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Exercise 1

A material point M moves according to the parametric equations: $x = a\cos(\omega t)$, $y = a\sin(\omega t)$, $z = bt$, where $a, b, \omega > 0$.

1. Simplest polar equations in cylindrical and spherical coordinates:

   - *Cylindrical coordinates:* in Cylindrical system,

     $\begin{cases} x = rcos(\theta) \\ y = rin(\theta) \\ z = z \end{cases} \quad x^2 + y^2 = r^2 = a^2 \implies r = a,\ z = bt \text{ and } \theta = wt$

   - *Spherical coordinates:*

     $\begin{cases} x = rsin(\phi)cos(\theta) \\ y = rsin(\phi)sin(\theta) \\ *z = rcos(\phi) \end{cases}$

     $r = \sqrt{x^2 + y^2 + z^2} = \sqrt{a^2 + (bt)^2},$

$$
\begin{aligned}
x^2 + y^2 &= r^2\sin^2(\phi) \\
\sqrt{x^2 + y^2} &= r\sin(\phi) \\
\frac{\sqrt{x^2 + y^2}}{z} &= \tan(\phi) \\
\phi = tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \\
\phi = tan^{-1}\left(\frac{a}{bt}\right)
\end{aligned}
$$

$Hence, \phi = tan^{-1}\left(\frac{a}{bt}\right),$

$$\frac{y}{x} = \tan(\theta)$$
$$\theta = tan^{-1}\left(\frac{y}{x}\right)$$
$$\theta = tan^{-1}\left(\frac{a\sin(wt)}{acos(wt)}\right)$$
$$\theta = tan^{-1}\left(\frac{\sin(wt)}{cos(wt)}\right)$$

$\theta = wt$

2. Calculate, in each case, the components of the infinitesimal displacement vector as a function of time.

   - Components in cylindrical :
     The parametric equations are: $x = a\cos(\omega t), \quad y = a\sin(\omega t), \quad z = bt.$

     In cylindrical coordinates: $r = a, \quad \theta = \omega t, \quad z = bt.$

     $\implies dr = 0, \quad d\theta = \omega\, dt, \quad dz = b\, dt.$

     The infinitesimal displacement vector in cylindrical coordinates is: $dr = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{e}_z.$

     Substituting the results: $dr = 0\,\hat{e}_r + (a\omega\, dt)\,\hat{e}_\theta + (b\, dt)\,\hat{e}_z.$

     Thus, the components are: $\mathbf{dr} = 0\,\hat{e}_r + a\omega\, dt\,\hat{e}_\theta + b\, dt\,\hat{e}_z.$

   - Components in Spherical Coordinates:
     The parametric equations are: $x = a\cos(\omega t), \quad y = a\sin(\omega t), \quad z = bt.$
     In spherical coordinates:
     $r = \sqrt{a^2 + (bt)^2}, \quad \theta = \omega t, \quad \phi = \tan^{-1}\left(\frac{a}{bt}\right).$
     $\implies dr = \frac{b^2 t}{\sqrt{a^2 + (bt)^2}}\, dt, \quad d\theta = \omega\, dt,\ d\phi = -\frac{ab}{a^2 + b^2t^2}\, dt.$
     The infinitesimal displacement vector in spherical coordinates is: $dr = dr\,\hat{e}_r + r d\theta\,\hat{e}_\theta + r\sin\phi\, d\phi\,\hat{e}_\phi.$
     Substituting the results: $dr = \left(\frac{b^2 t}{\sqrt{a^2 + (bt)^2}}\, dt\right)\hat{e}_r + \left(\sqrt{a^2 + (bt)^2}\cdot \omega\, dt\right)\hat{e}_\theta + \left(\sqrt{a^2 + (bt)^2}\cdot \sin\phi \cdot -\frac{ab}{a^2 + b^2t^2}\, dt\right)\hat{e}_\phi.$

3. Show that the velocity vector forms a constant angle α with the z-axis.

The velocity vector is the time derivative of the position vector $\mathbf{r}(t)$:

$r(t) = a\cos(\omega t)\hat{\imath} + a\sin(\omega t)\hat{\jmath} + bt\hat{k}$.

The velocity vector $v(t)$ is:

$v(t) = \dfrac{dr(t)}{dt} = -a\omega\sin(\omega t)\hat{\imath} + a\omega\cos(\omega t)\hat{\jmath} + b\hat{k}$.

The angle α between the velocity vector and the z-axis is given by:

$\cos(\alpha) = \dfrac{v(t)\cdot\hat{k}}{|v(t)|}$.

The dot product $v(t)\cdot\hat{k}$ is:

$v(t)\cdot\hat{k} = b$.

The magnitude of $v(t)$ is:

$|v(t)| = \sqrt{(a\omega)^2 + b^2}$.

Thus, the cosine of the angle α is:

$\cos(\alpha) = \dfrac{b}{\sqrt{(a\omega)^2 + b^2}}$.

Since this expression is constant, the velocity vector forms a constant angle α with the z-axis.

4. The trajectory of M is said to be helical because it forms a helix wrapped around a circular cylinder. The pitch h of the helix is the distance between two successive positions of the moving point on the same generator. Establish the relationship between b and h.

   The trajectory of the point M is helical, meaning it forms a helix wrapped around a circular cylinder. The pitch h of the helix is defined as the vertical distance between two successive positions of the moving point on the same generator of the cylinder.

   The vertical velocity is given by $v_z = b$, and the time taken for one complete revolution around the cylinder is: $t_{\text{rev}} = \dfrac{2\pi}{\omega}$. During this time, the vertical displacement is: $h = \Delta z = v_z \cdot t_{\text{rev}} = b \cdot \dfrac{2\pi}{\omega}$. Thus, the relationship between the pitch h and the parameter b is: $h = \dfrac{2\pi b}{\omega}$.

5. Calculate the tangential and normal accelerations and determine the radius of curvature R of the trajectory.

   The velocity vector $v(t)$ is: $v(t) = -a\omega\sin(\omega t)\hat{\imath} + a\omega\cos(\omega t)\hat{\jmath} + b\hat{k}$.

   The magnitude of the velocity vector is: $|v(t)| = \sqrt{(a\omega\sin(\omega t))^2 + (a\omega\cos(\omega t))^2 + b^2} = \sqrt{a^2\omega^2 + b^2}$.

   The acceleration vector $a(t)$ is the time derivative of $v(t)$: $a(t) = \dfrac{dv(t)}{dt} = -a\omega^2\cos(\omega t)\hat{\imath} - a\omega^2\sin(\omega t)\hat{\jmath}$.

   The magnitude of the acceleration vector is: $|a(t)| = \sqrt{(a\omega^2\cos(\omega t))^2 + (a\omega^2\sin(\omega t))^2} = a\omega^2$.

The tangential acceleration a_t is the component of $a(t)$ in the direction of the velocity vector $v(t)$: $a_t = \frac{v(t) \cdot a(t)}{|v(t)|}$. The normal acceleration a_n is given by: $a_n = \sqrt{|a(t)|^2 - a_t^2}$.
The radius of curvature R is: $R = \frac{|v(t)|^2}{|a(t)|}$.

The radius of curvature R is given by: $R = \frac{|v(t)|^2}{|a(t)|} = \frac{a^2\omega^2 + b^2}{a\omega^2} = a + \frac{b^2}{a\omega^2}$.

Thus, the radius of curvature is: $R = a + \frac{b^2}{a\omega^2}$.

Exercise 2

A particle, initially at rest at point x0, moves in a straight line with an acceleration given by the algebraic expression: $\gamma = -Cx^3$
where $C > 0$ is a constant, and x is the position of the particle at time t ($x > 0$ and $x_0 > x$).

1. Calculate the velocity of the particle as a function of its position x.

   Using the relation between acceleration and velocity: $\gamma = \dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = -Cx^3$

$$\begin{aligned}
\frac{dv}{dx}\frac{dx}{dt} &= -Cx^3 \\
v\frac{dv}{dx} &= -cx^3 \\
vdv &= -Cx^3dx \\
\int vdv &= \int -cx^3dx \\
\frac{1}{2}v^2 &= -\frac{c}{4}x^4 + K
\end{aligned}$$

   we know that at rest point x_0 $V_0 = 0 \implies 0 = -\frac{c}{4}x_0^4 + K \implies K = \dfrac{c}{4}x_0^4$

   so our equation becomes,
   $\dfrac{1}{2}v^2 = -\dfrac{c}{4}x^4 + \dfrac{c}{4}x_0^4 \implies v = \sqrt{\frac{c}{2}(x_0^4 - x^4)}$

2. Assume $x_0 \to \inf$. Determine the time-dependent equation of motion $x = x(t)$.

   we know that $x_0 \to \inf$
   $v = \sqrt{\dfrac{c}{2}x^4} = x^2\sqrt{\dfrac{c}{2}}$

$$\begin{aligned}
\frac{dx}{dt} &= x^2\sqrt{\frac{c}{2}} \\
dx &= x^2\sqrt{\frac{c}{2}}dt
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{x^2} &= \int \sqrt{\frac{c}{2}}dt \\
-\frac{1}{x} &= \sqrt{\frac{c}{2}}t + K_0
\end{aligned}$$

   and we know that at $t = 0$, $x_0 \to \inf$ that means $\frac{1}{x_0} \to 0$ and $K_0 = 0$

   and we have $-\dfrac{1}{x} = \sqrt{\dfrac{c}{2}}t \implies x = -\frac{1}{\sqrt{\frac{c}{2}}t}$

Exercise 3

A material point describes a plane curve whose equation, in polar coordinates (r, θ), is: $r = \dfrac{r_0(1 + cos(\theta))}{2}$ where r_0 is a given length.

1. Question 1 :

   (a) What is the general shape of the trajectory? Specify the points of intersection of this trajectory with the Cartesian axes Ox and Oy.

   The equation of the trajectory in polar coordinates is given by: $r(\theta) = \dfrac{r_0(1 + \cos\theta)}{2}$
   This represents a curve known as a *limacon of Pascal*, which is a type of curve that has a single loop. The curve intersects the x-axis at points where $\theta = 0$ and $\theta = \pi$, and the y-axis at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.
   The intersections with the axes are as follows:

   - For the x-axis, set $\theta = 0$ and $\theta = \pi$: $r(0) = \dfrac{r_0(1 + 1)}{2} = r_0, \quad r(\pi) = \dfrac{r_0(1 - 1)}{2} = 0$ Thus, the intersection points with the x-axis are at $r = r_0$ and $r = 0$.
   - For the y-axis, set $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$: $r\left(\dfrac{\pi}{2}\right) = \dfrac{r_0(1 + 0)}{2} = \dfrac{r_0}{2}, \quad r\left(\dfrac{3\pi}{2}\right) = \dfrac{r_0(1 - 0)}{2} = \dfrac{r_0}{2}$ Thus, the intersection points with the y-axis are at $r = \frac{r_0}{2}$.

   (b) Express the curvilinear abscissa s of the point, measured from point A corresponding to $\theta = 0$, as a function of the polar angle θ. For which polar angle does s $= r_0$? Denote the corresponding position of the point as B.

   The curvilinear abscissa s is the arc length along the curve from point A (corresponding to $\theta = 0$) to a general point corresponding to an angle θ. It is given by: $s(\theta) = \displaystyle\int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ where $r(\theta) = \frac{r_0(1+\cos\theta)}{2}$.

   Differentiating $r(\theta)$: $\dfrac{dr}{d\theta} = \dfrac{r_0(-\sin\theta)}{2}$

   Thus, the curvilinear abscissa becomes: $s(\theta) = \displaystyle\int_0^\theta \sqrt{\left(\frac{r_0(1 + \cos\theta)}{2}\right)^2 + \left(\frac{r_0(-\sin\theta)}{2}\right)^2}\, d\theta$

   Simplify the expression inside the square root: $s(\theta) = \displaystyle\int_0^\theta \frac{r_0}{2}\sqrt{(1 + \cos\theta)^2 + \sin^2\theta}\, d\theta$ $= \displaystyle\int_0^\theta \frac{r_0}{2}\sqrt{2 + 2\cos\theta}\, d\theta$ Now, to find the value of θ for which $s = r_0$, solve for θ such that: $s(\theta) = r_0$ This gives the corresponding polar angle θ_B at which $s = r_0$.

   (c) Deduce the perimeter of the closed trajectory studied here.

   The perimeter of the closed trajectory is given by: $P = \displaystyle\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ Substitute the expressions for $r(\theta)$ and $\frac{dr}{d\theta}$, and perform the integration to find the total length of the closed curve.

2. Take the time origin as the instant when the point passes through A, and assume that the trajectory is described with a constant angular velocity w, which is known.

   Let the time origin correspond to the instant when the point passes through A, i.e. when $\theta = 0$. Since the point moves with constant angular velocity ω, the angle θ as a function of time is: $\theta(t) = \omega t$ Thus, the position of the particle at time t is given by: $r(t) = \dfrac{r_0(1+\cos(\omega t))}{2}$

3. Question 3

   (a) Express the linear velocity of the particle as a function of time. The linear velocity $v(t)$ is: $v(t) = r(t) \cdot \omega = \dfrac{r_0(1+\cos(\omega t))}{2} \cdot \omega$

   (b) Find this linear velocity as a function of the polar radius r.

   The linear velocity as a function of r is: $v = \omega \cdot r(\theta) = \omega \cdot \dfrac{r_0(1+\cos\theta)}{2}$

4. Determine the radial (γ_r) and orthoradial (γ_θ) components of the acceleration. Deduce its magnitude.

   The radial and orthoradial components of acceleration are given by: $\gamma_r = \ddot{r} - r\dot{\theta}^2, \quad \gamma_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

5. Let α and β be the angles formed, respectively, by the tangent (T) with the Ox-axis and the normal (N) with the polar radius OM, corresponding to the polar angle θ.

   (a) Establish the expression of the normal acceleration γ_N in terms of γ_r, γ_θ, and β only.

   The normal acceleration is given by: $\gamma_N = \sqrt{\gamma_r^2 + \gamma_\theta^2}$

   (b) Calculate the angles *alpha* and *beta* as functions of θ.

   The angles α and β are: $\alpha = \text{atan2}(v_y, v_x), \quad \beta = \text{atan2}(r, \dot{r})$

   (c) Substitute $\theta(t) = \omega t$ to express γ_N as a function of time.

   (d) Radius of Curvature

   The radius of curvature is: $R = \dfrac{v^2}{|\gamma_N|}$

6. Numerical application: $r_0 = 50cm$, $w = 3.2rad/s$. Calculate the velocity, acceleration, and radius of curvature of the trajectory at points A and B.

   Given: $r_0 = 50\,\text{cm} = 0.5\,\text{m}, \quad \omega = 3.2\,\text{rad/s}$

   - Linear Velocity:
     - At point A ($\theta = 0$): $r_A = \dfrac{r_0(1+\cos 0)}{2} = r_0 = 0.5\,\text{m}$ $v_A = r_A \cdot \omega = 0.5 \cdot 3.2 = 1.6\,\text{m/s}$
     - At point B ($\theta = \pi$): $r_B = \dfrac{r_0(1+\cos\pi)}{2} = 0\,\text{m}$ $v_B = 0\,\text{m/s}$
   - Radial and Orthoradial Acceleration
     - At point A ($\theta = 0$): $\gamma_r = 0, \quad \gamma_\theta = r_A\omega^2 = 0.5 \cdot 3.2^2 = 5.12\,\text{m/s}^2$

- At point B ($\theta = \pi$): $\gamma_r = 0, \quad \gamma_\theta = 0$

- Radius of Curvature
  - At point A ($\theta = 0$): $R_A = \dfrac{v_A^2}{\gamma_\theta} = \dfrac{1.6^2}{5.12} = 0.5\,\text{m}$
  - At point B ($\theta = \pi$): $R_B = \infty$
    Thus, at point A, the radius of curvature is $0.5\,\text{m}$, and at point B, the radius of curvature is infinite.