Mathematic Problem Solving Assignment 2

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September 14, 2024

Problem 7:

Let n be a positive integer. Suppose you have P_n points in the space. Each pair of points is connected by a line, and each line is colored with one of n different colors. Find the least value of P_n so that there is at least one triangle with sides of the same color.

Solution

let's try for each value of n :

for $n = 1$, the best possible value P_1 can be taken it is 3 because with three we have one Monochromatic triangle with the same color(one color) because we have one color(n=1)

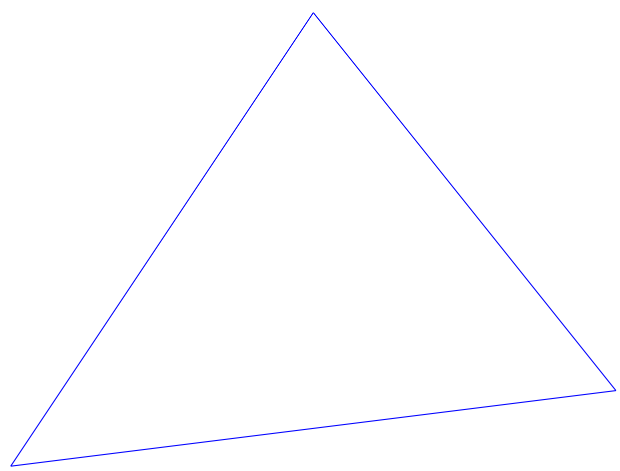

Figure 1: Monochromatic triangle for n=1 and $P_1 = 3$

Now let us take the case where n=2, find the first P_2 such that we can find the monochromatic triangle whatever the arrangement of the two colors

- for $p_2 = 4$ we found a case such that we are not able to find a monochromatic triangle (for this case we choose two colors: red and blue):

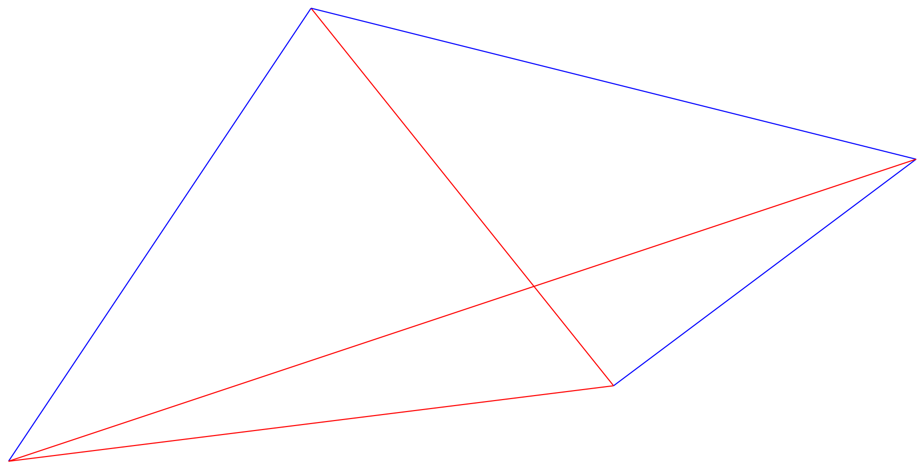

Figure 2: for n=2 and $P_2 = 4$

- Let's try with $p_2 = 5$ we found one case such that we aren't able to find one Monochromatic triangle (for this case, we choose two colors: red and blue):

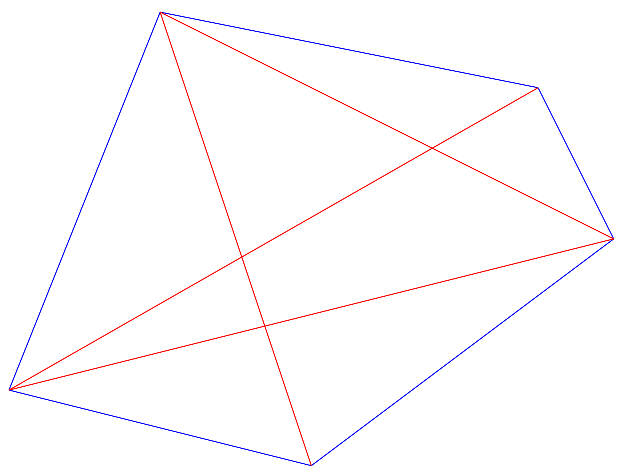

Figure 3: for n=2 and $P_2 = 4$

- Let's try with $p_2 = 6$ we found it's always possible to find a Monochromatic triangle independent of the distribution or arrangement of colors. So,

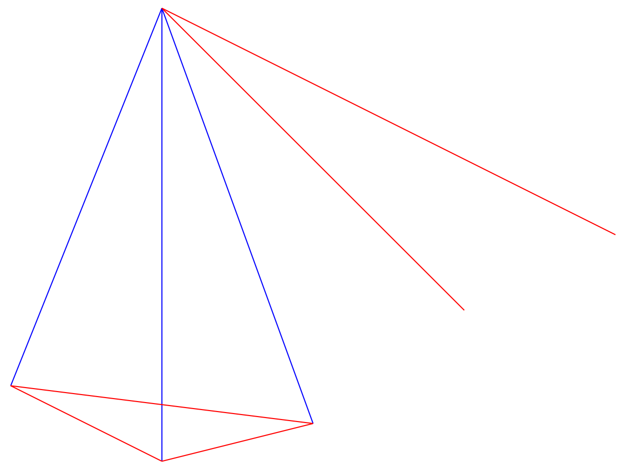

Figure 4: for n=2 and $p_2 = 6$

Based on the graphic below, we can affirm that it's always possible to get a Monochromatic triangle independent of the distribution or arrangement of colors. What strategy do we use to distribute colors? We also fix one point and match it with three points with the same colors (blue) and two points with another color (red) and follow the same strategy for the first three colors. We show that, for, $p_2 = 6$ it is always possible to find a monochromatic triangle independent of distribution or arrangement of colors.

Now let's find the relationship between P_1 and P_2:

$$P_2 = (1+1)P_1 - 1 + 1$$

Let's find P_3 based on the previous conjecture, we can affirm that:

$$\begin{aligned} P_3 &= (1+2)P_2 - 2 + 1 \\ &= 3 * 6 - 1 \\ &= 17 \end{aligned}$$

Now let's write the relation between P_{n+1} and P_n :

$$P_{n+1} = (1+n)P_n - n + 1$$

Let's prove that $P_n = \lceil n!e \rceil + 1$

$$\begin{aligned} P_{n+1} &= (n+1)P_n - n + 1 \\ P_{n+1} - 1 &= (n+1)P_n - n \\ P_{n+1} - 1 &= (n+1)P_n - n - 1 + 1 \\ &= (n+1)P_n - (n+1) + 1 \\ &= (n+1)(P_n - 1) + 1 \end{aligned}$$

Let's denote $A_n = P_n - 1$, now we have $A_{n+1} = (n+1)A_n + 1$

$$
\begin{aligned}
A_{n+1} &= A_n + 1 \\
\frac{A_{n+1}}{(n+1)!} &= \frac{(n+1)A_n}{(n+1)!} + \frac{1}{(n+1)!} \\
\frac{A_{n+1}}{(n+1)!} &= \frac{A_n}{n!} + \frac{1}{(n+1)!}
\end{aligned}
$$

Let's denote by $B_n = \frac{A_n}{n!}$, now we have $B_{n+1} = B_n + \frac{1}{(n+1)!}$ and by using conjecture we can express B_n as :

$$
\begin{aligned}
\cancel{B_2} - B_1 &= \frac{1}{2!} \\
\cancel{B_3} - \cancel{B_2} &= \frac{1}{3!} \\
\cancel{B_4} - \cancel{B_3} &= \frac{1}{4!} \\
\vdots &= \vdots \\
B_n - \cancel{B_{n-1}} &= \frac{1}{n!} \\
\hline
B_n - B_1 &= \sum_{k=2}^{n} \frac{1}{k!}
\end{aligned}
$$

$$
\begin{aligned}
\implies B_n &= b_1 + \sum_{k=2}^{n} \frac{1}{k!} \\
&= 2 + \sum_{k=2}^{n} \frac{1}{k!} \\
&= \frac{1}{0!} + \frac{1}{1!} + \sum_{k=2}^{n} \frac{1}{k!} \\
&= \sum_{k=0}^{n} \frac{1}{k!} \\
\frac{A_n}{n!} &= \sum_{k=0}^{n} \frac{1}{k!} \\
\implies A_n &= n! \left(\sum_{k=0}^{n} \frac{1}{k!} \right) \\
A_n &= n! \left(\sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=n+1}^{\infty} \frac{1}{k!} \right) \\
&= n! \left(\sum_{k=0}^{\infty} \frac{1}{k!} \right) - n! \left(\sum_{k=n+1}^{\infty} \frac{1}{k!} \right) \\
&= n!e - n! \left(\sum_{k=n+1}^{\infty} \frac{1}{k!} \right) with \quad e = \sum_{k=0}^{\infty} \frac{1}{k!} \\
A_n &= n!e - \left(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots \right)
\end{aligned}
$$

Based on bellow expression, we can assume that $An \leq n!e$ and $n!e - f(n) \leq A_n \leq n!e$ with $f(n) = \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$

Let's prove that $0 < f(n) < 1$

$f(n)$ is a decreasing series, so $f(n) < f(1)$

$$
\begin{aligned}
f(1) &= \frac{1}{(2)} + \frac{1}{2*3} + \frac{1}{(2*3*4} + \dots \\
&= \frac{1}{(2!)} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!} + \dots \\
&= \sum_{k=2}^{\infty} \frac{1}{k} \\
&= -2 + 2 + \sum_{k=2}^{\infty} \frac{1}{k} \\
&= -2 + \frac{1}{0!} + \frac{1}{1!} + \sum_{k=2}^{\infty} \frac{1}{k} \\
&= -2 + \sum_{k=0}^{\infty} \frac{1}{k} \\
f(1) &= -2 + e < 1
\end{aligned}
$$

Based on the previous resultant, we can affirm that:

$$
\begin{aligned}
n!e - f(n) \leq A_n &\leq n!e \\
n!e - 1 \leq A_n &\leq n!e \quad \textit{because} \quad f(n) < f(1) < e - 2 < 1 \\
n!e - 1 \leq P_n - 1 &\leq n!e \\
n!e \leq P_n &\leq n!e + 1
\end{aligned}
$$

by this inequality, we have just shown that: $P_n = \lceil n!e \rceil + 1$