Numerical Linear Algebra: Assignment 1

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Question 1

Let$A \in \mathbb{R}^{n\times n}$. We will show that the following conditions are equivalent:

1. $\text{rank}(A) = n$,
2. $\text{range}(A) = \mathbb{R}^n$,
3. $\text{null}(A) = \{0\}$,
4. 0 is not an eigenvalue of$A$,
5. 0 is not a singular value of$A$,
6. $\det(A) \neq 0$.

In order to answer to this question, let's prove that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 1$.

1. **If** A has an inverseA^{-1} **then** $\text{rank}(A) = n$:
   - If A is invertible, then the column vectors of$A$ span $\mathbb{R}^n$. Thus, the rank of$A$ must be$n$.
2. **If** $\text{rank}(A) = n$ **then** $\text{range}(A) = \mathbb{R}^n$:
   - If the rank of A is n, then the dimension of the range (column space) is also n, meaning that the range of A spans $\mathbb{R}^n$.
3. **If** $\text{range}(A) = \mathbb{R}^n$ **then** $\text{null}(A) = \{0\}$:
   - By the Rank-Nullity Theorem, we have $\text{rank}(A) + \text{null}(A) = dim(\mathbb{R}^n) = n$. Since $\text{range}(A) = \mathbb{R}^n$ implies $\text{rank}(A) = n$, we find $\text{null}(A) = 0$.
4. **If** $\text{null}(A) = \{0\}$ **then** 0 is not an eigenvalue of$A$:
   - If the null space is just $\{0\}$, the only solution to$Ax = 0$ is$x = 0$, meaning0 cannot be an eigenvalue.
5. **If** 0 is not an eigenvalue of A **then** 0 is not a singular value of A:
   - Singular values are the square roots of the eigenvalues of A^TA. If 0 is not an eigenvalue of A, it cannot be an eigenvalue of A^TA either.
6. **If** 0 is not a singular value of A **then** $\det(A) \neq 0$:
   - A matrix is non-singular if and only if its determinant is non-zero.
7. **If** $\det(A) \neq 0$ **then** A has an inverse A^{-1}:
   - A fundamental property of determinants states that a matrix is invertible if and only if its determinant is non-zero.

$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 1$.
Consequently, all seven conditions are equivalent.

Question 2

Show that if a matrix A is both triangular and unitary, then it is diagonal.

To show that a matrix A that is both triangular and unitary must be diagonal, we first define what we mean by triangular and unitary.

1. **Triangular Matrix**: A matrix A is upper triangular if all entries below the main diagonal are zero, i.e., $a_{ij} = 0$ for $i > j$. It is lower triangular if all entries above the main diagonal are zero.

2. **Unitary Matrix**: A matrix A is unitary if $A^*A = I$, where A^* is the conjugate transpose of A and I is the identity matrix.

Assume A is an upper triangular unitary matrix.

Step 1: Structure of A

Since A is upper triangular, we can write it in the form: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$

Step 2: Unitary Condition

Since A is unitary, we have: $A^*A = I$ Calculating A^*, we find: $A^* = \begin{pmatrix} \overline{a_{11}} & 0 & 0 & \cdots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & 0 & \cdots & 0 \\ \overline{a_{13}} & \overline{a_{23}} & \overline{a_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \overline{a_{1n}} & \overline{a_{2n}} & \overline{a_{3n}} & \cdots & \overline{a_{nn}} \end{pmatrix}$

Step 3: Compute A^*A

Now, we compute A^*A: $A^*A = \begin{pmatrix} \overline{a_{11}} & 0 & 0 & \cdots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & 0 & \cdots & 0 \\ \overline{a_{13}} & \overline{a_{23}} & \overline{a_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \overline{a_{1n}} & \overline{a_{2n}} & \overline{a_{3n}} & \cdots & \overline{a_{nn}} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$

The entry in the (i, j)-th position of A^*A is given by:

$(A^*A)_{ij} = \sum_{k=1}^{n} \overline{a_{ik}} a_{kj}$

For $i = j$ (diagonal entries): $(A^*A)_{ii} = \sum_{k=1}^{n} \overline{a_{ik}} a_{ki} = |a_{ii}|^2$ Since A is unitary, we have $|a_{ii}|^2 = 1$ for all i.

For $i \neq j$ (off-diagonal entries): $(A^*A)_{ij} = \sum_{k=1}^{n} \overline{a_{ik}} a_{kj}$

If $i < j$, then $a_{kj} = 0$ for all $k < j$ (due to upper triangularity), leading to:

$(A^*A)_{ij} = 0$

Similarly, if $i > j$, then $a_{ik} = 0$ for all $k \geq i$, also leading to: $(A^*A)_{ij} = 0$

Conclusion

Thus, $A^*A = I$ implies: $|a_{ii}|^2 = 1 \quad \text{for all } i \quad \text{and} \quad a_{ij} = 0 \quad \text{for all } i \neq j$. This shows that A must be a diagonal matrix with $|a_{ii}| = 1$. Therefore, we conclude:

If a matrix A is both triangular and unitary, then it is diagonal.

Question 3

Let $S \in \mathbb{C}^{n*n}$ be skew-hermitian, i.e. $S^* = -S$.

1. Show by using the previous exercise that the eigenvalues of S are pure imaginary.

   let's denote by λ an eigenvalues of S and v its corresponding eigenvectors, so we have: $Sv = \lambda v$

$$\begin{aligned}
(Sv)^* &= (\lambda v)^* \\
\implies v^* S^* &= \lambda^* v^* \\
\implies v^* - S &= \lambda^* v^* \\
\implies v^* - Sv &= \lambda^* v^* v \\
\implies v^* - \lambda v &= \lambda^* v^* v \\
\implies -\lambda &= \frac{\lambda^* v^* v}{v^* v} \\
\implies -\lambda &= \lambda^* \\
\implies -a - b.i &= a - b.i \\
\implies a &= 0
\end{aligned}$$

   so we have $-\lambda = \lambda^* \implies \lambda$ is pure imaginary

2. Show that $I - S$ is nonsingular

   $I - S$ is nonsingular, then 0 cannot be an eigenvalue. let's prove this by contradiction, let's suppose that 0 is a eigenvalue of $I - S$

   we have $(I - S)v = 0 * v = 0 \implies Sv = v$

   we know that v is a eigenvector of S and its corresponding eigenvalue is $\Lambda = 1$, or we previously prove that λ is pure imaginary(this is contradiction)

   so $I - S$ is nonsingular

3. Show that the matrix $Q = (I - S)^{-1}(I + S)$, known as the Caley transform of S, is unitary.

   in order to show that let's prove that $Q^* Q = I_n$

$$\begin{aligned}
Q^* Q &= [(I - S)^{-1}(I + S)]^* [(I - S)^{-1}(I + S)] \\
&= [(I + S)^* (I - S)^{-*}][(I - S)^{-1}(I + S)] \\
&= [(I + S^*)(I - S)^{*-}][(I - S)^{-1}(I + S)] \\
&= [(I - S)(I - S)^{*-}][(I - S)^{-1}(I + S)] \\
&= [(I - S)(I - S)^{-*}][(I - S)^{-1}(I + S)] \\
&= [(I - S)(I - S^*)^{-}][(I - S)^{-1}(I + S)] \\
&= (I - S)(I + S)^{-1}(I - S)^{-1}(I + S) \\
&= (I + S)^{-1}(I - S)(I - S)^{-1}(I + S) \\
&= I
\end{aligned}$$

   so the matrix $Q = (I - S)^{-1}(I + S)$ is unitary

Question 4

If u, and v are m-vectors, the matrix $A = I + uv^*$ is known as a rank-one perturbation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I - \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is the null(A)?

if A is singular there are exist A^{-1} such as $A * A^{-1} = I$, let's prove now its inverse is $A^{-1} = I - \alpha uu^*$ and give the expression for *alpha* that A is nonsingular matrix

$$
\begin{aligned}
A^{-1}A &= [I - \alpha uv^*][I + uv^*] \\
&= I + uv^* - \alpha uv^* - \alpha uv^* uv^* \\
&= I - \alpha uv^* - \alpha uv^* uv^* + uv^* \\
&= I + [1-\alpha]uv^* - \alpha uv^* uv^* \\
&= I + [1-\alpha]uv^* - \alpha u(v^*u)v^* \\
&= I + [1 - \alpha v^* u]uv^* = I
\end{aligned}
$$

$\implies 1 - \alpha v^* u \implies \alpha = \frac{1}{v^* u}$

For what u and v is A singular?
A singualr means the determinant is equal to 0

$$
\begin{aligned}
det(A) = det(I - uv^*) &= (1 + v^* u) det(I) \\
&= (1 + v^* u) = 0
\end{aligned}
$$

$\implies v^* u = -1$

If it is singular, what is the null(A)?
the $Null(A) = \forall x, Ax = 0$

$$
\begin{aligned}
Ax &= 0 \\
(I + uv*) &= 0 \\
x + uv^* x &= 0 \\
x &= -uv^* x \\
v^* x &= -v^*(uv^* x) \\
v^* x &= -v^* u(v^* x) \\
v^* x(1 + v^* x) &= 0 \\
v^* x(1 - 1) &= 0
\end{aligned}
$$

$\implies v^* x$ can be any scalar

if we take $x = cv$ where $c \in \mathbb{R}$ $Ax = Acv = (cv + cuv^* v)$

$$
\begin{aligned}
Ax = Acv = cv + cuv^* v &= cv + c(v^* u)^* v \\
&= cv + c(-1)^* v \\
&= cv - cv & = 0
\end{aligned}
$$

That means x = cv is null space of A $Null(A) = span(v)$

$Null(A) = \{\forall x \in \mathbb{R}^m / x = cv, \forall c \in \mathbb{R}\}$

Question 5

Find the singular value decomposition (SVD) of the matrix

$$
B = \frac{1}{15} \begin{pmatrix} 14 & 2 \\ 4 & 22 \\ 16 & 13 \end{pmatrix}
$$

in order to find this we need to express B as $B = U \sum V^T$

- let's compute $\sum$:

  $B^T = \frac{1}{15}\begin{pmatrix} 14 & 2 & 16 \\ 2 & 22 & 13 \end{pmatrix}$

  $B^TB = \frac{1}{15}\begin{pmatrix} 14 & 2 & 16 \\ 2 & 22 & 13 \end{pmatrix} \frac{1}{15}\begin{pmatrix} 14 & 2 \\ 4 & 22 \\ 16 & 13 \end{pmatrix} = \begin{pmatrix} 2.08 & 1.44 \\ 1.44 & 2.92 \end{pmatrix}$

$$
\begin{aligned}
dte(B^TB - \lambda I) = 0 \\
\implies \begin{pmatrix} 2.08-\lambda & 1.44 \\ 1.44 & 2.92-\lambda \end{pmatrix} = 0 \\
\implies (2.08-\lambda)(2.92-\lambda) - 1.44 * 1.44 = 0 \\
\implies \lambda^2 - 5\lambda + 4 = 0 \\
\lambda_1 = 4 \qquad \text{or} \qquad \lambda_2 = 1
\end{aligned}
$$

  $\sum = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

- let's computer V^T

  we know that V is a matrix of eigenvector of B^TB:

  for $\lambda = 4$ we have $(B^TB - 4I)x = 0$

$$
\begin{cases} -1.92x_1 + 1.44x_2 = 0 \\ 1.44x_1 - 1.08x_2 = 0 \end{cases}
$$

  $\implies x_1 = \frac{3}{4}x_2$ and $x = (z_1, x_2) = (\frac{3}{4}x_2, x_2) = x_2(\frac{3}{4}, 1)$

  for $\Lambda = 1$ we have $(B^TB - 1I)x = 0$

$$
\begin{cases} 1.08x_1 + 1.44x_2 = 0 \\ 1.44x_1 + 1.92x_2 = 0 \end{cases}
$$

  $\implies x_1 = -\frac{4}{3}x_2$ and $x = (z_1, x_2) = (-\frac{4}{3}x_2, x_2) = x_2(-\frac{4}{3}, 1)$

  so

  $V = \left(\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}\right)$ and $V^T = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

- let's compute U

  we know that U is a matrix of eigenvector of BB^T

  $BB^T = \frac{1}{15}\begin{pmatrix} 14 & 2 \\ 4 & 22 \\ 16 & 13 \end{pmatrix} \frac{1}{15}\begin{pmatrix} 14 & 4 & 16 \\ 2 & 22 & 13 \end{pmatrix} = \frac{1}{225}\begin{pmatrix} 200 & 100 & 250 \\ 100 & 500 & 350 \\ 250 & 350 & 425 \end{pmatrix}$

  and the matrix of eigenvactor of BB^T is $U = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

the singular value decomposition of B is

$$
B = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}
$$