Exact and Approximate Solutions of Source Localization Problems

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***Abstract*—We consider least squares (LS) approaches for locating a radiating source from range measurements (which we call R-LS) or from range-difference measurements (RD-LS) collected using an array of passive sensors. We also consider LS approaches based on squared range observations (SR-LS) and based on squared range-difference measurements (SRD-LS). Despite the fact that the resulting optimization problems are nonconvex, we provide exact solution procedures for efficiently computing the SR-LS and SRD-LS estimates. Numerical simulations suggest that the exact SR-LS and SRD-LS estimates outperform existing approximations of the SR-LS and SRD-LS solutions as well as approximations of the R-LS and RD-LS solutions which are based on a semidefinite relaxation.**

***Index Terms*—Efficiently and globally optimal solution, generalized trust region subproblems (GTRS), least squares, nonconvex, quadratic function minimization, range measurements, range-difference measurements, single quadratic constraint, source localization, squared range observations.**

I. Introduction

THE problems of locating a radiating source from range measurements or from range-difference measurements collected using a network (or array) of passive sensors have received significant attention in the signal processing literature owing to their importance to many applications including teleconferencing, wireless communications, surveillance, navigation, and geophysics [1]–[10]. In this paper, which considers both problems, the main focus is on efficient computation of least squares (LS) estimates of the source's coordinate vector. The models that we consider for the said vector are based on the assumption that the sensor network can be used, along with some form of preprocessing, to obtain (noisy) range or range-difference measurements. From a practical standpoint, this is a simplifying assumption (e.g., in nonline-of-sight scenarios), albeit one commonly made in [7] and [8]. Even so, the resulting location estimation problems are *nonconvex* and, therefore, rather difficult to solve globally, which explains why only approximate solutions to them have appeared in [1], [2], [5], and [7]. We should also mention here the family of data fusion methods [10] in which linear least squares problems are constructed via subtraction of equations. However, these methods do not provide optimal solutions since they implicitly assume the existence of an error-free measurement.

In this paper, we first consider the problem of source localization from range measurements. In Section II we provide a result that explains why a recently proposed semidefinite relaxation (SDR) [7] of the R-LS approach to this problem may yield an accurate approximation; however, we also show that the SDR may lead to a poor approximation. For lack of a good solution to the R-LS problem, we then turn our attention to an SR-LS approach. Although the latter approach also leads to a nonconvex problem, we show that this problem can be efficiently and globally solved.

Then we go on to consider the source localization problem from range-difference measurements in Section III. Our main results here concern an SRD-LS approach to this problem. In particular, we show that despite the fact that the said SRD-LS problem is also nonconvex, it can be efficiently solved, and we provide the details of an algorithm that computes the global solution of this problem. We end Section III by remarking that an SDR approach applied to a corresponding RD-LS criterion leads to extremely poor solutions.

Several numerical examples suggest that the exact SR-LS and SRD-LS solutions can be more accurate by several orders of magnitude than existing approximate SR-LS and SRD-LS estimates, and than SDR-based approximations of the R-LS and RD-LS solutions.

Notation: Vectors are denoted by boldface lowercase letters, e.g., $\mathbf{y}$, and matrices by boldface uppercase letters, e.g., $\mathbf{A}$. The ith component of a vector $\mathbf{y}$ is written as y_i. The identity matrix of order n is denoted by $\mathbf{I}_n$ and the all-zero matrix of order $n\times k$ is denoted by $\mathbf{0}_{n\times k}$. Given two matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}\succ\mathbf{B}(\mathbf{A}\succeq\mathbf{B})$ means that $\mathbf{A}-\mathbf{B}$ is positive definite (semidefinite). Given a positive definite $n\times n$ matrix $\mathbf{B}$ and a symmetric $n\times n$ matrix $\mathbf{A}$ the generalized eigenvalues of the matrix pair $(\mathbf{A},\mathbf{B})$ are given by $\lambda_i(\mathbf{A},\mathbf{B})=\lambda_i(\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}), i=1,\ldots,n$, where for a given symmetric matrix $\mathbf{M}$, $\lambda_i(\mathbf{M})$ denotes the ith eigenvalue of $\mathbf{M}$ (ordered decreasingly).

II. Source Localization From Range Measurements

Consider an array of m sensors, and let $\mathbf{a}_i\in\mathbb{R}^n$ denote the coordinates of the ith sensor (in practical applications $n=2$ or 3). Let $\mathbf{x}\in\mathbb{R}^n$ denote the source's coordinate vector. Finally, let r_i denote a noisy observation of the range between the source and the ith sensor

$$r_i=\|\mathbf{x}-\mathbf{a}_i\|+\varepsilon_i,\quad i=1,\ldots,m. \tag{1}$$

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Here $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)^T$ denotes the unknown noise vector. We assume that the noisy measurements r_i are positive ($r_i > 0$). The source localization problem consists of estimating $\mathbf{x}$ from the observed ranges r_i. In this section, we consider an R-LS approach and an SR-LS approach to this problem. Both methodologies lead to nonconvex optimization problems. For the R-LS approach, we show that a recently proposed semidefinite relaxation (SDR) [7] can produce inaccurate solutions to the R-LS problem, in spite of the fact that the SDR solution is shown to satisfy at least one of the conditions required for tightness of the relaxation. The SR-LS problem—although also nonconvex—can be globally solved by transforming it into a generalized trust region subproblem [11]. We will end this section with a numerical comparison of the R-LS and SR-LS methods.

A. The R-LS Approach

One approach for estimating the source location $\mathbf{x}$ is via the minimization of the least squares criterion

$$(\mathrm{R-LS}): \quad \min_{\mathbf{x}} \sum_{i=1}^{m} (r_i - \|\mathbf{x} - \mathbf{a}_i\|)^2. \tag{2}$$

The solution to (2) is called the *range-based least squares* (R-LS) estimate. Note that when $\boldsymbol{\varepsilon}$ follows a Gaussian distribution with a covariance matrix proportional to the identity matrix, the R-LS solution is in fact a maximum likelihood estimator [7]. Problem (2) is nonconvex and, thus, finding its exact solution is in principle a difficult task. For this reason, the approach advocated in [7] is to construct a semidefinite relaxation (SDR) of (2). This is done by first rewriting (2) as

$$\begin{aligned} \min_{\mathbf{x},\mathbf{g}} \quad & \sum_{i=1}^{m} (r_i - g_i)^2 \\ \text{s.t.} \quad & g_i^2 = \|\mathbf{x} - \mathbf{a}_i\|^2, \quad i = 1, \ldots, m \end{aligned} \tag{3}$$

and then making the change of variables

$$\mathbf{G} = \begin{pmatrix} \mathbf{g} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{g}^T & 1 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}^T & 1 \end{pmatrix} \tag{4}$$

which transforms (3) into

$$\begin{aligned} \min_{\mathbf{X},\mathbf{G}} \quad & \sum_{i=1}^{m} \left(G_{ii} - 2r_i G_{m+1,i} + r_i^2 \right) \\ \text{s.t.} \quad & G_{ii} = \mathrm{Tr}(\mathbf{C}_i \mathbf{X}), \quad i = 1, \ldots, m \\ & \mathbf{G} \succeq \mathbf{0}, \mathbf{X} \succeq \mathbf{0} \\ & G_{m+1,m+1} = X_{n+1,n+1} = 1 \\ & \mathrm{rank}(\mathbf{X}) = \mathrm{rank}(\mathbf{G}) = 1 \end{aligned} \tag{5}$$

where

$$\mathbf{C}_i = \begin{pmatrix} \mathbf{I} & -\mathbf{a}_i \\ -\mathbf{a}_i^T & \|\mathbf{a}_i\|^2 \end{pmatrix}, \quad i = 1, \ldots, m. \tag{6}$$

Finally, by dropping the rank constraints in (5), we obtain the following SDR of (2):

$$\begin{aligned} \min_{\mathbf{X},\mathbf{G}} \quad & \sum_{i=1}^{m} \left(G_{ii} - 2r_i G_{m+1,i} + r_i^2 \right) \\ \text{s.t.} \quad & G_{ii} = \mathrm{Tr}(\mathbf{C}_i \mathbf{X}), \quad i = 1, \ldots, m \\ & \mathbf{G} \succeq \mathbf{0}, \mathbf{X} \succeq \mathbf{0} \\ & G_{m+1,m+1} = X_{n+1,n+1} = 1. \end{aligned} \tag{7}$$

The SDR (7) can be solved efficiently via interior point methods [12]. By its nature, the SDR is not guaranteed to have the same optimal value as the original R-LS problem. The values of both problems coincide if and only if there exists an optimal solution of the SDR (7) for which both matrices $\mathbf{X}$ and $\mathbf{G}$ have rank one. The numerical experiments reported in [7] indicate that the SDR can provide an *"accurate approximation"* of the R-LS problem. A partial theoretical explanation lending some support to this empirical observation is given in the following result which states that at any optimal solution $(\mathbf{X}, \mathbf{G})$ of the SDR, the matrix $\mathbf{G}$ must have rank one.

Lemma 2.1: Let $(\mathbf{G}, \mathbf{X})$ be an optimal solution of (7). Then $\mathbf{G}$ has rank 1.

Proof: Every principal submatrix of a positive semidefinite matrix is itself a positive semidefinite matrix. In particular, this implies that

$$G_{ii} G_{m+1,m+1} \geq G_{m+1,i}^2, \quad i = 1, \ldots, m \tag{8}$$

so that

$$G_{m+1,i} \leq \sqrt{G_{ii}}, \quad i = 1, \ldots, m. \tag{9}$$

We will prove that in fact $G_{m+1,i} = \sqrt{G_{ii}}$ for every $i = 1, \ldots, m$. To show this, suppose there exists $1 \leq i_0 \leq m$ for which

$$G_{m+1,i_0} < \sqrt{G_{i_0 i_0}}. \tag{10}$$

Consider the rank-one matrix $\widetilde{\mathbf{G}}$ defined by $\widetilde{\mathbf{G}} \equiv (\sqrt{G_{ii} G_{jj}})_{i,j=1}^{m+1}$. Since $\widetilde{G}_{ii} = G_{ii}$ for $i = 1, \ldots, m+1$ it follows that $(\widetilde{\mathbf{G}}, \mathbf{X})$ is a feasible solution of (7). Moreover, using $\widetilde{G}_{m+1,i} = \sqrt{G_{ii}}$ along with (9) and (10), we obtain

$$\begin{aligned} \sum_{i=1}^{m} & \left(G_{ii} - 2r_i G_{m+1,i} + r_i^2 \right) \\ & > \sum_{i=1}^{m} \left(G_{ii} - 2r_i \sqrt{G_{i,i}} + r_i^2 \right) \\ & = \sum_{i=1}^{m} \left(\widetilde{G}_{ii} - 2r_i \widetilde{G}_{m+1,i} + r_i^2 \right) \end{aligned}$$

so that $(\widetilde{\mathbf{G}}, \mathbf{X})$ is a feasible solution of (7) with a strictly smaller value of the objective function than the value of the optimal solution $(\mathbf{G}, \mathbf{X})$, which is a contradiction to the optimality of $(\mathbf{G}, \mathbf{X})$. We thus conclude that $G_{m+1,i} = \sqrt{G_{ii}}$ for every $i = 1, \ldots, m$.

Finally, we will show that $\mathbf{G} = \widetilde{\mathbf{G}}$. For every $1 \leq i < j \leq m$, the positive semidefinitness of the principal submatrix corresponding to rows and columns $i, j, m+1$ yields

$$\begin{pmatrix} G_{ii} & G_{ij} & \sqrt{G_{ii}} \\ G_{ij} & G_{jj} & \sqrt{G_{jj}} \\ \sqrt{G_{ii}} & \sqrt{G_{jj}} & 1 \end{pmatrix} \succeq \mathbf{0} \tag{11}$$

By a simple property of Schur's complements, (11) implies that

$$\begin{pmatrix} G_{ii} & G_{ij} \\ G_{ij} & G_{jj} \end{pmatrix} - \begin{pmatrix} \sqrt{G_{ii}} \\ \sqrt{G_{jj}} \end{pmatrix} \begin{pmatrix} \sqrt{G_{ii}} & \sqrt{G_{jj}} \end{pmatrix} \succeq \mathbf{0} \tag{12}$$

which is the same as

$$\begin{pmatrix} 0 & G_{ij} - \sqrt{G_{ii}G_{jj}} \\ G_{ij} - \sqrt{G_{ii}G_{jj}} & 0 \end{pmatrix} \succeq \mathbf{0}. \tag{13}$$

The latter condition is satisfied if and only if $G_{ij} = \sqrt{G_{ii}G_{jj}}$, which implies that $\mathbf{G} = \widetilde{\mathbf{G}}$ is a rank-one matrix. ■

Note that in spite of Lemma 2.1, (7) is only a *relaxation* of the R-LS problem and not an exact reformulation. The matrix $\mathbf{G}$ is indeed guaranteed to have rank one but the matrix $\mathbf{X}$ might have rank larger than one, see for instance Example 1 below in which the SDR produces a poor solution.

Example 1: Consider an array of $m = 5$ sensors in the plane ($n = 2$) whose coordinates are $\mathbf{a}_1 = (6,4)^T, \mathbf{a}_2 = (0,-10)^T, \mathbf{a}_3 = (5,-3)^T, \mathbf{a}_4 = (1,-4)^T$ and $\mathbf{a}_5 = (3,-3)^T$. The source's coordinates are $\mathbf{x}_S = (-2,3)^T$. The exact distances ($\|\mathbf{x} - \mathbf{a}_i\|$) and the observed noisy distances ($r_i = \|\mathbf{x} - \mathbf{a}_i\| + \varepsilon_i$) are given by the matrix shown at the bottom of the page. Each noise component ε_i is a realization of a Gaussian distributed random variable with mean zero and standard deviation 0.1. We have solved the SDR using SeDuMi [13]. In this example, the SDR is not tight: the value of the objective function of the SDR is 0.0656 while the value of the R-LS problem is 0.10477 (the R-LS problem was solved here by a very inefficient grid search on the plane). The optimal matrix $\mathbf{G}$ is of rank one (as guaranteed by Lemma 2.1), while the optimal matrix $\mathbf{X}$ is not of rank one (it has the eigenvalues 15.13, 4.72, and 0.35). The R-LS solution is $(-1.9907, 3.0474)^T$, which is a good approximation to the true source location.

Since the matrix $\mathbf{X}$ is not of rank one, it is not straightforward to compare the R-LS solution to the "SDR solution." The question of course is how to generate an approximate solution *vector* from the optimal matrix $\mathbf{X}$. A standard approach[1] is to use a rank one approximation based on the maximum eigenvalue of $\mathbf{X}$. This approach results in the solution $(-1.7718, 3.36655)^T$, which evidently is much less accurate than the R-LS solution. ■

[1]There are of course other approaches to derive an approximation vector. For example, another known technique is to define the vector as $(X_{1,3}, X_{2,3})^T$. In this paper, we consider the eigenvalue approximation which we observed empirically to be superior to the last-column approach.

B. The SR-LS Approach

A different approach to the source localization problem is to apply a least squares methodology to the squared range measurements, see, e.g., [5] and the references therein. In this approach, we seek a vector $\mathbf{x}$, called the *squared-range-based least squares* (SR-LS) estimate, which is a solution to

$$(\mathrm{SR-LS}): \quad \min_{\mathbf{x}} \sum_{i=1}^{m} \left(\|\mathbf{x} - \mathbf{a}_i\|^2 - r_i^2 \right)^2. \tag{14}$$

Note that the SR-LS approach is suboptimal in the maximum likelihood sense. This is due to the fact that the covariance matrix of the squared errors in the squared range domain is not proportional to the identity matrix [9].

The SR-LS (14), like the R-LS (2), is nonconvex. However, as opposed to the R-LS problem, we will show that a *global* solution of (14) can be computed efficiently. To do so, we transform (14) into a constrained minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n, \alpha\in\mathbb{R}} \left\{ \sum_{i=1}^{m} \left(\alpha - 2\mathbf{a}_i^T\mathbf{x} + \|\mathbf{a}_i\|^2 - r_i^2 \right)^2 : \|\mathbf{x}\|^2 = \alpha \right\} \tag{15}$$

which can also be written as (using the substitution $\mathbf{y} = (\mathbf{x}^T, \alpha)^T$)

$$\min_{\mathbf{y}\in\mathbb{R}^{n+1}} \{ \|\mathbf{A}\mathbf{y} - \mathbf{b}\|^2 : \mathbf{y}^T\mathbf{D}\mathbf{y} + 2\mathbf{f}^T\mathbf{y} = 0 \} \tag{16}$$

where

$$\mathbf{A} = \begin{pmatrix} -2\mathbf{a}_1^T & 1 \\ \vdots & \vdots \\ -2\mathbf{a}_m^T & 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} r_1^2 - \|\mathbf{a}_1\|^2 \\ \vdots \\ r_m^2 - \|\mathbf{a}_m\|^2 \end{pmatrix} \tag{17}$$

and

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{pmatrix}, \mathbf{f} = \begin{pmatrix} \mathbf{0} \\ -0.5 \end{pmatrix}. \tag{18}$$

We assume that $\mathbf{A}$ has full column rank which in particular implies that $\mathbf{A}^T\mathbf{A}$ is nonsingular.

Problem (16) is also nonconvex and, thus, existing methods for solving (16) or (14) can produce only suboptimal solutions. One approach to obtain an approximation of the solution of (16) is to discard the quadratic constraint [14], [1], [5]. This gives rise to the *unconstrained squared-range-based LS estimate* (USR-LS) in which the linear LS problem

$$\min_{\mathbf{y}\in\mathbb{R}^{n+1}} \{ \|\mathbf{A}\mathbf{y} - \mathbf{b}\|^2 \} \tag{19}$$

is solved. The solution of (19) is given by $\mathbf{y}^* = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$ and the corresponding estimate of $\mathbf{x}$ is the vector comprised of the first n components of $\mathbf{y}^*$.

exact	8.0622	13.1529	9.2195	7.6157	7.8102
noisy	8.0051	13.0112	9.1138	7.7924	8.0210.

Next, we show how to compute an *exact* solution of the SR-LS (16). Note that (16) belongs to the class of problems consisting of minimizing a quadratic function subject to a single quadratic constraint. Problems of this type are called generalized trust region subproblems (GTRS) [11]. GTRS problems, although usually nonconvex, possess necessary and sufficient optimality conditions from which efficient solution methods can be derived. In particular, by [11, Theorem 3.2], $\mathbf{y} \in \mathbb{R}^{n+1}$ is an optimal solution of (16) if and only if there exists $\lambda \in \mathbb{R}$ such that

$$(\mathbf{A}^T\mathbf{A} + \lambda\mathbf{D})\mathbf{y} = \mathbf{A}^T\mathbf{b} - \lambda\mathbf{f} \tag{20}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\mathbf{f}^T\mathbf{y} = 0 \tag{21}$$

$$\mathbf{A}^T\mathbf{A} + \lambda\mathbf{D} \succeq \mathbf{0}. \tag{22}$$

It follows that the optimal solution of (16) is given by

$$\hat{\mathbf{y}}(\lambda) = (\mathbf{A}^T\mathbf{A} + \lambda\mathbf{D})^{-1}(\mathbf{A}^T\mathbf{b} - \lambda\mathbf{f}) \tag{23}$$

where λ is the unique solution of

$$\varphi(\lambda) = 0, \quad \lambda \in I \tag{24}$$

and the function φ is defined by

$$\varphi(\lambda) \equiv \hat{\mathbf{y}}(\lambda)^T\mathbf{D}\hat{\mathbf{y}}(\lambda) + 2\mathbf{f}^T\hat{\mathbf{y}}(\lambda). \tag{25}$$

The interval I consists of all λ for which $\mathbf{A}^T\mathbf{A} + \lambda\mathbf{D}$ is positive definite, which immediately implies that

$$I = \left(-\frac{1}{\lambda_1(\mathbf{D}, \mathbf{A}^T\mathbf{A})}, \infty\right). \tag{26}$$

Moreover, it is known by ([11, Theorem 5.2]) that $\varphi(\lambda)$ is strictly decreasing over I and therefore a simple bisection algorithm can be used to find the optimal λ over the interval I.

Note that we have limited the discussion to the case in which $\mathbf{A}^T\mathbf{A} + \lambda\mathbf{D}$ is strictly positive definite, which is equivalent to saying that the optimal λ is different from $-1/\lambda_1(\mathbf{D}, \mathbf{A}^T\mathbf{A})$. The case in which the optimal λ is equal to $-1/\lambda_1(\mathbf{D}, \mathbf{A}^T\mathbf{A})$ is the so-called "hard case" of the GTRS problem [15], which can also be treated by a more refined analysis. However, the value $\lambda = -1/\lambda_1(\mathbf{D}, \mathbf{A}^T\mathbf{A})$ is very unlikely to occur both theoretically and practically (it never occurred in the tens of thousands of simulations we have performed). Therefore, for the sake of simplicity, we have tacitly assumed that $\mathbf{A}^T\mathbf{A} + \lambda\mathbf{D}$ is positive definite.

The procedure for calculating the SR-LS estimate is summarized next.

Procedure SR-LS:

a) Use a bisection algorithm to obtain a solution λ to (24).
b) The SR-LS estimate is given by the first n components of the vector $\hat{\mathbf{y}}(\lambda)$ in (23).

TABLE I
MEAN SQUARED POSITION ERROR OF THE SDR, SR-LS, AND USR-LS METHODS

σ	N_σ	SDR	SR-LS	USR-LS
1e-3	264	3.0e-6	**1.8e-6**	5.2e-6
1e-2	433	2.6e-4	**2.0e-4**	4.9e-4
1e-1	481	2.3e-2	**2.0e-2**	5.5e-2
1e+0	556	**2.2e+0**	2.4e+0	5.2e+0

An alternative approach for computing the SR-LS estimate is considered in [9]. The method in [9] consists of finding *all* the five roots of the equation $\varphi(\lambda) = 0$ over the real line. This is done by invoking a root finding procedure for polynomials of degree five. In a second stage, the global optimal solution of SR-LS is chosen as the best of the derived possible solutions. Interestingly, in the above we proved that there is no need to find all the roots of $\varphi(\lambda) = 0$ and that a simple bisection algorithm is sufficient.

Example 1 (Cont'd): The SR-LS solution is $(-2.018, 2.9585)^T$, which gives a relatively good approximation of the true source location. The USR-LS method yields the poor quality solution $(-1.3450, 2.6238)^T$. ■

Example 2: In this example, the SR-LS estimate is compared with the USR-LS estimate as well as the SDR solution using an array of five sensors in the plane. We performed Monte Carlo runs where in each run the sensor locations $\mathbf{a}_i$ and the source location $\mathbf{x}_S$ were randomly generated from a uniform distribution over the square $[-10, 10] \times [-10, 10]$. The observed distances r_i are given by (1) with ε_i being generated from a normal distribution with mean zero and standard deviation σ. In our experiments σ takes on four different values: $1, 10^{-1}, 10^{-2}$, and 10^{-3} (for these values of σ, the observed distances r_i were always positive). The numbers in the three right columns of Table I are the average of the squared position error $||\hat{\mathbf{x}} - \mathbf{x}_S||^2$ over 1000 realizations, where $\hat{\mathbf{x}}$ is the SDR solution, the SR-LS solution, or the USR-LS solution. For every σ, the number N_σ denotes the number of runs out of the 1000 in which the SDR was "tight enough;" the term "tight enough" describes a run in which the sum of the absolute values of the eigenvalues of $\mathbf{X}$ excluding the maximum eigenvalue is less than 10^{-3}. The best result for each possible value of σ is marked by boldface. Note that the SR-LS estimate outperforms the SDR and USR-LS solutions for $\sigma = 10^{-3}, 10^{-2}$ and 10^{-1}, while the SDR is slightly better for $\sigma = 1$. Also, there is a considerable amount of runs in which the SDR problem was not tight. In order to understand the effect of this nontightness on the performance of the SDR solution, we screened out all the runs that were not "tight enough" and measured the averaged squared position error of the three methods only over the "tight enough" runs (see Table II).

It is evident that the SDR solution now outperforms the SR-LS and USR-LS estimates for all values of σ. This is due to the fact that for "tight enough" runs the SDR solution is an excellent approximation of the R-LS solution, which is also a maximum likelihood estimator.

TABLE II
MEAN SQUARED POSITION ERROR AVERAGED OVER THE "TIGHT ENOUGH" RUNS

σ	SDR	SR-LS	USR-LS
1e-3	**1.0e-6**	1.8e-6	3.3e-6
1e-2	**1.1e-4**	1.9e-4	4.7e-4
1e-1	**1.3e-1**	2.0e-1	5.9e-1
1e+0	**1.5e+0**	2.2e+0	5.7e+0

To summarize this example, we note that the R-LS solution appears to perform better than the SR-LS and USR-LS estimators. However, no exact method is known that can be used to efficiently calculate the R-LS solution, and the approximate SDR solution can be less accurate than the SR-LS solution. ■

We end this section by noting that the GTRS methodology can be used to devise an algorithm for solving a weighted version of the SR-LS problem

$$\min_{\mathbf{x}} \sum_{i=1}^{m} w_i \left(\|\mathbf{x}-\mathbf{a}_i\|^2 - r_i^2 \right)^2$$

where $w_1, \ldots, w_m$ are positive constants. The corresponding GTRS problem is given by

$$\min_{\mathbf{y}\in\mathbb{R}^{n+1}} \{ \|\mathbf{\Gamma}(\mathbf{A}\mathbf{y}-\mathbf{b})\|^2 : \mathbf{y}^T\mathbf{D}\mathbf{y} + 2\mathbf{f}^T\mathbf{y} = 0 \}$$

where $\mathbf{\Gamma} = \mathrm{diag}(\sqrt{w_1}, \ldots, \sqrt{w_m})$.

III. SOURCE LOCALIZATION FROM RANGE-DIFFERENCE MEASUREMENTS

Suppose that there exists an additional sensor (sensor 0) located at the origin and that the range-difference measurements between sensor i and sensor 0 are given by

$$d_i = \|\mathbf{x}-\mathbf{a}_i\| - \|\mathbf{x}\|, \quad i = 1, \ldots, m \tag{27}$$

which yield the following equations in the vector $\mathbf{x}$:

$$-2d_i\|\mathbf{x}\| - 2\mathbf{a}_i^T\mathbf{x} = d_i^2 - \|\mathbf{a}_i\|^2, \quad i = 1, \ldots, m. \tag{28}$$

The latter equations hold only approximately due to measurement errors and sensor calibration errors. Consequently, a reasonable way to estimate $\mathbf{x}$ based on (28) is via the minimization of the following LS criterion:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{i=1}^{m} \left(-2\mathbf{a}_i^T\mathbf{x} - 2d_i\|\mathbf{x}\| - g_i \right)^2 \tag{29}$$

where $g_i = d_i^2 - \|\mathbf{a}_i\|^2$. The solution of this problem will be called the *squared-range-difference-based least squares* (SRD-LS) estimate. Similarly to the SR-LS solution, the SRD-LS estimate is not optimal in the maximum-likelihood sense [16]–[18].

Problem (29) was considered in several works [1], [3], [14]. However, an exact solution of this problem was never derived.

In order to solve this problem, we use a similar approach to the one advocated in Section II-B and reformulate (29) as a constrained LS problem (with $\mathbf{y} = (\mathbf{x}^T, \|\mathbf{x}\|)^T$)

$$\min_{\mathbf{y}\in\mathbb{R}^{n+1}} \{ \|\mathbf{B}\mathbf{y}-\mathbf{g}\|^2 : \mathbf{y}^T\mathbf{C}\mathbf{y} = 0, y_{n+1} \geq 0 \} \tag{30}$$

where

$$\mathbf{B} = \begin{pmatrix} -2\mathbf{a}_1^T & -2d_1 \\ \vdots & \vdots \\ -2\mathbf{a}_m^T & -2d_m \end{pmatrix}$$
$$\mathbf{C} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{pmatrix}. \tag{31}$$

The matrix $\mathbf{B}$ is assumed to have full column rank. We note that the constraint $y_{n+1} \geq 0$ was overlooked in previous works [3], [14] and as a result the solution obtained in these works is incorrect in some cases. Similarly to Section II-B, the *unconstrained SRD-LS* (USRD-LS) approximate solution of the above problem is the vector comprised of the first n components of $(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{g}$, which is the minimizer of (30) after discarding the two constraints. It was shown in [14] that the USRD-LS estimate coincides with the spherical interpolation method of [1], [5] and the subspace minimization solution of [5].

Mathematically, the key difference between (30) and (16) is that (30) has *two* quadratic constraints (recall that a linear constraint is a special case of a general quadratic constraint), while (16) involves only a single quadratic constraint. As opposed to GTRS problems, there are no known necessary and sufficient optimality conditions for nonconvex quadratic optimization problems with two quadratic constraints. In fact, it is still an open question whether or not nonconvex quadratic problems with two constraints can be solved efficiently [19], [20].

In this section we will show, based on the special structure of (30), that an efficient algorithm can be devised to find the global solution of this problem. We begin by introducing sufficient optimality conditions for (30) that are similar to the necessary and sufficient optimality conditions (20), (21), and (22) for (16).

Lemma 3.1 (Sufficient Optimality Conditions): Let $\tilde{\mathbf{y}} \in \mathbb{R}^{n+1}$. If there exists $\lambda \in \mathbb{R}$ such that

$$(\mathbf{B}^T\mathbf{B} + \lambda\mathbf{C})\tilde{\mathbf{y}} = \mathbf{B}^T\mathbf{g} \tag{32}$$
$$\mathbf{B}^T\mathbf{B} + \lambda\mathbf{C} \succeq \mathbf{0} \tag{33}$$
$$\tilde{\mathbf{y}}^T\mathbf{C}\tilde{\mathbf{y}} = 0, \tilde{y}_{n+1} \geq 0 \tag{34}$$

then $\tilde{\mathbf{y}}$ is an optimal solution of (30).

Proof: It follows from (32) and (33) that $\tilde{\mathbf{y}}$ is an optimal solution of

$$\min_{\mathbf{y}\in\mathbb{R}^{n+1}} \{ \|\mathbf{B}\mathbf{y}-\mathbf{g}\|^2 + \lambda\mathbf{y}^T\mathbf{C}\mathbf{y} \}. \tag{35}$$

Therefore, for every feasible solution $\mathbf{y}$ of (30) we have

$$\|\mathbf{B}\mathbf{y}-\mathbf{g}\|^2 = \|\mathbf{B}\mathbf{y}-\mathbf{g}\|^2 + \lambda\mathbf{y}^T\mathbf{C}\mathbf{y} \tag{36}$$
$$\geq \|\mathbf{B}\tilde{\mathbf{y}}-\mathbf{g}\|^2 + \lambda\tilde{\mathbf{y}}^T\mathbf{C}\tilde{\mathbf{y}} \tag{37}$$
$$= \|\mathbf{B}\tilde{\mathbf{y}}-\mathbf{g}\|^2 \tag{38}$$

where the first and last equalities are valid due to the feasibility of $\mathbf{y}$ and $\tilde{\mathbf{y}}$, respectively, and the inequality follows from the fact that $\tilde{\mathbf{y}}$ is an optimal solution of (35). ■

Motivated by the sufficient conditions of Lemma 3.1, we can now define a solution procedure similar to procedure SR-LS defined in Section II-B.

Procedure Prototype:

a) Find the solution λ^* to

$$\tilde{\mathbf{y}}(\lambda)^T\mathbf{C}\tilde{\mathbf{y}}(\lambda)=0, \quad \lambda\in I_1 \tag{39}$$

where

$$\tilde{\mathbf{y}}(\lambda)=(\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C})^{-1}\mathbf{B}^T\mathbf{g}. \tag{40}$$

b) Set $\mathbf{z}=\tilde{\mathbf{y}}(\lambda^*)$.

The function $\phi(\lambda)\equiv\tilde{\mathbf{y}}(\lambda)^T\mathbf{C}\tilde{\mathbf{y}}(\lambda)$ is strictly decreasing over I_1 so that a simple bisection algorithm can be invoked in step (a).

By Lemma 3.1 it follows that if $\mathbf{z}$ satisfies $z_{n+1}\geq 0$, then $\mathbf{z}$ is a global optimal solution of (30). However, there is no guarantee that z_{n+1} will indeed be nonnegative as the following example demonstrates.

Example 3: Consider an array of $m=5$ sensors in the plane ($n=2$) whose coordinates are given by $\mathbf{a}_1=(-5,-13)^T,\mathbf{a}_2=(-12,1)^T,\mathbf{a}_3=(-1,-5)^T,\mathbf{a}_4=(-9,-12)^T$ and $\mathbf{a}_5=(-3,-12)^T$. The source's coordinates are $\mathbf{x}_S=(-5,11)^T$. The exact range-differences ($\|\mathbf{x}_S-\mathbf{a}_i\|-\|\mathbf{x}_S\|$) and their noisy observations are given by

exact	11.9170	0.1235	4.4094	11.2622	11.0037
noisy	11.8829	0.1803	4.6399	11.2402	10.8183.

The observed range-differences were obtained by adding a white Gaussian noise with standard deviation 0.2 to the exact range-differences. The solution to (39) is $\lambda^*=0.3347$ which, when used in (40), gives $\tilde{\mathbf{y}}(\lambda^*)=(-7.1645,-12.2497,-14.1910)^T$. The last component of this solution is negative, which implies that this solution is meaningless. ■

Later on, we will demonstrate by numerical simulations (see Example 4) that procedure Prototype usually fails for problems with "high" noise levels. In order to derive a solution for (30) *for every possible instance*, we will use *necessary* optimality conditions, as devised in the following theorem.

Theorem 3.1 (Necessary Optimality Conditions): The optimal solution to (30) is either $\mathbf{y}=0$ or has the form

$$\tilde{\mathbf{y}}(\lambda)=(\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C})^{-1}\mathbf{B}^T\mathbf{g} \tag{41}$$

where λ is such that

$$\tilde{\mathbf{y}}(\lambda)^T\mathbf{C}\tilde{\mathbf{y}}(\lambda)=0 \tag{42}$$

and $\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}$ has at most one negative eigenvalue.

Proof: Problem (30) can be rewritten as

$$\min_{\mathbf{y}\in\mathbb{R}^{n+1}}\{f_0(\mathbf{y}):f_1(\mathbf{y})=0,f_2(\mathbf{y})\leq 0\} \tag{43}$$

where $f_0(\mathbf{y})=\|\mathbf{B}\mathbf{y}-\mathbf{g}\|^2, f_1(\mathbf{y})=\mathbf{y}^T\mathbf{C}\mathbf{y}$ and $f_2(\mathbf{y})=-y_{n+1}$.

Let $\mathbf{y}$ be a nonzero optimal solution of (43). Since $\mathbf{y}\neq\mathbf{0}$ it follows that the second constraint is not active, i.e., $f_2(\mathbf{y})<0$, and that $\nabla f_1(\mathbf{y})\neq\mathbf{0}$, which implies that $\mathbf{y}$ is a regular vector[2]. Thus, by the first order optimality conditions [21], it follows that there exist $\lambda\in\mathbb{R}$ and $\mu\geq 0$ such that

$$\nabla f_0(\mathbf{y})+\lambda\nabla f_1(\mathbf{y})+\mu\nabla f_2(\mathbf{y})=0 \tag{44}$$

$$f_1(\mathbf{y})=0, f_2(\mathbf{y})\leq 0 \tag{45}$$

$$\mu f_2(\mathbf{y})=0. \tag{46}$$

Furthermore, by the second-order optimality conditions [21]

$$\mathbf{v}^T(\nabla^2 f_0(\mathbf{y})+\lambda\nabla^2 f_1(\mathbf{y})+\mu\nabla^2 f_2(\mathbf{y}))\mathbf{v}\geq 0 \tag{47}$$

for all $\mathbf{v}$ such that $\mathbf{v}^T\nabla f_1(\mathbf{y})=0$. Combining the fact that $f_2(\mathbf{y})<0$ with (46) implies $\mu=0$, which, together with (44) yields

$$(\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C})\mathbf{y}=\mathbf{B}^T\mathbf{g}.$$

Finally, by the Courant-Fischer min-max theorem ([22, p. 116]) along with (47), we conclude that the second smallest eigenvalue λ_n of $\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}$ satisfies

$$\lambda_n=\max_{\mathbf{w}\neq\mathbf{0}}\min_{\mathbf{v}^T\mathbf{w}=0}\frac{\mathbf{v}^T(\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C})\mathbf{v}}{\|\mathbf{v}\|^2}$$
$$\geq\min_{\mathbf{v}^T\nabla f_1(\mathbf{y})=0}\frac{\mathbf{v}^T(\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C})\mathbf{v}}{\|\mathbf{v}\|^2}\geq 0.$$

Therefore, $\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}$ has at most one negative eigenvalue. ■

Before proceeding, we introduce the following essential notation:

$$\alpha_i=-\frac{1}{\lambda_i(\mathbf{C},\mathbf{B}^T\mathbf{B})}, \quad i=1,\ldots,n \tag{48}$$

$$\alpha_0=-\frac{1}{\lambda_{n+1}(\mathbf{C},\mathbf{B}^T\mathbf{B})}. \tag{49}$$

Since $\mathbf{B}^T\mathbf{B}$ is positive definite and $\mathbf{C}$ has one negative eigenvalue and n strictly positive eigenvalues, it follows that α_0 is positive while α_i for $i=1,\ldots,n$ are negative:

$$\alpha_n\leq\alpha_{n-1}\leq\ldots\leq\alpha_1<0<\alpha_0. \tag{50}$$

We will also make use of the intervals

$$I_0=(\alpha_0,\infty), I_1=(\alpha_1,\alpha_0), I_2=(\alpha_2,\alpha_1).$$

The following simple Lemma, which will be used later on in the analysis, characterizes the number of positive and negative eigenvalues of $\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}$ over the intervals I_0, I_1, and I_2.

Lemma 3.2: (i). I_1 is the set of all λ for which $\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}$ is positive definite. (ii). The union of intervals $I_0\cup I_2$ is the set of all λ for which $\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}$ has exactly one negative eigenvalue and n positive eigenvalues.

[2]Given an optimization problem, a feasible point is regular if the gradients of the active constraints are linearly independent [21].

Proof: First note that

$$\begin{aligned}&\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}\\&=(\mathbf{B}^T\mathbf{B})^{1/2}(\mathbf{I}+\lambda(\mathbf{B}^T\mathbf{B})^{-1/2}\mathbf{C}(\mathbf{B}^T\mathbf{B})^{-1/2})\\&\quad\times(\mathbf{B}^T\mathbf{B})^{1/2}.\end{aligned}\tag{51}$$

Therefore, the numbers of positive, negative and zero eigenvalues of $\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}$ are the same as the numbers of positive, negative, and zero eigenvalues of $\mathbf{I}+\lambda(\mathbf{B}^T\mathbf{B})^{-1/2}\mathbf{C}(\mathbf{B}^T\mathbf{B})^{-1/2}$, respectively. The eigenvalues of the latter matrix are given by $\{1+\lambda\lambda_i(\mathbf{C},\mathbf{B}^T\mathbf{B}):i=1,\ldots,n+1\}$, which, by using the notation of α_i (48), can be written as $\{1-\lambda/\alpha_i:i=0,\ldots,n\}$. We now proceed to prove the two parts of the lemma by investigating the signs of the numbers $1-\lambda/\alpha_i, i=0,\ldots,n$.

i) $\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}\succ\mathbf{0}$ if and only if the set of inequalities $1-\lambda/\alpha_i>0, i=0,\ldots,n$, holds true. By (50), the latter set of inequalities is equivalent to $\lambda\in I_1$.

ii) In order to prove this part of the lemma, we split our analysis into two cases. If $\lambda\geq 0$ then $1-\lambda/\alpha_i>0$ for every $i=1,\ldots,n$. Therefore, in this case there is only one negative eigenvalue if and only if $1-\lambda/\alpha_0>0$, that is $\lambda\in I_0$. If $\lambda\leq 0$ then $1-\lambda/\alpha_0>0$ and we also have the following:

$$1-\lambda/\alpha_1\leq 1-\lambda/\alpha_2\leq\cdots\leq 1-\lambda/\alpha_n.$$

Therefore, for the case $\lambda\leq 0$, the matrix $\mathbf{B}^T\mathbf{B}+\lambda\mathbf{C}$ has one negative eigenvalue and n positive eigenvalues if and only if

$$1-\lambda/\alpha_1<0<1-\lambda/\alpha_2,$$

which is equivalent to $\lambda\in I_2$. ∎

Using the Theorem 3.1, we can now define a procedure that is *guaranteed* to find the global optimal solution of (29):

Procedure SRD-LS:

a) Apply procedure Prototype and obtain a vector $\mathbf{z}\in\mathbb{R}^{n+1}$.
   - If $z_{n+1}\geq 0$ then STOP. The output of the procedure is the vector made of the first n components of $\mathbf{z}$.
   - If $z_{n+1}<0$, then perform steps (b), (c), and (d).

b) Find all roots $\lambda_1,\ldots,\lambda_p$ of

$$\tilde{\mathbf{y}}(\lambda)^T\mathbf{C}\tilde{\mathbf{y}}(\lambda)=0,\lambda\in I_0\cup I_2\tag{52}$$

for which the $(n+1)$th component of $\tilde{\mathbf{y}}(\lambda_i)$ is nonnegative.

c) Let $\mathbf{z}$ be the vector with the smallest objective function among the vectors $\mathbf{0},\tilde{\mathbf{y}}(\lambda_1),\ldots,\tilde{\mathbf{y}}(\lambda_p)$.

d) The output of the procedure is the vector made of the first n components of $\mathbf{z}$.

Example 3 (Cont'd): The relevant intervals are given by

$$\begin{aligned}&I_0=(0.605,\infty), I_1=(-276.901,0.605)\\&I_2=(-1071.135,-276.901).\end{aligned}$$

We have already observed that procedure Prototype fails here. Therefore, by Theorem 3.1, the optimal solution is either the all-zero vector or is of the form $\tilde{\mathbf{y}}(\lambda)$ where λ is a solution to (52). In this example, (52) has only one root, which happens to lie in the interval I_0, and that is equal to $\lambda=0.8773$. The corresponding global solution of (30) is $(-4.9797,10.2784,11.4211)^T$. A simple calculation shows that this solution has a smaller objective value than the all-zero vector and thus that $(-4.9797,10.2784)^T$ is the global optimal solution of (29). ∎

The only remaining implementation issue is how to find all the roots of (52). Note that since $\mathbf{B}^T\mathbf{B}$ is positive definite it follows that $\mathbf{B}^T\mathbf{B}$ and $\mathbf{C}$ can be simultaneously diagonalized, i.e., there exists a nonsingular $(n+1)\times(n+1)$ matrix $\mathbf{P}$ for which

$$\begin{aligned}\mathbf{P}^T\mathbf{B}^T\mathbf{B}\mathbf{P}&=\operatorname{diag}(\gamma_1,\ldots,\gamma_{n+1})\\\mathbf{P}^T\mathbf{C}\mathbf{P}&=\operatorname{diag}(\delta_1,\ldots,\delta_{n+1}).\end{aligned}$$

Therefore, (52) reads

$$\sum_{j=1}^{n+1}\frac{f_j^2\delta_j}{(\gamma_j+\lambda\delta_j)^2}=0$$

where $\mathbf{f}=\mathbf{P}^T\mathbf{B}^T\mathbf{g}$. Multiplying this equation by the product of all the denominators $\prod_{j=1}^{n+1}(\gamma_j+\lambda\delta_j)^2$ transforms it into a polynomial equation of order $2n$

$$\sum_{j=1}^{n+1}f_j^2\delta_j\prod_{k=1,k\neq j}^{n+1}(\gamma_k+\lambda\delta_k)^2=0.$$

For $n=2$ this is a polynomial equation of order 4 and, thus, explicit algebraic expressions for its roots exist. For $n=3$ this is a polynomial equation of order 6: the roots of this polynomial equation can be found by standard root finding routines. For example, an efficient and stable procedure for finding all the roots is by calculating the eigenvalues of the companion matrix associated with the polynomial in question (this technique is implemented for example in the MATLAB function roots).

Example 4: In this example, we consider an array with 11 sensors (including sensor 0). In each run, the coordinates of the 10 sensors that are not located at the origin were randomly generated from a uniform distribution over the square $[-200,-190]\times[-200,-190]$ and the coordinates of the source were randomly generated from a uniform distribution over the square $[-10,10]\times[-10,10]$. The observed range-difference measurements were obtained by adding a normal random variable with mean zero and variance σ^2 to the exact range-differences. Table III describes the results for five values of σ. For each value of σ, 10 000 runs were performed. The first two columns give the location of the optimal λ (if it exists): the column $I_0(I_1)$ shows the number of runs for which the optimal λ belonged to $I_0(I_1)$. The column *zero* describes the number of runs in which the optimal solution was the all-zero vector $\mathbf{0}$. Finally, the two last columns show the mean squared position error of the SRD-LS and USRD-LS solutions (averaged over 10 000 realizations).

Note that for high noise levels, the optimal λ had a good chance to reside in I_0 (in which case, procedure Prototype fails).

TABLE III
COMPARISON BETWEEN THE SRD-LS AND USRD-LS ESTIMATES IN THE RANGE-DIFFERENCE SETTING

σ	I_1	I_0	zero	SRD-LS	USRD-LS
1e-4	10000	0	0	**7.1e-3**	1.9e-1
1e-3	9999	1	0	**3.9e-1**	1.9e+1
1e-2	9575	385	40	**5.4e+0**	3.1e+3
1e-1	3805	6047	148	**3.6e+1**	3.8e+4
1e+0	0	9476	524	**1.2e+2**	7.1e+4

For some unknown reason, the optimal λ never belonged to I_2 in our experiments. Furthermore, for high noise levels, there is also a chance that the optimal solution is the all-zero vector.

Evidently, the SRD-LS estimate outperforms the USRD-LS estimate by several orders of magnitude. The reason for the extremely poor performance of the USRD-LS estimate is that for some realizations of the positions of the sensors, source and errors, the matrix $\mathbf{B}$ was ill-conditioned. In these cases, the USRD-LS solution is unstable and can have a huge norm, which causes the average to be very large. In that respect, the SRD-LS solution might be viewed as a regularization of the USRD-LS estimate. ■

As a final remark we note that an alternative approach to the source localization problem from range-difference measurements is to use a least squares criterion similar to the one used in Section II-A. More precisely, the estimate $\mathbf{x}$ can be chosen to be the solution of

$$\min_{\mathbf{x}} \sum_{i=1}^{m} (d_i + ||\mathbf{x}|| - ||\mathbf{x} - \mathbf{a}_i||)^2. \tag{53}$$

An SDR of this nonconvex problem can be devised by following a construction similar to the one described in Section II-A. However, we have observed through numerical experiments that the so obtained SDR estimate is an extremely poor approximation of the true source location (far worse than the SRD-LS and USRD-LS estimates) and that it usually provides a meaningless solution.

IV. CONCLUSION

The results of this paper suggest that the problems of source localization from range measurements or from range-difference measurements should be approached using the SR-LS and SRD-LS methods, for which we have provided computationally efficient algorithms. The exact SRD-LS estimate of the source's coordinate vector can be more accurate by several orders of magnitude than the approximate least squares solutions proposed in the previous literature (see the numerical examples in the paper), a fact that appears to offset completely the slight computational advantage of the latter estimates. We have also shown that the SR-LS approach outperforms an SDR-based approximation of the R-LS solution, which is a maximum likelihood estimator under the Gaussian white noise assumption (in the cases when the R-LS solution can be found exactly, it usually provides more accurate solutions). We are thus led to the conclusion that the ability to find an *exact* solution of the SR-LS problem is the source of the attractiveness of this method. It is an open question whether exact solutions of the R-LS problem can be computed efficiently.

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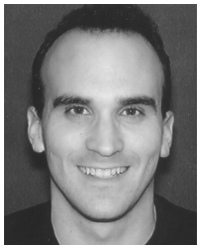

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