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A Study of a Non-Standard Eigenvalue Problem and its Application to Three-Layer Immiscible Porous Media and Hele-Shaw Flows with Exponential Viscous Profile

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Abstract. We consider a non-standard eigenvalue problem arising in stability studies of 3-layer immiscible porous media and Hele-Shaw flows which contain the viscous profile of the middle layer as a coefficient in the eigenvalue problem. We characterize the eigenvalues and eigenfunctions of this eigenvalue problem. We then apply this characterization to an exponential viscous profile and numerically compute the associated eigenvalues and eigenfunctions. We provide an explicit sequence of numbers that give upper and lower bounds on the eigenvalues. We also discuss the limiting cases when either the length of the middle layer approaches zero or the exponential viscous profile approaches a constant viscosity profile.

Mathematics Subject Classification. 76E17, 34L10, 34L15.

Keywords. Three-Layer Hele-Shaw Flow, Non-standard Eigenvalue Problem, Sturm-Liouville Problem, Pseudo-spectral Method.

1. Introduction

Three phase flows in porous media play an important role in many technical applications ranging from oil recovery to ground water remediation to storage of carbon dioxide. In such flows, hydrodynamic stability of the displacement process drives some of the phenomena whose control can maximize the performance of these displacement processes and requires an in-depth study. For immiscible displacement processes, of which we are interested in this paper, instability of the individual layers due to a non-uniform mobility profile in the layers interacts with that of the individual interfaces which are unstable due to jumps in mobilities at the interfaces, leading to complex fingering patterns on the interfaces as well as complex flow patterns in the layers. This study aims to quantify some of these complex stability issues by analyzing them at the linear stage of the development of these patterns. We do so here by studying porous media flow within a Hele-Shaw model, which is justified, especially when studied in the presence of a mobility profile in the layers; see Daripa [4,5] for a discussion on the similarity between these flows. Therefore, we study the problem of three-phase flow within a Hele-Shaw cell. An interesting direct application of this flow is three immiscible phase flow in thin fracture in porous media with intermediate transition region.

There have been numerous theoretical and numerical studies on two-layer immiscible Hele-Shaw flow since the early 1950s, starting with the work of Saffman and Taylor [13]. There are many review articles on these studies, for example see [10,12]. These studies were originally motivated by displacement processes arising in secondary oil recovery, even though these studies have much wider appeal in the sciences and engineering. In the late 1970s, tertiary displacement processes involved in chemical enhanced oil recovery generated interest in three-layer and multi-layer Hele-Shaw flows (see [3,4,6,9]). In this paper, we briefly describe the three-layer case from [3] before moving onto our studies of the associated eigenvalue problem.

Three regions of fluid in the Hele-Shaw cell (Fig. 1) are separated by sharp interfaces that are initially at $x = -L$ and $x = 0$ along which there is interfacial tension given by the values T_1 and T_0, respectively. The fluid upstream ($-\infty < x < -L$) has a constant viscosity μ_l and a velocity $\mathbf{u} = (U, 0)$ as $x \to -\infty$.

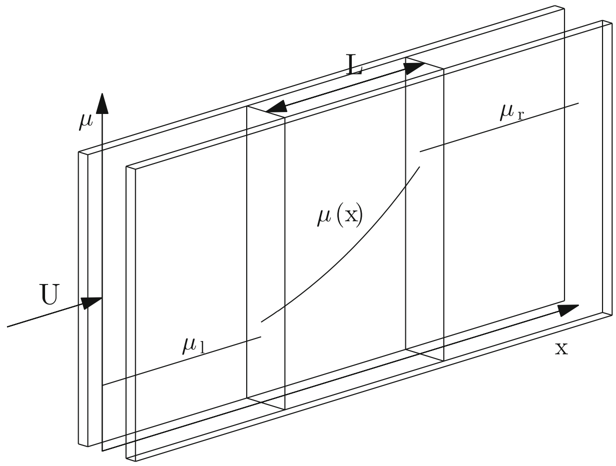

FIG. 1. Three-layer rectilinear Hele-Shaw flow in which the middle layer has a smooth viscous profile. The physical set-up as well as the viscous profile are shown in this figure

The fluid downstream ($0 < x < \infty$) has a constant viscosity μ_r. The middle layer, which has length L, contains a fluid of viscosity $\mu(x,t)$ where $\mu_l < \mu(x,t) < \mu_r$ for all $x \in (-L, 0)$. We assume here that $\mu(x,t)$ and its spatial derivative are continuous.

The governing equations for the system are

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla p = -\mu\,\mathbf{u}, \quad \frac{\partial \mu}{\partial t} + \mathbf{u} \cdot \nabla \mu = 0. \tag{1}$$

Equation $(1)_1$ is the continuity equation for incompressible flow, Eq. $(1)_2$ is Darcy's Law, and Eq. $(1)_3$ is an advection equation for viscosity, which holds when viscosity is an invertible function of the concentration of polymer.

This system admits a simple basic solution in which all of the fluid moves with constant velocity $\mathbf{u} = (U, 0)$ and the interfaces remain planar. The pressure, $p(x)$, of the basic solution is found by integrating $(1)_2$. In a moving frame with velocity U, the basic solution is stationary. We perturb the basic solution by $(\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\mu})$. The linearized equations for $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$, $\tilde{p}$ and $\tilde{\mu}$ are

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \quad \nabla \tilde{p} = -\mu \tilde{\mathbf{u}} - \tilde{\mu}(U, 0), \quad \frac{\partial \tilde{\mu}}{\partial t} + \tilde{u}\frac{d\mu}{dx} = 0. \tag{2}$$

We decompose the disturbances into normal modes. They take the form

$$(\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\mu}) = (f(x), \tau(x), \psi(x), \phi(x))e^{iky+\sigma t}, \tag{3}$$

where k is the wavenumber and σ is the growth rate of the disturbances. This ansatz is used in the linearized equations (2) along with linearized kinematic and dynamic boundary conditions to derive an eigenvalue problem for $f(x)$. More details on this derivation can be found in Daripa [3]. The eigenvalue problem is

$$\left.\begin{array}{ll}(\mu f')' - (k^2\mu - \frac{k^2U}{\sigma}\mu')f = 0, & -L < x < 0 \\ \mu(-L)f'(-L) = \left(\mu_l k - \frac{E_1}{\sigma}\right) f(-L) & \\ -\mu(0)f'(0) = \left(\mu_r k - \frac{E_0}{\sigma}\right) f(0), & \end{array}\right\} \tag{4}$$

where $E_0 = k^2U(\mu_r - \mu(0)) - T_0k^4$ and $E_1 = k^2U(\mu(-L) - \mu_l) - T_1k^4$.

In order to simplify our analysis of these equations, we use the variable $\lambda = \frac{1}{\sigma}$. Then, the above equations can be written as

$$\left.\begin{array}{ll}(\mu f')' - (k^2\mu - k^2U\mu'\lambda)f = 0, & -L < x < 0 \\ \mu(-L)f'(-L) = (\mu_l k - E_1\lambda)f(-L) & \\ -\mu(0)f'(0) = (\mu_r k - E_0\lambda)f(0). & \end{array}\right\} \tag{5}$$

Equation $(5)_1$ looks like a typical Sturm–Liouville problem, but note that the boundary conditions $(5)_2$ and $(5)_3$ contain the spectral parameter, λ. Therefore, much of the classical theory does not apply.

The maximum value of the growth rate, σ, determines the stability of the system. Therefore, it is of physical significance to understand the minimum value of λ and its dependence on the parameters. To this end, we study the nature of the spectrum of the above differential operator. A complete understanding of the eigenvalues and eigenfunctions can shed light on strategies to stabilize the flow through control of the physical quantities.

2. Characterization of the Eigenvalues and Eigenfunctions

We now investigate the nature of the eigenvalues and eigenfunctions associated with the eigenvalue problem (5). This section follows the techniques of Churchill [2].

Theorem 1. *Let $f(x)$ solve (5). Let E_0, E_1, U, k, μ_l, $\mu_r > 0$. Let $\mu(x)$ be a positive, strictly increasing function in $C^1([-L, 0])$. Then the eigenvalue problem has a countably infinite number of real eigenvalues that can be ordered*

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

with the property that for the corresponding eigenfunctions, $\{f_i\}_{i=0}^{\infty}$, f_i has exactly i zeros in the interval $(-L, 0)$. Additionally, the eigenfunctions are continuous with a continuous derivative.

Proof. The fact that there are a countably infinite number of real eigenvalues that can be ordered and corresponding eigenfunctions with the prescribed number of zeros is proven by Ince [11, p. 232–233] in Theorem I and Theorem II. The regularity of the eigenfunctions comes from the existence theorem of Ince [11, p. 73]. It remains to show that all of the eigenvalues are both real and positive. Let (f, λ) satisfy the eigenvalue problem. We take the inner product of $(5)_1$ with $f^*(x)$, the complex conjugate of $f(x)$.

$$\int_{-L}^{0} (\mu(x) f'(x))' f^*(x) dx - k^2 \int_{-L}^{0} (\mu(x) - U\mu'(x)\lambda) |f(x)|^2 dx = 0.$$

We then perform integration by parts on the first integral and use the boundary conditions $(5)_2$ and $(5)_3$.

$$\begin{aligned} &-(\mu_r k - E_0\lambda)|f(0)|^2 - (\mu_l k - E_1\lambda)|f(-L)|^2 - \int_{-L}^{0} \mu(x)|f'(x)|^2 dx \\ &- k^2 \int_{-L}^{0} (\mu(x) - U\mu'(x)\lambda)|f(x)|^2 dx = 0. \end{aligned}$$

Solving for λ,

$$\lambda = \frac{\mu_r k|f(0)|^2 + \mu_l k|f(-L)|^2 + \int_{-L}^{0} \mu(x) \left\{|f'(x)|^2 + k^2|f(x)|^2\right\} dx}{E_0|f(0)|^2 + E_1|f(-L)|^2 + k^2 U \int_{-L}^{0} \mu'(x)|f(x)|^2}. \tag{6}$$

Note that all terms are real and positive. Therefore, $\lambda > 0$ [7]. □

2.1. An Orthogonality Property of the Eigenfunctions

We now note the following property of the eigenfunctions for later use. Let f_i and f_j be eigenfunctions of (5). Then

$$\left(\mu f_i' f_j - \mu f_i f_j'\right)' = (\mu f_i')' f_j - (\mu f_j')' f_i = (\lambda_j - \lambda_i) k^2 U \mu' f_i f_j. \tag{7}$$

Using the boundary conditions $(5)_2$ and $(5)_3$,

$$\begin{aligned} (\lambda_j - \lambda_i) \int_{-L}^{0} f_i f_j (k^2 U \mu') dx &= \left(\mu f_i' f_j - \mu f_i f_j'\right) \Big|_{-L}^{0} \\ &= (\lambda_i - \lambda_j) \left\{E_0 f_i(0) f_j(0) + E_1 f_i(-L) f_j(-L)\right\}, \end{aligned}$$

and therefore, if $\lambda_i \neq \lambda_j$,

$$\int_{-L}^{0} f_i f_j (k^2 U \mu') dx + E_0 f_i(0) f_j(0) + E_1 f_i(-L) f_j(-L) = 0. \tag{8}$$

2.2. Transformation to a Regular Sturm–Liouville Problem

We now wish to connect the eigenvalue problem (5) to a related eigenvalue problem whose properties are known. Since $f_0(x)$ is non-zero on $[-L, 0]$, we can define the function, for each integer $i \geq 1$,

$$F_i(x) = \mu(x) \frac{d}{dx} \left(\frac{f_i(x)}{f_0(x)} \right). \tag{9}$$

We use the following lemma.

Lemma 1. *Let E_0, E_1, U, k, μ_l, $\mu_r > 0$ and $\mu(x)$ be a positive, strictly increasing function in $C^1([-L, 0])$. Additionally, let $\mu(x)$ be twice differentiable. Let $\{F_i\}_{i=1}^{\infty}$ be the set of functions defined by (9) where $\mu(x)$ is a strictly increasing, positive function on $[-L, 0]$ and $\{f_i\}_{i=0}^{\infty}$ is the set of eigenfunctions of (5) corresponding to the eigenvalues $\{\lambda_i\}_{i=0}^{\infty}$. Then for each $i \in \mathbb{N}$, (F_i, λ_i) is a solution to the regular Sturm-Liouville problem*

$$\begin{cases} \left(\frac{f_0^2}{\mu'} F' \right)' + \left\{ \frac{2}{(\mu')^2} (\mu' f_0 f_0'' - \mu' (f_0')^2 + \mu'' f_0 f_0') + \frac{k^2 U f_0^2}{\mu} (\lambda - \lambda_0) \right\} F = 0 \\ E_1 f_0(-L) F'(-L) = \{ k^2 U \mu'(-L) f_0(-L) - 2 E_1 f_0'(-L) \} F(-L) \\ -E_0 f_0(0) F'(0) = \{ k^2 U \mu'(0) f_0(0) + 2 E_0 f_0'(0) \} F(0). \end{cases} \tag{10}$$

Furthermore, there are no other solutions to (10).

Proof. Let $i \in \mathbb{N}$. By using the quotient rule on Eqs. (9) as well as (7), we get

$$(f_0^2 F_i)' = (\mu f_0 f_i' - \mu f_i f_0')' = (\lambda_0 - \lambda_i) k^2 U \mu' f_0 f_i. \tag{11}$$

Therefore,

$$\left(\frac{1}{k^2 U \mu' f_0^2} (f_0^2 F_i)' \right)' = \left((\lambda_0 - \lambda_i) \frac{f_i}{f_0} \right)',$$

which can be rewritten as

$$\left(\frac{1}{k^2 U \mu' f_0^2} (f_0^2 F_i)' \right)' + (\lambda_i - \lambda_0) \frac{F_i}{\mu} = 0. \tag{12}$$

But after some simple calculations, this reduces to

$$\left(\frac{f_0^2}{\mu'} F_i' \right)' + \left\{ \frac{2}{(\mu')^2} (\mu' f_0 f_0'' - \mu' (f_0')^2 + \mu'' f_0 f_0') + \frac{k^2 U f_0^2}{\mu} (\lambda_i - \lambda_0) \right\} F_i = 0, \tag{13}$$

which is the Eq. $(10)_1$. Next, we find the boundary conditions satisfied by the F_i. It follows from relation (9) that

$$\mu f_i' = f_0 F_i + f_i \frac{\mu f_0'}{f_0}. \tag{14}$$

Replacing the left-hand side of (14) with the boundary condition $(5)_2$, we obtain

$$f_0(-L) F_i(-L) + \left\{ \frac{\mu(-L) f_0'(-L)}{f_0(-L)} - (\mu_l k - E_1 \lambda_i) \right\} f_i(-L) = 0.$$

Using the boundary condition $(5)_2$ for f_0,

$$f_0(-L) F_i(-L) - E_1 (\lambda_0 - \lambda_i) f_i(-L) = 0. \tag{15}$$

Using (11), we have that

$$(\lambda_0 - \lambda_i) f_i(-L) = \frac{1}{k^2 U \mu'(-L)} \{ f_0(-L) F_i'(-L) + 2 f_0'(-L) F_i(-L) \}. \tag{16}$$

Combining Eqs. (15) and (16),

$$f_0(-L)F_i(-L) - \frac{E_1}{k^2U\mu'(-L)}\{f_0(-L)F_i'(-L) + 2f_0'(-L)F_i(-L)\} = 0,$$

and therefore

$$E_1 f_0(-L)F_i'(-L) = \{k^2U\mu'(-L)f_0(-L) - 2E_1 f_0'(-L)\}F_i(-L), \tag{17}$$

which is the boundary condition $(10)_2$. Similarly, we obtain the boundary condition $(10)_3$ by repeating the same process using (14) and $(5)_3$. Therefore, F_i and λ_i satisfy the system (10).

It remains to show that the set $\{(F_i, \lambda_i)\}_{i=1}^{\infty}$ defined by (9) is all of the solutions to (10). Let $(G(x), \alpha)$ solve (10). We will show that $(G, \alpha) = (F_i, \lambda_i)$ for some i. Define the function

$$g(x) = f_0(x)\left[\int_{-L}^{x} \frac{G(t)}{\mu(t)}dt + C\right], \tag{18}$$

where C is given by the expression

$$C = \frac{f_0^2(-L)G'(-L) + 2f_0(-L)f_0'(-L)G(-L)}{(\lambda_0 - \alpha)k^2U\mu'(-L)f_0^2(-L)}, \qquad \alpha \neq \lambda_0. \tag{19}$$

Claim: $g(x)$ and α satisfy (5). We prove this below.

Note that

$$g' = f_0\frac{G}{\mu} + f_0'\left[\int_{-L}^{x} \frac{G(t)}{\mu(t)}dt + C\right]. \tag{20}$$

Therefore,

$$(\mu g')' = f_0G' + 2f_0'G + (\mu f_0')'\left[\int_{-L}^{x} \frac{G(t)}{\mu(t)}dt + C\right]. \tag{21}$$

Using (18),

$$(\mu g')' = \frac{1}{f_0}\left\{(f_0^2G)' + g(\mu f_0')'\right\}.$$

We wish to show that $\frac{(\mu g')' - (k^2\mu - k^2U\mu'\alpha)g}{k^2U\mu' f_0}$ is a constant. Using the above equality,

$$\begin{aligned}&\frac{(\mu g')' - (k^2\mu - k^2U\mu'\alpha)g}{k^2U\mu' f_0}\\ &\quad= \frac{1}{k^2U\mu' f_0^2}(f_0^2G)' + \frac{1}{k^2U\mu' f_0}\left\{\frac{1}{f_0}(\mu f_0')' - (k^2\mu - k^2U\mu'\lambda_0) - k^2U\mu'(\lambda_0 - \alpha)\right\}g.\end{aligned}$$

Since (f_0, λ_0) satisfies $(5)_1$, we obtain

$$\frac{(\mu g')' - (k^2\mu - k^2U\mu'\alpha)g}{k^2U\mu' f_0} = \frac{1}{k^2U\mu' f_0^2}(f_0^2G)' + (\alpha - \lambda_0)\frac{g}{f_0}.$$

If we take the derivative of this expression, we see that

$$\left(\frac{1}{k^2U\mu' f_0^2}(f_0^2G)' + (\alpha - \lambda_0)\frac{g}{f_0}\right)' = \left(\frac{1}{k^2U\mu' f_0^2}(f_0^2G)'\right)' + (\alpha - \lambda_0)\frac{G}{\mu}.$$

But this is zero by the equivalence of (12) and (13) and the fact that (G, α) solves $(10)_1$. Therefore, the original expression is equal to some constant, D. That is,

$$\frac{(\mu g')' - (k^2\mu - k^2U\mu'\alpha)g}{k^2U\mu' f_0} = D,$$

and therefore,

$$(\mu g')' - (k^2\mu - k^2U\mu'\alpha)g = Dk^2U\mu' f_0, \qquad \forall x \in [-L, 0]. \tag{22}$$

We now show that $D = 0$ and therefore $g(x)$ and α solve $(5)_1$. We replace g in this equation by using our original definition of g, (18), along with Eq. (21).

$$f_0 G' + 2f_0' G + (\mu f_0')' \left[\int_{-L}^{x} \frac{G(t)}{\mu(t)} dt + C \right] - (k^2\mu - k^2 U\mu'\alpha) f_0 \left[\int_{-L}^{x} \frac{G(t)}{\mu(t)} dt + C \right] = Dk^2 U\mu' f_0.$$

Using some algebraic manipulation and the fact that f_0 satisfies $(5)_1$,

$$G' + 2\frac{f_0'}{f_0} G - k^2 U\mu'(\lambda_0 - \alpha) \left[\int_{-L}^{x} \frac{G(t)}{\mu(t)} dt + C \right] = Dk^2 U\mu',$$

therefore,

$$D = \frac{1}{k^2 U\mu'} \left(G' + 2\frac{f_0'}{f_0} G \right) - (\lambda_0 - \alpha) \left[\int_{-L}^{x} \frac{G(t)}{\mu(t)} dt + C \right]. \tag{23}$$

This expression holds for all values of $x \in [-L, 0]$, so we may choose $x = -L$. Then

$$D = \frac{1}{k^2 U\mu'(-L)} \left(G'(-L) + 2\frac{f_0'(-L)}{f_0(-L)} G(-L) \right) - (\lambda_0 - \alpha)C. \tag{24}$$

Using our choice of C from (19), we get $D = 0$ as long as $\alpha \neq \lambda_0$.

We now show that $\alpha \neq \lambda_0$ by contradiction. Assume that $\alpha = \lambda_0$. Then, by (23)

$$D = \frac{f_0^2(x)G'(x) + 2f_0(x)f_0'(x)G(x)}{k^2 U\mu'(x) f_0^2(x)}, \qquad \forall x \in [-L, 0]. \tag{25}$$

Note that the numerator above can be expressed as $\left(f_0^2(x)G(x)\right)'$. Recall that $E_0, E_1 \geq 0$. Therefore, we can consider four separate cases:

1. $E_0, E_1 \neq 0$

   First note that Eq. $(10)_1$ along with the initial conditions $F(c) = \alpha$ and $F'(c) = \beta$ for some point $c \in [-L, 0]$ and some constants α and β has a unique solution [11, p. 73]. By boundary condition $(10)_2$, if $G(-L) = 0$ and $E_1 \neq 0$, then $G'(-L) = 0$. Therefore, $G(x) \equiv 0$, which contradicts that G is an eigenfunction of (10). Therefore, we can conclude that since $E_1 \neq 0$, $G(-L) \neq 0$. Likewise, since $E_0 \neq 0$, $G(0) \neq 0$.

   When $E_1 \neq 0$, we can rearrange the boundary condition $(10)_2$ to get

$$\frac{f_0^2(-L)G'(-L) + 2f_0(-L)f_0'(-L)G(-L)}{k^2 U\mu'(-L) f_0^2(-L)} = \frac{G(-L)}{E_1}.$$

   But by (25), the left-hand side of the above equation is D. Therefore, for all $x \in [-L, 0]$,

$$\frac{\left(f_0^2(x)G(x)\right)'}{k^2 U\mu'(x) f_0^2(x)} = \frac{G(-L)}{E_1}. \tag{26}$$

   Multiplying by $k^2 U\mu'(x) f_0^2(x)$ and integrating from $-L$ to 0, we get

$$f_0^2(0)G(0) - f_0^2(-L)G(-L) = k^2 U \frac{G(-L)}{E_1} \int_{-L}^{0} \mu'(x) f_0^2(x) dx,$$

   and therefore

$$f_0^2(0)G(0) = G(-L) \left\{ f_0^2(-L) + \frac{k^2 U}{E_1} \int_{-L}^{0} \mu'(x) f_0^2(x) dx \right\}. \tag{27}$$

   Note that the coefficients of $G(-L)$ and $G(0)$ are both positive. Therefore, $G(-L)$ and $G(0)$ must have the same sign.

   When $E_0 \neq 0$, we can rearrange the boundary condition $(10)_3$ to get

$$\frac{f_0^2(0)G'(0) + 2f_0(0)f_0'(0)G(0)}{k^2 U\mu'(0) f_0^2(0)} = -\frac{G(0)}{E_0}.$$

Again, the left-hand side is equal to D. Therefore, we can combine this with (26) to get

$$\frac{G(-L)}{E_1} = -\frac{G(0)}{E_0}. \tag{28}$$

This tells us that $G(-L)$ and $G(0)$ have opposite signs, which is a contradiction.

2. $E_0 = 0$ and $E_1 \neq 0$
When $E_0 = 0$, the boundary condition $(10)_3$ for $G(x)$ becomes

$$k^2U\mu'(0)f_0(0)G(0) = 0,$$

which can only be true if $G(0) = 0$. Since $E_1 \neq 0$, Eqs. (26) and (27) still hold. Also, as seen in the previous case, $E_1 \neq 0$ implies that $G(-L) \neq 0$. However, (27) cannot be true if $G(0) = 0$ and $G(-L) \neq 0$. Thus, we have a contradiction.

3. $E_0 \neq 0$ and $E_1 = 0$
When $E_1 = 0$, the boundary condition $(10)_2$ for $G(x)$ becomes

$$k^2U\mu'(-L)f_0(-L)G(-L) = 0,$$

which can only be true if $G(-L) = 0$. Using this fact along with Eq. (25), we get that for all $x \in [-L, 0]$,

$$\frac{\left(f_0^2(x)G(x)\right)'}{k^2U\mu'(x)f_0^2(x)} = \frac{f_0^2(-L)G'(-L) + 2f_0(-L)f_0'(-L)G(-L)}{k^2U\mu'(-L)f_0^2(-L)} = \frac{G'(-L)}{k^2U\mu'(-L)}.$$

Multiplying by $k^2U\mu'(x)f_0^2(x)$ and integrating from $-L$ to 0, we get

$$f_0^2(0)G(0) - f_0^2(-L)G(-L) = \frac{G'(-L)}{\mu'(-L)}\int_{-L}^{0}\mu'(x)f_0^2(x)dx,$$

and therefore

$$f_0^2(0)G(0) = \frac{G'(-L)}{\mu'(-L)}\int_{-L}^{0}\mu'(x)f_0^2(x)dx. \tag{29}$$

Note that by the uniqueness theorem stated in Case 1 and the fact that $G(x) \not\equiv 0$, $G'(-L) \neq 0$. Since $E_0 \neq 0$, we know from Case 1 that $G(0) \neq 0$. Also, for all $x \in [-L, 0]$

$$\frac{f_0^2(x)G'(x) + 2f_0(x)f_0'(x)G(x)}{k^2U\mu'(x)f_0^2(x)} = -\frac{G(0)}{E_0}.$$

In particular, this is true at $x = -L$. Therefore,

$$\frac{G'(-L)}{k^2U\mu'(-L)} = -\frac{G(0)}{E_0}. \tag{30}$$

However, Eq. (29) implies that $G(0)$ and $G'(-L)$ are of the same sign and Eq. (30) implies that $G(0)$ and $G'(-L)$ have opposite signs, which is a contradiction.

4. $E_0 = E_1 = 0$
When $E_0 = E_1 = 0$, $G(-L) = G(0) = 0$. Since $E_1 = 0$, Eq. (29) still holds. Therefore, $G'(-L) = 0$. But then, by the uniqueness theorem, $G(x) \equiv 0$ which is a contradiction.

Therefore, since all cases lead to a contradiction, we have shown that $\alpha \neq \lambda_0$. Therefore, $(g(x), \alpha)$ solves $(5)_1$.

We claim that $(g(x), \alpha)$ also satisfies the boundary conditions $(5)_2$ and $(5)_3$. From $(10)_2$, we know that

$$E_1f_0(-L)G'(-L) = \{k^2U\mu'(-L)f_0(-L) - 2E_1f_0'(-L)\}G(-L),$$

and therefore

$$f_0(-L)G(-L) - \frac{E_1}{k^2U\mu'(-L)}\{f_0(-L)G'(-L) + 2f_0'(-L)G(-L)\} = 0. \tag{31}$$

Since $G(x) = \mu\left(\frac{g(x)}{f_0(x)}\right)$ and $g(x)$ solves $(5)_1$, we can follow the steps used to derive (11) to get

$$(f_0^2G)' = (\mu f_0g' - \mu gf_0')' = (\lambda_0 - \alpha)k^2U\mu'f_0g.$$

Dividing by f_0 and evaluating at $x = -L$ yields

$$f_0(-L)G'(-L) + 2f_0'(-L)G(-L) = (\lambda_0 - \alpha)k^2 U\mu'(-L)g(-L).$$

Substituting this into (31), we get

$$f_0(-L)G(-L) - E_1(\lambda_0 - \alpha)g(-L) = 0.$$

We use the boundary condition $(5)_2$ for the function f_0 to get that

$$\frac{\mu(-L)f_0'(-L)}{f_0(-L)} - (\mu_l k - E_1\lambda_0) = 0.$$

Multiplying this by $g(-L)$ and combining with the previous expression gives

$$f_0(-L)G(-L) - E_1(\lambda_0 - \alpha)g(-L) + \left\{\frac{\mu(-L)f_0'(-L)}{f_0(-L)} - (\mu_l k - E_1\lambda_0)\right\}g(-L) = 0.$$

Using (20) and (18) evaluated at $x = -L$ and rearranging terms yields

$$(\mu_l k - E_1\alpha)g(-L) = \mu(-L)g'(-L).$$

Therefore, $(g(x), \alpha)$ satisfies $(5)_2$. Following the same process, we can see that $(g(x), \alpha)$ also satisfies $(5)_3$. Therefore, $(g(x), \alpha)$ satisfies (5), which proves our claim.

Since $(g(x), \alpha)$ solves (5), $g \equiv f_i$ for some i and $\alpha = \lambda_i$. This means that $G \equiv F_i$. □

Lemma 1 shows us that the set $\{(F_i, \lambda_i)\}_{i=1}^{\infty}$ is the set of solutions to a regular Sturm–Liouville problem. Therefore, the set $\{F_i\}_{i=1}^{\infty}$ forms an orthonormal basis of the space

$$L_w^2(-L, 0) = \left\{ f(x) \,\middle|\, \int_{-L}^{0} |f(x)|^2 w(x)dx < \infty \right\},$$

where $w(x) = \frac{k^2 U f_0^2(x)}{\mu(x)}$. In addition, it verifies the fact that the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ are real and only have a limit point at infinity.

We now wish to show that a certain class of functions can be written as a linear combination of the eigenfunctions, $\{f_i\}_{i=0}^{\infty}$. Since $\{(F_i, \lambda_i)\}_{i=1}^{\infty}$ is an orthonormal basis of $L_w^2(-L, 0)$, any function $f(x) \in L_w^2(-L, 0)$ can be expanded as

$$f(x) = \sum_{i=1}^{\infty} c_i F_i,$$

where $c_i = \int_{-L}^{0} f(x)F_i(x)w(x)dx$.

We define the bilinear form

$$B(f, g) = \int_{-L}^{0} fg(k^2 U\mu')dx + E_0 f(0)g(0) + E_1 f(-L)g(-L). \tag{32}$$

Recall from (8) that for any distinct eigenfunctions f_i and f_j of (5), $B(f_i, f_j) = 0$. Using this bilinear form, we may now expand any function in terms of the eigenfunctions using the following theorem.

Theorem 2. *Let E_0, E_1, U, k, μ_l, $\mu_r > 0$ and $\mu(x)$ be a twice differentiable, positive, strictly increasing function in $C^1([-L, 0])$. Let $\{f_i\}_{i=0}^{\infty}$ be the eigenfunctions of (5). Let $w(x) = \frac{k^2 U f_0^2(x)}{\mu(x)}$. Let*

$$H_w^1(-L, 0) = \left\{ f(x) \in L_w^2(-L, 0) | f'(x) \in L_w^2(-L, 0) \right\}.$$

Then for any function $f(x) \in H_w^1(-L, 0)$,

$$f(x) = \sum_{i=0}^{\infty} A_i f_i(x), \tag{33}$$

where equality is in the sense of $L_w^2(-L, 0)$ and the constants A_i are given by

$$A_i = \frac{B(f, f_i)}{B(f_i, f_i)}. \tag{34}$$

Proof. Let $f \in H_w^1(-L,0)$. Then, since μ and f_0 are in $C^1([-L,0])$, $\mu\left(\frac{f}{f_0}\right)' \in L_w^2(-L,0)$. Since the set $\{F_i\}_{i=1}^{\infty}$ is complete in $L_w^2(-L,0)$, we can write

$$\mu\left(\frac{f}{f_0}\right)' = \sum_{i=1}^{\infty} A_i F_i, \tag{35}$$

where

$$A_i = \int_{-L}^{0} \mu\left(\frac{f}{f_0}\right)' F_i w(x)dx. \tag{36}$$

Dividing (35) by μ and integrating gives us that for any $x \in [-L,0]$,

$$\int_{-L}^{x}\left(\frac{f}{f_0}\right)' dt = \sum_{i=1}^{\infty} A_i \int_{-L}^{x} \frac{F_i}{\mu} dt.$$

Using that $\frac{F_i}{\mu} = \left(\frac{f_i}{f_0}\right)'$, we get

$$\frac{f(x)}{f_0(x)} - \frac{f(-L)}{f_0(-L)} = \sum_{i=1}^{\infty} A_i\left[\frac{f_i(x)}{f_0(x)} - \frac{f_i(-L)}{f_0(-L)}\right],$$

and therefore

$$f(x)f_0(-L) - f(-L)f_0(x) = \sum_{i=1}^{\infty} A_i[f_i(x)f_0(-L) - f_i(-L)f_0(x)] \quad \forall x \in [-L,0]. \tag{37}$$

Let $\tilde{w}(x) = k^2 U\mu'(x)$. Multiply the above equation by $f_0\tilde{w}$ and integrate from $-L$ to 0. Then

$$\begin{aligned} f_0(-L)&\int_{-L}^{0} f(x)f_0(x)\tilde{w}(x)dx - f(-L)\int_{-L}^{0} f_0^2(x)\tilde{w}(x)dx \\ &= \sum_{i=1}^{\infty} A_i\left[f_0(-L)\int_{-L}^{0} f_i(x)f_0(x)\tilde{w}(x)dx - f_i(-L)\int_{-L}^{0} f_0^2(x)\tilde{w}(x)dx\right]. \end{aligned}$$

Recall the bilinear form (32)

$$B(f,g) = \int_{-L}^{0} fg\tilde{w}dx + E_0 f(0)g(0) + E_1 f(-L)g(-L),$$

and, from (8), that for all $i \neq j$

$$B(f_i, f_j) = 0.$$

If we replace the integrals above using that

$$\int_{-L}^{0} fg\tilde{w}dx = B(f,g) - E_0 f(0)g(0) - E_1 f(-L)g(-L),$$

and cancel like terms, we get

$$\begin{aligned} f_0(-L)B(f,f_0) &- f(-L)B(f_0,f_0) - E_0 f_0(0)\left[f(0)f_0(-L) - f(-L)f_0(0)\right] \\ &= -B(f_0,f_0)\sum_{i=1}^{\infty} A_i f_i(-L) - E_0 f_0(0)\sum_{i=1}^{\infty} A_i\left[f_i(0)f_0(-L) - f_i(-L)f_0(0)\right]. \end{aligned} \tag{38}$$

Equation (37) with $x = 0$ gives

$$f(0)f_0(-L) - f(-L)f_0(0) = \sum_{i=1}^{\infty} A_i[f_i(0)f_0(-L) - f_i(-L)f_0(0)].$$

Plug this into (38) to get

$$f(-L) = \sum_{i=1}^{\infty} A_i f_i(-L) + f_0(-L)\frac{B(f,f_0)}{B(f_0,f_0)}.$$

If we define

$$A_0 = \frac{B(f,f_0)}{B(f_0,f_0)},$$

then

$$f(-L) = \sum_{i=0}^{\infty} A_i f_i(-L). \tag{39}$$

If we now plug this into Eq. (37), we get that $\forall x \in [-L,0]$

$$f(x)f_0(-L) - \sum_{i=0}^{\infty} A_i f_i(-L) f_0(x) = \sum_{i=1}^{\infty} A_i [f_i(x)f_0(-L) - f_i(-L)f_0(x)]$$

Solving for $f(x)$ gives

$$f(x) = \sum_{i=0}^{\infty} A_i f_i(x). \tag{40}$$

It remains to show that

$$A_i = \frac{B(f,f_i)}{B(f_i,f_i)}, \qquad \text{for } i \neq 0.$$

Recall that these coefficients came from the expression (36). Consider a function $h \in H_w^1(-L,0)$. Using integration by parts and (11), we have that

$$\int_{-L}^{0} \left(\frac{h}{f_0}\right)' f_0^2 F_i dx = \left[\frac{h}{f_0}(\mu f_0 f_i' - \mu f_0' f_i)\right]_{-L}^{0} + (\lambda_i - \lambda_0)\int_{-L}^{0} h f_i k^2 U \mu' dx. \tag{41}$$

Using the boundary conditions $(5)_2$ and $(5)_3$ to replace the derivatives,

$$\left[\frac{h}{f_0}(\mu f_0 f_i' - \mu f_0' f_i)\right]_{-L}^{0} = (\lambda_i - \lambda_0)\{E_0 h(0) f_i(0) + E_1 h(-L) f_i(-L)\}.$$

Therefore, using this in (41),

$$\int_{-L}^{0} \left(\frac{h}{f_0}\right)' f_0^2 F_i dx = (\lambda_i - \lambda_0)\left\{\int_{-L}^{0} h f_i k^2 U \mu' dx + E_0 h(0) f_i(0) + E_1 h(-L) f_i(-L)\right\},$$

or

$$\int_{-L}^{0} \left(\frac{h}{f_0}\right)' f_0^2 F_i dx = (\lambda_i - \lambda_0) B(h, f_i). \tag{42}$$

In particular, using $h = f$,

$$\int_{-L}^{0} \left(\frac{f}{f_0}\right)' f_0^2 F_i dx = (\lambda_i - \lambda_0) B(f, f_i). \tag{43}$$

On the other hand, it follows from (35) that

$$\int_{-L}^{0} \mu \left(\frac{f}{f_0}\right)' \frac{f_0^2 F_i}{\mu} dx = \sum_{j=1}^{\infty} A_j \int_{-L}^{0} \frac{f_0^2 F_i F_j}{\mu} dx.$$

But since $\{F_i\}_{i=1}^{\infty}$ is orthonormal in $L_w^2(-L,0)$,

$$\int_{-L}^{0} \mu \left(\frac{f}{f_0}\right)' \frac{f_0^2 F_i}{\mu} dx = A_i \int_{-L}^{0} \frac{f_0^2 F_i^2}{\mu} dx.$$

Equating this with (43) yields

$$A_i \int_{-L}^{0} \frac{f_0^2 F_i^2}{\mu} dx = (\lambda_i - \lambda_0) B(f, f_i). \tag{44}$$

But using (42) with $h = f_i$ and recalling from the definition of F_i that $\frac{F_i}{\mu} = \left(\frac{f_i}{f_0}\right)'$,

$$\int_{-L}^{0} \frac{f_0^2 F_i^2}{\mu} dx = \int_{-L}^{0} \left(\frac{f_i}{f_0}\right)' f_0^2 F_i dx = (\lambda_i - \lambda_0) B(f_i, f_i).$$

Combining this with (44), we get

$$A_i (\lambda_i - \lambda_0) B(f_i, f_i) = (\lambda_i - \lambda_0) B(f, f_i).$$

which leads to our desired result, (34). This concludes the proof of Theorem 2. □

3. Exponential Viscous Profile

We now apply the above theory to the case where the viscosity of the middle layer follows an exponential profile where $\mu(-L) < \mu(0)$. Note that this meets the condition of the previous section since $\mu(x)$ is positive, strictly increasing, and smooth. So for all k such that $E_0, E_1 > 0$, there are infinitely many positive values of σ which can be ordered $\sigma_1 > \sigma_2 > \cdots$ with a limit point at 0 and any function in $H_w^1(-L, 0)$ can be expanded in terms of the eigenfunctions. The viscous profile can be written as

$$\mu(x) = \mu(-L) e^{\alpha(x+L)}, \quad -L < x < 0, \tag{45}$$

where $\alpha = \frac{1}{L} \ln\left(\frac{\mu(0)}{\mu(-L)}\right)$. Therefore, $\mu'(x) = \alpha\mu(x)$. Plugging this into Eq. $(5)_1$, for $-L < x < 0$,

$$f''(x) + \alpha f'(x) + k^2 \left(U\alpha\lambda - 1\right) f(x) = 0.$$

This is a homogeneous, constant coefficient, second order differential equation. Therefore, the fundamental solutions are $e^{r_1(\lambda)x}$ and $e^{r_2(\lambda)x}$ for

$$r_1(\lambda) = \frac{-\alpha}{2} + i\beta, \qquad r_2(\lambda) = \frac{-\alpha}{2} - i\beta, \tag{46}$$

where

$$\beta^2 = k^2(U\alpha\lambda - 1) - \frac{\alpha^2}{4}. \tag{47}$$

The general solution can be written as

$$f(x) = e^{-\frac{\alpha x}{2}} (A\cos(\beta x) + B\sin(\beta x)). \tag{48}$$

This holds except when $r_1 = r_2$ (i.e. when $\beta = 0$). We will consider this special case later. For now, assume that $\beta \neq 0$. Then

$$f'(x) = -\frac{\alpha}{2} f(x) + \beta e^{-\frac{\alpha x}{2}} (-A\sin(\beta x) + B\cos(\beta x)). \tag{49}$$

Therefore

$$f(0) = A, \qquad f(-L) = e^{\frac{\alpha L}{2}} (A\cos(\beta L) - B\sin(\beta L)),$$

and

$$f'(0) = -\frac{\alpha}{2} f(0) + \beta B, \qquad f'(-L) = -\frac{\alpha}{2} f(-L) + \beta e^{\frac{\alpha L}{2}} (A\sin(\beta L) + B\cos(\beta L)).$$

Plugging these into the boundary condition $(5)_3$,

$$\left(\frac{\mu_r k - E_0\lambda}{\mu(0)} - \frac{\alpha}{2}\right) A + \beta B = 0.$$

Likewise, from the boundary condition $(5)_2$,

$$\left\{-\left(\frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\alpha}{2}\right)\cos(\beta L) + \beta\sin(\beta L)\right\} A + \left\{\left(\frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\alpha}{2}\right)\sin(\beta L) + \beta\cos(\beta L)\right\} B = 0.$$

This gives us a matrix equation of the form $\mathbf{Mx} = \mathbf{0}$ where

$$\mathbf{M} = \begin{pmatrix} \frac{\mu_r k - E_0\lambda}{\mu(0)} - \frac{\alpha}{2} & \beta \\ -\left(\frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\alpha}{2}\right)\cos(\beta L) + \beta\sin(\beta L) & \left(\frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\alpha}{2}\right)\sin(\beta L) + \beta\cos(\beta L) \end{pmatrix},$$

and

$$\mathbf{x} = \begin{pmatrix} A \\ B \end{pmatrix}.$$

This equation has a nontrivial solution if and only if the determinant of $\mathbf{M}$ is zero. Let $H(\lambda, k) = \det(\mathbf{M})$. Then

$$H(\lambda, k) = H_1(\lambda, k)\sin(\beta L) + \beta H_2(\lambda, k)\cos(\beta L), \tag{50}$$

where

$$H_1(\lambda, k) = \left(\frac{\mu_r k - E_0\lambda}{\mu(0)} - \frac{\alpha}{2}\right)\left(\frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\alpha}{2}\right) - \beta^2, \tag{51}$$

and

$$H_2(\lambda, k) = \left\{\frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\mu_r k - E_0\lambda}{\mu(0)}\right\}. \tag{52}$$

The roots of $H(\lambda, k)$ are the values of λ that are solutions to the eigenvalue problem. However, note that $\beta = 0$ implies $H(\lambda, k) = 0$. As stated above, the analysis used to derive H does not hold when $\beta = 0$. We treat this case next.

We define the number

$$\gamma_0 := \frac{\alpha^2 + 4k^2}{4k^2 U\alpha}. \tag{53}$$

By examining (47), it is seen that $\beta = 0 \iff \lambda = \gamma_0$. Since the characteristic equation now has repeated roots, the eigenfunctions will be of the form $f(x) = Ae^{-\frac{\alpha}{2}x} + Bxe^{-\frac{\alpha}{2}x}$. Then

$$f'(x) = e^{-\frac{\alpha}{2}x}\left\{-\frac{\alpha}{2}A + \left(1 + -\frac{\alpha}{2}x\right)B\right\}.$$

Therefore,

$$f(0) = A, \qquad f(-L) = e^{\frac{\alpha L}{2}}(A - BL),$$

and

$$f'(0) = -\frac{\alpha}{2}A + B, \qquad f'(-L) = e^{\frac{\alpha L}{2}}\left\{-\frac{\alpha}{2}A + \left(1 + \frac{\alpha L}{2}\right)B\right\}.$$

Pluggin these into the boundary condition $(5)_2$,

$$\left(\frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\alpha}{2}\right)A - \left\{L\left(\frac{\mu_l k - E_1\lambda}{\mu(-L)}\right) + 1 + \frac{\alpha L}{2}\right\}B = 0.$$

Likewise, using the boundary condition $(5)_3$,

$$\left(\frac{\mu_r k - E_0\lambda}{\mu(0)} - \frac{\alpha}{2}\right)A + B = 0.$$

This gives us a matrix equation of the form $\widetilde{\mathbf{M}}\mathbf{x} = \mathbf{0}$ where

$$\widetilde{\mathbf{M}} = \begin{pmatrix} \frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\alpha}{2} & -\left\{L\left(\frac{\mu_l k - E_1\lambda}{\mu(-L)}\right) + 1 + \frac{\alpha L}{2}\right\} \\ \frac{\mu_r k - E_0\lambda}{\mu(0)} - \frac{\alpha}{2} & 1 \end{pmatrix}.$$

Again, solutions occur when $\det(\widetilde{\mathbf{M}}) = 0$. Let $\widetilde{H}(\lambda, k) = det(\widetilde{\mathbf{M}})$. Then

$$\widetilde{H}(\lambda, k) = \left\{ \frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\mu_r k - E_0\lambda}{\mu(0)} + L\left(\frac{\mu_r k - E_0\lambda}{\mu(0)} - \frac{\alpha}{2}\right)\left(\frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\alpha}{2}\right)\right\}. \tag{54}$$

Recall that when $\beta \neq 0$, we seek values of λ such that $H(\lambda, k) = 0$. However, when $\beta = 0$, λ is fixed ($\lambda = \gamma_0$). Therefore, this is only an eigenvalue of the problem for a wavenumber k such that $\widetilde{H}(\gamma_0, k) = 0$.

We now return to the case when $\beta \neq 0$. Note that when $\lambda > \gamma_0$, β is real-valued, but when $\lambda < \gamma_0$, β is imaginary. In the latter case, using that $\sinh(ix) = i\sin(x)$ and $\cosh(ix) = \cos(x)$,

$$H(\lambda, k) = i\left\{H_1(\lambda, k)\sinh(|\beta|L) + |\beta|H_2(\lambda, k)\cosh(|\beta|L)\right\}, \tag{55}$$

and H is purely imaginary. Therefore, when $\lambda < \gamma_0$ we can find the zeros of $Im(H)$. In summary, we have that

$$H(\lambda, k) = \begin{cases} \text{real, if } \lambda > \gamma_0 \\ \text{imaginary, if } \lambda < \gamma_0. \end{cases} \tag{56}$$

In order to investigate the zeros of H, we define the sequence of positive numbers $\{\gamma_n\}_{n=0}^{\infty}$ by

$$\gamma_n := \frac{4n^2\pi^2 + \alpha^2L^2 + 4k^2L^2}{4k^2L^2U\alpha}. \tag{57}$$

Here, the definition of γ_0 coincides with (53). Note that $\gamma_n > \gamma_0$ for any $n \geq 1$, which implies that β will be real when $\lambda = \gamma_n$. Also, for any $n \geq 1$, if $\lambda = \gamma_n$, then $\beta = n\pi/L$. Therefore $\sin(\beta L) = 0$ and

$$H(\gamma_n, k) = \begin{cases} -\frac{n\pi}{L}H_2(\gamma_n, k), & n \text{ odd} \\ \frac{n\pi}{L}H_2(\gamma_n, k), & n \text{ even}. \end{cases} \tag{58}$$

Therefore, if $H_2(\gamma_n, k) = 0$, then γ_n is an eigenvalue of the system (5). More generally, if $n \geq 1$ and $H_2(\gamma_n, k)$ and $H_2(\gamma_{n+1}, k)$ have the same sign, then $H(\gamma_n, k)$ and $H(\gamma_{n+1}, k)$ will have opposite signs. Therefore, $H(\lambda, k) = 0$ for some $\gamma_n < \lambda < \gamma_{n+1}$.

This knowledge allows us to understand the behavior of H. In particular, we will show that for any k, H has infinitely many zeros with a limit point at infinity. Consider the function H_2. Using (52), $H_2 = 0$ when

$$\lambda = \frac{\frac{\mu_l}{\mu(-L)} + \frac{\mu_r}{\mu(0)}}{\frac{E_1}{\mu(-L)} + \frac{E_0}{\mu(0)}}k. \tag{59}$$

Note that for a fixed k, there is only one value of λ such that $H_2 = 0$. Let $\lambda^*(k)$ denote this value. Using the definitions of E_0 and E_1,

$$\lambda^*(k) = \frac{\frac{\mu_l}{\mu(-L)} + \frac{\mu_r}{\mu(0)}}{kU\left(\frac{\mu_r}{\mu(0)} - \frac{\mu_l}{\mu(-L)}\right) - k^3\left(\frac{T_1}{\mu(-L)} + \frac{T_0}{\mu(0)}\right)}. \tag{60}$$

There will be at most one value of n such that $\lambda^*(k) \in [\gamma_n, \gamma_{n+1})$. For all values of n such that $\lambda^*(k) \notin [\lambda_n, \lambda_{n+1}]$, we have the following lemma.

Lemma 2. *Fix k and let $\lambda^*(k)$ be defined by (60). For all $n \geq 1$ such that $\lambda^*(k) \notin [\gamma_n, \gamma_{n+1}]$, problem (5) has an eigenvalue λ such that*

$$\gamma_n < \lambda < \gamma_{n+1}, \tag{61}$$

and the corresponding eigenfunction f has either n or $n+1$ zeros on the interval $(0, L)$.

Proof. Since $\lambda^*(k) \notin [\gamma_n, \gamma_{n+1}]$, $H_2(\lambda, k)$ has no zeros in $[\gamma_n, \gamma_{n+1}]$. Therefore, $H_2(\gamma_n, k)$ and $H_2(\gamma_{n+1}, k)$ have the same sign. By (58), $H(\gamma_n, k)$ and $H(\gamma_{n+1}, k)$ have opposite signs, and therefore, $H(\lambda, k) = 0$ for some $\lambda \in (\gamma_n, \gamma_{n+1})$. So λ is an eigenvalue of (5).

Recall that the eigenfunctions are of the form $f(x) = e^{-\frac{\alpha x}{2}}(A\cos(\beta x) + B\sin(\beta x))$ for some constants A and B. If $\lambda \in (\gamma_n, \gamma_{n+1})$, then $\frac{n\pi}{L} < \beta < \frac{(n+1)\pi}{L}$. Therefore, the oscillatory part of f has between $\frac{n}{2}$ and $\frac{n+1}{2}$ periods on the interval $(-L, 0)$. Therefore, f must have either n or $n+1$ zeros in the interval. $\square$

This provides an infinite sequence of unbounded, increasing eigenvalues, as predicted by Theorem 1.

We now wish to characterize the relationship between the wavenumber, k, and the value of n such that $\lambda^*(k) \in [\gamma_n, \gamma_{n+1})$. If we add a condition to the parameters $\mu_r, \mu(0), \mu(-L), \mu_l, T_0$, and T_1, then we get the following fact.

Lemma 3. *Let μ_r, $\mu(0)$, $\mu(-L)$, μ_l, T_0, and T_1 be such that there exists a value k_c such that E_0 and E_1 are positive for $k < k_c$ and $E_0 = E_1 = 0$ when $k = k_c$. Then there is a sequence of wavenumbers $\{k_n\}_{n=1}^{\infty}$ such that*

1. *For all n, k_n is the maximum wavenumber such that $0 < k_n < k_c$ and $H_2(\gamma_n, k_n) = 0$.*
2. *$k_1 < k_2 < k_3 < k_4 < \cdots$*
3. *$lim_{n\to\infty} k_n = k_c$.*
4. *For all n such that $k_n \geq \frac{k_c}{\sqrt{3}}$ and all k such that $k_n \leq k < k_{n+1}$, $\gamma_n \leq \lambda^*(k) < \gamma_{n+1}$ (where $\gamma_n = \lambda^*(k) \iff k_n = k$).*
5. *For all n such that $k_n \geq \frac{k_c}{\sqrt{3}}$ and all k such that $k_n < k < k_{n+1}$, there is an eigenvalue λ such that $\gamma_j < \lambda < \gamma_{j+1}$ for all $j \neq n$.*

Proof. 1. Fix a value of n. Recall (60) which gives the value $\lambda^*(k)$ such that $H_2(\lambda^*(k), k) = 0$. Also recall that γ_n depends on k and is given by (57) as $\gamma_n(k) = \frac{4n^2\pi^2+\alpha^2L^2+4k^2L^2}{4k^2L^2U\alpha}$. Therefore, we must show that there is a value $k_n \in (0, k_c)$ such that $\gamma_n(k_n) = \lambda^*(k_n)$. Note that $\gamma_n(k) = \mathcal{O}(\frac{1}{k^2})$ as $k \to 0$, and $\lambda^*(k) = \mathcal{O}(\frac{1}{k})$ as $k \to 0$. Therefore, $\lambda^*(k) < \gamma_n(k)$ for small enough k. However, as $k \to k_c$, $\lambda^*(k) \to \infty$. This comes from the expression (59) for $\lambda^*(k)$ along with the fact that $E_0, E_1 \to 0$ as $k \to k_c$. In contrast, $\gamma_n(k)$ has a finite limit as $k \to k_c$. Therefore, $\lambda^*(k) > \gamma_n(k)$ when k is sufficiently close to k_c and there must be at least one value of $k \in (0, k_c)$ such that $\lambda^*(k) = \gamma_n(k)$. Since both $\lambda^*(k)$ and $\gamma_n(k)$ are rational functions of k, there will be finitely many such points. Therefore, we choose k_n to be the maximum number in the interval $(0, k_c)$ such that $\lambda^*(k) = \gamma_n(k)$.

2. Note that $\gamma_{n+1}(k) > \gamma_n(k)$ for all n and k. Therefore, $\gamma_{n+1}(k_n) > \gamma_n(k_n) = \lambda^*(k_n)$ for all n. But as we saw above, $\lambda^*(k) > \gamma_{n+1}(k)$ for k sufficiently close to k_c. Therefore, there is a $k \in (k_n, k_c)$ such that $\gamma_{n+1}(k) = \lambda^*(k)$. This proves that $k_n < k_{n+1}$.
3. Fix $k < k_c$. Since $lim_{n\to\infty}\gamma_n = \infty$, we may choose an N large enough so that $\gamma_n(k) > \lambda^*(k)$ for all $n > N$. Let $n > N$. Since $\lim_{k\to k_c} \lambda^*(k) = \infty$ and $\lim_{k\to k_c} \gamma_n(k)$ is finite, there is a $\tilde{k} \in (k, k_c)$ such that $\gamma_n(\tilde{k}) = \lambda^*(\tilde{k})$, and therefore, $k_n > k$. Therefore, we have shown that $k_n > k$ for all $n > N$.
4. Let $k_n \geq \frac{k_c}{\sqrt{3}}$ and $k_n \leq k < k_{n+1}$. The fact that $\gamma_n \leq \lambda^*(k)$ with equality only when $k = k_n$ holds for all values of n (not just when $k_n \geq \frac{k_c}{\sqrt{3}}$) and follows from our choice of k_n as a maximum in item 1. It remains to show that $\lambda^*(k) < \gamma_{n+1}$. Note that $\gamma_{n+1}(k_n) > \gamma_n(k_n) = \lambda^*(k_n)$. Also note that γ_{n+1} is a decreasing function of wavenumber. For wavenumbers in $[\frac{k_c}{\sqrt{3}}, k_c)$, λ^* is an increasing function of wavenumber. Therefore, there is at most one wavenumber in (k_n, k_c) such that $\gamma_{n+1} = \lambda^*$. k_{n+1} is this unique value. Therefore, $\lambda^*(k) < \gamma_{n+1}$.
5. This follows from item 4 and Lemma 2.

□

3.1. Numerical Results

We now choose values for the parameters and investigate the behavior of the system. Let

$$\mu_l = 2, \quad \mu(-L) = 4, \quad \mu(0) = 8, \quad \mu_r = 10, \quad U = 1, \quad L =, 1 \quad T_0 = T_1 = 1.$$

Using these values, E_0 and E_1 are positive for $0 < k < \sqrt{2}$. Therefore, these are the wavenumbers for which our theory in Sect. 2 holds. In particular, for each k there are infinitely many values of σ which are positive, can be put in decreasing order, and have zero as a limit point. Figure 2 shows a plot of the fifteen largest values of σ using a pseudo-spectral method (see "Appendix A.1"). For $0 < k < \sqrt{2}$, the values of σ behave as expected. Starting near $k = \sqrt{2}$, some values of σ become negative as expected.

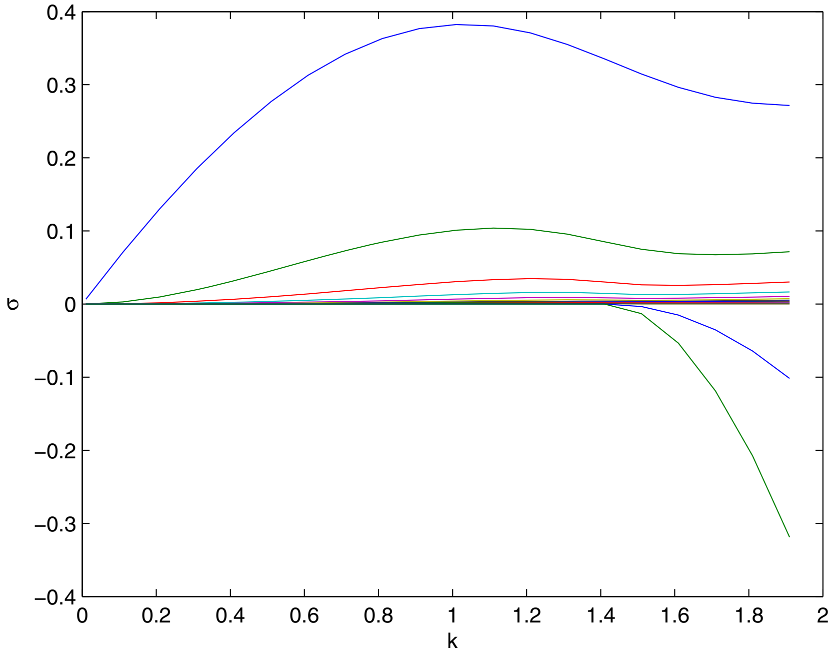

FIG. 2. (Color plot online) A plot of the dispersion *curves* for the fifteen largest values of σ

TABLE 1. *The fifteen largest values of σ for several different values of k*

	$k = 0.05$	$k = 0.20$	$k = 0.70$	$k = 1.20$	$k = 1.35$
σ_0	3.2860×10^{-2}	1.2499×10^{-1}	3.3920×10^{-1}	3.7208×10^{-1}	3.4761×10^{-1}
σ_1	5.9690×10^{-4}	8.7902×10^{-3}	7.1555×10^{-2}	1.0256×10^{-1}	9.1769×10^{-2}
σ_2	1.0464×10^{-4}	1.6506×10^{-3}	1.7766×10^{-2}	3.4771×10^{-2}	3.2629×10^{-2}
σ_3	3.6704×10^{-5}	5.8457×10^{-4}	6.7805×10^{-3}	1.5787×10^{-2}	1.5734×10^{-2}
σ_4	1.7881×10^{-5}	2.8548×10^{-4}	3.3990×10^{-3}	8.6584×10^{-3}	9.0778×10^{-3}
σ_5	1.0430×10^{-5}	1.6667×10^{-4}	2.0071×10^{-3}	5.3685×10^{-3}	5.8527×10^{-3}
σ_6	6.7943×10^{-6}	1.0862×10^{-4}	1.3156×10^{-3}	3.6204×10^{-3}	4.0651×10^{-3}
σ_7	4.7651×10^{-6}	7.6200×10^{-5}	9.2600×10^{-4}	2.5935×10^{-3}	2.9778×10^{-3}
σ_8	3.5222×10^{-6}	5.6332×10^{-5}	6.8597×10^{-4}	1.9435×10^{-3}	2.2700×10^{-3}
σ_9	2.7074×10^{-6}	4.3305×10^{-5}	5.2805×10^{-4}	1.5080×10^{-3}	1.7848×10^{-3}
σ_{10}	2.1451×10^{-6}	3.4312×10^{-5}	4.1880×10^{-4}	1.2028×10^{-3}	1.4384×10^{-3}
σ_{11}	1.7409×10^{-6}	2.7849×10^{-5}	3.4014×10^{-4}	9.8092×10^{-4}	1.1829×10^{-3}
σ_{12}	1.4409×10^{-6}	2.3050×10^{-5}	2.8167×10^{-4}	8.1486×10^{-4}	9.8921×10^{-4}
σ_{13}	1.2121×10^{-6}	1.9391×10^{-5}	2.3705×10^{-4}	6.8742×10^{-4}	8.3906×10^{-4}
σ_{14}	1.0337×10^{-6}	1.6537×10^{-5}	2.0222×10^{-4}	5.8755×10^{-4}	7.2038×10^{-4}

The values of σ are given for several different values of k in Table 1.

Our choice of parameters satisfies the assumptions of Lemma 3 with $k_c = \sqrt{2}$. Therefore, Lemma 3 ensures a sequence $\{k_n\}_{n=1}^{\infty}$. There is also a unique wavenumber $k_0 \in (0, \sqrt{2})$ such that $\widetilde{H}(\gamma_0, k_0) = 0$. The first several values of k_n are given below. Note that $k_1 > \frac{k_c}{\sqrt{3}}$. Therefore, parts 4 and 5 of Lemma 3 hold for all $n \geq 1$.

$$k_0 = 0.126, k_1 = 1.282, k_2 = 1.375, k_3 = 1.396. \tag{62}$$

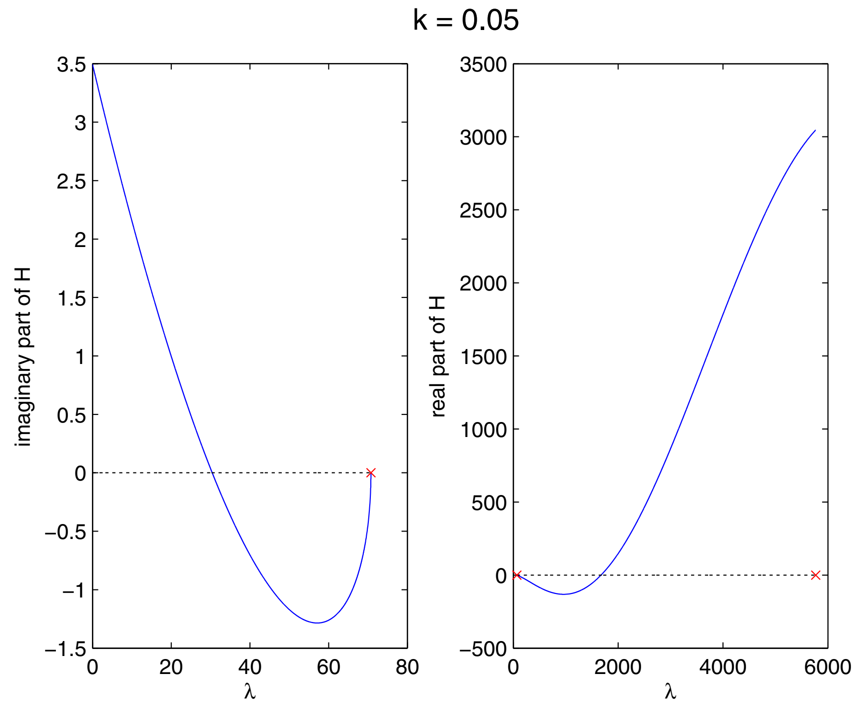

FIG. 3. (Color plot online) Plots of $H(\lambda, k)$ when $k = 0.05$. The *left plot* shows the range of λ for which $\lambda < \gamma_0$ and the *right plot* shows the range of λ for which $\lambda > \gamma_0$. The x's denote γ_0 and γ_1

The eigenvalues exhibit different behaviors depending on the wavenumber relative to these values. We will now explain the behavior in each region and plot the function $H(\lambda, k)$ for some particular k in that region.

First consider when $k < k_0$. This is the only region for which there is an eigenvalue λ such that $\lambda < \gamma_0$. Figure 3 shows a plot of H versus λ for $k = 0.05$. The plot on the left is the region in which H is imaginary (i.e. when $\lambda < \gamma_0$). Note that H has one zero in this region. Since this is the smallest value of λ which satisfies $H(\lambda, k) = 0$, this is the λ_0 given by Theorem 1. Therefore, the associated eigenfunction has no zeros on $(-L, 0)$.

The plot on the right is the region in which H is real (i.e. when $\lambda > \gamma_0$). The x's on the λ axis correspond to $\lambda = \gamma_0$ and $\lambda = \gamma_1$. We see that H has a zero between these two values and the eigenfunction corresponding to this eigenvalue must have one zero in $(-L, 0)$. So this is λ_1 given by Theorem 1. Note that with these values of the parameters, $\lambda^*(0.05) < \gamma_n$ for all $n \geq 1$. Therefore, Lemma 2 gives us upper and lower bounds for an infinite sequence of eigenvalues. We claim that this is all of the remaining eigenvalues. To see this, consider the eigenvalue given by Lemma 2 with $n = 1$, that is, $\gamma_1 < \lambda < \gamma_2$. The eigenfunction corresponding to this eigenvalue must have either one or two zeros. So this value must be λ_1 or λ_2. But we already know that $\lambda_1 < \gamma_1$. Therefore, it must be λ_2. Likewise, for any n, the λ given by Lemma 2 must correspond to λ_{n+1}. To show this behavior, we plotted H for larger values of λ in Fig. 4. The x's denote the values of γ_n. In the plot, we see that H has a zero between each value of γ_n and γ_{n+1} as H continues to oscillate.

The behavior we see for $k = 0.05$ holds for all values of k such that $k < k_0$. In summary, we have

$$\lambda_0 < \gamma_0 < \lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 < \lambda_3 < \gamma_3 < \cdots$$

In order to illustrate these results, Table 2 shows the first fifteen values of γ_i and λ_i. Additionally, we plotted several of the eigenfunctions corresponding to $k = 0.05$ in Fig. 5.

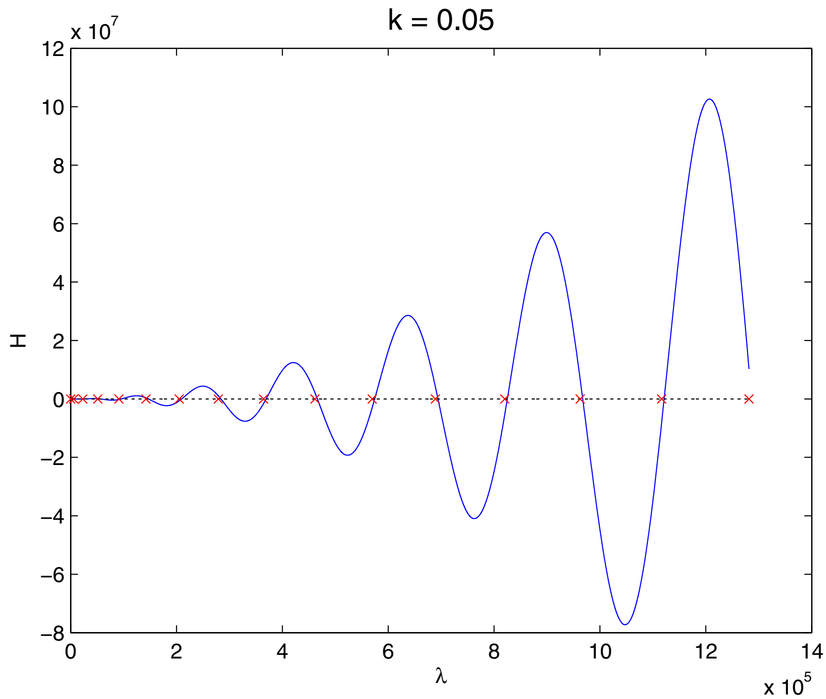

FIG. 4. (Color plot online) A plot of $H(\lambda, k)$ when $k = 0.05$ in the range of λ for which $\lambda > \gamma_0$. The x's denote the values of γ_n

TABLE 2. *The values of* γ_i *and* λ_i *for* $0 < i < 14$ *for* $k = 0.05$

i	γ_i	λ_i
0	7.0757×10^1	3.0432×10^1
1	5.7663×10^3	1.6753×10^3
2	2.2853×10^4	9.5569×10^3
3	5.1331×10^4	2.7245×10^4
4	9.1199×10^4	5.5925×10^4
5	1.4246×10^5	9.5879×10^4
6	2.0511×10^5	1.4718×10^5
7	2.7915×10^5	2.0986×10^5
8	3.6458×10^5	2.8391×10^5
9	4.6141×10^5	3.6936×10^5
10	5.6962×10^5	4.6619×10^5
11	6.8923×10^5	5.7441×10^5
12	8.2023×10^5	6.9402×10^5
13	9.6262×10^5	8.2502×10^5
14	1.1164×10^6	9.6741×10^5

Next, we investigate the case when $k_0 < k < k_1$. For k in this region, $\lambda^*(k) < \gamma_n$ for all $n \geq 1$. Therefore, there will be an eigenvalue λ such that $\gamma_n < \lambda < \gamma_{n+1}$ for all $n \geq 1$. However, there is no eigenvalue that is less than γ_0. To see this, consider Fig. 6 in which we plot $H(\lambda, k)$ for $k = 1.2$. The plot on the left shows that H has no zeros in this region. However, H has two zeros in the region between γ_0 and γ_1. These two eigenvalues are λ_0 and λ_1. Therefore, following the argument above, for $n \geq 1$, the

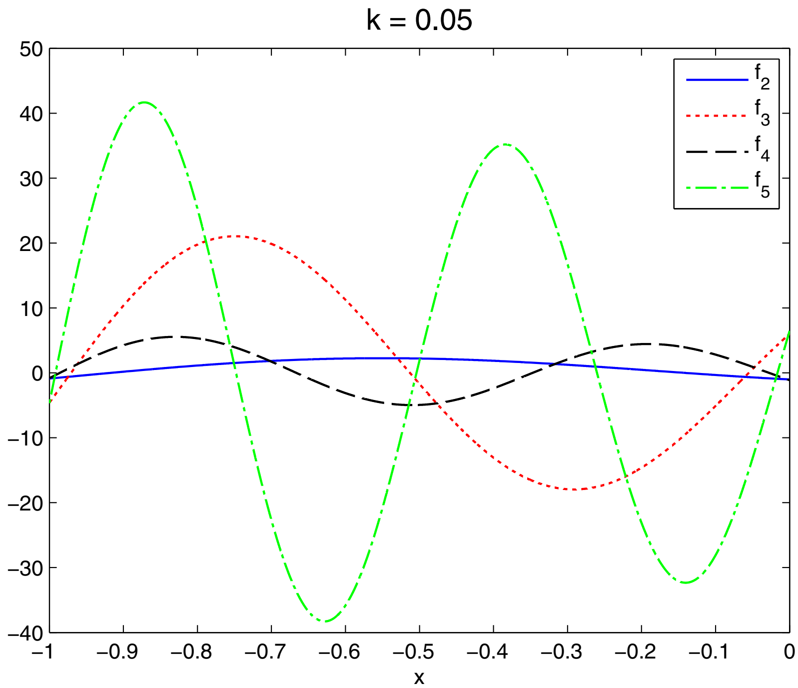

FIG. 5. (Color plot online) A plot of several eigenfunctions when $k = 0.05$

eigenvalue between γ_n and γ_{n+1} is λ_{n+1}. So

$$\gamma_0 < \lambda_0 < \lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 < \lambda_3 < \gamma_3 < \cdots$$

The first fifteen values of γ_i and λ_i when $k = 1.2$ are given in Table 3. Several eigenfunctions are plotted in Fig. 7.

For values of k such that $k_n < k < k_{n+1}$ for some $n \geq 1$, Lemma 3 tells us that $\gamma_n < \lambda^*(k) < \gamma_{n+1}$. Therefore, there will be exactly one eigenvalue between γ_i and γ_{i+1} except when $i = n$. In this case, we always get two eigenvalues. We see this in Fig. 8 in which we plotted $H(\lambda, k)$ for $k = 1.35$. Note that this falls in the range $k_1 < k < k_2$. Again, there are no eigenvalues that are less than γ_0. λ_0 is between γ_0 and γ_1. Then there are two eigenvalues between γ_1 and γ_2. So

$$\gamma_0 < \lambda_0 < \gamma_1 < \lambda_1 < \lambda_2 < \gamma_2 < \lambda_3 < \gamma_3 < \cdots$$

The first fifteen values of γ_i and λ_i for $k = 1.35$ are given in Table 4. Several eigenfunctions are plotted in Fig. 9.

In general, for $n \geq 2$ and $k_n < k < k_{n+1}$

$$\gamma_0 < \lambda_0 < \cdots < \gamma_{n-1} < \lambda_{n-1} < \gamma_n < \lambda_n < \lambda_{n+1} < \gamma_{n+1} < \lambda_{n+1} < \cdots$$

As we've seen, for $k > k_1$, the first positive eigenvalue, λ_0 is between γ_0 and γ_1. As k increases, λ_0 gets closer to γ_1. Recall that $\gamma_1 = \frac{4\pi^2+\alpha^2L^2+4k^2L^2}{4k^2L^2U\alpha}$. Therefore, as $k \to \infty$, $\gamma_1 \to \frac{1}{U\alpha}$. Therefore, the growth rate of the most dangerous mode for large wavenumbers approaches $\frac{1}{U\alpha}$. This is seen in Fig. 10 in which we plot the largest value of $\sigma = \frac{1}{\lambda}$ verses k as well as $\frac{1}{\gamma_1}$ and $U\alpha$.

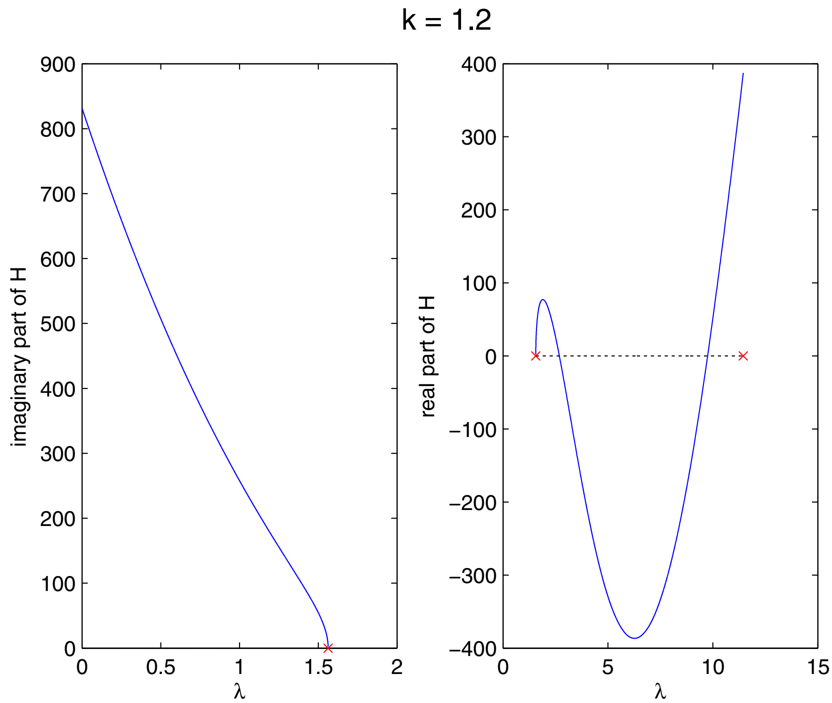

FIG. 6. (Color plot online) Plots of $H(\lambda, k)$ when $k = 1.2$. The *left plot* shows the range of λ for which $\lambda < \gamma_0$ and the *right plot* shows the range of λ for which $\lambda > \gamma_0$. The x's denote γ_0 and γ_1

TABLE 3. *The values of* γ_i *and* λ_i *for* $0 < i < 14$ *for* $k = 1.20$

i	γ_i	λ_i
0	1.5630×10^0	2.6876×10^0
1	1.1451×10^1	9.7506×10^0
2	4.1115×10^1	2.8760×10^1
3	9.0556×10^1	6.3341×10^1
4	1.5977×10^2	1.1549×10^2
5	2.4876×10^2	1.8627×10^2
6	3.5753×10^2	2.7621×10^2
7	4.8608×10^2	3.8558×10^2
8	6.3440×10^2	5.1453×10^2
9	8.0250×10^2	6.6313×10^2
10	9.9037×10^2	8.3142×10^2
11	1.1980×10^3	1.0194×10^3
12	1.4254×10^3	1.2272×10^3
13	1.6726×10^3	1.4547×10^3
14	1.9396×10^3	1.7020×10^3

3.2. Limiting Cases

We now investigate several limiting cases. First, we will look at the case when the viscous gradient in the middle layer vanishes ($\alpha \to 0$, see (45)). There are two different physical situations in which this can happen. The first is for the middle layer to maintain a constant, finite length while the viscosities at

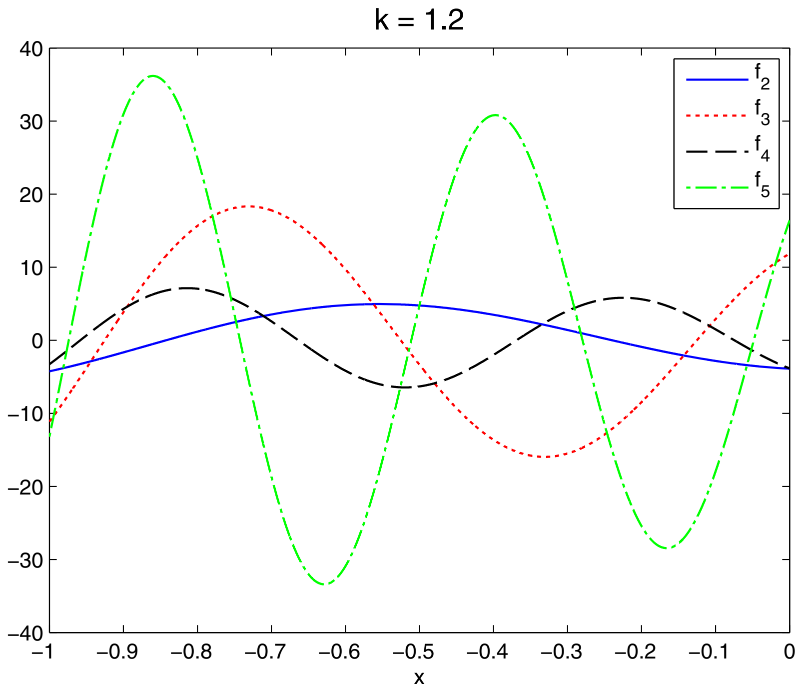

FIG. 7. (Color plot online) A plot of several eigenfunctions when $k = 1.20$

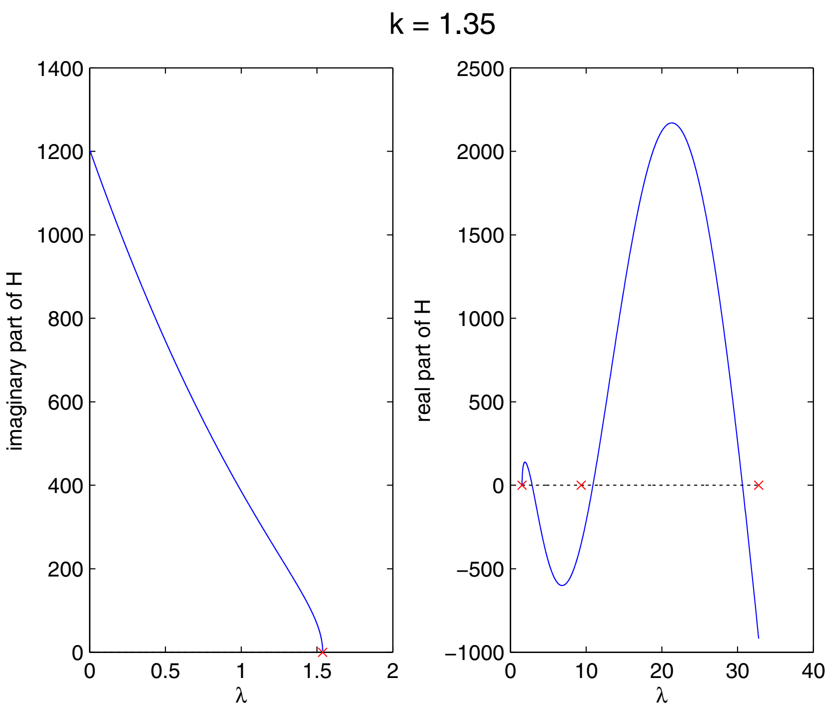

FIG. 8. (Color plot online) Plots of $H(\lambda, k)$ when $k = 1.35$. The *left plot* shows the range of λ for which $\lambda < \gamma_0$ and the *right plot* shows the range of λ for which $\lambda > \gamma_0$. The x's denote γ_0 and γ_1

TABLE 4. *The values of* γ_i *and* λ_i *for* $0 < i < 14$ *for* $k = 1.35$

i	γ_i	λ_i
0	1.5378×10^0	2.8768×10^0
1	9.3506×10^0	1.0897×10^1
2	3.2789×10^1	3.0648×10^1
3	7.1853×10^1	6.3556×10^1
4	1.2654×10^2	1.1016×10^2
5	1.9686×10^2	1.7086×10^2
6	2.8280×10^2	2.4600×10^2
7	3.8437×10^2	3.3582×10^2
8	5.0156×10^2	4.4053×10^2
9	6.3437×10^2	5.6028×10^2
10	7.8282×10^2	6.9521×10^2
11	9.4689×10^2	8.4539×10^2
12	1.1266×10^3	1.0109×10^3
13	1.3219×10^3	1.1918×10^3
14	1.5328×10^3	1.3882×10^3

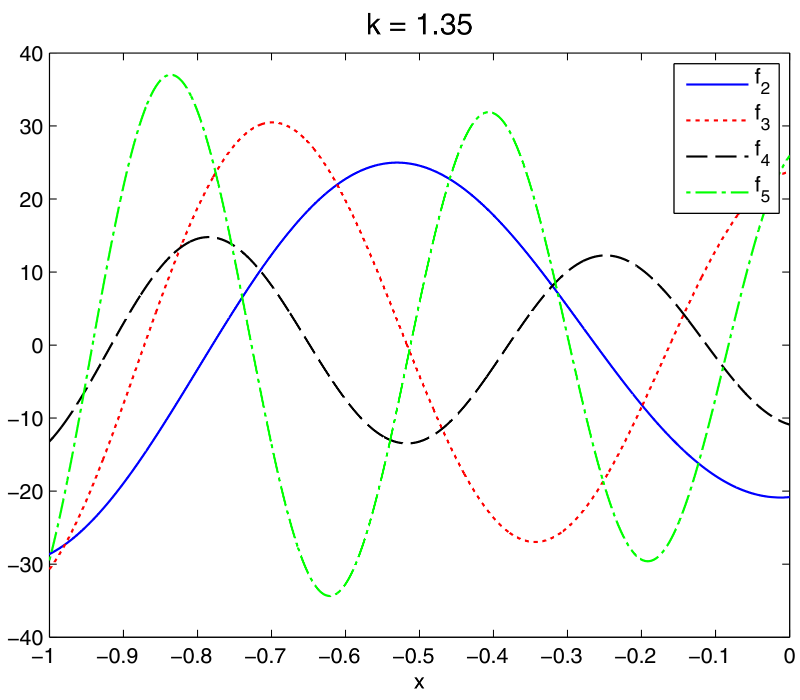

FIG. 9. (Color plot online) A plot of several eigenfunctions when $k = 1.35$

the endpoints of the middle layer approach each other ($\mu(-L) \to \mu(0)$). In the limit, this amounts to a finite middle layer with constant viscosity. The other physical situation is for the viscosity at each end of the layer to remain the same, but the length of the middle layer to increase to infinity. In this limit, the effects of the two interfaces are decoupled. We investigate both of these cases in Sect. 3.2.1.

The other limiting case we investigate is when the length of the middle layer goes to zero. We handle this case in Sect. 3.2.2.

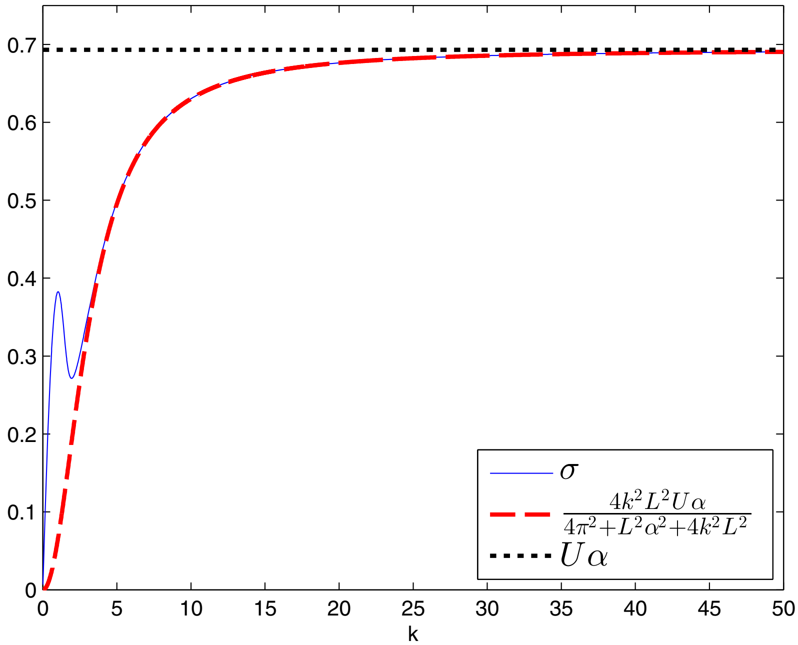

FIG. 10. (Color plot online) For large k, the largest value of σ behaves like $\frac{1}{\gamma_1}$

3.2.1. $\lim_{\alpha\to 0}$ Case. We first consider the limit as $\alpha \to 0$. Considering (47), $\beta^2 \to -k^2$ as $\alpha \to 0$. Recall that the cutoff value between real and complex values of H is at $\gamma_0 = \frac{\alpha^2+4k^2}{4k^2U\alpha}$ which goes to ∞ like $\frac{1}{\alpha}$ as $\alpha \to 0$. Therefore, as α vanishes, the infinite sequence of eigenvalues found in the previous section become arbitrarily large. In particular, the values of λ that occur when H is real are bounded below by $\frac{1}{U\alpha}$ (and therefore the corresponding σ values bounded above by $U\alpha$). Now, consider the function $H(\lambda, k)$ as $\alpha \to 0$ in the region $\lambda < \gamma_0$. Note that as $\alpha \to 0$,

$$H_1(\lambda, k) \to \left(\frac{\mu_r k - E_0\lambda}{\mu(0)}\right)\left(\frac{\mu_l k - E_1\lambda}{\mu(-L)}\right) + k^2, \tag{63}$$

and H_2 is independent of α. Recall

$$H_2(\lambda, k) = \left\{\frac{\mu_l k - E_1\lambda}{\mu(-L)} + \frac{\mu_r k - E_0\lambda}{\mu(0)}\right\}. \tag{64}$$

Using (63) and (64) in (55), and with some algebraic manipulation,

$$H(\lambda, k) \to i\left(\tilde{c}\lambda^2 + \tilde{b}\lambda + \tilde{a}\right),$$

where

$$\begin{aligned}
\tilde{c} &= \frac{k^2}{2\mu(0)\mu(-L)}\frac{E_0E_1}{k^2}\left(e^{kL} - e^{-kL}\right),\\
\tilde{b} &= -\frac{k^2}{2\mu(0)\mu(-L)}\left\{\left(\frac{\mu_r E_1}{k} + \frac{\mu_l E_0}{k}\right)\left(e^{kL} - e^{-kL}\right) + \left(\frac{\mu(0)E_1}{k} + \frac{\mu(-L)E_0}{k}\right)\left(e^{kL} + e^{-kL}\right)\right\},\\
\tilde{a} &= \frac{k^2}{2\mu(0)\mu(-L)}\left\{\left[\mu_r\mu_l + \mu(0)\mu(-L)\right]\left(e^{kL} - e^{-kL}\right) + \left[\mu_l\mu(0) + \mu_r\mu(-L)\right]\left(e^{kL} + e^{-kL}\right)\right\}.
\end{aligned}$$

Therefore, using that $\lambda = 1/\sigma$, the condition $H(\lambda, k) = 0$ is equivalent to the quadratic equation $a\sigma^2 + b\sigma + c = 0$ where

$$\begin{aligned} a =& - e^{kL}(\mu_r + \mu(0))(\mu_l + \mu(-L)) + e^{-kL}(\mu_r - \mu(0))(\mu_l - \mu(-L)), \\ b =& \left\{(\mu_r + \mu(0))e^{kL} + (\mu(0) - \mu_r)e^{-kL}\right\} \left(\frac{E_1}{k}\right) + \left\{(\mu_l + \mu(-L))e^{kL} + (\mu(-L) - \mu_l)e^{-kL}\right\} \left(\frac{E_0}{k}\right), \\ c =& \frac{E_0E_1}{k^2} \left(e^{-kL} - e^{kL}\right). \end{aligned} \tag{65}$$

There are two different ways in which $\alpha \to 0$. The first is for $\mu(-L) \to \mu(0)$. In this case, the middle layer is of finite length, but the viscosity of the middle layer is essentially constant. If we denote $\mu \equiv \mu(-L) = \mu(0)$, then a, b, and c correspond to the coefficients found for a constant viscosity middle layer [4]. Therefore, the exponential viscous profile reduces to the constant viscosity case.

The other way in which $\alpha \to 0$ is to preserve the size of the viscous jumps at the interfaces, but let $L \to \infty$. Then

$$\begin{aligned} c \to& - \frac{E_0}{k}\frac{E_1}{k}e^{kL}, \\ b \to& \left[(\mu_r + \mu(0))\frac{E_1}{k} + (\mu_l + \mu(-L))\frac{E_0}{k}\right] e^{kL}, \\ a \to& - (\mu_r + \mu(0))(\mu_l + \mu(-L))e^{kL}. \end{aligned}$$

The two solutions to the quadratic equation, $\sigma_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, are given by

$$\sigma_+ = \frac{E_0}{k(\mu_r + \mu(0))}, \qquad \sigma_- = \frac{E_1}{k(\mu_l + \mu(-L))}.$$

These are the usual Saffman-Taylor growth rates of each interface [13].

3.2.2. $\lim_{L\to 0}$ Case. Next, we consider the limit as $L \to 0$. Recall that $\alpha = \frac{1}{L}\ln\left(\frac{\mu(0)}{\mu(-L)}\right)$. Therefore, $\alpha \to \infty$ at a rate of $\frac{1}{L}$ as $L \to 0$. Using (47), $\beta^2 \to -\frac{\alpha^2}{4}$. Like the previous case, $\gamma_0 \to \infty$ as $L \to 0$, but this time, $\gamma_0 \to \frac{\alpha}{4k^2U}$. Therefore, the values of λ that occur when H is real are bounded below by $\frac{\alpha}{4k^2U}$ (and therefore the corresponding σ values bounded above by $\frac{4k^2U}{\alpha}$). Now, consider the function $H(\lambda, k)$ as $L \to 0$ in the region $\lambda < \gamma_0$. Note that $H_2(\lambda, k)$ is independent of L. As $L \to 0$,

$$H_1(\lambda, k) \to \frac{\alpha}{2}\left(\frac{\mu_r k - E_0\lambda}{\mu(0)} - \frac{\mu_l k - E_1\lambda}{\mu(-L)}\right). \tag{66}$$

Therefore, using (66) and our estimate for β,

$$H(\lambda, k) \to \frac{i\alpha}{2}e^{-\frac{\alpha L}{2}}\left\{\left(\frac{\mu_r k - E_0\lambda}{\mu(0)}\right)e^{\alpha L} + \left(\frac{\mu_l k - E_1\lambda}{\mu(-L)}\right)\right\}.$$

Since $e^{\alpha L} = \frac{\mu(0)}{\mu(-L)}$, $H(\lambda, k) = 0$ if and only if

$$\lambda = \frac{(\mu_r + \mu_l)}{kU(\mu_r - \mu(0) + \mu(-L) - \mu_l) - k^3(T_0 + T_1)}.$$

Therefore, the growth rate is

$$\sigma = \frac{kU(\mu_r - \mu(0) + \mu(-L) - \mu_l) - k^3(T_0 + T_1)}{(\mu_r + \mu_l)}, \tag{67}$$

which is the Saffman–Taylor growth rate for a single interface with a viscosity jump equal to the sum of the viscosity jumps at the two interfaces and with interfacial tension equal to the sum of the interfacial tensions of the two interfaces. This implies that even an infinitely small middle layer will be less unstable than the two layer flow.

4. Conclusions

We studied the spectrum of a non-standard eigenvalue problem that arises from the linear stability analysis of three-layer Hele-Shaw flows with a variable viscosity middle layer. This problem differs from regular Sturm-Liouville problems because of the presence of the eigenvalue in the boundary conditions. However, we were able to show that there is an infinite set of discrete eigenvalues and that the corresponding eigenfunctions are complete in a certain Hilbert space. We then applied this theory to the case of an exponential viscous profile. Not only were we able to verify the theoretical results of the previous section, but also provide a sequence of numbers, $\{\gamma_n\}$, that alternate with the eigenvalues of the system. We verified this with numerical computation of the eigenvalues using a pseudo-spectral method. Finally, we investigated several limiting cases. The first was when the viscous profile of the middle layer approaches a constant viscosity, both in the case of a fixed-length middle layer and also as the length of the middle layer goes to infinity. The second limiting case was when the length of the middle layer approaches zero.

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Appendix A. Computation of the Eigenvalues and Eigenfunctions

A.1. Computing the Eigenvalues

In order to numerically compute the eigenvalues, we use a pseudo-spectral method. We first describe the method and then its application to our specific problem. For a more detailed treatment and proofs of convergence rates, see [1,15].

Let $T_n(y)$ denote the n^{th} Chebyshev polynomial, which can be defined in terms of trigonometric functions as

$$T_n(y) = \cos(n \cos^{-1}(y)), \qquad y \in [-1, 1]. \tag{68}$$

The Chebyshev polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} \frac{T_n(y)T_m(y)}{\sqrt{1-y^2}} dy = C_n \delta_{nm}, \tag{69}$$

where $C_0 = \pi$ and $C_n = \frac{\pi}{2}$ for $n \neq 0$. Additionally, the Chebyshev polynomials form a complete set with respect to the weight function $w(y) = \frac{1}{\sqrt{1-y^2}}$. Therefore, for any $f \in L^2_w([-1, 1])$, we may expand f as

$$f(y) = \sum_{n=0}^{\infty} a_n T_n(y), \qquad a_n = \frac{1}{\sqrt{C_n}} \int_{-1}^{1} \frac{f(y)T_n(y)}{\sqrt{1-y^2}} dy. \tag{70}$$

In order to use this expansion to solve our eigenvalue problem, we approximate the solution as the finite sum of the first N Chebyshev polynomials

$$f(y) \approx \sum_{n=0}^{N} a_n T_n(y). \tag{71}$$

In order to optimize the rate of convergence, we evaluate these at the extremal values of the Chebyshev polynomials (the Gauss–Chebyshev–Lobatto points), which are given by

$$y_j = \cos\left(\frac{j\pi}{N}\right), \qquad j = 0, \dots, N. \tag{72}$$

Using these points, $T_n(y_j) = \cos\left(\frac{nj\pi}{N}\right)$. In order to solve an eigenvalue problem, we also need an expansion for the derivatives of f. We write the kth derivative of f as

$$f^{(k)}(y) = \sum_{n=0}^{N} a_n T_n^{(k)}(y). \tag{73}$$

Using the change of variables $y = cos(\theta)$ and (68), we get $T_n(y) = \cos(n\theta)$. Therefore

$$T_n'(y) = \frac{n\sin(n\theta)}{\sin(\theta)}. \tag{74}$$

Using some trigonometric identities in (74), we arrive at the recurrence relation

$$T_n'(y) = 2nT_{n-1}(y) + \left(\frac{n}{n-2}\right) T_{n-2}'(y).$$

In general, for $k \geq 1$,the kth derivative satisfies the recurrence relation

$$T_0^{(k)}(y) = 0, \qquad T_1^{(k)}(y) = T_0^{(k-1)}(y), \qquad T_n^{(k)}(y) = 2nT_{n-1}^{(k-1)}(y) + \left(\frac{n}{n-2}\right) T_{n-2}^{(k)}(y). \tag{75}$$

We may use this relation to build differentiation matrices in the following way. Let $\mathbf{a} = \{a_0, \ldots, a_N\}^T$ where the a_i's are the coefficients from (71). Let $\mathbf{D_0}$ be an $(N+1)\times(N+1)$ matrix such that the entry in row i and column j is given by

$$(\mathbf{D_0})_{i,j} = T_{j-1}(y_{i-1}). \tag{76}$$

Then $\mathbf{D_0 a} = \mathbf{f}$ where $\mathbf{f} = \{f(y_0), f(y_1), \ldots, f(y_N)\}^T$. We denote the k^{th} differentiation matrix by $\mathbf{D_k}$. Using (75), we can recursively build $\mathbf{D_k}$ from $\mathbf{D_{k-1}}$ using

$$(\mathbf{D_k})_{i,j} = T_{j-1}^{(k)}(y_{i-1}) = \begin{cases} 0, & j = 1, \\ (\mathbf{D_{k-1}})_{i,j-1}, & j = 2, \\ 2(j-1)(\mathbf{D_{k-1}})_{i,j-1} + \left(\frac{j-1}{j-3}\right)(\mathbf{D_k})_{i,j-2}, & 3 \leq j \leq N+1. \end{cases} \tag{77}$$

Then, for any $k \geq 0$, $\mathbf{D_k a} = \mathbf{f_k}$ where $\mathbf{f_k} = \{f^{(k)}(y_0), f^{(k)}(y_1), \ldots, f^{(k)}(y_N)\}^T$. For an explicit example of a MATLAB program that builds these matrices, see Schmid and Henningson [14, p. 491–492].

With these matrices, we can solve the eigenvalue problem (5). Recall equation $(5)_1$:

$$(\mu f')' - (k^2\mu - k^2 U\mu'\lambda)f = 0, \quad -L < x < 0.$$

Note that the Gauss–Chebyshev–Lobatto points are in the interval $[-1, 1]$. We map these points to the interval $[-L, 0]$ using the affine map $x = \frac{L}{2}(y-1)$. Therefore, our collocation points are $x_i = \frac{L}{2}(y_i - 1)$. Additionally, since $\frac{d}{dx} = \frac{2}{L}\frac{d}{dy}$, we let $\mathbf{D_k^x} = \left(\frac{2}{L}\right)^k \mathbf{D_k}$. Note that $(5)_1$ can be rewritten as

$$-\mu(x)f''(x) - \mu'(x)f'(x) + k^2\mu(x)f(x) = \lambda k^2 U\mu'(x)f(x). \tag{78}$$

We require that this equation hold at each collocation point, x_i, which gives a system of $N+1$ equations. Let $\mathbf{V}$ and $\mathbf{V'}$ be the matrices defined by

$$(\mathbf{V})_{i,j} = \begin{cases} \mu(x_i), & j = i, \\ 0, & \text{otherwise} \end{cases}, \qquad (\mathbf{V'})_{i,j} = \begin{cases} \mu'(x_i), & j = i, \\ 0, & \text{otherwise} \end{cases}. \tag{79}$$

Then the i^{th} entry of the vector $\mathbf{V D_k^x a}$ is $\mu(x_i)f^{(k)}(x_i)$ and likewise for $\mathbf{V' D_k^x a}$. Therefore, the condition that (78) holds for each x_i is given by the matrix equation

$$-\mathbf{V D_2^x a} - \mathbf{V' D_1^x a} + k^2 \mathbf{V D_0^x a} = \lambda k^2 U \mathbf{V' D_0^x a}. \tag{80}$$

Let $\mathbf{A} = -\mathbf{V D_2^x} - \mathbf{V' D_1^x} + k^2 \mathbf{V D_0^x}$ and $\mathbf{B} = k^2 U \mathbf{V' D_0^x}$. Then we have the generalized eigenvalue problem $\mathbf{Aa} = \lambda \mathbf{Ba}$. However, we must enforce the boundary conditions by amending the first and last rows of $\mathbf{A}$

and $\mathbf{B}$, which correspond to $x_0 = 0$ and $x_N = -L$, respectively. The boundary conditions $(5)_2$ and $(5)_3$ can be rewritten as

$$\mu(0)f'(0) + \mu_r k f(0) = E_0\lambda f(0),$$
$$\mu(-L)f'(-L) - \mu_l k f(-L) = -E_1\lambda f(-L).$$

Therefore, the first and last rows of $\mathbf{A}$ and $\mathbf{B}$ are

$$(\mathbf{A})_{1,j} = \mu(0)(\mathbf{D^x_1})_{1,j} + \mu_r k(\mathbf{D^x_0})_{1,j}, \qquad (\mathbf{B})_{1,j} = E_0(\mathbf{D^x_0})_{1,j},$$
$$(\mathbf{A})_{N+1,j} = \mu(-L)(\mathbf{D^x_1})_{N+1,j} - \mu_l k(\mathbf{D^x_0})_{N+1,j}, \qquad (\mathbf{B})_{N+1,j} = -E_1(\mathbf{D^x_0})_{N+1,j}.$$

We solve the generalized eigenvalue problem using MATLAB's "eig" command.

A.2. Finding the Eigenfunctions

Once the eigenvalues are known, we can compute the eigenfunctions from the general form (48):

$$f(x) = e^{-\frac{\alpha x}{2}}(A\cos(\beta x) + B\sin(\beta x)).$$

This can be rewritten in terms of the values at the endpoints as,

$$f(x) = e^{-\frac{\alpha x}{2}}\left(f(0)\cos(\beta x) + \frac{f(0)\cos(\beta L) - f(-L)e^{-\frac{\alpha L}{2}}}{\sin(\beta L)}\sin(\beta x)\right). \tag{81}$$

Therefore,

$$f'(x) = -\frac{\alpha}{2}f(x) + \beta e^{-\frac{\alpha x}{2}}\left(-f(0)\sin(\beta x) + \frac{f(0)\cos(\beta L) - f(-L)e^{-\frac{\alpha L}{2}}}{\sin(\beta L)}\cos(\beta x)\right). \tag{82}$$

Using these expressions in the boundary condition $(5)_3$ and rearranging terms,

$$\frac{e^{-\frac{\alpha L}{2}}}{\sin(\beta L)}f(-L) = \left(\frac{\mu_r k - E_0\lambda}{\mu(0)\beta} - \frac{\alpha}{2\beta} + \frac{\cos(\beta L)}{\sin(\beta L)}\right)f(0). \tag{83}$$

Using (83) in (81),

$$f(x) = f(0)e^{-\frac{\alpha x}{2}}\left(\cos(\beta x) + \left(\frac{\alpha}{2} - \frac{\mu_r k - E_0\lambda}{\mu(0)}\right)\frac{\sin(\beta x)}{\beta}\right). \tag{84}$$

This gives the eigenfunction up to an arbitrary constant, $f(0)$. We choose this constant so that

$$\int_{-L}^{0} f(x)dx = 1. \tag{85}$$

Using that

$$\int_{-L}^{0} e^{-\frac{\alpha x}{2}}\cos(\beta x)dx = \frac{-\frac{\alpha}{2} + e^{\frac{\alpha L}{2}}\left(\frac{\alpha}{2}\cos(\beta L) + \beta\sin(\beta L)\right)}{\frac{\alpha^2}{4} + \beta^2},$$

and

$$\int_{-L}^{0} e^{-\frac{\alpha x}{2}}\sin(\beta x)dx = \frac{-\beta + e^{\frac{\alpha L}{2}}\left(\beta\cos(\beta L) - \frac{\alpha}{2}\sin(\beta L)\right)}{\frac{\alpha^2}{4} + \beta^2},$$

condition (85) yields

$$f(0) = \frac{\frac{\alpha^2}{4} + \beta^2}{-\alpha + e^{\frac{\alpha L}{2}}\left(\alpha\cos(\beta L) + \left(\beta^2 - \frac{\alpha^2}{4}\right)\frac{\sin(\beta L)}{\beta}\right) + \left(\frac{\mu_r k - E_0\lambda}{\mu(0)}\right)\left(1 + e^{\frac{\alpha L}{2}}\left(\frac{\alpha}{2}\frac{\sin(\beta L)}{\beta} - \cos(\beta L)\right)\right)}. \tag{86}$$

Plugging (86) into (84) gives the normalized eigenfunction. Note that λ appears explicitly in the expression for the eigenfunctions in addition to the fact that β depends on λ. When we have obtained the eigenvalues $\{\lambda_i\}$, we get $f_i(x)$ by plugging λ_i into (86) and (84).

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