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1. (a) Solve the boundary value problem

$$u^{''} + \lambda u = 0 \qquad 0 < x < 1$$
$$\frac{12}{10}u^{'}(0) + u(0) = 0, u(1) = 0$$

- case $\lambda = 0$,
  $U(x) = Ax + B$ with boundary condition let's find A and B
  $\begin{cases} \frac{12}{10}u^{'}(0) + u(0) = 0 \\ U^{'}(1) = 0 \end{cases} \implies \begin{cases} \frac{10}{12}B + A = 0 \\ A + B = 0 \end{cases}$
  after solving this equation we have $A = B = 0$ the only solution is $U = 0$
- case $\lambda = -\alpha^2 < 0$, $U(x) = Ae^{\alpha x} + Be^{-\alpha x}$
  by applying boundary conditions we have:
  $\begin{cases} \frac{12}{10}u^{'}(0) + u(0) = 0 \\ U^{'}(1) = 0 \end{cases} \implies \begin{cases} \frac{12}{10}(\alpha A - B\alpha) + A + B = 0 \\ Ae^{\alpha} + Be^{-\alpha} = 0 \end{cases}$
  $\begin{cases} \frac{12}{10}(\alpha A - B\alpha) + A + B = 0 \\ A = -Be^{-2\alpha} \end{cases}$
  after substitute $A = -Be^{-2\alpha}$ in $\frac{12}{10}(\alpha A - B\alpha) + A + B = 0$ we have

  $$B\left[\frac{12}{10}\alpha e^{-2\alpha} - \frac{12}{10}\alpha - e^{-2\alpha} + 1\right] = 0$$

  Since $B \neq 0$ the value of alpha can only be determined numerically
- case $\lambda = \alpha^2 > 0$
  The solution of the equation is given by: $u(x) = Acos(\alpha x) + Bsin(\alpha x)$,
  $u^{'}(x) = -A\alpha sin(\alpha x) + B\alpha cos(\alpha x)$
  by applying the boundary condition:
  $\begin{cases} \frac{12}{10}u^{'}(0) + u(0) = 0 \\ U^{'}(1) = 0 \end{cases} \implies \begin{cases} \frac{12}{10}(B\alpha) + A = 0 \\ Acos(\alpha) + Bsin(\alpha) = 0 \end{cases} \implies \begin{cases} A = -\frac{12}{10}B\alpha \\ Acos(\alpha) = -Bsin(\alpha) \end{cases}$
  $\begin{cases} A = -\frac{12}{10}B\alpha \\ tan(\alpha) = -\frac{B}{A} \end{cases}$

$\implies tan(\alpha) = \frac{10}{12\alpha} = \frac{5}{6\alpha}$
the equation of U(x) can be written with cosh and sinh()
$U(x) = Ccosh(\alpha x) + Dsinh(\alpha x)$
where $cosh(\alpha x) = \frac{e^{-\alpha x}+e^{-\alpha x}}{2}$ and $sinh(x) = \frac{e^{\alpha x}-e^{-\alpha x}}{2}$ and $C = A + B$ and $D = A - B$

(b) Estimate λ_n for large n (remember the eigenvalues are ordered $\lambda_1 < \lambda_2 <$).
when n is larger the $tan(\alpha) = \frac{10}{12\alpha}$ can be approximate by $\alpha_n = n\pi$
Hence $\lambda = \alpha_n^2 = (n\pi)^2$

2. Consider the advection–diffusion equation $\begin{cases} \partial_t U - 2\partial_x U = \partial_{xx]U} \\ \frac{22}{10}U(0,t) + U_x(0,t) = 0, \qquad U(1,t) = 0 \\ U(x,0) = e^{-x} \end{cases}$

(a) Show that the above PDE can be brought into the form

$$\partial_x w = \partial_{xx} w$$

using the transformation $u(x,t) = e^{-x-t}w(x,t)$
$w(x,t) = e^{x+t}u(x,t)$

$$\begin{aligned} \partial_x w(x,t) &= e^{x+t}u(x,t) + e^{x+t}\partial_x u(x,t) \\ \partial_{xx} w(x,t) &= e^{x+t}u(x,t) + e^{x+t}\partial_x u(x,t) + e^{x+t}\partial_x u(x,t) + e^{x+t}\partial_{xx} u(x,t) \quad = e^{x+y}u(x,t) + 2e^{x+t}\partial_x \\ \partial_t w(x,t) &= e^{x+t}u(x,t) + e^{x+t}\partial_t u(x,t) \end{aligned}$$

$$\begin{aligned} \partial_t W &= \partial_{xx} W \\ e^{x+t}u(x,t) + e^{x+t}\partial_t u(x,t) &= e^{x+y}u(x,t) + 2e^{x+t}\partial_x u(x,t) + e^{x+t}\partial_{xx} u(x,t) \\ \partial_t e^{x+t}u(x,t) - 2e^{x+t}\partial_x u(x,t) &= e^{x+t}\partial_{xx} u(x,t) \\ \partial_t u(x,t) - 2\partial_x u(x,t) &= \partial_{xx} u(x,t) \end{aligned}$$

(b) Determine the boundary conditions on w(0, t), w(1, t) and the initial data w(x, 0).
using the boundary condition on U we have

$$\begin{aligned} \frac{22}{10}u(0,t) + U_x(0,t) &= 0 & U(1,t) &= 0 \\ \frac{22}{10}e^{-t}w(0,t) + \left[-e^{-t}w(0,t) + e^{-t}W_x(0,t)\right] &= 0 & e^{-1-t}w(1,t) &= 0 \\ \frac{12}{10}e^{-t}w(0,t) + e^{-t}w_x(0,t) &= 0 & e^{-1-t}w(1,t) &= 0 \\ e^{-t}\left[w(0,t) + w_x(0,t)\right] &= 0 & e^{-1-t}w(1,t) &= 0 \\ \left[w(0,t) + w_x(0,t)\right] &= 0 & w(1,t) &= 0 \end{aligned}$$

and by using other boundary conditions,

$$\begin{aligned} u(x,0) &= e^{-x} \\ e^{-x}w(x,0) &= e^{-x} \\ w(x,0) &= 1 \end{aligned}$$

The boundary condition is: $\begin{cases} w(0,t) + w_x(0,t) = 0, \qquad\qquad w(1,t) = 0 \\ w(x,0) = 1 \end{cases}$

(c) Use the separation of variables ansatz $w(x,t) = X(x)T(t)$ to derive the equations

$$\begin{aligned} w_t &= X(x)T^{'}(t) \\ w_{xx} &= X^{''}(x)T(t) \end{aligned}$$

$$\begin{aligned} w_t &= lW_{xx} \\ X(x)T^{'}(t) &= X^{''}(x)T(t) \\ \frac{T^{'}(t)}{T(t)} &= \frac{X^{''}(x)}{X(x)} = -\lambda \end{aligned}$$

we can rewrite this equation as :

$$\begin{aligned} T^{'}(t) + \lambda T(t) &= 0 \\ X^{''}(x) + \lambda X(x) &= 0 \end{aligned}$$

let's now determine the boundary condition on X(x)
using the boundary condition on W we have:

$$\begin{aligned} X(0)T(t) + X^{'}(0)T(t) &= 0 \qquad\qquad X(1)T(t) = 0 \\ X(x)T(0) &= 1 \end{aligned}$$

$\Longrightarrow$

$$X(0) + X^{'}(0) = 0, \qquad\qquad X(1) = 0$$

(d) Find the general solution for w(x, t). You do not need to determine the exact expression for the coefficients.
In the equation $X^{''}(x) + \lambda X(x) = 0$

- case $\lambda = 0$, X(x) = Ax + B with boundary conditions we have
$\begin{cases} X^{'}(0) + X(0) = 0 \\ X(1) = 0 \end{cases} \implies \begin{cases} A + B = 0 \\ A + B = 0 \end{cases} \implies A = -B$
Hence $X(x) = Ax - A = A(x-1)$
- case $\lambda = -\alpha^2 < 0$
In the equation $X^{''}(x) + \lambda X(x) = 0$ the solution of this equation is : $X(x) = Ae^{\alpha x} + Be^{-\alpha x}$ and $X^{'}(x) = A\alpha e^{\alpha x} - B\alpha e^{-\alpha x}$
with the initial condition we have,
$\begin{cases} X^{'}(0) + X(0) = 0 \\ X(1) = 0 \end{cases} \implies \begin{cases} A + B + A\alpha - B\alpha = 0 \\ Ae^{\alpha} + Be^{-\alpha} = 0 \end{cases}$ we have $A = -Be^{-2\alpha}$
and $-Be^{-2\alpha} + B - B\alpha e^{-2\alpha} - B\alpha = 0 \implies$
$B(-e^{-2\alpha} + 1 - e^{-2\alpha} - \alpha) = 0$
Since $B \neq 0$ we have $-2\alpha + 1 - e^{-2\alpha} - \alpha = 0$ this solution can only be determined numerically
- $\lambda = \alpha^2 > 0$
In this case, the solution of X can be written as $X(x) = Acos(\alpha x) + Bsin(\alpha x)$
$X^{'}(x) = -A\alpha sin(\alpha x) + B\alpha cos(\alpha x)$
using the boundary condition we have
$\begin{cases} X^{'}(0) + X(0) = 0 \\ X(1) = 0 \end{cases} \implies \begin{cases} A + B\alpha = 0 \\ Acos(\alpha) + Bsin(\alpha) = 0 \end{cases}$
So $tan(\alpha) = -\frac{B}{A} = -\frac{B}{-B\alpha} = \frac{1}{\alpha}$
λ_n is the eigenvalue of and $\lambda_n = \alpha_n^2$
Hence the solution of the question is :
$T^{'}(x) + \lambda T(x) = 0$ and the solution of the equation is $T_n(t) = A_n e^{-\lambda_n t}$

$$\begin{aligned} w(x,t) &= \sum_{n\geq 1} T_n(t) X_n(t) \\ &= \sum_{n\geq 1} A_n e^{-\lambda_n t} X_n(x) \end{aligned}$$

(e) Use your previous answer to write down an expression for the solution u(x, t). For a fixed finite value of x, comment on the behavior of u(x, t) as $t \to \infty$.

$$\begin{aligned} u(x,t) &= e^{-x-t} w(x,t) \\ &= e^{-x-t} \sum_{n\geq 1} A_n e^{-\lambda_n t} X_n(x) \\ &= e^{-x} \sum_{n\geq 1} A_n e^{-(\lambda_n+1)t} X_n(x) \end{aligned}$$

Using the previous equation we can see that when x is fixed and t goes to infinity $u(x,t) \to 0$ that means the solution decays exponentially with the time

(f) (a) Check that $U(x,t) = 1 - x^2 - 2t$ is a solution to the heat equation $u_t = u_{xx}$.
let's compute the both side : $u_t = -2$ and $u_x = -2x$ and $u_{xx} = -2$

we have $u_t = u_{xx} = -2$ which means $u(x,t) = 1 - x^2 - 2t$ is the solution of the heat equation.

(g) (b) Denote the space-time rectangle

$$\bar{R} = 0 \leq x \leq 1 \quad x \quad 0 \leq t \leq 1$$

Use the weak maximum principle to determine $max_{R^-}U(x,t)$ and $min_{R^-}U(x,t)$ using the maximum principle we have :

$$max(u(x,t))_R = max(max(f(t)), max(g(t)), max(h(t)))$$

So $f(t) = u(x,0) = 1 - x^2$ and $max(f(t)) = 1$ in $[0,1]$ and $g(t) = u(0,t) = 1 - t$ $max(g(t)) = 1$ over $[0,1]$
Hence $max(u(x,t))_R = 1$

$$min(u(x,t))_R = min(min(f(t)), min(g(t)), min(h(t)))$$

Hence $min(u(x,t)) = -1$

(h) Plot the function U(x, t) in $\bar{R}$ and interpret the position of the hottest point compared to the coldest point.

(i) Let $T > 0$ and consider instead the rectangle

$$\bar{R}_T = \{0 \leq x \leq 1\} x \{0 \leq x \leq 1\}$$

Define $f(T) = min(\hat{R}_T)U(x,t)$. Is f(T) decreasing or increasing as a function of T?