A Class of Model Equations for Bi-directional Propagation of Capillary-Gravity Waves

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Abstract

A class of model equations that describe the bi-directional propagation of small amplitude long waves on the surface of shallow water is derived from two-dimensional potential flow equations at various orders of approximation in two small parameters, namely the amplitude parameter $\alpha = a/h_0$ and wavelength parameter $\beta = (h_0/l)^2$, where a and l are the actual amplitude and wavelength of the surface wave, and h_0 is the height of the undisturbed water surface from the flat bottom topography. These equations are also characterized by the surface tension parameter, namely the Bond number $\tau = \Gamma/\rho g h_0^2$, where Γ is the surface tension coefficient, ρ is the density of water, and g is the acceleration due to gravity.

The traveling solitary wave solutions are explicitly constructed for a class of lower order Boussinesq system. From the Boussinesq equation of higher order, the appropriate equations to model solitary waves are derived under appropriate scaling in two specific cases: (i) $\beta \ll (1/3-\tau) \leq 1/3$ and (ii) $(1/3-\tau) = O(\beta)$. The case (i) leads to the classical Boussinesq equation whose fourth-order dispersive term vanishes for $\tau = 1/3$. This emphasizes the significance of the case (ii) that leads to a sixth-order Boussinesq equation, which was originally introduced on a heuristic ground by Daripa and Hua [Appl. Math. Comput. **101**, 159–207, 1999] as a dispersive regularization of the ill-posed fourth-order Boussinesq equation.

Keywords: Capillary-gravity waves, Bi-directional wave propagation, Boussinesq systems and Boussinesq equations, Solitary waves.

1 Introduction

The classical (fourth-order) Boussinesq equation

$$\eta_{tt} = \eta_{xx} + (\eta^2)_{xx} + \eta_{xxxx}, \tag{1.1}$$

describes the bi-directional propagation of small amplitude and long wavelength capillary-gravity waves on the surface of shallow water [1, 2] as well as lattice waves in non-linear lattices [3, 4]. The model equation (1.1) was originally derived by Boussinesq [5] from the Euler's equation of motion for

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two-dimensional potential flow beneath a free surface by introducing appropriate approximations for small amplitude long waves. Later, Korteweg-deVries [6] derived an equivalent model equation for uni-directional wave propagation, known as the classical (third-order) KdV equation, by using the far-field analysis in addition to the Boussinesq's approximations. The third-order KdV equation has become more popular because of its mathematical simplicity and well-posedness.

Ill-posedness of equation (1.1) is a deficiency in the model and raises questions about the physical relevance of this equation in the context of modeling bi-directional propagation of small amplitude long water waves. Even otherwise, this equation poses mathematical and numerical difficulties [7, 8]. One of the well-known wellposed variant of this equation which is valid at the same order is to replace the last term in equation (1.1) by η_{xxtt}. There are other ways to regularize this equation in a physically consistent way, e.g. by including the effect of surface tension which will lead to a higher order model than the Boussinesq equation (1.1). Even though the surface tension itself may not be that important for long waves, their inclusion in deriving equations which model propagation of such waves may be important, in particular in nonlinear models that may otherwise generate dangerous short waves such as equation (1.1). In this paper, through a systematic derivations of higher order Boussinesq equation models by including second order terms in perturbation theory, we show that the a higher order Boussinesq model, originally introduced by Daripa et. al [8], is a physically relevant model of bi-directional propagation of small amplitude long waves on the surface of shallow water.

The ill-posed interfacial model equations are often regularized by adding the effect of surface tension [8, 15]. When the effect of surface tension is included, the solutions to the water wave equations are characterized by the Bond number $\tau = \Gamma/\rho g h_0^2$, in addition to the amplitude parameter $\alpha = a/h_0$ and wavelength parameter $\beta = (h_0/l)^2$. Here, Γ is the surface tension coefficient, ρ is the density of water, g is the acceleration due to gravity, h_0 is the height of the undisturbed water surface, a is the amplitude of the surface wave, and l is the wavelength of the surface wave. The inclusion of the surface tension effect, in some cases, leads to the higher order model equations, such as the fifth-order KdV equation [16] and the sixth-order Boussinesq equation [8, 17]. These equations are physically relevant model equations for shallow water waves in the limit $\tau \uparrow 1/3$ (i.e., when the Bond number τ is less than but very close to 1/3). It has been proved that these equations do not possess classical local solitary wave solutions, but admit weakly non-local solitary wave solutions characterized by oscillatory tails at the far-field [8, 16, 17, 18, 19, 20, 21, 22, 23].

Based on a general theory of noncanonical perturbations of Hamiltonian systems, Olver [9, 10] derived some new Hamiltonian model equations for both uni- and bi-directional propagation of small amplitude long waves on the surface of shallow water. Later, Olver and Kichenassamy (see Olver [22] and Kichenassamy [23]) have analytically studies these higher order model equations for water waves including the well-known fifth-order KdV equation and investigated the issue of existence of solitary wave solutions for those equations. Recently, Bona et al. [14] derived a number of variants of classical Boussinesq system for such bi-directional wave propagation problems and presented their higher order generalizations including their relevance to experiments and observations (also see the references there in).

Higher order models that support waves propagating in both directions, the topic of this paper, are of interest in practical situations where waves can propagate in either directions. One obvious situation where two-way propagation is desirable is when the flow is bounded by walls. There is no sensible way to study wall-reflection in either KdV or its fifth-order generalization. However, the higher order boussinesq equations circumvents this problem and allow the possibility of such studies. In [8], the sixth order Boussinesq equation was introduced and numerically studied for the first time with its simplest aspect by modeling a single solitary wave propagating in one direction. Next meaningful step would be to study this equation with wall-reflection which we hope to take up in the future.

In this paper, we derive a class of higher order Boussinesq systems and Boussinesq equations which are appropriate for the description of bi-directional wave propagation of small amplitude and long wavelength on the surface of shallow water. This is an important and interesting development because the widely used first-order models make sense only if no new qualitative features have been thrown away with the second-order terms. However, the primary objective of this paper is to include the effect of surface tension in the models of shallow water waves which has often been neglected in the earlier derivations in literature [11, 12, 13, 14]. The secondary objective is to derive and establish the physical relevance of the sixth-order Boussinesq equation [8, 17] in the context of shallow water waves.

In section 2, generalized higher order Boussinesq systems correct up to $O(\alpha^2, \alpha\beta, \beta^2)$ are derived from two-dimensional potential flow equations governing the shallow water waves under gravity. In section 3, various lower order Boussinesq systems are derived as special cases and the traveling solitary wave solutions are constructed explicitly. In section 4, a higher order Boussinesq equation is derived which is of the same order of approximation as the generalized higher order Boussinesq system derived in section 2. From this equation, the appropriate equations to model solitary waves in two cases, namely when the Bond number τ satisfies (i) $\beta \ll (1/3 - \tau) \leq 1/3$; or (ii) $(1/3 - \tau) = O(\beta)$, are derived. In section 5, the behavior of the solutions to the model equations derived in section 4 is discussed and concluding remarks are made.

2 Generalized Higher Order Boussinesq Systems

Let $z = 0$ be the bottom topography, $z = h_0$ be the height of the undisturbed water surface, and a be the amplitude of the surface wave. So, if $\eta(x,t)$ represents the free surface elevation from its undisturbed location, then $z = h(x,t) = h_0 + a\eta(x,t)$ represents the free water surface. Let l denotes the wavelength of the surface wave and ϕ denotes the potential function. We introduce the following non-dimensionalization

$$x \to lx, \quad z \to h_0 z, \quad t \to \frac{l}{\sqrt{gh_0}}t, \quad \phi \to \frac{la\sqrt{gh_0}}{h_0}\phi, \tag{2.1}$$

where g is the acceleration due to gravity. In non-dimensional form, the governing equation and boundary conditions for water waves (see [1, 2] for more details) are then given by

$$\beta\phi_{xx} + \phi_{zz} = 0, \tag{2.2}$$

and

$$\left.\begin{array}{l} \phi_z = 0 \quad \text{at} \quad z = 0, \\ \eta_t + \alpha\eta_x\phi_x - \frac{1}{\beta}\phi_z = 0 \quad \text{at} \quad z = 1 + \alpha\eta, \\ \phi_t + \frac{1}{2}\alpha\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_z^2 + \eta - \beta\tau\frac{\eta_{xx}}{[1+\alpha^2\beta\eta_x^2]^{3/2}} = 0 \quad \text{at} \quad z = 1 + \alpha\eta. \end{array}\right\} \tag{2.3}$$

Here $\alpha = a/h_0$ (amplitude parameter), $\beta = (h_0/l)^2$ (wavelength parameter), and $\tau = \Gamma/\rho g h_0^2$ (Bond number); Γ is the surface tension coefficient and ρ is the density of water. These full Hamiltonian water wave equations (2.2) and (2.3) with zero and non-zero τ were also considered by Olver [9, 10] in the studies of deriving Hamiltonian and non-Hamiltonian model equations for water waves. Now we seek the solution for the potential function ϕ in the form [1, 2, 14]

$$\phi = \sum_{k=0}^{\infty}(-1)^k\beta^k\frac{z^{2k}}{(2k)!}\frac{\partial^{2k}f}{\partial x^{2k}}, \tag{2.4}$$

where $f = f(x,t)$ is the value of the potential function ϕ at bottom $z = 0$. The solution for ϕ can also be expressed in terms of the potential function $\psi = \psi(x,t) = \phi(x,\theta,t)$ at an arbitrary height $z = \theta$, $0 \le \theta \le 1$, as in Olver [9, 10]. Equation (2.4) suggests that the horizontal velocity ϕ_x is of $O(1)$ and the vertical velocity ϕ_z is of $O(\beta)$. Also, ϕ satisfies the Laplace equation (2.2) and the bottom boundary condition in equation (2.3). Now substituting expansion (2.4) in the free surface boundary conditions of equation (2.3), and rearranging the series, we obtain

$$\eta_t + \alpha\eta_x\sum_{k=0}^{\infty}(-1)^k\beta^k\frac{(1+\alpha\eta)^{2k}}{(2k)!}\frac{\partial^{2k+1}f}{\partial x^{2k+1}} + \sum_{k=0}^{\infty}(-1)^k\beta^k\frac{(1+\alpha\eta)^{2k+1}}{(2k+1)!}\frac{\partial^{2k+2}f}{\partial x^{2k+2}} = 0, \tag{2.5}$$

and

$$\left[\sum_{k=0}^{\infty}(-1)^k\beta^k\frac{(1+\alpha\eta)^{2k}}{(2k)!}\frac{\partial^{2k+1}f}{\partial t\partial x^{2k}}\right] + \frac{\alpha}{2}\left[\sum_{k=0}^{\infty}(-1)^k\beta^k\frac{(1+\alpha\eta)^{2k}}{(2k)!}\frac{\partial^{2k+1}f}{\partial x^{2k+1}}\right]^2$$
$$+ \frac{\alpha\beta}{2}\left[\sum_{k=0}^{\infty}(-1)^k\beta^k\frac{(1+\alpha\eta)^{2k+1}}{(2k+1)!}\frac{\partial^{2k+2}f}{\partial x^{2k+2}}\right]^2 + \eta - \beta\tau\eta_{xx}\left[1+\alpha^2\beta\eta_x^2\right]^{-3/2} = 0. \tag{2.6}$$

Retaining the terms up to $O(\alpha^2, \alpha\beta, \beta^2)$ in equations (2.5) and (2.6), we get the following higher order model equations for water waves involving the free surface elevation $\eta(x,t)$ and the potential function $f(x,t)$ at the bottom $z = 0$:

$$\eta_t + \alpha\eta_x(f_x - \frac{\beta}{2}f_{xxx}) + (1+\alpha\eta)f_{xx} - \frac{\beta}{6}(1+3\alpha\eta)f_{xxxx} + \frac{\beta^2}{120}f_{xxxxxx} = 0, \tag{2.7}$$

and

$$f_t - \frac{\beta}{2}(1+2\alpha\eta)f_{xxt} + \frac{\beta^2}{24}f_{xxxxt} + \frac{\alpha}{2}(f_x^2 - \beta f_x f_{xxx}) + \frac{\alpha\beta}{2}f_{xx}^2 + \eta - \beta\tau\eta_{xx} = 0. \tag{2.8}$$

Differentiating equation (2.8) w.r.t. x and writing $u_0 = f_x$ (horizontal velocity at bottom $z = 0$), we obtain the model equations (2.7) and (2.8) in the following equivalent forms:

$$\eta_t + u_{0x} + \alpha(\eta u_0)_x - \frac{\beta}{6} u_{0xxx} - \frac{\alpha\beta}{2}(\eta u_{0xx})_x + \frac{\beta^2}{120} u_{0xxxxxx} = 0, \tag{2.9}$$

and

$$\begin{aligned} \eta_x + u_{0t} + \alpha u_0 u_{0x} - \beta\big(\tau\, \eta_{xxx} + \frac{1}{2} u_{0xxt}\big) + \frac{\beta^2}{24} u_{0xxxxt} \\ - \alpha\beta\big[(\eta u_{0xt})_x + \frac{1}{2}(u_0 u_{0xx})_x - u_{0x} u_{0xx}\big] = 0. \end{aligned} \tag{2.10}$$

Equations (2.9) and (2.10) constitute the higher order Boussinesq system. We now derive a class of model equations all of which are formally equivalent to the system (2.9) and (2.10). If u denotes the horizontal velocity ϕ_x at a height $z = \theta,\ 0 \le \theta \le 1$, then we can relate u_0 to u with the help of equation (2.4) as

$$u = u_0 - \beta \frac{\theta^2}{2} u_{0xx} + \beta^2 \frac{\theta^4}{24} u_{0xxxx} + O(\beta^3). \tag{2.11}$$

By inverting equation (2.11), we obtain

$$u_0 = u + \beta \frac{\theta^2}{2} u_{xx} + \beta^2 \frac{5\theta^4}{24} u_{xxxx} + O(\beta^3). \tag{2.12}$$

Substituting expression (2.12) for u_0 into equations (2.9) and (2.10), and simplifying, we get the following higher order Boussinesq system:

$$\begin{aligned} \eta_t + u_x + \alpha(\eta u)_x + \frac{\beta}{6}(3\theta^2 - 1) u_{xxx} + \frac{\alpha\beta}{2}(\theta^2 - 1)(\eta u_{xx})_x \\ + \frac{\beta^2}{120}(5\theta^2 - 1)^2 u_{xxxxxx} = 0, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \eta_x + u_t + \alpha u u_x + \frac{\beta}{2}\Big[(\theta^2 - 1) u_{xxt} - 2\tau\, \eta_{xxx}\Big] + \frac{\alpha\beta}{2}\Big[(\theta^2 + 1) u_x u_{xx} \\ + (\theta^2 - 1) u u_{xxx} - 2(\eta u_{xt})_x\Big] + \frac{\beta^2}{24}(5\theta^2 - 1)(\theta^2 - 1) u_{xxxxt} = 0, \end{aligned} \tag{2.14}$$

which matches exactly with the one of the second-order Boussinesq system derived by Olver [10] (see equations (4.7) and (4.8) in page 283) for bi-directional water waves. From equations (2.13) and (2.14), we obtain the following lower order approximations:

$$\left.\begin{aligned} u_x &= \lambda_1 u_x - (1 - \lambda_1)\Big[\eta_t + \alpha(\eta u)_x + \tfrac{\beta}{6}(3\theta^2 - 1) u_{xxx}\Big] \ +\ O(\alpha\beta, \beta^2), \\ u_x &= \lambda_2 u_x - (1 - \lambda_2)\eta_t \ +\ O(\alpha, \beta), \end{aligned}\right\} \tag{2.15}$$

and

$$\left.\begin{aligned} u_t &= \lambda_3 u_t - (1 - \lambda_3)\Big[\eta_x + \alpha u u_x + \tfrac{\beta}{2}(\theta^2 - 1) u_{xxt} - \beta\tau\eta_{xxx}\Big] \ +\ O(\alpha\beta, \beta^2), \\ u_t &= \lambda_4 u_t - (1 - \lambda_4)\eta_x \ +\ O(\alpha, \beta), \end{aligned}\right\} \tag{2.16}$$

for any arbitrary $\lambda_i \in \mathbb{R},\ i = 1, 2, 3, 4,$ and $0 \le \theta^2 \le 1$. Using the lower order approximations (2.15) and (2.16) in the dispersion terms of the higher order Boussinesq system (2.13) and (2.14), we obtain the following generalized higher order Boussinesq systems:

$$\eta_t + u_x + \alpha(\eta u)_x + \beta\Big[C_1 u_{xxx} + C_3\eta_{xxt}\Big] + \alpha\beta\Big[C_3(\eta u)_{xxx} + \frac{1}{2}(\theta^2-1)(\eta u_{xx})_x\Big] + \beta^2\Big[C_2 u_{xxxxx} + C_4\eta_{xxxxt}\Big] = 0, \tag{2.17}$$

and

$$\eta_x + u_t + \alpha u u_x + \beta\Big[C_5 u_{xxt} + \big(C_7 - \tau\big)\eta_{xxx}\Big] + \alpha\beta\Big[C_7(uu_x)_{xx} + \frac{1}{2}(\theta^2+1)u_x u_{xx} + \frac{1}{2}(\theta^2-1)uu_{xxx} - (\eta u_{xt})_x\Big] + \beta^2\Big[C_6 u_{xxxxt} + \big(C_8 - C_7\,\tau\big)\eta_{xxxxx}\Big] = 0, \tag{2.18}$$

where the constants $C_1, C_2, \cdots, C_8$ are given by

$$\begin{aligned}
C_1 &= \frac{1}{2}\Big(\theta^2 - \frac{1}{3}\Big)\lambda_1, & C_2 &= -\frac{1}{4}\Big(\theta^2 - \frac{1}{3}\Big)^2(1-\lambda_1) + \frac{5}{24}\Big(\theta^2 - \frac{1}{5}\Big)^2\lambda_2,\\
C_3 &= -\frac{1}{2}\Big(\theta^2 - \frac{1}{3}\Big)(1-\lambda_1), & C_4 &= -\frac{5}{24}\Big(\theta^2 - \frac{1}{5}\Big)^2(1-\lambda_2),\\
C_5 &= \frac{1}{2}(\theta^2-1)\lambda_3, & C_6 &= \frac{5}{24}\Big(\theta^2 - \frac{1}{5}\Big)(\theta^2-1)\lambda_4 - \frac{1}{4}(\theta^2-1)^2(1-\lambda_3),\\
C_7 &= -\frac{1}{2}(\theta^2-1)(1-\lambda_3), & C_8 &= -\frac{5}{24}\Big(\theta^2 - \frac{1}{5}\Big)(\theta^2-1)(1-\lambda_4).
\end{aligned} \tag{2.19}$$

So the model equations (2.17) and (2.18) describe a four-parameter family of Boussinesq systems for bi-directional propagation of water waves. The second-order Boussinesq system derived by Olver [10] (see equations (4.8) and (4.9) in page 283) can also be deduced from equations (2.17) and (2.18) under the special case $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 0$ and $\lambda_4 = 0$. In this case, the terms of $O(\alpha\beta)$ and $O(\beta^2)$ in equation (2.18) are reduced respectively to $\big[(2-\theta^2)u_x u_{xx} + (\eta\eta_{xx})_x\big]$ and $\big[\frac{1}{24}(\theta^2-5)(\theta^2-1) + \frac{1}{2}(\theta^2-1)\tau\big]\eta_{xxxxx}$. Now using the following scaling transformations

$$x \to \beta^{1/2}\,x, \quad t \to \beta^{1/2}\,t, \quad \eta \to \beta^{-1}\,\eta, \quad u \to \beta^{-1}\,u, \tag{2.20}$$

the generalized higher order Boussinesq system (2.17) and (2.18) can be put into the following canonical form:

$$\eta_t + u_x + S(\eta u)_x + \Big[C_1 u_{xxx} + C_3\eta_{xxt}\Big] + S\Big[C_3(\eta u)_{xxx} + \frac{1}{2}(\theta^2-1)(\eta u_{xx})_x\Big] + \Big[C_2 u_{xxxxx} + C_4\eta_{xxxxt}\Big] = 0, \tag{2.21}$$

and

$$\eta_x + u_t + Suu_x + S\Big[C_7(uu_x)_{xx} + \frac{1}{2}(\theta^2+1)u_x u_{xx} + \frac{1}{2}(\theta^2-1)uu_{xxx} - (\eta u_{xt})_x\Big] + \Big[C_5 u_{xxt} + (C_7-\tau)\eta_{xxx}\Big] + \Big[C_6 u_{xxxxt} + (C_8 - C_7\,\tau)\eta_{xxxxx}\Big] = 0, \tag{2.22}$$

where $S = \alpha/\beta$ is the Stokes number. Since $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, are arbitrary, and $0 \le \theta^2 \le 1$, we can obtain various variants of the higher order Boussinesq system by suitably choosing λ_i's and θ in equations (2.21) and (2.22). For modeling solitary water waves, we need to have a balance between the effects of non-linearity and the effects of dispersion. In this case, the two small parameters α and β are treated as proportional to each other, i.e., $\alpha = O(\beta)$ as $\beta \to 0$; therefore, $S = O(1)$.

3 Generalized Lower Order Boussinesq Systems

If we consider the generalized Boussinesq system (2.17) and (2.18), retain terms up to $O(\alpha, \beta)$, scale the variables using the transformation (2.20), and take $\alpha = O(\beta)$ as $\beta \to 0$ (say $\alpha = \beta$), we obtain the following lower order Boussinesq systems

$$\eta_t + u_x + (\eta u)_x + D_1\, u_{xxx} - D_2\, \eta_{xxt} = 0, \tag{3.1}$$

and

$$\eta_x + u_t + u u_x + (D_3 - \tau)\, \eta_{xxx} - D_4\, u_{xxt} = 0, \tag{3.2}$$

where D_1, D_2, D_3 and D_4 are given in terms of $\lambda = \lambda_1$, $\mu = (1 - \lambda_3)$, and θ as follows:

$$\begin{aligned} D_1 &= C_1 = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda, \qquad & D_2 &= -C_3 = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1 - \lambda), \\ D_3 &= C_7 = -\frac{1}{2}(\theta^2 - 1)\mu, \qquad & D_4 &= -C_5 = -\frac{1}{2}(\theta^2 - 1)(1 - \mu). \end{aligned} \tag{3.3}$$

The lower order Boussinesq system (3.1) and (3.2) reduces to the Boussinesq system considered in Chen [12, 13] and Bona et al. [14] in the special case $\tau = 0$, and to the Boussinesq system derived in Olver [9] (see equation (4.7) in page 244) in the special case $\tau = 0$, $\lambda = 1$ $\mu = 0$. However, the presence of the surface tension parameter τ (Bond number) can make the above Boussinesq system (3.1) and (3.2) numerically well-posed as initial value problems. So, these systems can be considered as regularized Boussinesq systems.

It is worth noting that various choices of the parameter values for λ, μ and θ in system (3.1) and (3.2) will give various systems all of which will be of the same order, and hence, will be formally equivalent to each other. Some examples are given below. The origin of the some of these systems with $\tau = 0$ is discussed in Chen [12] and Bona et al. [14]. All of these systems are mentioned here for the sake of completeness.

• **Case I**. If $\theta^2 = 0$, $\lambda = 0$, $\mu = 0$, then we have $D_1 = 0$, $D_2 = -1/6$, $D_3 = 0$, $D_4 = 1/2$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x + \tfrac{1}{6}\eta_{xxt} = 0, \\ &\eta_x + u_t + u u_x - \tau \eta_{xxx} - \tfrac{1}{2} u_{xxt} = 0. \end{aligned}\right\} \tag{3.4}$$

• **Case II**. If $\theta^2 = 0,\ \lambda = 0,\ \mu = 1$, then we have $D_1 = 0,\ D_2 = -1/6,\ D_3 = 1/2,\ D_4 = 0$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x + \tfrac{1}{6}\eta_{xxt} = 0, \\ &\eta_x + u_t + uu_x + (\tfrac{1}{2} - \tau)\eta_{xxx} = 0. \end{aligned}\right\} \tag{3.5}$$

• **Case III**. If $\theta^2 = 0,\ \lambda = 1,\ \mu = 0$, then we have $D_1 = -1/6,\ D_2 = 0,\ D_3 = 0,\ D_4 = 1/2$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x - \tfrac{1}{6}u_{xxx} = 0, \\ &\eta_x + u_t + uu_x - \tau\eta_{xxx} - \tfrac{1}{2}u_{xxt} = 0. \end{aligned}\right\} \tag{3.6}$$

This system with $\tau = 0$ is derived in Whitham [1] (equation 13.101).

• **Case IV**. If $\theta^2 = 0,\ \lambda = 1,\ \mu = 1$, then we have $D_1 = -1/6,\ D_2 = 0,\ D_3 = 1/2,\ D_4 = 0$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x - \tfrac{1}{6}u_{xxx} = 0, \\ &\eta_x + u_t + uu_x + (\tfrac{1}{2} - \tau)\eta_{xxx} = 0. \end{aligned}\right\} \tag{3.7}$$

• **Case V**. If $\theta^2 = 1/3$, λ is arbitrary, $\mu = 0$, then we have $D_1 = 0,\ D_2 = 0,\ D_3 = 0,\ D_4 = 1/3$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x = 0, \\ &\eta_x + u_t + uu_x - \tau\eta_{xxx} - \tfrac{1}{3}u_{xxt} = 0. \end{aligned}\right\} \tag{3.8}$$

This system with $\tau = 0$ was originally derived by Boussinesq [5].

• **Case VI**. If $\theta^2 = 1/3$, λ is arbitrary, $\mu = 1$, then we have $D_1 = 0,\ D_2 = 0,\ D_3 = 1/3,\ D_4 = 0$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x = 0, \\ &\eta_x + u_t + uu_x + (\tfrac{1}{3} - \tau)\eta_{xxx} = 0. \end{aligned}\right\} \tag{3.9}$$

• **Case VII**. If $\theta^2 = 2/3,\ \lambda = 0,\ \mu = 0$, then we have $D_1 = 0,\ D_2 = 1/6,\ D_3 = 0,\ D_4 = 1/6$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x - \tfrac{1}{6}\eta_{xxt} = 0, \\ &\eta_x + u_t + uu_x - \tau\eta_{xxx} - \tfrac{1}{6}u_{xxt} = 0. \end{aligned}\right\} \tag{3.10}$$

This system with $\tau = 0$ is the regularized Boussinesq system considered by Bona and Chen [11].

• **Case VIII**. If $\theta^2 = 2/3,\ \lambda = 0,\ \mu = 1$, then we have $D_1 = 0,\ D_2 = 1/6,\ D_3 = 1/6,\ D_4 = 0$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x - \tfrac{1}{6}\eta_{xxt} = 0, \\ &\eta_x + u_t + uu_x + (\tfrac{1}{6} - \tau)\eta_{xxx} = 0. \end{aligned}\right\} \tag{3.11}$$

This system with $\tau = 0$ is known as the coupled-regularized-KdV system [12].

• **Case IX**. If $\theta^2 = 2/3,\ \lambda = 1,\ \mu = 0$, then we have $D_1 = 1/6,\ D_2 = 0,\ D_3 = 0,\ D_4 = 1/6$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x + \tfrac{1}{6}u_{xxx} = 0,\\ &\eta_x + u_t + uu_x - \tau\eta_{xxx} - \tfrac{1}{6}u_{xxt} = 0. \end{aligned}\right\} \tag{3.12}$$

This system with $\tau = 0$ is also known as the coupled-regularized-KdV system [12].

• **Case X**. If $\theta^2 = 2/3,\ \lambda = 1,\ \mu = 1$, then we have $D_1 = 1/6,\ D_2 = 0,\ D_3 = 1/6,\ D_4 = 0$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x + \tfrac{1}{6}u_{xxx} = 0,\\ &\eta_x + u_t + uu_x + (\tfrac{1}{6} - \tau)\eta_{xxx} = 0. \end{aligned}\right\} \tag{3.13}$$

This system with $\tau = 0$ is known as the coupled-KdV system [12].

• **Case XI**. If $\theta^2 = 1,\ \lambda = 0,\ \mu$ is arbitrary, then we have $D_1 = 0,\ D_2 = 1/3,\ D_3 = 0,\ D_4 = 0$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x - \tfrac{1}{3}\eta_{xxt} = 0,\\ &\eta_x + u_t + uu_x - \tau\eta_{xxx} = 0. \end{aligned}\right\} \tag{3.14}$$

• **Case XII**. If $\theta^2 = 1,\ \lambda = 1,\ \mu$ is arbitrary, then we have $D_1 = 1/3,\ D_2 = 0,\ D_3 = 0,\ D_4 = 0$. So, we obtain the Boussinesq system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x + \tfrac{1}{3}u_{xxx} = 0,\\ &\eta_x + u_t + uu_x - \tau\eta_{xxx} = 0. \end{aligned}\right\} \tag{3.15}$$

This system with $\tau = 0$ is the integrable version of Boussinesq system considered by Krishnan [24].

All these systems are equivalent and describe the bi-directional propagation of capillary-gravity waves within the same order of approximation $O(\alpha, \beta)$. It is found that the generalized lower order Boussinesq system (3.1) and (3.2) supports the traveling-solitary-wave solutions which are discussed in the following subsection.

3.1 Traveling Solitary Wave Solutions

In this section, we will look for the exact traveling wave solutions of the generalized lower order Boussinesq system (3.1) and (3.2) under the condition that $\eta(x,t)$ and $u(x,t)$ are proportional to each other and that they approach to zero as $|x| \to \infty$. Therefore, we express

$$\eta(x,t) = \eta(X) \quad \text{and} \quad u(x,t) = u(X); \quad X = x - c_s t, \tag{3.16}$$

where c_s is the velocity of the wave. Then we have the system (3.1) and (3.2) in the form

$$\left.\begin{aligned} &-c_s\eta' + u' + (\eta u)' + D_1u''' + D_2c_s\eta''' = 0,\\ &\eta' - c_su' + uu' + (D_3 - \tau)\eta''' + D_4c_su''' = 0, \end{aligned}\right\} \tag{3.17}$$

where the prime denotes the differentiation w.r.t. X. Integrating the system (3.17) once, we get

$$\left.\begin{array}{l} -c_s\eta + u + \eta u + D_1 u'' + D_2 c_s \eta'' = 0, \\ \eta - c_s u + \frac{u^2}{2} + (D_3 - \tau)\eta'' + D_4 c_s u'' = 0, \end{array}\right\} \tag{3.18}$$

where the constants of integration have been set to zero, since we are looking for solitary wave solutions. If u and η are proportional to each other, namely $u(X) = B\eta(X)$, $B(\neq 0) \in \mathbb{R}$, then we obtain the system (3.18) in the form

$$\left.\begin{array}{l} (B^2 - c_s B)\eta + B^2\eta^2 + (D_1 B^2 + D_2 c_s B)\eta'' = 0, \\ 2(1 - c_s B)\eta + B^2\eta^2 + 2(D_3 - \tau + D_4 c_s B)\eta'' = 0. \end{array}\right\} \tag{3.19}$$

In order for the system (3.19) to have a non-trivial solitary wave solution, it is necessary that the two equations in system (3.19) are identical, which implies

$$\left.\begin{array}{l} B^2 + c_s B - 2 = 0, \\ D_1 B^2 + (D_2 - 2D_4)c_s B - 2(D_3 - \tau) = 0. \end{array}\right\} \tag{3.20}$$

Thus we have a linear system (3.20) for B^2 and $c_s B$, the solution of which will depend on the values of D_1, D_2, D_3, D_4 and τ as follows:

• **Case I:** If $D_2 - 2D_4 - D_1 \neq 0$ (i.e., the determinant of the coefficient matrix in system (3.20) is non-zero), then there is an unique solution of the system (3.20) given by

$$\left.\begin{array}{l} B^2 = \frac{2(D_2 - 2D_4 - D_3 + \tau)}{D_2 - 2D_4 - D_1}, \\ c_s B = 2 - B^2 = \frac{2(D_3 - \tau - D_1)}{D_2 - 2D_4 - D_1}. \end{array}\right\} \tag{3.21}$$

• **Case II:** If $D_2 - 2D_4 - D_1 = 0$ and $D_1 = D_3 - \tau$, then there are infinitely many solutions of the system (3.20) given by

$$c_s B = 2 - B^2, \quad B^2 = \text{arbitrary}. \tag{3.22}$$

• **Case III:** If $D_2 - 2D_4 - D_1 = 0$ and $D_1 \neq D_3 - \tau$. There is no solution.

So, in cases I and II, we expect the solitary wave solutions to exist. Since both the equations in system (3.19) are identical, we can consider any one of these two to find the solitary wave solution. Therefore, we have the differential equation for η as

$$2(B^2 - 1)\eta + B^2\eta^2 + \big[(D_1 - D_2)B^2 + 2D_2\big]\eta'' = 0. \tag{3.23}$$

Differentiating equation (3.23) once and writing it in standard form, we have

$$\frac{1 - B^2}{B^2}\,\eta' \; - \; \frac{(D_1 - D_2)B^2 + 2D_2}{2B^2}\,\eta''' \; = \; \eta\eta'. \tag{3.24}$$

The following lemma [25] assures the existence of solitary wave solutions to equation (3.24).

• **Lemma 1:** Let R_1 and R_2 be two real constants. Then the equation

$$R_1\eta'(X) - R_2\eta'''(X) = \eta(X)\eta'(X), \tag{3.25}$$

has a solitary wave solution if $R_1R_2 > 0$. Moreover, the solitary wave solution is given by

$$\eta(X) \;=\; 3R_1 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{R_1}{R_2}}(X+X_0)\right), \tag{3.26}$$

where X_0 is an arbitrary constant. ■

It follows from the above lemma that equation (3.24) will admit solitary wave solutions if

$$\left(\frac{1-B^2}{B^2}\right)\left(\frac{(D_1-D_2)B^2+2D_2}{2B^2}\right) \;>\; 0 \quad \text{or} \quad (1-B^2)((D_1-D_2)B^2+2D_2) \;>\; 0. \tag{3.27}$$

Also, we need to have $B^2 > 0$, since B has to be a non-zero real constant. The solitary wave solution of equation (3.24) is given by

$$\eta(X) \;=\; A\operatorname{sech}^2\big[\gamma(X+X_0)\big], \tag{3.28}$$

where

$$A = \frac{3(1-B^2)}{B^2} \quad \text{and} \quad \gamma = \frac{1}{2}\left[\frac{2(1-B^2)}{(D_1-D_2)B^2+2D_2}\right]^{1/2}. \tag{3.29}$$

The solitary wave solution for $u(X)$ is then given by

$$u(X) = A\,B\operatorname{sech}^2\big[\gamma(X+X_0)\big]. \tag{3.30}$$

We can summarize the above results in the following theorem:

• **Theorem 1:** The system of equations

$$\left.\begin{aligned} &\eta_t + u_x + (\eta u)_x + D_1 u_{xxx} - D_2\eta_{xxt} = 0,\\ &\eta_x + u_t + uu_x + (D_3-\tau)\eta_{xxx} - D_4 u_{xxt} = 0, \end{aligned}\right\} \tag{3.31}$$

admits a pair of solitary wave solution of the form

$$\left.\begin{aligned} \eta(x,t) &= A\operatorname{sech}^2\big[\gamma(x+x_0-c_st)\big],\\ u(x,t) &= A\,B\operatorname{sech}^2\big[\gamma(x+x_0-c_st)\big], \end{aligned}\right\} \tag{3.32}$$

where

$$\left.\begin{aligned} B &= \begin{cases} \pm\left[\frac{2(D_2-2D_4-D_3+\tau)}{D_2-2D_4-D_1}\right]^{1/2}, & \text{if} \quad D_2-2D_4-D_1 \neq 0\\ \text{arbitrary}, & \text{if} \quad D_2-2D_4-D_1 = 0 \text{ and } D_1 = D_3-\tau \end{cases}\\ c_s &= \tfrac{2-B^2}{B}\\ A &= \tfrac{3(1-B^2)}{B^2}\\ \gamma &= \tfrac{1}{2}\left[\tfrac{2(1-B^2)}{(D_1-D_2)B^2+2D_2}\right]^{1/2} \end{aligned}\right\} \tag{3.33}$$

under the condition

$$(1-B^2)((D_1-D_2)B^2+2D_2) \;>\; 0 \quad \text{and} \quad B^2 \;>\; 0. \tag{3.34}$$

If $D_1 - D_2 + 2D_4 = 0$ but $D_1 \neq D_3 - \tau$, then there does not exist solitary wave solutions to the above system. ■

This general theorem provides an easy way to test the existence and non-existence of solitary wave solutions for the special Boussinesq systems (3.4) through (3.15).

4 Higher Order Boussinesq Equation

The Boussinesq equations of various orders for surface elevation $\eta(x,t)$ can be obtained by eliminating the horizontal velocity u from the Boussinesq systems derived in section 2. We choose the Boussinesq system consisting of equations (2.9) and (2.10). From these equations, we obtain the lower order approximated systems as

$$\eta_t + u_{0x} = O(\alpha,\beta), \quad \eta_x + u_{0t} = O(\alpha,\beta), \tag{4.1}$$

and

$$\left.\begin{aligned} &\eta_t + u_{0x} + \alpha(\eta u_0)_x - \tfrac{\beta}{6} u_{0xxx} = O(\alpha\beta,\beta^2),\\ &\eta_x + u_{0t} + \alpha u_0 u_{0x} - \tfrac{\beta}{2}(2\tau-1)\eta_{xxx} = O(\alpha\beta,\beta^2). \end{aligned}\right\} \tag{4.2}$$

It follows from equations (4.1a) and (4.2a) that

$$u_0 = -\int_{-\infty}^{x} \eta_t dx + O(\alpha,\beta), \quad u_0 = -\int_{-\infty}^{x} \eta_t dx + \alpha\eta \int_{-\infty}^{x} \eta_t dx - \frac{\beta}{6}\eta_{xt} + O(\alpha\beta,\beta^2). \tag{4.3}$$

Using the lower order approximations (4.1) and (4.2) in the resulting equation from ($\frac{\partial}{\partial t}$ [Eqn. (2.9)] – $\frac{\partial}{\partial x}$ [Eqn. (2.10)]), we obtain the following higher order Boussinesq equation:

$$\begin{aligned} \eta_{tt} - \eta_{xx} + \alpha[(\eta u_0)_t - u_0 u_{0x}]_x + \frac{\beta}{3}(3\tau-1)\eta_{xxxx} + \frac{\beta^2}{15}(5\tau-2)\eta_{xxxxxx} \\ + \frac{\alpha\beta}{6}[(u_0 u_{0xxx})_x - 6(u_{0x}u_{0xx})_x + 3(\eta\eta_{xxx})_x - 6(\eta\eta_{xx})_{xx}] = 0. \end{aligned} \tag{4.4}$$

We need to eliminate u_0 from equation (4.4). Substituting the expressions for u_0 given by equation (4.3) in equation (4.4) and using the lower order approximations (4.1b) and (4.2b), we obtain

$$\begin{aligned} \eta_{tt} - \eta_{xx} - \alpha\left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t dx\right)^2\right]_{xx} - \beta\left[\frac{1}{3} - \tau\right]\eta_{xxxx} + \alpha^2\left[\eta\left(\int_{-\infty}^{x} \eta_t dx\right)^2\right]_{xx} \\ - \alpha\beta\left[\frac{2}{3}(\eta_t^2)_{xx} + (\eta\eta_{xx})_{xx} - \tau(\eta\eta_{xxx})_x\right] - \frac{\beta^2}{3}\left[\frac{2}{5} - \tau\right]\eta_{xxxxxx} = 0, \end{aligned} \tag{4.5}$$

which is the required higher order Boussinesq equation appropriate for small values of α and β. It follows from equation (4.5) that the effect of non-linearity appears at $O(\alpha), O(\alpha^2)$ and $O(\alpha\beta)$ terms, where as, the effect of dispersion appears at $O(\beta)$ and $O(\beta^2)$ terms. The leading order dispersion term is $\beta(1/3-\tau)\eta_{xxxx}$. The following two cases are worth considering.

• **Case I:** If $\beta \ll (1/3-\tau) \leq 1/3$, that is, $(1/3-\tau) = K_1 = O(1)$, then a balance between the non-linearity and dispersion, which is necessary to model a solitary wave, requires $\alpha = O(\beta)$ as $\beta \to 0$, that is, $\alpha = K_2\beta$ as $\beta \to 0$ (K_2 fixed). Then we have the Boussinesq equation (4.5), correct up to $O(\alpha) = O(\beta)$, as

$$\eta_{tt} - \eta_{xx} - \alpha\left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t dx\right)^2\right]_{xx} - \frac{K_1}{K_2}\alpha\,\eta_{xxxx} = 0, \tag{4.6}$$

This case is appropriate for $0 \leq \tau \ll 1/3$.

• **Case II:** If $(1/3 - \tau) = O(\beta)$ as $\beta \to 0$, that is, $(1/3 - \tau) = K_1\beta$ as $\beta \to 0$ (K_1 fixed), then a balance between the non-linearity and dispersion requires $\alpha = O(\beta^2)$ as $\beta \to 0$, that is, $\alpha = K_2\beta^2$ as $\beta \to 0$ (K_2 fixed). Then we have the Boussinesq equation (4.5) correct up to $O(\alpha) = O(\beta^2)$ as

$$\eta_{tt} - \eta_{xx} - \alpha \left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t dx\right)^2\right]_{xx} - \frac{K_1}{K_2}\,\alpha\,\eta_{xxxx} - \frac{\alpha}{45K_2}\,\eta_{xxxxxx} = 0. \tag{4.7}$$

This case is appropriate for $\tau \uparrow 1/3$ (Bond number less than but very close to 1/3).

• **Co-ordinate Transformation and Transformed Equations:** If we use the transformation

$$\left.\begin{array}{l} X = \sqrt{\frac{K_2}{K_1}}\left(x + \alpha \int_{-\infty}^{x} \eta(x,t)dx\right), \\ T = \sqrt{\frac{K_2}{K_1}}\, t, \end{array}\right\} \tag{4.8}$$

and substitute

$$N = \frac{3}{2}(\eta - \alpha\eta^2), \tag{4.9}$$

then we obtain the fourth-order and sixth-order Boussinesq equations (4.6) and (4.7) in the forms

$$N_{TT} - N_{XX} - \alpha\,(N^2)_{XX} - \alpha\,N_{XXXX} = 0, \tag{4.10}$$

and

$$N_{TT} - N_{XX} - \alpha\,(N^2)_{XX} - \alpha\,N_{XXXX} - \epsilon^2\,\alpha\,N_{XXXXXX} = 0, \tag{4.11}$$

where $\epsilon^2 = \frac{K_2}{45K_1^2}$. It is worth summarizing here that both the equations (4.10) and (4.11) represent the bi-directional propagation of small amplitude (weakly non-linear) and long (weakly dispersive) capillary-gravity waves on the surface of shallow water; equation (4.10) being appropriate when $0 \leq \tau \ll 1/3$, where as, equation (4.11) being appropriate when $\tau \uparrow 1/3$. It is to be noted that $\tau \uparrow 1/3$ can hold true in the joint limit $K_1 \to \infty$ and $\beta \to 0$. So, we will have ϵ^2 as a small parameter independent of the amplitude parameters α. Therefore, equation (4.11) can be considered as a singular perturbation of equation (4.10).

4.1 Conversion of Boussinesq Equations into KdV Equations

The above Boussinesq equations (4.10) and (4.11) can be converted into the corresponding KdV equations by using the far-field co-ordinate transformations

$$\xi = X - T \quad \text{and} \quad \tau = \alpha\,T. \tag{4.12}$$

The transformation (4.12) describes a wave which changes slowly in a reference frame moving with velocity one (the non-dimensional shallow water velocity). The leading order terms in the transformed equations correspond to the following KdV equations:

$$N_\tau + NN_\xi + \frac{1}{2}N_{\xi\xi\xi} = 0, \tag{4.13}$$

and

$$N_\tau + N\,N_\xi + \frac{1}{2}N_{\xi\xi\xi} + \frac{K_2}{90K_1^2}N_{\xi\xi\xi\xi\xi} = 0. \tag{4.14}$$

If we further use the change of variables

$$\xi \to \frac{\xi}{\sqrt{2\delta}}, \quad \tau \to \frac{\tau}{\delta\sqrt{2\delta}}, \quad N \to \delta N, \tag{4.15}$$

where δ is an arbitrary scaling parameter, then the KdV equations (4.13) and (4.14) reduce to the following forms:

$$N_\tau + NN_\xi + N_{\xi\xi\xi} = 0, \tag{4.16}$$

and

$$N_\tau + N\,N_\xi + N_{\xi\xi\xi} + \frac{2K_2\delta}{45K_1^2}N_{\xi\xi\xi\xi\xi} = 0. \tag{4.17}$$

Equations (4.14) and (4.17) are two versions of the fifth-order KDV equation originally derived by Hunter and Scherule [16]. Equation (4.17) is exactly same as equation (2.29) in Hunter and Scherule [16] if we use the notation $\epsilon^2 = \frac{2K_2\delta}{45K_1^2}$. The other generalized higher order KdV equations [22] can be obtained from the generalized higher order Boussinesq systems and Boussinesq equations by restricting the waves to a submanifold traveling only in one direction, as in Olver [9, 10].

5 Notes on Fourth-Order and Sixth-Order Boussinesq Equations

The fourth-order Boussinesq equation (4.10) is completely integrable, supports an one-parameter family of solitary wave solutions [1, 2], and has Lax pair [26]. However, this equation is severely ill-posed, and therefore, is unlikely to have classical solutions for long time for arbitrary initial data. In fact, a numerical study [7, 8] of this equation as an initial value problem reveals the difficulty in constructing even known exact solutions because of the dreadful illposedness. A physically meaningful regularization procedure follows when one recognizes from equation (4.10) that $N_{TT} = N_{XX} + O(\alpha)$. Therefore, one can rewrite equation (4.10) in the form

$$N_{TT} - N_{XX} - \alpha\,(N^2)_{XX} - \alpha\,N_{XXTT} = 0. \tag{5.1}$$

The resulting equation (5.1) is well-posed, and therefore, can be used to describe the unsteady bi-directional wave propagation. The fourth-order Boussinesq equations (4.10) and (5.1) can be written in more standard forms

$$\eta_{tt} - \eta_{xx} - (\eta^2)_{xx} - \eta_{xxxx} = 0, \tag{5.2}$$

and

$$\eta_{tt} - \eta_{xx} - (\eta^2)_{xx} - \eta_{xxtt} = 0, \tag{5.3}$$

by using the following change of variables

$$X \to \alpha^{1/2}x, \quad T \to \alpha^{1/2}t, \quad N \to \alpha^{-1}\eta. \tag{5.4}$$

The traveling solitary wave solutions of equations (5.2) and (5.3) are given by

$$\eta = \frac{3}{2}(c_s^2 - 1)\,\mathrm{sech}^2\Big[\frac{\sqrt{c_s^2-1}}{2}(X - X_0)\Big], \tag{5.5}$$

and

$$\eta = \frac{3}{2}(c_s^2 - 1)\,\mathrm{sech}^2\Big[\frac{\sqrt{c_s^2-1}}{2c_s}(X - X_0)\Big]. \tag{5.6}$$

It is observed that both the solitary wave solutions (5.5) and (5.6) have same amplitude, $3(c_s^2-1)/2$, but different widths, $(\sqrt{c_s^2-1})/2$ and $(\sqrt{c_s^2-1})/2c_s$.

The sixth-order (singularly perturbed) Boussinesq equation (4.11) was originally introduced on a heuristic ground by Daripa and Hua [8] as a dispersive regularization of the ill-posed fourth-order Boussinesq equation (4.10). The physical relevance of this equation in the context of water waves is established here. In fact, it is seen that the singularly perturbed Boussinesq equation (4.11) actually describes the bi-directional propagation of small amplitude long capillary-gravity waves on the surface of shallow water for Bond number τ less than but very close to 1/3 (i.e., $\tau \uparrow 1/3$).

The initial value calculations of Daripa and Hua [8] for the regularized sixth-order Boussinesq equation (4.11) with initial data in the form of local solitary waves resulted in solutions which contain a hump at its core and small amplitude oscillations far from the core. Motivated by their work, Daripa and Dash [17] recently studied this equation both analytically and numerically using the technique of asymptotics beyond all orders and the perturbation analysis in the Fourier domain of the wave and numerically using the Newton-Kantorovich pseudospectral (collocation) method to find the traveling wave solutions. They proved the non-existence of local solitary wave solutions and showed the existence of weakly non-local solitary wave solutions for this equation which are characterized by small amplitude fast oscillations in the far-field. This behavior confirms the numerical prediction of Daripa and Hua [8]. It is well known that these types of solutions also exist for the fifth-order KdV equation [16, 18, 19, 20, 21, 23]. Most recently, Kichenassamy [23] devised the method of variational calculus to obtain this type of weakly nonlocal solitary wave solutions of the fifth-order KdV equation with negative speed and exponentially decaying tails. This method can also be applied to the sixth-order Boussinesq equation to find rigorously the weakly nonlocal solitary wave solutions with small amplitude fast oscillations in the far-field.

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