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1 Übung 2.1

1.1 Übung 2.1.(a)

(a) $n^3 \in \Omega(5n^2 + 7n + 6)$

$\exists c \in \mathbb{R}_{>0}, \exists n_0 \in \mathbb{N}\ \forall n \geq n_0,\ n^3 \geq c(5n^2 + 7n + 6)$ für alle $n \geq n_0$
Für Ω-Notation müssen wir zeigen, dass es Konstanten $c > 0$ und n_0 gibt, sodass:
$n^3 \geq c(5n^2 + 7n + 6)$ für alle $n \geq n_0$

Vereinfachung der rechten Seite:
$c(5n^2 + 7n + 6) < c(5n^2 + 7n^2 + 6n^2) = 18cn^2$ für alle $n \geq 1$

$c = \frac{1}{18}$, dann brauchen wir:
$n^3 \geq n^2$. Also, $18cn^2 \leq 18\frac{1}{18}n^2 \leq n^3$

Dies ist wahr für alle $n \geq 1$, mit $n_0 = 1$.

Also Die Aussage ist WAHR. $n^3 \in \Omega(5n^2 + 7n + 6)$, weil n^3 asymptotisch mindestens so schnell wächst wie $5n^2 + 7n + 6$.

1.2 Übung 2.1.(b)

(b) $\lfloor \frac{1}{2}n \rfloor \in o(n)$

$\forall c \in \mathbb{R}_{>0}, \exists n_0 \in \mathbb{N}\ \forall n \geq n_0,\ \lfloor \frac{1}{2}n \rfloor \leq cn$
Für litte-o-Notation müssten wir zeigen, dass für jedes $c > 0$ ein n_0 existiert, sodass:
$\lfloor \frac{1}{2}n \rfloor \leq cn$ für alle $n \geq n_0$

Es gilt: $\lfloor \frac{1}{2}n \rfloor \leq \frac{1}{2}n$

Wähle $c = 0.6$:
$\lfloor \frac{1}{2}n \rfloor \leq \frac{1}{2}n \leq 0.6n$ für alle $n > 0$. Für $c = 0.4$ die Ungleichung wird falsch sein.

Da $\frac{1}{2}n$ proportional zu n ist und die Abrundungsfunktion dies nur verkleinert, kann $\lfloor \frac{1}{2}n \rfloor$ nicht in $o(n)$ liegen. Also, $\lim_{n\to\infty} \frac{\lfloor \frac{1}{2}n \rfloor}{n} = \frac{1}{2} \neq 0$

Also Die Aussage ist FALSCH. $\lfloor \frac{1}{2}n \rfloor \notin o(n)$, weil $\lfloor \frac{1}{2}n \rfloor$ asymptotisch proportional zu n ist und damit in $\Theta(n)$ liegt, nicht in $o(n)$.

1.3 Übung 2.1.(c)

We need to prove the following proposition:
$\exists c \in \mathbb{R}_{>0}, \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0,\ f(n)g(n) \geq c \max\{f(n), g(n)\}$ for all $f, g : \mathbb{N} \to \mathbb{N}$

We know that:

$$\max\{f(n), g(n)\} = \begin{cases} f(n) & \text{if } f(n) \geq g(n), \\ g(n) & \text{if } g(n) > f(n). \end{cases}$$

(1). Lets suppose $f(n) \geq g(n)$: $\max\{f(n), g(n)\} = f(n)$

then we need to prove that $f(n)g(n) \geq cf(n)$. Since $f(n) \neq 0$ ($f(n) \in \mathbb{N}$), we can divide $f(n)$ from both sides and we have: $g(n) \geq c$.

Let suppose c=1. Then we need to prove that: $g(n) \geq 1$, which is true $\forall n \geq n_0$, with a fixed $n_0 \in \mathbb{N}$, since $g(n) \in \mathbb{N}$ ($g(n) \in \{1, 2, 3, ...\}$), $\forall n \in \mathbb{N}$

(2). Lets suppose $g(n) > f(n)$: $\max\{f(n), g(n)\} = g(n)$

then we need to prove that $f(n)g(n) \geq cg(n)$. Since $g(n) \neq 0$ ($g(n) \in \mathbb{N}$), $f(n) \geq c$.

Let suppose c=1. Then we need to prove that: $f(n) \geq 1$, which is true $\forall n \geq n_0$, with a fixed $n_0 \in \mathbb{N}$, since $f(n) \in \mathbb{N}$ ($f(n) \in \{1, 2, 3, ...\}$), $\forall n \in \mathbb{N}$

So, for all $f, g : \mathbb{N} \to \mathbb{N}$, $\exists c \in \mathbb{R}_{>0}$ $(c = 1), \exists n_0 \in \mathbb{N}$ (a fixed natural number) such that $\forall n \geq n_0$ $f(n)g(n) \geq c \max\{f(n), g(n)\}$. We can conclude that for all $f, g : \mathbb{N} \to \mathbb{N}$, $f \cdot g \in \Omega(\max\{f, g\})$

1.4 Übung 2.1.(d)

$2^n \in \Omega(n!)$. We need to prove the following proposition:

$\exists c \in \mathbb{R}_{>0}, \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $2^n \geq cn!$, which will be false.

We also know from the course that if $f, g : \mathbb{N} \to \mathbb{R}$, the following holds:

$$f \in \Omega(g) \iff \liminf_{n\to\infty} \frac{f(n)}{g(n)} > 0 \quad (*)$$

and $\liminf_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)}{g(n)}$, if the series $\frac{f(n)}{g(n)}$ is convergent.

In our case, $a_n = \frac{2^n}{n!}$ is absolutely convergent, since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ (**)

afterwards we use the following obeservation:

$$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdot ... \cdot 2}{1 \cdot 2 \cdot 3 \cdot ... \cdot n} = \frac{2}{1} \cdot \frac{2}{2} \cdot ... \cdot \frac{2}{n-1} \cdot \frac{2}{n} = \frac{4}{n} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot ... \cdot \frac{2}{n-1} \leq \frac{4}{n}$$

$$0 < \frac{2^n}{n!} \leq \frac{4}{n}$$

.

Using the squeeze theorem and the fact that the series is converges (**), we can find the limit.

$$\liminf_{n\to\infty} \frac{2^n}{n!} = \lim_{n\to\infty} \frac{2^n}{n!} = 0$$

Using (*) we can conclude that $2^n \notin \Omega(n!)$. So, the proposition will be FALSE

2 Übung 2.2

2.1 Explication of the algorithm

To solve the problem we are going to use the fast multiplication algorith which alows us to compute a^b faster that the normal method.

We are going to rely on the folowing observation.

- if $b = 0$ we return 1, since any number raised to the power of 0 is 1.
- if b is even we compute $a^{b/2}$ recursively, then square the result to get a^b

  $a^b = \left(a^{\frac{b}{2}}\right)^2$
- if b is odd we compute $a^{\frac{b-1}{2}}$ recursively, then square it and after multypling it by a to get a^b

  $a^b = a\left(a^{\frac{b-1}{2}}\right)^2$

Algorithm 1

Data: a and b natural numbers
Result: a^b

Power(a,b)

```
if b = 0 then
    return 1
else if b mod 2 = 0 then
    aux ← Power(a, b/2)
    return aux · aux
else
    aux ← Power(a, (b − 1)/2)
    Return aux · aux · a
end if
```

2.2 Time complexity

The algorithm reduces the exponent b by approximately half in each recursive call:

- Each time b is even, the function calls itself with $b/2$.
- Each time b is odd, the function calls itself with $(b-1)/2$.

Thus, the algorithm makes $O(\log b)$ recursive calls. Since we assume that addition and multiplication are constant-time operations, the overall time complexity of the algorithm is $O(\log b)$.