end{aligned}$$

which is not always positive semidefinite. Hence $F(\mathbf{x})$ is not convex.

Source Localization From Range Measurements

Weighted Squared Range Least Squares Formulation

- Following [BSL2008], we convert (WSR) into a GTRS as

$$\underset{\mathbf{y} \in \mathbb{R}^{n+1}}{\text{minimize}} \|\mathbf{A}_w \mathbf{y} - \mathbf{b}_w\|^2 \quad (1a)$$

$$\text{subject to: } \mathbf{y}^T \mathbf{D} \mathbf{y} + 2\mathbf{f}^T \mathbf{y} = 0 \quad (1b)$$

where $\mathbf{y} = [\mathbf{x}^T \ \alpha]^T$, $\alpha = \|\mathbf{x}\|$, $\mathbf{A}_w = \mathbf{\Gamma} \mathbf{A}$ and $\mathbf{b}_w = \mathbf{\Gamma} \mathbf{b}$ with fixed $\mathbf{\Gamma} = \text{diag}(\sqrt{w_1}, \dots, \sqrt{w_m})$, and

$$\mathbf{A} = \begin{pmatrix} -2\mathbf{a}_1^T & 1 \\ \vdots & \vdots \\ -2\mathbf{a}_m^T & 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} r_1^T - \|\mathbf{a}_1\|^T \\ \vdots \\ r_m^T - \|\mathbf{a}_m\|^T \end{pmatrix} \quad (2)$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}_{n \times n} & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & 0 \end{pmatrix}, \mathbf{f} = \begin{pmatrix} \mathbf{0} \\ -0.5 \end{pmatrix} \quad (3)$$

Source Localization From Range Measurements

The Algorithm

- 1 Input data: Sensor locations $\{\mathbf{a}_i, i = 1, \dots, m\}$, range measurements $\{r_i, i = 1, \dots, m\}$, maximum number of iterations k_{max} and convergence tolerance ζ .
- 2 Generate data set $\mathbf{A}, \mathbf{b}, \mathbf{D}, \mathbf{f}$ using (2) and (3). Set $k = 1, w_i^{(1)} = 1$ for $i = 1, \dots, m$.
- 3 Set $\mathbf{\Gamma}_k = \text{diag} \left(\sqrt{w_1^{(k)}}, \dots, \sqrt{w_m^{(k)}} \right)$, $\mathbf{A}_w = \mathbf{\Gamma}_k \mathbf{A}$ and $\mathbf{b}_w = \mathbf{\Gamma}_k \mathbf{b}$.
- 4 Solve the WSR-LS problem (IRWSR) via (1) to obtain its global solution \mathbf{x}_k .
- 5 If $k = k_{max}$ or $\|\mathbf{x}_k - \mathbf{x}_{k-1}\| < \zeta$, terminate and output \mathbf{x}_k as the solution; otherwise, set $k = k + 1$, update weights $\{w_i^{(k)}, i = 1, \dots, m\}$ and repeat from Step 3).

Source Localization From Range-Difference Measurements

Weighted Squared Range-Difference Least Squares Formulation

- By introducing new variable $\mathbf{y} = [\mathbf{x}^T \|\mathbf{x}\|]^T$ and noticing nonnegativity of the component y_{n+1} problem (WSRD) is converted to

$$\underset{\mathbf{y} \in \mathbb{R}^{n+1}}{\text{minimize}} \|\mathbf{B}_w \mathbf{y} - \mathbf{g}_w\| \quad (4a)$$

$$\text{subject to: } \mathbf{y}^T \mathbf{C} \mathbf{y} = 0 \quad (4b)$$

$$y_{n+1} \geq 0 \quad (4c)$$

- where $\mathbf{B}_w = \mathbf{\Gamma} \mathbf{B}$, $\mathbf{g}_w = \mathbf{\Gamma} \mathbf{g}$, $\mathbf{\Gamma} = \text{diag}\{\sqrt{w_1}, \dots, \sqrt{w_m}\}$, $\mathbf{g} = [g_1 \dots g_m]^T$ and

$$\mathbf{B} = \begin{pmatrix} -2\mathbf{a}_1^T & -2d_1 \\ \vdots & \vdots \\ -2\mathbf{a}_m^T & -2d_m \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & -1 \end{pmatrix} \quad (5)$$

Source Localization From Range Difference Measurements

The Algorithm

- 1 Input data: Sensor locations $\{\mathbf{a}_i, i = 0, 1, \dots, m\}$ with $\mathbf{a}_0 = \mathbf{0}$, range-difference measurements $\{d_i, i = 1, \dots, m\}$, maximum number of iterations k_{max} and convergence tolerance ξ .
- 2 Generate data set $\{\mathbf{B}, \mathbf{g}, \mathbf{C}\}$ using (5). Set $k = 1$, $w_i^{(1)} = 1$ for $i = 1, \dots, m$.
- 3 Set $\mathbf{\Gamma}_k = \text{diag}\left(\sqrt{w_1^{(k)}}, \dots, \sqrt{w_m^{(k)}}\right)$, $\mathbf{B}_w = \mathbf{\Gamma}_k \mathbf{B}$ and $\mathbf{g}_w = \mathbf{\Gamma}_k \mathbf{g}$.
- 4 Solve WSRD-LS problem (4) to obtain its global solution \mathbf{x}_k .
- 5 If $k = k_{max}$ or $\|\mathbf{x}_k - \mathbf{x}_{k-1}\| < \xi$, terminate and output \mathbf{x}_k as the solution; otherwise, set $k = k + 1$, update weights $\{w_i^{(k)}, i = 1, \dots, m\}$ and repeat from Step 3).

We express the objective in (R) as $F(\mathbf{x}) = f(\mathbf{x}) - g(\mathbf{x})$ with

$$f(\mathbf{x}) = m\mathbf{x}^T \mathbf{x} - 2\mathbf{x}^T \sum_{i=1}^m \mathbf{a}_i \quad \text{and} \quad g(\mathbf{x}) = 2 \sum_{i=1}^m r_i \|\mathbf{x} - \mathbf{a}_i\|$$

Then, we replace $\nabla g(\mathbf{x}_k)$ by a subgradient of $g(\mathbf{x})$ at \mathbf{x}_k :

$$\partial g(\mathbf{x}_k) = 2 \sum_{i=1}^m r_i \partial \|\mathbf{x}_k - \mathbf{a}_i\|,$$

where

$$\partial \|\mathbf{x}_k - \mathbf{a}_i\| = \begin{cases} \frac{\mathbf{x}_k - \mathbf{a}_i}{\|\mathbf{x}_k - \mathbf{a}_i\|}, & \text{if } \mathbf{x}_k \neq \mathbf{a}_i \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

Hence $\hat{g}(\mathbf{x}, \mathbf{x}_k)$ can be formed as:

$$\begin{aligned}\hat{g}(\mathbf{x}, \mathbf{x}_k) &= g(\mathbf{x}_k) + \nabla g(\mathbf{x}_k)^T (\mathbf{x} - \mathbf{x}_k) \\ &= 2 \sum_{i=1}^m r_i \|\mathbf{x}_k - \mathbf{a}_i\| + 2(\mathbf{x} - \mathbf{x}_k)^T \sum_{i=1}^m r_i \partial \|\mathbf{x}_k - \mathbf{a}_i\| \\ &= 2\mathbf{x}^T \sum_{i=1}^m r_i \partial \|\mathbf{x}_k - \mathbf{a}_i\| + c\end{aligned}$$

where c is a constant given by

$$c = -2 \sum_{i=1}^m r_i \mathbf{a}_i^T \partial \|\mathbf{x}_k - \mathbf{a}_i\|.$$

PCCP-based LS Algorithm for Source Localization

Step 1: Input sensor locations $\{\mathbf{a}_i, i = 1, \dots, m\}$, range measurements $\{r_i, i = 1, \dots, m\}$, $\mathbf{x}_0, K_{max}, \tau_0, \tau_{max}, \mu > 0, \gamma, \sigma$, and set $k = 0$.

Step 2: Form \mathbf{v}_k and solve PCCP. Denote the solution as $(\mathbf{s}^*, \hat{\mathbf{s}}^*, \mathbf{x}^*)$.

Step 3: Update $\tau_{k+1} = \min(\mu\tau_k, \tau_{max})$, set $k = k + 1$.

Step 4: If $k = K_{max}$, terminate and output \mathbf{x}^* as the solution; otherwise, set $\mathbf{x}_k = \mathbf{x}^*$ and repeat from Step 2.

Sequential Convex Relaxation for Range-Difference Localization

Step 1: Input data:

- sensor locations $\{\mathbf{a}_i, i = 1, \dots, m\}$,
- range-difference measurements $\{d_i, i = 1, \dots, m\}$,
- initial point \mathbf{x}_0 ,
- initial weight τ_0 and upper limit of weight τ_{max} ,
- increment bound β
- convergence tolerance ϵ . Set iteration count to $k = 0$.

Form \mathbf{S} , \mathbf{C} and \mathbf{q} as

$$\mathbf{S} = [\mathbf{0}_{m \times 1} \quad -\mathbf{1}_{m \times 1} \quad -\mathbf{I}_m], \mathbf{C} = \begin{bmatrix} \mathbf{I}_{m+3} \\ -\mathbf{I}_{m+3} \end{bmatrix}, \mathbf{q} = \beta \mathbf{e}$$

Sequential Convex Relaxation for Range-Difference Localization I

Step 2: Form y_k and \mathbf{z}_k as $y_k = \|\mathbf{x}_k\|$, $\mathbf{z}_k = \begin{bmatrix} \|\mathbf{x}_k - \mathbf{a}_1\| \\ \vdots \\ \|\mathbf{x}_k - \mathbf{a}_m\| \end{bmatrix}$

Form $\mathbf{A}_k, \tilde{\mathbf{d}}_k, \mathbf{b}_k, \mathbf{C}_k$ and solve

$$\begin{aligned} & \underset{\tilde{\boldsymbol{\delta}}}{\text{minimize}} && f(\tilde{\boldsymbol{\delta}}) + \tau_k \sum_{i=1}^{m+1} (u_i + v_i) + \tau_k w \\ & \text{subject to} && \mathbf{A}_k \tilde{\boldsymbol{\delta}} - \mathbf{b}_k = \mathbf{u} - \mathbf{v} \\ & && \mathbf{C} \tilde{\boldsymbol{\delta}} - \mathbf{q} \leq w \mathbf{e} \\ & && \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}, w \geq 0 \end{aligned}$$

Denote the solution as $\tilde{\boldsymbol{\delta}}_k = (\delta_x^*, \delta_y^*, \delta_z^*)$.

Sequential Convex Relaxation for Range-Difference Localization II

Step 3: Update $\tau_{k+1} = \min(1.5\tau_k, \tau_{max})$, set $k = k + 1$. Update $\tilde{\mathbf{x}}^*$ to

$$\mathbf{x}^* = \mathbf{x}^k + \delta_x^*$$

$$y^* = y^k + \delta_y^*$$

$$\mathbf{z}^* = \mathbf{z}^k + \delta_z^*$$

Step 4: If $\|\tilde{\delta}_k\| \leq \epsilon$, terminate and output \mathbf{x}^* as the solution; otherwise, set $\tilde{\mathbf{x}}_k = \mathbf{x}^*$ and repeat from Step 2.

Sequential Convex Relaxation for Range-Based Localization I

Step 1: Input data:

- sensor locations $\{\mathbf{a}_i, i = 1, \dots, m\}$,
- range measurements $\{r_i, i = 1, \dots, m\}$,
- initial point \mathbf{x}_0 , initial relaxation parameter γ_0 ,
- the number of iterations to be executed K_{max} .

Set iteration count to $k = 0$.

Sequential Convex Relaxation for Range-Based Localization II

Step 2: Solve

$$\begin{aligned} & \underset{\mathbf{x}, \mathbf{z}}{\text{minimize}} && \sum_i^m (z_i - r_i)^2 \\ & \text{subject to:} && \|\mathbf{x} - \mathbf{a}_i\| \leq (1 + \gamma)z_i \end{aligned}$$




$$-\|\mathbf{x}_k - \mathbf{a}_i\| - \partial\|\mathbf{x}_k - \mathbf{a}_i\|^T(\mathbf{x} - \mathbf{a}_i) \leq -(1 - \gamma)z_i, \quad i = 1, 2, \dots, m$$

Denote the solution as $\tilde{\mathbf{x}}_k = (\mathbf{x}^*, \mathbf{z}^*)$.

Step 3: Update $\gamma_{k+1} = f(\gamma_k)$ linearly or quadratically. Set $k = k + 1$.

Step 4: If $k = K_{\max}$, terminate and output \mathbf{x}^* as the solution; otherwise, set $\mathbf{x}_k = \mathbf{x}^*$ and repeat from Step 2.

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