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Exercise 1

For the Markov Chain find the hitting probabilities $hi4, i = 1, 2, 3, 4$ by directly applying the theorem presented in class. Make sure you justify and check every condition stated in the theorem.

$$\begin{cases} h_{14} = 0 \\ h_{24} = \frac{1}{2}h_{24} + \frac{1}{2}h_{14} \\ h_{34} = \frac{1}{2} + \frac{1}{2}h_{24} \\ h_{44} = 1 \end{cases} \implies \begin{cases} h_{14} = 0 \\ h_{24} = \frac{1}{2}h_{34} \\ h_{34} = \frac{1}{2} + \frac{1}{2}h_{24} \\ h_{44} = 1 \end{cases}$$

$h_{34} = \frac{2}{3}$ and $h_{24} = \frac{1}{3}$

Exercise 2

A Markov Chain has transition matrix:

$$\begin{pmatrix} 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \end{pmatrix}$$

1. Describe the set of stationary distribution for the chain. the set of stationary distribution is the set of solution of this equation : $\pi P = \pi$ where $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$

   In order to find this we need to solve this system of equation :

   $$(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) \begin{pmatrix} 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \end{pmatrix} = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$$

   after this equation we have this system of equation :

$$\begin{cases} \frac{3}{4}\pi_2 + \frac{1}{4}\pi_5 = \pi_1 \\ \frac{1}{4}\pi_1 + \frac{3}{4}\pi_5 = \pi_2 \\ \pi_3 = \pi_3 \\ \pi_4 = \pi_4 \\ \frac{3}{4}\pi_1 + \frac{1}{4}\pi_2 = \pi_5 \end{cases}$$

after solve this equation we have this: $\pi_2 = \pi_5 = \pi_1$ and

the set of stationary distribution is $\pi = (\pi_1, \pi_1, \pi_3, \pi_4, \pi_1)$ such as $\pi_1+\pi_2+\pi_3+\pi_4+\pi_5 = 1$ $\implies 3\pi_1 + \pi_3 + \pi_4 = 1$

2. Find $\lim_{n\to\infty} P^n$ (Use technology). Explain the long behavior of the chain.

   In order to find the long term behavior of the Markov chain we just take a largest value of n computer the matrix numerically to identifier the pattern. and we see that after

$$P^n = \begin{pmatrix} 0.1478667 & 0.4149415 & 0.09554803 & 0.2164030 & 0.1252407 \\ 0.1478315 & 0.4149254 & 0.09556069 & 0.2163803 & 0.1253020 \\ 0.1478257 & 0.4149183 & 0.09556400 & 0.2163711 & 0.1253208 \\ 0.1478328 & 0.4149285 & 0.09555956 & 0.2163842 & 0.1252950 \\ 0.1478265 & 0.4149060 & 0.09556717 & 0.2163562 & 0.1253441 \end{pmatrix}$$

3. Does the chain have a limiting distribution? Justify your answer.

   Yes. since all columns and rows of the the previous question have the same that means this chains have the limiting distribution

Exercise 3

Let P be a stochastic matrix.

1. If P is regular, is P^2 regular? by definition of the regular If P is regular that means there exists a k such as $(P^k)_{ij} > 0$ for all i, j which also means that after the step k the transition matrix all the probability is strictly greater that 0.

   and we know that $P^2 = P*P$ and $P^2 = P*P$ and $P^k * P^k = P^{2k}$ and we know that $2k > k$ which means for P^2 we can always find the $k' = 2k$ such as all entries are non-negative.

2. If P is the transition matrix of an irreducible Markov Chain, is P^2 the transition matrix of an irreducible Markov Chain?

   If P is the transition matrix of an irreducible Markov Chain, by definition, for any two states i and j, there exists some n such that $(P^n)_{ij} > 0$. In other words, after some finite number of steps, state j can be reached from state i.

   Now, consider the matrix P^2, which is the result of multiplying P by itself: $P^2 = P \cdot P$. The entry $(P^2)_{ij}$ represents the probability of transitioning from state i to state j in exactly 2 steps. Specifically, $(P^2)_{ij} = \sum_{k=1}^{n} P_{ik}P_{kj}$, where the sum is over all possible intermediate states k.

   To prove that P^2 is the transition matrix of an irreducible Markov Chain, we need to show that for any pair of states i and j, there exists some n such that $(P^2)_{ij} > 0$.

Since P is irreducible, we know that for any two states i and j, there exists some m such that $(P^m)_{ij} > 0$, meaning state j is reachable from state i in m steps.

- For P^2 to have a nonzero entry $(P^2)_{ij}$, we need to check if we can reach state j from state i in exactly 2 steps. This can be done through an intermediate state k, where k can be any state (including i and j).

- If P is irreducible, then for any pair of states i and j, there exists some k such that $P_{ik} > 0$ and $P_{kj} > 0$. This means that state j can be reached from state i via state k, which implies $(P^2)_{ij} > 0$.

Thus, P^2 will have positive entries for all pairs of states i and j, meaning that all states are reachable from all other states in exactly two steps.

Exercise 4:

1. Find the transition matrix P

   $$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

2. What is the expected time to get from vertex 1 to vertex 2?

   let's denote by A the set 2 in order to find the expected time from vertex 1 to reach A.

   $$\begin{cases} m_{22} = 0 \\ m_{1A} = 1 + p_{11}m_{1A} + p_{13}m_{3A} \\ m_{3A} = 1 + p_{31}m_{1A} + p_{33}m_{3A} \end{cases}$$

   after solving this equation we have $m_{1A} = m_{3A} = 2$

   Hence, the expected time to get from vertex 1 to vertex 2 is 2

Exercise 5

1. Identify the classes.

   we have this communications classes : $\{3, 4\}$, $\{7, 8\}$, $\{5\}$, $\{6\}$, $\{2\}$, $\{1\}$

2. Find the transient and recurrent states. the following states are transient : $\{5, 1, 4, 3, 2\}$ and the following states are recurrent: $\{6, 7, 8\}$