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Course: Partial Differential Equations
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1. Consider the ODE.

$$u^{''}(x) + u^{'}(x) = g(x) \tag{1}$$

and g(x) some function.

with boundary conditions

$$u^{'}(0) = u(0) = \frac{1}{2}(u^{'}(1) + u(1)) \tag{2}$$

(a) Does a solution necessarily exist, or is there a condition that g(x) must satisfy for existence? Hint: integrate (1) over $x \in [0, 1]$.

$$\begin{aligned}
\int_0^1 \left(u^{''}(x) + u^{'}(x)\right) &= \int_0^1 g(x)dx \\
\left[u^{'}(x) - u^{'}(x)\right]_0^1 + [u(x) - u(x)]_0^1 &= \int_0^1 g(x)dx \\
\left[u^{'}(1) - u^{'}(0)\right] + [u(1) - u(0)] &= \int_0^1 g(x)dx \\
u^{'}(1) + u(1) - u^{'}(0) - u(0) &= \int_0^1 g(x)dx \\
2u^{'}(0) - 2u(0) &= \int_0^1 g(x)dx \\
0 &= \int_0^1 g(x)dx
\end{aligned}$$

The solution of the ODE (1) exist when $\int_0^1 g(x)dx$

(b) Explain why solutions to (1) are not necessarily unique. let's denote by h_h and h_p the homogeneous and particular solution of the ODE, The general solution is :

$$u(x) = u_h(x) + u_p(x),$$

the characteristic solution of the homogeneous part is :

$r^2 + r = 0$ and the solution is $r = 0$ or $r = -1$ Hence, the homogeneous solution of the ODE can be written as :

$$u_h(x) = A + Be^{-x},$$

where A and B are constants that can be determined with the boundary condition. $u_h^{'}(x) = -Be^{-x}$

$$\begin{aligned} u_h^{'}(0) = u_h(0) &= \frac{1}{2}(u_h^{'}(1) + u_h(1)) \\ -B = A + B &= \frac{1}{2}(-Be^1 + A + Be^1) \\ -B = A + B &= \frac{1}{2}A \end{aligned}$$

The above equation has an infinity solution, which that means with boundary conditions is not possible to determine all the constants of the homogeneous part This means our ODE doesn't have a unique solution.
Hence, The actual boundary condition doesn't provide us with sufficient conditions to determine all the constants of the solution which means our solution still has many solutions. So, the solution of our differential equation is not unique.

2. Consider the transport equation with both initial and boundary conditions

$$u_t + 4u_x = 0, t > 0, x > 0 \qquad u(x, 0) = e^{-x^2}, \qquad u(0, t) = \frac{1}{1+t} \tag{3}$$

Use the initial and boundary conditions to determine the solution u(x, t) to (3). Hint: you should use that the solution u is constant along the characteristic lines $\zeta = x - 4t$.

$$\begin{aligned} u_t + 4u_x &= 0 \\ (1, 4).\nabla U &= 0 \end{aligned}$$

$$\begin{aligned} \frac{dx}{dt} &= 4 \\ x &= 4t + \zeta \\ \zeta &= x - 4t \end{aligned}$$

$$u(x, t) = f(\zeta) = f(x - 4t)$$

with the initial condition:

we have :

$$
\begin{aligned}
u(x,0) &= f(x) = e^{-x^2} \\
u(0,t) = f(-4t) &= \frac{t}{1+t} \\
f(t) &= \frac{1}{1-\frac{1}{4}t} \\
f(t) &= \frac{4}{4-t}
\end{aligned}
$$

when $t = 0$,

$$u(x,t) = f(\zeta) = e^{-(x-4t)^2}$$

when $x = 0$,

$$u(x,t) = f(\zeta) = \frac{4}{4-(x-4t)}$$

Using a matching condition at the intersection of these domains ensures a consistent solution. Hence, the solution to the transport equation becomes:

$u(x,t) = \begin{cases} e^{-(x-4t)^2} & \textbf{if} \quad x \geq 4t \\ \frac{4}{4-(x-4t)} & \textbf{if} \quad x < 4t \end{cases}$

3. Consider a mountain range with height given by h(x). Let u(x, t) be the depth of water moving with speed $-h^{'}(x)$ and flux $\phi = -h^{'}(x)u$ across the mountains.

   (a) Derive, using the conservation equation, the PDE
   the conservation equation is given by :

$$u_t + \phi_x = 0$$

   where $u(x,t)$ is the depth of water, $\phi(x,t)$ is the flux of water.
   and known that $\phi = -h^{'}(x)u$, when we substitute in the preview expression:

$$
\begin{aligned}
u_t + (-h^{'}(x)u)_x &= 0 \\
u_t - h^{''}(x)u - h^{'}u_x &= 0
\end{aligned}
$$

   Hence $u_t - h^{'}(x)u_x = h^{''}(x)u$

(b) Explain (for example with a diagram showing some example surfaces h(x)) why it only makes sense for the water speed to be $-h^{'}(x)$ and not $+h^{'}(x)$.

we know that h(x) represents, the height of the mountain range and water naturally moves from higher elevations to lower elevations due to gravity. to answer this question let's take two cases:

$f^{'}(a) = \lim a \rightarrow 0 \frac{f(a+h) - f(a)}{h}$

- when $h^{'}(x) > 0$, means f is an increase function, if we consider ($-h^{'}$ as a speed), means water flows go to the opposite direction of x.
- when $h^{'}(x) < 0$ means f is a decrease function, which means water flows go the same direction with x

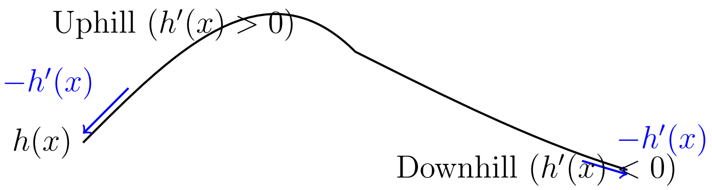

Figure 1: Example of situation

(c) Explain the physical meaning of the term on the RHS of (4). Hint: think about $sign(h^{''})$.

In Order to explain this, the $sign(h^{''})$ tell us about if the function h(x) is concave or convex function

- when $h^{''}(x) > 0$, means h(x) is concave (the slope of the mountain increases) if $h^{''}(x)$ increases the value of the second member of U in the equation a possible physical interpretation is that this corresponds to the steep part of the mountain where water will tend to accumulate more quickly
- when $h^{''}(x) < 0$ means h(x) is convex (the slope of the mountain decreases) if $h^{''}(x)$ decreases the value of the second member of U in the PDE, this corresponds to the surface where the surface is almost flattened, means the u(x,t) go slowly.

For the rest of this question, we consider (4) in the case of a single valley with $h(x) = x^2$.

(d) Show that the characteristic curves are given by $xe^{2t} = constant$. Transform the PDE from (x, t) coordinates to characteristic coordinates (ζ, Θ) defined by $\zeta = xe^{2t}$, $\tau = t$

$$\begin{aligned}
\frac{dx}{dt} &= -2x \\
\frac{dx}{x} &= -2dt \\
\int \frac{dx}{x} &= \int -2dt \\
ln(x) &= -2t + c \\
x &= \zeta e^{-2t} \\
\zeta &= xe^{2t}
\end{aligned}$$

ζ is a constant, that means $xe^{2t}$is the characteristic curves.
let's find the general solution:

$$\begin{aligned}
u(\zeta, \tau) = u(x,t) \qquad &\text{and} \qquad \tau = t \\
\tau_t = 2xe^{2t} \qquad &\text{and} \qquad \tau_x = e^{2t} \\
u_x = u_\zeta \zeta_t + u_\tau \tau_t &= 2xe^{2t} u_\zeta \\
u_t = u_\zeta \zeta_t + u_\tau \tau_t &= 2xe^{2t} u_\zeta + u_\tau
\end{aligned}$$

The PDE becomes,
$h^{'}(x) = 2x$ and $h^{''}(x) = 2$

$$\begin{aligned}
u_t - 2xu_x &= 2u \\
2xe^{2t} u_\zeta + u_\tau - 2xe^{2t} u_\zeta &= 2u \\
u_\tau &= 2u \\
\frac{du}{d\tau} &= 2u \\
\frac{du}{u} &= 2d\tau \\
\int \frac{du}{u} &= \int 2d\tau \\
ln(u) &= 2\tau + c \\
u &= f(\zeta) e^{2\tau}
\end{aligned}$$

Hence, $u(\zeta, \tau) = f(\zeta) e^{2\tau}$

(e) Use the initial condition

$$u(x,0) = 1 \qquad if \qquad |x| \leq 1 \qquad \text{and} \qquad u(x,0) = 0 \qquad if \qquad |x| \neq 1$$

to find the exact solution of the PDE.

- if $|x| \leq 1$:

$$u(x,0) = f(x)e^0 = 1$$
$$f(x) = 1$$

  Hence the solution is $u(x,t) = e^{2t}$ when $|xe^{2t}| \leq 1$

- if $|x| > 1$:

$$u(x,0) = f(x) = 0$$
$$f(x) = 0$$

  Hence, u(x,t) $= 0$ when $|xe^{2t}| > 1$

hence,

$$u(x,t) = \begin{cases} e^{2t} & \text{if} \quad |xe^{2t}| \leq 1 \\ 0 & \text{if} \quad |xe^{2t}| > 1 \end{cases}$$

(f) Interpret $\int u(x)dx$ in light of the solution you just derived in part (e).

we know that this, integration doesn't depend on the value of t, which means the integration $\int u(x)dx$, represents the total water depth over the region.

$$\int u(x)dx = \int_{-e^(2t)}^{e^{2t}} e^{2t}dt + \int 0dt = 2$$

The above result shows us, that water depth remains constant (with a value equal to 2) over time, even though the water spreads or concentrates in different regions.