

Mathematic Problem Solving Assignment 1

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7 September 2024

Problem 1: Sum of p-th powers

1. Show that: $\sum_{k=1}^n k = \frac{1}{2}n(n+1)$

let's show by using induction:

- Initialization: for $n = 1$ we have $\sum_{k=1}^1 k = 1$ and $\frac{1}{2} * 1 * (1 + 1) = 1$
thus, $\sum_{k=1}^1 = \frac{1}{2} * 1 * (1 + 1)$
The property is true for $n = 1$
- hypothesis: let's suppose that the property is true at rank n and let's prove that is also true at rank $n + 1$

$$\begin{aligned}\sum_{k=n+1}^1 k &= \sum_{k=n+1}^1 k + (n+1) \\ &= \frac{1}{2}n(n+1) + (n+1) \quad \text{by using hypothesis of induction} \\ &= \frac{1}{2}(n+1)(n+2)\end{aligned}$$

as it is also true at rank $n+1$ by induction $\sum_{k=1}^n k = \frac{1}{2}n(n+1)$

2. Prove that: $\sum_{k=1}^n [(k+1)^3 - k^3] = n(n^2 + 3n + 3)$ and $\sum_{k=1}^{n+1} [(k+1)^3 - k^3] = 3 \sum_{k=1}^n [k^2 + k + \frac{1}{3}]$

Firstly, let's prove that: $\sum_{k=1}^n [(k+1)^3 - k^3] = n(n^2 + 3n + 3)$ we can prove this by induction:

- Initialization: for $n = 1$ we have,

$$\sum_{k=1}^n [(k+1)^3 - k^3] = (1+1)^3 - 1^3 = 2^3 - 1 = 7 \text{ and } 1(1^2 + 2 * 1 + 3) = 7$$

$$\text{thus, } \sum_{k=1}^n [(k+1)^3 - k^3] = 1(1^2 + 2 * 1 + 3)$$

so the property is true at rank $n = 1$

- Hypothesis: let's suppose that the property is true at rank n and let's prove it is also true at rank $n+1$

$$\begin{aligned}\sum_{k=1}^{n+1} [(k+1)^3 - k^3] &= \sum_{k=1}^n [(k+1)^3 - k^3] + (n+2)^3 - (n+1)^3 \\ &= n(n^2 + 3n + 3) + (n+2)^3 - (n+1)^3 \quad \text{by using the hypothesis of induction} \\ &= n^3 + 6n^2 + 12n + 7\end{aligned}$$

and

$$\begin{aligned}(n+1)[(n+1)^3 + 3(n+1) + 3] &= (n+1)[n^2 + 2n + 1 + 3n + 3 + 3] \\ &= (n+1)(n^2 + 5n + 7) \\ &= n^3 + 6n^2 + 12n + 7\end{aligned}$$

so, $\sum_{k=1}^{n+1} [(k+1)^3 - k^3] = (n+1) [(n+1)^3 + 3(n+1) + 3]$

the property is also true at rank $n+1$, by induction $\sum_{k=1}^n [(k+1)^3 - k^3] = n(n^2 + 3n + 3)$

Secondly, let's prove that $\sum_{k=1}^{n+1} [(k+1)^3 - k^3] = 3 \sum_{k=1}^n [k^2 + k + \frac{1}{3}]$.

To prove that, we can use direct proof:

$$\begin{aligned} \sum_{k=1}^{n+1} [(k+1)^3 - k^3] &= \sum_{k=1}^{n+1} [k^3 + 3k^2 + 3k + 1 - k^3] \\ &= \sum_{k=1}^{n+1} [3k^2 + 3k + 1] \\ &= \frac{1}{3} \sum_{k=1}^{n+1} \left[k^2 + k + \frac{1}{3} \right] \end{aligned}$$

3. Deduce that: $\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$

by using the previous result:

$$\begin{aligned} \sum_{k=1}^{n+1} [(k+1)^3 - k^3] &= \frac{1}{3} \sum_{k=1}^{n+1} \left[k^2 + k + \frac{1}{3} \right] = n(n+1)(2n+1) \\ \implies 3 \sum_{k=1}^n k^2 + 3 \sum_{k=1}^n k + 3 \sum_{k=1}^n \frac{1}{3} &= n(n^2 + 3n + 3) \\ \implies \sum_{k=1}^n k^2 &= \frac{1}{3} \left[n(n^2 + 3n + 3) - \frac{3}{1}n(n+1) - n \right] \\ &= \frac{1}{6}n [2(n^2 + 3n + 3) - 3(n+1) - 2] \\ &= \frac{1}{6}n [2n^2 + 6n + 6 - 3n - 3 - 2] \\ &= \frac{1}{6}n [2n^2 + 3n + 1] \end{aligned}$$

4. find polynomial P_4 of degree 5 such that $\sum_{k=1}^n k^4 = P_4(n)$

let's calculate $P_3(n)$:

$$\begin{aligned} (n+1)^4 - n^4 &= 4n^3 + 6n^2 + 4n + 1 \\ 1^4 - 0^4 &= 4 * 0^3 + 6 * 0^2 + 4 * 0 + 1 \\ 2^4 - 1^4 &= 4 * 1^3 + 6 * 1^2 + 4 * 1 + 1 \\ 3^4 - 2^4 &= 4 * 2^3 + 6 * 2^2 + 4 * 2 + 1 \\ &\vdots \\ (n+1)^4 - n^4 &= 4n^3 + 6n^2 + 4n + 1 \end{aligned}$$

$$(n+1)^4 = 4 \sum_{k=0}^n k^3 + 6 \sum_{k=0}^n k^2 + 4 \sum_{k=0}^n k + \sum_{k=0}^n 1$$

$$\begin{aligned}
\Rightarrow \sum_{k=0}^n k^3 &= \frac{1}{4} \left[(n+1)^4 - 6 \sum_{k=0}^n k^2 + 4 \sum_{k=0}^n nk - (n+1) \right] \\
&= \frac{1}{4} [(n+1)^4 - n(n+)(2n+1) - 2n(n+1) - (n+1)] \\
&= \frac{1}{4} (n+1) [(n+1)^3 - n(2n+1) - 2n-1] \\
&= \frac{1}{4} (n+1) [(n+1)^3 - (2n+1)(n+1)] \\
&= \frac{1}{4} (n+1)^2 [(n+1)^2 - (2n+1)] \\
&= \frac{1}{4} n^2 (n+1)^2
\end{aligned}$$

let's calculate $P_4(n)$:

$$\begin{aligned}
(n+1)^5 - n^5 &= 5n^4 + 10n^3 + 10n^2 + 5n + 1 \\
\cancel{n^5} - \cancel{n^5} &= 5 * 0^4 + 10 * 0^3 + 10 * 0^2 + 5 * 0 + 1 \\
\cancel{2^5} - \cancel{2^5} &= 5 * 1^4 + 10 * 1^3 + 10 * 1^2 + 5 * 1 + 1 \\
\cancel{3^5} - \cancel{3^5} &= 5 * 2^4 + 10 * 2^3 + 10 * 2^2 + 5 * 2 + 1 \\
&\vdots \\
(n+1)^5 - \cancel{n^5} &= 4n^4 + 6n^3 + 4n^2 + 4n + 1
\end{aligned}$$

$$(n+1)^5 = 5 \sum_{k=0}^n k^4 + 10 \sum_{k=0}^n k^3 + 10 \sum_{k=0}^n k^2 + 5 \sum_{k=0}^n k + \sum_{k=0}^n 1$$

$$\begin{aligned}
\Rightarrow \sum_{k=1}^n k^4 &= \frac{1}{5} \left[(n+1)^5 - 10 \sum_{k=0}^n k^3 - 10 \sum_{k=0}^n k^2 - 5 \sum_{k=0}^n k - \sum_{k=0}^n 1 \right] \\
&= \frac{1}{5} \left[(n+1)^5 - \frac{10}{4} n^2 (n+1)^2 - \frac{10}{6} n(n+1)(2n+1) - \frac{5}{2} n(n+1) - (n+1) \right] \\
&= \frac{1}{5} (n+1) \left[(n+1)^4 - \frac{5}{2} n^2 (n+1) - \frac{5}{3} n(2n+1) - \frac{5}{2} n - 1 \right] \\
&= \frac{1}{30} (n+1) [6(n+1)^4 - 15n^2(n+1) - 10n(2n+1) - 15n - 6] \\
&= \frac{1}{30} (n+1)(2n+1)(3n^2 + 3n - 1)
\end{aligned}$$

5. shows that in general, we can find a polynomial P of degree p+1 such that $\sum_{k=1}^n k^p = P_p(n)$

By using the binomial expression, we have,

$$\begin{aligned}
(n+1)^{p+1} - n^{p+1} &= \sum_{k=0}^{p+1} C_k^{p+1} n^k 1^{p+1-k} - n^{p+1} \\
&= \cancel{n^{p+1}} + \sum_{k=0}^p C_k^{p+1} n^k - \cancel{n^{p+1}} \\
&= \sum_{k=0}^p C_k^{p+1} n^k \\
&= (p+1)n^p + C_{p-1}^{p+1} n^{p-1} + C_{p-2}^{p+1} n^{p-2} + \dots + 1
\end{aligned}$$

By using conjecture,

$$\begin{aligned}
(n+1)^{p+1} - n^{p+1} &= (p+1)n^p + C_{p-1}^{p+1}n^{p-1} + C_{p-2}^{p+1}n^{p-2} + \dots + 1 \\
\cancel{1^{p+1}} - \cancel{0^{p+1}} &= (p+1) * 0^p + C_{p-1}^{p+1} * 0^{p-1} + C_{p-2}^{p+1} * 0^{p-2} + \dots + 1 \\
\cancel{2^{p+1}} - \cancel{1^{p+1}} &= (p+1) * 1^p + C_{p-1}^{p+1} * 1^{p-1} + C_{p-2}^{p+1} * 1^{p-2} + \dots + 1 \\
\cancel{3^{p+1}} - \cancel{2^{p+1}} &= (p+1) * 2^p + C_{p-1}^{p+1} * 2^{p-1} + C_{p-2}^{p+1} * 2^{p-2} + \dots + 1 \\
&\vdots \\
1^{p+1} - \cancel{0^{p+1}} &= (p+1)n^p + C_{p-1}^{p+1}n^{p-1} + C_{p-2}^{p+1}n^{p-2} + \dots + 1
\end{aligned}$$

$$\begin{aligned}
(n+1)^{p+1} &= (p+1) \sum_{k=1}^n k^p + C_{p-1}^{p+1} \sum_{k=1}^n k^{p-1} + C_{p-2}^{p+1} \sum_{k=1}^n k^{p-2} + \dots + \sum_{k=0}^n 1 \\
&= (p+1)P_p(n) + C_{p-1}^{p+1}P_{p-1}(n) + C_{p-2}^{p+1}P_{p-2}(n) + \dots + (n+1)
\end{aligned}$$

$$\Rightarrow P_p(n) = \frac{1}{p+1} \left[(n+1)^{p+1} - (n+1) - \sum_{k=1}^{p-1} C_{p-k}^{p+1} P_{p-k}(n) \right]$$

Expression below established a relationship between $P_p(n)$ and $P_1(n), \dots, P_{p-1}(n)$.

now let's prove that $P_p(n)$ is a polynomial of degree p+1. let's prove it by induction

- Initialization: for p=1, we have $P_1(n) = \sum_{k=1}^n nk = \frac{1}{2}n(n+1)$
 $Degree(P_1(n)) = 2 = 1 + 1$
The property is true for p = 1
- Hypothesis: let's suppose that the degree of $P_p(n)$ is p+1 and let's prove that the degree of P_{p+1} is p+2.
 $P_{p+1}(n) = \frac{1}{p+2} [(n+1)^{p+2} - (n+1) - \sum_{k=1}^p C_{p+1-k}^{p+2} P_{p+1-k}(n)]$

we know, by using the hypothesis of induction $Degree(P_{p+1-k}) \leq (p+1), \forall k \in [1, p]$.

thus, the $Degree(P_{p+1}(n)) = Degree((n+1)^{p+2}) = p+2$

by induction, we prove that it is possible to find a polynomial $P_p(n)$ such that his degree is p+1

Let's give the answer for p = 5,

$$\begin{aligned}
P_5(n) &= \frac{1}{6} \left[(n+1)^6 - (n+1) - \sum_{k=1}^4 C_{5-k}^6 P_{5-k}(n) \right] \\
&= \frac{1}{6} [(n+1)^6 - (n+1) - C_4^6 P_4(n) - C_3^6 P_3(n) - C_2^6 P_2(n) - C_1^6 P_1(n)] \\
&= \frac{1}{6} [(n+1)^6 - (n+1) - 15P_4(n) - 20P_3(n) - 15P_2(n) - 6P_1(n)] \\
&= \frac{1}{6} \left[(n+1)^6 - (n+1) - \frac{1}{2}(n+1)(2n+1)(3n^2+3n-1) - 5n^2(n+1)^2 - \frac{5}{2}n(n+1)(2n+1) - 3n(n+1) \right] \\
&= \frac{1}{6}(n+1) \left[(n+1)^5 - 1 - \frac{1}{2}(2n+1)(3n^2+3n-1) - 5n^2(n+1) - \frac{5}{2}n(2n+1) - 3n \right] \\
&= \frac{1}{12}(n+1) [2(n+1)^5 - 2 - (2n+1)(3n^2+3n-1) - 10n^2(n+1) - 5n(2n+1) - 6n] \\
&= \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)
\end{aligned}$$