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Course: Stochastic Processes

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1. Gambler's Ruin: Assume that the gambler's initial stake is \$3 and the gambler plays until either gaining \$8 or going bankrupt. At each play, the gambler wins \$1, with probability 0.6, or loses \$1, with probability 0.4.

   (a) Find the one-step transition matrix.

$$\begin{cases} P_{i,i+1} = 0.6 \\ P_{i,i-1} = 0.4 \\ p_{0,0} = 0 = P_{8,8} \end{cases}$$

$$P = \begin{bmatrix} 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   (b) Find the four-step transition matrix.

$$P^4 = \begin{bmatrix} 0.192 & 0.288 & 0.384 & 0.32 & 0.24 & 0.16 & 0.08 & 0 & 0 \\ 0.288 & 0.384 & 0.384 & 0.384 & 0.32 & 0.24 & 0.16 & 0.08 & 0 \\ 0.384 & 0.384 & 0.384 & 0.384 & 0.384 & 0.32 & 0.24 & 0.16 & 0.08 \\ 0.32 & 0.384 & 0.384 & 0.384 & 0.384 & 0.384 & 0.32 & 0.24 & 0.16 \\ 0.24 & 0.32 & 0.384 & 0.384 & 0.384 & 0.384 & 0.384 & 0.32 & 0.24 \\ 0.16 & 0.24 & 0.32 & 0.384 & 0.384 & 0.384 & 0.384 & 0.384 & 0.32 \\ 0.08 & 0.16 & 0.24 & 0.32 & 0.384 & 0.384 & 0.384 & 0.384 & 0.384 \\ 0 & 0.08 & 0.16 & 0.24 & 0.32 & 0.384 & 0.384 & 0.384 & 0.384 \\ 0 & 0 & 0.08 & 0.16 & 0.24 & 0.32 & 0.384 & 0.384 & 1 \end{bmatrix}$$

   (c) Find the gambler's expected fortune after four plays.

   since the fact that the game start with \$3 our initial state vector: $\pi_0 = [0, 0, 1, 0, 0, 0, 0, 0]$ the expected fortune of the gambler's the the initial state multiplier by the the transition matrix of of four-step transition matrix and also by vector state. let's X represent the variable of the fortune after four plays.

$E(X) = \pi_0 * p^4 * V$ where v is vector state $v = [1, 2, 3, 4, 5, 6, 7, 8]$
$E(X) = 3.65$

2. Andrei Andreyevich Markov, the Russian mathematician who introduced Markov chains over 100 years ago, first applied them in the analysis of the poem Eugene Onegin by Alexander Pushkin. In the first 20,000 letters of the poem, Markov counted (by hand!) 8,638 vowels and 11,362 consonants. He also tallied pairs of successive letters. Of the 8,638 pairs that start with vowels, 1,104 pairs are vowel–vowel. Of the 11,362 pairs that start with consonants, 3,827 are consonant–consonant. Markov treated the succession of letter types as a random sequence.

   summary of our dataset:

   - voyels:
     - Total number of vowels: 8,638
     - Pairs starting with a vowel: 8,638 pairs
     - Number of vowel–vowel pairs: 1,104
   - Consonants
     - Total number of consonants: 11,362
     - Pairs starting with a consonant: 11,362 pairs
     - Number of consonant–consonant pairs: 3,827

   let's now find the transition probability of our Markov chains, the state of our dataset is C, V where S is for Consonant and V is for Vowel.

$$P_{V\to V} = \frac{Number of vowel\text{–}vowel pairs}{\text{Total pairs starting with vowels}} = \frac{1104}{8638} = 0.128 \tag{1}$$
$$P_{V\to C} = 1 - P(V - V) = 0.872 \tag{2}$$

$$P_C = \frac{Number of consonant\text{–}consonant pairs}{\text{Total pairs starting with consonant}} = \frac{3827}{11362} = 0.337 \tag{3}$$
$$P_{C\to V} = 1 - P(C \to C) = 0.663 \tag{4}$$

   and the transition matrix give us $P = \begin{pmatrix} P(V \to V) & P(V \to C) \\ P(C \to V) & P(C \to C) \end{pmatrix} = \begin{pmatrix} 0.128 & 0.872 \\ 0.663 & 0.337 \end{pmatrix}$

3. Assume that $X_0, X_1, ...$ is a two-state Markov chain on $S = 0, 1$ with transition matrix $\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$ One can model a bi-variate process that looks back two time-periods by the following construction. Let $Zn = (X_{n-1}, X_n)$, for $n \neq 1$. The sequence $Z_1, Z_2, ...$ is a Markov chain with state space $SxS = (0, 0), (0, 1), (1, 0), (1, 1)$. Give the transition matrix of the new chain.

   in this bivariate transition matrix we need to find the Probability of $P(X_n/Z_{n-1})$ and this probability can be written as $P(X_n/Z_{n-1} = P(X_n = (i, j)/z_{n-1} = (m, n) = p(X_{n-1} = i, X_n = j/X_{n-2} = m, X_{n-1} = n)$

We define a new bivariate process $Z_n = (X_{n-1}, X_n)$, where the state space is $S \times S = \{(0,0),(0,1),(1,0),(1,1)\}$. We want to find the transition matrix of this new bivariate chain.

The transition matrix for Z_n is given by:

$P_Z((m,n),(i,j)) = P(X_n = i, X_{n+1} = j \mid X_{n-1} = m, X_n = n)$

Using the chain rule, we can decompose this probability as:

$P_Z((m,n),(i,j)) = P(X_n = i \mid X_{n-1} = n) \cdot P(X_{n+1} = j \mid X_n = i)$

Given the transition matrix P_X, we can now compute the transition probabilities for the bivariate chain. For example:

- From $(0,0)$ to $(0,0)$: $P_Z((0,0),(0,0)) = P(X_n = 0 \mid X_{n-1} = 0) \cdot P(X_{n+1} = 0 \mid X_n = 0) = (1-p)(1-p)$
- From $(0,0)$ to $(0,1)$:
  $P_Z((0,0),(0,1)) = P(X_n = 0 \mid X_{n-1} = 0) \cdot P(X_{n+1} = 1 \mid X_n = 0) = (1-p)p$
- From $(0,0)$ to $(1,0)$:
  $P_Z((0,0),(1,0)) = P(X_n = 1 \mid X_{n-1} = 0) \cdot P(X_{n+1} = 0 \mid X_n = 1) = p(1-q)$
- From $(0,0)$ to $(1,1)$:
  $P_Z((0,0),(1,1)) = P(X_n = 1 \mid X_{n-1} = 0) \cdot P(X_{n+1} = 1 \mid X_n = 1) = pq$

The transition matrix P_Z for the bivariate Markov chain $Z_n = (X_{n-1}, X_n)$ is given by:

$$P_Z = \begin{pmatrix} (1-p)^2 & (1-p)p & p(1-q) & pq \\ q(1-p) & qp & (1-p)(1-q) & (1-p)q \\ (1-q)p & (1-q)(1-p) & (1-p)^2 & p(1-p) \\ pq & p(1-q) & (1-p)p & (1-p)^2 \end{pmatrix}$$

4. The behavior of dolphins in the presence of tour boats in Patagonia, Argentina is studied in Dans et al. (2012). A Markov chain model is developed, with state space consisting of five primary dolphin activities (socializing, traveling, milling, feeding, and resting). The following transition matrix is obtained.

$$A = \begin{pmatrix} 0.84 & 0.11 & 0.01 & 0.04 & 0.00 \\ 0.03 & 0.80 & 0.04 & 0.10 & 0.03 \\ 0.01 & 0.15 & 0.70 & 0.07 & 0.07 \\ 0.03 & 0.19 & 0.02 & 0.75 & 0.01 \\ 0.03 & 0.09 & 0.05 & 0.00 & 0.83 \end{pmatrix}$$

Estimate the long-term distribution of dolphin activity.

in order to find the long term distribution we need to find $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ such that $\pi A = \pi$

let's solve the equation:

$\pi_1 = 0.84\pi_1 + 0.03\pi_2 + 0.01\pi_3 + 0.03\pi_4 + 0.03\pi_5$

$\pi_2 = 0.11\pi_1 + 0.80\pi_2 + 0.15\pi_3 + 0.19\pi_4 + 0.09\pi_5$

$\pi_3 = 0.01\pi_1 + 0.04\pi_2 + 0.70\pi_3 + 0.02\pi_4 + 0.05\pi_5$

$\pi_4 = 0.04\pi_1 + 0.10\pi_2 + 0.07\pi_3 + 0.75\pi_4 + 0.00\pi_5$

$\pi_5 = 0.00\pi_1 + 0.03\pi_2 + 0.07\pi_3 + 0.01\pi_4 + 0.83\pi_5$

important to note the $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$

After solve this equation numerically we got : Stationary distribution: [0.2 0.2 0.2 0.2 0.2]

5. Assume that a student can be in 1 of 4 states: Rich, Average, Poor, In Debt Assume the following transition probabilities:

   - If a student is Rich, in the next time step the student will be Average with probability 0.75
   - If a student is Rich, in the next time step the student will be Poor with probability 0.2
   - If a student is Rich, in the next time step the student will be In Debt with probability 0.05
   - If a student is Average, in the next time step the student will be Rich with probability 0.05
   - If a student is Average, in the next time step the student will be Average with probability 0.2
   - If a student is Average, in the next time step the student will be In Debt with probability 0.45
   - If a student is Poor, in the next time step the student will be Average with probability 0.4
   - If a student is Poor, in the next time step the student will be Poor with probability 0.3
   - If a student is Poor, in the next time step the student will be In Debt with probability 0.2
   - If a student is In Debt, in the next time step the student will be Average with probability 0.15
   - If a student is In Debt, in the next time step the student will be Poor with probability 0.3
   - If a student is In Debt, in the next time step the student will be In Debt with probability 0.55 Model the above as a discrete Markov chain

   (a) Draw the corresponding Markov chain and obtain the corresponding stochastic matrix.

   based on the data above we see that we have four state R, A, P, I and we can write the transition matrix as follow based on the description text above:

$$P = \begin{pmatrix} 0 & 0.75 & 0.2 & 0.05 \\ 0.05 & 0.2 & 0.3 & 0.45 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0 & 0.15 & 0.3 & 0.55 \end{pmatrix}$$

   we can draw the transition matrix above as follow :

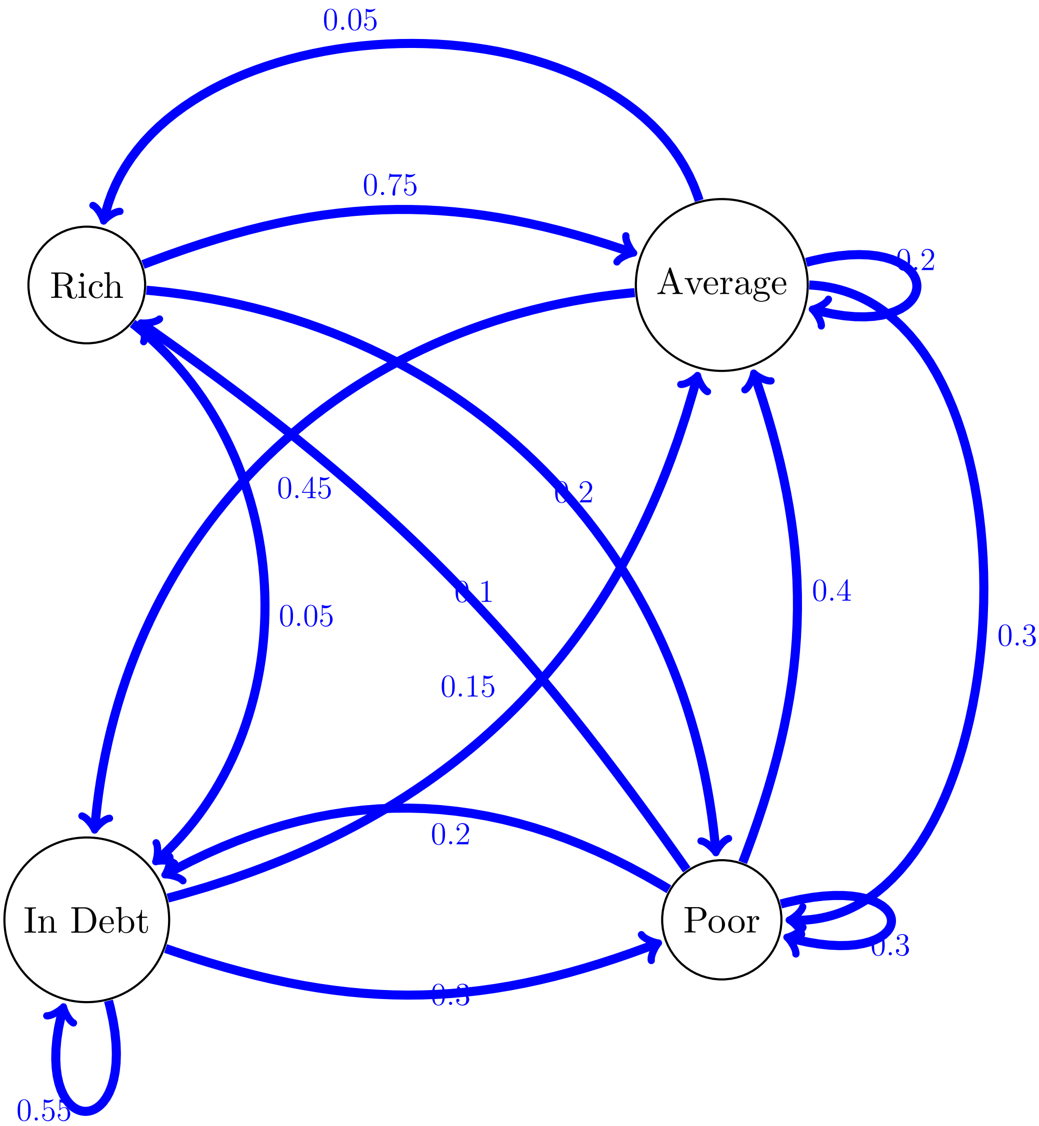

Figure 1: Diagram of the state transition matrix

(b) If a student starts at the "Average" state, what will be the probability of them getting ot state "Rich" after

- 1 time step? the path is direct from average to rich the probability is 0.05
- 2 time steps? in two step the path could be : Average - Average - Rich the probability give $P = 0.2 * 0.05 = 0.01$
- 3 time steps? in three step the path could be Average - Average - Average - Rich so the probability is $P = (0.2)^3 * (0.05) = 4 * 10^4$

(c) For a Markov chain $X_0, X_1, ...,$ with an initial distribution α and transition matrix P, justify the joint probability using conditional probability and the Markov property.
$P(X_6 = i, X_{10} = j, X_{13} = k, X_{25} = l)$
$P(X_6 = i, X_{10} = j, X_{13} = k, X_{25} = l) = (\alpha * P^6)_i (P^4)_{ij} (P^3)_{jk} (P^{12})_{kl}$