AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

(AIMS RWANDA, KIGALI)

Name: Darix SAMANI SIEWE
Course: Algorithms and Data Structure
Assignment Number: 2
Date: November 27, 2024

Problem 1: Heaps (4 points)

In both cases please justify your answers.

1. Let A be a max-heap indexed from 1 to n that contains n distinct elements. At which indexes can the smallest element in the heap be located?

   Before depth into the prove let's take some example to find the pattern and generalize our solution for any size:

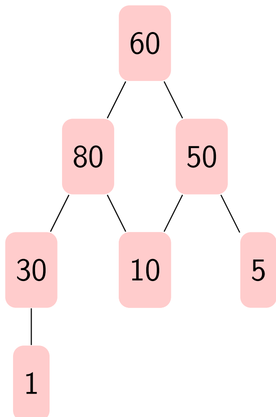

Figure 1: example for max-heap with $n = 7$

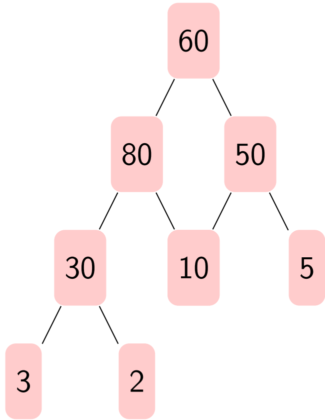

Figure 2: example for max-heap with $n = 8$

By the definition of max-heap, let's denote by i the position of each element in a heap $(1 \leq i \leq n)$:

- if $2i \leq n$, then $A[2i] \leq A[i]$
- if $(2i + 1) \leq n$, then $A[2i + 1] \leq A[i]$

The definition above, means that for each parent node in the max-heap, the left child and the right child is also a max-heap.

Let's find the parent position of max-map that contains the smallest element in max-heap. the exact parent position of sub-max-heap that contains the smallest element in max-heap is the position $\lfloor n/2 \rfloor$

That means the index position where we can find the smallest elements is the element from $\lfloor n/2 \rfloor$ to n

2. The height of an element in a heap is its distance from the root. The height of the heap is maximum over heights of all its elements. Let A be a heap of height h. What are the smallest and largest possible number of elements contained in A?

   To answer to this question let's try with some example in order to find the pattern and generalize the formation to find the smallest and largest number of element contained in A:

   let's denote by $min(A_h)$ and $max(A_h)$ the smallest and largest number of elements contained in A respectively

   - if $h = 0$, we have just one element in the heap, $min(A_0) = 1 = 2^0$ and $max(A_0) = 1 = 2^{0+1} - 1$
   - if $h = 1$ the number of element in the heap can be 2 or 3 that means the $min(A_1) = 2 = 2^1$ and $max(A_1) = 3 = 2 * 2 - 1$
   - if $h = 2$, $min(A_2) = 4 = 2^2 = 2 * min(A_2)$ and $max(A_2) = 7 = 2 * 4 - 1 = 2 * min(A_2) - 1$
   - if $h = 3$, $min(A_3) = 8 = 2^3 = 2 * min(A_2)$ and $max(A_3) = 15 = 2 * 8 - 1 = 2 * min(A_3) - 1$

based on the result above, by conjecture, $\forall h > 0$ $min(A_h) = 2^2$ and there are exist a relationship between $max(A_h)$ and $min(A_h)$: $max(A_h) = 2 * min(A_h) - 1$, that means $max(A_h) = 2 * 2^h - 1 = 2^{h+1} - 1$

Hence, By conjecture, $min(A_h) = 2^h$
and $max(A_h) = 2^{h+1} - 1$

Once that we found by conjecture the expression of smallest and largest possible number of elements contained in A: let's now prove by induction. Because we need to prove the recurrence expression the base case on induction is $h = 0$

- base case h = 0: we know that when h=0, that means there are just two step in the max-heap:
  and $min(A_1) = 1 = 2^0$ and $max(A_1) = 1 = 2^{0+1} - 1$
  Hence, the base case is true,
- Induction step: let's assume the $\forall h \geq 1$ the statement is true and let's prove that is also true at $h + 1$,

$$min(A_{h+1}) = 2 * min(A_h) = 2 * 2^h = 2^{h+1}$$

  so the statement is true

$$max(A_{h+1}) = 2 * min(A_{h+1}) - 1 = 2 * 2^{h+1} - 1 = 2^{(h+1)+1} - 1$$

  so the statement is for induction step is true,
- conclusion:
  we have prove above that, the base case is true and the induction step is also true, by induction the statement is true.

Problem 2: Paths and adjacency matrix (4 points)

In the following, let $G = (V, E)$ be a simple graph with adjacency matrix M . Let u, $v \in V$ and $k \in N$.

1. A matrix is called nonnegative if all its entries are nonnegative. Show that M^k is non-negative for every k.

   let's prove this by induction,

   by definition of adjacent matrix M: $M_{(i,j)} = \begin{cases} 1 & if \quad (i,j) \in E(G) \\ 0 & otherwise \end{cases}$

   - base case k = 1, $M^1 = M_{(i,j)} = \begin{cases} 1 & if \quad (i,j) \in E(G) \\ 0 & otherwise` \end{cases}$

     all element of M can between 1 and 0 that means all elements is nonnegative, Hence base case for $k = 1$ is true.

- k = 2,
  $M^2 = M * M = a_{(i,j)}$ where $a_{(i,j)} = \sum_{z=1}^{n} M_{iz}M_{zj}$ we known that M_{iz} and M_{zj} is nonnegative by the definition of adjacent matrix, so the product $M_{zj} * M_{zj}$ for z in $[1, n]$.
  Hence, the base case when k=2, is also True
- Induction Step; let's assume that $\forall k > 2$, M^K is nonnegative matrix, let's prove that M^{k+1} is nonnegative matrix
  $M^{k+1} = M^k{*}M$, Hence by the hypothesis of induction M^K is nonnegative matrix and by the base case M is also the nonnegative matrix, the production of two nonnegative matrix is also a nonnegative matrix,
- Conclusion: By induction, M^k is nonnegative matrix

2. Show that there exists a walk from u to v of length k in G if and only if $M^k_{u,v} > 0$

   in order to show this let's show this in two side:

   - Let's prove that if there are exists a walk from u to v of length k $\implies$ $M^k_{(u,v)} > 0$
     let's prove this induction:
     - base case $k = 1$, that means there exist a walk from u to v of length 1, prove that $M^1_{(u,v)} > 0$:
       it length from u to v is 1 that means (u, v) is a edges of G, so $M^1(u, v) = 1 > 0$,
       Hence, the base case is true,
     - induction step: let's suppose that there are exists a walk from u to v of length k $\implies$ $M^K_{(u,v)} > 0$ let's prove that there are exists a walk from u to v of length (k+1) $\implies$ $M^{k+1}_{(u,v)} > 0$
       A walk of length $k+1$ from u to v can be decomposed into a walk from u to some intermediate vertex w of length k, followed by an edge from w to v. Specifically, there is some vertex w such that:
       * There is a walk of length k from u to w.
       * There is an edge from w to v, i.e., $(w, v) \in E(G)$.

       The entry $M^{k+1}_{u,v}$ is given by:

       $$M^{k+1}_{u,v} = \sum_{w=1}^{n} M^k_{u,w} \cdot M_{wv}$$

       Since $M_{wv} \geq 0$ and $M^k_{u,w} > 0$ (by the inductive hypothesis), this sum is positive.
       Hence, $M^{k+1}_{u,v} > 0$, completing the inductive step.

     Therefore, by induction, if there is a walk from u to v of length k, then $M^k_{u,v} > 0$.
   - let's prove that if $M^k_{u,v} > 0 \implies$ there are exists a walk from u to v of length k in G.
     if $M^k_{(u,v)} > 0$ that means there are exists w such as length of walk from u to w is $(k-1)$ by using the question 1 we have $M^{k-1}_{(u,w)} > 0$ by induction there exist some vertices such as the length of walk from u to $u_0, U_1, \dots, w$ is respectively $1, 2, \dots, k-1$,
     Hence There exist a walk from u to v of length k

Hence, we have prove the two parts of equivalent, Hence I have prove the equivalent

3. Show an example of a graph and u, v, $k > 1$ where the length of the shortest path from u to v is at most k, but $M^k_{u,v} = 0$

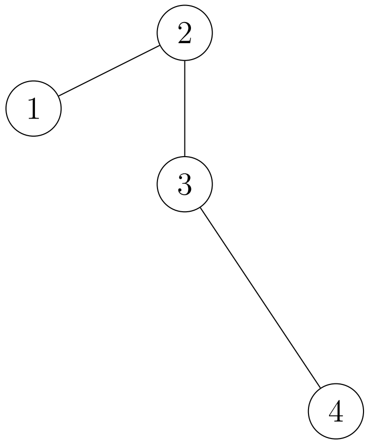

Figure 3: example of simple graph