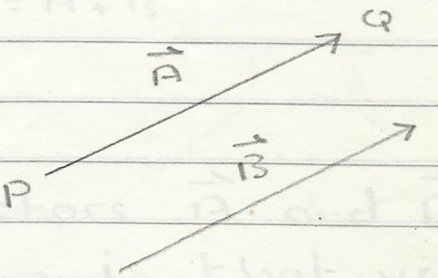
Chapter 1 : Vector Analysis

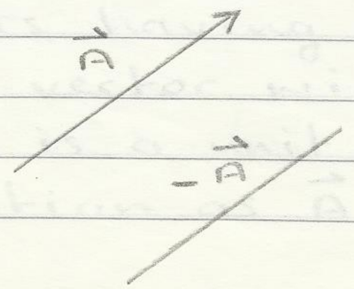
A "Vector" is a directed line segment $\vec{PQ}$ from one point P called the "initial point" to another point Q called the "terminal point". Vectors are denoted by bold faced letters or letters with an arrow over them. The "magnitude" or "length" of the vector is denoted by $|\vec{PQ}|$.



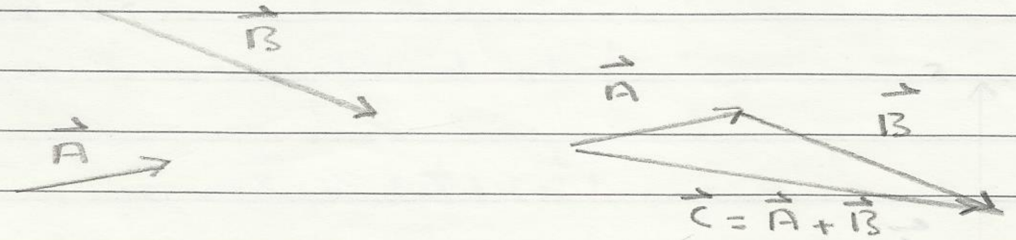
Vector Algebra:

1- Two vectors $\vec{A}$ and $\vec{B}$ are "equal" if they have the same magnitude and direction regardless of their initial points. Thus $\vec{A} = \vec{B}$ in the figure above.

2- A vector having direction opposite to that of vector $\vec{A}$ but with the same magnitude is denoted by $-\vec{A}$.



3- The "sum" or "resultant" of vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C}$ formed by placing the initial point of $\vec{B}$ on the terminal point of $\vec{A}$ and joining the initial point of $\vec{A}$ to the terminal point of $\vec{B}$.



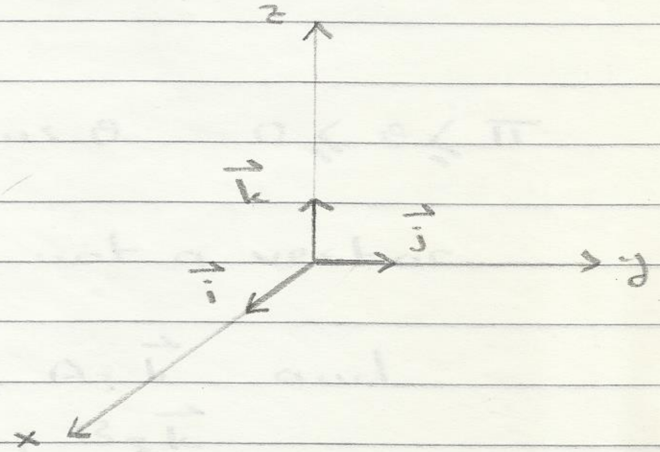
4- The "difference" of vectors $\vec{A}$ and $\vec{B}$, represented by $\vec{A} - \vec{B}$, is that vector $\vec{C}$ which added to $\vec{B}$ gives $\vec{A}$. Equivelantly, $\vec{A} - \vec{B}$ may be defined as $\vec{A} + (-\vec{B})$.

5- Multiplication of a vector $\vec{A}$ by a scalar m produces a vector $m\vec{A}$ with magnitude $|m|$ times the magnitude of $\vec{A}$ and direction the same as or opposite to that of $\vec{A}$ according as m is positive or negative.

Unit Vectors:

"Unit Vectors" are vectors having unit length. If $\vec{A}$ is any vector with length $|\vec{A}| > 0$, then $\vec{A} / |\vec{A}|$ is a unit vector having the same direction as $\vec{A}$.

Rectangular Unit Vectors:
The rectangular unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors having the direction of the positive x, y and z axes of a rectangular coordinate system. We use right-handed rectangular coordinate systems.



Components of a Vector:
Any vector can be represented with initial point at the origin O of a rectangular coordinate system. Let (A_1, A_2, A_3) be the coordinates of the terminal point of vector $\vec{A}$ with initial point at O. The vectors $A_1\vec{i}$, $A_2\vec{j}$ and $A_3\vec{k}$ are called the "component vectors" of $\vec{A}$ in the x-, y- and z- directions respectively. We write

$$\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

The magnitude of $\vec{A}$ is $|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$

The "position vector" $\vec{r}$ from O to the point (x, y, z) is written

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

and has magnitude $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Dot or Scalar Product :

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta \qquad 0 \leqslant \theta \leqslant \pi$$

$\vec{A} \cdot \vec{B}$ is a scalar and not a vector.

If $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ and
$\vec{B} = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$

Then $\vec{A} \cdot \vec{B} = A_1B_1 + A_2B_2 + A_3B_3$
$\vec{A} \cdot \vec{A} = A_1^2 + A_2^2 + A_3^2$
$\vec{B} \cdot \vec{B} = B_1^2 + B_2^2 + B_3^2$.

Cross or Vector Product :

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \, \vec{u} \qquad 0 \leqslant \theta \leqslant \pi$$

$\vec{A} \times \vec{B}$ is a vector whose direction is indicated by the unit vector $\vec{u}$ which is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and $\vec{A}$, $\vec{B}$ and $\vec{u}$ form a right-handed system.

If $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ and
$\vec{B} = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$,

Then $\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$

Vector Functions:

If corresponding to each value of a scalar "u" we associate a vector $\vec{A}$, then $\vec{A}$ is called a "function of u" denoted by $A(u)$. We can write $\vec{A}(u) = A_1(u)\,\vec{i} + A_2(u)\,\vec{j} + A_3(u)\,\vec{k}$.

The function concept is easily extended. Thus, if to each point (x, y, z) there corresponds a vector $\vec{A}$, then $\vec{A}$ is a function of (x, y, z) indicated by

$$\vec{A}(x,y,z) = A_1(x,y,z)\,\vec{i} + A_2(x,y,z)\,\vec{j} + A_3(x,y,z)\,\vec{k}.$$

A vector function $\vec{A}(x,y,z)$ defines a "vector field" since it associates a vector with each point of a region. Similarly $\phi(x,y,z)$ defines a "scalar field" since it associates a scalar with each point of a region.

Limits, Continuity and Derivatives of Vector Functions:

1- The vector function $\vec{A}(u)$ is said to be "continuous" at u_0 if given any positive number ϵ, we can find some positive number δ such that $|\vec{A}(u) - \vec{A}(u_0)| < \epsilon$ whenever $|u - u_0| < \delta$. This is equivalent to the statement $\lim_{u \to u_0} \vec{A}(u) = \vec{A}(u_0)$.

2- The derivative of $\vec{A}(u)$ is defined as

$$\frac{d\vec{A}}{du} = \lim_{\Delta u \to 0} \frac{\vec{A}(u+\Delta u) - \vec{A}(u)}{\Delta u}$$

provided this limit exists.

In case $\vec{A}(u) = A_1(u)\,\vec{i} + A_2(u)\,\vec{j} + A_3(u)\,\vec{k}$;

then $$\frac{d\vec{A}}{du} = \frac{dA_1}{du}\,\vec{i} + \frac{dA_2}{du}\,\vec{j} + \frac{dA_3}{du}\,\vec{k}$$

3- If $\vec{A}(x,y,z) = A_1(x,y,z)\,\vec{i} + A_2(x,y,z)\,\vec{j} + A_3(x,y,z)\,\vec{k}$,

then $$d\vec{A} = \frac{\partial \vec{A}}{\partial x}\,dx + \frac{\partial \vec{A}}{\partial y}\,dy + \frac{\partial \vec{A}}{\partial z}\,dz$$

is the "differential" of $\vec{A}$.

4- Derivatives of products obey rules similar to those for scalar functions. However, when cross products are involved the order may be important.

a- $$\frac{d}{du}(\phi\vec{A}) = \phi\frac{d\vec{A}}{du} + \frac{d\phi}{du}\vec{A}$$

b- $$\frac{\partial}{\partial y}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{\partial \vec{B}}{\partial y} + \frac{\partial \vec{A}}{\partial y}\cdot\vec{B}$$

c- $$\frac{\partial}{\partial z}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{\partial \vec{B}}{\partial z} + \frac{\partial \vec{A}}{\partial z}\times\vec{B}.$$

Example:

If $\vec{A}$ and $\vec{B}$ are differentiable functions of a scalar u, prove:

$$\frac{d}{du}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{d\vec{B}}{du} + \frac{d\vec{A}}{du}\cdot\vec{B}$$

Method (1)

$$\frac{d}{du}(\vec{A}\cdot\vec{B}) = \lim_{\Delta u\to 0}\frac{(\vec{A}+\Delta\vec{A})\cdot(\vec{B}+\Delta\vec{B}) - \vec{A}\cdot\vec{B}}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\frac{\vec{A}\cdot\Delta\vec{B} + \Delta\vec{A}\cdot\vec{B} + \Delta\vec{A}\cdot\Delta\vec{B}}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\left(\vec{A}\cdot\frac{\Delta\vec{B}}{\Delta u} + \frac{\Delta\vec{A}}{\Delta u}\cdot\vec{B} + \frac{\Delta\vec{A}}{\Delta u}\cdot\Delta\vec{B}\right)$$

$$= \vec{A}\cdot\frac{d\vec{B}}{du} + \frac{d\vec{A}}{du}\cdot\vec{B}$$

Method (2)

Let $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$,
$\vec{B} = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$.

Then
$$\frac{d}{du}(\vec{A}\cdot\vec{B}) = \frac{d}{du}(A_1B_1 + A_2B_2 + A_3B_3)$$

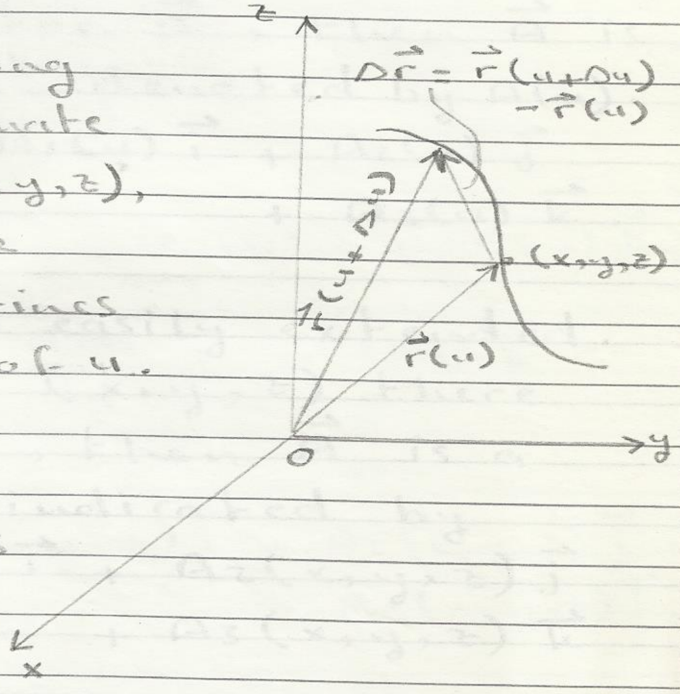
$$= \left(A_1\frac{dB_1}{du} + A_2\frac{dB_2}{du} + A_3\frac{dB_3}{du}\right)$$

$$+ \left(\frac{dA_1}{du}B_1 + \frac{dA_2}{du}B_2 + \frac{dA_3}{du}B_3\right)$$

$$= \vec{A}\cdot\frac{d\vec{B}}{du} + \frac{d\vec{A}}{du}\cdot\vec{B}$$

Geometric Interpretation of a Vector Derivative:

If $\vec{r}$ is the vector joining the origin O of a coordinate system and the point (x, y, z), then specification of the vector function $\vec{r}(u)$ defines x, y and z as functions of u.



As u changes, the terminal point of $\vec{r}$ describes a "space curve" having parametric equations
$x = x(u)$, $y = y(u)$, $z = z(u)$.

If the parameter u is the arc length s measured from some fixed point on the curve, then

$$\frac{d\vec{r}}{ds} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} + \frac{dz}{ds}\vec{k} = \vec{T}$$

is a unit vector in the direction of the tangent to the curve and is called the "unit tangent vector".

If u is the time t, then $\frac{d\vec{r}}{dt}$ represents the "velocity" $\vec{v}$ with which the terminal point of $\vec{r}$ describes the curve. Similarly, $\frac{d^2\vec{r}}{dt^2}$ represents its "acceleration" $\vec{a}$ along the curve.

Gradient, Divergence and Curl :

The vector differential operator ∇ (del), is defined by

$$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} .$$

This vector operator possesses properties analogous to those of ordinary vectors. It is useful in defining three quantities which arise in practical applications and are known as the "gradient", the "divergence" and the "curl".

1- Gradient. Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e ϕ defines a differentiable scalar field). Then the "gradient" of ϕ is defined by

$$\text{grad}\,\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

An interesting interpretation is that if $\phi(x,y,z) = c$ is the equation of a surface, then $\nabla\phi$ is a normal to this surface.

2- Divergence. Let $\vec{A}(x,y,z) = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. $\vec{A}$ defines a differentiable vector field). Then the "divergence" of $\vec{A}$ is defined by

$$\operatorname{div}\vec{A} = \nabla\cdot\vec{A} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(A_1\vec{i} + A_2\vec{j} + A_3\vec{k}\right)$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

3- Curl. If $\vec{A}(x,y,z)$ is a differentiable vector field then the "curl" of $\vec{A}$, is defined by

$$\operatorname{curl}\vec{A} = \nabla\times\vec{A} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(A_1\vec{i} + A_2\vec{j} + A_3\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\vec{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\vec{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\vec{k}$$

In the expansion of the determinant, the operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ must precede A_1, A_2, A_3.

Formulas Involving ∇:

If the partial derivatives of $\vec{A}$, $\vec{B}$, u and v are assumed to exist, then

1- $\nabla(u+v) = \nabla u + \nabla v$ or $\text{grad}(u+v) = \text{grad}\,u + \text{grad}\,v$.

2- $\nabla \cdot (\vec{A}+\vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$ or $\text{div}(\vec{A}+\vec{B}) = \text{div}\vec{A} + \text{div}\vec{B}$

3- $\nabla \times (\vec{A}+\vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$ or $\text{curl}(\vec{A}+\vec{B}) = \text{curl}\,\vec{A} + \text{curl}\,\vec{B}$.

4- $\nabla \cdot (u\vec{A}) = (\nabla u) \cdot \vec{A} + u(\nabla \cdot \vec{A})$

5- $\nabla \times (u\vec{A}) = (\nabla u) \times \vec{A} + u(\nabla \times \vec{A})$.

6- $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

7- $\nabla \cdot (\nabla u) = \nabla^2 u \equiv \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

is called the "Laplacian" of u

and $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the "Laplacian operator".

8- $\nabla \times (\nabla u) = 0$. The curl of the gradient of u is zero.

9- $\nabla \cdot (\nabla \times \vec{A}) = 0$. The divergence of the curl of $\vec{A}$ is zero.

10- $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$.

Example (1):

If $\phi = x^2 y z^3$ and $\vec{A} = xz\,\vec{i} - y^2\,\vec{j} + 2x^2 y\,\vec{k}$,
find a- $\nabla\phi$, b- $\nabla \cdot \vec{A}$, c- $\nabla \times \vec{A}$, d- $\mathrm{div}(\phi\vec{A})$
e- $\mathrm{curl}(\phi\vec{A})$.

a- $\nabla\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\phi$

$= \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$

$= 2xyz^3\,\vec{i} + x^2 z^3\,\vec{j} + 3x^2 y z^2\,\vec{k}$.

b- $\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (xz\,\vec{i} - y^2\,\vec{j} + 2x^2 y\,\vec{k})$

$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2 y)$

$= z - 2y$.

c- $\nabla \times \vec{A} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times (xz\,\vec{i} - y^2\,\vec{j} + 2x^2 y\,\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y^2 & 2x^2 y \end{vmatrix}$$

$= (2x^2 - 0)\,\vec{i} - (4xy - x)\,\vec{j} + (0 - 0)\,\vec{k}$

$= 2x^2\,\vec{i} + (x - 4xy)\,\vec{j}$

d- $\text{div}(\phi \vec{A}) = \nabla \cdot (\phi \vec{A})$

$$= \nabla \cdot (x^3 y z^4 \vec{i} - x^2 y^3 z^3 \vec{j} + 2x^4 y^2 z^3 \vec{k})$$

$$= \frac{\partial}{\partial x}(x^3 y z^4) + \frac{\partial}{\partial y}(-x^2 y^3 z^3) + \frac{\partial}{\partial z}(2x^4 y^2 z^3)$$

$$= 3x^2 y z^4 - 3x^2 y^2 z^3 + 6x^4 y^2 z^2$$

e- $\text{curl}(\phi \vec{A}) = \nabla \times (\phi \vec{A})$

$$= \nabla \times (x^3 y z^4 \vec{i} - x^2 y^3 z^3 \vec{j} + 2x^4 y^2 z^3 \vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 y z^4 & -x^2 y^3 z^3 & 2x^4 y^2 z^3 \end{vmatrix}$$

$$= (4x^4 y z^3 + 3x^2 y^3 z^2)\vec{i}$$

$$+ (4x^3 y z^3 - 8x^3 y^2 z^3)\vec{j}$$

$$- (2x y^3 z^3 + x^3 z^4)\vec{k}$$

Example (2):

Prove that $\nabla \phi$ is a vector perpendicular to the surface $\phi(x, y, z) = c$, where "c" is a constant.

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector to any point $P(x, y, z)$ on the surface.

Then $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$ lies in the plane tangent to the surface at P.

But $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0$

or $\left(\frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k} \right) \cdot \left(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k} \right) = \vec{0}$

i.e. $\nabla \phi \cdot d\vec{r} = 0$ so that $\nabla \phi$ is perpendicular to $d\vec{r}$ and therefore to the surface.

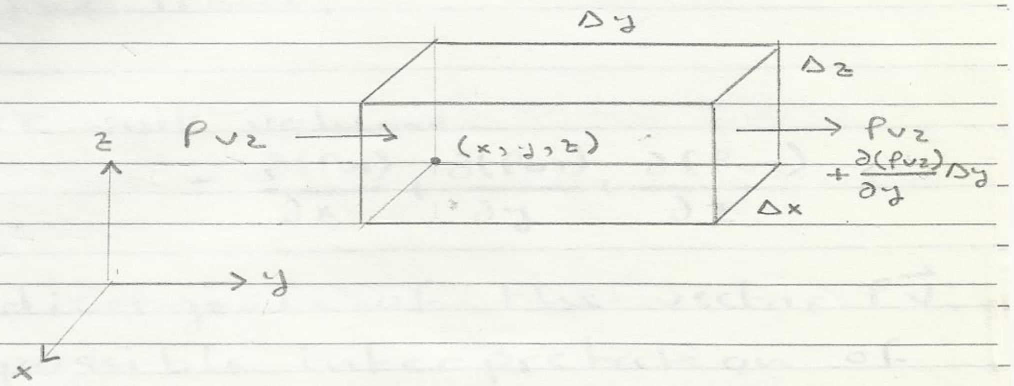
$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2.$

ii. $\nabla \cdot \nabla\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right).$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi.$$



To give a physical interpretation for the divergence:
Consider a typical volume element in a region filled with a moving fluid.

Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be a vector function representing at each point the instantaneous velocity of the particle of fluid.

The loss of fluid through each face

Right face : $\left[\rho v_2 + \dfrac{\partial(\rho v_2)}{\partial y}\Delta y\right]\Delta x\,\Delta z\,\Delta t$

Left face : $-\rho v_2\,\Delta x\,\Delta z\,\Delta t$ (Density × Volume = Mass)

Front face : $\left[\rho v_1 + \dfrac{\partial(\rho v_1)}{\partial x}\Delta x\right]\Delta y\,\Delta z\,\Delta t$

Rear face : $-\rho v_1\,\Delta y\,\Delta z\,\Delta t$

Top face : $\left[\rho v_3 + \dfrac{\partial(\rho v_3)}{\partial z}\Delta z\right]\Delta x\,\Delta y\,\Delta t$

Bottom face : $-\rho v_3\,\Delta x\,\Delta y\,\Delta t$

Adding, and dividing by $\Delta V \Delta t \equiv \Delta x \Delta y \Delta z \Delta t$, we obtain in the limit

Rate of loss per unit volume

$$= \frac{\partial(\rho v_1)}{\partial x} + \frac{\partial(\rho v_2)}{\partial y} + \frac{\partial(\rho v_3)}{\partial z}$$

which is the divergence of the vector $\rho\vec{v}$. Therefore one possible interpretation of the divergence is the rate of loss of fluid per unit volume.

For an incompressible fluid, the density ρ is constant, and there is neither gain nor loss in a general element. Hence

$$\nabla \cdot (\rho\vec{v}) = \rho\, \nabla \cdot \vec{v} = 0$$

which is known as the "equation of continuity" for incompressible fluids.

To interpret the "curl" physically:

If $\vec{v} = \vec{\omega} \times \vec{r}$

Where $\vec{v}$ is the linear velocity vector

$\vec{\omega}$ is the angular velocity vector.

Then $\text{curl } \vec{v} = \nabla \times \vec{v} = \nabla \times (\vec{\omega} \times \vec{r})$

$$= \nabla \times \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[(\omega_2 z - \omega_3 y)\,\vec{i} + (\omega_3 x - \omega_1 z)\,\vec{j} + (\omega_1 y - \omega_2 x)\,\vec{k} \right].$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

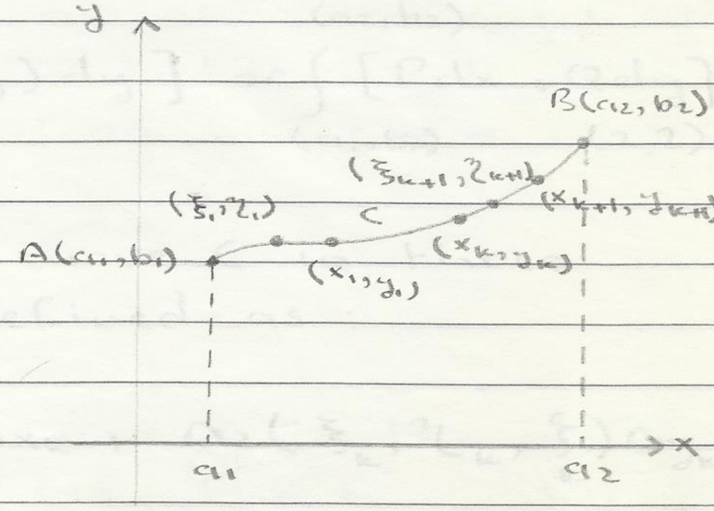
$$= 2(\omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}) = 2\vec{\omega}$$

i.e. $\vec{\omega} = \frac{1}{2} \nabla \times \vec{v} = \frac{1}{2} \text{curl } \vec{v}$

i.e. The angular velocity of a rotating body is equal to one-half the curl of the linear velocity of any point of the body.

Chapter 2: Line Integrals and Green's Theorem in the Plane.

Line Integrals:



Let C be a curve in the xy-plane which connects points $A(a_1, b_1)$ and $B(a_2, b_2)$. Let $P(x,y)$ and $Q(x,y)$ be single-valued functions defined at all points of C. Subdivide C into n parts by choosing $(n-1)$ points on it given by $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1})$. Call $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$, $k = 1, 2, \ldots, n$ where $(a_1, b_1) \equiv (x_0, y_0)$, $(a_2, b_2) \equiv (x_n, y_n)$ and suppose that points (ξ_k, η_k) are chosen so that they are situated on C between points (x_{k-1}, y_{k-1}) and (x_k, y_k). Form the sum

$$\sum_{k=1}^{n} \{ P(\xi_k, \eta_k) \Delta x_k + Q(\xi_k, \eta_k) \Delta y_k \} \qquad (2.1)$$

The limit of this sum as $n \to \infty$ in such a way that all the quantaties $\Delta x_k, \Delta y_k$ approach zero, is called a "line integral" along C and is denoted by

$$\int_C [P(x,y)\,dx + Q(x,y)\,dy] \quad \text{or} \int_{(a_1,b_1)}^{(a_2,b_2)} [P\,dx + Q\,dy] \qquad -(2.2)$$

A line integral along a curve C in three dimensional space is defined as:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \{ A_1(\xi_k, \eta_k, \zeta_k)\,\Delta x_k + A_2(\xi_k, \eta_k, \zeta_k)\,\Delta y_k + A_3(\xi_k, \eta_k, \zeta_k)\,\Delta z_k \}$$

$$= \int_C [A_1\,dx + A_2\,dy + A_3\,dz] \qquad -(2.3)$$

where A_1, A_2 and A_3 are functions of x, y and z.

Vector Notation for Line Integrals:

It is often convenient to express a line integral in vector form as an aid in physical or geometric understanding as well as for brevity of notation.

$$\int_C [A_1\,dx + A_2\,dy + A_3\,dz]$$

$$= \int_C (A_1\vec{i} + A_2\vec{j} + A_3\vec{k}) \cdot (dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \int_C \vec{A} \cdot d\vec{r} \qquad -(2.4)$$

If C is given in parametric form $x = \phi(t)$, $y = \psi(t)$, the line integral becomes

$$\int_{t_1}^{t_2} [P\{\phi(t), \psi(t)\}\phi'(t)\,dt + Q\{\phi(t), \psi(t)\}\psi'(t)\,dt] \quad \text{--- (2.7)}$$

where t_1 and t_2 denote the values corresponding to points A and B respectively.

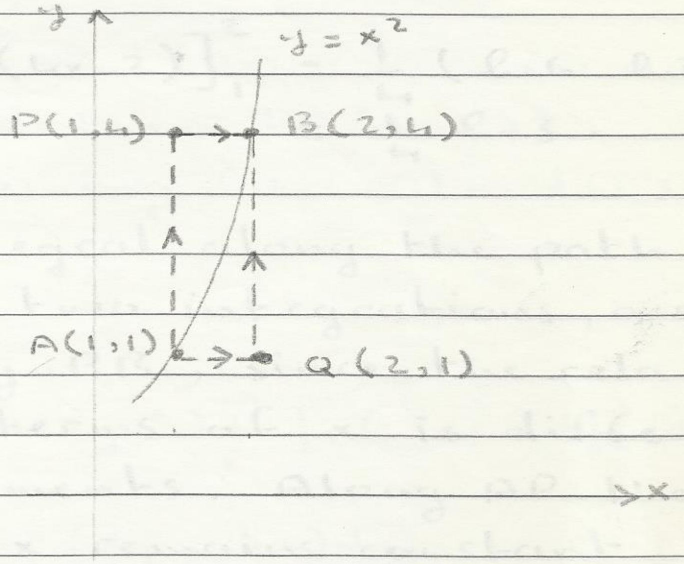
Properties of Line Integrals:

1. $$\int_C [P(x,y)\,dx + Q(x,y)\,dy] = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

2. $$\int_{(a_1,b_1)}^{(a_2,b_2)} [P\,dx + Q\,dy] = -\int_{(a_2,b_2)}^{(a_1,b_1)} [P\,dx + Q\,dy]$$

3. $$\int_{(a_1,b_1)}^{(a_2,b_2)} [P\,dx + Q\,dy] = \int_{(a_1,b_1)}^{(a_3,b_3)} [P\,dx + Q\,dy] + \int_{(a_3,b_3)}^{(a_2,b_2)} [P\,dx + Q\,dy]$$

Example (1):
What is the value of $\int_A^B [1/(x+y)]\,dx$ along each of the paths shown in figure.



Before this integral can be evaluated, y must be expressed in terms of x.
To do this, we recall from the definition of a line integral that the integrand is always to be evaluated ALONG THE PATH OF INTEGRATION.
Along the parabolic arc joining A and B, we have $y = x^2$

$$\int_1^2 \frac{dx}{x + x^2} = \int_1^2 \left(\frac{1}{x} - \frac{1}{1+x} \right) dx$$

$$= \Big[\ln x - \ln(1+x) \Big]_1^2 = \ln\left(\frac{4}{3}\right)$$

Along the straight-line path from A to B, we have $y = 3x - 2$, and making the substitution in the integrand, we obtain the ordinary definite integral

$$\int_1^2 \frac{dx}{x+(3x-2)} = \frac{1}{4}\left[\ln(4x-2)\right]_1^2 = \frac{1}{4}(\ln 6 - \ln 2)$$
$$= \frac{1}{4}\ln 3 .$$

To compute the line integral along the path APB, we must perform two integrations, one along AP and one along PB, since the relation expressing y in terms of x is different on these two segments. Along AP the integral is zero, since x remains constant and therefore in the sum leading to the integral each Δx_k is zero. Along PB, on which $y = 4$, we have the integral

$$\int_1^2 \frac{dx}{x+4} = \left[\ln(x+4)\right]_1^2 = \ln\frac{6}{5} .$$

Along the path AQB we again have two integrations to perform. Along AQ, on which $y = 1$, we have the integral

$$\int_1^2 \frac{dx}{x+1} = \left[\ln(x+1)\right]_1^2 = \ln\frac{3}{2}$$

Along the vertical segment QB the integral is again zero.

Example (2):

If $\vec{A} = (3x^2 - 6yz)\vec{i} + (2y + 3xz)\vec{j} + (1 - 4xyz^2)\vec{k}$, evaluate $\int_c \vec{A}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C:

a- $x = t$, $y = t^2$, $z = t^3$.

b- The straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(1,1,1)$.

c- The straight line joining $(0,0,0)$ and $(1,1,1)$.

$$\int \vec{A}\cdot d\vec{r} = \int_c [(3x^2 - 6yz)\vec{i} + (2y + 3xz)\vec{j} + (1 - 4xyz^2)\vec{k}]\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_c [(3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz].$$

a- If $x = t$, $y = t^2$, $z = t^3$, points $(0,0,0)$ and $(1,1,1)$ correspond to $t = 0$ and $t = 1$ respectively.

Then,

$$\vec{A} = (3t^2 - 6t^5)\vec{i} + (2t^2 + 3t^4)\vec{j} + (1 - 4t^9)\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$$

$$d\vec{r} = (\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt$$

$$\int_c \vec{A}\cdot d\vec{r} = \int_0^1 [(3t^2 - 6t^5)\,dt + (4t^3 + 6t^5)\,dt + (3t^2 - 12t^{11})\,dt]$$

$$= 2$$

b- Along the straight line from $(0,0,0)$ to $(0,0,1)$, $x=0$, $y=0$, $dx=0$, $dy=0$, while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1} [\{3(0)^2 - 6(0)(z)\}0 + \{2(0) + 3(0)(z)\}0 + \{1 - 4(0)(0)(z^2)\} dz]$$

$$= \int_{z=0}^{1} dz = 1$$

Along the straight line from $(0,0,1)$ to $(0,1,1)$, $x=0$, $z=1$, $dx=0$, $dz=0$, while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1} [\{3(0)^2 - 6(y)(1)\}0 + \{2y + 3(0)(1)\} dy + \{1 - 4(0)(y)(1)^2\}0]$$

$$= \int_{y=0}^{1} 2y\, dy = 1$$

Along the straight line from $(0,1,1)$ to $(1,1,1)$, $y=1$, $z=1$, $dy=0$, $dz=0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} [\{3x^2 - 6(1)(1)\} dx + \{2(1) + 3x(1)\}0 + \{1 - 4x(1)(1)^2\}0]$$

$$= \int_{x=0}^{1} (3x^2 - 6)\, dx = -5$$

Adding, $\int_C \vec{A}\cdot d\vec{r} = 1+1-5 = -3$

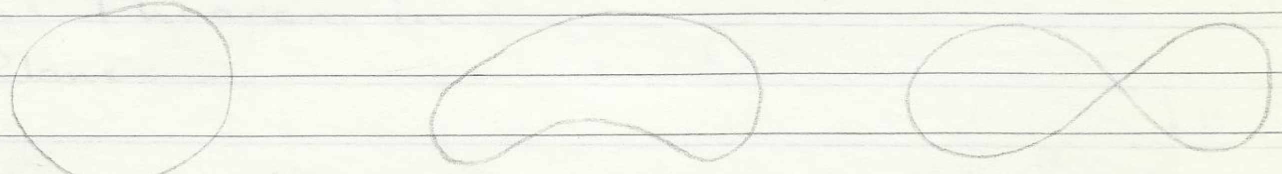
c. The straight line joining $(0,0,0)$ and $(1,1,1)$ is given in parametric form by $x = t$, $y = t$, $z = t$.

Then

$$\int_C \vec{A}\cdot d\vec{r} = \int_{t=0}^{t=1} \left[(3t^2 - 6t^2)\,dt + (2t + 3t^2)\,dt + (1 - 4t^4)\,dt \right] = \frac{6}{5}.$$

Simple Closed Curves. Simply and Multiply-Connected Regions:

A "simple closed curve" is a closed curve which does not intersect itself anywhere.



Simple   Simple   Not simple

If a plane region has the property that any closed curve in it can be continuously shrunk to a point without leaving the region, then the region is called "simply connected", otherwise it is called "multiply connected".

Examples of the former are: interior of circle, exterior of circle. Example of the latter is the area between two concentric circles.

The "positive direction" around a closed curve is defined as the direction in which an observer would move if he traversed the curve in such a way that the area of R was always on his left.

According to this definition, if R is the interior of a simple closed curve C, then the positive direction around C is the counterclockwise direction. If R is the region exterior to a simple closed curve C, then the positive direction around C is clockwise.

Green's Theorem in the Plane:

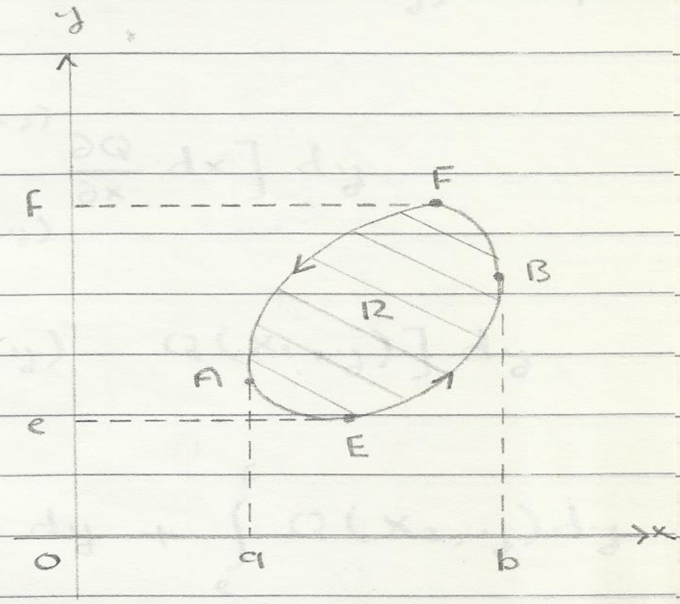
Let P, Q, $\partial P/\partial y$, $\partial Q/\partial x$ be single-valued and continuous in a simply-connected region R bounded by a simple closed curve C. Then

$$\boxed{\oint_C [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad \text{—— (2.8)}}$$

where $\oint_C$ is used to emphasize that C is closed and that it is described in the positive direction.

This theorem is also true for regions bounded by two or more closed curves (i.e. multiply-connected regions).

To prove Green's theorem in the plane if C is a closed curve which has the property that any straight line parallel to the coordinate axes cuts C in at most two points.



Let the equations of the curves AEB and AFB be $y = Y_1(x)$ and $y = Y_2(x)$ respectively. If R is the region bounded by C, we have

$$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{x=b}\left[\int_{y=Y_1(x)}^{y=Y_2(x)} \frac{\partial P}{\partial y}\,dy\right]dx$$

$$= \int_{x=a}^{x=b} \Big[P(x,y)\Big]_{y=Y_1(x)}^{y=Y_2(x)}\,dx$$

$$= \int_a^b \left[P(x,Y_2) - P(x,Y_1)\right] dx$$

$$= -\underset{C_1}{\int_a^b} P(x,Y_1)\,dx - \underset{C_2}{\int_b^a} P(x,Y_2)\,dx = -\oint_C P\,dx$$

Then $\oint_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dx\,dy$ ——(2.9)

Similarly let the equations of curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$ respectively. Then

$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_{y=e}^{y=f} \left[\int_{x=X_1(y)}^{x=X_2(y)} \frac{\partial Q}{\partial x}\,dx \right] dy$$

$$= \int_e^f \left[Q(X_2, y) - Q(X_1, y) \right] dy$$

$$= \int_f^e Q(X_1, y)\,dy + \int_e^f Q(X_2, y)\,dy$$

$$= \oint_C Q\,dy$$

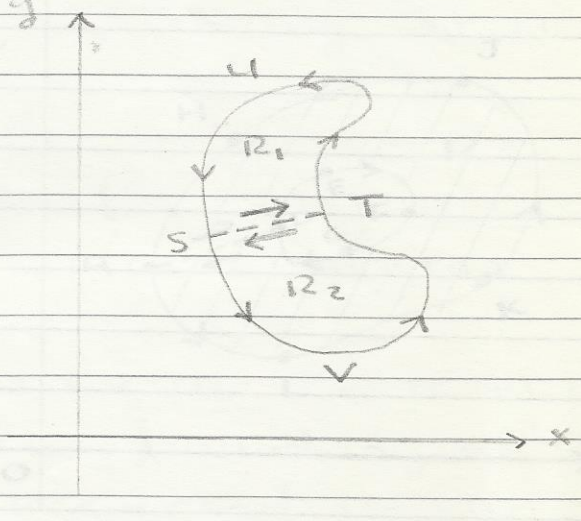
$$\oint_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dx\,dy \quad \text{——(2.10)}$$

Adding (2.9) and (2.10),

$$\oint_C [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

To extend the proof to curves for which lines parallel to the coordinate axes may cut the curve in more than two points.

Consider a closed curve C such as shown in figure. By constructing line ST the region is divided into two regions which are of the type considered in the previous proof and for which Green's theorem applies i.e.



$$\int_{STUS} [P\,dx + Q\,dy] = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad \text{—(2.11)}$$

$$\int_{SVTS} [P\,dx + Q\,dy] = \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad \text{—(2.12)}$$

Adding the left hand sides

$$\int_{STUS} + \int_{SVTS} = \cancel{\int_{ST}} + \int_{TUS} + \int_{SVT} + \cancel{\int_{TS}} = \int_{TUS} + \int_{SVT} = \int_{TUSVT}$$

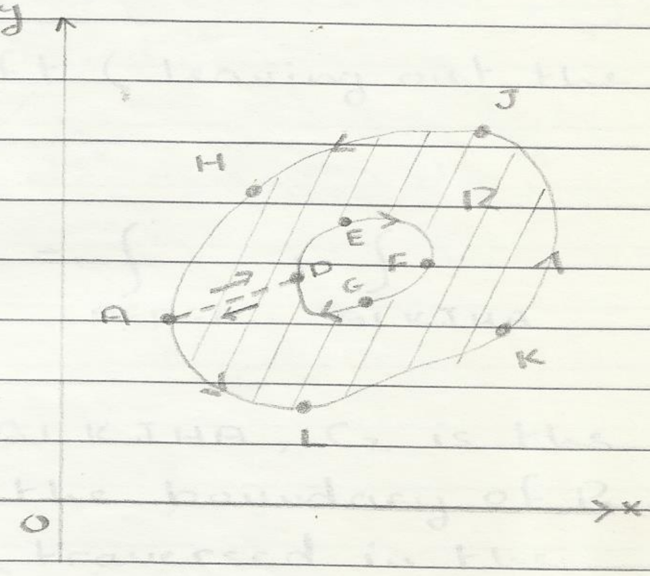
Adding the right hand sides

$$\iint_{R_1} + \iint_{R_2} = \iint_{R}$$

Then $\displaystyle\int_{TUSVT} [P\,dx + Q\,dy] = \iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$

To extend the theorem to multiply-connected regions, consider the shaded region shown in the figure.

The region is multiply-connected since not every closed curve can be shrunk to a point without leaving R, as is observed by considering a curve surrounding DEFGD for example.



The boundary of R, which consists of the exterior boundary AHJKLA and the interior boundary DEFGD is to be traversed in the positive direction, so that a person traveling in this direction always has the region on his left. It is seen that the positive directions are those indicated in the figure.

In order to establish the theorem, construct a line, such as AD, called a "cross-cut", connecting the exterior and interior boundaries. The region bounded by ADEFGDALKJHA is simply-connected, and so Green's theorem is valid.

Then

$$\int_{ADEFGDLKHJA} [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

But the integral on the left (leaving out the integrand) is equal to

$$\cancel{\int_{AD}} + \int_{DEFGD} + \cancel{\int_{DA}} + \int_{ALKJHA} = \int_{DEFGD} + \int_{ALKJHA}$$

Thus if C_1 is the curve ALKJHA, C_2 is the curve DEFGD and C is the boundary of R consisting of C_1 and C_2 (traversed in the positive directions), then $\int_{C_1} + \int_{C_2} = \int_C$

and so

$$\int_C [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

To express Green's theorem in the plane in vector notation:

We have $P\,dx + Q\,dy = (P\vec{i} + Q\vec{j}) \cdot (dx\vec{i} + dy\vec{j})$

$= \vec{A} \cdot d\vec{r}$

where $\vec{A} = P\vec{i} + Q\vec{j}$

and $\vec{r} = x\vec{i} + y\vec{j}$, $d\vec{r} = dx\vec{i} + dy\vec{j}$.

Also,

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$= -\frac{\partial Q}{\partial z}\vec{i} + \frac{\partial P}{\partial z}\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$(\nabla \times \vec{A}) \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Then Green's theorem in the plane can be written

$$\int_C \vec{A} \cdot d\vec{r} = \iint_R (\nabla \times \vec{A}) \cdot \vec{k}\,dx\,dy \quad \text{—— (2-13)}$$

The area bounded by a simple closed curve C is given by $\frac{1}{2}\int_C [x\,dy - y\,dx]$.

Putting $P = -y$, $Q = x$.

Then $\int_C [x\,dy - y\,dx] = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dx\,dy$

$$= 2\iint_R dx\,dy = 2A$$

where A is the required area.

Thus $A = \frac{1}{2}\int_C [x\,dy - y\,dx]$.

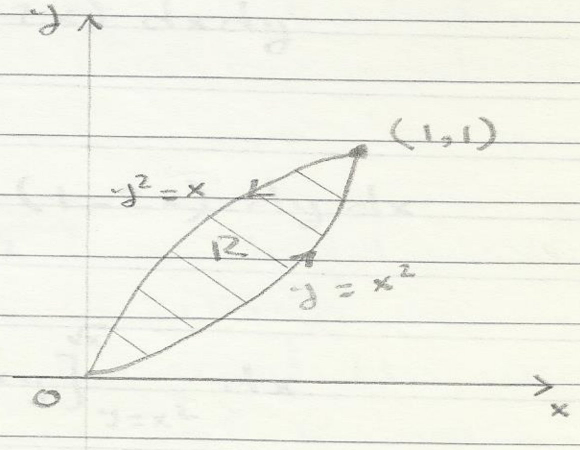
Example:

Verify Green's theorem in the plane for

$$\int_C [(2xy - x^2)\,dx + (x + y^2)\,dy]$$

where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

The plane curves $y = x^2$ and $y^2 = x$ intersect at (0,0) and (1,1). The positive direction in traversing C is as shown in figure.



Along $y = x^2$, the line integral equals

$$\int_{x=0}^{1} [\{(2x)(x^2) - x^2\} dx + \{x + (x^2)^2\} d(x^2)]$$

$$= \int_0^1 (2x^3 + x^2 + 2x^5)\, dx = \frac{7}{6}$$

Along $y^2 = x$, the line integral equals

$$\int_{y=1}^{0} [\{2(y^2)(y) - (y^2)^2\} d(y^2) + \{y^2 + y^2\} dy]$$

$$= \int_1^0 (4y^4 - 2y^5 + 2y^2)\, dy = -\frac{17}{15}$$

The required line integral $= \frac{7}{6} - \frac{17}{15} = \frac{1}{30}$.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x+y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right] dx\,dy$$

$$= \iint_R (1-2x)\, dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (1-2x)\, dy\,dx$$

$$= \int_{x=0}^{1} \left[y - 2xy\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 (x^{1/2} - 2x^{3/2} + x^2 + 2x^3)\, dx$$

$$= \frac{1}{30}$$

Green's theorem is verified.

Independence of the Path for a Line Integral:

A necessary and sufficient condition for $\int_C [P\,dx + Q\,dy]$ to be independent of the path C joining any two given points in a region R is that in R

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{—— (2.14)}$$

where it is supposed that these partial derivatives are continuous in R.

The condition (2.14) is also the condition that $P\,dx + Q\,dy$ is an exact differential, i.e. that there exists a function $\phi(x, y)$ such that $P\,dx + Q\,dy = d\phi$. In such case if the end points of curve C are (x_1, y_1) and (x_2, y_2), the value of the line integral is given by

$$\int_{(x_1,y_1)}^{(x_2,y_2)} [P\,dx + Q\,dy] = \int_{(x_1,y_1)}^{(x_2,y_2)} d\phi = \phi(x_2, y_2) - \phi(x_1, y_1) \quad \text{—— (2.15)}$$

In particular if (2.14) holds and C is closed, we have $x_1 = x_2$, $y_1 = y_2$ and

$$\oint_C [P\,dx + Q\,dy] = 0 \quad \text{—— (2.16)}$$

roof:

Sufficiency:

Suppose $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Then by Green's theorem,

$$\int_C [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 0$$

where R is the region bounded by C.]

ecessity:

Suppose $\int_C [P\,dx + Q\,dy] = 0$ around every closed path C in R and that $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ at some point of R. In particular suppose $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} > 0$ at the point (x_0, y_0).

By hypothesis $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous in R,

o that there must be some region τ contain-ng (x_0, y_0) as an interior point for which $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} > 0$. If Γ is the boundary of τ,

hen by Green's theorem (sign convention for τ and Γ)

$$\int_\Gamma [P\,dx + Q\,dy] = \iint_\tau \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy > 0$$

hich contradicts the hypothesis that $\int [P\,dx + Q\,dy] = 0$ for ALL closed curves in R.

Thus $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ cannot be positive.

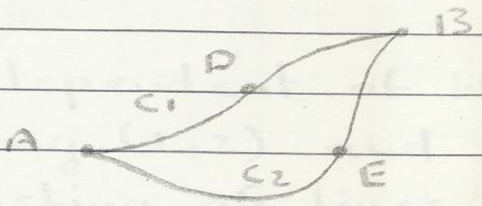
Similarly $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ cannot be negative, and

it follows that it must be identically zero, i.e. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ identically in R.

To prove that the condition for $\int_A^B [P\,dx + Q\,dy]$

to be independent of the path in R joining points A and B is that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ identically in R.

If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then (from above)



$\int_{ADBEA} [P\,dx + Q\,dy] = 0$ (closed path)

$\int_{ADB} + \int_{BEA} = 0$

$\int_{ADB} = -\int_{BEA} = \int_{AEB}$

and $\int_{C1} = \int_{C2}$

i.e. the integral is independent of the path.

Example:

Prove that $\int_{(1,2)}^{(3,4)} [(6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy]$

is independent of the path joining (1,2) and (3,4). Evaluate the integral.

$P = 6xy^2 - y^3$, $Q = 6x^2y - 3xy^2$.

$$\frac{\partial P}{\partial y} = 12xy - 3y^2 = \frac{\partial Q}{\partial x}$$

Therefore, the line integral is independent of the path.

To evaluate the integral

Method (1):

Since the line integral is independent of the path, choose any path joining (1,2) and (3,4), for example that consisting of lines from (1,2) to (3,2) [along which $y = 2$, $dy = 0$] and then (3,2) to (3,4) [along which $x = 3$, $dx = 0$]. Then the required integral equals

$$\int_{x=1}^{3} (24x - 8)\,dx + \int_{y=2}^{4} (54y - 9y^2)\,dy$$

$$= 80 + 156 = 236$$

Method (2):

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we must have

$$\frac{\partial \phi}{\partial x} = 6xy^2 - y^3 \quad , \quad \frac{\partial \phi}{\partial y} = 6x^2y - 3xy^2$$

From which

$$\phi = 3x^2y^2 - xy^3 + f(y) \text{ and } \phi = 3x^2y^2 - xy^3 + g(x)$$

The only way in which these two expressions for ϕ are equal is if $f(y) = g(x) = c$, a constant. Hence $\phi = 3x^2y^2 - xy^3 + c$. Then

$$\int_{(1,2)}^{(3,4)} \left[(6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy \right]$$

$$= \int_{(1,2)}^{(3,4)} d(3x^2y^2 - xy^3 + c)$$

$$= \left[3x^2y^2 - xy^3 + c \right]_{(1,2)}^{(3,4)} = 236$$

Complex Variables
and Conformal Mapping:

Complex Numbers:
A complex number is a number of the form $a+bi$ where "a" and "b" are real numbers and "i", which is called the "imaginary unit", has the property that $i^2 = -1$.
If $z = a+bi$, then "a" is called the "real part" of "z" and "b" is called the "imaginary part" of "z". The symbol "z", which represents any set of complex numbers, is called a "complex variable".

Two complex numbers $(a+bi)$ and $(c+di)$ are "equal" if and only if $a=c$ and $b=d$.

The "complex conjugate", or "conjugate", of a complex number $(a+bi)$ is $(a-bi)$. The complex conjugate of a complex number "z" is often indicated by $\bar{z}$.

Fundamental Operations
with Complex Numbers:

1- Addition:

$$(a+bi)+(c+di) = a+bi+c+di$$
$$= (a+c)+(b+d)i$$

2- Subtraction:

$$(a+bi)-(c+di) = a+bi-c-di$$
$$= (a-c)+(b-d)i$$

3- Multiplication:

$$(a+bi)+(c+di) = ac + adi + bci + bdi^2$$
$$= (ac-bd) + (ad+bc)\,i$$

4- Division:

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}$$
$$= \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2}$$
$$\frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\,i$$

Example (1):

a- $(-7-i)+(3+2i) = -7+3-i+2i = -4+i$

b- $(5+3i)+[(-1+2i)+(7-5i)]$

$= (5+3i)+[-1+2i+7-5i] = (5+3i)+(6-3i)$
$= 11$

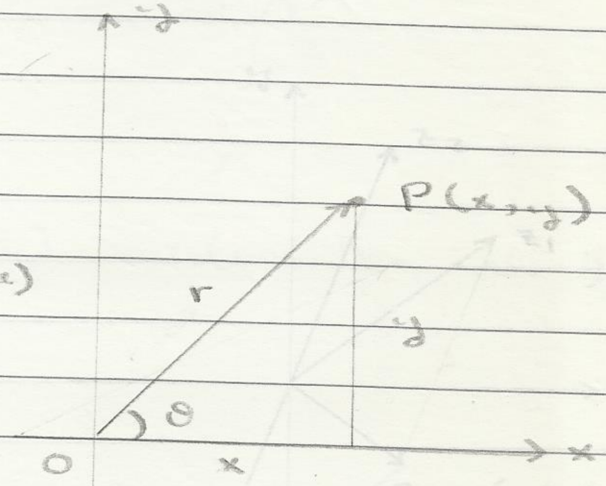
c- $(2-3i)(4+2i) = 2(4+2i)-3i(4+2i)$
$= 8+4i-12i+6 = 14-8i$

d- $\frac{3-2i}{-1+i} = \frac{3-2i}{-1+i} \cdot \frac{-1-i}{-1-i}$

$$= \frac{-3-3i+2i+2i^2}{1-i^2} = \frac{-5-i}{2} = -\frac{5}{2} - \frac{1}{2}\,i$$

Geometric Representation of Complex Numbers:

A complex number $z = x + iy$ can be represented as a point P with coordinates (x, y) on a rectangular xy plane called the "complex plane" or "Argand diagram". "z" can also be represented as a vector which joins the origin to P.



$x = r\cos\theta$, $y = r\sin\theta$

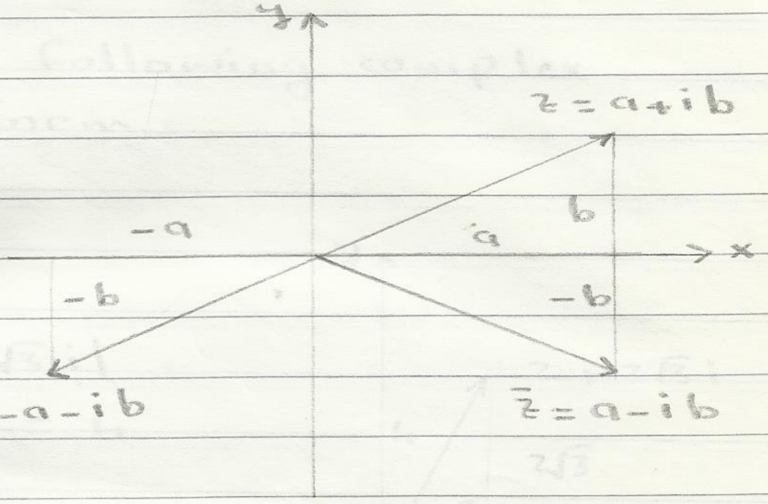
$r = \sqrt{x^2 + y^2}$ - modulus (absolute value)

$\theta = \tan^{-1}\frac{y}{x}$ - argument (amplitude)

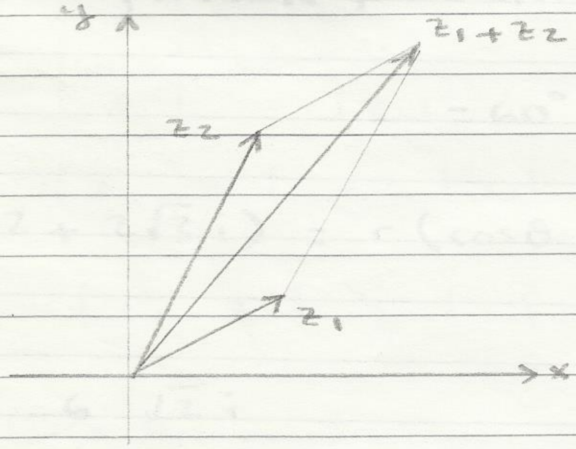
$z = x + iy$

$z = r\cos\theta + ir\sin\theta$
$= r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$

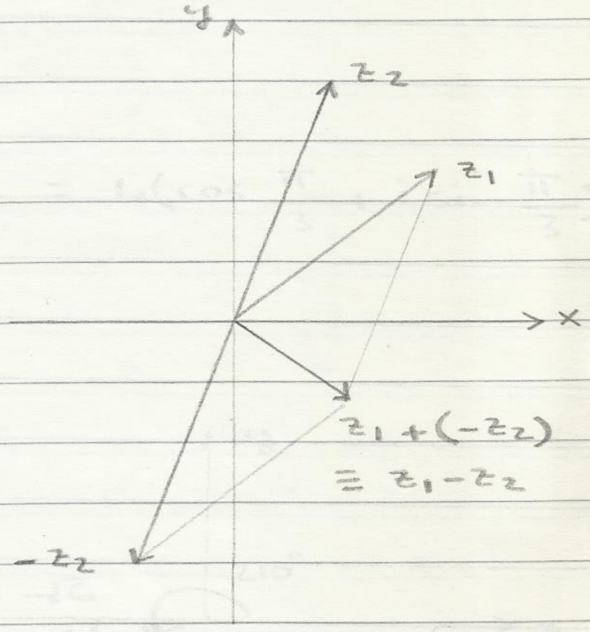
This is known as the "polar form" of a complex number.



y
$z = a + ib$
b
a
-a
-b
-b
x
$-z = -a - ib$
$\bar{z} = a - ib$



y
$z_1 + z_2$
z_2
z_1
x

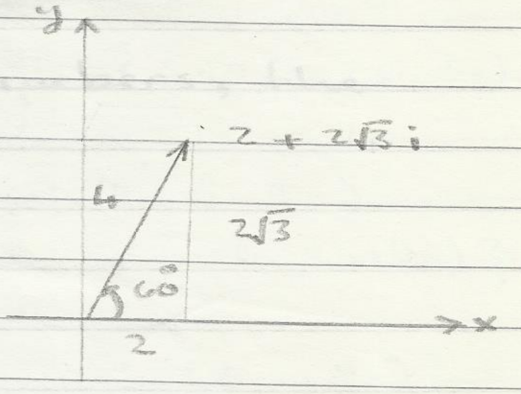


y
z_2
z_1
x
$z_1 + (-z_2)$
$= z_1 - z_2$
$-z_2$

Example:

Express each of the following complex numbers in polar form

a- $2 + 2\sqrt{3}\, i$



Modulus, $r = |2 + 2\sqrt{3}\, i|$

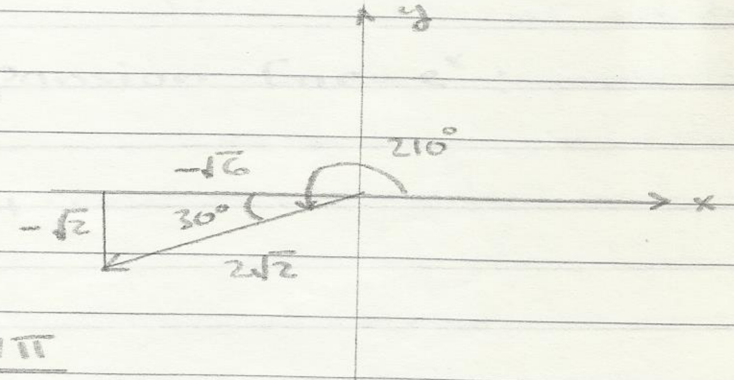
$= \sqrt{4 + 12} = 4$

Amplitude, $\theta = \sin^{-1} \frac{2\sqrt{3}}{4}$

$= 60° = \frac{\pi}{3}$.

$(2 + 2\sqrt{3}\, i) = r(\cos\theta + i \sin\theta) = 4(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$.

b- $-6 - \sqrt{2}\, i$



$r = |-\sqrt{6} - \sqrt{2}\, i|$

$= \sqrt{6 + 2} = 2\sqrt{2}$

$\theta = 180° + 30° = 210° = \frac{7\pi}{6}$

$-\sqrt{6} - \sqrt{2}\, i = 2\sqrt{2}\,(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6})$

Absolute Value:

The "absolute value" or "modulus" of a complex number $a+bi$ is defined as

$$|a+bi| = \sqrt{a^2+b^2}.$$

If z_1 and z_2 are complex numbers, the following properties hold:

1- $|z_1 z_2| = |z_1||z_2|$

2- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

3- $|z_1 + z_2| \leq |z_1| + |z_2|$

4- $|z_1 - z_2| \geqslant |z_1| - |z_2|$.

Euler's Formula:

The Taylor series expansion for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Substituting $x = i\theta$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \dots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \dots\right)$$

$e^{i\theta} = \cos\theta + i\sin\theta$ which is called Euler's formula.

De Moivre's Theorem:

If $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
and $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

Then $z_1 z_2 = \{r_1(\cos\theta_1 + i\sin\theta_1)\}\{r_2(\cos\theta_2 + i\sin\theta_2)\}$

$$= r_1 r_2 \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$$

$$= r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$$

$$z_1 z_2 \dots z_n = r_1 r_2 \dots r_n \{\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n)\}$$

If $z_1 = z_2 = \dots = z_n = z$

$$\boxed{z^n = \{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)}$$

This is known as De Moivre's theorem.

Example:
Find all values of z for which $z^5 = -32$, and locate these values in the complex plane.

In polar form

$$-32 = 32\{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\},$$

$k = 0, \pm 1, \pm 2, \dots$

Let $z = r(\cos\theta + i\sin\theta)$. Then by De Moivre's theorem,

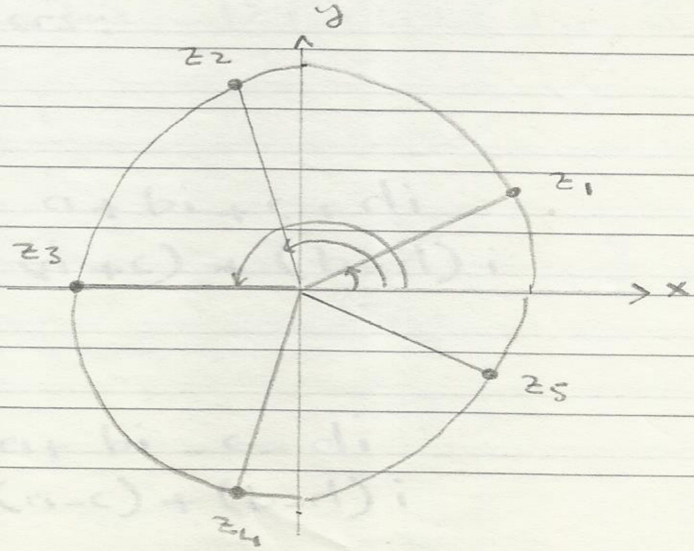
$$z^5 = r^5(\cos 5\theta + i\sin 5\theta)$$
$$= 32\{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\}$$

$$r^5 = 32, \quad 5\theta = \pi + 2k\pi,$$
$$r = 2, \quad \theta = (\pi + 2k\pi)/5.$$

$$z = 2\left\{\cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)\right\}$$

If $k = 0$, $z = z_1 = 2(\cos\pi/5 + i\sin\pi/5)$
If $k = 1$, $z = z_2 = 2(\cos 3\pi/5 + i\sin 3\pi/5)$
If $k = 2$, $z = z_3 = 2(\cos 5\pi/5 + i\sin 5\pi/5) = -2$
If $k = 3$, $z = z_4 = 2(\cos 7\pi/5 + i\sin 7\pi/5)$
If $k = 4$, $z = z_5 = 2(\cos 9\pi/5 + i\sin 9\pi/5)$.

By considering $k = 5, 6, \ldots$ as well as negative values, $-1, -2, \ldots$, repetitions of the above five values of z are obtained. Hence these are the "only solutions" or "roots" of the given equation. These five roots are called the "fifth roots of -32".



Variables and Functions:

A symbol, such as "z", which can stand for any one of a set of complex numbers is called a "complex variable".
If to each value which a complex variable "z" can assume there corresponds one or more values of a complex variable "w", we say that "w" is a "function" of "z" and write $w = f(z)$ or $w = G(z)$, etc. A function is "single-valued" if for each value of "z" there corresponds only one value of "w"; otherwise it is "multiple-valued". Unless otherwise specified we shall assume that $f(z)$ is single-valued. In general we can write $w = f(z) = u(x,y) + i\,v(x,y)$, where "u" and "v" are real functions of "x" and "y".

Example (1):

$$w = z^2 = (x+iy)^2$$
$$= x^2 - y^2 + 2ixy = u + iv$$

so that $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$.

These are called the "real" and "imaginary" parts of $w = z^2$ respectively.

Example (2):

$$w = \frac{1}{1-z} = \frac{1}{1-(x+iy)}$$

$$= \frac{1}{1-x-iy} \cdot \frac{1-x+iy}{1-x+iy} = \frac{1-x+iy}{(1-x)^2 - i^2 y^2}$$

$$= \frac{1-x+iy}{(1-x)^2+y^2}.$$

$$u(x,y) = \frac{1-x}{(1-x)^2+y^2}, \quad v(x,y) = \frac{y}{(1-x)^2+y^2}$$

Limits:

$f(z)$ is said to have the limit "l" as z approaches "z_0" and we write $\lim\limits_{z \to z_0} f(z) = l$, if, for any positive number ϵ (however small) we can find some positive number δ, such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Continuity:

The function $f(z)$ is said to be "continuous" at $z = z_0$ if $\lim\limits_{z \to z_0} f(z) = f(z_0)$. This implies three conditions in order that $f(z)$ be continuous at $z = z_0$:

1. $\lim\limits_{z \to z_0} f(z) = l$ must exist
2. $f(z_0)$ must exist, i.e. $f(z)$ is defined at z_0
3. $l = f(z_0)$.

erivatives:

If $f(z)$ is single-valued in some region R of the z-plane, the "derivative" of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \text{--- (1)}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$.

Analytic Functions:

If the derivative $f'(z)$ exists at all points z of a region R, then $f(z)$ is said to be "analytic in R".

Cauchy-Riemann Equations:

A necessary condition that $w = f(z) = u(x,y) + i\,v(x,y)$ be analytic in a region R is that, in R, u and v satisfy the Cauchy-Riemann equations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \quad \text{--- (2)}$$

If the second partial derivatives of u and v with respect to x and y exist and are continuous, we find by differentiating (2)

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0} \quad \text{--- (3)}$$

thus the real and imaginary parts satisfy Laplace's equation in two dimensions. Functions satisfying Laplace's equation are called "harmonic functions".
*(Proof)

Example (3):

If $w = f(z) = \dfrac{1+z}{1-z}$, find a- $\dfrac{dw}{dz}$ and b- determine where $f(z)$ is non-analytic.

a- $$\frac{d}{dz}\left(\frac{1+z}{1-z}\right) = \frac{(1-z)\frac{d}{dz}(1+z) - (1+z)\frac{d}{dz}(1-z)}{(1-z)^2}$$

$$= \frac{(1-z)(1) - (1+z)(-1)}{(1-z)^2} = \frac{2}{(1-z)^2}.$$

b- The function $f(z)$ is analytic for all finite values of z except $z=1$ where the derivative does not exist and the function is non-analytic.

Proof:

Since $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$, we have

$$f'(z+\Delta z) = f[x+\Delta x + i(y+\Delta y)]$$
$$= u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y).$$

Then

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + i\{v(x+\Delta x, y+\Delta y) - v(x,y)\}}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, the required limit is

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\left\{\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right\}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

If $\Delta x = 0$, the required limit is

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \left\{\frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}\right\}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

If the derivative is to exist, these two special limits must be equal, i.e.,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

and $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

xample(4):
For $f(z) = \bar{z} = x - iy$, we have $u = x$ and $v = -y$.

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = 0 \;, \quad \frac{\partial v}{\partial y} = -1$$

only the second of the Cauchy-Riemann equations is satisfied. Hence, there is no point in the z-plane where $f'(z)$ exists.

Example (5):
For $f(z) = z\bar{z} = x^2 + y^2$, we have $u = x^2 + y^2$ and $v = 0$.

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = 2y \quad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0$$

The Cauchy-Riemann equations are:

$$2x = 0 \quad \text{and} \quad 2y = 0$$

are satisfied only at the origin. Hence $z = 0$ is the only point at which $f'(z)$ exists, and therefore $f(z) = z\bar{z}$ is nowhere analytic.

xample (6):

For $f(z) = z^2 = (x^2 - y^2) + 2ixy$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial x} = 2y \quad \frac{\partial v}{\partial y} = 2x$$

The Cauchy - Riemann equations are identically satisfied. Hence the derivative $f'(z)$ exists at all points of the z-plane, and its value

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 2x + 2iy$$
$$= 2z$$

This is exactly what formal differentiation according to the power rule would give.

Example (7):

Prove that in polar form the Cauchy-Reimann equations can be written

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in terms of r, θ by chain rule.

Apply Cauchy-Riemann equations and solve simultaneous equations.

Conformal Mapping:

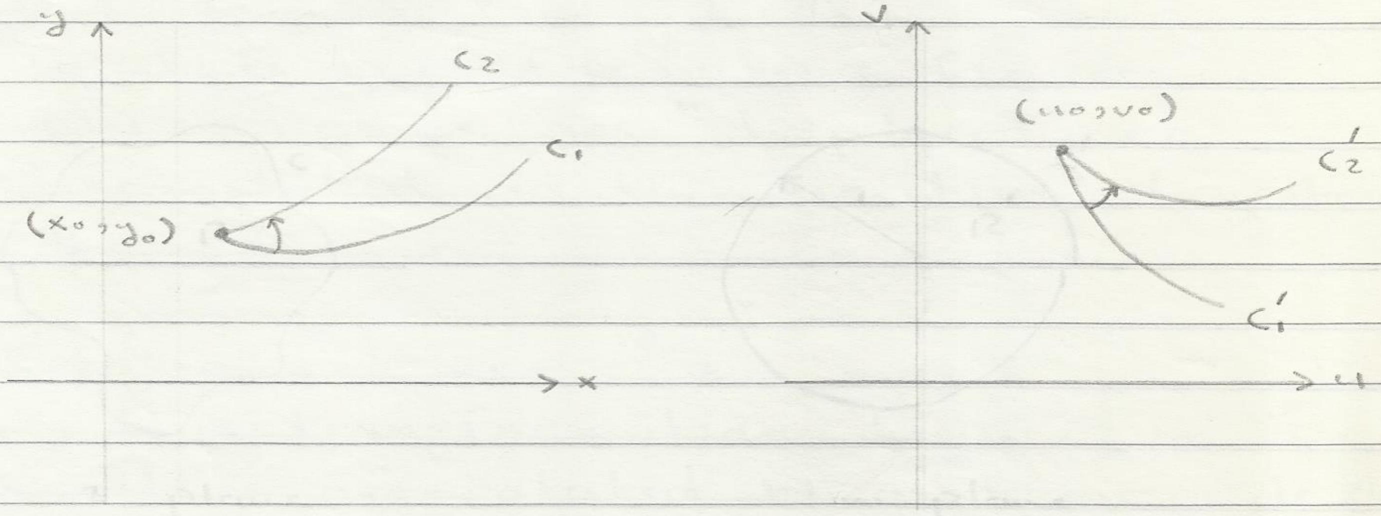
Transformations or Mappings:

The set of equations

$$u = u(x, y) \qquad \text{—— (4)}$$
$$v = v(x, y)$$

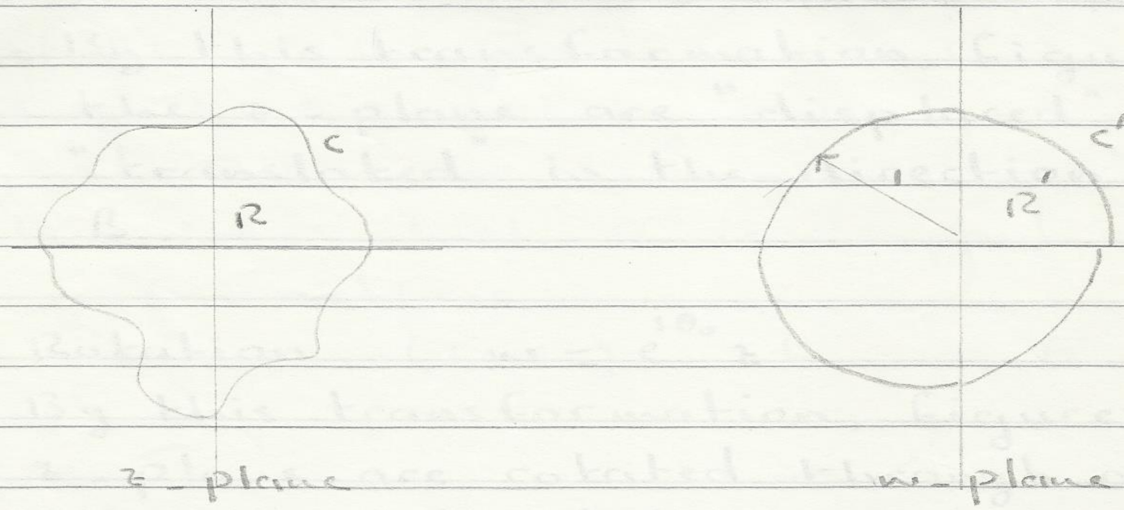
defines in general, a "transformation" or "mapping" which establishes a correspondence between points in the uv- and xy- planes. The equations (4) are called "transformation equations". If to each point of the "uv" plane there corresponds one and only one point of the "xy" plane, we speak of a "one to one" transformation or mapping. In such a case a set of points in the xy- plane is "mapped" into a set of points in the uv- plane and conversely. The corresponding sets of points in the two planes are often called "images" of each other.

Conformal Mapping:
Suppose that under transformation (4) point (x_0, y_0) of the xy- plane is mapped into point (u_0, v_0) of the uv- plane, while curves C_1 and C_2 [intersecting at (x_0, y_0)] are mapped respectively into curves C_1' and C_2' [intersecting at (u_0, v_0)]



Then if the transformation is such that the angle at (x_0, y_0) between C_1 and C_2 is equal to the angle at (u_0, v_0) between C_1' and C_2' both in magnitude and sense, the transformation or mapping is said to be "conformal" at (x_0, y_0).
The following theorem is fundamental.
Theorem:
If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R, then the mapping $w = f(z)$ is conformal at all points of R.

Riemann's Mapping Theorem:
Let C be a simple closed curve in the z-plane forming the boundary of a region R. Let C' be a circle of radius one and centre at the origin "the unit circle" forming the boundary of region R' in the w-plane. The region R' is sometimes called the "unit disk".



Then "Riemann's mapping theorem" states that there exists a function $w = f(z)$, analytic in R, which maps each point of R into a corresponding point of R' and each point of C into a corresponding point of C', the correspondence being one to one.
This function $f(z)$ contains three arbitrary real constants which can be determined by making the centre of C' correspond to some given point in R, while some point on C' corresponds to a given point on C.
It should be noted that while Riemann's mapping theorem demonstrates the "existence"

of a mapping function, it does not actually produce this function.

Some General Transformations:
In the following α, β are given complex constants while a, θ_0 are real constants.

1- Translation. $w = z + \beta$
By this transformation, figures in the z-plane are "displaced" or "translated" in the direction of vector β.

2- Rotation. $w = e^{i\theta_0} z$
By this transformation, figures in the z-plane are rotated through an angle θ_0. If $\theta_0 > 0$ the rotation is counterclockwise, while if $\theta_0 < 0$ the rotation is clockwise.

3- Stretching. $w = az$
By this transformation, figures in the z-plane are stretched (or contracted) in the direction z if $a > 1$ (or $0 < a < 1$). We consider contraction as a special case of stretching.

4- Inversion. $w = 1/z$

5- Linear Transformation. $w = \alpha z + \beta$
This is a combination of the transformations of translation, rotation and stretching.

6- Bilinear or Fractional Transformation. $w = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma \neq 0$

This is a combination of the transformations of translation, rotation, stretching and inversion.

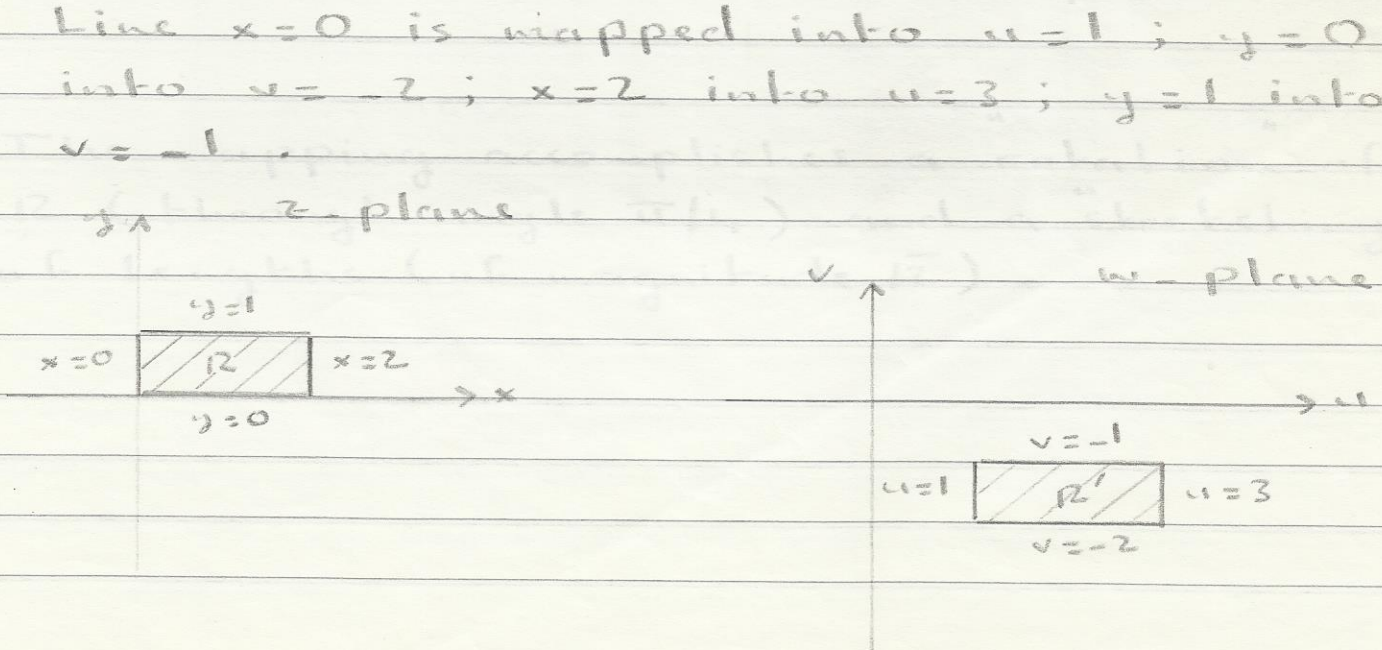
Example (1):
Let the rectangular region R in the z-plane be bounded by $x=0$, $y=0$, $x=2$, $y=1$. Determine the region R' of the w plane into which R is mapped under the transformations:

a- $w = z + (1-2i)$ b- $w = \sqrt{2}\, e^{\pi i/4} z$

a-

If $w = z + (1-2i)$, then
$u + iv = x + iy + 1 - 2i$
$= (x+1) + i(y-2)$
and $u = x+1$, $v = y-2$.

Line $x=0$ is mapped into $u=1$; $y=0$ into $v=-2$; $x=2$ into $u=3$; $y=1$ into $v=-1$.



The transformation accomplishes a "translation" of the rectangle. In general, $w = z + \beta$ accomplishes a translation of any region.

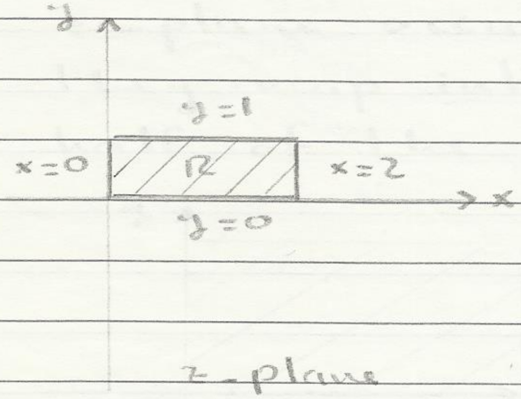
b- If $w = \sqrt{2}\, e^{\pi i/4} z$,

then $u + iv = (1+i)(x+iy)$

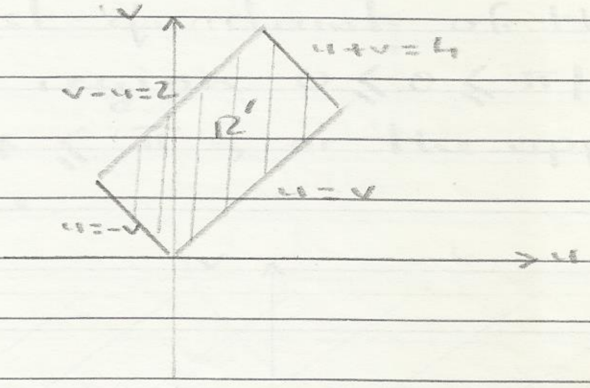
$= x - y + i(x+y)$

and $u = x - y$, $v = x + y$.

Line $x = 0$ is mapped into $u = -y$, $v = y$ or $u = -v$; $y = 0$ into $u = x$, $v = x$ or $u = v$; $x = 2$ into $u = 2 - y$, $v = 2 + y$ or $u + v = 4$; $y = 1$ into $u = x - 1$, $v = x + 1$ or $v - u = 2$.



z-plane



w-plane

The mapping accomplishes a "rotation" of R (through angle $\pi/4$) and a "stretching" of lengths (of magnitude $\sqrt{2}$).

Example (2):

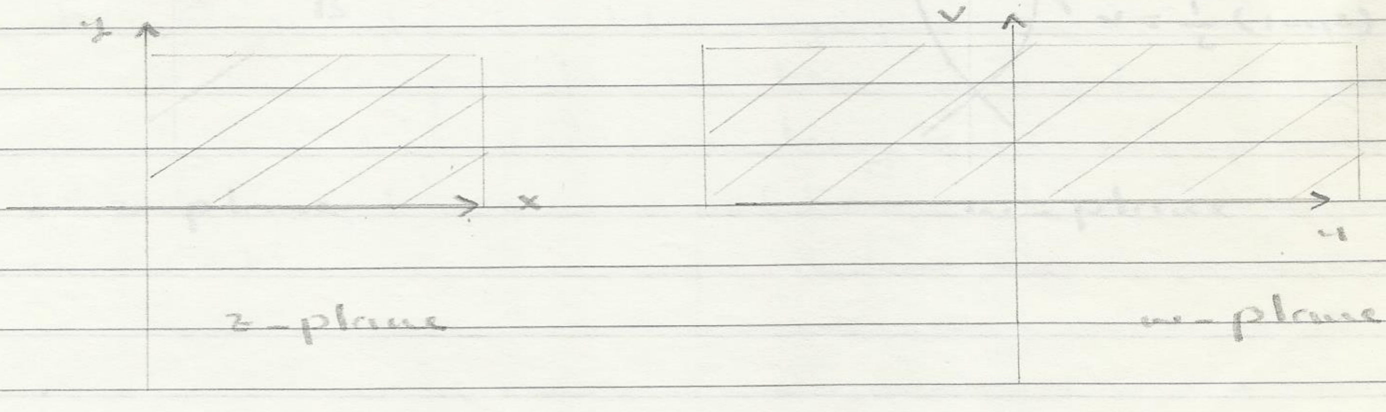
Determine the region of the w-plane into which each of the following is mapped by the transformation $w = z^2$.

a- First quadrant of the z-plane.

Let $z = re^{i\theta}$, $w = \rho e^{i\phi}$.

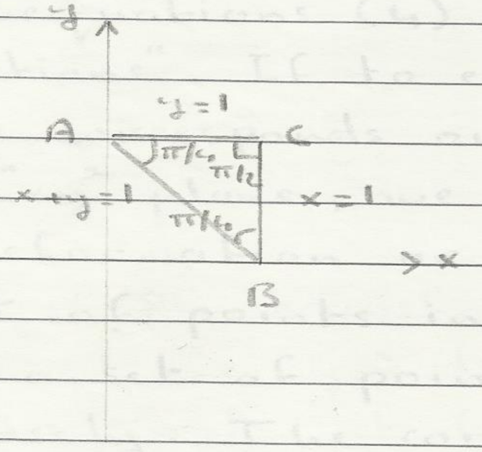
Then $\rho e^{i\phi} = r^2 e^{2i\theta}$ and $\rho = r^2$, $\phi = 2\theta$.

Thus points in the z-plane at (r, θ) are rotated through angle 2θ. Since all points in the first quadrant of the z-plane occupy the region $0 \leq \theta \leq \pi/2$, they map into $0 \leq \phi \leq \pi$, or the upper half of the w-plane.

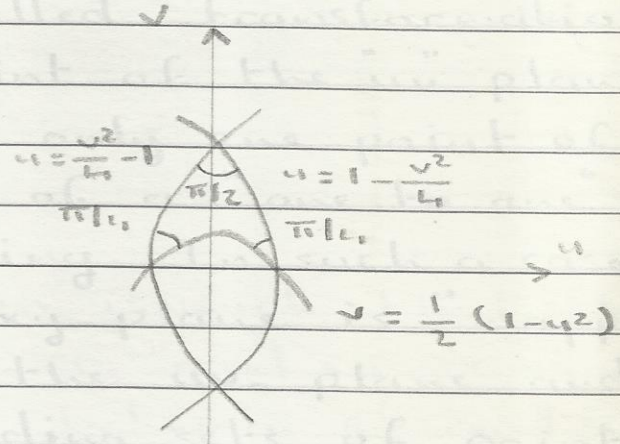


b. Region bounded by $x=1$, $y=1$ and $x+y=1$.

Since $w=z^2$ is equivalent to $u+iv=(x+iy)^2=x^2-y^2+2ixy$, we see that $u=x^2-y^2$, $v=2xy$.
Then line $x=1$ maps into $u=1-y^2$, $v=2y$ or $u=1-v^2/4$; line $y=1$ into $u=x^2-1$, $v=2x$ or $u=v^2/4-1$, line $x+y=1$ or $y=1-x$ into $u=x^2-(1-x)^2=2x-1$, $v=2x(1-x)=2x-2x^2$ or $v=\frac{1}{2}(1-u^2)$ on eliminating x.



z-plane



w-plane

Physical Applications of Conformal Mapping:

An important property of analytic functions is that they are "harmonic" i.e. they satisfy Laplace's equation. Another important characteristic is that " a harmonic function remains harmonic in conformal mapping", which makes conformal mapping useful in solving engineering problems.

Suppose, for example, that $\phi(x,y)$ is a solution of

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{— (3.5)}$$

Then, when an analytic function $z = x+iy = f(u+iv)$ transforms ϕ into a function of u and v, $\phi(u,v)$ will satisfy

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0 \qquad \text{— (3.6)}$$

wherever the mapping is conformal.

The imaginary part gives the streamlines:

$$\psi = c\left(r - \frac{1}{r}\right)\sin\theta = \text{constant.} \quad \text{— (3.16)}$$

the streamline $\psi = 0$ consists of the circle $r = 1$ and the x-axis. For very large values of r, the streamline is given approximately by $r\sin\theta = y = \text{const.}$

Hence, the complex potential

$$F = \Phi + i\Psi = cu + icv = cw .$$

The analytic function

$$w = z + \frac{1}{z} \quad \text{— (3.14)}$$

maps the upper half of the w-plane onto the upper half of the z-plane:

By substituting $x + iy$ for z, $v = 0$ when $y = 0$ or when $x^2 + y^2 = 1$. Consequently, the u-axis maps onto the x-axis for $x < -1$ and $x > 1$ and onto the upper half of the circle $x^2 + y^2 = 1$ when $-1 \leqslant x \leqslant 1$. Also for very large values of z, w is nearly equal to z. So streamlines at large distances from the origin in the z-plane will be practically unaltered in their condition in the w-plane.

Using eq. (3.14) for the transformation of $F = cw$, we get, with $z = re^{i\theta}$

$$F = c\left(z + \frac{1}{z}\right) = c\left[\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\right] \quad \text{— (3.15)}$$

must lie outside the circle. At large distances from the cylinder, the streamlines should be parallel or nearly parallel to the x-axis.

The "complex potential" of the flow is given by the analytic function

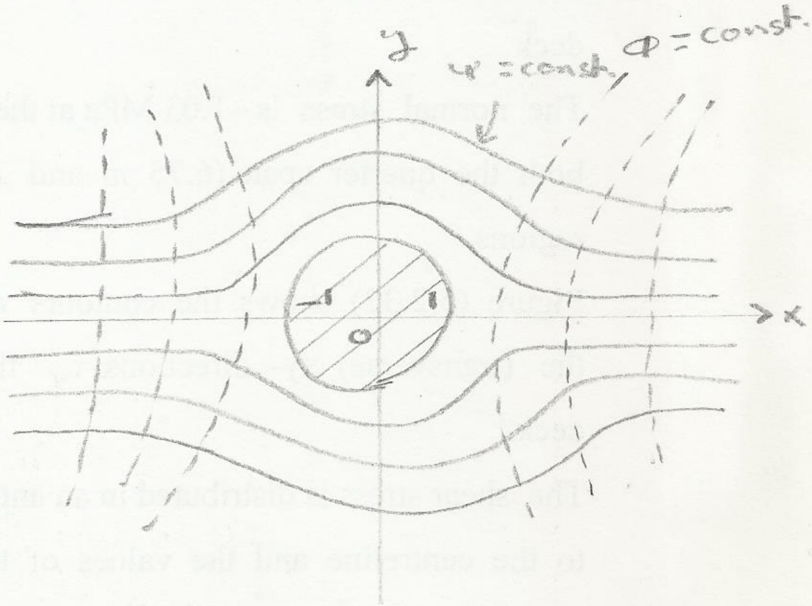
$$F(z) = \phi(x,y) + i\psi(x,y).$$

Taking advantage of symmetry, only the half of the z-plane above the x-axis needs to be considered. The required solution then is the imaginary part of $F(z)$ in the upper half of the z plane.

Because of the shape of the boundary, it is difficult to obtain the solution directly. However, the solution is known for unobstructed uniform flow. It is assumed that it takes place in the upper half of the w-plane, parallel to the u-axis. Then $V_1 = c$ and $V_2 = 0$, and since $V_1 = \dfrac{\partial\psi}{\partial v}$, $V_2 = -\dfrac{\partial\psi}{\partial u}$, the stream function is $\psi = cv$, where c is a real constant. The velocity potential must be $\phi = cu$ to satisfy the Cauchy-Riemann equations.

orthogonal families called respectively the "equipotential lines" and "streamlines" of the flow. In steady motion, streamlines represent the actual paths of fluid particles in the flow pattern.

Example:
Determine the paths of the particles of an ideal fluid approaching with uniform flow a long circular cylinder perpendicular to the direction of flow.



Take the flow of the fluid parallel to the x-axis.

Let the cylinder be represented in the z-plane by the circle $|z| = 1$, with centre at the origin of coordinates. Our objective is to determine $\psi(x, y)$ to satisfy Laplace's equation with the boundary condition that a streamline $\psi = k$ lies along the x-axis for $x < -1$ and $x > 1$ and along the circle for $-1 \leqslant x \leqslant 1$. Also, all other streamlines

The Complex Potential:

From (3.7) and (3.8) it is seen that the velocity potential (ϕ) is harmonic

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Consequently, (ϕ) must have a harmonic conjugate $\psi(x, y)$ to which it is related by the Cauchy-Riemann equations. The function $\psi(x, y)$ is called the "stream function".

As ϕ and ψ are harmonic conjugates, they define an analytic function

$$F(z) = \phi(x, y) + i\psi(x, y) \quad \text{— (3.9)}$$

This function is called the "complex potential" of the flow. By differentiation

$$\frac{d}{dz} F(z) = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y} = V_1 - iV_2 \quad \text{— (3.10)}$$

and the velocity is given by

$$V = V_1 + iV_2 \quad \text{— (3.11)}$$

with magnitude $|V| = \sqrt{V_1^2 + V_2^2}$. — (3.12)

Equipotential Lines and Streamlines:

The families of curves

$$\phi(x, y) = c \quad , \quad \psi(x, y) = k \quad \text{— (3.13)}$$

where c and k are constants, are

for the region R into a corresponding one for a simpler region.

ii- Solve the problem for the simpler region.

iii- Use the solution in (ii) to solve the given problem by employing the inverse mapping function.

Applications to Fluid Flow:

The solution of many important problems in fluid flow is often achieved by complex variable methods under the following assumptions.

1- The fluid flow is two-dimensional.

2- The flow is steady, i.e. the velocity of the fluid at any point depends only on the position (x,y) and not on time.

3- There exists a function ϕ called the "velocity potential", such that

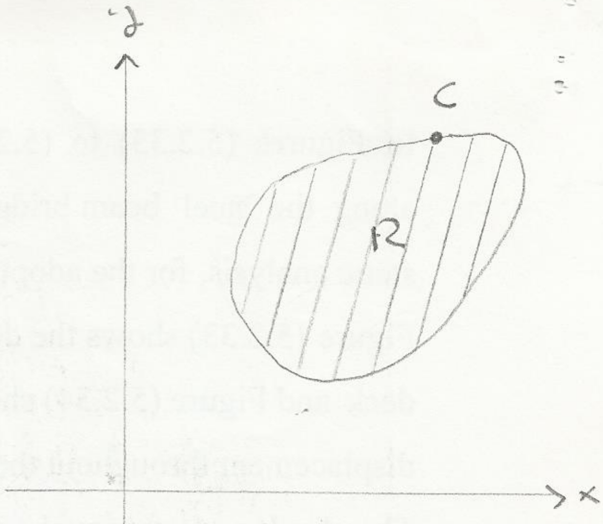
$$V_1 = \frac{\partial \phi}{\partial x}, \quad V_2 = \frac{\partial \phi}{\partial y}. \qquad (3.7)$$

4- The fluid is incompressible, i.e. the mass per unit volume of the fluid is constant, (3.8)

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = 0 \quad \text{equation of continuity.}$$

5- The fluid is non-viscous.

Dirichlet and Neumann Problems:



Let R be a simply-connected region bounded by a simple closed curve C. Two types of boundary-value problems are of great importance.

i- Dirichlet problems seek the determination of a function Φ which satisfies Laplace's equation in R and takes prescribed values on the boundary C.

ii- Neumann problems seek the determination of a function Φ which satisfies Laplace's equation in R and whose normal derivative $\frac{\partial \Phi}{\partial n}$ takes prescribed values on the boundary C.

Solutions for Problems by Conformal Mapping:

Boundary-value problems can be solved for any simply-connected region R which can be mapped conformally by an analytic function on to a simpler region. The basic procedure is the following:

i- Use the mapping function to transform the boundary-value problem