A Numerical Study of an Ill-posed Boussinesq Equation Arising in Water Waves and Nonlinear Lattices: Filtering and Regularization Techniques

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We consider all ill-posed Boussinesq equation which arises in shallow water waves and nonlinear lattices. This equation has growing and decaying modes in the linear as well as nonlinear regimes and its linearized growth rate σ for short-waves of wave-number k is given by $\sigma \sim k^2$. Previous numerical studies have addressed numerical difficulties and construction of approximate solutions for ill-posed problems with short-wave instability up to $\sigma \sim k$, e.g. Kelvin-Helmholtz ($\sigma \sim k$) and Rayleigh-Taylor ($\sigma \sim \sqrt{k}$) instabilities. These same issues are addressed and critically examined here for the present problem which has more severe short-wave instability. In order to develop numerical techniques for constructing good approximate solutions of this equation, we use a finite difference scheme to investigate the effect of this short-wave instability on the numerical accuracy of the exact solitary wave solution of this equation. Computational evidence is presented which indicates that numerical accuracy of the solutions is lost very quickly due to severe growth of numerical errors, roundoff as well as truncation. We use both filtering and regularization techniques to control growth of these errors and to provide better approximate solutions of this equation. In the filtering technique, numerical experiments with three types of spectral filters of increasing order of regularity are performed. We examine the role of regularity of these filters on the accuracy of the numerical solutions. Numerical evidence is provided which indicates that the regularity of a filter plays an important role in improving the accuracy of the solutions. In the regularization technique, the ill-posed equation is regularized by adding a higher order term to the equation. Two types of higher order terms are discussed: (i) one that diminishes the growth rate of all modes below a cutoff wavenumber and sets the growth rate of all modes above it to zero; and (ii) the other one diminishes the growth rate of all modes and the growth rate asymptotically approaches to zero as the wavenumber approaches infinity. We have argued in favor of the first type of regularization and numerical results using a finite difference scheme are presented. Numerical evidence is provided which suggests that regularization in combination with the most regular (C^2 here) spectral filter for small values of the regularization parameter can provide good approximate solutions of the ill-posed Boussinesq equation for longer time than possible otherwise. Some of the ideas presented here can possibly be utilized for solving other ill-posed problems with severe short-wave instabilities and may have an important role to play in numerical studies of their solutions.

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