Localization Algorithms in Passive Sensor Networks

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Outline

Introduction

- Navigation: outdoor; indoor
- Surveillance
- Localization of emergency callers
- Emergency and rescue operations / first responders
- Self-organizing networks
- Asset monitoring and tracking
- Other commercial location-based services
- ...

Introduction

- Ranging methods
  - range measurements (Time Of Arrival)
  - range-difference measurements (Time-Difference of Arrival)
  - received signal strength
- Angle Of Arrival Techniques
- Survey-Based Systems (fingerprinting)
  - memoryless systems (SVM, NN)
  - memory systems (Bayesian inference, grid-based Markov)
  - channel impulse response fingerprinting non-RF features

Basic Localization Systems and Methods

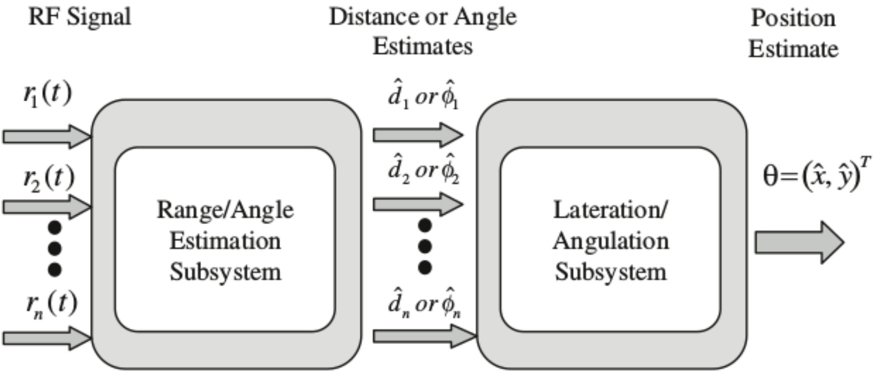

Figure: Classical geolocation system. Range or angle information is extracted from received RF signals. Location is then estimated by lateration/angulation techniques [GeoLoc].

Time Of Arrival Localization (TOA)

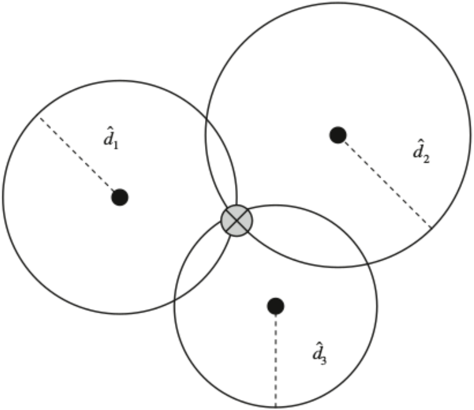

Figure: TOA-based trilateration. Range measurements to at least three BS make up a set of nonlinear equations that can be solved to estimate the position of a signal source [GeoLoc].

Time-Difference Of Arrival Localization (TDOA)

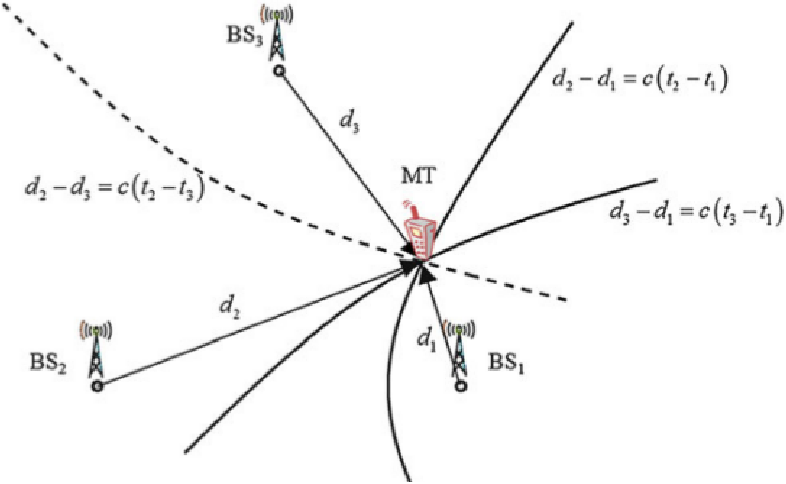

Figure: Example of observed time-difference of arrival (O-TDOA) method [GeoLoc].

Why Least Squares

- Least squares (LS) algorithms for range-based localization:
  - geometrically meaningful
  - provide low complexity solutions with competitive accuracy

- However:
  - the error measure is non-convex
  - excludes many local methods, that are iterative

- Solutions obtained using global localization techniques such as semidefinite programming (SDP) are not optimal in LS sense.

Iterative Re-Weighting Least-Squares Methods for Source Localization

Iterative Re-Weighting Least-Squares Methods for Source Localization

- Methods developed by A. Beck, P. Stoica, J. Li [BSL2008] for *squared* range LS (SR-LS) and *squared* range difference LS (SDR-LS) problems allow us to obtain exact and *global* solutions.

- The results produced are merely approximations of the original LS problems because SR-LS and SRD-LS are no longer ML solutions.

- Proposed iterative procedure where the SR-LS (or SRD-LS) algorithm is applied to a *weighted* sum of squared terms and special weights construction allow to obtain a solution which is considerably closer to the original range-based (or range-difference-based) LS solution.

Source Localization From Range Measurements

Measurement Model

- Throughout it is assumed that *range measurements* obey the model

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m.$$

where $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ - given array of m sensors;
$\boldsymbol{a}_i \in R^n$ contains n coordinates of the ith sensor in space R^n;
r_i - received noisy distance reading from the ith sensor;
ε_i - unknown noise associated with measurement from the ith sensor.

- The problem can be stated as to estimate the exact source location $\boldsymbol{x} \in R^n$ from noisy range measurements $\boldsymbol{r} = [r_1 \; r_2 \ldots r_m]^T$.

Source Localization From Range Measurements

LS Formulations

- The range-based least squares (R-LS) estimate refers to the solution of the problem

$$\underset{\boldsymbol{x}}{\text{minimize}}\, f(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \tag{R}$$

- If $\varepsilon \sim N(0, \boldsymbol{\Sigma})$ and $\boldsymbol{\Sigma} \propto \boldsymbol{I}$, then the R-LS solution of problem (R) is identical to the ML location estimator.

- The objective in (R) is highly non-convex with many local minimizers even for small-scale systems.

LS Formulations

- Alternatively, location estimate can be obtained by solving the *squared range based LS* (SR-LS) problem [BSL2008]

$$\underset{\boldsymbol{x}}{\text{minimize}} \sum_{i=1}^{m} (\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2)^2 \tag{SR}$$

- The SR-LS estimate is no longer an ML solution.
- To reduce the gap between the two solutions we propose a weighted SR-LS (WSR-LS) problem:

$$\underset{\boldsymbol{x}}{\text{minimize}} \sum_{i=1}^{m} w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2\right)^2 \tag{WSR}$$

An Iterative Re-Weighting Strategy

- The main idea is to use the weights $w_i, i = 1, \ldots, m$ to tune the objective in (WSR) toward the objective in (R).

$$\underbrace{w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2\right)^2}_{\text{in (WSR)}} \leftrightarrow \underbrace{\left(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i\right)^2}_{\text{in (R)}}$$

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- By writing the ith term in (WSR) as

$$w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2 \right)^2 = w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\| + r_i \right)^2 \underbrace{\left(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i \right)^2}_{\text{same as in (R)}}$$

  we note that the objective in (WSR) would be the same as in (R) if the weight w_i was assigned to $1/\left(\|\boldsymbol{x} - \boldsymbol{a}_i\| + r_i \right)^2$.

- Evidently, such weight assignments cannot be realized.

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- We solve a weighted SR-LS sub-problem, where at each iteration the weights are fixed:

$$\underset{x}{\text{minimize}} \sum_{i=1}^{m} w_i^{(k)} \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2 \right)^2 \qquad \text{(IRWSR)}$$

- for $k = 1$ all weights $\{w_i^{(1)}, i = 1, \ldots, m\}$ are set to unity;
- for $k \geq 2$ the weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ are assigned using the previous iterate $\boldsymbol{x}_{k-1}$ as

$$w_i^{(k)} = \frac{1}{\left(\|\boldsymbol{x}_{k-1} - \boldsymbol{a}_i\| + r_i\right)^2}.$$

Source Localization From Range-Difference Measurements

Problem Statement

- It is assumed that the range-difference measurements obey the model:

$$d_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x} - \boldsymbol{a}_0\| = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x}\|, \quad i = 1, \ldots, m$$

  where $\boldsymbol{a}_0$ - reference sensor placed at the origin.

- The standard range-difference LS (RD-LS) problem is formulated as

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}}\, F(\boldsymbol{x}) = \sum_{i=1}^{m} \left(d_i + \|\boldsymbol{x}\| - \|\boldsymbol{x} - \boldsymbol{a}_i\|\right)^2 \qquad \text{(RD)}$$

Source Localization From Range-Difference Measurements

SRD-LS and WSRD-LS formulations

- An approximation of the RD-LS solution can be obtained by solving the *squared range difference based LS* (SRD-LS) problem.
- We re-write the measurements model as $d_i + \|\boldsymbol{x}\| = \|\boldsymbol{x} - \boldsymbol{a}_i\|$ and square both sides to obtain

$$-2d_i\|\boldsymbol{x}\| - 2\boldsymbol{a}_i^T\boldsymbol{x} = g_i, \quad i = 1, \ldots, m$$

where $g_i = d_i^2 - \|\boldsymbol{a}_i\|^2$. The SRD-LS solution can be found as

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}} \sum_{i=1}^{m} \left(-2\boldsymbol{a}_i^T\boldsymbol{x} - 2d_i\|\boldsymbol{x}\| - g_i\right)^2$$

Improved Solution Using Iterative Re-weighting

- We consider the weighted SRD-LS problem

$$\underset{\boldsymbol{x}\in R^n}{\text{minimize}} \sum_{i=1}^{m} w_i \left(-2\boldsymbol{a}_i^T\boldsymbol{x} - 2d_i\|\boldsymbol{x}\| - g_i\right)^2 \qquad \text{(WSRD)}$$

where weights w_i for $i = 1, \ldots, m$ are *fixed* nonnegative constants.

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- The ith term of the objective function in (WSRD) can be written as:

$$
\begin{aligned}
& w_i \left(-2d_i \|\boldsymbol{x}\| - 2\boldsymbol{a}_i^T \boldsymbol{x} - g_i \right)^2 \\
= & w_i \left(d_i + \|\boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{a}_i\| \right) \underbrace{\left(d_i + \|\boldsymbol{x}\| - \|\boldsymbol{x} - \boldsymbol{a}_i\| \right)}_{\text{same as in RD}}
\end{aligned}
$$

- If weights w_i were set to $1/\left(d_i + \|\boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{a}_i\|\right)^2$ the objective in (WSRD) would be the same as in (RD).

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- We employ an iterative procedure where the weights in the kth iteration are assigned to

$$w_i^{(k)} = \frac{1}{(d_i + \|\boldsymbol{x}_{k-1}\| + \|\boldsymbol{x}_{k-1} - \boldsymbol{a}_i\|)^2}, i = 1, \dots, m$$

  with $\{w_i^{(1)} = 1, i = 1, \dots, m\}$.

- We will refer to the derived problem as the iterative re-weighted SRD-LS (WSRD-LS) problem and the solution obtained as IRWSRD-LS solution.

Performance Evaluation for SR-LS and IRWSR-LS

Table: Averaged MSE for SR-LS and IRWSR-LS methods by noise level

σ	**SR - LS**	**IRWSR-LS**	**Improvement (%)**
1e-03	2.03251062e-06	1.19962894e-06	41
1e-02	1.83717590e-04	1.24797437e-04	32
1e-01	1.83611315e-02	1.22233840e-02	33

Performance Evaluation for SRD-LS and IRWSRD-LS

Table: Averaged MSE for SRD-LS and IRWSRD-LS methods by noise level

σ	SRD - LS	IRWSRD-LS	Improvement (%)
1e-04	1.38301598e-08	8.22705918e-09	40
1e-03	1.60398717e-06	1.03880406e-06	35
1e-02	1.11632818e-04	6.67785604e-05	40
1e-01	1.20947651e-02	7.20891487e-03	40
1e+0	1.57050323e+00	9.70756420e-01	40

Penalty Convex-Concave Procedure for Source Localization

Problem Statement

Measurement Model

- The *range measurements* model is assumed to be given by

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m.$$

$\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ - given array of m sensors;
r_i - received noisy distance reading from sensor i;
ε_i - unknown noise associated with measurement from the ith sensor.

- The range-based least squares estimate refers to the solution of

$$\underset{\boldsymbol{x}}{\text{minimize}}\, F(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \qquad \text{(R)}$$

Penalty Convex-Concave Procedure for Source Localization

- We frame the localization problem as difference-of-convex-functions (DC) program.

- Proposed formulation:
  - based on a penalty convex-concave procedure (PCCP)
  - accepts infeasible initial points
  - additional constraints that enforce the algorithm's iteration path towards the LS solution
  - strategies to secure good initial points

Basic Convex-Concave Procedure (CCP)

- The CCP finds local optima of *nonconvex* problems of the form

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) - g(\boldsymbol{x}) \\ \text{subject to:} \quad & f_i(\boldsymbol{x}) \leq g_i(\boldsymbol{x}) \quad \text{for: } i = 1, 2, \ldots, m \end{aligned}$$

  where $f(\boldsymbol{x}), g(\boldsymbol{x}), f_i(\boldsymbol{x}), g_i(\boldsymbol{x})$ for $i = 1, 2 \ldots, m$ are convex.

- It is a descent algorithm that requires a *feasible* initial point $\boldsymbol{x}_0$, i.e. $f_i(\boldsymbol{x}) - g_i(\boldsymbol{x}) \leq 0$ for $i = 1, 2 \ldots, m$.

Basic Convex-Concave Procedure (CCP)

- The basic CCP algorithm is an iterative procedure including two key steps (in the k-th iteration):

1. Convexify: form $\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) = g(\boldsymbol{x}_k) + \triangledown g(\boldsymbol{x}_k)^T(\boldsymbol{x} - \boldsymbol{x}_k)$

   and $\hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = g_i(\boldsymbol{x}_k) + \triangledown g_i(\boldsymbol{x}_k)^T(\boldsymbol{x} - \boldsymbol{x}_k)$
   for $i = 1, 2 \ldots, m$

2. Solve the convex problem:

$$
\begin{array}{ll}
\underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) - \hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) \\
\text{subject to:} & f_i(\boldsymbol{x}) - \hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) \leq 0 \\
 & \text{for: } i = 1, 2, \ldots, m
\end{array}
$$

An example of the basic CCP procedure

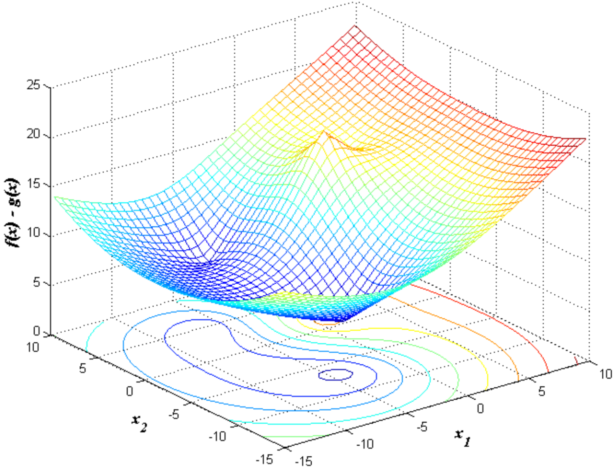

Figure: A nonconvex function in the form of the difference of two convex functions and its contour plot.

An example of the basic CCP procedure

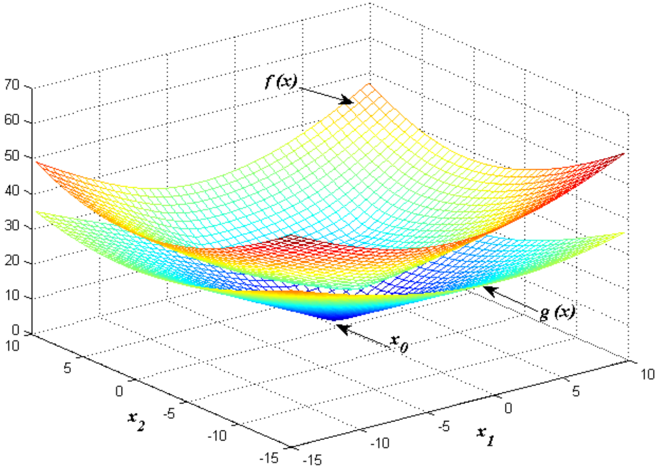

Figure: Separation of the nonconvex function into two convex functions.

An example of the basic CCP procedure

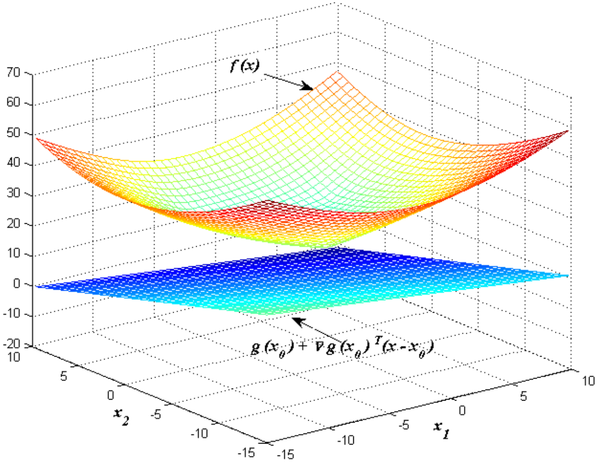

Figure: First order approximation of $g(\boldsymbol{x})$.

An example of the basic CCP procedure

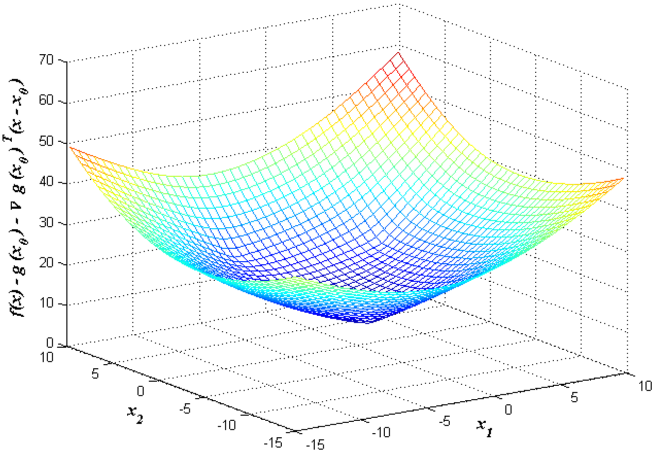

Figure: A convex approximation of the original nonconvex function at $\boldsymbol{x}_0 = (0, 0)$.

Range-Based Localization Revisited

- The range-based least squares (R-LS) estimate:

$$\underset{\boldsymbol{x}}{\text{minimize}}\, F(\boldsymbol{x}) = \sum_{i=1}^{m} \left(r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|\right)^2 \qquad \text{(R)}$$

Problem Reformulation

- We begin by re-writing the objective $F(\boldsymbol{x})$ up to a constant as:

$$\sum_{i=1}^{m}(r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\sum_{i=1}^{m}\boldsymbol{a}_i - 2\sum_{i=1}^{m} r_i\|\boldsymbol{x} - \boldsymbol{a}_i\|$$

which allows to formulate it in a basic CCP form $F(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\boldsymbol{x})$ with

$$f(\boldsymbol{x}) = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\sum_{i=1}^{m}\boldsymbol{a}_i \quad \text{- convex}$$

$$g(\boldsymbol{x}) = 2\sum_{i=1}^{m} r_i\|\boldsymbol{x} - \boldsymbol{a}_i\| \quad \text{- convex.}$$

Problem Reformulation

- Since $g(\boldsymbol{x})$ is not differentiable at the point where $\boldsymbol{x} = \boldsymbol{a}_i$ for some $1 \leq i \leq m$, we replace $\triangledown g(\boldsymbol{x}_k)$ by a subgradient of $g(\boldsymbol{x})$ at $\boldsymbol{x}_k$ as

$$\partial g(\boldsymbol{x}_k) = 2\sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|$$

where

$$\partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\| = \begin{cases} \dfrac{\boldsymbol{x}_k - \boldsymbol{a}_i}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|}, & \text{if } \boldsymbol{x}_k \neq \boldsymbol{a}_i \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

Problem Reformulation

- Up to a multiplicative factor $1/m$ and an additive constant term the objective in (R) can be written as

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \hat{F}(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{x} - 2\boldsymbol{x}^T \boldsymbol{v}_k$$

where

$$\boldsymbol{v}_k = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|, \quad \bar{\boldsymbol{a}} = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{a}_i$$

- Given $\boldsymbol{x}_k$ (in the k-th iteration) the solution of the quadratic problem can be obtained as

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|$$

Imposing Error Bounds

- The algorithm can be enhanced by imposing a bound on each squared measurement error

$$(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i)^2 \le \delta_i^2$$

which leads to

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \le 0 & \quad (C1) \\ r_i - \delta_i \le \|\boldsymbol{x} - \boldsymbol{a}_i\|, \quad \text{for } 1 \le i \le m. & \quad (C2) \end{aligned}$$

Both sets of constraints can be written in a form $f_i(\boldsymbol{x}) \le g_i(\boldsymbol{x})$.

- Constraints in (C1) are convex, with $f_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i$, and $g_i(\boldsymbol{x}) = 0$.

Imposing Error Bounds

- In case of (C2): define $f_i(\boldsymbol{x}) = r_i - \delta_i$ and $g_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\|$. Replace $g_i(\boldsymbol{x})$ with its approximation

$$\hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k)$$

  This allows to convexify constraints $r_i - \delta_i \leq \|\boldsymbol{x} - \boldsymbol{a}_i\|$ as

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \leq 0$$

- Summarizing, the problem in the k-th iteration can be stated as

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k \\ \text{subject to:} \quad & \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \leq 0 \end{aligned}$$

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \leq 0$$

Penalty CCP (PCCP)

- Technical problem: the formulation requires a feasible initial point $\boldsymbol{x}_0$.
- Solution approach: allow *infeasible* initial points by introducing slack variables $s_i \geq 0, \hat{s}_i \geq 0$, $1 \leq i \leq m$ into constraints (C1) and (C2) and penalizing the sum of violations.
- This leads to a *penalty* CCP:

$$\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{s},\hat{\boldsymbol{s}}}{\text{minimize}} \quad & \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k + \tau_k \sum_{i=1}^{m}(s_i + \hat{s}_i) \\
\text{subject to:} \quad & \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \leq s_i \\
-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - & \frac{(\boldsymbol{x}_k - \boldsymbol{a}_i)^T}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|}(\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \leq \hat{s}_i \\
& s_i \geq 0, \hat{s}_i \geq 0, \text{ for: } i = 1, 2, \ldots, m
\end{aligned}$$

where $0 \leq \tau_k \leq \tau_{max}$.

The Algorithm: Input parameters

Bound δ_i on the measurement error

- Lower δ_i leads to a "tighter" solution.
- Larger δ_i makes the algorithm less sensitive to outliers.
- If ε obeys a Gaussian distribution with zero mean and $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, then $\delta_i = \gamma\sigma_i$, where γ determines the width of confidence interval.
- For example, for $\gamma = 3$ we have the probability $Pr\{|\varepsilon_i| \leq 3\sigma_i\} \approx 0.99$.

The Algorithm: Input parameters

Initial point x_0

Techniques to select a good initial point:

- select the initial point uniformly randomly over the same region as the unknown source;
- set the initial point to the origin;
- run the algorithm from a set of candidate initial points and identify the solution as the one with lowest LS error;
- apply a *global* localization algorithm to generate an approximate LS solution, then take it as the initial point.

Numerical Results

System setup

- Sensors: $\{\boldsymbol{a}_i, i = 1, 2, \ldots, 5\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$
- Source: $\boldsymbol{x}_s$, located randomly in $\{\boldsymbol{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$
- Noise: $\{\varepsilon_i, i = 1, \ldots, m\}$ was modelled as i.i.d random variables with zero mean and variance σ^2, $\sigma \in \{10^{-3}, 10^{-2}, 10^{-1}, 1\}$
- $\gamma = 3$, $K_{max} = 20$

Numerical Results

Table: Averaged MSE for SR-LS and PCCP methods

σ	MLE	SR - LS	PCCP	R.I.
1e-03	6.0159e-01	1.3394e-06	**9.5243e-07**	29%
1e-02	3.5077e-01	1.4516e-04	**9.5831e-05**	34%
1e-01	3.7866e-01	1.2058e-02	**8.7107e-03**	28%
1e+0	1.4470e+00	1.3662e+00	**1.2346e+00**	10%

Least Squares Localization by Sequential Convex Relaxation

Range-Difference Localization

Problem Statement

- Assumed measurement model:

$$d_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x} - \boldsymbol{a}_0\| + \varepsilon_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x}\| + \varepsilon_i, \quad i = 1, \dots, m$$

where $\boldsymbol{a}_0$ - reference sensor placed at the origin.

- The standard range-difference LS (RD-LS) problem

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}}\, F(\boldsymbol{x}) = \sum_{i=1}^{m} \left(\|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x}\| - d_i\right)^2 \qquad \text{(RD)}$$

- Re-write the unconstrained problem (RD) as a constrained problem

$$\begin{array}{ll} \underset{\boldsymbol{x},y,\boldsymbol{z}}{\text{minimize}} & \sum_{i=1}^{m} (z_i - y - d_i)^2 \\ \text{subject to:} & \|\boldsymbol{x} - \boldsymbol{a}_i\| = z_i, \quad i = 1, 2, \ldots m \\ & \|\boldsymbol{x}\| = y \end{array}$$

- Assume the kth iterate known $\{\boldsymbol{x}_k, y_k, \boldsymbol{z}_k\}$. Let the next iterate be $\{\boldsymbol{x}_k + \boldsymbol{\delta}_x, y_k + \delta_y, \boldsymbol{z}_k + \boldsymbol{\delta}_z\}$, i.e. constraints become

$$\begin{aligned} \|\boldsymbol{x}_k + \boldsymbol{\delta}_x - \boldsymbol{a}_i\| &\approx z_i^k + \boldsymbol{\delta}_{z_i}, \quad i = 1, 2, \ldots, m \\ \|\boldsymbol{x}_k + \boldsymbol{\delta}_x\| &\approx y_k + \boldsymbol{\delta}_y \end{aligned}$$

Sequential Convex Relaxation

- Replace constraints by their affine approximations

$$\begin{aligned} \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial_x^T \|\boldsymbol{x}_k - \boldsymbol{a}_i\| \boldsymbol{\delta}_x &\approx z_i^k + \boldsymbol{\delta}_{z_i}, \quad i = 1, 2, \dots, m \\ \|\boldsymbol{x}_k\| + \partial_x^T \|\boldsymbol{x}_k\| \boldsymbol{\delta}_x &\approx y_k + \boldsymbol{\delta}_y \end{aligned}$$

- The objective can be written as

$$\begin{aligned} F(\boldsymbol{x}_{k+1}) &= \sum_{i=1}^{m} \left(z_i^{(k)} + \delta_{z_i} - (y_k + \delta_y) - d_i \right)^2 \\ &= \sum_{i=1}^{m} \left(-\delta_y + \delta_{z_i} - \tilde{d}_i^{(k)} \right)^2 \end{aligned}$$

where $\tilde{d}_i^{(k)} = d_i - y_k - z_i^{(k)}$

Sequential Convex Relaxation

- In kth iteration we solve the problem

$$
\begin{aligned}
\underset{\tilde{\boldsymbol{\delta}}}{\text{minimize}} \quad & f(\tilde{\boldsymbol{\delta}}) = \sum_{i=1}^{m} \left(-\delta_y + \delta_{z_j} - \tilde{d_i^{(k)}} \right)^2 \\
\text{subject to:} \quad & \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \frac{(\boldsymbol{x}_k - \boldsymbol{a}_i)^T \boldsymbol{\delta}_x}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|} = z_i^{(k)} + \delta_{z_j}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots m \\
& \|\boldsymbol{x}_k\| + \frac{\boldsymbol{x}_k^T \boldsymbol{\delta}_x}{\|\boldsymbol{x}_k\|} = y_k + \delta_y \\
& \begin{bmatrix} -\beta \mathbf{1}_2 \\ -\beta \\ -\beta \mathbf{1}_m \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{\delta}_x \\ \delta_y \\ \boldsymbol{\delta}_z \end{bmatrix} \leq \begin{bmatrix} \beta \mathbf{1}_2 \\ \beta \\ \beta \mathbf{1}_m \end{bmatrix}
\end{aligned}
$$

Sequential Convex Relaxation

- Express the problem in a standard form as

$$\begin{aligned} \underset{\boldsymbol{\delta}}{\text{minimize}} \quad & f\left(\tilde{\delta}\right) \\ \text{subject to} \quad & \boldsymbol{A}_k \tilde{\delta} = \boldsymbol{b}_k \\ & \boldsymbol{C}\tilde{\delta} \leq \boldsymbol{q} \end{aligned}$$

- Relax the constraints in order for the problem to be solvable

$$\begin{aligned} \underset{\boldsymbol{\delta}}{\text{minimize}} \quad & f\left(\tilde{\delta}\right) + \tau \sum_{i=1}^{m+1} (u_i + v_i) + \tau w \\ \text{subject to} \quad & \boldsymbol{A}_k \tilde{\delta} - \boldsymbol{b}_k = \boldsymbol{u} - \boldsymbol{v} \\ & \boldsymbol{C}\tilde{\delta} - \boldsymbol{q} \leq w\boldsymbol{e} \\ & \boldsymbol{u} \geq \boldsymbol{0}, \boldsymbol{v} \geq \boldsymbol{0}, w \geq 0 \end{aligned}$$

The Algorithm: Input parameters

- Bound β on the increment vector $\tilde{\boldsymbol{\delta}} = (\boldsymbol{\delta}_x, \delta_y, \boldsymbol{\delta}_z)$.
- The initial point $\boldsymbol{x}_0$.
- Initial weight for penalty terms τ_0.
- Upper limit of the weight τ_{max}.
- Convergence tolerance ϵ.

Numerical Results

System setup

- Sensors: $\{\boldsymbol{a}_i, i = 1, 2, \ldots, 11\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$, $\boldsymbol{a}_0 = \boldsymbol{0}$ placed at the origin.
- Source: $\boldsymbol{x}_s$, located randomly in $\{\boldsymbol{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$
- Noise: $\{\varepsilon_i, i = 1, \ldots, m\}$ was modelled as i.i.d random variables with zero mean and variance σ^2, $\sigma \in \{10^{-3}, 10^{-2}, 10^{-1}, 1\}$.
- $\beta = 3$; penalty terms $\tau_0 = 10, \tau_{max} = 10000$.
- Convergence tolerance $\epsilon = 10^{-6}$.

Table: MSE of position estimation for SRD-LS and SCR-RDLS methods

σ	SRD - LS	SCR-RDLS	(R.I.,%)
1e-03	1.2655e-06	8.4626e-07	33
1e-02	1.4492e-04	6.8385e-05	52
1e-01	1.3329e-02	7.1676e-03	46
1e+0	1.6077e+00	9.5371e-01	40

Measurement Model

- The *range measurements* model

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m.$$

$\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ - given array of m sensors;
r_i - received noisy distance reading from sensor i;
ε_i - unknown noise associated with measurement from the ith sensor.

- The range-based LS estimate refers to the solution of

$$\underset{\boldsymbol{x}}{\text{minimize}}\, F(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \tag{R}$$

Sequential Relaxation

- Equivalent constrained problem

$$\begin{aligned} \underset{\boldsymbol{x},\boldsymbol{z}}{\text{minimize}} \quad & \sum_{i}^{m} (z_i - r_i)^2 \\ \text{subject to:} \quad & \|\boldsymbol{x} - \boldsymbol{a}_i\| = z_i, \quad i = 1, 2, \ldots, m \\ & \boldsymbol{z} \geq \boldsymbol{0} \end{aligned}$$

- Relax the constraints

$$\begin{aligned} \underset{\boldsymbol{x},\boldsymbol{z}}{\text{minimize}} \quad & \sum_{i}^{m} (z_i - r_i)^2 \\ \text{subject to:} \quad & \|\boldsymbol{x} - \boldsymbol{a}_i\| \leq (1+\gamma) z_i \\ & \|\boldsymbol{x} - \boldsymbol{a}_i\| \geq (1-\gamma) z_i, \quad i = 1, 2, \ldots, m \end{aligned}$$

- $\gamma > 0$ is sequentially and monotonically decreasing, $\gamma_0 \in (0, 0.5)$

Sequential Relaxation

- Some constraints are non-convex

$$\begin{aligned} & \|\boldsymbol{x} - \boldsymbol{a}_i\| \leq (1+\gamma) z_i \qquad (\mathit{convex}) \\ \|\boldsymbol{x} - \boldsymbol{a}_i\| \geq (1-\gamma) z_i \iff & \underbrace{-\|\boldsymbol{x} - \boldsymbol{a}_i\|}_{\mathit{nonconvex}} \leq -(1-\gamma) z_i \quad (\mathit{nonconvex}) \end{aligned}$$

$i = 1, 2, \ldots, m.$

- Replace non-convex constraints with affine approximation ($\boldsymbol{x}_k$ is *known*)

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) \leq -(1-\gamma) z_i$$

Sequential Relaxation

- In the kth iteration ($\boldsymbol{x}_k$ is known), solve an SOCP problem

$$\begin{aligned} \underset{\boldsymbol{x},\boldsymbol{z}}{\text{minimize}} \quad & \sum_i^m (z_i - r_i)^2 \\ \text{subject to:} \quad & \|\boldsymbol{x} - \boldsymbol{a}_i\| \leq (1+\gamma) z_i \end{aligned}$$

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T(\boldsymbol{x} - \boldsymbol{x}_k) \leq -(1-\gamma) z_i$$

$i = 1, 2, ..., m$. Update γ

$$\begin{aligned} \gamma_{k+1} &= \gamma_0 - k\frac{\gamma_0}{K_{max}-1} && \text{linearly} \\ \gamma_{k+1} &= \gamma_0 \frac{(K_{max}-1-k)^2}{(K_{max}-1)^2} && \text{quadratically} \end{aligned}$$

Numerical Results

System setup

- Sensors: $\{\boldsymbol{a}_i, i = 1, 2, \dots, 5\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$.
- Source: $\boldsymbol{x}_s$, located randomly in $\{\boldsymbol{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$
- Noise: $\{\varepsilon_i, i = 1, \dots, m\}$ was modelled as i.i.d random variables with zero mean and variance σ^2, $\sigma \in \{10^{-3}, 10^{-2}, 10^{-1}, 1\}$.
- Initial relaxation parameter $\gamma_0 = 0.2$
- Number of iterations $K_{max} = 9$.

Numerical Results

Table: MSE of position estimation for SR-LS and SCR-RLS methods

σ	SRD - LS	SCR-RDLS	(R.I.,%)
1e-02	2.5360e-04	2.0596e-04	18
1e-01	1.8696e-02	1.4802e-02	21
1e+0	1.4440e+00	9.6327e-01	33

Conclusions

- New iterative methods for locating a radiating source based on noisy range and range-difference measurements.
- The iterative re-weighting methods are developed by transforming the SR-LS and SRD-LS algorithms [BSL2008] into an iterative procedure so that a weighted SR-LS (SRD-LS) objective assymptotically approaches the original R-LS objective.
- Convex minimization method based on PCCP that can be efficiently solved with an infeasible initial point.
- Proposed algorithms are found to outperform the existing methods.

Future Work

- Study and mitigation of the influence of sensor geometry on the accuracy of the developed methods (for example, geometric dilusion of precision).
- Multiple source localization in wireless sensor networks.

Q & A

Appendix

Nonconvexity of the R-LS objective

Given the objective

$$F(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2$$

its Hessian for points $\boldsymbol{x}$ that are not coincided with $\boldsymbol{a}_i$ for $1 \le i \le m$, is given by

$$\bigtriangledown^2 F(\boldsymbol{x}) = 2m\boldsymbol{I} + 2\sum_{i=1}^{m} \frac{r_i}{\|\boldsymbol{x} - \boldsymbol{a}_i\|^3} \cdot \left((\boldsymbol{x} - \boldsymbol{a}_i)(\boldsymbol{x} - \boldsymbol{a}_i)^T - \|\boldsymbol{x} - \boldsymbol{a}_i\|^2 \boldsymbol{I} \right)$$

which is not always positive semidefinite. Hence $F(\boldsymbol{x})$ is not convex.

Source Localization From Range Measurements

Weighted Squared Range Least Squares Formulation

- Following [BSL2008], we convert (WSR) into a GTRS as

$$\underset{\boldsymbol{y}\in R^{n+1}}{\text{minimize}} \; \|\boldsymbol{A}_w \boldsymbol{y} - \boldsymbol{b}_w\|^2 \tag{1a}$$

$$\text{subject to:} \quad \boldsymbol{y}^T \boldsymbol{D} \boldsymbol{y} + 2\boldsymbol{f}^T \boldsymbol{y} = 0 \tag{1b}$$

where $\boldsymbol{y} = [\boldsymbol{x}^T \;\; \alpha]^T$, $\alpha = \|\boldsymbol{x}\|$, $\boldsymbol{A}_w = \boldsymbol{\Gamma}\boldsymbol{A}$ and $\boldsymbol{b}_w = \boldsymbol{\Gamma}\boldsymbol{b}$ with fixed $\boldsymbol{\Gamma} = \text{diag}\left(\sqrt{w_1}, \ldots, \sqrt{w_m}\right)$, and

$$\boldsymbol{A} = \begin{pmatrix} -2\boldsymbol{a}_1^T & 1 \\ \vdots & \vdots \\ -2\boldsymbol{a}_m^T & 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} r_1^T - \|\boldsymbol{a}_1\|^T \\ \vdots \\ r_m^T - \|\boldsymbol{a}_m\|^T \end{pmatrix} \tag{2}$$

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{I}_{n\times n} & \boldsymbol{0}_{n\times 1} \\ \boldsymbol{0}_{1\times n} & 0 \end{pmatrix}, \boldsymbol{f} = \begin{pmatrix} \boldsymbol{0} \\ -0.5 \end{pmatrix} \tag{3}$$

The Algorithm

1. Input data: Sensor locations $\{\boldsymbol{a}_i, i = 1, \ldots, m\}$, range measurements $\{r_i, i = 1, \ldots, m\}$, maximum number of iterations k_{max} and convergence tolerance ζ.
2. Generate data set $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{D}, \boldsymbol{f}$ using (2) and (3). Set $k = 1, w_i^{(1)} = 1$ for $i = 1, \ldots, m$.
3. Set $\boldsymbol{\Gamma}_k = \text{diag}\left(\sqrt{w_1^{(k)}}, \ldots, \sqrt{w_m^{(k)}}\right)$, $\boldsymbol{A}_w = \boldsymbol{\Gamma}_k \boldsymbol{A}$ and $\boldsymbol{b}_w = \boldsymbol{\Gamma}_k \boldsymbol{b}$.
4. Solve the WSR-LS problem (IRWSR) via (1) to obtain its global solution $\boldsymbol{x}_k$.
5. If $k = k_{max}$ or $\|\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\| < \zeta$, terminate and output $\boldsymbol{x}_k$ as the solution; otherwise, set $k = k + 1$, update weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ and repeat from Step 3).

Source Localization From Range-Difference Measurements

Weighted Squared Range-Difference Least Squares Formulation

- By introducing new variable $\boldsymbol{y} = [\boldsymbol{x}^T \ \|\boldsymbol{x}\|]^T$ and noticing nonnegativity of the component y_{n+1} problem (WSRD) is converted to

$$\underset{\boldsymbol{y} \in R^{n+1}}{\text{minimize}} \|\boldsymbol{B}_w \boldsymbol{y} - \boldsymbol{g}_w\| \tag{4a}$$

$$\text{subject to: } \boldsymbol{y}^T \boldsymbol{C} \boldsymbol{y} = 0 \tag{4b}$$

$$y_{n+1} \geq 0 \tag{4c}$$

- where $\boldsymbol{B}_w = \boldsymbol{\Gamma}\boldsymbol{B}$, $\boldsymbol{g}_w = \boldsymbol{\Gamma}\boldsymbol{g}$, $\boldsymbol{\Gamma} = \text{diag}\{\sqrt{w_1}, \ldots, \sqrt{w_m}\}$, $\boldsymbol{g} = [g_1 \ldots g_m]^T$ and

$$\boldsymbol{B} = \begin{pmatrix} -2\boldsymbol{a}_1^T & -2d_1 \\ \vdots & \vdots \\ -2\boldsymbol{a}_m^T & -2d_m \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{0}_{n\times 1} \\ \boldsymbol{0}_{1\times n} & -1 \end{pmatrix} \tag{5}$$

Source Localization From Range Difference Measurements

The Algorithm

1. Input data: Sensor locations $\{\boldsymbol{a}_i, i = 0, 1, \ldots, m\}$ with $\boldsymbol{a}_0 = \mathbf{0}$, range-difference measurements $\{d_i, i = 1, \ldots, m\}$, maximum number of iterations k_{max} and convergence tolerance ξ.
2. Generate data set $\{\boldsymbol{B}, \boldsymbol{g}, \boldsymbol{C}\}$ using (5). Set $k = 1$, $w_i^{(1)} = 1$ for $i = 1, \ldots, m$.
3. Set $\boldsymbol{\Gamma}_k = \text{diag}\left(\sqrt{w_1^{(k)}}, \ldots, \sqrt{w_m^{(k)}}\right)$, $\boldsymbol{B}_w = \boldsymbol{\Gamma}_k \boldsymbol{B}$ and $\boldsymbol{g}_w = \boldsymbol{\Gamma}_k \boldsymbol{g}$.
4. Solve WSRD-LS problem (4) to obtain its global solution $\boldsymbol{x}_k$.
5. If $k = k_{max}$ or $\|\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\| < \xi$, terminate and output $\boldsymbol{x}_k$ as the solution; otherwise, set $k = k + 1$, update weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ and repeat from Step 3).

PCCP - Problem Reformulation

We express the objective in (R) as $F(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\boldsymbol{x})$ with

$$f(\boldsymbol{x}) = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\sum_{i=1}^{m}\boldsymbol{a}_i \quad \text{and} \quad g(\boldsymbol{x}) = 2\sum_{i=1}^{m} r_i\|\boldsymbol{x} - \boldsymbol{a}_i\|$$

Then, we replace $\bigtriangledown g(\boldsymbol{x}_k)$ by a subgradient of $g(\boldsymbol{x})$ at $\boldsymbol{x}_k$:

$$\partial g(\boldsymbol{x}_k) = 2\sum_{i=1}^{m} r_i \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|,$$

where

$$\partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\| = \begin{cases} \dfrac{\boldsymbol{x}_k - \boldsymbol{a}_i}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|}, & \text{if } \boldsymbol{x}_k \neq \boldsymbol{a}_i \\ \boldsymbol{0}, & \text{otherwise} \end{cases}$$

PCCP - Problem Reformulation

Hence $\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k)$ can be formed as:

$$\begin{aligned}\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) &= g(\boldsymbol{x}_k) + \bigtriangledown g(\boldsymbol{x}_k)^T(\boldsymbol{x} - \boldsymbol{x}_k) \\ &= 2\sum_{i=1}^{m} r_i \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + 2\,(\boldsymbol{x} - \boldsymbol{x}_k)^T \sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\| \\ &= 2\boldsymbol{x}^T \sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + c\end{aligned}$$

where c is a constant given by

$$c = -2\sum_{i=1}^{m} r_i \boldsymbol{a}_i^T \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|.$$

PCCP-based LS Algorithm for Source Localization

Step 1: Input sensor locations $\{\boldsymbol{a}_i, i = 1, \ldots, m\}$, range measurements $\{r_i, i = 1, \ldots, m\}$, $\boldsymbol{x}_0, K_{max}, \tau_0, \tau_{max}, \mu > 0, \gamma, \sigma$, and set $k = 0$.

Step 2: Form $\boldsymbol{v}_k$ and solve PCCP. Denote the solution as $(\boldsymbol{s}^*, \hat{\boldsymbol{s}}^*, \boldsymbol{x}^*)$.

Step 3: Update $\tau_{k+1} = \min\,(\mu\tau_k, \tau_{max})$, set $k = k + 1$.

Step 4: If $k = K_{max}$, terminate and output $\boldsymbol{x}^*$ as the solution; otherwise, set $\boldsymbol{x}_k = \boldsymbol{x}^*$ and repeat from Step 2.

Sequential Convex Relaxation for Range-Difference Localization

Step 1: Input data:
- sensor locations $\{\boldsymbol{a}_i, i = 1, \ldots, m\}$,
- range-difference measurements $\{d_i, i = 1, \ldots, m\}$,
- initial point $\boldsymbol{x}_0$,
- initial weight τ_0 and upper limit of weight τ_{max},
- increment bound β
- convergence tolerance ϵ. Set iteration count to $k = 0$.

Form $\boldsymbol{S}, \boldsymbol{C}$ and $\boldsymbol{q}$ as

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{0}_{m\times 1} & -\boldsymbol{1}_{m\times 1} & -\boldsymbol{I}_m \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} \boldsymbol{I}_{m+3} \\ -\boldsymbol{I}_{m+3} \end{bmatrix}, \boldsymbol{q} = \beta \boldsymbol{e}$$

Sequential Convex Relaxation for Range-Difference Localization I

Step 2: Form y_k and $\boldsymbol{z}_k$ as $y_k = \|\boldsymbol{x}_k\|, \boldsymbol{z}_k = \begin{bmatrix} \|\boldsymbol{x}_k - \boldsymbol{a}_1\| \\ \vdots \\ \|\boldsymbol{x}_k - \boldsymbol{a}_m\| \end{bmatrix}$

Form $\boldsymbol{A}_k, \tilde{\boldsymbol{d}}_k, \boldsymbol{b}_k, \boldsymbol{C}_k$ and solve

$$\begin{aligned} \underset{\boldsymbol{\delta}}{\text{minimize}} \quad & f\left(\tilde{\delta}\right) + \tau_k \sum_{i=1}^{m+1} (u_i + v_i) + \tau_k w \\ \text{subject to} \quad & \boldsymbol{A}_k \tilde{\delta} - \boldsymbol{b}_k = \boldsymbol{u} - \boldsymbol{v} \\ & \boldsymbol{C}\tilde{\delta} - \boldsymbol{q} \leq w\boldsymbol{e} \\ & \boldsymbol{u} \geq \boldsymbol{0}, \boldsymbol{v} \geq \boldsymbol{0}, w \geq 0 \end{aligned}$$

Denote the solution as $\tilde{\boldsymbol{\delta}}_k = (\boldsymbol{\delta}_x^*, \delta_y^*, \boldsymbol{\delta}_z^*)$.

Sequential Convex Relaxation for Range-Difference Localization II

Step 3: Update $\tau_{k+1} = \min\,(1.5\tau_k, \tau_{max})$, set $k = k + 1$. Update $\tilde{\boldsymbol{x}}^*$ to

$$\boldsymbol{x}^* = \boldsymbol{x}^k + \boldsymbol{\delta}_x^*$$
$$y^* = y^k + \delta_y^*$$
$$\boldsymbol{z}^* = \boldsymbol{z}^k + \boldsymbol{\delta}_z^*$$

Step 4: If $\|\tilde{\boldsymbol{\delta}}_k\| \leq \epsilon$, terminate and output $\boldsymbol{x}^*$ as the solution; otherwise, set $\tilde{\boldsymbol{x}}_k = \boldsymbol{x}^*$ and repeat from Step 2.

Sequential Convex Relaxation for Range-Based Localization I

Step 1: Input data:

- sensor locations $\{\boldsymbol{a}_i, i = 1, \ldots, m\}$,
- range measurements $\{r_i, i = 1, \ldots, m\}$,
- initial point $\boldsymbol{x}_0$, initial relaxation parameter γ_0,
- the number of iterations to be executed K_{max}.

Set iteration count to $k = 0$.

Sequential Convex Relaxation for Range-Based Localization II

Step 2: Solve

$$\underset{\boldsymbol{x},\boldsymbol{z}}{\text{minimize}} \quad \sum_{i}^{m} (z_i - r_i)^2$$

$$\text{subject to:} \quad \|\boldsymbol{x} - \boldsymbol{a}_i\| \leq (1+\gamma) z_i$$

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T(\boldsymbol{x} - \boldsymbol{a}_i) \leq -(1-\gamma) z_i, \;\; i = 1, 2, ..., m$$

Denote the solution as $\tilde{\boldsymbol{x}}_k = (\boldsymbol{x}^*, \boldsymbol{z}^*)$.

Step 3: Update $\gamma_{k+1} = f(\gamma_k)$ linearly or quadratically. Set $k = k + 1$.

Step 4: If $k = K_{max}$, terminate and output $\boldsymbol{x}^*$ as the solution; otherwise, set $\boldsymbol{x}_k = \boldsymbol{x}^*$ and repeat from Step 2.

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