## Another Approach to the Problem in (4.7)

$$\begin{aligned} &\underset{\boldsymbol{x},y,\boldsymbol{z}}{\text{minimize}} && \sum_{i=1}^{m} (z_i - y - d_i)^2 \\ &\text{subject to:} && \| \boldsymbol{x} - \boldsymbol{a}_i \| = z_i, \quad i = 1, 2, \ldots, m \\ & && \| \boldsymbol{x} \| = y \end{aligned} \tag{4.7a-c}$$

In the $k$th iteration, the $k$th iterate $\{\boldsymbol{x}_k, y_k, \boldsymbol{z}_k\}$. Let

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{x}_{k+1} + \boldsymbol{\delta}_x \\ y_{k+1} &= y_k + \delta_y \\ \boldsymbol{z}_{k+1} &= \boldsymbol{z}_k + \boldsymbol{\delta}_z \end{aligned} \tag{4.8a-c}$$

where $\{\boldsymbol{\delta}_x, \delta_y, \boldsymbol{\delta}_z\}$ are such that the constraints in (4.7b) and (4.7c) are better approximated at $\{\boldsymbol{x}_{k+1}, y_{k+1}, \boldsymbol{z}_{k+1}\}$ in the sense that

$$\| \boldsymbol{x}_{k+1} - \boldsymbol{a}_i \| \approx z_i^{(k+1)}, \quad i = 1, 2, \ldots, m$$

$$\| \boldsymbol{x}_{k+1} \| \approx y_{k+1}$$

namely,

$$\| \boldsymbol{x}_k + \boldsymbol{\delta}_x - \boldsymbol{a}_i \| \approx z_i^{(k)} + \delta_{z_i}, \quad i = 1, 2, \ldots, m$$

$$\| \boldsymbol{x}_k + \boldsymbol{\delta}_x \| \approx y_k + \delta_y$$

By replacing the left-hand sides of the above equations with their first-order Taylor approximations, we obtain

$$\| \boldsymbol{x}_k - \boldsymbol{a}_i \| + \partial^T \| \boldsymbol{x}_k - \boldsymbol{a}_i \| \boldsymbol{\delta}_x \approx z_i^{(k)} + \delta_{z_i}, \quad i = 1, 2, \ldots, m$$

$$\| \boldsymbol{x}_k \| + \partial_x^T \| \boldsymbol{x}_k \| \boldsymbol{\delta}_x \approx y_k + \delta_y$$

where $\partial_x$ is the subdifferential operator with respect to variable $\boldsymbol{x}$. Assuming $\boldsymbol{x}_k \neq \boldsymbol{a}_i$ and $\boldsymbol{x}_k$ is nonzero, then

$$\partial_x \| \boldsymbol{x}_k - \boldsymbol{a}_i \| = \frac{\boldsymbol{e}}{\| \boldsymbol{x}_k - \boldsymbol{a}_i \|} \quad \text{and} \quad \partial_x \| \boldsymbol{x}_k \| = \frac{\boldsymbol{e}}{\| \boldsymbol{x}_k \|}$$

where $\boldsymbol{e}$ is the all-one vector. Hence

$$\| \boldsymbol{x}_k - \boldsymbol{a}_i \| + \frac{\boldsymbol{e}^T \boldsymbol{\delta}_x}{\| \boldsymbol{x}_k \|} \approx z_i^{(k)} + \delta_{z_i}, \quad i = 1, 2, \ldots, m \qquad (4.9\text{a})_{\text{new}}$$

$$\| \boldsymbol{x}_k \| + \frac{\boldsymbol{e}^T \boldsymbol{\delta}_x}{\| \boldsymbol{x}_k \|} \approx y_k + \delta_y \qquad (4.9\text{b})_{\text{new}}$$

Based *aaa* on this, the problem to be solved in the *k*th iteration is formulated as

$$\begin{aligned} \underset{\tilde{\delta}}{\text{minimize}} \quad & f(\tilde{\boldsymbol{\delta}}) = \sum_{i=1}^{m} (-\delta_y + \delta_{z_i} - \tilde{d}_i^{(k)})^2 \\ \text{subject to:} \quad & \| \boldsymbol{x}_k - \boldsymbol{a}_i \| + \frac{\boldsymbol{e}^T \boldsymbol{\delta}_x}{\| \boldsymbol{x}_k \|} = z_i^{(k)} + \delta_{z_i}, \quad i = 1, 2, \ldots, m \\ & \| \boldsymbol{x}_k \| + \frac{\boldsymbol{e}^T \boldsymbol{\delta}_x}{\| \boldsymbol{x}_k \|} = y_k + \delta_y \\ & \begin{bmatrix} -\beta \mathbf{1}_2 \\ -\min\{\beta, y_k\} \\ -\min\{\beta, \boldsymbol{z}_k\} \end{bmatrix} \le \begin{bmatrix} \boldsymbol{\delta}_x \\ \delta_y \\ \boldsymbol{\delta}_z \end{bmatrix} \le \begin{bmatrix} \beta \mathbf{1}_2 \\ \beta \\ \beta \mathbf{1}_m \end{bmatrix} \end{aligned} \qquad (4.10\text{a-d})_{\text{new}}$$

The constraints in (4.10d) not only assure that the magnitude of each component in $\{\boldsymbol{\delta}_x, \delta_y, \boldsymbol{\delta}_z\}$ is no greater than $\beta$, but also they assure that all components of $\{y_{k+1}, \boldsymbol{z}_{k+1}\}$ are nonnegative as long as $\{y_k, \boldsymbol{z}_k\}$ are nonnegative, which are natural to impose as can be seen from (4.7b) and (4.7c) because they are vector norms. Obviously, the problem in (4.10) is a convex QP problem. One technical difficulty that may occur in solving problem (4.10) is that the feasible region defined by (4.10b), (4.10c), and (4.10d) may be empty. In such a case the constraints in problem (4.10) much be adequately relaxed in order for the problem to be solvable. To this end we rewrite (4.10) as

$$\begin{aligned} \underset{\tilde{\delta}}{\text{minimize}} \quad & f(\tilde{\boldsymbol{\delta}}) \\ \text{subject to:} \quad & \boldsymbol{A}\tilde{\boldsymbol{\delta}} = \boldsymbol{b} \\ & \boldsymbol{C}\tilde{\boldsymbol{\delta}} \le \boldsymbol{d} \end{aligned} \qquad (4.11)_{\text{new}}$$

By introducing nonnegative slack variables ***u***, ***v***, and ***w***, we relax the problem in (4.11) to

$$\begin{aligned} \underset{\tilde{\delta}}{\text{minimize}} \quad & f(\tilde{\boldsymbol{\delta}}) + \tau \sum_{i=1}^{m+1} (u_i + v_i) + \tau \sum_{j=1}^{2(m+3)} w_j \\ \text{subject to:} \quad & \boldsymbol{A}\tilde{\boldsymbol{\delta}} - \boldsymbol{b} = \boldsymbol{u} - \boldsymbol{v} \\ & \boldsymbol{C}\tilde{\boldsymbol{\delta}} - \boldsymbol{d} \le \boldsymbol{w} \\ & \boldsymbol{u} \ge \mathbf{0}, \boldsymbol{v} \ge \mathbf{0}, \boldsymbol{w} \ge \mathbf{0} \end{aligned} \qquad (4.12\text{a-d})_{\text{new}}$$

where $\tau > 0$ is a sufficiently large scalar. It is easy to verify that the feasible region defined by (4.12b) – (4.12d) is always nonempty. For example, if we fix $\tilde{\boldsymbol{\delta}} = \tilde{\boldsymbol{\delta}}_0$ arbitrarily, then obviously the point $\{\tilde{\boldsymbol{\delta}}_0, \boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{w}_0\}$ with

$$\boldsymbol{u}_0 = \max\{\mathbf{0}, \boldsymbol{A}\tilde{\boldsymbol{\delta}}_0 - \boldsymbol{b}\}, \boldsymbol{v}_0 = \max\{\mathbf{0}, -(\boldsymbol{A}\tilde{\boldsymbol{\delta}}_0 - \boldsymbol{b})\}\text{, and } \boldsymbol{w}_0 = \max\{\mathbf{0}, \boldsymbol{C}\tilde{\boldsymbol{\delta}}_0 - \boldsymbol{d}\}$$

Is a feasible point for problem (4.12). The penalty term tries to reduce the magnitudes of the slack variables while minimizing the original objective function $f(\tilde{\boldsymbol{\delta}})$. If the solution slack variables turn out to be all zero, then the solution $\tilde{\boldsymbol{\delta}}$ of (4.12) also solves problem (4.11). Otherwise, we conclude that problem (4.11) in not solvable and the solution obtained by solving (4.12) is a reasonable candidate for the kth iteration to update $\{\boldsymbol{x}_k, y_k, \boldsymbol{z}_k\}$.