On a Class of Boussinesq Equations for Shallow Water Waves

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Abstract. The Euler's equations describing the dynamics of capillary-gravity water waves in two-dimensions are considered in the limits of small-amplitude and long-wavelength under appropriate boundary conditions. Using a double-series perturbation analysis, a general Boussinesq type of equation is derived involving the small-amplitude and long-wavelength parameters. A recently introduced sixth-order Boussinesq equation by Daripa and Hua [Appl. Math. Comput. **101** (1999), 159–207] is recovered from this equation in the 1/3 Bond number limit (from below) when the above parameters bear a certain relationship as they approach zero.

Keywords: Shallow Water Waves, Boussinesq Equations, Bi-directional Wave Propagation.

1 Introduction

Theoretical models of shallow water waves are often derived under the idealized assumptions facilitating analysis and numerical computation. The hope is that these models are accurate enough for intended purpose. There are numerous models because no single model can capture all the phenomena associated with the shallow water waves. For example, the family of KdV equations describes the uni-directional propagation of waves, whereas the family of Boussinesq equations describes the bi-directional propagation of waves in shallow water (see Johnson [10] and Whitham [13] for details). Each model within each family has its own range of applicability. For example, with surface tension effects included, the third-order KdV and fourth-order Boussinesq equations are appropriate for Bond number greater than 1/3 (see Hunter and Vanden-Broeck [9]), whereas the fifth-order KdV and sixth-order Boussinesq equations are appropriate for Bond number less than but very close to 1/3 (see Hunter and Scherule [8]).

It is well known that the velocity field in the shallow water is actually more complicated than these models would seem to indicate. This is not so surprising as these models are valid when small-amplitude and long-wavelength parameters bear a certain relationship as they approach zero. These restrictions are too rigid

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and very unlikely to hold in general in shallow water for arbitrary values of these parameters, however small these might be. More appropriate models for shallow water that can more accurately predict actual velocity field and other associated quantities can be obtained by incorporating the effect of these two parameters. In fact, this strategy can be used in a straight forward manner to derive shallow water models that would lead to KdV equation.

The evolution and physical significance of various higher order model equations for water waves and the existence and non-existence of solitary wave solutions for these equations are discussed in Kichenassamy and Olver [11]. Recently, Bona and Chen [2] and Chen [4] have devised and studied various Boussinesq systems, some of which are of the same order of approximations as the ill-posed fourth-order Boussinesq equation but are well-posed as initial value problems. In these works, the effect of surface tension was neglected.

Here we are concerned with more general Boussinesq type of model equations containing both the amplitude and wavelength parameters in addition to the Bond number (surface tension parameter). Such a model will have extended range of applicability than the ones with only one parameter. In section 2, the Euler's equations describing the dynamics of capillary-gravity water waves in two-dimensions are considered in the limits of small-amplitude and long-wavelength under appropriate boundary conditions. In section 3, using a double-series perturbation analysis, a general Boussinesq type of equation containing both the small-amplitude and long-wavelength parameters is derived. In section 4, the fourth-order and sixth-order Boussinesq equations (Daripa and Hua [5]) are recovered when these parameters bear certain relationships as they approach zero.

2 Formulation of the Water Wave Problem

Let $z = 0$ represent the bottom topography and $z = h(x,t) = h_0 + a\eta(x,t)$ represent the free water surface, where h_0 is the height of the undisturbed water surface, a is the amplitude of the surface wave and $\eta(x,t)$ is the free surface elevation from its undisturbed location. Let (u,w) represent the velocity field in (x,z) co-ordinate. We use the following non-dimensionalization

$$\begin{aligned} &x \to lx,\ z \to h_0 z,\ t \to \tfrac{l}{\sqrt{gh_0}}t,\ u \to \tfrac{a}{h_0}\sqrt{gh_0}u, \\ &w \to \left(\tfrac{a}{h_0}\right)\left(\tfrac{h_0}{l}\right)\sqrt{gh_0}w,\ p \to p_a + \rho g(h_0 - z) + \tfrac{a}{h_0}(\rho g h_0)p, \end{aligned} \tag{2.1}$$

where l is the wavelength of the surface wave, g is the acceleration due to gravity, ρ is the density of the fluid, p is the pressure field, and p_a is the atmospheric pressure. In non-dimensional form, the Euler's equations of motion governing the capillary-gravity shallow water waves (see Johnson [10]) are given by

$$\begin{aligned} &u_t + \alpha(uu_x + wu_z) = -p_x, \\ &\beta\big[w_t + \alpha(uw_x + ww_z)\big] = -p_z, \\ &u_x + w_z = 0. \end{aligned} \tag{2.2}$$

The corresponding kinematic and dynamic boundary conditions are given by

$$\begin{aligned} w &= 0 \quad \text{at} \quad z = 0, \\ w &= \eta_t + \alpha\, u\, \eta_x \quad \text{at} \quad z = 1 + \alpha\eta, \\ p &= \eta - \beta\tau \frac{\eta_{xx}}{[1+\alpha^2\beta\eta_x^2]^{3/2}} \quad \text{at} \quad z = 1 + \alpha\eta. \end{aligned} \tag{2.3}$$

Here $\alpha = a/h_0$ (amplitude parameter), $\beta = (h_0/l)^2$ (wavelength parameter) and $\tau = \Gamma/\rho g h_0^2$ (Bond number) where Γ is the surface tension coefficient.

Before proceeding for analysis, we rewrite the boundary conditions (2.3) by expressing these at $z = 0$ and $z = 1$ through a Taylor series expansion for u, w and p. So, we have

$$\begin{aligned} & w = 0 \quad \text{at} \quad z = 0, \\ & w + \alpha\eta w_z + \frac{\alpha^2\eta^2}{2} w_{zz} = \eta_t + \alpha\eta_x(u + \alpha\eta u_z) + O(\alpha^3) \quad \text{at} \quad z = 1, \\ & p + \alpha\eta p_z + \frac{\alpha^2\eta^2}{2} p_{zz} = \eta - \beta\tau\eta_{xx} + O(\alpha^3, \alpha^2\beta^2) \quad \text{at} \quad z = 1. \end{aligned} \tag{2.4}$$

In the section below, we derive a general Boussinesq type of equation for η from governing equations (2.2) and boundary conditions (2.4) by suitably eliminating u, w and p through a double-series perturbation analysis.

3 Double Series Perturbation Analysis

We express the solutions of the unknown variables $q = (u, w, p, \eta)$ in the problem through a double power series of the form

$$q = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha^i \beta^j q_{ij} = q_{00} + \alpha q_{10} + \beta q_{01} + \alpha^2 q_{20} + \beta^2 q_{02} + \alpha\beta q_{11} + \cdots. \tag{3.1}$$

Upon substituting expansions (3.1) for u, w, p and η into the governing equations (2.2) and boundary conditions (2.4), we obtain the following equations and boundary conditions of various orders as the coefficients of $\alpha^i\beta^j$, $i = 0, 1, \cdots,\ j = 0, 1, \cdots$

Equations of $O(1)$:

$$\left\{ \begin{aligned} & u_{00t} = -p_{00x} \\ & p_{00z} = 0 \\ & u_{00x} + w_{00z} = 0 \end{aligned} \right\} \tag{3.2}$$

$$\left\{ \begin{aligned} & w_{00} = 0 \text{ at } z = 0 \\ & w_{00} = \eta_{00t} \text{ at } z = 1 \\ & p_{00} = \eta_{00} \text{ at } z = 1 \end{aligned} \right\} \tag{3.3}$$

Equations of $O(\alpha)$:

$$\left\{ \begin{aligned} & u_{10t} + u_{00}u_{00x} + w_{00}u_{00z} \\ & \qquad = -p_{10x} \\ & p_{10z} = 0 \\ & u_{10x} + w_{10z} = 0 \end{aligned} \right\} \tag{3.4}$$

$$\left\{ \begin{aligned} & w_{10} = 0 \text{ at } z = 0 \\ & w_{10} + \eta_{00}w_{00z} = \eta_{10t} \\ & \qquad + \eta_{00x}u_{00} \text{ at } z = 1 \\ & p_{10} + \eta_{00}p_{00z} = \eta_{10} \text{ at } z = 1 \end{aligned} \right\} \tag{3.5}$$

Equations of $O(\beta)$:

$$\left\{\begin{array}{l} u_{01t} = -p_{01x} \\ w_{00t} = -p_{01z} \\ u_{01x} + w_{01z} = 0 \end{array}\right\} \quad (3.6) \qquad \left\{\begin{array}{l} w_{01} = 0 \text{ at } z = 0 \\ w_{01} = \eta_{01t} \text{ at } z = 1 \\ p_{01} = \eta_{01} - \tau\eta_{00xx} \text{ at } z = 1 \end{array}\right\} \quad (3.7)$$

Equations of $O(\alpha^2)$:

$$\left\{\begin{array}{l} u_{20t} + (u_{00}u_{10})_x + w_{00}u_{10z} \\ \qquad + w_{10}u_{00z} = -p_{20x} \\ p_{20z} = 0 \\ u_{20x} + w_{20z} = 0 \end{array}\right\} \quad (3.8) \qquad \left\{\begin{array}{l} w_{20} = 0 \text{ at } z = 0 \\ w_{20} + \eta_{00}w_{10z} + \eta_{10}w_{00z} + \frac{\eta_{00}^2}{2}w_{00zz} \\ \qquad = \eta_{20t} + \eta_{00x}u_{10} + \eta_{10x}u_{00} \\ \qquad + \eta_{00x}\eta_{00}u_{00z} \text{ at } z = 1 \\ p_{20} + \eta_{00}p_{10z} + \eta_{10}p_{00z} + \frac{\eta_{00}^2}{2}p_{00z} \\ \qquad = \eta_{20} \text{ at } z = 1 \end{array}\right\} \quad (3.9)$$

Equations of $O(\beta^2)$:

$$\left\{\begin{array}{l} u_{02t} = -p_{02x} \\ w_{01t} = -p_{02z} \\ u_{02x} + w_{02z} = 0 \end{array}\right\} \quad (3.10) \qquad \left\{\begin{array}{l} w_{02} = 0 \text{ at } z = 0 \\ w_{02} = \eta_{02t} \text{ at } z = 1 \\ p_{02} = \eta_{02} - \tau\eta_{01xx} \text{ at } z = 1 \end{array}\right\} \quad (3.11)$$

Equations of $O(\alpha\beta)$:

$$\left\{\begin{array}{l} u_{11t} + (u_{00}u_{01})_x + w_{00}u_{01z} \\ \qquad + w_{01}u_{00z} = -p_{11x} \\ w_{10t} + u_{00}w_{00x} + w_{00}w_{00z} \\ \qquad = -p_{11z} \\ u_{11x} + w_{11z} = 0 \end{array}\right\} \quad (3.12) \qquad \left\{\begin{array}{l} w_{11} = 0 \text{ at } z = 0 \\ w_{11} + \eta_{00}w_{01z} + \eta_{01}w_{00z} = \eta_{11t} \\ \qquad + \eta_{00x}u_{01} + \eta_{01x}u_{00} \text{ at } z = 1 \\ p_{11} + \eta_{00}p_{01z} + \eta_{01}p_{00z} = \eta_{11} \\ \qquad - \tau\eta_{10xx} \text{ at } z = 1 \end{array}\right\} \quad (3.13)$$

Below we derive canonical equations governing η_{00}, η_{10}, η_{01}, η_{20}, η_{02}, and η_{11} from the above set of equations by eliminating the other variables, namely u, w, and p. These equations are then combined through the perturbation series (3.1) to obtain the appropriate Boussinesq equation for η.

- **Equation for $\eta_{00}(x,t)$:** It is easy to see from equations (3.2) and (3.3) that

$$p_{00} = \eta_{00}, \quad u_{00t} = -\eta_{00x}, \quad w_{00} = -zu_{00x}, \quad u_{00x} = -\eta_{00t}, \tag{3.14}$$

and hence we obtain the equation for η_{00} as

$$\eta_{00tt} - \eta_{00xx} = 0. \tag{3.15}$$

Therefore, the solution of η_{00} will be of traveling wave form $E_{00}(x-t)+F_{00}(x+t)$ for some arbitrary functions E_{00} and F_{00}.

- **Equation for $\eta_{10}(x,t)$:** It is easy to see from equation (3.4b) and condition (3.5c) that

$$p_{10} = \eta_{10}. \tag{3.16}$$

Substituting equation (3.16) in equation (3.4a) and noting that u_{00} is independent of z, we obtain

$$u_{10t} = -\eta_{10x} - \frac{1}{2}(u_{00}^2)_x. \tag{3.17}$$

Thus u_{10} is also independent of z, and therefore, direct integration of equation (3.4c) with the help of condition (3.5a) gives

$$w_{10} = -zu_{10x}. \tag{3.18}$$

By matching equation (3.18) with condition (3.5b) and then using equation (3.14), we get

$$u_{10x} = -\eta_{10t} - (\eta_{00}u_{00})_x. \tag{3.19}$$

Equations (3.17) and (3.19) with the help of equation (3.14) now give

$$\eta_{10tt} - \eta_{10xx} - \left[\frac{1}{2}\eta_{00}^2 + u_{00}^2\right]_{xx} = 0. \tag{3.20}$$

Again from equation (3.14) by noting that $u_{00} = -\int_{-\infty}^{x}\eta_{00t}dx$, we obtain the equation for η_{10} as

$$\eta_{10tt} - \eta_{10xx} - \left[\frac{1}{2}\eta_{00}^2 + \left(\int_{-\infty}^{x}\eta_{00t}dx\right)^2\right]_{xx} = 0. \tag{3.21}$$

Therefore, the solution of η_{10} will be of traveling wave form $E_{10}(x-t)+F_{10}(x+t) + tG_{10}(x-t) + tH_{10}(x+t)$ for some arbitrary functions E_{10} and F_{10}; the functions G_{10} and H_{10} are dependent on E_{00} and F_{00}.

• **Equations for** $\eta_{01}(x,t)$: Using equation (3.14) in equation (3.6b) and integrating the resulting equation with the help of condition (3.7c), we obtain

$$p_{01} = \left[\eta_{01} + \left(\frac{1}{2} - \tau\right)\eta_{00xx}\right] - \frac{1}{2}\eta_{00xx}\, z^2. \tag{3.22}$$

Equations (3.6a,c) and (3.22) then give

$$w_{01zt} = -u_{01xt} = p_{01xx} = \left[\eta_{01xx} + \left(\frac{1}{2} - \tau\right)\eta_{00xxxx}\right] - \frac{1}{2}\eta_{00xxxx}\, z^2. \tag{3.23}$$

On integrating equation (3.23) with the help of condition (3.7a) and then matching its value at $z = 1$ with the t-derivative of condition (3.7b), we obtain the equation for η_{01} as

$$\eta_{01tt} - \eta_{01xx} - \left(\frac{1}{3} - \tau\right)\eta_{00xxxx} = 0. \tag{3.24}$$

Therefore, the solution of η_{01} will be of traveling wave form $E_{01}(x-t)+F_{01}(x+t) + tG_{01}(x-t) + tH_{01}(x+t)$ for some arbitrary functions E_{01} and F_{01}; the functions G_{01} and H_{01} are dependent on E_{00} and F_{00}.

• **Equations for** $\eta_{20}(x,t)$**:** It is easy to see from equation (3.8b) and condition (3.9c) that

$$p_{20} = \eta_{20}. \tag{3.25}$$

Substituting equation (3.25) in equation (3.8a) and using the fact that u_{00} and u_{10} are independent of z, we obtain

$$u_{20t} = -[\eta_{20x} + (u_{00}u_{10})_x]. \tag{3.26}$$

Thus u_{20} is also independent of z, and therefore, direct integration of equation (3.8c) with the help of condition (3.9a) gives

$$w_{20} = -zu_{20x}. \tag{3.27}$$

By matching equation (3.27) with condition (3.9b) and then using equation (3.14), we get

$$u_{20x} = -[\eta_{20t} + (\eta_{00}u_{10})_x + (\eta_{10}u_{00})_x]. \tag{3.28}$$

Equations (3.26) and (3.28) with the help of equations (3.14), (3.17) and (3.19) now give

$$\eta_{20tt} - \eta_{20xx} - (\eta_{00}\eta_{10})_{xx} - (2u_{00x}u_{10})_x + (2\eta_{10t}u_{00})_x + (\eta_{00x}u_{00}^2)_x = 0. \tag{3.29}$$

Again from equation (3.19) by noting that $u_{10} = -\int_{-\infty}^{x}\eta_{10t}dx - \eta_{00}u_{00}$, we obtain the equation for η_{20} as

$$\eta_{20tt}-\eta_{20xx}-(\eta_{00}\eta_{10})_{xx}-2\left[\int_{-\infty}^{x}\eta_{00t}dx\int_{-\infty}^{x}\eta_{10t}dx\right]_{xx}+\left[\eta_{00}\Big(\int_{-\infty}^{x}\eta_{00t}dx\Big)^2\right]_{xx}=0. \tag{3.30}$$

Therefore, the solution of η_{20} will be of traveling wave form $E_{20}(x-t)+F_{20}(x+t) + tG_{20}(x-t) + tH_{20}(x+t) + t^2P_{20}(x-t) + t^2Q_{20}(x+t)$ for some arbitrary functions E_{20} and F_{20}; the functions G_{20} and H_{20} are dependent on E_{00} and F_{00}; the functions P_{20} and Q_{20} are dependent on $E_{00}, F_{00}, E_{10}, F_{10}, G_{10}$, and H_{10}.

• **Equations for** $\eta_{02}(x,t)$**:** Using the expression for w_{01t} from equation (3.23) in equation (3.10b) and integrating the resulting equation with the help of condition (3.11c), we obtained

$$\begin{aligned} p_{02} &= \left[\eta_{02} + \Big(\frac{1}{2}-\tau\Big)\eta_{01xx} + \frac{1}{2}\Big(\frac{5}{12}-\tau\Big)\eta_{00xxxx}\right] \\ &\quad - \frac{1}{2}\left[\eta_{01xx} + \Big(\frac{1}{2}-\tau\Big)\eta_{00xxxx}\right]z^2 + \frac{1}{24}\eta_{00xxxx}\, z^4. \end{aligned} \tag{3.31}$$

Equations (3.10a,c) and (3.31) then give

$$\begin{aligned} w_{02zt} = -u_{02xt} = p_{02xx} &= \left[\eta_{02xx} + \Big(\frac{1}{2}-\tau\Big)\eta_{01xxxx} + \frac{1}{2}\Big(\frac{5}{12}-\tau\Big)\eta_{00xxxxxx}\right] \\ &\quad - \frac{1}{2}\left[\eta_{01xxxx} + \big(\frac{1}{2}-\tau\big)\eta_{00xxxxxx}\right]z^2 + \frac{1}{24}\eta_{00xxxxxx}\, z^4. \end{aligned} \tag{3.32}$$

On integrating equation (3.32) with the help of condition (3.11a) and then matching its value at $z = 1$ with the t-derivative of condition (3.11b), we obtain the equation for η_{02} as

$$\eta_{02tt} - \eta_{02xx} - \Big(\frac{1}{3} - \tau\Big)\eta_{01xxxx} - \frac{1}{3}\Big(\frac{2}{5} - \tau\Big)\eta_{00xxxxxx} = 0. \tag{3.33}$$

Therefore, the solution of η_{02} will be of traveling wave form $E_{02}(x-t) + F_{02}(x+t) + tG_{02}(x-t) + tH_{02}(x+t) + t^2P_{02}(x-t) + t^2Q_{02}(x+t)$ for some arbitrary functions E_{02} and F_{02}; the functions G_{02} and H_{02} are dependent on E_{00} and F_{00}; the functions P_{02} and Q_{02} are dependent on $E_{00}, F_{00}, E_{01}, F_{01}, G_{01}$, and H_{01}.

• **Equations for $\eta_{11}(x,t)$:** Using equations (3.14), (3.17) and (3.18) in equation (3.12b) and integrating the resulting equation with the help of condition (3.13c), we obtain

$$p_{11} = \Big[\eta_{11} + \Big(\frac{1}{2} - \tau\Big)\eta_{10xx} + \frac{1}{2}\eta_{00t}^2 + \eta_{00}\eta_{00xx}\Big] - \frac{1}{2}\big[\eta_{10xx} + \eta_{00t}^2\big]\, z^2. \tag{3.34}$$

Equations (3.12a,c), (3.14) and (3.34) then give

$$\begin{aligned} w_{11zt} = -u_{11xt} &= -p_{11xx} + (u_{00}u_{01})_{xx} + (w_{00}u_{01z})_x \\ &= \Big[(u_{00}u_{01})_{xx} + (w_{00}u_{01z})_x + \eta_{11xx} + \Big(\frac{1}{2} - \tau\Big)\eta_{10xxxx} \\ &\quad + \frac{1}{2}(\eta_{00t}^2)_{xx} + (\eta_{00}\eta_{00xx})_{xx}\Big] - \frac{1}{2}\big[\eta_{10xxxx} + (\eta_{00t}^2)_{xx}\big] z^2. \end{aligned} \tag{3.35}$$

On integrating equation (3.35) with the help of condition (3.13a) and then matching its value at $z = 1$ with the t-derivative of condition (3.13b), we obtain

$$\begin{aligned} &\int_0^1 \big[(u_{00}u_{01})_{xx} + (w_{00}u_{01z})_x\big] dx + \eta_{11xx} + \big(\frac{1}{3} - \tau\big)\eta_{10xxxx} \\ &\quad + \frac{1}{3}(\eta_{00t}^2)_{xx} + (\eta_{00}\eta_{00xx})_{xx} = \eta_{11tt} + (\eta_{00}u_{01})_{xt}\Big|_{z=1} + (\eta_{01}u_{00})_{xt}. \end{aligned} \tag{3.36}$$

By the help of calculations based on $O(1)$ and $O(\beta)$ equations, it is easy to see that

$$\begin{aligned} \int_0^1 \big[(u_{00}u_{01})_{xx} + (w_{00}u_{01z})_x\big] dx &= -u_{00x}\big[3w_{01}(1) - w_{01z}(1)\big] \\ &\quad - u_{00}w_{01x}(1) - u_{00xx}\Big[2\int_{-\infty}^x \eta_{01t}dx + u_{01}(1)\Big]. \end{aligned} \tag{3.37}$$

So, equation (3.36) with the help of equations (3.37), (3.2), (3.3), (3.6) and (3.7) is reduced to the following equation for η_{11}:

$$\begin{aligned} &\eta_{11tt} - \eta_{11xx} - (\eta_{00}\eta_{01})_{xx} - 2\Big(\int_{-\infty}^x \eta_{00t}dx \int_{-\infty}^x \eta_{01t}dx\Big)_{xx} - \Big(\frac{1}{3} - \tau\Big)\eta_{10xxxx} \\ &\quad - \frac{2}{3}(\eta_{00t}^2)_{xx} - (\eta_{00}\eta_{00xx})_{xx} + \tau(\eta_{00}\eta_{00xxx})_x = 0. \end{aligned} \tag{3.38}$$

Therefore, the solution of η_{11} will be of traveling wave form $E_{11}(x-t)+F_{11}(x+t)+tG_{11}(x-t)+tH_{11}(x+t)+t^2P_{11}(x-t)+t^2Q_{11}(x+t)$ for some arbitrary functions E_{11} and F_{11}; the functions G_{11} and H_{11} are dependent on E_{00} and F_{00}; the functions P_{11} and Q_{11} are dependent on $E_{00}, F_{00}, E_{01}, F_{01}, E_{10}, F_{10}, G_{01}, H_{01}, G_{10}, H_{10}$.

• **Equation for $\eta(x,t)$:** Combining equations (3.15), (3.21), (3.24), (3.30), (3.33) and (3.38) using the series expansion (3.1), we obtain the following equation for η accurate up to $O(\alpha^2, \beta^2, \alpha\beta)$:

$$\eta_{tt} - \eta_{xx} - \alpha\Big[\frac{1}{2}\eta^2 + \Big(\int_{-\infty}^{x} \eta_t dx\Big)^2\Big]_{xx} - \beta\Big[\frac{1}{3} - \tau\Big]\eta_{xxxx} + \alpha^2\Big[\eta\Big(\int_{-\infty}^{x} \eta_t dx\Big)^2\Big]_{xx}$$
$$- \alpha\beta\Big[\frac{2}{3}(\eta_t^2)_{xx} + (\eta\eta_{xx})_{xx} - \tau(\eta\eta_{xxx})_x\Big] - \frac{\beta^2}{3}\Big[\frac{2}{5} - \tau\Big]\eta_{xxxxxx} = 0. \qquad (3.39)$$

This is a general Boussinesq type of equation valid for small values of α and β. It describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water. It is to be noted that the solutions for the perturbed components $\eta_{10}, \eta_{01}, \eta_{20}, \eta_{02}$, and η_{11} contains "secular terms" (the functions multiplied with t or t^2) which will grow in time. So, these solutions will be unbounded as $t \to \infty$. Therefore, the perturbation series approximation for η will not be uniformly valid for all t. However, an examination of the secular terms indicates that the approximation for η will be valid up to a time t for which both αt and βt are less than 1, that is, for all $0 \le t < 1/\epsilon$ where $\epsilon = \max(\alpha, \beta)$.

4 Fourth-Order and Sixth-Order Boussinesq Equations

The fourth-order and sixth-order Boussinesq equations can be derived from equation (3.39) under appropriate scalings. It follows from equation (3.39) that the effect of non-linearity appears at $O(\alpha), O(\alpha^2)$ and $O(\alpha\beta)$ terms, where as, the effect of dispersion appears at $O(\beta)$ and $O(\beta^2)$ terms. The leading order dispersion term is $\beta\big(\frac{1}{3} - \tau\big)\eta_{xxxx}$. Therefore, it is important to consider the two special cases (i) $\big(\frac{1}{3} - \tau\big) \gg \beta$ and (ii) $\big(\frac{1}{3} - \tau\big) = O(\beta)$. The case (i) leads to the fourth-order Boussinesq equation whose fourth-order dispersive term vanishes for $\tau = \frac{1}{3}$. This emphasizes the significance of the case (ii) which leads to the sixth-order Boussinesq equation. These are briefly presented here.

• **Case I:** If $\big(\frac{1}{3} - \tau\big) \gg \beta$, that is, $\big(\frac{1}{3} - \tau\big) = K_1$ and $\frac{1}{3} \ge K_1 \gg \beta$, then a balance between the non-linearity and dispersion, which is necessary to model a solitary wave, requires $\alpha = O(\beta)$ as $\beta \to 0$, that is, $\alpha = K_2\beta$ as $\beta \to 0$ (K_2 fixed). Then we have the Boussinesq equation (3.39), correct up to $O(\alpha) = O(\beta)$, as

$$\eta_{tt} - \eta_{xx} - \alpha\left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t dx\right)^2\right]_{xx} - \frac{K_1}{K_2}\alpha\,\eta_{xxxx} = 0, \qquad (4.1)$$

This case is appropriate for $0 \le \tau \ll \frac{1}{3}$.

• **Case II:** If $\left(\frac{1}{3}-\tau\right) = O(\beta)$ as $\beta \to 0$, that is $\left(\frac{1}{3}-\tau\right) = K_1\beta$ as $\beta \to 0$ (K_1 fixed), then a balance between the non-linearity and dispersion requires $\alpha = O(\beta^2)$ as $\beta \to 0$, that is, $\alpha = K_2\beta^2$ as $\beta \to 0$ (K_2 fixed). Then we have the Boussinesq equation (3.39) correct up to $O(\alpha) = O(\beta^2)$ as

$$\eta_{tt} - \eta_{xx} - \alpha\left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x}\eta_t dx\right)^2\right]_{xx} - \frac{K_1}{K_2}\,\alpha\,\eta_{xxxx} - \frac{\alpha}{45K_2}\,\eta_{xxxxxx} = 0. \quad (4.2)$$

This case is appropriate for $\tau \uparrow \frac{1}{3}$ (Bond number less than but very close to $\frac{1}{3}$).

• **Co-ordinate Transformation and Transformed Equations:** At first sight, the fourth-order and sixth-order Boussinesq equations (4.1) and (4.2) look rather complicated. But, if we use the following co-ordinate transformation

$$\left.\begin{array}{l} X = \sqrt{\frac{K_2}{K_1}}\left(x + \alpha\int_{-\infty}^{x}\eta(x,t)dx\right) \\ T = \sqrt{\frac{K_2}{K_1}}\,t \end{array}\right\}, \quad (4.3)$$

and substitute

$$N = \frac{3}{2}(\eta - \alpha\eta^2), \quad (4.4)$$

then the fourth-order and sixth-order Boussinesq equations (4.1) and (4.2) are transformed to the following canonical forms

$$N_{TT} - N_{XX} - \alpha\,(N^2)_{XX} - \alpha\,N_{XXXX} = 0, \quad (4.5)$$

and

$$N_{TT} - N_{XX} - \alpha\,(N^2)_{XX} - \alpha\,N_{XXXX} - \epsilon^2\,\alpha\,N_{XXXXXX} = 0, \quad (4.6)$$

where we have again neglected terms of $O(\alpha^2)$ and higher. Here $\epsilon^2 = \frac{K_2}{45K_1^2}$. It is worth summarizing here that both the equations (4.5) and (4.6) represent the bi-directional propagation of small amplitude (weakly non-linear) and long wavelength (weakly dispersive) capillary-gravity waves on the surface of shallow water; equation (4.5) being appropriate when $0 \le \tau \ll 1/3$, and that, equation (4.6) being more appropriate when $\tau \uparrow 1/3$ (i.e., Bond number is less than but very close to 1/3). It is to be noted that $\tau \uparrow 1/3$ can hold true in the joint limit $K_1 \to \infty$ and $\beta \to 0$. So, we will have ϵ^2 as a small parameter independent of the amplitude parameters α. Therefore, equation (4.6) can be considered as a singular perturbation of equation (4.5). It is also closely related to the fifth-order KdV equation (see Hunter and Scherule [8]) which is restricted only to the uni-directional propagation of such waves. The fourth-order and sixth-order Boussinesq equations (4.5) and (4.6) can be written even in more standard forms using the scaling $X \to \alpha^{1/2}X$, $T \to \alpha^{1/2}T$, and $N \to \alpha^{-1}N$.

It is to be noted that the fourth-order Boussinesq equation (4.5) supports an one-parameter family of traveling solitary wave solutions. However, it is severely ill-posed as an initial value problem (see Daripa and Hua [5]). The sixth-order

Boussinesq equation (4.6), introduced by Daripa and Hua [5] on a heuristic ground, provides dispersive regularization of the ill-posed fourth-order Boussinesq equation (4.5). Here, we established the physical relevance of this equation in the context of water waves.

Daripa and Dash [6] recently studied the sixth-order Boussinesq equation (4.6) both analytically and numerically. It is established that, unlike the local solitary waves, the traveling wave solutions of this equation do not vanish in the far-field. Instead, such waves possess small amplitude fast oscillations at distances far from the core of the waves extending up to infinity. So, these solutions are similar to the weakly non-local solitary wave solutions of the fifth-order KdV equation (Akylas and Yang [1], Boyd [3], Grimshaw and Joshi [7], Hunter and Scherule [8], Pomeau et al. [12]).

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