Applied Numerical Mathematics 34 (2000) 373–379
APPLIED NUMERICAL MATHEMATICS
www.elsevier.nl/locate/apnum

The fastest smooth Taylor bubble

Prabir Daripa
Department of Mathematics, Texas A&M University, College Station, TX-77843, USA

Abstract

The complicated nature of singularities associated with topological transition in the plane Taylor-bubble problem is briefly discussed in the context of estimating the speed of the fastest smooth Taylor-bubble in the absence of surface tension. Previous numerical studies were able to show the presence of a stagnation point at the tip of the bubbles for dimensionless speed $F < 0.357$ but were incomplete in characterizing the topology of these bubbles at the tip for values of $F > 0.29$ due to difficulties in obtaining numerical solutions with well-rounded profiles at the apex. These difficulties raise the question whether the bubbles rising at a speed $F \in (0.29, 0.357)$ are smooth, pointed or spurious. This issue has led us to carefully scrutinize certain asymptotic behavior of the Fourier spectrums of the numerical solutions for a wide range of values of F and to extend these results in an appropriate limiting sense. Our findings indicate that these plane bubbles with $F < 0.35784$ (accurate up to four decimal places) have well-rounded profiles at the apex. The purpose of this paper is to describe our approach and its use in arriving at the above conclusion.

Keywords: Bubble; Free boundary; Singularity; Fourier series method

1. Introduction

Topological transitions are common phenomena in many areas of fluid mechanics. Such transitions are usually nonlinear and involve one or both of the geometrical and mathematical singularities. For some of these problems, these singularities and nonlinearity can pose significant analytical and numerical challenges. One such problem arises in the context of a plane Taylor-bubble [6,13] rising through an incompressible fluid under gravity g at a speed U in a two-dimensional tube of width h. The bubble is infinitely long and is more like a propagating finger with the tails of the bubble asymptotically approaching the walls far downstream at infinity. Details on this problem can be found in [1].

This problem has a one-parameter family of solutions characterized by the dimensionless speed $F = U/\sqrt{gh}$, called the Froude number. These solutions are divided into three regimes depending on whether the bubbles are smooth, pointed (with 120° angle at the corner) or cusped [9]. At the transition points (i.e., the values of F) where these distinct topological changes occur, the mathematical singularity

E-mail address: prabir.daripa@math.tama.edu (P. Daripa).

0168-9274/00/$20.00
PII: S0168-9274(99)00037-9

at the tip of these bubbles undergoes sudden change. Due to the complicated nature of the singularity, it has been difficult using straight-forward numerical approaches to obtain solutions with prescribed tip angle for values of F well before the transition takes place. Some relevant remarks related to this problem are given below.

The asymptotic calculation of Garabedian [8] finds that the fastest smooth bubble rises at a speed $F \sim 0.24$. This speed is close to the speed of the experimentally observed smooth bubbles [2,7,12] and to the numerically computed speed of the zero surface tension limit bubble [15]. However, recent numerical calculations show that all plane bubbles with $F \leqslant 0.357$ (approximate) have stagnation points at the tip (see [14]). Vanden-Broeck [14] could obtain smooth bubbles for F up to 0.29 (approximate) and a pointed bubble at $F = 0.357$. Therefore, it would seem necessary to somehow provide some indication as to whether the bubbles in the approximate regime $0.29 < F < 0.357$ are smooth, pointed or spurious bubbles. This is what we attempt to do in this paper. We essentially revisit the calculations of Vanden-Broeck [14] and analyze certain asymptotic behavior of the Fourier spectrums of the numerical solutions for various values of F, and extend results in an appropriate limiting sense. Our findings seem to indicate that plane bubbles with $F < 0.35784$ (accurate up to four decimal places) have well-rounded profiles at the apex. Additionally, we also sketch a possible scenario of topology transition: how does smooth bubble become pointed and pointed bubble become cusped as F crosses the transition points?

In order to address these issues, we first need to briefly describe the formulation of this problem [1,3, 14]. With respect to the reference frame attached to the bubble, the fluid upstream in a tube of width h has a speed U downward. With appropriate normalization (speed by U and time by (h/U)), far upstream (i.e., $x \to -\infty$) $q = 1,\ \theta = 0$ where q is the speed and θ is the flow direction. The apex of the bubble is located at $x = y = 0$. It is useful to deal with this problem in an auxiliary circle plane, $|\sigma| \leqslant 1$, which is obtained by a conformal mapping of the potential plane image of the flow in the physical plane. This maps the bubble surface onto the upper semi-circle $\sigma = \mathrm{e}^{i\alpha}$, $\alpha \in [0, \pi]$, the walls on $(-1, 1)$ and the flow domain onto the interior of the domain bounded by the upper semi-circle and the real axis. The image of the apex of the bubble is $\sigma = \mathrm{i}$ and that of the tail of the bubble is $\sigma = \mp 1$.

The complex function $\tau = \nu - \mathrm{i}\theta$, where $\nu = \ln q$, is an analytic function of σ within the semi-circle and is continuous and real on the real axis since $\theta = 0$ on the walls. An appropriate representation of $\tau(\sigma)$ in $|\sigma| \leqslant 1$ is then given by

$$\mathrm{e}^{\tau(\sigma)} = \left(1+\sigma^2\right)^{\gamma}\left[-\ln C\left(1-\sigma^2\right)\right]^{1/3}[-\ln C]^{-1/3}\mathrm{e}^{g(\sigma;\theta_t)}, \tag{1}$$

$$g(\sigma;\theta_t) = \sum_{n=1}^{\infty} a_n(\theta_t)\sigma^{2n}, \tag{2}$$

where $0 < C < 0.5,\ \gamma = \theta_t/\pi \geqslant 0$ and the Fourier coefficients a_n are real. Since the values of these Fourier coefficients depend on the value of θ_t, we have explicitly shown this here using the notation $a_n(\theta_t)$. Explicit dependence of these coefficients on F is suppressed from this notation. A derivative form of the Bernoulli's equation on the bubble interface in the circle plane is given by [3]

$$\pi \tan\alpha\, \mathrm{e}^{2\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\alpha} + \frac{\mathrm{e}^{-\nu}}{F^2}\cos\theta = 0, \quad 0 \leqslant \alpha < \frac{\pi}{2}. \tag{3}$$

It is generally accepted that this nonlinear eigenvalue problem has a spectrum (F) which is divided into three regimes depending on whether bubbles are smooth, pointed ($\theta_t = 120^\circ$) or cusped.

For numerical purposes, Eq. (3) is expressed in terms of N number of Fourier coefficients a_n using Eq. (1) and truncated version of Eq. (2). This equation with prescribed values of F and θ_t is then applied to N equi-spaced points: $\alpha_I = (\pi/2N)(I - 1/2),\ I = 1, \ldots, N$, to determine N unknown Fourier coefficients.

The futility of computations with the above method for certain values of F in the neighborhood of transition points is well known [5,14]. The reasons for such dismal performance can be explained as follows.

Consider that there are smooth bubbles for $F < F_l$, pointed bubbles for $F_l \leqslant F \leqslant F_u$ and cusped bubbles for $F > F_u$. It then follows from (1) that

$$g(\sigma;\pi) \to -\tfrac{1}{3}\ln\bigl(1+\sigma^2\bigr) + g\bigl(\sigma;\tfrac{2}{3}\pi\bigr), \quad \text{as } F \uparrow F_l, \tag{4}$$

$$g(\sigma;0) \to -\tfrac{2}{3}\ln\bigl(1+\sigma^2\bigr) + g\bigl(\sigma;\tfrac{2}{3}\pi\bigr), \quad \text{as } F \downarrow F_u, \tag{5}$$

where $g(\sigma;\pi)$, $g(\sigma;\frac{2}{3}\pi)$ and $g(\sigma;0)$ are all bounded function in $|\sigma| \leqslant 1$. This implies that function $g(\sigma;\theta_t)$ has to develop a logarithmic branch point of a different order at the tip of the pointed bubbles at each of the transition points. Therefore, behavior of the function $g(\sigma;\theta_t)$ has to be complicated in the neighborhood of the transition points. This undoubtedly will cause numerical difficulties in computing solutions for values of F close to F_l and F_u. Therefore, there is little hope of estimating the value of F_l, the speed of the fastest smooth bubble by directly computing this bubble. One of the goals of this paper is to explain how a meaningful method to estimate this speed can be arrived at by probing the qualitative properties and some limiting behavior (in some sense, to be made precise below) of the Fourier spectrums of the numerical solutions. It is worth pointing out here that any attempt to estimate this speed by computing the flow angle or some other equivalent quantities at or near the tip results in dismal failure because of the difficulties in computing the flow variables accurately for values of F near F_l for reasons mentioned earlier (see also [5]). Another approach to estimating this speed will perhaps require a more flexible representation of the flow at the singular apex or inclusion of singular terms outside the physical domain of flow as in Longuet-Higgins [11], a topic for future research.

Since the Taylor series of the logarithmic terms in (4) and (5) about the origin converge for $|\sigma| \leqslant 1,\ \sigma \neq \mp \mathrm{i}$, and have coefficients which alternate in sign from odd to even mode, the Fourier spectrums of the solutions (see (2)) for values of F well before the transition may show signs of such oscillatory behavior. Since oscillations in the Fourier spectrum should also appear at transition in order for the series (2) to account for the jump in the apex angle at transition, there arises the problem of properly diagnosing such oscillations in the Fourier spectrums of the solutions. Moreover, such oscillations will also appear if the prescribed value of θ_t is not appropriate for the prescribed value of F. Therefore, we briefly analyze the source of such oscillations in the Fourier spectrums from an equivalent perspective.

Consider that computations use θ_t when a bubble with an apex angle θ_a ($\theta_a \neq \theta_t$) is the right solution of the problem at a specific value of F. Then the function $g(\sigma;\theta_t)$ in equation (1) will diverge logarithmically at the stagnation point and the series (2) for this function will converge only for $|\sigma| \leqslant 1,\ \sigma \neq \mp \mathrm{i}$. To be precise, we have from (1)

$$g(\sigma;\theta_t) = g(\sigma;\theta_a) + \varepsilon \ln\bigl(1+\sigma^2\bigr), \quad |\sigma| \leqslant 1, \tag{6}$$

where $\varepsilon = (\theta_a - \theta_t)/\pi$. The Taylor-series of $g(\sigma,\theta_t)$ will now converge only for $|\sigma| \leqslant 1,\ \sigma \neq \mp \mathrm{i}$, because of the logarithmic branch point at $\sigma = \mathrm{i}$. The Taylor series

$$\ln\bigl(1+\sigma^2\bigr) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sigma^{2n}, \tag{7}$$

for the logarithmic term in (6) converges slowly, in fact very slowly, at points in the neighborhood of $\sigma = \mp \mathrm{i}, \mp 1$ and also at $\sigma = \mp 1$. It then follows from (2), (6) and (7) that the Taylor series coefficients $a_n(\theta_t)$ of the function $g(\sigma, \theta_t)$ in terms of the Taylor series coefficients $a_n(\theta_a)$ of $g(\sigma, \theta_a)$ are given by

$$a_n(\theta_t) = a_n(\theta_a) + \varepsilon \frac{(-1)^n}{n}, \quad n = 1, \ldots, \infty. \tag{8}$$

Since $g(\sigma, \theta_a)$ is bounded and continuous on the unit circle, the most conservative estimate of the asymptotic behavior of Fourier coefficients $a_n(\theta_a)$ is given by

$$a_n(\theta_a) = \mathrm{o}\left(\frac{1}{n}\right), \quad \text{as } n \to \infty. \tag{9}$$

Therefore Eq. (8) becomes

$$a_n(\theta_t) \sim \varepsilon \frac{(-1)^n}{n}, \quad \text{as } n \to \infty, \tag{10}$$

indicating that amplitudes of the coefficients $a_n(\theta_t)$ for choice of $\theta_t \neq \theta_a$ will alternate in sign from odd to even n and will slowly decay to zero as $n \to \infty$.

To summarize, we have the following result which we refer below as "main result".

- *Absence of persistent oscillations in the values of a_n at large n as $n \to \infty$ must indicate that the corresponding solution is the correct solution, i.e., the apex angle of the solution at that value of F must be the value of θ_t used for computation.*

We make use of this result to validate numerical solutions by scrutinizing the qualitative asymptotic behavior of the Fourier spectrums as $N \to \infty$. This is to be inferred here from observing the trend in the behavior of the Fourier spectrums of the solutions obtained from using a sequence of modest values of N. Having established certain facts in the finite dimensional situations, i.e., for some finite values of N, we use the results to formulate the corresponding facts for the continuous problem, i.e., for the infinite-dimensional situation. It will be desirable to validate the final results by yet another method or direct proof.

It is appropriate here to introduce the following notations for two critical values of Froude numbers that we will often encounter below: $F_1 = 0.234$ and $F_\mathrm{C} = 0.3578$.

Results of numerical simulations in [4] with $\theta_t = 120°$ and several values of F between 0 and 1 showed that oscillations envelope the entire Fourier spectrum and do not disappear at large wavenumber as $n \to \infty$ for values of $F \neq F_\mathrm{C}$. This led us to conclude that a legitimate pointed bubble exists at $F = F_\mathrm{C}$. We also find that the Fourier modal amplitudes at F_C decay asymptotically as $a_n \sim n^{-1.35}$ at large n without oscillations.

In order to conclude that numerical solutions with $F < F_\mathrm{C}$ are all smooth bubbles, it is necessary to rule out the possibility of spurious pointed bubbles as well as to provide evidence that these solutions indeed have well-rounded profile at the apex. Therefore, we have carried out numerical simulations with the prescribed value of $\theta_t = 180°$ in (1). We briefly describe most pertinent numerical results below (see [5] for an exhaustive account of these and many other numerical results).

Reliable numerical solutions were obtained for $F \leqslant 0.234$ with $N = 31$, for F up to 0.3 (approximately) with $N = 121$ and for F up to 0.34 (approx) with $N = 251$. (High accuracy was determined by comparing these solutions with $N = 31$ and $N = 121$ with those obtained with $N = 121$ and $N = 251$, respectively. In fact, values of the Fourier coefficients obtained with $N = 31$ are accurate up to 10^{-6} (10^{-5}) when compared with their values obtained with $N = 121$ ($N = 251$).) All these bubbles satisfy the theoretical estimate of the shape of these bubbles at the tails.

Fourier coefficients of all solutions with $F \leqslant 0.234$ are found to decay monotonically with $N = 31$, 121 and 251. The asymptotic decay rate appears to be $a_n \sim n^{-1.4}$. Above analysis and our main result imply that these are legitimate smooth bubbles. This is also reflected in other quantities of interest (see [5]).

As soon as F exceeds 0.234, oscillations of very small amplitude develop for low wavenumber modes in the Fourier spectrum. At a fixed N and with increasing F, these oscillations increase in amplitude as well as propagate further out into the spectrum until the entire spectrum is enveloped with oscillations at some F. But with increasing N at a fixed F in the regime $0.234 < F < 0.30$, these oscillations disappear at large wavenumbers and the Fourier modal amplitudes decay asymptotically like $a_n \sim n^{-1.4}$ without oscillations. We needed N up to 251 to arrive this conclusion. Our main result then implies that these are also legitimate smooth bubbles.

At $F = 0.31$, oscillations appear in the entire Fourier spectrum (i.e., Fourier modal amplitudes change trend (increase/decrease) from odd to even n) with any N up to 251. Oscillations of very small amplitude of the order of 10^{-7} appears in the far end of the spectrum at this value of F. With increasing F, decay rate of a_n slowly decreases and the amplitude of the oscillations gradually increases. For example, the amplitude of these oscillations at $n > 200$ is of the order of 10^{-4} at $F = 0.34$. We believe these oscillations are genuine (i.e., not an effect of discretization) but will not persist asymptotically (i.e., as $n \to \infty$) if calculations are done with higher values of N. Our calculations with higher values of N were not successful due to the complicated nature of the singularities which are not adequately accounted for in our numerical method. Nonetheless, we can still estimate the range of values of F for which smooth bubbles exist by analyzing the limit behavior of the Fourier spectrums and applying our main result.

Fig. 1(a) displays the frequency of oscillations (defined as the number of oscillations as a fraction of $N - 2$) in the Fourier spectrum as a function of F for three choices of N. The sudden emergence of oscillations in the Fourier spectrum is clearly visible as soon as 'F' exceeds a critical value of 0.234. These oscillations first develop in the low-wavenumber modes and then propagate towards higher wavenumber modes (never the other way) with increasing F until the entire spectrum is enveloped with persistent oscillations at some N-dependent value $F_O(N)$ where $F_1 < F_O(N) < F_C$. For $F \in (F_O(N), F_C)$, all modes participate in this oscillatory behavior.

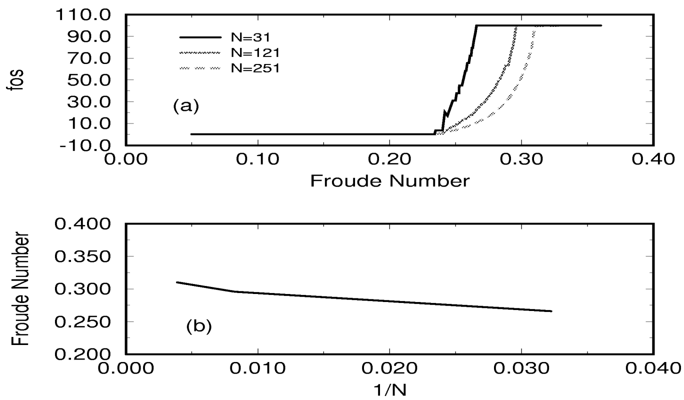

Fig. 1. (a) Plots of fos (frequency of oscillations) of the Fourier coefficients a_n vs. F for three different choices of number of Fourier modes N. (b) Plot of minimum values of F with 100% oscillations in the Fourier spectrum vs. $(1/N)$.

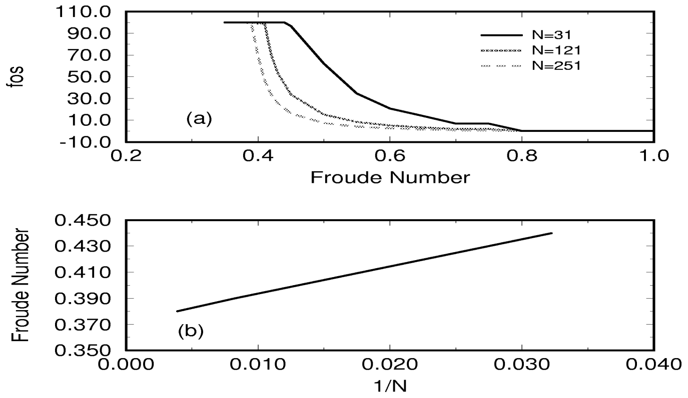

Fig. 2. (a) Plots of frequency of oscillations of the Fourier coefficients a_n vs. F for three different choices of number of Fourier modes N. (b) Plot of minimum values of F with 100% oscillations in the Fourier spectrum vs. $(1/N)$.

Many of the remarks made earlier is captured in Fig. 1. But the key thing to note is that $F_O(N)$ increases with increasing N. Fig. 1(b) shows a plot of $F_O(N)$ vs. $(1/N)$. Fig. 1 indicates that $F_O(N)$ will most likely approach a value close to F_C as $N \to \infty$, an issue discussed in more detail below. In other words, oscillations at large wavenumber in the Fourier spectrums of the solutions will probably disappear as $N \to \infty$ only for $F < F_C$. But, these oscillations will persist at $F = F_C$. Our main result then implies that all solutions with $F < F_C$ must be well-rounded at the apex and that a singularity of different order very likely arises at $F = F_C$, indicating this to be a transition point from smooth to non-smooth bubbles. As we have seen before, indeed a pointed bubble exists at the value of F_C.

We have also carried out numerical simulations with $\theta_t = 0°$ for cusped bubbles and similar difficulties are observed for values of F close to but greater than F_C. Some plots similar to Fig. 1 is shown in Fig. 2 for the case of cusped bubbles. Clearly, a similar argument as presented before leads us to conclude the existence of cusped bubbles for $F > F_C$. The Fourier modal amplitudes for $F > 0.4$ are found to decay asymptotically as $a_n \sim n^{-1.3}$ at large n without oscillations.

The following transition scenario seems compatible with the results of our simulations (see [5] for more details). A rounded nose of vanishing size appears at the tip as soon as F decreases below F_C. The radius of curvature of this nascent smooth profile at the tip gradually increases with decreasing F (and approaches infinity as $F \to 0$ which corresponds to a flat interface separating the heavy liquid from the gas). Similarly, a cusp of vanishing size appears at the tip as soon as F exceeds F_C. The size of the stem at the cusp gradually increases with increasing F and approaches infinity as $F \to \infty$. A plot of the free boundary in three critical cases is shown in Fig. 3.

Our findings definitely suggest that the fastest zero surface tension bubble is neither the zero surface tension limit bubble nor the one that is observed in practice. It appears that surface tension acts here as a destabilizing force. Moreover, the transition points from smooth-to-pointed bubbles and pointed-to-cusped bubbles are the same (i.e., $F_l = F_u = F_C$). Therefore, it is reasonable to expect (see (4) and (5)) that the singular expansion of the complex velocity in F in the neighborhood of F_C is very complicated. This complicated expansion could be at the root of the discrepancy between the asymptotic result of Garabedian and the numerical results presented here and elsewhere [5,16].

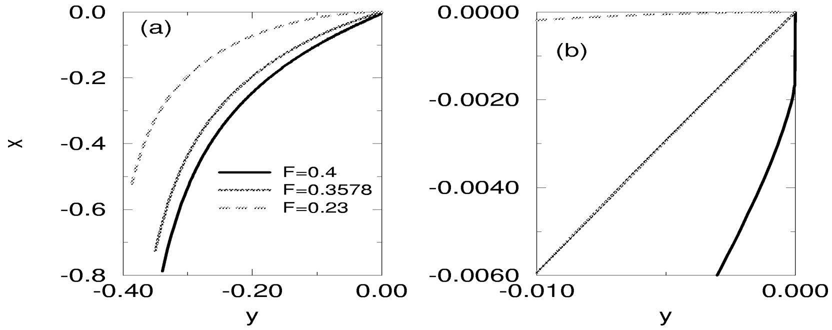

Fig. 3. (a) A plot of the free boundary for three critical cases: $F = 0.23$ (smooth), $F = 0.3578$ (pointed), and $F = 0.4$ (cusped). (b) Magnified view of the bubble profiles near the apex.

Acknowledgements

We thank the referees for their helpful suggestions. We acknowledge the financial support of the Texas Advanced Research Program (Grant No. TARP-97010366-030).

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