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Problem 1: O() notation

In this problem, you can always assume that f, g, h : $N_{>0} \to R_{>0}$. That is, the functions always take strictly positive real values.

(a) If $f \in o(g)$ and $g \in o(h)$, then $f \in o(h)$. by definition, $f \in o(g) \implies \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0$ and $g \in o(h) \implies \lim_{n\to\infty}\left(\frac{g(n)}{h(n)}\right) = 0$.

we kwon that, $\forall n \in \mathbb{N}, g(n) > 0, \lim_{n\to\infty}\left(\frac{f(n)}{h(n)}\right) = \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\frac{f(n)}{g(n)}\right) = 0 * 0 = 0$

Hence by definition, $f \in o(h)$

(b) If $exp(f) \in O(exp(g))$, then $f \in O(g)$.

by definition, if $exp(f) \in O(exp(g))$, $\implies \exists C > 0, n_0 \in \mathbb{N}, /\forall n \geq n_0, exp(f) < Cexp(g)$

$$\begin{aligned} exp(f) &\leq Cexp(g) \\ \implies \log(exp(f)) &\leq \log(Cexp(g)) \\ \implies f &< \log(C) + g \end{aligned}$$

let's take case by case:

- if $C \in [0, 1]$, $log(C) < 0$ and $f < log(C) + g < g$
  Hence, $f \in O(g)$
- $C > 1$, we can always find K such as $f < log(C) + g < Kg$
  Hence, $f \in O(g)$

In conclusion, The statement is true

(c) If $g(n) \in o(f(n))$, then $f(n)g(n) = O(f(n))$. this statement is false let's prove by the counterexample.

if $f(n) = n^2$ and $g(n) = n$ and $lim_{n\to\infty}\frac{g(n)}{f(n)} = \lim_{n\to\infty}\frac{n}{n^2} = 0 \implies g \in o(f)$ and the other hand

$f(n)g(n) = n^3 \notin O(n^2)$, Hence are found the counterexample such as the statement is not true

That mean that the statement is not always true

(d) There exist f and g such that $f \in o(g)$ and $g \in o(f)$. prove that is statement is false by Contradiction, let's suppose that, $f \in o(g)$ by definition, $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0$ and

$\lim_{n\to\infty}\dfrac{g(n)}{f(n)} = \lim_{n\to\infty}\dfrac{1}{\frac{f(n)}{g(f)}} = \infty \implies g \notin o(f)$ this is a contradiction because when $f \in o(g)$ $g \notin o(f)$

(e) There exist f and g such that $f \notin O(g)$ and $g \notin O(f)$.

this statement is true because we found a one case that statement is true.

let's $f(n) = 1$ and $g(n) = |nsin(n)|$ we want to prove that $f \notin O(g)$ and $g \notin O(f)$

let's prove this by contradiction,

let's suppose that $f \in O(g)$ and $g \in O(f)$, by definition:

- let's assume that $f \in O(g) \implies \exists C > 0, n_0 \in \mathbb{N}, /\forall n \geq n_0 f(n) \leq Cg(n) \implies 1 \leq C|nsin(n)|$
  we know that $\forall n \in \mathbb{N}, 0 \leq |nsin(n)| \leq n$
  that for any given C, we can always find value of $n \geq n_0$ such as $|nsin(n)| < \frac{1}{n}$ because $\implies f > Cg(n)$ this contracts our assumption, Hence $f \notin O(g)$
- let's assume that $g \in O(f) \implies \exists C > 0, n_0 \in \mathbb{N}/g(n) \leq Cf(n) \implies |nsin(n)| \leq C$
  we know that $\forall n \in \mathbb{N}, 0 \leq |nsin(n)| \leq n$
  that for any given C, we can always find value of $n \geq n_0$ such as $|nsin(n)| > c$ $\implies g(n) > cf(n)$ this contradicts our assumptions, Hence $f \notin O(g)$

The statement is true because we take a one example that the statement is true and prove by contradiction

(f) $log(n!) = O(n^2)$

we want to prove that $\exists C > 0, n_0 \in \mathbb{N}/\forall n \geq n_0, log(n!) \leq Cn^2$

$$\begin{aligned} &\forall n \in \mathbb{N}, n! \leq n^n \\ \implies &\log(n!) \leq n\log(n) \leq n^2 \end{aligned}$$

just take $n_0 = 1$ and $c = 1$, that means the statement is true

Problem 2: A recurrence

Let T : $N_{>0} \to N_{>0}$ and $C > 0$. Assume that T satisfies the following recurrence relation for every $n > 2$:

$$T(n) \leq 2T([\frac{n}{3}]) + Cn$$

Use induction and write a careful proof that $T(n) \in O(n)$.

- base case n=3 :

  $T(3) \leq 2T[1] + C3 \leq 2T(1) + C3 \leq 3T(1) + C3 \leq 3(T(1) + C)$

  based on the expression bellow,

  $\forall n \geq 3$, if we take $k = (T(1) + c)$ we have $T(3) \leq 3(T(1) + C)$ that means $T(3) \in O(3)$

  Hence, the base case is true

- Induction step: let's suppose that $\forall n > 3$, $T(n) \in O(n)$ and let's prove that $T(n+1) \in O(n+1)$

  we want to prove that $\exists C > 0, n_o \in \mathbb{N}, T(n+1) \leq C(n+1)$

  we know that:

$$T(n+1) \leq 2T[\frac{n+1}{3}] + C(n+1) \leq 2T[(\frac{n}{3} + \frac{1}{3}] + C(n+1) \quad (1)$$

  and we know that $\forall n > 3, [(\frac{n}{3}+\frac{1}{3}] < n$ that mean by the hypothesis of induction $T[(\frac{n}{3}+\frac{1}{3}] \in O([(\frac{n}{3} + \frac{1}{3}])$

  and $O([(\frac{n}{3} + \frac{1}{3}]) \in O(n) \in O(n+1)$ by the theorem of transitivity of big O $T[(\frac{n}{3} + \frac{1}{3}] \in O(n+1)$

  $\forall n > 3, T[(\frac{n}{3} + \frac{1}{3}] \in O(n+1)$

  $\implies \exists K > 0, n_0 \in \mathbb{N}, \forall n \geq n_0, \qquad T[(\frac{n}{3} + \frac{1}{3}] \leq K(n+1)$

  (1) become,

$$\begin{aligned} T(n+1) &\leq 2k(n+1) + C(n+1) \\ &\leq 3k(n+1) + C(n+1) \\ &\leq (n+1)(3k+C) \qquad \leq K'(n+1) \qquad \text{Where} \quad K' = 3K + C \end{aligned}$$

  Hence, $\forall n > 3 \exists K' = 3k + C > 0, n_0 = n_0 \in \mathbb{N}, \forall n \geq n_0, \qquad T(n+1) \leq K'(n+1)$ by the definition of Big O , $T(n+1) \in O(n+1)$

  Hence, the induction step is also true,

- Conclusion,

  The base case is true and the induction step is also true, by induction $T(n) \in O(n)$