

Mathematic Problem Solving Assignment 1

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Problem 1: Sum of p-th powers

1. Show that: $\sum_{k=1}^n k = \frac{1}{2}n(n + 1)$

let's show by using induction:

- Initialization: for $n = 1$ we have $\sum_{k=1}^1 k = 1$ and $\frac{1}{2} * 1 * (1 + 1) = 1$
thus, $\sum_{k=1}^1 k = \frac{1}{2} * 1 * (1 + 1)$
The property is true for $n = 1$
- hypothesis: let's suppose that the property is true at rank n and let's prove that is also true at rank $n + 1$

$$\begin{aligned}\sum_{k=n+1}^1 k &= \sum_{k=n+1}^1 k + (n + 1) \\ &= \frac{1}{2}n(n + 1) + (n + 1) \quad \text{by using hypothesis of induction} \\ &= \frac{1}{2}(n + 1)(n + 2)\end{aligned}$$

as it is also true at rank $n + 1$ by induction $\sum_{k=1}^n k = \frac{1}{2}n(n + 1)$

2. Prove that: $\sum_{k=1}^n [(k + 1)^3 - k^3] = n(n^2 + 3n + 3)$ and $\sum_{k=1}^{n+1} [(k + 1)^3 - k^3] = 3 \sum_{k=1}^n [k^2 + k + \frac{1}{3}]$

Firstly, let's prove that: $\sum_{k=1}^n [(k + 1)^3 - k^3] = n(n^2 + 3n + 3)$ we can prove this by induction:

- Initialization: for $n = 1$ we have,

$$\sum_{k=1}^n [(k + 1)^3 - k^3] = (1 + 1)^3 - 1^3 = 2^3 - 1 = 7 \text{ and } 1(1^2 + 2 * 1 + 3) = 7$$

$$\text{thus, } \sum_{k=1}^n [(k + 1)^3 - k^3] = 1(1^2 + 2 * 1 + 3)$$

so the property is true at rank $n = 1$

- Hypothesis: let's suppose that the property is true at rank n and let's prove it is also true at rank $n + 1$

$$\begin{aligned}\sum_{k=1}^{n+1} [(k + 1)^3 - k^3] &= \sum_{k=1}^n [(k + 1)^3 - k^3] + (n + 2)^3 - (n + 1)^3 \\ &= n(n^2 + 3n + 3) + (n + 2)^3 - (n + 1)^3 \quad \text{by using the hypothesis of induction} \\ &= n^3 + 6n^2 + 12n + 7\end{aligned}$$

and

$$\begin{aligned}(n + 1)[(n + 1)^3 + 3(n + 1) + 3] &= (n + 1)[n^2 + 2n + 1 + 3n + 3 + 3] \\ &= (n + 1)(n^2 + 5n + 7) \\ &= n^3 + 6n^2 + 12n + 7\end{aligned}$$

so, $\sum_{k=1}^{n+1} [(k+1)^3 - k^3] = (n+1) [(n+1)^3 + 3(n+1) + 3]$

the property is also true at rank $n+1$, by induction $\sum_{k=1}^n [(k+1)^3 - k^3] = n(n^2 + 3n + 3)$

Secondly, let's prove that $\sum_{k=1}^{n+1} [(k+1)^3 - k^3] = 3 \sum_{k=1}^n [k^2 + k + \frac{1}{3}]$.

To prove that, we can use direct proof:

$$\begin{aligned}\sum_{k=1}^{n+1} [(k+1)^3 - k^3] &= \sum_{k=1}^{n+1} [k^3 + 3k^2 + 3k + 1 - k^3] \\ &= \sum_{k=1}^{n+1} [3k^2 + 3k + 1] \\ &= \frac{1}{3} \sum_{k=1}^{n+1} \left[k^2 + k + \frac{1}{3} \right]\end{aligned}$$

3. Deduce that: $\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$

by using the previous result:

$$\begin{aligned}\sum_{k=1}^{n+1} [(k+1)^3 - k^3] &= \frac{1}{3} \sum_{k=1}^{n+1} \left[k^2 + k + \frac{1}{3} \right] = n(n+1)(2n+1) \\ \implies 3 \sum_{k=1}^n k^2 + 3 \sum_{k=1}^n k + 3 \sum_{k=1}^n \frac{1}{3} &= n(n^2 + 3n + 3) \\ \implies \sum_{k=1}^n k^2 &= \frac{1}{3} \left[n(n^2 + 3n + 3) - \frac{3}{1}n(n+1) - n \right] \\ &= \frac{1}{6}n [2(n^2 + 3n + 3) - 3(n+1) - 2] \\ &= \frac{1}{6}n [2n^2 + 6n + 6 - 3n - 3 - 2] \\ &= \frac{1}{6}n [2n^2 + 3n + 1]\end{aligned}$$

4. find polynomial P_4 of degree 5 such that $\sum_{k=1}^n k^4 = P_4(n)$

let's calculate $P_3(n)$:

$$\begin{aligned}(n+1)^4 - n^4 &= 4n^3 + 6n^2 + 4n + 1 \\ 1^4 - 0 &= 4 * 0^3 + 6 * 0^2 + 4 * 0 + 1 \\ 2^4 - 1^4 &= 4 * 1^3 + 6 * 1^2 + 4 * 1 + 1 \\ 3^4 - 2^4 &= 4 * 2^3 + 6 * 2^2 + 4 * 2 + 1 \\ &\vdots = \vdots \\ (n+1)^4 - n^4 &= 4n^3 + 6n^2 + 4n + 1\end{aligned}$$

$$(n+1)^4 = 4 \sum_{k=0}^n k^3 + 6 \sum_{k=0}^n k^2 + 4 \sum_{k=0}^n k + \sum_{k=0}^n 1$$

$$\begin{aligned}
\implies \sum_{k=0}^n k^3 &= \frac{1}{4} \left[(n+1)^4 - 6 \sum_{k=0}^n k^2 + 4 \sum_{k=0}^n nk - (n+1) \right] \\
&= \frac{1}{4} [(n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - (n+1)] \\
&= \frac{1}{4}(n+1) [(n+1)^3 - n(2n+1) - 2n - 1] \\
&= \frac{1}{4}(n+1) [(n+1)^3 - (2n+1)(n+1)] \\
&= \frac{1}{4}(n+1)^2 [(n+1)^2 - (2n+1)] \\
&= \frac{1}{4}n^2(n+1)^2
\end{aligned}$$

let's calculate $P_4(n)$:

$$\begin{aligned}
(n+1)^5 - n^5 &= 5n^4 + 10n^3 + 10n^2 + 5n + 1 \\
\cancel{\chi} - \emptyset &= 5 * 0^4 + 10 * 0^3 + 10 * 0^2 + 5 * 0 + 1 \\
\cancel{\chi} - \cancel{\chi} &= 5 * 1^4 + 10 * 1^3 + 10 * 1^2 + 5 * 1 + 1 \\
\cancel{\chi} - \cancel{\chi} &= 5 * 2^4 + 10 * 2^3 + 10 * 2^2 + 5 * 2 + 1 \\
&\vdots = \vdots \\
(n+1)^5 - \cancel{\chi} &= 4n^3 + 6n^2 + 4n + 1
\end{aligned}$$

$$(n+1)^5 = 5 \sum_{k=0}^n k^4 + 10 \sum_{k=0}^n k^3 + 10 \sum_{k=0}^n k^2 + 5 \sum_{k=0}^n k + \sum_{k=0}^n 1$$

$$\begin{aligned}
\implies \sum_{k=1}^n k^4 &= \frac{1}{5} \left[(n+1)^5 - 10 \sum_{k=0}^n k^3 - 10 \sum_{k=0}^n k^2 - 5 \sum_{k=0}^n k - \sum_{k=0}^n 1 \right] \\
&= \frac{1}{5} \left[(n+1)^5 - \frac{10}{4}n^2(n+1)^2 - \frac{10}{6}n(n+1)(2n+1) - \frac{5}{2}n(n+1) - (n+1) \right] \\
&= \frac{1}{5}(n+1) \left[(n+1)^4 - \frac{5}{2}n^2(n+1) - \frac{5}{3}n(2n+1) - \frac{5}{2}n - 1 \right] \\
&= \frac{1}{30}(n+1) [6(n+1)^4 - 15n^2(n+1) - 10n(2n+1) - 15n - 6] \\
&= \frac{1}{30}(n+1)(2n+1)(3n^2 + 3n - 1)
\end{aligned}$$

5. shows that in general, we can find a polynomial P of degree p+1 such that $\sum_{k=1}^n k^p = P_p(n)$

By using the binomial expression, we have,

$$\begin{aligned}
(n+1)^{p+1} - n^{p+1} &= \sum_{k=0}^{p+1} C_k^{p+1} n^k 1^{p+1-k} - n^{p+1} \\
&= n^{p+1} + \sum_{k=0}^p C_k^{p+1} n^k - n^{p+1} \\
&= \sum_{k=0}^p C_k^{p+1} n^k \\
&= (p+1)n^p + C_{p-1}^{p+1} n^{p-1} + C_{p-2}^{p+1} n^{p-2} + \dots + 1
\end{aligned}$$

By using conjecture,

$$\begin{aligned}
(n+1)^{p+1} - n^{p+1} &= (p+1)n^p + C_{p-1}^{p+1}n^{p-1} + C_{p-2}^{p+1}n^{p-2} + \dots + 1 \\
\underline{1^{p+1}} - \underline{0^{p+1}} &= (p+1)*0^p + C_{p-1}^{p+1}*0^{p-1} + C_{p-2}^{p+1}*0^{p-2} + \dots + 1 \\
\underline{2^{p+1}} - \underline{1^{p+1}} &= (p+1)*1^p + C_{p-1}^{p+1}*1^{p-1} + C_{p-2}^{p+1}*1^{p-2} + \dots + 1 \\
\underline{3^{p+1}} - \underline{2^{p+1}} &= (p+1)*2^p + C_{p-1}^{p+1}*2^{p-1} + C_{p-2}^{p+1}*2^{p-2} + \dots + 1 \\
&\vdots = \vdots \\
1^{p+1} - 0^{p+1} &= (p+1)n^p + C_{p-1}^{p+1}n^{p-1} + C_{p-2}^{p+1}n^{p-2} + \dots + 1
\end{aligned}$$

$$\begin{aligned}
(n+1)^{p+1} &= (p+1) \sum_{k=1}^n k^p + C_{p-1}^{p+1} \sum_{k=1}^n k^{p-1} + C_{p-2}^{p+1} \sum_{k=1}^n k^{p-2} + \dots + \sum_{k=0}^n 1 \\
&= (p+1)P_p(n) + C_{p-1}^{p+1}P_{p-1}(n) + C_{p-2}^{p+1}P_{p-2}(n) + \dots + (n+1)
\end{aligned}$$

$$\implies P_p(n) = \frac{1}{p+1} \left[(n+1)^{p+1} - (n+1) - \sum_{k=1}^{p-1} C_{p-k}^{p+1} P_{p-k}(n) \right]$$

Expression below established a relationship between $P_p(n)$ and $P_1(n), \dots, P_{p-1}(n)$.

now let's prove that $P_p(n)$ is a polynomial of degree $p+1$. let's prove it by induction

- Initialization: for $p=1$, we have $P_1(n) = \sum_{k=1}^n nk = \frac{1}{2}n(n+1)$
 $Degree(P_1(n)) = 2 = 1 + 1$
The property is true for $p = 1$
- Hypothesis: let's suppose that the degree of $P_p(n)$ is $p+1$ and let's prove that the degree of P_{p+1} is $p+2$.
 $P_{p+1}(n) = \frac{1}{p+2} [(n+1)^{p+2} - (n+1) - \sum_{k=1}^p C_{p+1-k}^{p+2} p + 1 - k P_{p+1-k}(n)]$

we know, by using the hypothesis of induction $Degree(P_{p+1-k}) \leq (p+1), \forall k \in [1, p]$.

thus, the $Degree(P_{p+1}(n)) = Degree((n+1)^{p+2}) = p+2$

by induction, we prove that it is possible to find a polynomial $P_p(n)$ such that his degree is $p+1$

Let's give the answer for $p = 5$,

$$\begin{aligned}
P_5(n) &= \frac{1}{6} \left[(n+1)^6 - (n+1) - \sum_{k=1}^4 C_{5-k}^6 P_{5-k}(n) \right] \\
&= \frac{1}{6} [(n+1)^6 - (n+1) - C_4^6 P_4(n) - C_3^6 P_3(n) - C_2^6 P_2(n) - C_1^6 P_1(n)] \\
&= \frac{1}{6} [(n+1)^6 - (n+1) - 15P_4(n) - 20P_3(n) - 15P_2(n) - 6P_1(n)] \\
&= \frac{1}{6} \left[(n+1)^6 - (n+1) - \frac{1}{2}(n+1)(2n+1)(3n^2+3n-1) - 5n^2(n+1)^2 - \frac{5}{2}n(n+1)(2n+1) - 3n(n+1) \right] \\
&= \frac{1}{6}(n+1) \left[(n+1)^5 - 1 - \frac{1}{2}(2n+1)(3n^2+3n-1) - 5n^2(n+1) - \frac{5}{2}n(2n+1) - 3n \right] \\
&= \frac{1}{12}(n+1) [2(n+1)^5 - 2 - (2n+1)(3n^2+3n-1) - 10n^2(n+1) - 5n(2n+1) - 6n] \\
&= \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)
\end{aligned}$$