The paper “Linear instability of the displacement of an Oldroyd-B fluid by air in a Hele-Shaw cell” that follows in the succeeding pages was originally submitted to the SIAM Journal of Applied Mathematics in March of 2014. It has not been possible to revise this paper since then. This paper remains unpublished.

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Linear instability of the displacement of an Oldroyd-B fluid by air in a Hele-Shaw cell

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Abstract

We perform linear instability analysis for the displacement of an incompressible Oldroyd-B fluid by air in a Hele-Shaw cell. We derive the perturbation equations from the full flow equations and average the dynamic boundary condition at the interface across the gap of the Hele-Shaw cell. A scaling procedure is used which allows us to neglect some terms in the divergence-free condition, flow equations and constitutive relations. The novelty of this paper is an explicit formula for the growth rate of perturbations given by a ratio: the denominator contains a term which depends on the relaxation and the retardation (time) constants, denoted by λ_1 and λ_2 respectively. When ($\lambda_1 = \lambda_2$), the growth rate is quite similar to the Saffman-Taylor formula for a Newtonian liquid displaced by air. According to this new formula, the displacement of an Oldroyd-B fluid by air is more unstable than that of a Newtonian liquid by air. In particular, we find that the maximum growth rate increases monotonically with increasing values of $(\lambda_1 - \lambda_2) < c$ where c depends on the problem data and is usually very small. This result is in qualitative agreement with the numerical results of Wilson (1990). We find that the growth rate is unbounded for either one or two values of the wavenumber depending on whether $(\lambda_1 - \lambda_2) = c$ or $(\lambda_1 - \lambda_2) > c$. We discuss relevance of such singularities in σ to fractures usually observed in Hele-Shaw flows of complex fluids.

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1 Introduction

Chemical enhanced oil recovery process often uses complex displacing fluids. These complex fluids are described by rheological properties very different from that of a Newtonian fluid. An important problem is then to develop a model for a displacement process of two complex fluids in a porous medium or an equivalent approximation. A simple model that has proven useful for flow of a Newtonian fluid in a porous medium is the flow of a Newtonian fluid in a Hele-Shaw cell which is a device consisting of two planar plates separated by a distance, b, very small compared to the dimensions of the plates. Flow of a viscous fluid confined in such a Hele-Shaw cell is commonly known as the Hele-Shaw flow. If the fluid is Newtonian, then it is simple to work out the flow velocity within lubrication approximation which neglects flow in the direction perpendicular to the plates and the velocity gradients in the plane parallel to the plates of the Hele-Shaw cell. Averaging this flow velocity in the gap then shows that the average velocity is given by the familiar Darcy's law of fluid flow in homogeneous porous medium with an equivalent permeability given by $b^2/(12\mu)$ where μ is the viscosity of the fluid. Much of the interest in Hele-Shaw flows originally arose from this analogy between such flows and porous media flows.

Much of the later works on Hele-Shaw flows has involved more than one fluid: mainly displacement of one fluid by another less viscous fluid. One of the reasons behind interest in such immiscible displacement processes is due to their relevance to secondary oil recovery. Even though Darcy's law has been derived based upon single phase flow, it has been applied to such two phase flows with considerable success. One of the earliest works on this type of Hele-Shaw flows is that of Saffman and Taylor [26] in which these authors study instability of an interface separating a viscous fluid from a less viscous displacing fluid. This instability is commonly known as the Saffman-Taylor instability. Dynamics and stability of one or more interfaces in Hele-Shaw flows have been studied extensively over last few decades due to intrinsic interest in viscous fingering and nonlinear pattern formation aside from industrial relevance of such studies and its relation to other nonlinear phenomena. However, most of these studies on Hele-Shaw flows with the exception of few have been carried out with Newtonian fluids.

In recent years, fingering instability in Hele-Shaw cell has been studied experimentally and numerically when one or both the fluids is complex such as liquid crystals (Buka et. al. [6, 7]), polymer solutions and melts (Zhao & Maher [32]), clays, and foam (Lemarie et. al. [15]). It has been found that Saffman-Taylor instability arises in such displacement processes as well with strikingly different complex fingering patterns depending on the nature of the complex fluids. New effects which are otherwise absent in the Newtonian case have been observed by Nittman et. al. [21] in experiments of polymer solution being displaced by water, and by Daccord and Lenormand [9] and Van Damme [29] in experiments of slurries and clay pastes displaced by water. In many cases, unstable interfaces in such complex fluids exhibit a rich variety of “fractal” patterns including viscous fingers. Numerous studies have attempted to extend the approach of Saffman and Taylor for pattern formation of non-Newtonian fluids: see de Gennes [11], Corvera-Poiré and Ben Amar [2, 3, 8], and Lavermann and Procaccia [16]. The fingering instability for Oldroyd-B and power law fluids has been studied by Wilson [31] and for upper convected Maxwell and power law fluids by Mora and Manna [18, 19].

Non-Newtonian fluids can show one or more of the following properties: finite yield stress, shear dependent viscosity, and elasticity among many others. Depending on the extent to which these properties affect the flow and interfacial instabilities, the fingering patterns can show a range of complexity from finger formation of various widths to complex patterns of fractal type. Shear thinning property usually leads to narrower finger as opposed to viscoelastic effect which usually tends to widen the finger width; see Bonn et. al. [4, 5]. This effect of shear thinning property can be partially explained from results on multi-layer Hele-Shaw flows of Newtonian fluids; see Daripa [10]. Solutions containing polymer usually show both shear thinning and viscoelastic properties resulting in more complex patterns. Even more complex instabilities appear when, in addition to these, finite yield stress effects are present such as in pastes and gels. Understanding and disentangling the effect of each of these different non-Newtonian properties on the overall dynamics of instabilities is a formidable challenge. Progress in this direction requires a combination of theoretical, numerical and experimental approach.

Linear stability analysis of such displacement processes of Saffman-Taylor type where both the fluids (displaced and displacing) are Oldroyd-B has not been performed to-date to the best of the author’s knowledge. This is partly due to the analytical difficulties involved in extracting a closed form formula for the corresponding dispersion relation. Therefore, it makes sense first to make progress in this direction by considering only one of these two fluids as visco-elastic which is what we do in this paper. Even this problem has not been studied in depth. Interestingly, it is worth mentioning here that there has been many studies of linear stability involving visco-elastic fluids for shear flows, thin-film flows and parallel flows; for example see [1, 14, 24, 25, 27, 28].

In this paper, we perform systematically linear instability analysis of the displacement process of an incompressible Oldroyd-B fluid by air in a Hele-Shaw cell. The perturbation equations are derived from the full flow equations, the constitutive relations, and the divergence-free condition. A scaling procedure, based on the flow geometry of the Hele-Shaw cell and other relevant variables, is derived which is subsequently used to justifiably neglect some terms of the linearized equations in the small Deborah number regime. The perturbations are Fourier decomposed with amplitude $f(z)$ whose explicit formula is also derived. This allows us to perform a depth-average procedure in the dynamic boundary condition (i.e., Laplace law) as a final step. This gives an explicit formula for the dispersion relation which is given by a ratio; see the formula (83). The denominator of this formula depends on the difference, $(\lambda_1 - \lambda_2)$, between the relaxation and the retardation (time) constants which appear in the constitutive relation (2) of the Oldroyd-B fluid. Interestingly, when $\lambda_1 = \lambda_2$ this formula reduces to the Saffman-Taylor formula (with some qualifications; see section 4) for a Newtonian liquid. Towards this end, we mention that for the case of weak elasticity, some results related with the stability of a thin film flow with visco-elasticity effect are given in Saprykin, Koopmans and Kalliadasis [27].

Our dispersion relation shows that the displacement process of an Oldroyd-B fluid by air is more unstable than that of a Newtonian liquid by air, thereby showing that elasticity is destabilizing. In particular, this formula shows that the maximum growth rate increases monotonically with increasing values of $(\lambda_1 - \lambda_2)$ for $(\lambda_1 - \lambda_2) < c$ where c is a constant depending on the problem data, in qualitative agreement with the numerical results of Wilson [31]. However, it is worth mentioning here that Wilson [31] considered a different scaling procedure but obtained a quite similar set of perturbation equations (see the relations (23)-(31) in Wilson [31]). He numerically solved a reduced set of equations to obtain numerical values of the growth rate σ. When $(\lambda_1 - \lambda_2) \geq c$, we obtain from our formula either one or two values of the wavenumber k where the growth rate is unbounded. These potential singular points may be related with the fractures usually observed in Hele-Shaw flows of complex fluids [21, 30, 32, 20]. For values of k (wavenumber) near the singular points of the growth rate σ, perhaps a nonlinear stability theory will be more useful. The following quote is taken verbatim from Wilson [31]: "this is still not quite enough to make an analytical solution possible." The novelty of this paper is an explicit formula (83) for the growth rate of perturbations in terms of the problem data.

The paper is laid out as follows. In section 2, we present the flow and constitutive equations for Oldroyd-B fluids. In section 3, we present the scaling procedure and the basic flow about which the stability calculations are performed. In section 4, linear stability analysis is performed leading to an explicit dispersion relation which reduces to the Saffman-Taylor formula for the Newtonian fluid. Thus we provide a natural extension of the Saffman-Taylor formula to the Oldroyd-B fluid which quantifies the effect of elasticity of Oldroyd-B fluid. Finally, we conclude in section 5.

2 The Oldroyd-B fluid

The Oldroyd-B fluid was first introduced by J. Oldroyd [22]. This fluid model is an extension of the upper convected Maxwell fluid and is equivalent to a fluid with elastic head and spring dumbbells. Below we denote by $\tau, \mathbf{D}$ the extra-stress and strain-rate tensors respectively, by μ the fluid viscosity, and by λ_1, λ_2 the relaxation and retardation time constants respectively. Below $\mathbf{L}$ is the derivative matrix of velocity $\mathbf{v} = (u, v, w)$. The strain-rate tensor is given by $\mathbf{D} = (\mathbf{L} + \mathbf{L}^T)$ where $(\mathbf{L}_{ij})^T = \mathbf{L}_{ji}$. The steady Stokes flow equations appropriate for flow in a Hele-Shaw cell are

$$-\nabla p + \nabla \cdot \tau = 0. \tag{1}$$

We have the following constitutive relation

$$\tau + \lambda_1 \tau^{\nabla} = \mu[\mathbf{D} + \lambda_2 \mathbf{D}^{\nabla}], \tag{2}$$

where τ^{∇} and $\mathbf{D}^{\nabla}$ denote the upper convected derivatives given by the following formulas (see [13, 17, 23]).

$$\tau^{\nabla} = \tau_t + \mathbf{v} \cdot \nabla \tau - (\mathbf{L}\tau + \tau \mathbf{L}^T), \qquad \mathbf{D}^{\nabla} = \mathbf{D}_t + \mathbf{v} \cdot \nabla \mathbf{D} - (\mathbf{L}\mathbf{D} + \mathbf{D}\mathbf{L}^T). \tag{3}$$

We consider an incompressible fluid. Therefore, we have the divergence-free condition.

$$u_x + v_y + w_z = 0. \tag{4}$$

From (1) we get the following flow equations

$$p_x = \tau_{11,x} + \tau_{12,y} + \tau_{13,z}, \quad p_y = \tau_{21,x} + \tau_{22,y} + \tau_{23,z}, \quad p_z = \tau_{31,x} + \tau_{32,y} + \tau_{33,z}, \tag{5}$$

where $\tau_{ij,x}$ refers to the x derivative of the component ij of the extra-stress tensor τ. Throughout this paper, with slight abuse of notation perhaps, we have used the numbers 1, 2 and 3 as subscripts to the notation for components of the stress tensor τ to mean the x, y and z variables respectively. Thus, for example $\tau_{11,x}$ means $\frac{\partial}{\partial x}\tau_{xx}$. The no-slip boundary condition for the velocity components on the plates and the kinematic and the dynamic boundary conditions on the air-fluid interface are given later in section 4 (see the relations (37)-(38)).

3 The scaling procedure

Figure 1 shows a Hele-Shaw cell with air displacing Oldroyd-B fluid. We use the scaling procedure given by Mora and Manna [18, 19] which allows us to neglect some terms in the perturbed constitutive relations and the perturbed flow equations.

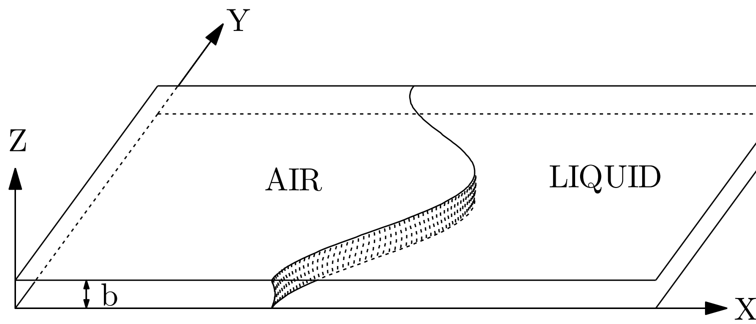

Figure 1: A Hele-Shaw cell: air displacing an Oldroyd-B fluid in the positive Ox direction.

3.1 Basic steady flow with two velocity components

Consider the flow driven by the constant pressure gradients in the x, y directions. The elements of this basic flow are denoted by the superscript ^{0}. The subscripts denote the partial derivatives with respect to x, y, z. The basic pressure and velocity are denoted by

$$\nabla p^0 = (p_x^0, p_y^0, 0), \quad \mathbf{v^0} = (u^0(z), v^0(z), 0). \tag{6}$$

The steady state basic solution depends only on z and satisfies the constitutive relations given by

$$\tau^0 - \lambda_1(\mathbf{L}^0\tau^0 + \tau^0\mathbf{L}^{0T}) = \mu\{\mathbf{D}^0 - \lambda_2(\mathbf{L}^0\mathbf{D}^0 + \mathbf{D}^0\mathbf{L}^{0T})\}, \tag{7}$$

$$\mathbf{L}^0 = \begin{pmatrix} 0 & 0 & u_z^0 \\ 0 & 0 & v_z^0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{D}^0 = \mathbf{L}^0 + \mathbf{L}^{0T} = \begin{pmatrix} 0 & 0 & u_z^0 \\ 0 & 0 & v_z^0 \\ u_z^0 & v_z^0 & 0 \end{pmatrix}. \tag{8}$$

Then the components τ_{ij}^0 of the basic extra-stress tensor are given in terms of the basic velocity components by the equations

$$\begin{pmatrix} \tau_{11}^0 & \tau_{12}^0 & \tau_{13}^0 \\ \tau_{21}^0 & \tau_{22}^0 & \tau_{23}^0 \\ \tau_{31}^0 & \tau_{32}^0 & \tau_{33}^0 \end{pmatrix} - \lambda_1 \begin{pmatrix} 2u_z^0\tau_{31}^0 & (u_z^0\tau_{32}^0 + \tau_{13}^0 v_z^0) & u_z^0\tau_{33}^0 \\ (u_z^0\tau_{32}^0 + \tau_{13}^0 v_z^0) & 2v_z^0\tau_{32}^0 & v_z^0\tau_{33}^0 \\ u_z^0\tau_{33}^0 & v_z^0\tau_{33}^0 & 0 \end{pmatrix} =$$

$$\mu \begin{pmatrix} 0 & 0 & u_z^0 \\ 0 & 0 & v_z^0 \\ u_z^0 & v_z^0 & 0 \end{pmatrix} - 2\mu\lambda_2 \begin{pmatrix} (u_z^0)^2 & u_z^0 v_z^0 & 0 \\ u_z^0 v_z^0 & (v_z^0)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

An immediate consequence of the above relation is $\tau_{33}^0 = 0$. Other components of the basic extra-stress tensor are

$$\left.\begin{aligned} \tau_{13}^0 &= \mu u_z^0, \quad \tau_{23}^0 = \mu v_z^0, \\ \tau_{11}^0 &= 2(\lambda_1 - \lambda_2)\mu (u_z^0)^2, \\ \tau_{12}^0 &= 2(\lambda_1 - \lambda_2)\mu u_z^0 v_z^0, \\ \tau_{22}^0 &= 2(\lambda_1 - \lambda_2)\mu (v_z^0)^2. \end{aligned}\right\} \tag{10}$$

The following basic flow equations hold

$$\left.\begin{aligned} p_x^0 &= \tau_{11,x}^0 + \tau_{12,y}^0 + \tau_{13,z}^0, \\ p_y^0 &= \tau_{21,x}^0 + \tau_{22,y}^0 + \tau_{23,z}^0, \\ p_z^0 &= \tau_{31,x}^0 + \tau_{32,y}^0 + \tau_{33,z}^0. \end{aligned}\right\} \tag{11}$$

As $\tau_{33}^0 = 0$ and $\tau^0 = \tau^0(z)$, the third relation above gives $p_z^0 = 0$. From the first two flow equations above, it follows

$$\left.\begin{aligned} p_x^0 &= \tau_{13,z}^0 = G = \mu u_{zz}^0, \\ p_y^0 &= \tau_{23,z}^0 = H = \mu v_{zz}^0, \end{aligned}\right\} \tag{12}$$

where G and H are negative constants. From (12), we get the velocity components

$$u^0 = \frac{1}{\mu} p_x^0 (z^2 - bz)/2, \quad v^0 = \frac{1}{\mu} p_y^0 (z^2 - bz)/2, \tag{13}$$

which are positive because $p_x^0, p_y^0 < 0$, and $z(z-b) < 0$ in the range $z \in (0, b)$. We use the relations (10) and (13) to obtain a scaling procedure for our particular flow geometry of thin Hele-Shaw cell starting from the basic pressure. Let l be a lateral length scale, and Q be a characteristic pressure determined in Appendix-A (also see condition (21)). We introduce the scaled coordinates denoted with a prime as follows.

$$x' = x/l, \quad y' = y/l, \quad z' = z/b, \quad \& \quad \epsilon = b/l << 1. \tag{14}$$

Below, the scaled basic dependent variables are denoted with a superscript ${}^{0'}$. Thus

$$p^{0'} = p^0/Q. \tag{15}$$

The scaled time coordinate t' is defined later in (19). It follows from (13) after introducing the above scalings,

$$\left.\begin{aligned} u^0 &= \left(\frac{\epsilon^2 Ql}{\mu}\right) u^{0'}, \text{ where } u^{0'} = p_{x'}^{0'} (z'^2 - z')/2, \\ v^0 &= \left(\frac{\epsilon^2 Ql}{\mu}\right) v^{0'}, \text{ where } v^{0'} = p_{y'}^{0'} (z'^2 - z')/2. \end{aligned}\right\} \tag{16}$$

From the formulas (10), (16) and applying the same procedure, we obtain

$$\left.\begin{array}{l}\tau^0_{11}=2(\lambda_1-\lambda_2)(\epsilon^2Q^2/\mu)\tau^{0'}_{11},\ \text{where}\ \tau^{0'}_{11}:=\{p^{0'}_{x'}(z'-1/2)\}^2,\\ \tau^0_{12}=2(\lambda_1-\lambda_2)(\epsilon^2Q^2/\mu)\tau^{0'}_{12},\ \text{where}\ \tau^{0'}_{12}:=p^{0'}_{x'}p^{0'}_{y'}(z'-1/2)^2,\\ \tau^0_{22}=2(\lambda_1-\lambda_2)(\epsilon^2Q^2/\mu)\tau^{0'}_{22},\ \text{where}\ \tau^{0'}_{22}:=\{p^{0'}_{y'}(z'-1/2)\}^2,\\ \tau^{0'}_{33}=0,\\ \tau^0_{13}=(\epsilon Q)\tau^{0'}_{13},\ \text{where}\ \tau^{0'}_{13}:=p^{0'}_{x'}(z'-1/2),\\ \tau^0_{23}=(\epsilon Q)\tau^{0'}_{23},\ \text{where}\ \tau^{0'}_{23}:=p^{0'}_{y'}(z'-1/2).\end{array}\right\} \tag{17}$$

The formula (16) gives the characteristic velocity

$$v_c=(\epsilon^2Ql/\mu). \tag{18}$$

Therefore we get the characteristic time $t_c=l/v_c=\mu/\epsilon^2Q$ and we have

$$t'=\frac{t}{(\mu/\epsilon^2Q)},\quad \partial/\partial t=\frac{\epsilon^2Q}{\mu}\times\partial/\partial t'. \tag{19}$$

We introduce Deborah number De_1 for the ratio of the relaxation time λ_1 to the characteristic time (μ/ϵ^2Q). Similarly, De_2 for the ratio of the retardation time λ_2 to the same characteristic time.

$$De_1=\epsilon^2\frac{\lambda_1Q}{\mu},\qquad De_2=\epsilon^2\frac{\lambda_2Q}{\mu}. \tag{20}$$

We consider the regime where

$$\lambda_iQ/\mu=O(1),\quad i=1,2. \tag{21}$$

Then $De_1,De_2=O(\epsilon^2)$ and our results hold in the regime of small Deborah numbers. Mora and Manna [18, 19] studied in the regime of Deborah number order one and larger but for a Maxwell upper-convected fluid. Wilson [31] considered the regime of large De_1 (see the Fig. 1 in Wilson [31]).

3.2 Basic steady flow with one velocity component.

We consider now the simpler case when the flow is due to the pressure gradient only in the direction Ox.

$$\nabla p^0=(p^0_x,0,0),\quad \mathbf{v}^0=(u^0(z),0,0). \tag{22}$$

Following the above procedure, we get only two non-zero components of the basic extra-stress tensor and only τ^0_{11} depends on λ_1 and λ_2.

$$\tau^0_{11}=2(\lambda_1-\lambda_2)\mu(u^0_z)^2,\quad \tau^0_{13}=\mu u^0_z. \tag{23}$$

We recall from (12) the following velocity-pressure relation which does not contain the constants λ_1 and λ_2.

$$p_x^0 = \tau_{13,z}^0 = G < 0, \quad u^0 = (1/\mu)G(z^2 - bz)/2. \tag{24}$$

We average the basic velocity profile given by (24) across the plates and obtain the Darcy's law:

$$< u^0 >= \frac{1}{b}\int_0^b u^0 dz = -\frac{b^2}{12\mu}p_x^0. \tag{25}$$

The basic steady interface air-liquid is then given by

$$x =< u^0 > t.$$

The basic pressure which can depend on time t is given by

$$p^0 = G(x- < u^0 > t), \quad x > \ < u^0 > t. \tag{26}$$

This is equivalent with the formula (7) given in Wilson [31].

4 Linear stability analysis

We perform linear stability of the basic flow defined by (22) and (23). With slight abuse of notation perhaps, the perturbations are denoted by u, v, w, p and τ, the same notations used in section 2 for the flow variables. This should not be the source of any confusion because we deal with only perturbed variables below. We scale these perturbations using the same scaling formulas (15) through(17) derived before. The scaled perturbations are denoted by a prime as a superscript. The perturbation equations obtained by inserting the perturbed variables in the divergence-free condition, in the constitutive relations, and in the flow equations are derived below starting with the divergence free condition.

We use the scalings (16) for u, v, w. Using the divergence-free condition we then obtain

$$\epsilon^2(Ql/\mu)[u'_{x'} + v'_{y'}](1/l) + \epsilon^2(Ql/\mu)w'_{z'}(1/b) = 0 \Rightarrow \epsilon[u'_{x'} + v'_{y'}] + w'_{z'} = 0. \tag{27}$$

It follows $w'_{z'} = 0$ and $u'_{x'} + v'_{y'} = 0$. The no-slip conditions on the plates give $w' = 0$ and thus we can neglect w. We perform normal mode analysis for linear stability by considering

$$u = f(z)\exp(\alpha x + \sigma t)\cos(ky), \quad v = f(z)\exp(\alpha x + \sigma t)\sin(ky). \tag{28}$$

From the divergence-free condition, it follows that

$$\alpha = -k. \tag{29}$$

The ansatz for the function $f(z)$ is given in equation (82) later. Next we derive and analyze the constitutive relations for perturbations. We insert the perturbations in the constitutive relation (2). Then

$$\tau^0 + \tau + \lambda_1(\tau^0 + \tau)^{\nabla} = \mu[\mathbf{D}^0 + \mathbf{D} + \lambda_2(\mathbf{D}^0 + \mathbf{D})^{\nabla}]. \tag{30}$$

It follows from $(3)_1$ and the fact that $\tau_x^0 = 0$,

$$(\tau^0 + \tau)^{\nabla} = \tau_t + u^0\tau_x - [\mathbf{L}^0\tau^0 + \tau^0\mathbf{L}^{0T}] - [\mathbf{L}^0\tau + \mathbf{L}\tau^0 + \tau^0\mathbf{L}^T + \tau\mathbf{L}^{0T}], \tag{31}$$

to linear order in perturbations. Similarly, it follows from $(3)_2$ that

$$(\mathbf{D}^0 + \mathbf{D})^{\nabla} = \mathbf{D}_t + u^0\mathbf{D}_x - [\mathbf{L}^0\mathbf{D}^0 + \mathbf{D}^0\mathbf{L}^{0T}] - [\mathbf{L}^0\mathbf{D} + \mathbf{L}\mathbf{D}^0 + \mathbf{D}^0\mathbf{L}^T + \mathbf{D}\mathbf{L}^{0T}], \tag{32}$$

to linear order in perturbations. Substituting (31) and (32) in (30) and using $\tau_x^0 = \tau_y^0 = \mathbf{D}_x^0 = \mathbf{D}_y^0 = 0$, we obtain

$$\tau + \lambda_1(\tau_t + u^0\tau_x - \mathbf{E}) = \mu\{\mathbf{D} + \lambda_2(\mathbf{D}_t + u^0\mathbf{D}_x - \mathbf{F})\}, \tag{33}$$

where $\mathbf{E}$ and $\mathbf{F}$, after using the fact that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $\tau^0, \mathbf{D}^0, \tau, \&\mathbf{D}$ are symmetric (i.e., $\tau^0\mathbf{L}^T = (\mathbf{L}\tau^0)^T, \tau\mathbf{L}^{0T} = (\mathbf{L}^0\tau)^T, \mathbf{D}^0\mathbf{L}^T = (\mathbf{L}\mathbf{D}^0)^T, \mathbf{D}\mathbf{L}^{0T} = (\mathbf{L}^0\mathbf{D})^T$), are given by

$$\begin{aligned}
\mathbf{E} : &= \mathbf{L}^0\tau + \mathbf{L}\tau^0 + \tau^0\mathbf{L}^T + \tau\mathbf{L}^{0T} = \mathbf{L}^0\tau + \mathbf{L}\tau^0 + (\mathbf{L}\tau^0)^T + (\mathbf{L}^0\tau)^T \\
&= \begin{pmatrix} 2(u_z^0\tau_{31} + u_x\tau_{11}^0 + \tau_{13}^0 u_z) & (u_z^0\tau_{32} + v_x\tau_{11}^0 + v_z\tau_{13}^0) & (u_z^0\tau_{33} + u_x\tau_{13}^0) \\ (u_z^0\tau_{32} + v_x\tau_{11}^0 + v_z\tau_{13}^0) & 0 & \tau_{13}^0 v_x \\ (u_z^0\tau_{33} + u_x\tau_{13}^0) & \tau_{13}^0 v_x & 0 \end{pmatrix}.
\end{aligned} \tag{34}$$

$$\mathbf{F} := \mathbf{L}^0\mathbf{D} + (\mathbf{L}^0\mathbf{D})^T + \mathbf{L}\mathbf{D}^0 + (\mathbf{L}\mathbf{D}^0)^T = \begin{pmatrix} 4u_z^0 u_z & 2u_z^0 v_z & u_x u_z^0 \\ 2u_z^0 v_z & 0 & v_x u_z^0 \\ u_x u_z^0 & v_x u_z^0 & 0 \end{pmatrix}. \tag{35}$$

The following flow equations for perturbations are obtained from (1).

$$\left.\begin{aligned}
p_x &= \tau_{11,x} + \tau_{12,y} + \tau_{13,z}, \\
p_y &= \tau_{21,x} + \tau_{22,y} + \tau_{23.z}, \\
p_z &= \tau_{31,x} + \tau_{32,y} + \tau_{33,z}.
\end{aligned}\right\} \tag{36}$$

Next we need the dynamic and kinematic boundary conditions. It follows from (26) (see also Wilson [31]) that the linearized dynamic boundary condition at the interface is given by

$$(G\eta + p) - \tau_{11} = \gamma\,(\eta_{yy} + \eta_{zz}), \tag{37}$$

where γ is the surface tension on the interface $x = \eta(y,z,t) + < u^0 > t$ and $(\eta_{yy} + \eta_{zz})$ is the total curvature of the interface to leading order in perturbation. The interface satisfies the kinematic boundary condition $\eta_t = u$ on the interface which together with the ansatz $(28)_1$ for $u(z)$ gives

$$\eta = u/\sigma. \tag{38}$$

4.1 Leading order equations for the flow variables and the stress tensor

The dispersion relation is obtained from the dynamic boundary condition (37) through an averaging procedure as we will see below. To do this, we first need to express the terms in (37) as functions of z either explicitly or implicitly so that averaging of the dynamic interfacial condition (37) across the gap of the Hele-Shaw cell can be carried out. This in turn requires us to first derive from the constitutive equations (33) and the perturbed flow equations (36), the leading order terms for $\tau_{33}, \tau_{31}, \tau_{32}, p_z, \tau_{11}, \tau_{12}, \tau_{22}, p_x, p_y$ in terms of (u,v) and the derivatives of (u,v). This is what we do next.

(i) **Formula for** τ_{33}:

From (33), the constitutive equation for the perturbation τ_{33} is given by

$$\tau_{33} + \lambda_1 \tau_{33,t} + \lambda_1 u^0 \tau_{33,x} = 0, \tag{39}$$

which after scaling becomes

$$K\tau'_{33} + \lambda_1 K \tau'_{33,t'} \left(\frac{\epsilon^2 Q}{\mu}\right) + \lambda_1 K \left(\frac{1}{l}\right)\left(\frac{\epsilon^2 Q l}{\mu}\right) u^{0'} \tau'_{33,x'} = 0, \tag{40}$$

where K is a suitable non-zero scaling for τ_{33} which suffices since the equation above is linear in τ_{33}. Using the condition (21) of section 3.1 we obtain to leading order $O(1)$, $\tau'_{33} = 0$ or equivalently, $\tau_{33} = 0$.

(ii) **Formula for** τ_{13}:

The constitutive equation for the perturbation τ_{31} is obtained from (33), (34) and (35).

$$\tau_{31} + \lambda_1 \tau_{31,t} + \lambda_1 u^0 \tau_{31,x} - \lambda_1 (u^0_z \tau_{33} + u_x \tau^0_{13}) = \mu \left(u_z + \lambda_2 u_{zt} + \lambda_2 u^0 u_{zx} - \lambda_2 u_x u^0_z\right). \tag{41}$$

We compare the terms of various orders in the above equation after using our scaling procedure (14), (16), (17), (19), the condition (21) and $\tau_{33} = 0$ which was just derived above. It is easy to see that

$$\left.\begin{aligned}
&\lambda_1\tau_{31,t} = \lambda_1(\epsilon Q)\tau'_{31,t'}(\epsilon^2 Q/\mu) \sim Q\cdot\epsilon^3\tau'_{31,t'} = O(\epsilon^3),\\
&\lambda_1 u^0\tau_{31,x} = \lambda_1(\epsilon^2 Ql/\mu)u^{0'}(\epsilon Q)\tau'_{31,x'}(1/l) \sim Q\cdot\epsilon^3 u^{0'}\tau'_{31,x'} = O(\epsilon^3),\\
&\lambda_1 u_x\tau^0_{31} = \lambda_1(\epsilon^2 Ql/\mu)u'_{x'}(1/l)(\epsilon Q)\tau^{0'}_{31} \sim Q\cdot\epsilon^3 u'_{x'}\tau^{0'}_{31} = O(\epsilon^3),
\end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned}
&\mu\lambda_2 u_{z,t} = \mu\lambda_2(\epsilon^2 Ql/\mu)u'_{z',t'}(1/b)(\epsilon^2 Q/\mu) \sim Q\cdot\epsilon^3 u'_{z',t'} = O(\epsilon^3),\\
&\mu\lambda_2 u_{zx}u^0 = \mu\lambda_2(\epsilon^2 Ql/\mu)^2 u'_{z'x'}(1/bl)u^{0'} \sim Q\cdot\epsilon^3 u'_{z'x'}u^{0'} = O(\epsilon^3),\\
&\mu\lambda_2 u_x u^0_z = \mu\lambda_2(\epsilon^2 Ql/\mu)^2 u'_{x'}u^{0'}_{z'}(1/bl) \sim Q\cdot\epsilon^3 u'_{x'}u^{0'}_{z'} = O(\epsilon^3),
\end{aligned}\right\} \tag{43}$$

$$\tau_{31} = (\epsilon Q)\tau'_{31} = O(\epsilon), \tag{44}$$

$$\mu u_z = \mu\left(\frac{\epsilon^2 Ql}{\mu}\right)u'_{z'}\frac{1}{b} = (\epsilon Q)u'_{z'} = O(\epsilon). \tag{45}$$

Substituting the above estimates in (41), we obtain to leading order $O(\epsilon)$,

$$\tau_{13} = \mu u_z. \tag{46}$$

(iii) **Formula for τ_{32}:**

From the relations (33), (34), and (35), it follows that the constitutive equation for τ_{32} is given by

$$\tau_{32} + \lambda_1\tau_{32,t} + \lambda_1 u^0\tau_{32,x} - \lambda_1\tau^0_{13}v_x = \mu\{v_z + \lambda_2(v_{zt} + u^0 v_{zx}) - \lambda_2 v_x u^0_z\}. \tag{47}$$

The scaled values (15) through (17) and the condition (21) give

$$\lambda_1\tau_{32,t}, \quad \lambda_1 u^0\tau_{32,x}, \quad \lambda_1\tau^0_{13}v_x = O(\epsilon^3), \tag{48}$$

$$\mu\lambda_2 v_{zt}, \quad \mu\lambda_2 u^0 v_{xz}, \quad \mu\lambda_2 v_x u^0_z = O(\epsilon^3), \tag{49}$$

$$\tau_{32} = (\epsilon Q)\tau'_{32} = Q\cdot O(\epsilon), \quad \mu v_z = \mu(\epsilon^2 Ql/\mu)v'_{z'}(1/b) = (\epsilon Q)v'_{z'} = O(\epsilon). \tag{50}$$

Substituting the above estimates in (47), we obtain the following equation accurate to leading order $O(\epsilon)$.

$$\tau_{32} = \mu v_z. \tag{51}$$

(iv) **Formula for p_z:**

Using (46), (51) and $\tau_{33} = 0$ in $(36)_3$, we obtain

$$p_z = \tau_{31,x} + \tau_{32,y} = \mu(u_{zx} + v_{zy}) = 0. \tag{52}$$

(v) **Formula for** τ_{11}:

We obtain the following constitutive equation for τ_{11} from (33), (34) and (35).

$$\tau_{11} + \lambda_1 \tau_{11,t} + \lambda_1 u^0 \tau_{11,x} - 2\lambda_1(u_z^0 \tau_{31} + u_x \tau_{11}^0 + \tau_{13}^0 u_z) = \mu\{2u_x + 2\lambda_2(u_{x,t} + u^0 u_{xx}) - 4\lambda_2 u_z^0 u_z\}. \tag{53}$$

From the scaling procedure (15)-(17), the equations $(10)_1$, (46), and the condition (21) we get the following estimates and formulas:

$$\left.\begin{aligned}
\lambda_1 \tau_{11,t} &= 2\lambda_1(\lambda_1 - \lambda_2)(\epsilon^2 Q^2/\mu)(\epsilon^2 Q/\mu)\tau'_{11,t'} = O(\epsilon^4),\\
\lambda_1 u^0 \tau_{11,x} &= 2\lambda_1(\lambda_1 - \lambda_2)(\epsilon^2 Ql/\mu)(\epsilon^2 Q^2/\mu)u^{0'}\tau'_{11,x'}(1/l) = O(\epsilon^4),\\
\lambda_1 u_z^0 \tau_{31} &= \lambda_1 \mu u_z^0 u_z = \lambda_1 \mu(\epsilon^2 Ql/\mu)^2 u_{z'}^{0'} u'_{z'}(1/b^2) = O(\epsilon^2),\\
\lambda_1 u_x \tau_{11}^0 &= \lambda_1(\epsilon^2 Ql/\mu)u'_{x'}(1/l)2(\lambda_1 - \lambda_2)(\epsilon^2 Q/\mu)\tau_{11}^{0'} = O(\epsilon^4),\\
\lambda_1 \tau_{13}^0 u_z &= \lambda_1 \mu u_z^0 u_z = \lambda_1 \mu(\epsilon^2 Ql/\mu)^2 u_{z'}^{0'} u'_{z'}(1/b^2) = O(\epsilon^2),
\end{aligned}\right\} \tag{54}$$

$$\lambda_2 \mu u_{x,t} = \lambda_2 \mu(\epsilon^2 Ql/\mu)(\epsilon^2 Q/\mu)(1/l)u'_{x',t'} = O(\epsilon^4), \tag{55}$$

$$\lambda_2 \mu u^0 u_{xx} = \lambda_2(\epsilon^2 Ql/\mu)^2 u^{0'} u'_{x'x'}(1/l^2) = O(\epsilon^4), \tag{56}$$

$$\mu\lambda_2 u_z^0 v_z = \mu\lambda_2(\epsilon^2 Ql/\mu)^2 u_{z'}^{0'} v'_{z'}(1/b^2) = (\epsilon^2)(\lambda_2 Q^2/\mu)u_{z'}^{0'} v'_{z'} = O(\epsilon^2), \tag{57}$$

$$\mu u_x = (\epsilon^2 Ql)(1/l)u'_{x'} = O(\epsilon^2), \quad \tau_{11} = 2(\lambda_1 - \lambda_2)(\epsilon^2 Q^2/\mu)\tau'_{11} = O(\epsilon^2). \tag{58}$$

Using the above estimates (54) through (58) in the equation (53), we fnd the following equation to leading order in ϵ^2.

$$2(\lambda_1 - \lambda_2)(\epsilon^2 Q^2/\mu)\tau'_{11} = 2(\epsilon^2 Ql)(1/l)u'_{x'} + 4\mu(\lambda_1 - \lambda_2)\frac{(\epsilon Q)^2}{\mu}u_{z'}^{0'} v'_{z'} \tag{59}$$

and returning to the unscaled quantities we obtain

$$\tau_{11} = 2\mu u_x + 4\mu(\lambda_1 - \lambda_2)u_z^0 u_z. \tag{60}$$

The term $2\mu u_x$ in the above equation do not appear when the lubrication approximation is used. Here we do not neglect the x, y derivatives of the velocity in comparison to the z derivatives, unlike in lubrication approximation.

(vi) **Formula for** τ_{12}:

The relations (33)-(34)-(35) give the following constitutive equation for τ_{12}:

$$\tau_{12} + \lambda_1 \tau_{12,t} + \lambda_1 u^0 \tau_{12,x} - \lambda_1(u_z^0 \tau_{32} + v_x \tau_{11}^0 + \tau_{13}^0 v_z) = \mu\{(u_y + v_x) + \lambda_2[(u_y + v_x)_t + u^0(u_y + v_x)_x] - 2\lambda_2 u_z^0 v_z\}. \tag{61}$$

The scaling procedure (15)-(17), the equations (10), (51) and the condition (21) give the following estimates and formulas:

$$\left.\begin{aligned}
&\lambda_1\tau_{12,t} = 2\lambda_1(\lambda_1-\lambda_2)(\epsilon^2Q^2/\mu)(\epsilon^2Q/\mu)\tau'_{12,t'} = O(\epsilon^4),\\
&\lambda_1u^0\tau_{12,x} = 2\lambda_1(\lambda_1-\lambda_2)(\epsilon^2Ql/\mu)(\epsilon^2Q^2/\mu)u^{0'}\tau'_{12,x'}(1/l) = O(\epsilon^4),\\
&\lambda_1v_x\tau^0_{11} = 2\lambda_1(\lambda_1-\lambda_2)(\epsilon^2Ql/\mu)(\epsilon^2Q^2/\mu)v'_{x'}(1/l)\tau^{0'}_{11} = O(\epsilon^4),\\
&\lambda_2\mu(u_y+v_x)_t = \lambda_2\mu(\epsilon^2Ql/\mu)(u'_{y'}+v'_{x'})_{t'}(1/l)(\epsilon^2Q/\mu) = O(\epsilon^4),\\
&\lambda_2\mu u^0(u_y+v_x)_x = \lambda_2\mu(\epsilon^2Ql/\mu)^2(u'_{y'}+v'_{x'})_{x'}(1/l^2) = O(\epsilon^4),
\end{aligned}\right\} \tag{62}$$

$$\lambda_1u^0_z\tau_{32} = \lambda_1u^0_z(\mu v_z) = \lambda_1\mu(\epsilon^2Ql/\mu)^2u^{0'}_{z'}v'_{z'}(1/b^2) = O(\epsilon^2), \tag{63}$$

$$\lambda_1v_z\tau^0_{13} = \lambda_1v_z(\mu u^0_z) = \lambda_1\mu(\epsilon^2Ql/\mu)^2v'_{z'}u^{0'}_{z'}(1/b^2) = O(\epsilon^2), \tag{64}$$

$$\mu\lambda_2u^0_zv_z = \mu\lambda_2(\epsilon^2Ql/\mu)^2u^{0'}_{z'}v'_{z'}(1/b^2) = (\epsilon^2)(\lambda_2Q^2/\mu)u^{0'}_{z'}v'_{z'} = O(\epsilon^2), \tag{65}$$

$$\left.\begin{aligned}
&\mu(u_y+v_x) = (\epsilon^2Ql)(u'_{y'}+v'_{x'})(1/l) = O(\epsilon^2),\\
&\tau_{12} = 2(\lambda_1-\lambda_2)(\epsilon^2Q^2/\mu)\tau'_{12} = O(\epsilon)^2.
\end{aligned}\right\} \tag{66}$$

We insert the above estimates (62) through (66) in the equation (61) and obtain the following equation to leading order ϵ^2.

$$\tau_{12} = \mu(u_y+v_x) + 2\mu(\lambda_1-\lambda_2)u^0_zv_z. \tag{67}$$

The term $\mu(u_y+v_x)$ does not appear when the lubrication approximation is used. Note again that we do not neglect the x, y derivatives of the velocity in comparison to z derivatives.

(vii) **Formula for** τ_{22}:

The relations (33), (34) and (35) give the following constitutive equation for τ_{22}:

$$\tau_{22} + \lambda_1\tau_{22,t} + \lambda_1u^0\tau_{22,x} = 2\mu v_y + \lambda_2v_{y,t} + \lambda_2u^0v_{y,x}. \tag{68}$$

From the scaling procedure (15) through (17) and the condition (21) we get

$$\lambda_1\tau_{22,t}, \quad \lambda_1u^0\tau_{22,x}, \quad \lambda_2v_{y,t}, \quad \lambda_2u^0v_{y,x} = O(\epsilon^4). \tag{69}$$

$$\tau_{22} = 2(\lambda_1-\lambda_2)(\epsilon^2Q^2/\mu)\tau'_{22} = O(\epsilon^2), \quad 2\mu v_y = \mu(\epsilon^2Ql/\mu)v'_{y'}(1/l) = O(\epsilon^2). \tag{70}$$

We use the above estimates in the equation (68) and obtain the following equation to leading order ϵ^2.

$$\tau_{22} = 2\mu v_y. \tag{71}$$

(viii) **Formula for** p_x:

From the perturbed flow equations $(36)_1$, (46), (60), and (67), we have

$$p_x = \tau_{11,x} + \tau_{12,y} + \tau_{13,z} = 2\mu u_{xx} + 4\mu(\lambda_1 - \lambda_2)u_z^0 u_{zx} + \mu(u_y + v_x)_y + 2\mu(\lambda_1 - \lambda_2)u_z^0 v_{zy} + \mu u_{zz}. \tag{72}$$

The free divergence condition gives

$$u_{zx} + v_{zy} = (u_x + v_y)_z = 0, \tag{73}$$

and from the normal mode ansatz (28), we have

$$u_{xx} = k^2 f(z)\exp(\alpha x + \sigma t)cos(ky), \quad u_{yy} = v_{xy} = -k^2 f(z)\exp(\alpha x + \sigma t)cos(ky). \tag{74}$$

Therefore from (72), (73), and (74), we obtain the following expression for p_x.

$$p_x = 2\mu(\lambda_1 - \lambda_2)u_z^0 u_{zx} + \mu u_{zz}. \tag{75}$$

(ix) **Formula for** p_y:

From $(36)_2$, (51), (67) and (71), we obtain

$$p_y = \tau_{21,x} + \tau_{22,y} + \tau_{23,z} = \mu(u_y + v_x)_x + 2\mu(\lambda_1 - \lambda_2)u_z^0 v_{zx} + 2\mu v_{yy} + \mu v_{zz}. \tag{76}$$

Using the ansatz (28) for u, v in the above equation, we obtain

$$p_y = 2\mu(\lambda_1 - \lambda_2)u_z^0 v_{zx} + \mu v_{zz}. \tag{77}$$

Remark 1. *It is worth mentioning here that to leading order we find that* $p_{xx} + p_{yy} = 0$*, same as for Newtonian fluid for Darcy flow. Indeed, as* $u_{zz,x} + v_{zz,y} = (u_x + v_y)_{zz} = 0, \quad u_{zx,x} + v_{zx,y} = (u_x + v_y)_{zx} = 0$*, the relations* (75) *and* (77) *directly lead us to this result.*

□

4.2 Dispersion relation

The term $(p - \tau_{11})$ which appears in the dynamic interfacial condition (37) is first obtained from the leading order equations given above as follows. The leading order equations (67) and (46) are first substituted in $(36)_1$ and then integrated which leads to

$$p - \tau_{11} = \left(-\frac{\mu}{k}\right)\left\{(u_y + v_x)_y + 2(\lambda_1 - \lambda_2)u_z^0 v_{zy} + u_{zz}\right\}. \tag{78}$$

Substituting (38) and (78) in the dynamic interfacial condition (37) and then using the ansatz (28) for u, v we obtain

$$G\frac{f(z)}{\sigma} - \frac{\mu}{k}\{-2k^2 f(z) + 2(\lambda_1 - \lambda_2)ku_z^0 f_z(z) + f_{zz}\} = \gamma\left(\frac{-k^2 f(z) + f_{zz}(z)}{\sigma}\right). \tag{79}$$

To obtain the dispersion relation from the depth average of the above equation for which we need the following averages: $< u_z^0 f_z >$ and $< f(z) >$, we first find the form of the amplitude $f(z)$ as follows. We use the estimates given by our scaling procedure. From (77) and (52) we get $(p_y)_z = 0$. Thus

$$2\mu(\lambda_1 - \lambda_2)(\epsilon^2 Ql/\mu)^2(u_{z'}^{0'} v'_{z'x'})_{z'}(1/b^3)(1/l) + \mu(\epsilon^2 Ql/\mu)v'_{z'z'z'}(1/b^3) = 0. \tag{80}$$

Then it follows after using the condition (21) that

$$\epsilon^2(u_{z'}^{0'} v'_{z'x'})_{z'} + v'_{z'z'z'} = 0. \tag{81}$$

Therefore we get to leading order $v'_{z'z'z'} = 0$. Then it follows $v_{zzz} = 0$ and we get $f_{zzz} = 0$. From the non-slip boundary conditions on the plates of the velocity u we get

$$f(z) = z(z - b). \tag{82}$$

Then the above formula (79) becomes

$$G\frac{f(z)}{\sigma} - \frac{2\mu}{k} + 2\mu k f(z) - 2\mu(\lambda_1 - \lambda_2)u_z^0 f_z(z) = \gamma\left(\frac{-k^2 f(z) + 2}{\sigma}\right).$$

Using the formula (82) for $f(z)$ and the form (24) of the basic velocity u^0 in terms of the basic pressure, we obtain

$$< f(z) >= (1/b)\int_0^b (z^2 - zb) = -b^2/6,$$

and

$$< u_z^0 f_z >= (G/\mu) < (z - b/2)(2z - b) >= (G/\mu)b^2/6.$$

The above averages are introduced in the depth average of the dynamic boundary condition (79). This leads to the following dispersion relations after using $G = p_x^0 = -12\mu < u^0 > /b^2 < 0$ (see $(12)_1$ and (25)).

$$\sigma = \frac{1}{\delta}\left[< u^0 > (1 - Ca^{-1})\, k - \left(\frac{b^2}{12}\right)\, \frac{\gamma}{\mu}k^3\right], \tag{83}$$

where $Ca = < u^0 > \mu/\gamma$ is the Capillary number, and

$$\delta = 1 - 2 < u^0 > (\lambda_1 - \lambda_2)k + b^2k^2/6. \tag{84}$$

The formula (83) for the growth rate is the generalized Saffman-Taylor formula which includes the effect of elasticity through the denominator δ which depends on $(\lambda_1-\lambda_2)$. We observe the following from and about this formula.

(i) The number of real roots of the denominator δ depends on the size of of $(\lambda_1-\lambda_2)$. As $(\lambda_1-\lambda_2)$ is considered positive, we have

$$\Delta = \{[< u^0 > (\lambda_1 - \lambda_2)]^2 - b^2/6\} > 0 \quad \text{if } (\lambda_1 - \lambda_2) > c = b/(< u^0 > \sqrt{6}).$$

We have following three cases.

$$(\lambda_1 - \lambda_2) < c \Rightarrow \delta > 0 \quad \forall\ k.$$

$$(\lambda_1 - \lambda_2) = c \Rightarrow \delta = 0 \quad for \quad k_1 =< u^0 > (\lambda_1 - \lambda_2)\frac{6}{b^2} = \sqrt{6}/b.$$

$$(\lambda_1 - \lambda_2) > c \Rightarrow \delta = 0 \quad for \quad k_{2,3} = \left\{< u^0 > (\lambda_1 - \lambda_2)^{+}_{-}\sqrt{\Delta}\right\}\frac{6}{b^2}.$$

It follows that for $(\lambda_1 - \lambda_2) = c$ we have a blow-up of σ at $k = k_1$ and for $(\lambda_1 - \lambda_2) > c$, σ blows up at $k = k_2$ and k_3.

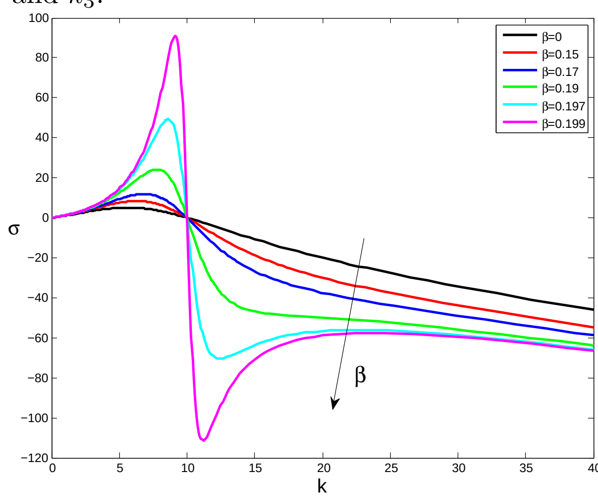

Figure 2: Plots of $\sigma = \frac{k-0.01k^3}{1-\beta k+0.01k^2}$ versus k. For values of $\beta = 0.1, 0.15, 0.17, 0.19, 0.197$ and 0.199, the maximum growth rate σ_m = 5, 8, 12, 22, 50, and 90 respectively.

(ii) Notice that the dispersion relation (83) for the growth rate is of the form

$$\sigma(k) = \frac{a_1\, k - a_2\, k^3}{1 - \beta k + a_3 k^2},$$

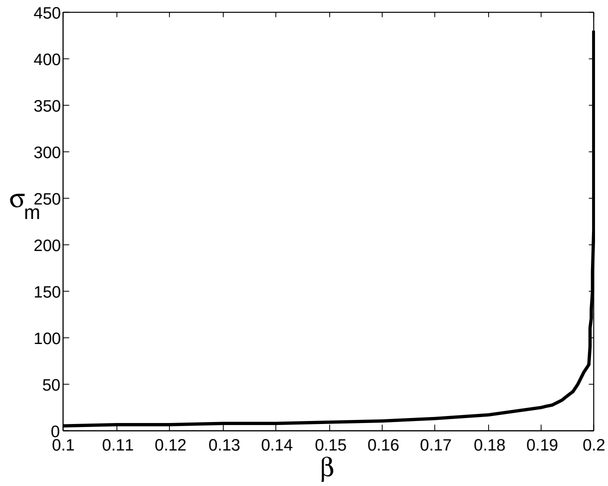

Figure 3: Plot of the maximum (over all wavenumbers) value σ_m of the growth rate versus β for the growth rate $\sigma = \frac{k-0.01k^3}{1-\beta k+0.01k^2}$. The growth rate σ_m diverges super exponentially as $\beta = 0.2$ is approached.

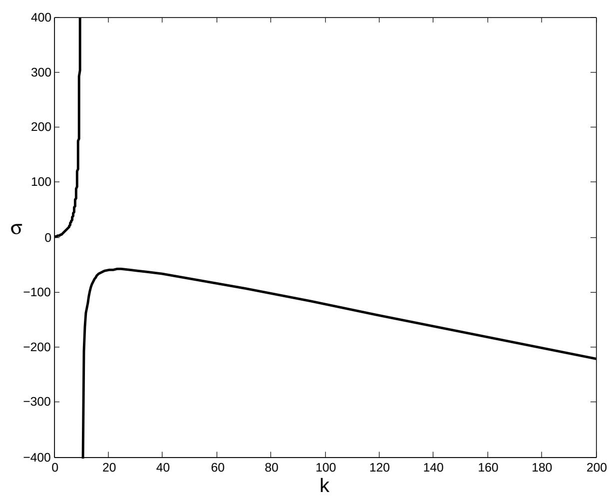

Figure 4: Plot of $\sigma = \frac{k-0.01k^3}{1-\beta k+0.01k^2}$ versus k when $\beta = 0.2$. We get only one blow-up of σ. This is the only value of β for which there is only one blow-up in σ.

where β is a multiple of the elasticity measure $(\lambda_1 - \lambda_2)$ and a_2, a_3 are very small constants of the order b^2. In Figures 2, 4 and 5, we plot σ versus k when $a_1 = 1, a_2 = a_3 = 0.01$. In the

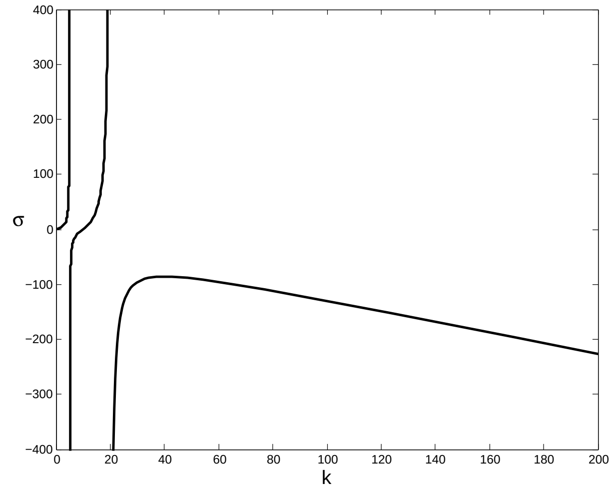

Figure 5: Plot of $\sigma = \frac{k-0.01k^3}{1-\beta k+0.01k^2}$ versus k when $\beta = 0.25$. We see two blow-ups of σ. This is the scenario for all values of $\beta > 0.2$.

Fig. 2, we plot σ for several increasing values of β starting with $\beta = 0$ corresponding to the Newtonian case $\lambda_1 = \lambda_2$ (see Appendix-C). For β increasing from 0.1 to 0.199, the maximum value σ_m of σ monotonically increases from 5 to 90 in this figure. Then σ_m grows very rapidly with β. This theoretical result is consistent with the numerical results of Wilson [31], who considered a Jeffrey's fluid. We prove in Appendix-B that the Jeffrey's fluid described by the constitutive relations (1)- (4) of Wilson [31] is in fact the Oldroyd-B fluid given by the constitutive relations (2) of this paper. In Fig. 3, we plot the maximum (over all wavenumber) value σ_m of the growth rate versus β. We see that σ_m diverges very rapidly, in fact super-exponentially, as $\beta = 0.2$ is approached. In Fig. 4, we see that the growth rate has a singularity (a vertical asymptotic) at $k = 10$ when $\beta = 0.2$. For $\beta > 0.2$, there are two singular values of k at which σ blows up which can be seen by examining the denominator in the expression for δ. Fig. 5 shows this scenario for $\beta = 0.25$. In all cases, $\sigma(k) \sim -(a_2/a_3)k$ asymptotically for large k.

(iii) The formula (83) is quite similar with the Saffman-Taylor formula [26]

$$\sigma_{\rm st} = \left[< u^0 > \, k - \left(\frac{b^2}{12} \right) \, \frac{\gamma}{\mu} k^3 \right], \tag{85}$$

except that the formula (83) contains one new term in the numerator and two new terms in the denominator. The new term $\gamma\, k/\mu$ in the numerator is due to the *total* curvature $(\eta_{yy} + \eta_{zz})$. This term will disappear if we used instead η_{yy} for the curvature which is a common practice in such type of stability calculations in Hele-Shaw flows (see [12, 26]). However, η_{zz} is related with our 3D approach and formula (28). One of the two terms in the denominator arises due to elasticity of the fluid which depends on the relaxation and retardation time constants. The other term $b^2k^2/6$ in the denominator appears due to the 3D approach which uses Fourier modes decomposition (28).

(iv) It is shown below in Appendix-C that the Oldroyd-B fluid is Newtonian if $\lambda_1 = \lambda_2$ for the basic flow given by (22) and (23). Then the formula (83) reduces to

$$\sigma = \left\{ \left(< u^0 > -\frac{\gamma}{\mu} \right) k - \left(\frac{b^2}{12} \right) \frac{\gamma}{\mu} k^3 \right\} / (1 + b^2k^2/6),$$

which is the new Saffman-Taylor growth rate formula based on our 3D approach. This differs from classical formula (85) in that the average velocity $< u^0 >$ is now shifted by γ/μ. As mentioned earlier, this shift arises due to the use of the total curvature in the dynamic boundary condition (37). We emphasize that this term does not arise in our approach as well if the total curvature is not used in the dynamic boundary condition.

5 Conclusions

We have obtained an explicit approximate formula (83) for the dispersion relation in terms of the problem data. This formula shows the following.

(i) The displacement process of an Oldroyd-B fluid by air is more unstable than that of a Newtonian liquid by air.

(ii) The maximum growth rate increases monotonically and the most dangerous wave-number changes very little with increasing values of $(\lambda_1 - \lambda_2)$ when $(\lambda_1 - \lambda_2) < c$, where c is a small value in terms of the problem data.

(iii) For $(\lambda_1 - \lambda_2) = c$ we have only one blow-up point and for $(\lambda_1 - \lambda_2) > c$, we obtain two blow-up points of σ. This result seems to be related with the fractures usually observed in Hele-Shaw flows of complex fluids. Even though we used a linear theory, very large values of the growth rate near the blow-up points should be studied using an appropriate nonlinear theory.

(iv) When $\lambda_1 = \lambda_2$, the growth rate according to our formula (83) is quite similar with the Saffman-Taylor formula for a Newtonian fluid displaced by air in a Hele-Shaw cell. In this case, due to our 3D approach and Fourier modes decomposition ansatz (28), the formula (83) contains the new term $\gamma\, k/\mu$ in the numerator and the new term $b^2k^2/6$ in the denominator.

In closing, we point out here some similarities and differences between our procedure and results and that of Wilson [31] and Mora & Manna[19] - the only two other works related to this problem available in the literature. In Wilson [31], perturbation equations are derived which are then solved numerically and numerical solutions for perturbations are averaged across the gap to find numerical values of growth rates which he plots. Wilson [31] and Mora & Manna [19] do not provide any formula for the growth rates. In Mora & Manna (2010), equations are first scaled and then simplified by dropping some terms in the regime of Deborah number of the order of one and larger. But the equations still remain non-linear. Then they perform linear stability analysis of this system. We do exactly the opposite for $De << 1$ regime. We first perform linear stability analysis of the system and obtain linearized equations. Then we scale these linearized equations and simplify these equations for $De << 1$ regime by keeping only leading order terms. We Fourier decompose the perturbations with amplitude $f(z)$ which is a crucial difference between our work and that of Mora and Manna (2010) in which such function $f(z)$ does not appear. Additionally we show how to get an explicit formula for $f(z)$ which allows us to average the dynamic boundary condition, namely Laplace's law, across the gap and thereby obtain an explicit dispersion relation.

Even though we have a small Deborah number, some similarities exits with Mora & Manna which is interesting: blow-up of the growth constant. On the other hand, Wilson did not obtain blow-up. Both Mora and Manna [19] and Wilson [31] found that the maximum value of the growth rate increases with increasing values of $(\lambda_1 - \lambda_2)$. We also obtain the same behaviour. Mora and Manna (2010) obtained multi-valued growth rate curves but we do not. We obtain only single-values growth rate curves, so does Wilson [31]. Therefore, it is possible that some of these phenomena are universal across entire range of Deborah number, in particular blow-up phenomena of growth rates whereas the multi-valued nature of the dispersion curves may be dependent on the regimes of Deborah number. It will be interesting to study the transition behavior from single-valued to multi-valued nature of the dispersion curves in the space of Deborah number. This is one of the topics of future research in this area.

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Appendix A: Scales for the pressure and the stresses

We first discuss the determination of the characteristic pressure Q which we have used above. This characteristic pressure Q is related to characteristics velocity v_c through (18) which can be taken as the maximum velocity $U = (-Gb^2)/8\mu$ of the basic velocity profile $u^0(z)$ (see $(13)_1$). Thus it follows from equating this maximum velocity to v_c that

$$Q = -Gl/8 = \mu Ul/b^2. \tag{86}$$

This determines the scale Q for the pressure which is merely the basic pressure gradient $G = p_x^0$ up to a multiplicative constant $(-l/8)$. Therefore the condition (21) is equivalent to

$$\frac{\lambda_i(-Gl)}{8\mu} = O(1).$$

The pressure gradient G which satisfies this condition gives the *Deborah* numbers De_1, De_2 of the order $O(\epsilon^2)$. Thus our results of this paper hold for this regime.

The scales for the velocity given in terms of Q in (16) can now be determined from the basic pressure gradient $G = p_x^0$ (or from U, the maximum of the basic velocity profile due to relation (86)). For example, it follows from $(16)_1$ that

$$u^0 = \left(-\frac{Gb^2}{8\mu}\right) u^{0'} = \epsilon^2 \left(-\frac{Gl^2}{8\mu}\right) u^{0'}.$$

We can also use the maximum velocity U for scaling the extra-stress tensor. As $Q = \mu Ul/b^2$, the condition (21) gives

$$(\lambda_1 - \lambda_2)(\epsilon^2 Q^2/\mu) \sim \epsilon^2 Q = \epsilon^2(\mu Ul/b^2) = \mu U/l, \quad \epsilon Q = (b/l)(\mu Ul/b^2) = \mu U/b.$$

Then the relations $(17)_1$ and $(17)_5$ become

$$\tau_{11}^0 = 2(\mu U/l)\tau_{11}^{0'}, \quad \tau_{13}^0 = 2(\mu U/b)\tau_{13}^{0'}$$

which are quite similar with the scalings (2.5) of Saprykin, Koopmans and Kalliadasis [27] given in Appendix A there.

Appendix B: Relation between Jeffrey's fluid in Wilson [31] and our Oldroyd-B fluid

Here we show that the Jeffrey's fluid considered in Wilson [31] is in fact same as the Oldroyd-B fluid considered in this paper. We recall the equations (1)-(2) of Wilson [31] and first relate these to the notations used in this paper.

$$\mathbf{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\mathbf{D}_{ij}, \quad \mathbf{e} = \frac{1}{2}(\mathbf{L}^T + \mathbf{L}) = \mathbf{D}/2, \tag{87}$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right), \quad \omega = \frac{1}{2}(\mathbf{L}^T - \mathbf{L}). \tag{88}$$

The constitutive equations (3)-(4) of Wilson [31] are

$$\tau + \lambda_1\{\widehat{\tau} - (\tau\mathbf{e} + \mathbf{e}\tau)\} = 2\mu\,\{\mathbf{e} + \lambda_2(\widehat{\mathbf{e}} - 2\mathbf{e}\mathbf{e})\}, \tag{89}$$

$$\widehat{\tau} = \tau_t + \mathbf{v}\nabla\tau + \omega\tau - \tau\omega, \quad \widehat{\mathbf{e}} = \mathbf{e}_t + \mathbf{v}\nabla\mathbf{e} + \omega\mathbf{e} - \mathbf{e}\omega, \tag{90}$$

where $\widehat{\tau}$ and $\widehat{\mathbf{e}}$ are the *corotational derivatives*. Some simple calculations show us that the above constitutive relations are equivalent with our constitutive relations (2). Indeed, from (87) and (88) we obtain

$$2[(\omega\tau - \tau\omega) - (\tau\mathbf{e} + \mathbf{e}\tau)] = (\mathbf{L}^T - \mathbf{L})\tau - \tau(\mathbf{L}^T - \mathbf{L}) - \tau(\mathbf{L}^T + \mathbf{L}) - (\mathbf{L}^T + \mathbf{L})\tau = -2(\mathbf{L}\tau + \tau\mathbf{L}^T),$$

which together with $(90)_1$ (also recall $(3)_1$) gives

$$\widehat{\tau} - (\tau\mathbf{e} + \mathbf{e}\tau) = \tau^{\nabla}. \tag{91}$$

Moreover, from (87) and (88) it also follows that

$$\begin{aligned} 2(\omega\mathbf{e} - \mathbf{e}\omega) - 4\mathbf{e}\mathbf{e} &= \frac{1}{2}(\mathbf{L}^T - \mathbf{L})(\mathbf{L}^T + \mathbf{L}) - \frac{1}{2}(\mathbf{L}^T + \mathbf{L})(\mathbf{L}^T - \mathbf{L}) - (\mathbf{L}^T + \mathbf{L})^2 \\ &= \mathbf{L}^T\mathbf{L} - \mathbf{L}\mathbf{L}^T - (\mathbf{L}^T)^2 - \mathbf{L}\mathbf{L}^T - \mathbf{L}^T\mathbf{L} - \mathbf{L}^2 = -2\mathbf{L}\mathbf{L}^T - \mathbf{L}^2 - (\mathbf{L}^T)^2. \end{aligned} \tag{92}$$

We rewrite $-(\mathbf{L}\mathbf{D} + \mathbf{D}\mathbf{L}^T)$ below in a suitable form.

$$-(\mathbf{L}\mathbf{D} + \mathbf{D}\mathbf{L}^T) = -\mathbf{L}(\mathbf{L}^T + \mathbf{L}) - (\mathbf{L}^T + \mathbf{L})\mathbf{L}^T = -2\mathbf{L}\mathbf{L}^T - \mathbf{L}^2 - (\mathbf{L}^T)^2. \tag{93}$$

Then from (92) and (93), we have $2(\omega\mathbf{e} - \mathbf{e}\omega) - 4\mathbf{e}\mathbf{e} = -(\mathbf{L}\mathbf{D} + \mathbf{D}\mathbf{L}^T)$ which together with $(90)_2$ (also recall $(3)_2$) gives

$$2(\widehat{\mathbf{e}} - 2\mathbf{e}\mathbf{e}) = \mathbf{D}^{\nabla}. \tag{94}$$

Using (91) and (94) in (89) gives our Oldroyd-B constitutive relation (2).

Appendix C: When $\lambda_1 = \lambda_2$

The basic steady flow corresponding to the constant pressure gradients and velocities described by relations (6)-(8) is Newtonian if $\lambda_1 = \lambda_2$ in the constitutive relations (7). Indeed, if $\lambda_1 = \lambda_2$ then the constitutive relations (7) become

$$\tau^0 - \lambda_1(\mathbf{L}^0\tau^0 + \tau^0\mathbf{L}^{0T}) = \mu\{\mathbf{D}^0 - \lambda_1(\mathbf{L}^0\mathbf{D}^0 + \mathbf{D}^0\mathbf{L}^{0T})\}. \tag{95}$$

We introduce $\mathbf{A} = \tau^0 - \mu\mathbf{D}^0 = a_{ij}$, then $\mathbf{A} = \mathbf{A}^T$ and we get

$$\mathbf{A} - \lambda_1(\mathbf{L}^0\mathbf{A} + \mathbf{A}\mathbf{L}^{0T}) = 0. \tag{96}$$

We claim that from (96) it follows $\mathbf{A} = 0$.

Indeed, we have (see also the left hand side of the relation (9)):

$$\mathbf{L}^0\mathbf{A} + \mathbf{A}\mathbf{L}^{0T} = \begin{pmatrix} 2u_z^0 a_{31} & (u_z^0 a_{32} + v_z^0 a_{31}) & u_z^0 a_{33} \\ (u_z^0 a_{32} + v_z^0 a_{31}) & 2v_z^0 a_{23} & v_z^0 a_{33} \\ u_z^0 a_{33} & v_z^0 a_{33} & 0 \end{pmatrix} \tag{97}$$

and (96) becomes

$$\begin{pmatrix} a_{11} - 2\lambda_1 u_z^0 a_{31} & a_{12} - \lambda_1(u_z^0 a_{32} + v_z^0 a_{31}) & a_{13} - \lambda_1 u_z^0 a_{33} \\ a_{21} - \lambda_1(u_z^0 a_{32} + v_z^0 a_{31}) & a_{22} - 2\lambda_1 v_z^0 a_{23} & a_{23} - \lambda_1 v_z^0 a_{33} \\ a_{31} - \lambda_1 u_z^0 a_{33} & a_{32} - \lambda_1 v_z^0 a_{33} & a_{33} \end{pmatrix} = 0. \tag{98}$$

First we get $a_{33} = 0$. The third row entries of the above relation give $a_{31} = a_{32} = 0$. Using these values in the second row entries gives $a_{21} = a_{22} = 0$. Finally, the first row entries give $a_{11} = 0$. Thus we have proved that $\mathbf{A} = 0$ which means $\tau^0 - \mu\mathbf{D}^0 = 0$. Then the basic flow is indeed Newtonian when $\lambda_1 = \lambda_2$. □

References

[1] J. Azaiez and G. Homsy, *Linear stability of free shear flow of viscoelastic liquids*, Journal of Fluid Mechanics, 268 (1994), pp. 37–69.

[2] M. Ben Amar, *Viscous fingering: A singularity in laplacian grown models*, Phys.Rev. E, 51 (1995), pp. 3819–3822.

[3] M. Ben Amar and E. Corvera Poir, *Pushing a non-newtonian fluid in a Hele-Shaw cell: From fingers to needles*, Physics of Fluids (1994-present), 11 (1999), pp. 1757–1767.

[4] D. Bonn, H. Kellay, M. Brunlich, M. Ben Amar, and J. Meunier, *Viscous fingering in complex fluids*, Physica A: Statistical Mechanics and its Applications, 220 (1995), pp. 60 – 73.

[5] D. Bonn and J. Meunier, *Viscous fingering and related instabilities in complex fluids*, Phil. Mag. B, 78 (1997), pp. 131 – 142.

[6] A. Buka, J. Kertesz, and T. Vicsek, *Transitions of viscous fingering patterns in nematic liquid crystals*, Nature, 323 (1986), pp. 424–425.

[7] A. Buka and P. Palffy-Muhoray, *Morphological phase transitions in viscous fingering patterns in the liquid crystal 8cb*, J. Phys. France, 49 (1988), pp. 1319–1323.

[8] E. Corvera Poire and M. Ben Amar, *Finger behavior of a shear thinning fluid in a Hele-Shaw cell*, Phys. Rev. Lett., 81 (1998), pp. 2048–2051.

[9] G. Daccord and R. Lenormand, *Fractal patterns from chemical dissolution*, Nature, 325 (1987), pp. 41–43.

[10] P. Daripa, *Hydrodynamic stability of multi-layer Hele-Shaw flows*, Journal of Statistical Mechanics, 12 (2008), p. 28.

[11] P. G. de Gennes, *Time effects in viscoelastic fingering*, EPL (Europhysics Letters), 3 (1987), p. 195.

[12] P. Drazin and W. H. Reid, *Hydrodynamic Stability*, Cambridge University Press, 1981.

[13] D. Joseph, *Fluid Dynamics of Viscoelastic Liquids*, no. v. 84 in Applied Mathematical Sciences, Springer, 1990.

[14] R. Kupferman, *On the linear stability of plane couette flow for an oldroyd-b fluid and its numerical approximation*, Journal of Non-Newtonian Fluid Mechanics, 127 (20055), pp. 169–190.

[15] E. Lemaire, P. Levitz, G. Daccord, and H. Van Damme, *From viscous fingering to viscoelastic fracturing in colloidal fluids*, Phys. Rev. Lett., 67 (1991), pp. 2009–2012.

[16] A. Levermann and I. Procaccia, *Bi-laplacian growth patterns in disordered media*, Phys. Rev. Lett., 89 (2002), p. 234501.

[17] C. Macosko, *Rheology: principles, measurements, and applications*, Advances in interfacial engineering series, VCH, 1994.

[18] S. Mora and M. Manna, *Saffman-Taylor instability for generalized newtonian fluids*, Phys. Rev. E, 80 (2009), p. 016308.

[19] ——, *Saffman-Taylor instability of viscoelastic fluids: From viscous fingering to elastic fractures*, Phys. Rev. E, 81 (2010), p. 026305.

[20] J. Nase, A. Lindner, and C. Creton, *On the respective roles of low surface tension and non-newtonian rheological properties in fractal fingering*, Phys. Rev. Lett., 101 (2008).

[21] J. Nittmann, G. Daccord, and H. E. Stanley, *Fractal growth viscous fingers: quantitative characterization of a fluid instability phenomenon*, Nature, 314 (1985), pp. 141–144.

[22] J. G. Oldroyd, *On the formulation of rheological equations of state*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 200 (1950), pp. pp. 523–541.

[23] P. J. Oliveira and F. Pinho, *Plane contraction flows of upper convected maxwell and phan-thien-tanner fluids as predicted by a finite-volume method*, Journal of Non-Newtonian Fluid Mechanics, 88 (1999), pp. 63 – 88.

[24] M. Renardy and Y. Renardy, *Linear stability of plane couette flow of an upper convected maxwell fluid*, Journal of Non-Newtonian Fluid Mechanics, 22 (1986), pp. 23–33.

[25] J. Ro and G. Homsy, *Viscoelastic free-surface flows — thin-film hydrodynamics of Hele–Shaw and dip coating flows*, Journal of Non-Newtonian Fluid Mechanics, 57 (1995), pp. 203–225.

[26] P. G. Saffman and G. Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 245 (1958), pp. 312–329.

[27] S. Saprykin, R. Koopmans, and S. Kalliadasis, *Free-surface thin-film flows over topography; influence of inertia and viscoelasticity*, J. Fluid Mech., 58 (2007), pp. 271–293.

[28] R. Sureshkumar and A. Beris, *Linear stability analysis of viscoelastic poiseuille flow using an arnoldi-based orthogonalization algorithm*, Journal of Non-Newtonian Fluid Mechanics, 56 (1995), pp. 151–182.

[29] H. Van Damme, E. Alsac, C. Laroche, and L. Gatineau, *On the respective roles of low surface tension and non-newtonian rheological properties in fractal fingering*, EPL (Europhysics Letters), 5 (1988), p. 25.

[30] H. Van Damme, H. Obrecht, P. Levitz, L. Gatineau, and C. Laroche, *Fractal viscous fingering in clay slurries*, Nature, 320 (1986), pp. 731–733.

[31] S. D. R. Wilson, *The Taylor-Saffman problem for a non-newtonian liquid*, Journal of Fluid Mechanics, 220 (1990), pp. 413–425.

[32] H. Zhao and J. V. Maher, *Associating-polymer effects in a Hele-Shaw experiment*, Phys. Rev. E, 47 (1993), pp. 4278–4283.