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Course: Topological Data Analysis — Date: January 11, 2025

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lesson 1

1. (Equivalent definition of continuity): A map f is continuous if and only if the preimage of any closed set is closed.

   A map $f : X \to Y$ between topological spaces is continuous if and only if the preimage of every closed set in Y is closed in X.

   proof this using the two sens:

   - If f is continuous, then the preimage of any closed set is closed:
     Let $C \subset Y$ be a closed set. By the continuity of f, the preimage $f^{-1}(C)$ is open in X. The complement of C, denoted $Y \setminus C$, is open in Y, and the preimage of an open set is open (by continuity). Therefore, the complement of $f^{-1}(C)$ is open, meaning $f^{-1}(C)$ is closed in X.
   - If the preimage of every closed set is closed, then f is continuous
     To show continuity, we need to prove that the preimage of every open set is open. Since open sets in Y are complements of closed sets, it follows that the preimage of an open set in Y is the complement of the preimage of a closed set. Since the preimage of a closed set is closed, its complement is open. Therefore, the preimage of every open set is open, which is exactly the definition of continuity.

   Thus, the condition that the preimage of every closed set is closed is equivalent to the usual definition of continuity.

2. Let $x \in R^n$, and $r > 0$. Let $y \in B(x, r)$. Show that $B(y, r - |x - y|) \in B(x, r)$
   $B(x, r) = \{y \in R^n, |x - y| < r\}$

$$\begin{aligned} |||x - z|| &\leq |||x - y| + ||y - z|| \\ \implies &< ||x - y|| + (r - |||x - y|) \\ &= r \end{aligned}$$

   so, $z \in B(x, r)$

3. Show that the open balls B(x, r) of R^n are open sets (with respect to the Euclidean topology).

   the definition of an open ball is : $z_0 \in B(x, r) \implies ||z_0 - r|| < r$

   we want to find $\epsilon > 0$ such that $B(z_0, \epsilon)$ and we know that to respect to the euclidean topology $d = r - ||z_0 - x||$.

   and d is an distance which means $d > 0$, just take $d = \epsilon$ let's consider the open ball $B(z_0,) = z \in R^n, |||z - z_0|| < \epsilon$

   next, we want to show that, $B(z0, \epsilon) \subset B(x, r)$, just taking any point $zI \in B(z_0, \epsilon)$ Hence

   $||y - y_0|| < \epsilon = r - ||z_0 - x||]$

   using the triangular inequality, we have:

   $||z - x|| \leq ||y - z_0|| + |||z_0 - x|$

   we know that $||z - z_0|| < \epsilon$, we have

   $||y - x|| < \epsilon + ||z_0 - x|| = (r - ||z_0 - x||) + ||y_0 - x|| = r$

   Hence we have proof that $y \in B(x, r)$

   Which means taking any $z \in B(z_0, \epsilon)$, we have shown that $B(z_0, \epsilon) \subset B(x, r)$,

   Hence for any point in $B(z\epsilon, \epsilon)$ we can always find the ϵ such that $B(z_0, \epsilon) \subset B(x, r)$ which is open the definition of open ball in Euclidean topology on R^n

4. Let x, y $\in R^n$, and $r = ||x - y||$. Show that $B(\frac{x+y}{2}, \frac{r}{2}) \subset B(x, r) \cap B(y, r)$

   in order to prove that let's take, we need to prove that for any point in $B(\frac{x+y}{2}, \frac{r}{2})$, this point in the intersection between B(x, r) and B(y, r)

   - let's prove that for $z \in B(\frac{x+y}{2}, \frac{r}{2})$ z is also $B(x, r)$
     by definition,

$$||z - (\frac{x+y}{2})|| < \frac{r}{2}$$

$$\begin{aligned} ||z - x|| &< ||z - \frac{x+y}{2} + \frac{x+y}{2} - x|| \\ &< ||z - \frac{x+y}{2}|| + ||\frac{x+y}{2} - x|| \\ &< \frac{r}{2} + |||x - y|| \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r \end{aligned}$$

     Hence, we have show that $z \in B(x, r)$

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\begin{aligned}
||z-y|| &< ||z-\frac{x+y}{2}+\frac{x+y}{2}-y|| \\
&< ||z-\frac{x+y}{2}||+||\frac{x+y}{2}-y|| \\
&< \frac{r}{2}+|||x-y|| \\
&< \frac{r}{2}+\frac{r}{2} \\
&= r
\end{aligned}
$$

Hence $z \in B(x,r)$

5. Show that the set of rational numbers is not an open subset of R.

   Let's consider $q \in Q$ an arbitrary rational number. We need to check if there is some open interval. $(q-\epsilon, q+\epsilon)$ that is entirely contained in Q.

   and we know that Q is dense in R which means for any open for any $\epsilon > 0$ the open interval contains both $(q-\epsilon, q+\epsilon)$ contains both rational and irrational numbers.

   Hence, there doesn't exist $\epsilon > 0$ such as the open interval $(q-\epsilon, q+\epsilon) \subset Q$.

6. Show that a map is continuous if and only if the preimage of closed sets are closed sets. in order to show that let's show it in two parts:

   - if the map is continuous and let's show that the preimage of closed sets are closed sets.

     Let's denote the map between two topologies, X and Y, by f: $X \to Y$.

     $C \subset Y$ to be a closed set in Y, we want to show that the preimage $f^{-}(C) \subset X$ is closed set

     and we know that f is continuous by the definition from the topological viewpoint for any open internal $O \subset Y$ the preimage $f^{-1}(O)$ is also an open set.

     from also the equivalent definition the preimage of a closed set is also a closed set using the notion of complement. we can prove that using the notion of complement we know that O is an open set and the complement of an open set is a closed set.

     $f^{-1}(O^c) = (f^{-1}(O))^c$ we know that $f^{-1}(O)$ is an open set and its complement is a closed set.

   - if the pre-image of closed sets are closed sets, the map f is continuous

     for any closed set in $O \in Y$ the pre-image $f^{-1}O$ is closed, which means the pre-image of the open set is an open set. for any open set in Y, its pre-image is also an open set, by definition f is a continuous set.

Lesson 2

1. Show that the following spaces are homeomorphic:

- Let $a \neq b$. Show that the intervals [0, 1] is homeomorphic to [a, b]. to show that let's take a map f(x) = (b-a)x + 1, and we know that this function by definition since f is a linear function
  $\lim_{x \to x_0} f(x) = f(x_0)$
  the inverse of this function is $g(x) = \frac{x-a}{b-a}$ this function is also linear but the function exists only if $a \neq b$.
- the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ is is homeomorphic to the interval (0, 1).
  let's take the function $f(x) = \frac{1}{2}(sin(\frac{\pi}{2}x)+1)$ we know that this function is continuous and this inverse($g(x) = \frac{2}{\pi})arsin(2y-1)$ is also continuous. and the function f is bijective.

  Hence the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ are homeomorphic to $(0, 1)$.
- The interval $(-1, 1)$ is homeomorphic to R ,
  In this case, just take the map $f(x) = tan(\frac{\pi}{2}x)$ this function is continuous and this inverse $g(x) = \frac{\pi}{2}tan(x)$ is also continuous and the function is bijective Hence we have shown that the interval $(-1, 1)$ is homeomorphic to R since the exists a map between this two function such as the map is continuous and this inverse is also continuous.

2. Show that the function f : $R^n \to [0, \infty)$ defined by $f(x) = ||x||$ is continuous.

   a function f is continuous if for every point $x_0 \in R^n$ and for every $\epsilon > 0$ there exist $\delta > 0$ such as for every $x \in R^n$
   $\|x - x_0\| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

   $$|f(x) - f(x_0)| = \||x| - |x_0|\| \leq \|x - x_0\| \qquad \text{using the triangulation law's}$$
   $$\||x| - |x_0|\| \leq \|x - x_0\| < \epsilon$$

   we need just to take $\delta = \epsilon$

   Hence, f is continuous

3. Consider the sets
   $D = x \in R^2 : \|x\| \leq 1$ and $S^2 = x \in R^3 : \|x\| = 1$

   - let $S^1 = x \in R^2 : \|x\| = 1$ show that f: $D\S^1 \to R^2$ defined by:
     $f(x) = \frac{x}{1-\|x\|}$ is homomorphism.
     using the fact that $\|h(x)\| = \frac{\|x\|}{1+\|x\|} < 1$ we see that $h : R^2 \to X\S^1$, this function is continuous because its coordinate are continuous.
     let's now find the inverse of this function

$$y = \frac{x}{1 - \|x\|}$$
$$y(1 - \|x\|) = x$$
$$\|y(1 - \|x\|)\| = \|x\|$$
$$\|y\|(1 - \|x\|) = \|x\|$$
$$\|x\|(1 + \|y\|) = \|y\|$$
$$\|x\| = \frac{\|y\|}{1 + \|y\|}$$

now we have found the inverse of this function $g(x) = \frac{x}{1+\|x\|}$ and this function also continues due to the continuity of these coordinates.

and fog = x Hence f is bijective

Since f is continuous, bijective, and has a continuous inverse, we conclude that f is a homomorphism

- Show that the map g : $S^2/\{0,0,1\} \to R^2$ defined by: $g(x) = \frac{1}{1-x_3}(x_1, x_2)$ is a homomorphism.

  f is a continuous function because all these coordinates is continuous,

  let's now find this inverse,

$$y = (y_1, y_2, y_3) = \frac{1}{1 - x_3(x_1, x_2)}$$

$\begin{cases} y_1 = \frac{1}{1-x_3} x1, \\ y_2 = \frac{1}{1-x_3} x_2, \end{cases}$

using the fact that $x_1^2 + x_2^2 + x_3^{\overline{=}}1$

submitting in the question y_1 and y_2 into the equation we have:

$y_1^2(1 - x_3)^2 + y_2^{(}1 - x_3^2) + x_3^2 = 1$

$(y_1^2 + y_2^2)(1 - x_3)^2 + x_3^2 = 1$

$x_3 = \frac{x_1^2+x_2^2-1}{x_1^2+x_{2+1}}$

and substitute it in $y_1 = \frac{2}{x_1^2+x_2^2+x_3^2}$ and $y_2 = \frac{2}{x_1^2+x_2^2+x_3^2}$

so the inverse of this function is defined by $g(x_1, x_2) = \left(\frac{2}{x_1^2+x_2^2+x_3^2}, \frac{2}{x_1^2+x_2^2+x_3^2}, \frac{x_1^2+x_2^2-1}{x_1^2+x_{2+1}}\right)$

Lesson 3

1. let X be a topological space and $x, y \in X$, the path between the points x and y is a continuous functionZ. if n : $[0, 1] \to X$ such that n(0)= x and n(1) = y

   show that the unit share $S^n = \{x \in R^{n+1}, \|x\| = 1\}$

2. **Value intermediate problem** Let X be a connected topological and let f : $X \to Y$ be a continuous function. if $a < b$ are points in R such that a = f(x) and b= f(y) for some $x, y \in X$ then for each $c \in (a, b)$ there exits $z \in X$ such as c = f(z)

   apply the intermediate value problem theorem to answer the following

   let f: $R \to R$ be a continous function such that $f(x).f(f(x)) = 2$ for all $x \in R$ and that f(3) = 10. find the value of f(5).

   $f(3).f(f(3)) = 2, \implies, 10.f(f(3)) = 2 \; f(f(3)) = \frac{2}{10} \implies f(10) = \frac{1}{5}$

   $5 \in [3, 10]$ that means there exist c = f(5) by apply the intermediate value problem.

3. Let f: $R^n \to R$ be a continous function. show that there exists point $x \in S^n$ such that f(x) = f(-x)¿ here if $x = (x_1, x_2, ..., x_{n+1})$, then $-x = (-x_1, -x_2, ..., -x_{n+1})$

   since f is continuous x, $-x \in S^n$ we need just to take the point $x = (0, 0, 0, 0, ..., 0)$ Hence $f(x_0) = f(-x_0)$, we have shown that there exists a point just take the point O.

4. Show that the intervals $[0, 1)$ and $(0, 1)$ are not homeomorphic.

   prove by contradiction, let's suppose that $[0, 1)$ and $(0, 1)$ are homeomorphic By definition, this means that there exists a map f : $[0, 1) \to (0, 1)$ which is continuous, invertible, and with continuous inverse.

5. Show that the closed interval [0, 1] is not homeomorphic to a cross.

   let's prove this by contradiction let's suppose that [0,1] is homeomorphic to a cross. this means that there exists a map f : $[0, 1) \to (0, 1)$ which is continuous, invertible, and with continuous inverse.

   $\forall x \in [0, 1]$ Consider the subsets $[0, 1]/\{x\} \subset [0, 1]$ and $cross\{x\} \subset cross$.

   [0,1]/{x} have two number of connected component and $cross\{x\} \subset cross$ have one number of connected component. this is a contradiction.

   Hence closed interval [0, 1] is not homeomorphic to a cross