Numerical Linear Algebra: Assignment 1

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Exercise 1

Write a nonrecursive algorithm in the spirit of row-oriented forward substitution that implements column-oriented forward substitution. Use a single array that contains b initially, stores intermediate results (e.g. $\hat{b}$,b) during the computation, and contains y at the end. Use your algorithm to write a Julia program that solves the system Ax = b where

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 2 & -4 & 0 \\ 1 & 2 & 3 \end{pmatrix}, B = \begin{pmatrix} -15 \\ -2 \\ 10 \end{pmatrix}$$

```
function non_recursive_column_oriented_forward(A, b)
    n = length(b)
    y = copy(b)
    for i in 1:n
        for j in 1:i-1
            y[i] = y[i] - A[i,j]y[j]
        end
        y[i] = y[i]/A[i,i]
    end
    return y
end

A = [5 0 0; 2 -4 0; 1 2 3]
b = Float64[-15 2 10]'
non_recursive_column_oriented_forward(A, b)
```

Exercise 2

Consider a system of n carts connected by springs. The stiffness of the i-th spring is k_i newtons/meter. Each cart experiences a steady force f_i newtons, causing displacements x_i meters.

(a) Write down a system of n linear equations Ax = b that could be solved for $x_, \dots, x_n$. Notice that if n is at all large, the vast majority of the entries of A will be zeros. Matrices with this property are called sparse. Since all of the nonzeros are confined to a narrow band around the main diagonal, A is also called banded. In particular, the nonzeros are confined to three diagonals, so A is tridiagonal.

- Formulation of the System:
  The force on each cart can be expressed as:
  $-k_{i-1}(x_i - x_{i-1}) + k_i(x_{i+1} - x_i) = f_i$
  Assuming all springs have uniform stiffness $k_i = 1$, the equation simplifies to:
  $-x_{i-1} + 2x_i - x_{i+1} = f_i$
  This leads to the matrix equation:
  $Ax = b$
  where:

A is a tridiagonal matrix defined as: $A = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & \\ -1 & 2 & -1 & 0 & \cdots & 0 & \\ 0 & -1 & 2 & -1 & \cdots & 0 & \\ 0 & 0 & -1 & 2 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$

b is the force vector given by:

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) Computation for $n = 20$

For $n = 20$, where $k_i = 1$ for all i, and forces $f_5 = 1$ newton and $f_{16} = -1$ newton, we can compute the displacement vector x.

```
using ToeplitzMatrices
using LinearAlgebra

# Number of carts
n = 20

# Create the tridiagonal matrix A
A = Tridiagonal(ones(n-1), 2 * ones(n), ones(n-1))

# Create the force vector b
b = zeros(n)
b[5] = 1.0    # f5 = 1
b[16] = -1.0  # f16 = -1

# Solve the system Ax = b
x = A \ b

# Round the displacement vector x in order to get the exact soltuion of our system
x_rounded = round.(x)

# Output the results
println("The displacement vector x of our system is : ", x_rounded)
```

Exercise 3

(a) The conductance C of a resistor (in siemens) is equal to the reciprocal of the resistance: $C = \frac{1}{R}$. Show that if two nodes with voltages x_i and x_j are connected by a resistor with conductance C, the power drawn by the resistor is $C(x_i - x_j)^2$, which can also be expressed as

$A = \begin{bmatrix} x_i, & X_j \end{bmatrix} \begin{bmatrix} C & -C \\ -C & C \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix}$

For which condition A is positive?

to answer to this question we need to check if $A = A^T$ (Symmetric: The matrix must be symmetric) and A is positive :

- Symmetric:
  $\begin{bmatrix} C & -C \\ -C & C \end{bmatrix} = \begin{bmatrix} C & -C \\ -C & C \end{bmatrix}^T$
- Quadratic form: A is positive if its quadractic form pistive :
  $A = C(x_i - x_j)^2 \geq 0 \implies C \geq 0$

A is positive when $C \geq 0$

(b) Consider the circuit in Figure 2. Show that the power drawn by this circuit is a sum of terms of the form $C(x_i - x_j)^2$ and is positive unless all of the nodal voltages $x_1, \ldots, x_6$ are equal.

we know that the power draw by the circuit is the sum of the power drawn by each resistor

- Power of the ones sides:
  let's find the power draw by each resistor :

  $$P = RI^2 = R(\frac{U}{R})^2 = \frac{1}{R}U^2$$

  and we know that U is equal do the difference between two born:

  $$P_{i,j} = C_{i,j}(x_i - x_j)^2$$

  and the total power is $P = \sum_{i,j} C_{i,j}(x_i - x_j)^2$
  where (i, j) denotes pairs of nodes connected by resistors with conductance C_{ij}.
- Formulation of the Power Expression:
  We can represent the power as a quadratic form. Define a vector $\mathbf{x}$ of nodal voltages:
  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$
  The total power can be represented in matrix form as:
  $P = \mathbf{x}^T \mathbf{K} \mathbf{x}$
  where $\mathbf{K}$ is a symmetric matrix representing the conductance between nodes. The entries of $\mathbf{K}$ will be:
  - $K_{ii} = \sum_{j \text{ connected to } i} C_{ij}$
  - and $K_{ij} = -C_{ij}$ if nodes i and j are connected.
- Positive Definiteness of the Matrix:
  The matrix $\mathbf{K}$ must be positive semi-definite because:
  - Each term $C(x_i - x_j)^2$ is non-negative since $C > 0$.
  - The total power P is a sum of non-negative terms, thus $P \geq 0$.

- Condition for $P = 0$

  The total power P is equal to zero if and only if:

  $\mathbf{x}^T\mathbf{K}\mathbf{x} = 0.$

  This occurs if all terms $C(x_i - x_j)^2 = 0$ for every connected pair (i, j). The only way for each square term to be zero is if:

  $x_i - x_j = 0 \quad \text{for all } i, j \quad \Rightarrow \quad x_1 = x_2 = x_3 = x_4 = x_5 = x_6.$

(c) Show that the power can be expressed as $x^T Hx$, where H is a 6 x 6 symmetric matrix. This matrix is positive semi-definite, as x $x^T Hx \geq 0$ for all x.

To show that the total power drawn by the circuit can be expressed as $P_{\text{total}} = x^T Hx$, where H is a 6×6 symmetric matrix, we proceed as follows:

- Constructing the Matrix H:

  The matrix H is a Laplacian matrix derived from the conductance in the circuit. Each element H_{ij} is defined as follows:

  - Diagonal Entries: H_{ii} is the sum of conductance connected to node i.
  - Off-Diagonal Entries: For $i \neq j$, $H_{ij} = -C_{ij}$, where C_{ij} is the conductance between nodes i and j. If there is no resistor between nodes i and j, then $H_{ij} = 0$.

- Conductance Values from the Circuit:

  The circuit in Figure 2 provides the following resistances (in ohms) and corresponding conductance:

$$
\begin{aligned}
C_{12} &= \frac{1}{1} = 1, & C_{13} &= \frac{1}{2} = 0.5, \\
C_{23} &= \frac{1}{1} = 1, & C_{34} &= \frac{1}{5} = 0.2, \\
C_{45} &= \frac{1}{1} = 1, & C_{46} &= \frac{1}{2} = 0.5.
\end{aligned}
$$

- Matrix H:

  Using these values, we construct H as follows: $H = \begin{bmatrix} 1.5 & -1 & -0.5 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -0.5 & -1 & 1.7 & -0.2 & 0 & 0 \\ 0 & 0 & -0.2 & 1.7 & -1 & -0.5 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & 0.5 \end{bmatrix}.$

- Explanation of Each Entry in H: firstly we need to compute the diagonal entry after compute the no diagonale entry :

  - Diagonal Entries:

$$
\begin{aligned}
H_{11} &= 1 + 0.5 = 1.5, \\
H_{22} &= 1 + 1 = 2, \\
H_{33} &= 0.5 + 1 + 0.2 = 1.7, \\
H_{44} &= 0.2 + 1 + 0.5 = 1.7, \\
H_{55} &= 1, \\
H_{66} &= 0.5.
\end{aligned}
$$

  - Off-Diagonal Entries:

$$
\begin{aligned}
H_{12} &= H_{21} = -1, \\
H_{13} &= H_{31} = -0.5, \\
H_{23} &= H_{32} = -1, \\
H_{34} &= H_{43} = -0.2, \\
H_{45} &= H_{54} = -1, \\
H_{46} &= H_{64} = -0.5.
\end{aligned}
$$

So, the matrix H is symmetric and positive semi-definite, and it represents the power dissipation in the circuit. Therefore, we can express the total power as: $P_{\text{total}} = x^T Hx$, where $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_6 \end{bmatrix}^T$ represents the voltages at each node in the circuit.

(d) Derive the linear system ($Ax = b$) for the circuit in Figure 2. Show that A is a submatrix of the matrix H from part (c). Deduce that A is positive definite. Use Julia to find a Cholesky factor of A.

In this part, we derive the matrix A for the system of linear equations $Ax = b$ by applying Kirchhoff's Current Law (KCL) at each node of the circuit, as shown in Figure 2.

- Applying Kirchhoff's Current Law (KCL)

  Kirchhoff's Current Law states that the sum of currents entering a node must be equal to the sum of currents leaving that node. In terms of voltages and conductance, we can express this for each node in the circuit.

  Let x_i denote the voltage at node i. The current flowing through a resistor between nodes i and j is given by $C_{ij}(x_i - x_j)$, where $C_{ij} = \frac{1}{R_{ij}}$ is the conductance of the resistor between nodes i and j.

  Using KCL for each node in Figure 2, we obtain the following equations:

  - Node x_1: $C_{12}(x_1 - x_2) + C_{13}(x_1 - x_3) = 0$
  - Node x_2: $C_{21}(x_2 - x_1) + C_{23}(x_2 - x_3) = 0$
  - Node x_3: $C_{31}(x_3 - x_1) + C_{32}(x_3 - x_2) + C_{34}(x_3 - x_4) = 0$
  - Node x_4: $C_{43}(x_4 - x_3) + C_{45}(x_4 - x_5) + C_{46}(x_4 - x_6) = 0$
  - Node x_5: $C_{54}(x_5 - x_4) = 6\text{V}$
  - Node x_6: $C_{64}(x_6 - x_4) = 0$

  Nodes x_5 and x_6 are connected to voltage sources, so we treat these as known values in vector b.

- Formulating the Matrix A

  From the above equations, we can rewrite the system in matrix form as $Ax = b$, where:

  - A is a submatrix of the Laplacian matrix H, excluding the rows and columns corresponding to nodes connected directly to voltage sources (x_5 and x_6.
  - The entries of A represent the conductance relationships between the nodes x_1 to x_4, where each diagonal entry A_{ii} is the sum of conductance connected to node i, and each off-diagonal entry A_{ij} is $-C_{ij}$ if there is a resistor between nodes i and j.

  Thus, A is given by: $A = \begin{bmatrix} 1.5 & -1 & -0.5 & 0 \\ -1 & 2 & -1 & 0 \\ -0.5 & -1 & 1.7 & -0.2 \\ 0 & 0 & -0.2 & 1.7 \end{bmatrix}$. And b is: $b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}$.

- Show that A is a submatrix of the matrix H from part (c).

  In part (c), we derived the symmetric matrix H based on Kirchhoff's Current Law (KCL) for all nodes x_1 through z_6 in the circuit. The matrix H is given by:

- Deduce that A is positive definite A

  Since A is a submatrix of the positive semi-definite matrix H and it corresponds to a reduced circuit with grounded nodes (due to the voltage sources), A is positive definite. This property guarantees that the system $Ax = b$ has a unique solution for x.

- Cholesky Factorization of A

  To solve $Ax = b$ efficiently, we can use the Cholesky decomposition of A. The Cholesky factorization expresses A as: $A = LL^T$, where L is a lower triangular matrix.

  In Julia, we can compute the Cholesky factorization as follows:

```
using LinearAlgebra

# Define matrix A
A = [
    1.5  -1   -0.5   0;
   -1     2   -1     0;
   -0.5  -1    1.7  -0.2;
    0     0   -0.2   1.7
]
```

```
# Perform Cholesky factorization
L = cholesky(A).L

# Display the result
println("The Cholesky factor L of A is:")
println(L)
```

The output will provide the Cholesky factor L such that $A = LL^T$, which can be used for efficient solving of the system $Ax = b$ using forward and backward substitution.

Exercise 5

Write your own Julia program that solves positive definite systems $Ax = b$ by calling subroutines to (a) calculate the Cholesky factor, (b) perform forward substitution, and (c) perform back substitution. Try out your program on the following problems.

i Find the exact solution of the problem (1) and (2). (with c = 1, d = 2, and f(x)= x).

our equation is $-U^{''}(x) + U^{'}(x) + 2U(x) = x$

In order to answer to this question let's find the general solution of homogenesis differential equation and after find the particular solution :

- Gene rale solution:

$$-U^{''}(x) + U^{'}(x) + 2U(x) = 0$$

and the question characteristic of this previous differential equation is :

$$-r^2 + r + 2 = 0$$

$d = (1) + 4(-1)(2) = 1 + 8 = 9$
$r_1 = \frac{-1+3}{2(-1)} = -1$ and $r_2 = \frac{-1-3}{2(-1)} = 2$
so, the general solution is $U(x) = C_1 e^{-x} + C_2 e^{2x}$

- particular solution: so f is a polynomial function of degree 1 that means our particular solution is $U_p(x) = Ax + B \begin{cases} U_p^{'}(x) = A \\ U_p^{''}(x) = 0 \end{cases}$ so i have $-0 + A + 2(Ax + B) = x \implies \begin{cases} 2A = 1 \\ A + 2B = 0 \end{cases} \implies \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{4} \end{cases}$

so the solution of our differential equation is :
$U(x) = C_1 e^{-x} + C_2 e^{2x} + \frac{1}{2}x - \frac{1}{4}$ let's find C_1 and C_2 using the initial condition
$\begin{cases} u(0) = 0 \\ U(1) = 0 \end{cases} \implies \begin{cases} C_1 + C_2 - \frac{1}{4} = 0 \\ C_1 e^{-1} + C_2 e^2 + \frac{1}{2} - \frac{1}{4} = 0 \end{cases} \implies \begin{cases} C_1 + C_2 = \frac{1}{4} \\ C_1 e^{-1} + C_2 e^2 = -\frac{1}{4} \end{cases} \implies \begin{cases} C_1 = \frac{1}{4} - C_2 \quad (1) \\ C_1 e^{-1} + C_2 e^2 = -\frac{1}{4} \quad (2) \end{cases}$

(1) and (2) we have $(\frac{1}{4} - C_2)e^{-1} + C_2 e^2 = -\frac{1}{4} \implies C_2 = \dfrac{-\frac{1}{4} - \frac{1}{4}e^{-1}}{-e^{-1} + e^2}$

$C_1 = \dfrac{1}{4} - \dfrac{-\frac{1}{4} - \frac{1}{4}e^{-}1}{-e^{-1} + e^2}$

so the solution is $U(x) = C_1 e^{-x} + C_2 e^{2x} + \dfrac{1}{2}x - \dfrac{1}{4}$ with $C_1 = \frac{1}{4} - \frac{-\frac{1}{4} - \frac{1}{4}e^{-}1}{-e^{-1}+e^2} = \frac{1}{4} + \frac{(1+e^{-1})}{4(e^2 - e^{-1})}$ and

$C_2 = \dfrac{-\frac{1}{4} - \frac{1}{4}e^{-1}}{-e^{-1} + e^2} = -\dfrac{(1 + e^{-1})}{4(e^2 - e^{-1})}$

ii Write the system of linear equations Ax = b for (a) m = 6, (b) m = 8, and (c) m = 20, give A and b in each case. The matrix A is a tridiagonal matrix (the nonzero elements are on the main diagonal and on the line above and below the main diagonal). In Julia, use the Tridiagonal function to learn an easy way to enter coefficients matrix A. Use these values c = 1, d = 2, and f(x) = x. (create a small function for f(x)).

let's write the equation of our system :

$$
\frac{-U_{i+1}+2U_i-U_{i-1}}{h^2}+\frac{U_{i+1}-U_{i-1}}{2h}+2U_i=x_i
$$
$$
\implies (-\frac{1}{h^2}+\frac{1}{2h})U_{i+1}+(\frac{2}{h^2}+2)U_i+(\frac{1}{h^2}-\frac{1}{2h})U_{i-1}=x_i
$$

so based on the previous result the element in main diagonal is $\frac{2}{h^2}+2$ and the element in bellow main diagonal is $(\frac{1}{h^2}-\frac{1}{2h})$ and the element in above the diagonal is $(-\frac{1}{h^2}+\frac{1}{2h})$

- for m = 6 ,
  $h=\frac{1}{6}$, $\frac{2}{h^2}+2=74$, $(\frac{1}{h^2}-\frac{1}{2h})=33$ and $(-\frac{1}{h^2}+\frac{1}{2h})=-33$
  $$
  A=\begin{pmatrix}74 & -33 & 0 & 0 & 0\\ 33 & 74 & -33 & 0 & 0\\ 0 & 33 & 74 & -33 & 0\\ 0 & 0 & 33 & 74 & -33\\ 0 & 0 & 0 & 33 & 74\end{pmatrix} \text{ and } b=\begin{pmatrix}\frac{1}{6}\\ \frac{1}{3}\\ \frac{1}{2}\\ \frac{2}{3}\\ \frac{5}{6}\end{pmatrix}
  $$
- for m = 8 ,
  $h=\frac{1}{8}$, $\frac{2}{h^2}+2=128$, $(\frac{1}{h^2}-\frac{1}{2h})=60$ and $(-\frac{1}{h^2}+\frac{1}{2h})=-60$
  $$
  A=\begin{pmatrix}128 & -60 & 0 & 0 & 0 & 0 & 0\\ 60 & 128 & -60 & 0 & 0 & 0 & 0\\ 0 & 60 & 128 & -60 & 0 & 0 & 0\\ 0 & 0 & 60 & 128 & -60 & 0 & 0\\ 0 & 0 & 0 & 60 & 128 & -60 & 0\\ 0 & 0 & 0 & 0 & 60 & 128 & -60\\ 0 & 0 & 0 & 0 & 0 & 60 & 128\end{pmatrix} \text{ and } b=\begin{pmatrix}\frac{1}{8}\\ \frac{1}{4}\\ \frac{3}{8}\\ \frac{1}{2}\\ \frac{5}{8}\\ \frac{3}{4}\\ \frac{7}{8}\end{pmatrix}
  $$
- for m = 20 ,
  $h=\frac{1}{20}$, $\frac{2}{h^2}+2=802$, $(\frac{1}{h^2}-\frac{1}{2h})=410$ and $(-\frac{1}{h^2}+\frac{1}{2h})=-390$
  $$
  A=\begin{pmatrix}802 & -390 & 0 & 0 & 0 & 0 & 0\ldots & \\ 410 & 802 & -390 & 0 & 0 & 0 & 0\ldots & \\ 0 & 410 & 802 & -390 & 0 & 0 & 0\ldots & \\ 0 & 0 & 410 & 802 & -390 & 0 & 0\ldots & \\ 0 & 0 & 0 & 410 & 802 & -390 & 0\ldots & \\ 0 & 0 & 0 & 0 & 410 & 802 & -390\ldots & \\ 0 & 0 & 0 & 0 & 0 & 410 & 802\ldots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{pmatrix} \text{ and } b=\begin{pmatrix}\frac{1}{20}\\ \frac{2}{20}\\ \frac{3}{20}\\ \frac{4}{20}\\ \frac{5}{20}\\ \frac{6}{20}\\ \frac{7}{20}\\ \frac{8}{20}\\ \frac{9}{20}\\ \frac{10}{20}\\ \frac{11}{20}\\ \frac{12}{20}\\ \frac{13}{20}\\ \frac{14}{20}\\ \frac{15}{20}\\ \frac{16}{20}\\ \frac{17}{20}\\ \frac{18}{20}\\ \frac{19}{20}\end{pmatrix}
  $$

iii Solve the system Ax = b for each case and plot all the 3 approximate solutions and the exact solution found in (i) in the same plot (plot of u against x). Comment and discuss what you see in the plot. You have to install the package Plots in Julia. (use the 2 functions plot and plot!)

```
using LinearAlgebra
using Plots

function exact_solution(x)
    C2 = -(1+exp(-1))/(4(exp(2) - exp(-1)))
    C1 = (1/4) - C2
```

```
    return C1*exp(-x) + C2*exp(2*x) + 0.5*x -1/4
end

function solve_finite_difference(m)
    h = 1.0 / m
    x = range(h, stop=1-h, length=m-1)  # Interior points x_i

    # Define the diagonals of matrix A

    main_diag = fill((2/h^2 + 2), m-1)
    sub_diag = fill((-1/h^2 - 1/2*h), m-2)
    super_diag = fill(-1/h^2 + 1/2*h, m-2)

    # Construct the tridiagonal matrix A
    A = Tridiagonal(sub_diag, main_diag, super_diag)

    # Define the vector f
    f = collect(x)  # f_i = x_i

    # Solve the linear system A * u = f
    u = A \ f

    # Return the solution with boundary values added
    return [0.0; u; 0.0], [0.0; f; 1.0]#
end

f = range(0, stop=1, length=10)
# Solve for m = 6, m = 8, and m = 20
u6, f6 = solve_finite_difference(6)
println(u6, f6)
u8, f8 = solve_finite_difference(8)
u20, f20 = solve_finite_difference(20)


p = plot(f, map(exact_solution, f), size = (1000, 500), title = "Solution of Differentiel equation",
plot!(p, f6, u6, label = "Approximation solution m=6")
plot!(p, f8, u8, label = "Approximation solution m=8")
plot!(p, f20, u20, label = "Approximation solution m=20")
xlabel!("x")
ylabel!("U")
```