AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

(AIMS RWANDA, KIGALI)

Name: Darix SAMANI SIEWE | Assignment Number: 2
Course: Classical Mechanics | Date: December 14, 2024

Exercise 1

In this exercise, we study the motion of a point particle of mass m moving along the axis Ox and experiencing a force deriving from the potential

$$V(x) = \sigma x^2 + \lambda x^4$$

with σ and λ some real parameters. This is the type of potential that appears, for example, for a pendulum in a magnetic field, with a magnet at its end. Let's begin with the study of the fixed points.

1. Determine the fixed points of the system, depending on the values of σ and λ.

   In order to determine the fixed point let's solve the equation $\frac{dV(x)}{dx} = 0$ and the constraint $\frac{d^2x}{dx^2} > 0$.

$$\frac{dV(x)}{dx} = 0$$
$$2\sigma x + 4\lambda x^3 = 0$$
$$x(2\sigma + 4\lambda x^2) = 0$$
$$x = 0 \qquad \text{or} \qquad x^2 = -\frac{2\lambda}{\sigma}$$

   if σ and λ have different sign, then $x = 0$ and $x =^{+}_{-} \sqrt{\frac{2\lambda}{\sigma}})$ is the fixed point.

2. What is the stability of these fixed points, depending on the values of σ and λ?

   in order to answer this question let's find the second derivative.

$$\frac{dV^2(x)}{dx^2} = 2\sigma + 8\lambda x = 0$$

for the first fixed point, we have the following:

- x = 0 :

$$\frac{dV^2(x)}{dx^2} = 2\sigma$$

  if $\sigma > 0$ then the point $x = 0$ is the fixed point is unstable, if $\sigma < 0$ then the fixed point is unstable.

- $x =^+_- \sqrt{\frac{2\lambda}{\sigma}}$, ($\sigma$ and λ have different sign)

$$\frac{dV^2(x)}{dx^2} = 2\sigma^+_- 8\lambda\sqrt{\frac{2\lambda}{\sigma}}$$

  if this case if $\sigma^3 > 32\lambda^3$ then this point is stable

- Sketch the potential V (x) and describe qualitatively the possible motions of the particle, considering different mechanical energies.
- Show that the equation of motion is of the form

$$\ddot{x} + Ax + Bx^3 = 0,$$

  with A and B constants to be determined as functions of m, σ, and λ.
  we know that $F = m\frac{dV(x)}{dt} = m\ddot{x} = -\nabla V(x) = -2\sigma x - 4\lambda x^3$
  $\Longrightarrow$

$$m\ddot{x} = -2\sigma x - 4\lambda x^3$$
$$\Longrightarrow m\ddot{x} + 2\sigma x + 4\lambda x^3 = 0$$
$$\ddot{x} + \frac{2\sigma}{m}x + \frac{4\lambda}{m}x^3 = 0$$

  by identification, $A = \frac{2\sigma}{m}$ and $B = \frac{4\lambda}{m}$

Exercise 2

A mass is placed on a frictionless horizontal table. The friction coefficient is given by μ. A spring which can be stretched or compressed is placed on the table. A mass m is attached to the end of the spring, and the other end is anchored to the wall. The equilibrium position is marked at zero. A student moves the mass out to x and releases it from rest. The mass oscillates as a simple harmonic oscillator.

1. Determine the equation of motion.

   by applying the first Newton law:

$$\begin{aligned}\sum \vec{F_{ext}} &= m\vec{a}\\ \vec{f_f} + \vec{f_s} + \vec{p} &= m\vec{a}\\ \text{on the x-axis}\\ -\mu mg - kx &= m\ddot{x}\\ m\ddot{x} + kx + \mu mg &= 0\\ \ddot{x} + \frac{k}{m}x + \mu g &= 0\end{aligned}$$

The equation of the motion is given by:

$$\ddot{x} + \frac{k}{m}x = -\mu g$$

2. Find the solution of this equation assuming that at $t = 0$, $x(0) = 0$ and $v(0) = v_0$.

   the equation characterized of the equation is given: $r^2 - \frac{k}{m} = 0$

   and there are two solutions $r_1 = -i\sqrt{\frac{k}{2m}}$ and $r_2 = i\sqrt{\frac{k}{2m}}$

   Hence, $x = A\cos(w_0 t) + B\sin(w_0 t)$ with $w_0 = \sqrt{\frac{k}{2m}}$ with the initial condition :

   $\dot{x} = -Aw_0\sin(w_0 t) + Bw_0\cos(w_0 t)$ This is the general solution of homogeneous solution since

   the second member is the constant of the particular solution is $x_p = C$

   the equation becomes $\frac{k}{m}C = -\mu g \implies C = -\frac{\mu gm}{k}$

   the general and the particular solution is :

   $x = A\cos(w_0 t) + B\sin(w_0 t) - \frac{\mu gm}{k}$

   $\begin{cases} x(0) = 0 \\ v(0) = 0 \end{cases} \implies \begin{cases} A = \frac{\mu gm}{k} \\ Bw_0 = v_0 \end{cases}$

   So $A = \frac{\mu gm}{k}$ and $B = \frac{v_0}{w_0}$

   the equation becomes $x = \frac{\mu gm}{k}\cos(w_0 t) + \frac{v_0}{w_0}sin(w_0 t) - \frac{\mu gm}{k}$

3. Give the solution in the case of:
   (a) Damped harmonic oscillator,
   (b) Weak damping,
   (c) Critical damping.

4. Consider the case of a driven damped oscillator with the external force given by $F = F_0 e^{iwt}$
   (a) Find the amplitude of vibration.
   (b) Suppose that m = 5 kg, $\mu = 0.45kg/s$, K = 10 N/m, $F_0 = 0.5N$, and $w = \pi rad/s$:
      - Plot the time histories of the system.
      - Plot the frequency resonance curve and comment.

Exercise 3

A particle of mass m is moving in the xy-plane with a position vector described by: $r = acos(wt)e_x + bsin(wt)e_y$,
where $a, b > 0$ and $a > b$.

1. Draw the motion of the particle and locate the points A and B, where the particle lies along the positive Ox and Oy axes, respectively. Give the name of this geometrical figure.

   - when the particle lies along the positive Ox: $y = bsin(wt) = 0 \implies wt = n\pi$ with $n \in \mathbb{Z}$ hence the point A = (a, 0)
   - when the particle lies along the positive axes Oy: $x = acos(wt) = 0 \implies cos(wt) = 0 \implies wt = \frac{\pi}{2} + n\pi$ with $n \in \mathbb{Z}$
     Hence the point $B = (0, b)$
     in order to draw the equation of the motion : $x = acos(wt)$ and $y = bsin(wt)$
     $\begin{cases} \frac{x}{a} = cos(wt) \\ \frac{y}{b} = sin(wt) \end{cases}$ Hence $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$ This is equation of ellipse

2. Compute the force $\vec{F}$ producing this motion, and show that it is directed towards the origin.

$$\vec{V}(t) = \frac{dr(t)}{dt} = -awsin(wt)\vec{e_x} + bwcos(wt)\vec{e_y}$$
$$\vec{a}(t) = \frac{dV(t)}{dt} = -aw^2cos(wt)\vec{e_x} + -bw^2sin(wt)\vec{e_y}$$

$\vec{F} = m\vec{a} = -mw^2(-acos(wt)\vec{e_y} + bsin(wt)\vec{e_y})$

3. Compute the kinetic energy of the particle at any instant of time t, and give its value at the points A and B.

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\sqrt{((aw)^2 + (bw)^2} \\ &= \frac{1}{2}mw\sqrt{(a^2 + b^2} \end{aligned}$$

   - At the point A = (a, 0) $v = aw$ and $K_a = \frac{1}{2}m(aw)^2$
   - At the point B = (0, 0) $v = aw$ and $K_B = \frac{1}{2}m(bw)^2$

4. Compute the work of the force \$$vecF$ between the points A and B, and show that it is equal to the difference of kinetic energies at these points.

$$
\begin{aligned}
W_{A-B} &= \int_A^B F_x dx + F_y dy \\
&= \int_A b - mw^2 acos(wt)dx - bmw^2 sin(wt)dy \\
&= \int_a^0 -mw^2 x dx + \int_0^b -mw^2 y dy \\
&= \left[-\frac{1}{2}mw^2x^2\right]_a^0 + \left[-\frac{1}{2}mw^2x^2\right]_0^b \\
&= +\frac{1}{2}mw^2a^2 - \frac{1}{2}mw^2b^2 \\
&= \frac{1}{2}mw^2\left(a^2 - b^2\right)
\end{aligned}
$$

$W_{A-B} = K_B - K_A = \frac{1}{2}mw^2b^2 - \frac{1}{2}mw^2a^2 = \frac{1}{2}mw^2(b^2 - a^2)$

5. Compute the work of the force $\vec{F}$ during a full period of the motion (i.e., when the particle starts and ends again at A, for example).

   when the particle starts and ends again at A, $W_{A-B} = W_{A-A} = 0$ we just replace $B = A$ in the previous result.

   Conclude from this that the force is conservative, and show that this agrees with the result of the calculation $rot\vec{f} = 0$. Find a potential V (x, y) which gives rise to this conservative force.

   we know that the work done in the full period (from A to A) is equal to zero which means the force is conservative, Hence The $rot\vec{f} = 0$

   now, let's calculate the potential V(x,y) $\vec{F} = -\nabla V(x, y)$

   $\begin{cases} F_x = -\frac{\partial V(x,y)}{\partial x} \\ F_y = -\frac{\partial V(x,y)}{\partial y} \end{cases}$

$$
\begin{aligned}
-amw^2cos(wt) &= -\frac{\partial V(x,y)}{\partial x} \\
-mw^2x &= \frac{\partial V(x,y)}{\partial x} \\
V(x,y) &= \int mw^2 x dx \\
V(x,y) &= \frac{1}{2}mw^2x^2 + V(y)
\end{aligned}
$$

   the the second terms we have :

$$
\begin{aligned}
-mw^2y &= -V'(y) \\
v(y) &= \frac{1}{2}mw^2y^2
\end{aligned}
$$

Hence $V(x, y) = \frac{1}{2}mw^2x^2 + \frac{1}{2}mw^2y^2 = \frac{1}{2}mw^2(x^2 + y^2)$