Math (5)

رسل ٤ع

If $A = (y-2x)i + (3x+2y)j$, Compute the Circulation about a Circle C in the xy-plane with Center at the origin and radius 2, if C is traversed in the positive direction.

$$\int A \cdot dr = \int [(y-2x)dx + (3x+2y)dy]$$

$x = r\cos\theta \rightarrow dx = -r\sin\theta d\theta$

$y = r\sin\theta \rightarrow dy = r\cos\theta d\theta$

$$\int_0^{2\pi} [(2\sin\theta - 4\cos\theta)(-2\sin\theta d\theta) + (6\cos\theta + 4\sin\theta)(2\cos\theta d\theta)]$$

$$\int_0^{2\pi} [-4\sin^2\theta d\theta + 8\sin\theta\cos\theta d\theta + 12\cos^2\theta d\theta + 8\sin\theta\cos\theta d\theta]$$

$$\int_0^{2\pi} [-4\sin^2\theta d\theta + 12\cos^2\theta d\theta + 16\sin\theta\cos\theta d\theta]$$

$$-2\int_0^{2\pi}(1-\cos 2\theta)d\theta + 6\int_0^{2\pi}(1+\cos 2\theta)d\theta + 8\int_0^{2\pi}\sin 2\theta d\theta$$

$$-2\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} + 6\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} - 4\cos 2\theta\Big|_0^{2\pi}$$

$$= -4\pi + 12\pi = 8\pi$$

$F = \nabla Q$

$\nabla \nabla f = 0$

$F = \nabla Q$

Evaluate $\int F \cdot dr$ where $F = (x-3y)i + (y-2x)j$ and C is the Closed Curve in the xy-plan, $x = 2\cos t$, $y = 3\sin t$ from $t = 0$ to $t = 2\pi$

$$\int F \cdot dr = \int [(x-3y)dx + (y-2x)dy]$$

$x = 2\cos t \longrightarrow dx = -2\sin t\, dt$

$y = 3\sin t \longrightarrow dy = 3\cos t\, dt$

$$\int_0^{2\pi} \left[(2\cos t - 9\sin t)(-2\sin t\, dt) + (3\sin t - 4\cos t)(3\cos t\, dt) \right]$$

$$\int_0^{2\pi} \left[-4\cos t \sin t\, dt + 18\sin^2 t\, dt + 9\sin t\cos t\, dt - 12\cos^2 t\, dt \right]$$

$$\int_0^{2\pi} \left[5\sin t\cos t\, dt + 18\sin^2 t\, dt - 12\cos^2 t\, dt \right]$$

$$\int_0^{2\pi} \frac{5}{2}\sin 2t\, dt + \int_0^{2\pi} 9(1-\cos 2t)\, dt - \int_0^{2\pi} 6(1+\cos 2t)\, dt$$

$$-\frac{5}{4}\cos 2t \Big|_0^{2\pi} + 9\left(\theta - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} - 6\left(\theta + \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi}$$

$= 6\pi$

Evaluate $\int_{(0,0)}^{(\pi,2)} (6xy - y^2)\,dx + (3x^2 - 2xy)\,dy$ along the Cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$

$dx = (1 - \cos\theta)\,d\theta$, $dy = \sin\theta\,d\theta$

$$\int_0^{\pi} (6(\theta - \sin\theta)(1 - \cos\theta) - (1 - \cos\theta)^2)(1 - \cos\theta)\,d\theta +$$
$$\int_0^{\pi} (3(\theta - \sin\theta)^2 - 2(\theta - \sin\theta)(1 - \cos\theta))\sin\theta\,d\theta$$

$$\int_0^{\pi} ((6\theta - 6\sin\theta)(1 - \cos\theta) - (1 - 2\cos\theta + \cos^2\theta))(1 - \cos\theta)\,d\theta +$$
$$\int_0^{\pi} (3\theta^2 - 6\theta\sin\theta + 3\sin^2\theta - 2(\theta - \theta\cos\theta - \sin\theta + \sin\theta\cos\theta))\sin\theta$$

Find the area bounded by one arch of the Cycloid
$x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $a > 0$ and the x-a

$$A = \frac{1}{2}\int (x\,dy - y\,dx)$$

$$x = a(\theta - \sin\theta) \Rightarrow dx = a(1 - \cos\theta)d\theta$$
$$y = a(1 - \cos\theta) \rightarrow dy = a\sin\theta\, d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi}\left[a^2(\theta - \sin\theta)\sin\theta\, d\theta - a^2(1-\cos\theta)(1-\cos\theta)d\theta\right.$$

$$A = \frac{1}{2}\int_0^{2\pi}\left[a^2\theta\sin\theta\, d\theta - a^2\sin^2\theta\, d\theta - a^2(1 - \cos\theta - \cos\theta + \cos^2\theta)\right]$$

$$A = \frac{a^2}{2}\int_0^{2\pi}\left[\theta\sin\theta\, d\theta - \sin^2\theta\, d\theta - d\theta + 2\cos\theta\, d\theta - \cos^2\theta\, d\theta\right]$$

$$A = \frac{a^2}{2}\int_0^{2\pi}\left[\theta\sin\theta\, d\theta - d\theta + 2\cos\theta\, d\theta - (\sin^2\theta + \cos^2\theta)d\theta\right]$$

$$A = \frac{a^2}{2}\int_0^{2\pi}\left[\theta\sin\theta\, d\theta + 2\cos\theta\, d\theta - 2d\theta\right]$$

$$\int u\,dv = u.v - \int v\,du$$

$$\int \theta\sin\theta\, d\theta \qquad u = \theta \rightarrow du = d\theta$$
$$dv = \sin\theta\, d\theta \rightarrow v = -\cos\theta$$

$$\int \theta\sin\theta\, d\theta = -\theta\cos\theta + \int \cos\theta\, d\theta = -\theta\cos\theta + \sin\theta$$

$$A = \frac{a^2}{2}\left\{-\theta\cos\theta\Big|_0^{2\pi} + \sin\theta\Big|_0^{2\pi} + 2\sin\theta\Big|_0^{2\pi} - 2\theta\Big|_0^{2\pi}\right.$$

$$A = \frac{a^2}{2}(-2\pi - 4\pi) \rightarrow A = 3a^2\pi$$

7. W

Let $A = t\,i - 3j + 2t\,k$, $B = i - 2j + 2k$, $C = 3i + t\,j - k$. Evaluate

(a) $\int_1^2 A.BXC\,dt$, (b) $\int_1^2 AX(BXC)\,dt$

(a)

$$BXC = \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ 3 & t & -1 \end{vmatrix}$$

$$BXC = (2-2t)\vec{i} + (6+1)\vec{j} + (t+6)\vec{k}$$
$$BXC = (2-2t)\vec{i} + 7\vec{j} + (t+6)\vec{k}$$

$$A.BXC = t(2-2t) - 21 + 2t(t+6) = 2t - 2t^2 - 21 + 2t^2 + 12t$$
$$= 14t - 21$$

$$\int_1^2 A.BXC\,dt = \int_1^2 (14t-21)\,dt = 7t^2 - 21t \Big|_1^2 = 0$$

$u = \theta$
$du = d\theta$
$\sin\theta = dv$
$v = -\cos\theta$

$-\theta\cos\theta - \int$
$= -\theta\cos\theta + s$

(b)

$$AX(BXC) = \begin{vmatrix} i & j & k \\ t & -3 & 2t \\ (2-2t) & 7 & (t+6) \end{vmatrix}$$

$$AX(BXC) = [-3(t+6) - 14t]\vec{i} + [2t(2-2t) - t(t+6)]\vec{j} + [7t + 3(2-2t)]$$

$$AX(BXC) = (-3t - 18 - 14t)\vec{i} + (4t - 4t^2 - t^2 - 6t)\vec{j} + (7t + 6 - 6t)\vec{k}$$
$$= (-17t - 18)\vec{i} + (-5t^2 - 2t)\vec{j} + (t+6)\vec{k}$$

$$\int_1^2 AX(BXC)\,dt = \int_1^2 (-17t - 18)\,dt\,\vec{i} + \int_1^2 (-5t^2 - 2t)\,dt\,\vec{j} + \int_0^2 (t+6)\,dt\,\vec{k}$$

$$= -\frac{87}{2}\vec{i} - \frac{44}{3}\vec{j} + \frac{15}{2}\vec{k}$$

$$28\int_0^{2\pi}\cos^2\theta\,d\theta = 14\int_0^{2\pi}(1+\cos 2\theta)\,d\theta = 14\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 28\pi$$

$$12\int_0^{2\pi}\sin\theta\cos\theta\,d\theta = 6\sin^2\theta\Big|_0^{2\pi} = 0$$

$$28\int_0^{2\pi}\sin\theta\cos\theta\,d\theta = 0$$

$$12\int_0^{2\pi}\sin^2\theta\,d\theta = 6\int_0^{2\pi}(1-\cos 2\theta)\,d\theta = 6\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 12\pi$$

$$-4\int_0^{2\pi}\cos^2\theta\,d\theta = -2\int_0^{2\pi}(1+\cos 2\theta)\,d\theta = -2\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = -4\pi$$

$$4\int_0^{2\pi}\sin^2\theta\,d\theta = 2\int_0^{2\pi}(1-\cos 2\theta)\,d\theta = 2\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = 4\pi$$

$$4\int_0^{2\pi}\sin\theta\cos\theta\,d\theta = 0, \quad 8\int_0^{2\pi}\sin\theta\cos\theta = 0$$

$$4\int_0^{2\pi}\cos\theta\,d\theta = 0, \quad 8\int_0^{2\pi}\sin\theta\,d\theta = 0$$

$$\int(A\times B)\times dr = 28\pi i + 12\pi j = 4\pi(7i+3j)$$

Z.W

if $A = (3x+y)i - xj + (y-2)k$ and $B = 2i - 3j + k$, evaluate $\oint_c (A \times B) \times$
around the Circle in the xy plane having Center at the origin and rad
2 traversed in the positive direction.

$$A \times B = \begin{vmatrix} i & j & k \\ (3x+y) & -x & (y-2) \\ 2 & -3 & 1 \end{vmatrix}$$

$$A \times B = [-x + 3(y-2)]\vec{i} + [2(y-2) - (3x+y)]\vec{j} + [-3(3x+y) + 2x]\vec{k}$$

$$A \times B = (-x+3y-6)\vec{i} + (2y-4-3x-y)\vec{j} + (-9x-3y+2x)\vec{k}$$

$x = 2\cos\theta$
$y = 2\sin\theta$

$$A \times B = (-x+3y-6)\vec{i} + (y-3x-4)\vec{j} + (-7x-3y)\vec{k}$$

$$(A \times B) \times dr = \begin{vmatrix} i & j & k \\ (-x+3y-6) & (y-3x-4) & (-7x-3y) \\ dx & dy & 0 \end{vmatrix}$$

$$= -(-7x-3y)dy\,\vec{i} + (-7x-3y)dx\,\vec{j} + [(-x+3y-6)dy - (y-3x-4)dx]\vec{k}$$

$$= \int_0^{2\pi} (14\cos\theta + 6\sin\theta)\,2\cos\theta\, d\theta\,\vec{i} + \int_0^{2\pi} (14\cos\theta + 6\sin\theta)\,2\sin\theta\, d\theta\,\vec{j} +$$

$$\int_0^{2\pi} [(-2\cos\theta - 2\sin\theta + 2)\,2\cos\theta\, d\theta + (2\sin\theta - 6\cos\theta - 4)\,2\sin\theta\, d\theta]\,\vec{k}$$

$12 + 28$

$$= \int_0^{2\pi} (28\cos^2\theta\, d\theta + 12\sin\theta\cos\theta\, d\theta)\vec{i} + \int_0^{2\pi} (28\cos\theta\sin\theta\, d\theta + 12\sin^2\theta\, d\theta)\vec{j}$$

$$\int_0^{2\pi} [(-4\cos^2\theta - 4\sin\theta\cos\theta\, d\theta + 4\cos\theta\, d\theta) + (4\sin^2\theta\, d\theta - 8\sin\theta\cos\theta\, d\theta - 8\sin$$

from (1,0,0) to (1,1,0) , x=1, dx=0, Z=0 , dZ=0 while y Va[ries]
from 0 to 1

$$\int \phi\, dr = \int_0^1 (2xy^2z + x^2y)\,dy\,\vec{j} = \int_0^1 (y)\,dy\,\vec{j} = \frac{1}{2}y^2\vec{j}\Big|_0^1 = \frac{1}{2}\vec{j}$$

from (1,1,0) to (1,1,1), x=1, dx=0, y=1, dy=0, while z Varies from 0 [to 1]

$$\int \phi\, dr = \int_0^1 (2xy^2z + x^2y)\,dz\,\vec{k} = \int_0^1 (2z+1)\,dz\,\vec{k} = (z^2+z)\vec{k}\Big|_0^1 = 2\vec{k}$$

$$\int \phi\, dr = \frac{1}{2}\vec{j} + 2\vec{k}.$$

C.W

if $\phi = 2xy^2z + x^2y$, evaluate $\int_C \phi\, dr$ where C

(a) is the Curve $x=t,\ y=t^2,\ z=t^3$. from $t=0$ to $t=1$.
(b) Consists of the Straightlines from (0,0,0) to (1,0,0), then to (1,1,0) and then to (1,1,1).

(a)

$$r = x\vec{i} + y\vec{j} + z\vec{k}$$
$$r = t\vec{i} + t^2\vec{j} + t^3\vec{k}$$
$$dr = (\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt$$
$$\phi = 2xy^2z + x^2y$$
$$\phi = 2t^8 + t^4$$

$$\int \phi\, dr = \int_0^1 (2t^8+t^4)(\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt$$
$$= \int_0^1 (2t^8+t^4)\,dt\,\vec{i} + \int_0^1 (2t^8+t^4)2t\,dt\,\vec{j} + \int_0^1 (2t^8+t^4)3t^2\,dt$$
$$= \int_0^1 (2t^8+t^4)\,dt\,\vec{i} + \int_0^1 (4t^9+2t^5)\,dt\,\vec{j} + \int_0^1 (6t^{10}+3t^6)\,dt\,\vec{k}$$
$$= \left(\frac{2}{9}t^9 + \frac{1}{5}t^5\right)\vec{i} + \left(\frac{2}{5}t^{10} + \frac{1}{3}t^6\right)\vec{j} + \left(\frac{6}{11}t^{11} + \frac{3}{7}t^7\right)\vec{k}\,\Big|_0^1$$
$$= \frac{19}{45}\vec{i} + \frac{11}{15}\vec{j} + \frac{75}{77}\vec{k}$$

(b)

$$dr = dx\vec{i} + dy\vec{j} + dz\vec{k}$$
$$\phi = 2xy^2z + x^2y$$

$$\int_C \phi\, dr = \int \left[(2xy^2z + x^2y)\,dx\,\vec{i} + (2xy^2z + x^2y)\,dy\,\vec{j} + (2xy^2z + x^2y)\,dz\,\vec{k}\right]$$

from (0,0,0) to (1,0,0), $y=0, dy=0, z=0, dz=0$ While x Varies from 0

$$\int \phi\, dr = 0$$

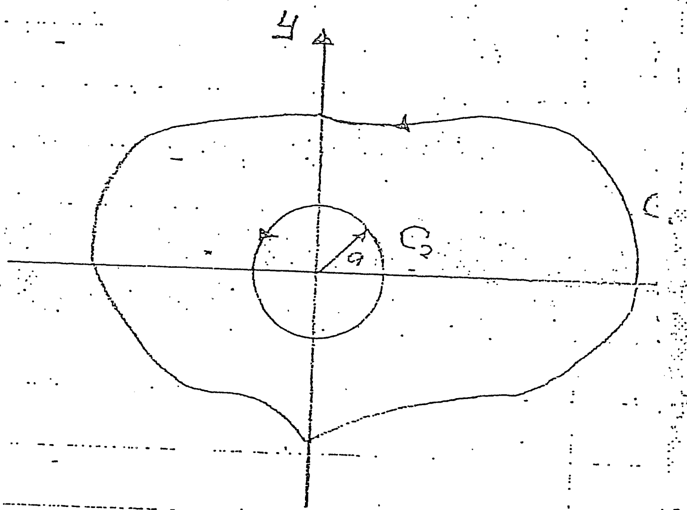
R: Consisting of C_1 and C_2 traversed in the positive dire

and So

$$\int_c [Pdx+Qdy] = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

ii-

$$\int F.dr = \int_{C_1} F.dr - \int_{C_2} F.dr$$



$$= \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy -$$
$$\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$= \iint_R \left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2}-\frac{\partial}{\partial y}\frac{-y}{x^2+y^2}\right)dxdy -$$
$$\iint_R \left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2}-\frac{\partial}{\partial y}\frac{-y}{x^2+y^2}\right)dxdy = 0$$

This gives us

$$\int_{C_1} F.dr = \int_{C_2} F.dr$$

For Circle $x = a\cos\theta \rightarrow dx = -a\sin\theta\, d\theta$

$y = a\sin\theta \rightarrow dy = a\cos\theta\, d\theta$

$$\int F.dr = \int_0^{2\pi} \frac{a^2\sin^2\theta\, d\theta + a^2\cos\theta\, d\theta}{a^2\cos^2\theta + a^2\sin^2\theta} = \int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} = 2\pi$$

التاريخ / / ٢٠ الموضوع :

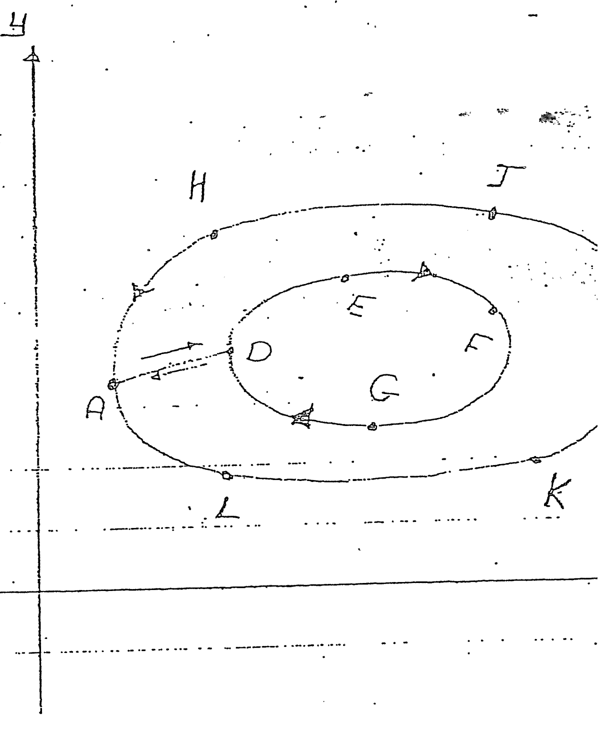
i- Prove Green's theorem in the plane for multiply connected region

ii- if C_1 is a simple closed curve enclosing the origin and C_2 is a circle of radius "a" centred at the origin such that C_2 is completely enclosed by C_1, evaluate the integral.

$$\int_{C_1} \frac{1}{(x^2+y^2)} (-y\,dx + x\,dy).$$

i-

The region R is multiply connected the boundary of R consist of the exterior boundary AHJKLA and the interior boundary DEFGD is to be traversed in the positive direction. In order to establish the theorem construct a line such as AD called cross cut connecting the exterior and interior boundary. The region bounded by ADEFGDALKJHA is simply connected and so Green theorem is valid.



$$\int_{ADEFGDALKJHA} [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

the integral on the left is equal to

$$\int_{AD} + \int_{DEFGD} + \int_{DA} + \int_{ALKJHA} = \int_{DEFGD} + \int_{ALKJHA}$$

if C_1 is the curve ALKJHA and C_2 is the curve DEFGD and C is the bo

Using Green's lemma, establish the formula.

$$\iint_R \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) dx\,dy = \int_c \frac{dF}{dn} ds$$

Where R is the region bounded by the Simple Closed Curve and dF/dn is the directional derivative of F in the direction of the outer normal to C.

By Green theorem

$$\int [P dx + Q dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

let $P = -\frac{\partial F}{\partial y}$, $Q = \frac{\partial F}{\partial x}$

$$\iint \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) dx\,dy = \iint \nabla^2 F\, dx\,dy$$

$$\int [P dx + Q dy] = \int \left[P \frac{dx}{ds} + Q \frac{dy}{ds}\right] ds$$

$$= \int \left[-\frac{\partial F}{\partial y}\frac{dx}{ds} + \frac{\partial F}{\partial x}\frac{dy}{ds}\right] ds$$

$$= \int (\nabla F \cdot n)\, ds = \int \frac{\partial F}{\partial n} ds$$

$$\therefore \iint \nabla^2 F\, dx\,dy = \int \frac{\partial F}{\partial n} ds$$

7. W

$$A=\frac{1}{2}\int_0^{2\pi} r^2\,d\theta$$

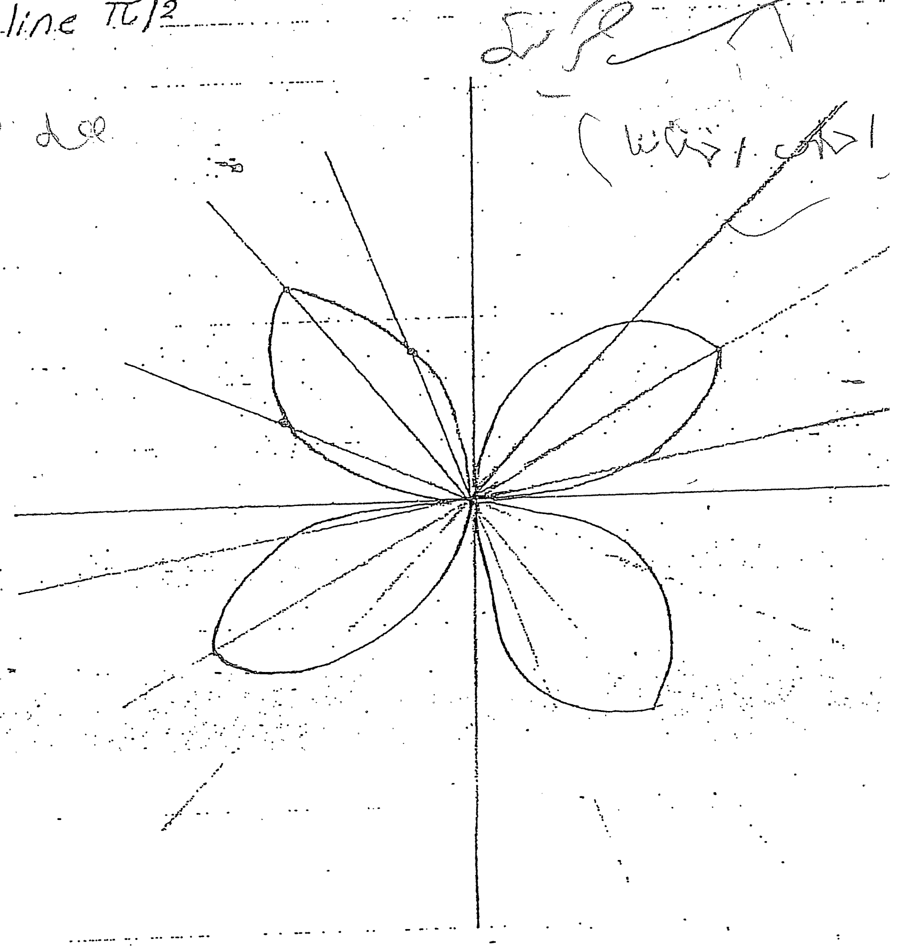
Find the area of a loop of the four leafed rose $\rho=3$

Symmetric about line $\pi/2$

$\rho=3\sin 2\theta$

$A=\frac{1}{2}\int \rho\, d\theta$

θ	ρ
0	0
$\pi/6$	2.6
$\pi/4$	3
$\pi/3$	2.6
$\pi/2$	0
$3\pi/2$	0
$5\pi/3$	-2.6
$7\pi/4$	-3
$11\pi/6$	-2.6
2π	0



$$A=\frac{1}{2}\int \rho^2\,d\phi$$

$$A=\frac{1}{2}\int_0^{\pi/2} 9\sin^2 2\phi\,d\phi=\frac{9}{4}\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta$$

$$=\frac{9}{4}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\pi/2}=\frac{9\pi}{8}$$

$$A=\frac{1}{2}\int r^2\,d\theta$$

Z.W

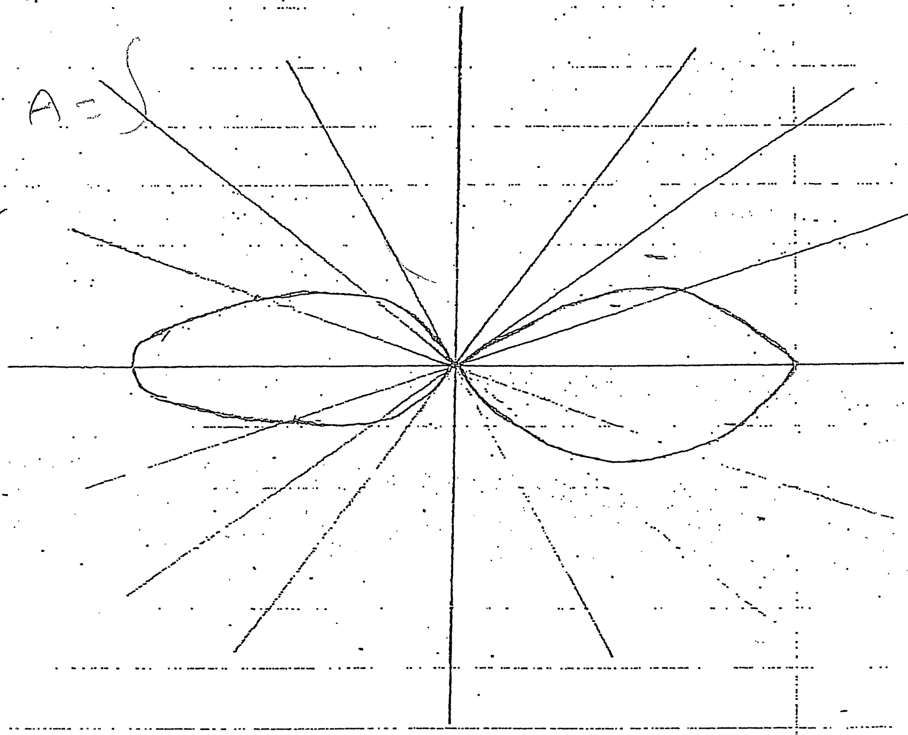
$\frac{1}{2}\int_0 r^2 d\theta$

Find the area of both loops of the lemniscate $\rho^2 = a^2\cos 2\phi$

the Symmetric about Polar axis

$\rho^2 = a^2\cos 2\phi \rightarrow \rho = a\sqrt{\cos 2\phi}$

θ	ρ
0	a
$\pi/6$	$0.71a$
$\pi/4$	0
$\pi/3$	—
$\pi/2$	—
$2\pi/3$	—
$3\pi/4$	0
$5\pi/6$	$0.71a$
π	a



$A = \frac{1}{2}\int \rho^2 d\phi$

$A = \frac{1}{2}\int_0^{\pi/4} a^2\cos 2\phi \, d\phi = \frac{a^2}{4}\sin 2\phi \Big|_0^{\pi/4} = \frac{a^2}{4}$

the total Integral $= 4 \times \frac{a^2}{4} = a^2$

$\frac{3\pi}{4}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, 0$

$A = \frac{1}{2}$

Discuss the transformation defined by the function $W = \sin$

$W = \sin z = \sin(x + yi) = \sin x \cos yi + \cos x \sin yi$

$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y \quad , \quad v = \cos x \sinh y$

$\sin^2 + \cos^2$ ... $\cosh^2 - \sinh^2$ ...

$\sin x = \dfrac{u}{\cosh y} \quad , \quad \cos x = \dfrac{v}{\sinh y}$

$\sin^2 x + \cos^2 x = \dfrac{u^2}{\cosh^2 y} + \dfrac{v^2}{\sinh^2 y} = 1$ Eq of Ellipse

or

$\cosh y = \dfrac{u}{\sin x} \quad , \quad \sinh y = \dfrac{v}{\cos x}$

$\cosh^2 y - \sinh^2 y = \dfrac{u^2}{\sin^2 x} - \dfrac{v^2}{\cos^2 x} = 1$ Eq of hyperbolic

$x^2 + y^2 = r$

$0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$

$A = \frac{1}{2}\int r^2 d\theta$

$A = \frac{1}{2}\int r^2 d\theta$

.W

* Prove that $w = z/(1-z)$ maps the upper half of the z plane onto the upper half of the w plane. What is the image of the Circle $|z| = 1$ under this transformation?

$$w = \frac{z}{1-z} = \frac{x+iy}{1-(x+iy)} = \frac{x+iy}{1-x-iy} \times \frac{1-x+iy}{1-x+iy}$$

$$A = \frac{1}{2}\int \rho^2 \, d\theta$$

$$A = \frac{1}{2}\int x\,dy - y\,dx$$

Z.W

Find the image of the Circle $|Z| = 2$ in the w-plane: it is mapped under the transformation $w = Z + 3 + 2i$

$w = Z + 3 + 2i$

$u + iv = x + yi + 3 + 2i$

$u + iv = x + 3 + (y + 2)\, i$

$u = x + 3 \quad , \quad v = y + 2$

For Circle $|Z| = 2 \rightarrow |x + iy| = 2 \rightarrow \sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = 4$

$u = x + 3 \rightarrow x = u - 3 \quad , \quad v = y + 2 \rightarrow y = v - 2$

$x^2 + y^2 = 4 \rightarrow (u - 3)^2 + (v - 2)^2 = 4$ $\quad u + iv = x + iy + 3 + 2i$

$\sqrt{x^2 + y^2} = 2$

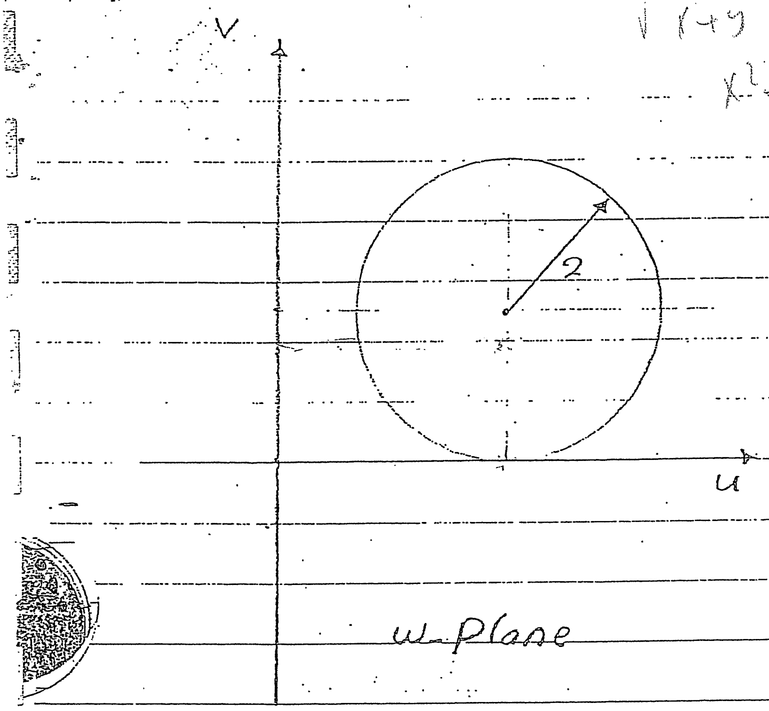
$x^2 + y^2 = 4$

$u = x + 3$

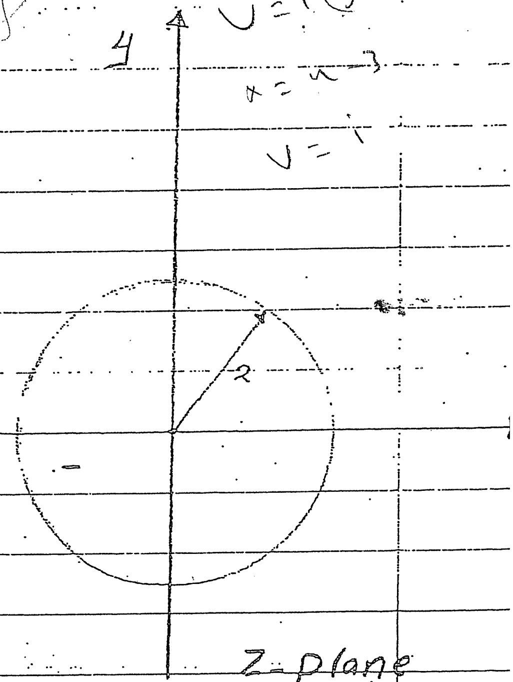
$v = i(y + 2)$

$x = u - 3$

$v = i$



w-plane



Z-plane

7..W

if u and v are harmonic in a region R, Show that

$$\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

is analytic in R

u, v are harmonic function

∴ Satisfy Laplace Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad , \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

$$U=\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right) \quad , \quad V=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x \partial y}-\frac{\partial^2 v}{\partial x^2} \quad , \quad \frac{\partial v}{\partial x}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial y \partial x} \quad , \quad \frac{\partial v}{\partial y}=\frac{\partial^2 u}{\partial y \partial x}+\frac{\partial^2 v}{\partial y^2}$$

By Using Cauchy Riemann Eq

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \rightarrow \frac{\partial^2 u}{\partial x \partial y}-\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 u}{\partial y \partial x}+\frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}+\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}\right) \rightarrow \frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \rightarrow \frac{\partial^2 u}{\partial y^2}-\frac{\partial v}{\partial y \partial x}=-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x \partial y}\right)$$

$$\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}\right)-\frac{\partial^2 v}{\partial y \partial x}=-\frac{\partial^2 v}{\partial x \partial y} \rightarrow \frac{\partial^2 v}{\partial y \partial x}=\frac{\partial^2 v}{\partial x \partial y}$$

∴ Analytic function

R.W

الموضوع : التاريخ : / / ٢٠

What is the Complex potential of an upward parallel flow in the direction of $y = 2x$.

$$\frac{d}{dz} f(z) = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y} = V_1 - i V_2$$

$V_1 = K$ (K positive real)

$V_1 = \frac{\partial \phi}{\partial x} \rightarrow V_1 = Kx$

$V_2 = \frac{\partial \phi}{\partial y} \rightarrow V_2 = Ky = K 2x$

$\frac{\partial \phi}{\partial x} = V_1 = K$

$\phi = Kx$

$u + iv = x + iy$

$u = \frac{\partial \phi}{\partial x}$

$\phi + i\psi = x + iy$

$\phi = x$

$\frac{\partial \phi}{\partial x} =$

$\frac{\partial \phi}{\partial y} = V_2$

$y = 2x$

$\frac{\partial \phi}{\partial x} = V_1 = K$

$\phi =$

$\phi = Kx + C$

$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$

$\phi = \int \frac{\partial \phi}{\partial x} dy = Ky$

$$\frac{\partial v}{\partial y} \qquad \frac{\partial V}{\partial y}$$

$$\frac{dw}{dz} = i\left(\frac{\partial U}{\partial r}\sin\theta + \frac{\partial U}{\partial \theta}\frac{1}{r}\cos\theta\right) + \frac{\partial V}{\partial r}\sin\theta + \frac{\partial V}{\partial \theta}\frac{1}{r}\cos\theta$$

$$= i\frac{\partial U}{\partial r}\sin\theta - i\frac{\partial U}{\partial \theta}\frac{1}{r}\cos\theta + \frac{\partial V}{\partial r}\sin\theta + \frac{\partial V}{\partial \theta}\frac{1}{r}\cos\theta$$

$$= -i\frac{\partial U}{\partial r}\sin\theta + \frac{\partial V}{\partial r}\sin\theta - i\frac{\partial U}{\partial \theta}\frac{1}{r}\cos\theta + \frac{\partial V}{\partial \theta}\frac{1}{r}\cos\theta$$

$$= \left(-i\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\right)\sin\theta + \left(-i\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial \theta}\right)\frac{1}{r}\cos\theta$$

$$= -i\left(\frac{\partial U}{\partial r} - \frac{1}{i}\frac{\partial V}{\partial r}\right)\sin\theta - i\left(\frac{\partial U}{\partial \theta} - \frac{1}{i}\frac{\partial V}{\partial \theta}\right)\frac{1}{r}\cos\theta$$

$$= -i\left(\frac{\partial U}{\partial r} + i\frac{\partial V}{\partial r}\right)\sin\theta - i\left(\frac{\partial U}{\partial \theta} + i\frac{\partial V}{\partial \theta}\right)\frac{1}{r}\cos\theta$$

$$= -i\sin\theta\frac{\partial w}{\partial r} - i\frac{1}{r}\cos\theta\frac{\partial w}{\partial \theta} \qquad (2)$$

$$\cos\theta\frac{\partial w}{\partial r} - \frac{1}{r}\frac{\partial w}{\partial \theta}\sin\theta = -i\sin\theta\frac{\partial w}{\partial r} - i\frac{1}{r}\cos\theta\frac{\partial w}{\partial \theta}$$

$$\cos\theta\frac{\partial w}{\partial r} + i\sin\theta\frac{\partial w}{\partial r} = \frac{1}{r}\frac{\partial w}{\partial \theta}\sin\theta - i\frac{1}{r}\cos\theta\frac{\partial w}{\partial \theta}$$

$$(\cos\theta + i\sin\theta)\frac{\partial w}{\partial r} = \frac{\sin\theta - i\cos\theta}{r}\frac{\partial w}{\partial \theta}$$

Z.W

$$\frac{dw}{dz}=\frac{\partial U}{\partial r}\cos\theta-\frac{\partial U}{\partial \theta}\frac{1}{r}\sin\theta+i\left(\frac{\partial V}{\partial r}\cos\theta-\frac{\partial V}{\partial \theta}\frac{1}{r}\sin\theta\right)$$

$$=\frac{\partial U}{\partial r}\cos\theta+i\frac{\partial V}{\partial r}\cos\theta-\left(\frac{\partial U}{\partial \theta}\frac{1}{r}\sin\theta+i\frac{\partial V}{\partial \theta}\frac{1}{r}\sin\theta\right)$$

$$=\left(\frac{\partial U}{\partial r}+i\frac{\partial V}{\partial r}\right)\cos\theta-\left(\frac{\partial U}{\partial \theta}+i\frac{\partial V}{\partial \theta}\right)\frac{1}{r}\sin\theta$$

$$=\cos\theta\frac{\partial}{\partial r}(u+iv)-\frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}(u+iv)$$

$$=\cos\theta\frac{\partial w}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial w}{\partial \theta} \longleftarrow (1)$$

$$\frac{dw}{dz}=-i\frac{\partial U}{\partial y}+\frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y}=\frac{\partial U}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial U}{\partial y}=\frac{\partial U}{\partial r}\frac{y}{r}+\frac{\partial U}{\partial \theta}\frac{x}{x^2+y^2}$$

$$\frac{\partial U}{\partial y}=\frac{\partial U}{\partial r}\frac{r\sin\theta}{r}+\frac{\partial U}{\partial \theta}\frac{r\cos\theta}{r^2}$$

$$\frac{\partial U}{\partial y}=\frac{\partial U}{\partial r}\sin\theta+\frac{\partial U}{\partial \theta}\frac{1}{r}\cos\theta$$

$$\frac{\partial V}{\partial y}=\frac{\partial V}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial y}$$

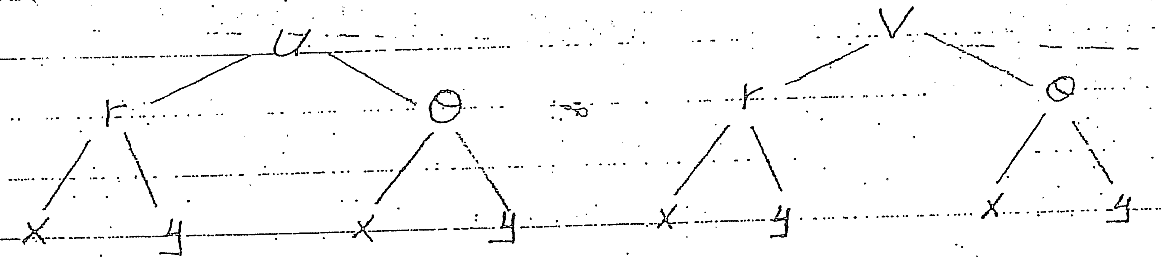
$$\frac{\partial V}{\partial y}=\frac{\partial V}{\partial r}\frac{y}{r}+\frac{\partial V}{\partial \theta}\frac{x}{x^2+y^2}$$

$$\frac{\partial V}{\partial y}=\frac{\partial V}{\partial r}\frac{r\sin\theta}{r}+\frac{\partial V}{\partial \theta}\frac{r\cos\theta}{r^2}$$

$$\frac{\partial V}{\partial y}=\frac{\partial V}{\partial r}\sin\theta+\frac{\partial V}{\partial \theta}\frac{1}{r}\cos\theta$$

u is an analytic function of $z = r(\cos\theta + i\sin\theta)$, Show that

$$\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} = -\frac{\sin\theta + i\cos\theta}{r}\frac{\partial w}{\partial \theta}$$



$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{r} + \frac{\partial u}{\partial \theta}\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{r\cos\theta}{r} - \frac{\partial u}{\partial \theta}\frac{r\sin\theta}{r^2}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{1}{r}\sin\theta}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{x}{r} - \frac{\partial v}{\partial \theta}\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{r\cos\theta}{r} - \frac{\partial v}{\partial \theta}\frac{r\sin\theta}{r^2}$$

$$\boxed{\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{1}{r}\sin\theta}$$

Z.W

$$\frac{dw}{dz} = \cancel{i\sin\theta\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)} = \cancel{i\sin\theta\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)}$$

$$\cancel{\frac{\partial}{\partial r}(u+iv)\, i\sin\theta} = -i\sin\theta\frac{\partial w}{\partial r}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial w}{\partial r}\sin\theta$$

$$= \frac{\partial w}{\partial r}(\cos\theta - i\sin\theta) = \frac{\partial w}{\partial r}e^{-i\theta}$$

$$\therefore \frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r}$$

$$\frac{\partial w}{\partial \alpha} = \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \alpha}$$

$$\frac{\partial w}{\partial \alpha} = \frac{\partial w}{\partial z}\, r(-\sin\alpha + i\cos\alpha)$$

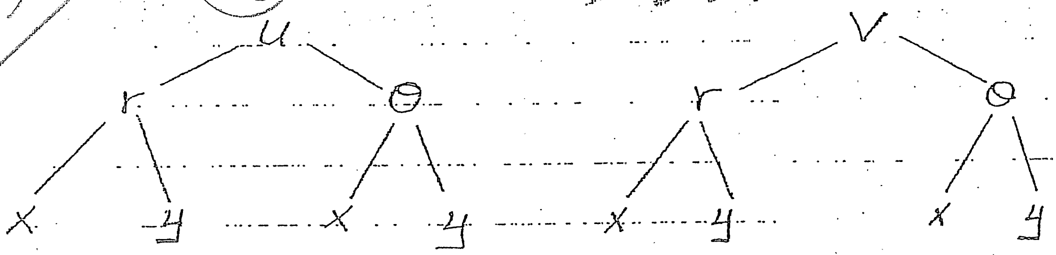
$$\frac{\partial w}{\partial z} = \frac{1}{r(-\sin\theta + i\cos\theta)}$$

$$-\sin\alpha - i\cos\alpha$$

التاريخ: / / الموضوع:

if $w = f(z)$ is analytic and express in polar Coordinates (r, θ)

Prove that $\dfrac{dw}{dz} = e^{-i\theta} \dfrac{\partial w}{\partial r}$



$$\boxed{\begin{aligned} \frac{dw}{dz} &= f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ \frac{dw}{dz} &= f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned}}$$

$x = r\cos\theta \;,\; y = r\sin\theta \;,\; r = \sqrt{x^2 + y^2} \;,\; \theta = \tan^{-1} \dfrac{y}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{r} = \frac{\partial u}{\partial r}\frac{r\cos\theta}{r} = \frac{\partial u}{\partial r}\cos\theta$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial v}{\partial r}\frac{x}{r} = \frac{\partial v}{\partial r}\frac{r\cos\theta}{r} = \frac{\partial v}{\partial r}\cos\theta$$

$$\frac{dw}{dz} = \frac{\partial u}{\partial r}\cos\theta + i\frac{\partial v}{\partial r}\cos\theta = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta$$

$$\frac{dw}{dz} = \frac{\partial}{\partial r}(u + iv)\cos\theta = \frac{\partial w}{\partial r}\cos\theta$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = \frac{\partial u}{\partial r}\frac{y}{r} = \frac{\partial u}{\partial r}\frac{r\sin\theta}{r} = \frac{\partial u}{\partial r}\sin\theta$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} = \frac{\partial v}{\partial r}\frac{y}{r} = \frac{\partial v}{\partial r}\frac{r\sin\theta}{r} = \frac{\partial v}{\partial r}\sin\theta$$

$$\frac{dw}{dz} = -i\frac{\partial u}{\partial r}\sin\theta + \frac{\partial v}{\partial r}\sin\theta$$

Z.W

Prove that if $w = f(z) = u + iv$ is analytic in a region R ...

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

$w = u + iv$

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} \qquad \frac{dw}{dz} = \cos\theta \frac{\partial w}{\partial r} - i\sin\theta \frac{\partial w}{\partial r}$$

$$= (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$$

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r} \qquad \frac{\partial w}{\partial x} = \frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r}$$

$$w = f(z) = re^{i\theta}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$u + iv$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial r}$$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad -\frac{\partial u}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + i\sin\theta$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \times \frac{1}{\cos\theta + i\sin\theta}$$

$$= \frac{\partial w}{\partial r} \times \frac{(\cos\theta - i\sin\theta)}{\cos\theta + i\sin\theta}$$

$$= \frac{\partial w}{\partial r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial u}{\partial r}\cos\theta \quad \frac{\partial w}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta \qquad \cos\theta\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= \cos\theta\frac{\partial w}{\partial r}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} = \boxed{\frac{\partial v}{\partial r}\sin\theta} \qquad \sin\theta\left(\frac{\partial v}{\partial r} + i\frac{\partial u}{\partial r}\right) \quad \sin\theta(v + iu)\frac{\partial w}{\partial r}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = \frac{\partial u}{\partial r}\sin\theta \qquad \sin\theta - i\frac{\partial w}{\partial r}$$

Z.W

$$\frac{\partial v}{\partial r}\sin\theta - i\frac{\partial u}{\partial r}\sin\theta$$

$$\sin\theta\left(\frac{\partial v}{\partial r} - i\frac{\partial u}{\partial r}\right)$$

if u and v are Conjugate harmonic function, proved

$$dv = \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx$$

$$v = f(x, y)$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

By Cauchy Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$dv = \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx$$

Z.W

The Straight lines $y = 2x$, $x + y = 6$ in the xy plane are mapped onto the w plane by means of the transformation $w = \frac{1}{z}$. Show graphically the images of the Straight lines in the w-plane

$$w = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{x - yi}{(x+yi)(x-yi)} = \frac{x - yi}{x^2 - xyi + xyi + y^2}$$

$$w = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i$$

$$u = \frac{x}{x^2 + y^2} \quad , \quad v = -\frac{y}{x^2 + y^2}$$

$$* \; y = 2x \longrightarrow v = -\frac{2x}{x^2 + 4x^2} \longrightarrow x = -\frac{v}{2}(x^2 + 4x^2)$$

$$u = \frac{x}{x^2 + y^2} \longrightarrow u = \frac{x}{x^2 + 4x^2} \longrightarrow \cancel{u(x^2 + 4x^2) = -\frac{v}{2}(x^2 + 4x^2)}$$

$$u = -\frac{v}{2} \qquad \text{O.K}$$

x	y	u	v
0	0	0	0
1	2	$\frac{1}{5}$	$-\frac{2}{5}$
-1	-2	$-\frac{1}{5}$	$\frac{2}{5}$

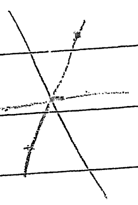
$$x + y = 6 \longrightarrow y = 6 - x$$

$$v = -\frac{y}{x^2 + y^2} \longrightarrow v = -\frac{(6-x)}{x^2 + (6-x)^2} \longrightarrow x^2 + (6-x)^2 = -\frac{(6-x)}{v}$$

$$u = \frac{x}{x^2 + y^2} \longrightarrow u = \frac{x}{x^2 + (6-x)^2} = \frac{x}{-\frac{(6-x)}{v}}$$

$$\cancel{-\frac{u}{v}(6-x) = x}$$

$$u = \frac{-\frac{u(6-x)}{v}}{-\frac{(6-x)}{v}}$$



Z.W

التاريخ / / ٢٠

الموضوع :

Discuss the transformation defined by $w=z^4$. Plot the image of the line $u=1$. What is the equation of the image of the line $x=1$?

$w=z^4$

$w=(x+iy)^2(x+iy)^2$

$w=(x^2+2xyi-y^2)(x^2+2xyi-y^2)$

$w=x^4+2x^3yi-x^2y^2+2x^3yi-4x^2y^2-2xy^3i-x^2y^2-2xy^3i+y^4$

$w=x^4-6x^2y^2+y^4+4x^3yi-4xy^3i$

$w=x^4-6x^2y^2+y^4+(4x^3y-4xy^3)i$

$u=x^4-6x^2y^2+y^4 \quad , \quad v=4x^3y-4xy^3$

$u=1 \Rightarrow x^4-6x^2y^2+y^4=1 \qquad x=1 \qquad v=4y-4y^3$

$1-6y^2+y^4=1$

$x+u=1 \qquad 6y^2=y^4 \qquad \Rightarrow \sqrt{6}=\frac{y^4}{y^2}=y^2$

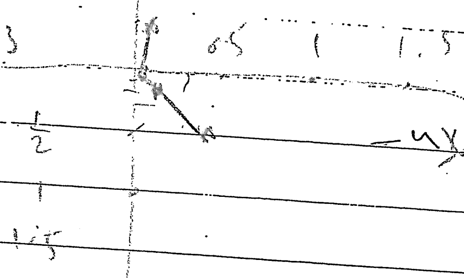
$v=4\sqrt{6}-4\sqrt{6}^3 = -20\sqrt{6}$

$\cancel{(x+yi)^2(x+yi)^2=(x^2-y^2+i2xy)(x^2-y^2+i2xy)}$

$\cancel{x^4-x^2y^2+i2x^3y-x^2y^2+y^4-i2xy^3+i2x^3y-i2xy^3}$

$y^4+x^4-6x^2y^2+\cancel{4x^3yi-4xy^3}$

$x^4+y^4-6x^2y^2=1$



$\cancel{-4x^2y^2}$

Discuss the transformation defined by the function $w = 1/Z$. Plot the ima[ge of] the Square Whose Vertices are the Points $Z = 1+i,\ 2+i,\ 2+2i,\ 1+2i$

$$w = \frac{1}{Z} = \frac{1}{Z}\,\frac{\bar{Z}}{\bar{Z}} = \frac{\bar{Z}}{Z\bar{Z}} = \frac{(x-yi)}{(x+yi)(x-yi)} = \frac{x-yi}{x^2-yxi+yxi+\ldots}$$

$$= \frac{x-yi}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\,i$$

$$u = \frac{x}{x^2+y^2}, \quad V = -\frac{y}{x^2+y^2}$$

$Z = 1+i \quad (x=1,\ y=1)$

$$u = \frac{1}{1^2+1^2} = \frac{1}{2}, \quad V = -\frac{1}{1^2+1^2} = -\frac{1}{2}$$

$Z = 2+i \quad (x=2,\ y=1)$

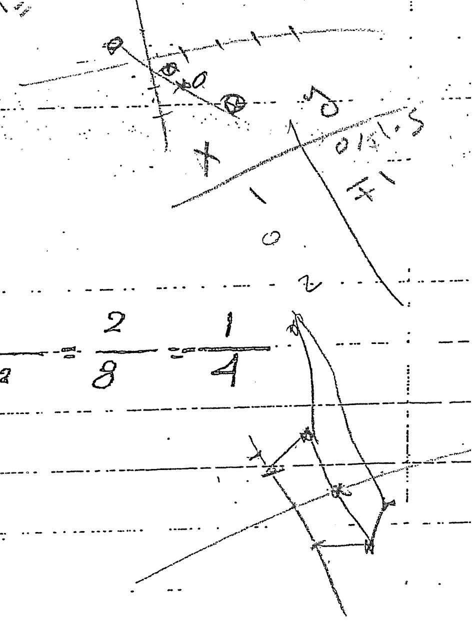
$$u = \frac{2}{2^2+1^2} = \frac{2}{5}, \quad V = -\frac{1}{2^2+1^2} = -\frac{1}{5}$$

$Z = 2+2i \quad (x=2,\ y=2)$

$$u = \frac{2}{2^2+2^2} = \frac{2}{8} = \frac{1}{4}, \quad V = -\frac{2}{2^2+2^2} = -\frac{2}{8} = -\frac{1}{4}$$

$Z = 1+2i \quad (x=1,\ y=2)$

$$u = \frac{1}{1^2+2^2} = \frac{1}{5}, \quad V = -\frac{2}{1^2+2^2} = -\frac{2}{5}$$



In polar form

$$w = \frac{1}{Z}, \quad w = \rho e^{i\phi}, \quad Z = re^{i\theta}$$

$$\rho e^{i\phi} = \frac{1}{r} e^{-i\theta} \rightarrow \rho = \frac{1}{r}, \quad \phi = -\theta$$

or $w = \frac{1}{Z} \rightarrow u + iv = \frac{1}{r} e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$u = \frac{1}{r}\cos\theta, \quad V = -\frac{1}{r}\sin\theta, \quad \cos\theta = ur, \quad \sin\theta = -Vr$

$\cos^2\theta = u^2r^2, \quad \sin\theta = V^2r^2, \quad \cos^2\theta + \sin^2\theta = 1 = u^2 + V^2 = \ldots$

Z.W

Show that the mapping $w = z^2$ transforms every straight line into a parabola.

$$w = z^2 = (x+yi)^2 = x^2 + 2xyi - y^2$$
$$= x^2 - y^2 + 2xyi$$
$$u = x^2 - y^2 \quad , \quad v = 2xy$$

line Parallel to the y-axis with equation $x = C_1$

$$v = 2C_1 y \longrightarrow y = \frac{v}{2C_1}$$

$$u = x^2 - y^2 \longrightarrow u = C_1^2 - \frac{1}{4}\frac{v^2}{C_1^2}$$

line Parallel to x-axis with equation $y = C_2$

$$v = 2xC_2 \longrightarrow x = \frac{v}{2C_2}$$

$$u = x^2 - y^2 = \frac{1}{4}\frac{v^2}{C_2^2} - C_2^2$$

$\therefore$ Every Straight line transform to the Parabola.

Z.W

: الموضوع

التاريخ

What is the image of the Circle $x^2+y^2=a^2$ under the mapping w

$\omega = Z^2 = (x+iy)^2 = x^2 + 2xyi - y^2$

$= x^2 - y^2 + 2xyi$

$\boxed{u = x^2 - y^2} \quad , \quad v = 2xy$

$u = x^2 - y^2 \quad v = 2a$

$= a^2\cos^2\alpha - a^2\sin^2\alpha$

for Circle $x^2+y^2=a^2$

$x = a\cos\theta \quad , \quad y = a\sin\theta$

$v = 2a^2\cos\alpha\sin\alpha$

$u = x^2 - y^2 = a^2\cos^2\theta - a^2\sin^2\theta$

$v = 2a^2 \Rightarrow a = \sqrt{\frac{v}{2}}$

$w = (x+iy)^2 = x^2 + 2xyi + y^2$

$u = x^2 - y^2 \quad v = 2xy$

$r^2\cos^2\alpha - r^2\sin^2\alpha$

$\Rightarrow a^2\sin^2\alpha$

وصولي

Z.W

i- Prove that in polar form the Cauchy-Riemann equations can b

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

ii- Hence prove that the real and imaginary parts of an analytic func of a Complex Variable when expressed in polar form Satisfy the equa

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

a.) $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Z.W

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\partial^2\phi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2$$

$$+\boxed{\frac{\partial^2\phi}{\partial u\partial v}\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}}+\boxed{\frac{\partial^2\phi}{\partial v\partial u}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}+\frac{\partial^2\phi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2$$

$$+\frac{\partial^2\phi}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2+\boxed{\frac{\partial^2\phi}{\partial v\partial u}\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}}+\boxed{\frac{\partial^2\phi}{\partial u\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}}+\frac{\partial^2\phi}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2$$

$$=\frac{\partial^2\phi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial^2\phi}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial^2\phi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial^2\phi}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2$$

$$=\frac{\partial^2\phi}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(-\frac{\partial v}{\partial x}\right)^2\right]+\frac{\partial^2\phi}{\partial v^2}\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$=\frac{\partial^2\phi}{\partial u^2}\,|f'(z)|^2+\frac{\partial^2\phi}{\partial v^2}\,|f'(z)|^2$$

$$=|f'(z)|^2\left(\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right)$$

التاريخ

الموضوع :

if $\phi(x,y)$ is a Solution of $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ prove that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=|f'(z)|^2\left(\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right)$$ where

$w=f(z)$ is analytic and $f'(z)\neq 0$

$\phi(x,y)$ is transform into a function u, v by a Conformal transforma

$\phi = u(x,y)+iv(x,y)$

$$\frac{\partial\phi}{\partial x}=\boxed{\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}}$$

$$\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial\phi}{\partial u}\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial^2\phi}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2\phi}{\partial u\partial v}\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial^2 v}{\partial x^2}$$
$$+\left(\frac{\partial^2\phi}{\partial v\partial u}\frac{\partial u}{\partial x}+\frac{\partial^2\phi}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x}$$

$$\frac{\partial^2\phi}{\partial y^2}=\frac{\partial\phi}{\partial u}\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial^2\phi}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2\phi}{\partial u\partial v}\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial^2 v}{\partial y^2}+$$
$$\left[\frac{\partial^2\phi}{\partial v^2}\frac{\partial v}{\partial y}+\frac{\partial^2\phi}{\partial v\partial u}\frac{\partial u}{\partial y}\right]\frac{\partial v}{\partial y}$$

By adding we obtain

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\partial\phi}{\partial u}\cancel{\frac{\partial^2 u}{\partial x^2}}+\left(\frac{\partial^2\phi}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2\phi}{\partial u\partial v}\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\cancel{\frac{\partial^2 v}{\partial x^2}}$$
$$+\left(\frac{\partial^2\phi}{\partial v\partial u}\frac{\partial u}{\partial x}+\frac{\partial^2\phi}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x}+\frac{\partial\phi}{\partial u}\cancel{\frac{\partial^2 u}{\partial y^2}}+\left(\frac{\partial^2\phi}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2\phi}{\partial u\partial v}\frac{\partial v}{\partial y}\right)$$
$$\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\cancel{\frac{\partial^2 v}{\partial y^2}}+\left[\frac{\partial^2\phi}{\partial v^2}\frac{\partial v}{\partial y}+\frac{\partial^2\phi}{\partial v\partial u}\frac{\partial u}{\partial y}\right]\frac{\partial v}{\partial y}$$

Z.W

2

Determine the equation of the image in the z-plane into which the Straight line $u = K$ (Const.) is mapped under the transform $w = \frac{z-1}{z+1}$. Sketch the Case $K = \frac{1}{2}$.

$$w = \frac{z-1}{z+1} = \frac{x+yi-1}{x+yi+1} = \frac{x-1+yi}{x+1+yi} \times \frac{x+1-yi}{x+1-yi}$$

$$= \frac{x^2-x+xyi+x-1+yi-xyi+yi+y^2}{x^2+x+xyi+x+1+yi-xyi-yi+y^2}$$

$$= \frac{x^2+y^2+2yi-1}{x^2+y^2+2x+1}$$

$$w = u(x,y) + iv(x,y)$$

$$u + iv = \frac{x^2+y^2-1}{x^2+y^2+2x+1} + \frac{2y\,i}{x^2+y^2+2x+1}$$

$$u = \frac{x^2+y^2-1}{x^2+y^2+2x+1} = K$$

$$Kx^2 + Ky^2 + 2Kx + K = x^2 + y^2 - 1 \qquad \times 2$$

$$2K(x^2+y^2+2x+1) = 2x^2+2y^2-2 \qquad @\ K = \tfrac{1}{2} \Rightarrow 2K = 1$$

$$\Rightarrow x^2+y^2+2x+1 = 2x^2+2y^2-2$$

$$x^2+y^2-2x-2-1 = 0 \rightarrow x^2+y^2-2x = 3$$

تحويل من w-plane إلى z-plane

$$x^2+y^2-2x+1-1 = 3$$

$$\boxed{(x-1)^2+y^2 = 4}$$

Z.W

Find the Velocities and paths of particles of an ideal fluid moving with uniform flow around a 90° bend (Consider the quadrant of the Z-plane). Use the mapping function $w = i$

$$w = F(z) = iZ^2 = i(x+iy)^2 = i(x^2+2xyi-y^2)$$
$$= x^2 i - 2xy - y^2 i$$
$$= (x^2-y^2)i - 2xy$$

$F(z) = \phi(x,y) + i\psi(x,y)$ — Complex potential

$$\phi = -2xy \quad , \quad \psi = x^2 - y^2$$

$$V_1 = \frac{\partial \phi}{\partial x} = -2y \quad , \quad V_2 = \frac{\partial \phi}{\partial y} = -2x$$

$$V = V_1 + iV_2 = -2y - i2x = -2(y+ix)$$

the magnitude of Velocity $|V| = \sqrt{V_1^2 + V_2^2}$

$$|V| = \sqrt{(-2y)^2 + (-2x)^2} = 2\sqrt{x^2+y^2}$$

Stream line	equipotential line
$\psi(x,y) = K$	$\phi(x,y) = C$
$x^2 - y^2 = K$	$-2xy = C$

$w = i(x^2 - y^2 + 2xyi)$

$= x^2 i - y^2 i - 2xy$

$\phi = -2xy$

$\psi = (x^2 - y^2)$

$V_1 = \frac{\partial \phi}{\partial x} = -2y$

$V_2 = -2x$

$V = -2y - i2x$

$\sqrt{4y^2 + 4x^2}$

Z.W

$(v_1 - iv_2)$

$v = v_1 + v_2$

$\cancel{2r\cos\theta + i2r\sin\theta + \frac{2}{r^3}\cos 3\theta + i\frac{2}{r^3}\sin 3\theta}$

$= 2r\cos\theta - \frac{2}{r^3}\cos 3\theta + i2r\sin\theta + i\frac{2}{r^3}\sin 3\theta$

$F(z) = \phi(x,y) + i\psi(x,y)$

$\phi = 2r\cos\theta - \frac{2}{r^3}\cos 3\theta \;,\; \psi = 2r\sin\theta + \frac{2}{r^3}\sin 3\theta$

$\frac{d}{dz}F(z) = 2r\cos\theta - \frac{2}{r^3}\cos 3\theta + i\left(2r\sin\theta + \frac{2}{r^3}\sin 3\theta\right) = V_1 - iV_2$

$V = V_1 + iV_2 = V_1 - iV_2 = 2r\cos\theta - \frac{2}{r^3}\cos 3\theta - i\left(2r\sin\theta + \frac{2}{r^3}\sin 3\theta\right)$

Streamline

$\psi(x,y) = K$

$\left(r^2 - \frac{1}{r^2}\right)\sin 2\theta = K$

$w = iz^2$

$= i(x^2 - y^2 + 2xyi)$

$= x^2 i - y^2 i - 2xy$

Streamline $\psi = 0$

$\phi = -2xy \quad \psi = x^2 - y^2$

$\left(r^2 - \frac{1}{r^2}\right)\sin 2\theta = 0$

$r^2 - \frac{1}{r^2} = 0 \rightarrow r^2 = \frac{1}{r^2} \rightarrow r = 1$

$v_1 = \frac{\partial\psi}{\partial x} = -2y = C$

or $\sin 2\theta = 0$

$v_2 = \frac{\partial\phi}{\partial y} = -2x$

equipotentialline

$\phi(x,y) = C$

$v = v_1 + iv_2 = -2y - i2x$

$\left(r^2 + \frac{1}{r^2}\right)\cos 2\theta = C$

$v = \sqrt{4x^2 + 4y^2}$

$\psi = -ix = C$

$\phi =$

Z.W

Find the Velocities and paths of particles of an ideal fluid mo... with uniform flow around the Cylinder (Consider the first quadr... Use the mapping function $w = z^2 + \frac{1}{z^2}$

$$w = f(z) = z^2 + \frac{1}{z^2} \qquad (z = re^{i\theta})$$

$$f(z) = (re^{i\theta})^2 + \frac{1}{(re^{i\theta})^2} = r^2 e^{i2\theta} + \frac{1}{r^2 e^{i2\theta}}$$

$$= r^2 e^{i2\theta} + \frac{1}{r^2} e^{-i2\theta}$$

$$= r^2(\cos 2\theta + i\sin 2\theta) + \frac{1}{r^2}(\cos 2\theta - i\sin 2\theta)$$

$$= r^2\cos 2\theta + ir^2\sin 2\theta + \frac{1}{r^2}\cos 2\theta - i\frac{1}{r^2}\sin 2\theta$$

$$= r^2\cos 2\theta + \frac{1}{r^2}\cos 2\theta + i\left(r^2\sin 2\theta - \frac{1}{r^2}\sin 2\theta\right)$$

$$= \left(r^2 + \frac{1}{r^2}\right)\cos 2\theta + i\left(r^2 - \frac{1}{r^2}\right)\sin 2\theta$$

$$F(z) = \phi(x, y) + i\psi(x, y)$$

$$\phi = \left(r^2 + \frac{1}{r^2}\right)\cos 2\theta \quad , \quad \psi = \left(r^2 - \frac{1}{r^2}\right)\sin 2\theta$$

$$\frac{d}{dz}F(z) = V_1 - iV_2$$

$$\frac{d}{dz}F(z) = \frac{d}{dz}\left(z^2 + \frac{1}{z^2}\right) = 2z - \frac{2}{z^3} = 2re^{i\theta} - \frac{2}{r^3 e^{3i\theta}}$$

$$= 2re^{i\theta} - \frac{2}{r^3} e^{-3i\theta}$$

$$= 2r(\cos\theta + i\sin\theta) - \frac{2}{r^3}(\cos 3\theta - i\sin 3\theta)$$

Z.W

$$1-\frac{1}{z^2}$$

$$\frac{d}{dZ}F(Z)=\frac{d}{dZ}\left(Z+\frac{1}{Z}\right)=1-\frac{1}{Z^2}=1-\frac{1}{r^2 e^{2i\theta}}$$

$$=1-\frac{1}{r^2}e^{-2i\theta}=1-\frac{1}{r^2}(\cos 2\theta-i\sin 2\theta)$$

$$=1-\frac{1}{r^2}\cos 2\theta+i\frac{1}{r^2}\sin 2\theta$$

فافهمه

$$\frac{d}{dZ}F(Z)=V_1-iV_2=1-\frac{1}{r^2}\cos 2\theta+i\frac{1}{r^2}\sin 2\theta$$

$$V=V_1+iV_2=\overline{V_1-iV_2}=1-\frac{1}{r^2}\cos 2\theta-i\frac{1}{r^2}\sin 2\theta$$

$$V=\sqrt{\left(1-\frac{1}{r^2}\cos 2\theta\right)^2+\left(-\frac{1}{r^2}\sin 2\theta\right)^2}$$

$$\frac{re^{-i\theta}}{re^{-2i\theta}}$$

شرح
حفظ
ملاحظة
كبير

$$V_1=1-\frac{1}{r^2}\cos 2\theta\quad,\quad V_2=-\frac{1}{r^2}\sin 2\theta$$

$e^{i\theta}$

Stream line

$$\psi(X,y)=K$$

$$\psi=\left(r-\frac{1}{r}\right)\sin\theta=K$$

$e^{-i\theta}$

Equipotential line

$$\phi(X,y)=C$$

$$\phi=\left(r+\frac{1}{r}\right)\cos\theta=C$$

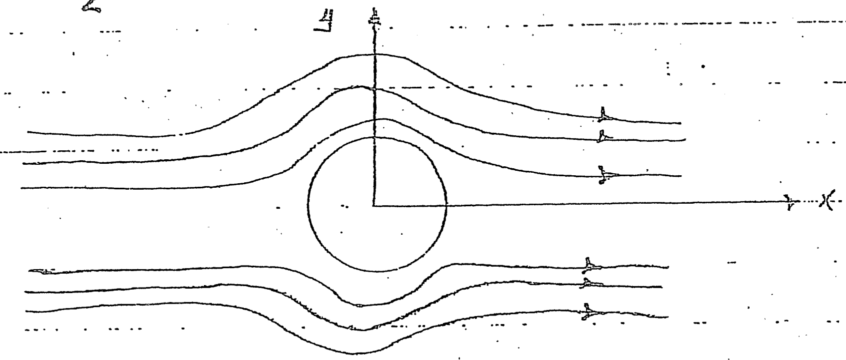
$$u=x^2+y^2+\frac{x^2+y^2}{(x^2+y^2)+(4x^2y^2)}$$

$$\frac{\partial u}{\partial x}=2x+\frac{(x^2+y^2)+4x^2y^2\ldots 2x}{\left((x^2+y^2)+4xy^2\right)^2}$$

Z.W

Find the Velocities and paths of particles of an ideal fluid mov... with uniform flow around the Cylinder. Use the mapping func...

$$w = Z + \frac{1}{Z}$$



المعطاة حل
دائرة فلازم نحول
الى ال
polar

$$F(z) = Z + \frac{1}{Z} \qquad (Z = re^{i\theta})$$

$$F(z) = re^{i\theta} + \frac{1}{re^{i\theta}} = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= r\cos\theta + ir\sin\theta + \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$

$$= r\cos\theta + \frac{1}{r}\cos\theta + ir\sin\theta - i\frac{1}{r}\sin\theta$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$F(z) = \phi(x, y) + i\psi(x, y)$$

$$\phi = \left(r + \frac{1}{r}\right)\cos\theta \quad , \quad \psi = \left(r - \frac{1}{r}\right)\sin\theta$$

$$\frac{d}{dz}F(z) = V_1 - iV_2$$

$V_1 - iV_2$

Z.W

$\frac{df(z)}{dz} = V_1 - iV_2$

Find the Velocities and paths of particles of an ideal fluid moving with uniform flow. Use the mapping function $F(z)=iZ^3$.

$$F(z)=iz^3=i(x+iy)^2(x+iy)=i(x^2+2xyi-y^2)(x+iy)$$
$$=(x^3+2x^2yi-xy^2+x^2yi-2xy^2-y^3i)i$$
$$=x^3i-2x^2y-xy^2i-x^2y-2xy^2i+y^3)$$
$$=(-2x^2y-x^2y+y^3)+(x^3-xy^2-2xy^2)i$$
$$=(-3x^2y+y^3)+(x^3-3xy^2)i$$

$$F(z)=\phi(x,y)+i\psi(x,y)$$

$$\phi=-3x^2y+y^3 \quad , \quad \psi=x^3-3xy^2$$

$$V_1=\frac{\partial\phi}{\partial x}=-6xy \quad , \quad V_2=\frac{\partial\phi}{\partial y}=-3x^2+3y^2$$

$$V=V_1+iV_2=-6xy+(-3x^2+3y^2)i$$

Streamline

$-3x^2+3y^2$

$\psi(x,y)=K$

$\psi=x^3-3xy^2=K$

$-3x^2y+y^3=K$

equipotential line

$\phi(x,y)=C$

$\phi=-3x^2y+y^3=C$

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial x}$$

$$\frac{\partial\phi}{\partial x}=v_1 \qquad \frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}$$

$$\frac{\partial\psi}{\partial y}=v_2$$

$$\frac{\partial\psi}{\partial x}$$

Z.W

* Show that the Stream lines of $F(z) = 1/z$ are Circles thru the origin.

$$F(z) = \frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{(x-iy)}{(x+iy)(x-iy)}$$

$$F(z) = \frac{x-iy}{x^2 - xyi + xyi + y^2} = \frac{x-iy}{x^2+y^2}$$

$$F(z) = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$$

$$\phi = \frac{x}{x^2+y^2}, \quad \psi = -\frac{y}{x^2+y^2}$$

$$\psi(x,y) = k$$

$$-\frac{y}{x^2+y^2} = k \longrightarrow x^2 + y^2 = -\frac{y}{k}$$ it is equation of Circle Center at origin.

حلو نثبت ان خطوط الجريان دوائر تمر بالمركز

[illegible]

Z.W

$f(z) = \phi(x,y) + i\psi(x,y)$

Find the Complex potential of a uniform flow parallel to the x-axis in the positive x-direction

$V = V_1 + iV_2$

$V = V_1 + iV_2$

$V_2 = 0$ because the flow in x-direction.

$V = V_1 = K$

$V_1 = \frac{\partial \phi}{\partial x}$, $V_2 = \frac{\partial \phi}{\partial y}$

$\frac{\partial \phi}{\partial x} = K \rightarrow \phi = Kx$

By Cauchy Riemann Equations

$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$, $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

$\frac{\partial \phi}{\partial x} = K$, $\frac{\partial \phi}{\partial y} = 0$

$\frac{\partial \psi}{\partial y} = K \rightarrow \psi = Ky + f(x) \rightarrow \frac{\partial \psi}{\partial x} = f'(x)$

$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \rightarrow f'(x) = 0 \rightarrow f(x) = 0$??

$\psi = Ky$ ثابت وليس صفر

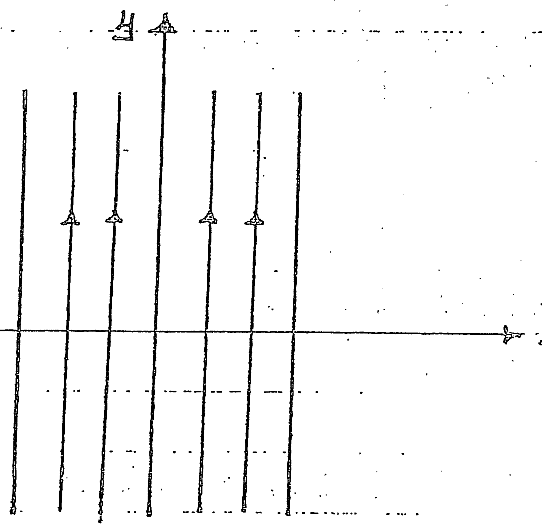
$F(z) = \phi(x,y) + i\psi(x,y)$ (Complex potential).

$F(z) = Kx + iKy = K(x+iy) = Kz$

$\therefore F(z) = Kz$

Z.W

Show that $F(z) = -iKz$ (K positive real) describes a uniform flow upward, which can be interpreted as a uniform flow between two parallel lines (parallel plane in three dimensional space). Find the Velocity Vector, the Stream lines and the equipotential line



$-iK(x+iy)$

$= f(z) = -iKx + Ky$

$\phi(x,y) = Ky$

$\psi(x,y) = \cancel{-K}\ -Kx$

~~$V_1 = \sqrt{V_1^2 + V_2^2} =$~~

$V_1 = \frac{\partial \phi}{\partial x} = 0 \quad V_2 = \frac{\partial \phi}{\partial y} = K$

$V = \sqrt{K^2} = K$

$F(z) = -iKz = -iK(x+yi) = -iKx + Ky$

$F(z) = \phi(x,y) + i\psi(x,y)$

$\phi = Ky \quad , \quad \psi = -Kx$

$V = V_1 + iV_2$

$V_1 = \frac{\partial \phi}{\partial x} = 0 \quad , \quad V_2 = \frac{\partial \phi}{\partial y} = K$

$V = iK$

Equipotential line

$\phi(x,y) = c$

$\phi = Ky = \text{Constant}$

Stream line

$\psi(x,y) = K$

$\psi = -Kx = \text{Constant}$

Z.W

$Vel. = V_1 + iV_2$

$\|Vel\| = \sqrt{V_1^2 + V_2^2}$

Find the Velocities and paths of particles of an ideal fluid moving with uniform flow around a 90° bend (Consider the first quad of the z-plane). Use the mapping function $w = z^2$

$$w = F(z) = z^2 = (x+iy)^2 = x^2 + 2xyi - y^2$$
$$= x^2 - y^2 + 2xy$$

$F(z) = \phi(x,y) + i\psi(x,y)$ (Complex potential)

$\phi = x^2 - y^2$, $\psi = 2xy$

$V_1 = \dfrac{\partial \phi}{\partial x}$, $V_2 = \dfrac{\partial \phi}{\partial y}$

$V_1 = 2x$, $V_2 = -2y$

$V = V_1 + iV_2 \rightarrow V = 2x - 2yi = 2(x - yi) = 2\bar{z}$

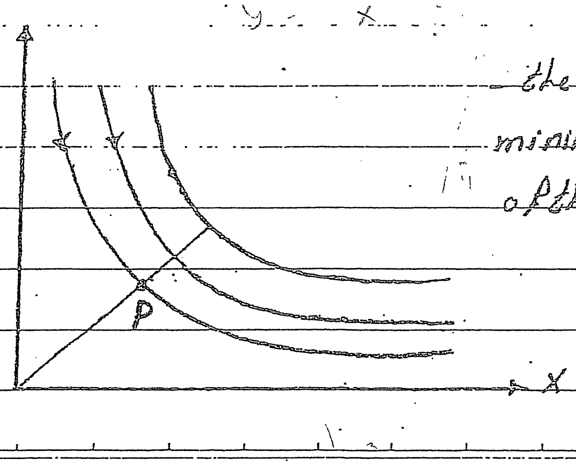
the magnitude of Velocity

$|V| = \sqrt{V_1^2 + V_2^2} = \sqrt{(2x)^2 + (-2y)^2} = 2\sqrt{x^2 + y^2}$

[illegible]

to find the Paths

$\psi(x,y) = K \longrightarrow xy = 1$



the Speed at point P is minimum where the Cross Section of the Channel is large.

Z.W

The Straight lines $y=2x$, $x+y=6$ in the xy plane are mappe onto the w plane by means of the transformation $w=Z^2$ Show graphically the images of the Straight lines in the w plane.

$$w=Z^2=(x+iy)^2=x^2+2xyi-y^2$$

$$=x^2-y^2+2xyi$$

$$u=x^2-y^2 \quad , \quad V=2xy$$

for line $y=2x$

$$V=2xy=2x(2x)=4x^2 \rightarrow V=4x^2 \rightarrow x^2=\frac{V}{4}$$

$$u=\frac{V}{4}-y^2=\frac{V}{4}-(2x)^2=\frac{V}{4}-4x^2=\frac{V}{4}-V$$

$$u=-\frac{3}{4}V$$

for line $x+y=6 \rightarrow y=6-x$

$$u=x^2-(6-x)^2=x^2-[36-12x+x^2]$$

$$u=x^2-36+12x-x^2 \rightarrow u=12x-36 \rightarrow u+36=12x$$

$$x=\frac{u+36}{12}$$

$$V=2xy \rightarrow V=2\,\frac{u+36}{12}\,y \rightarrow V=\frac{1}{6}(u+36)(6-x)$$

$$V=\frac{1}{6}(u+36)\left(6-\frac{u+36}{12}\right)=\frac{1}{6}(u+36)\left(\frac{72-u+36}{12}\right)$$

$$V=\frac{1}{6}(u+36)\frac{1}{12}(36-u)$$

$$\left(\frac{1}{6}u+6\right)\left(\frac{36}{12}-u\right)$$

$$\frac{1}{2}u-\frac{1}{6}u^2+18+6u$$

$$39u+u^2$$

Z.W

$$\frac{13}{2}u-\frac{1}{6}u^2+18=0$$

Determine the equation of the Curve in the w ... the Straight line $x+y=1$ is mapped under the tran...mation

(a) $w = z^2$ (b) $w = 1/z$

(a)

$$w = z^2 = (x+iy)^2 = x^2 + 2xyi - y^2$$
$$= x^2 - y^2 + 2xyi$$

$u = x^2 - y^2$, $v = 2xy$ (0, 1) (1, 0)

$x + y = 1 \longrightarrow y = 1 - x$

$u = x^2 - y^2 = x^2 - (1-x)^2 = x^2 - (1 - 2x + x^2)$

$u = \cancel{x^2} - 1 + 2x - \cancel{x^2}$

$\boxed{u = -1 + 2x} \longrightarrow u + 1 = 2x \longrightarrow x = \frac{u+1}{2}$ $2y(1-y)$

$v = 2xy = 2\left(\frac{u+1}{2}\right) y \longrightarrow \boxed{v = (u+1)\, y}$

$v = (u+1)(1-x) = (u+1)\left(1 - \frac{u+1}{2}\right) = (u+1)\left(\frac{2-u-1}{2}\right)$

$2v = (u+1)(1-u) \rightarrow 2v = u - u^2 + 1 - u$

$u^2 + 2v = 1$

(b) $w = \frac{1}{z} = \frac{1}{z} \cdot \frac{z'}{z'} = \frac{x - yi}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\, i$

$u = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$; $v = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}$ $2v = 1 - u^2$

$x + y = 1 \Longrightarrow y = 1 - x$

$u = \frac{x}{x^2 + 1 - 2x + x^2} = \frac{x}{2x^2 - 2x + 1}$ $\longrightarrow$ Discuss

$w = \frac{1}{z} \Longrightarrow w = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$\Longrightarrow w = \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$

$\therefore u = \frac{1}{r}\cos\theta$; $v = \frac{-1}{r}\sin\theta$

$\boxed{\frac{1}{r} = \frac{u}{\cos\theta}} \Longrightarrow v = \frac{-u}{\cos\theta}\sin\theta \Longrightarrow \boxed{v = -u\tan\theta}$

Z.W or $\cos\theta = ur \Rightarrow \cos^2\theta = u^2r^2 \Rightarrow \cos^2\theta + \sin^2\theta = u^2r^2 + v^2r^2 = 1 \Rightarrow u^2 + v^2 = \frac{1}{r^2}$

$\sin\theta = -vr \Rightarrow \sin^2\theta = v^2r^2$

التاريخ / /

Discuss the transformation defined by $w = z^3$; Plot the i[mage] of line $x=1$: What is the equation of the image of the line $x=$

$$w = z^3 = (x+iy)^3 = (x+iy)^2(x+iy)$$
$$= (x^2 + 2xyi - y^2)(x+iy)$$
$$= x^3 + 2x^2yi - xy^2 + x^2yi - 2xy^2 - y^3 i$$
$$= x^3 - xy^2 - 2xy^2 + (2x^2y + x^2y - y^3)i$$

$u = x^3 - xy^2 - 2xy^2$, $v = 2x^2y + x^2y - y^3$

$u = x^3 - 3xy^2$ $\qquad$ $v = 3x^2y - y^3$

at $x=1$, $y=0$, $y=-1$, $y=1$

$v = 3y - y^3 \longrightarrow 3v = 9y - 3y^3$

$u = 1 - 3y^2 \longrightarrow u = 1 - 3y$ $\therefore u = 1$

$\therefore 1 = 1 - 3y \Rightarrow 3y = 0 \Rightarrow y = 0$

$\therefore v = 0$ $\qquad$ $w = 1 - 3y^2 + i(3y - y^3)$

~~1-y~~ $\quad y = \frac{1-u}{3}$

$\therefore v = \frac{3-3u}{3} - \left(\frac{1-u}{3}\right)^3$

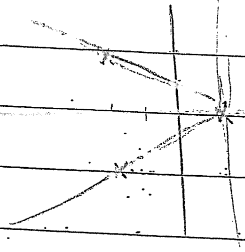
~~$x = 1$~~

$u = 1 \quad v = 0 \quad (1,0)$ $\qquad y = 0$

~~$u = -2 \quad v = 2 \quad (1,1)$~~ $\qquad (0,0) \quad u = 1 \quad v = 0$

~~$u = -2 \quad v = -2 \quad (1,-1)$~~ $\qquad (1,0) \quad u = 1 \quad v = 0$

$(-1,0) \quad u = 1 \quad v = 0$



Z.W

Q/ Discuss the transformation defined by $w = e^z$. What is the equation o[f] image of the line $x+y=1$

$w = e^z \rightarrow \rho e^{i\phi} = e^{x+iy} \rightarrow \rho e^{i\phi} = e^x e^{yi}$

$\rho = e^x \quad , \quad \phi = y$ e^{x+yi} at $x=0$ $y=1$, at $y=0$ $x=1$

for line $x+y=1$

at $x=0 \rightarrow y=1$ — $\rho = e^0 = 1$ $\rho = e^0$ $\phi = 1$

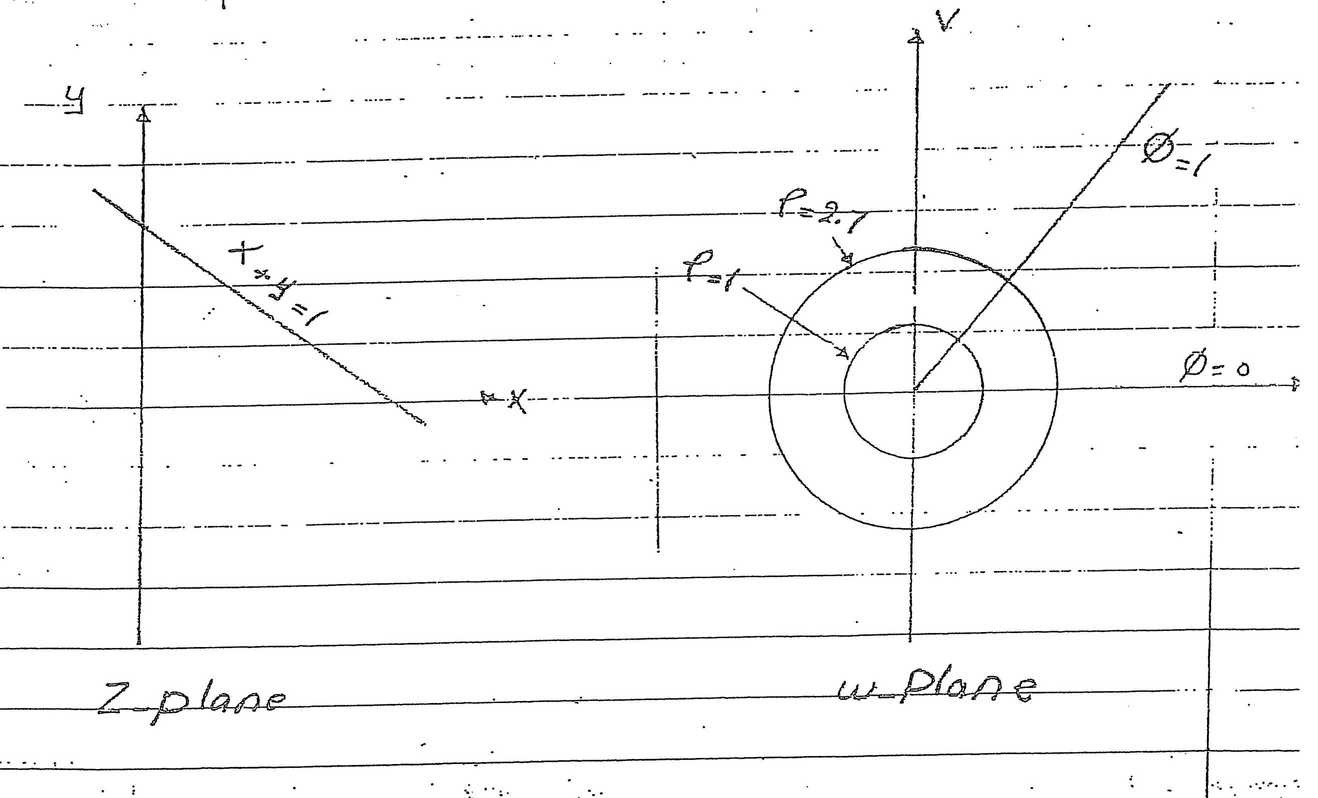
$\phi = 1 \quad , \quad \rho = e^0 = 1$ $\phi = 1$ $\rho = 1$ $\phi = 1$

$\rho = e^1$ $\phi =$

2.7

at $y=0 \rightarrow x=1$

$\phi = 0 \quad , \quad \rho = e^1 = 2.7$



Find the equations of the transformation defined by the func $(z-i)/z$, and show that every circle through the origin in the z-plane is transformed into a straight line in the w-plane.

$$w=\frac{z-i}{z}$$

$$w=\frac{z-i}{z}\cdot\frac{\bar{z}}{\bar{z}}=\frac{(z-i)\bar{z}}{z\bar{z}}=\frac{(x+iy-i)(x-iy)}{(x+iy)(x-iy)}$$

$$w=\frac{x^2+xyi-xi-xyi+y^2-y}{x^2-xyi+xyi+y^2}$$

$$w=\frac{x^2+y^2-y-xi}{x^2+y^2}$$

$$w=1-\frac{y}{x^2+y^2}-\frac{x}{x^2+y^2}i$$

$$u=1-\frac{y}{x^2+y^2}\quad,\quad v=\frac{x}{x^2+y^2}$$

for the circle center at origin $(x=0),(y=0)$

Eq. of circle $x^2+y^2=k^2 \Rightarrow u=1-\frac{y}{k^2}$, $v=\frac{-x}{k^2}$

$\therefore u=1$, $v=0$

$u=1$ Equation of line

$u=1$

Z.W

التاريخ : / / ٢٠ الموضوع :

Find $f(z)$ Such that $f'(z) = 4Z - 3$ and $f(1+i) = -3i$

to find $f(z)$: Integral $f'(z)$ with respect to Z

$$f(z) = \int f'(z) \longrightarrow f(z) = \int (4Z - 3)$$

$$f(z) = 2Z^2 - 3Z + C$$

to find C, $f(1+i) = -3i$

$$-3i = 2(1+i)^2 - 3(1+i) + C$$

$$-3i = 2(1+2i-1) - 3(1+i) + C$$

$$-3i = 2 + 4i - 2 - 3 - 3i + C \longrightarrow C = 3 - 4i$$

$$\therefore f(z) = 2Z^2 - 3Z + 3 - 4i$$

[illegible]

Prove that $f(z) = \dfrac{1}{Z-2}$ is analytic $\boxed{\text{in any region}}$ not including 2

$$f'(z) = \frac{(Z-2) \times 0 - 1 \times 1}{(Z-2)^2} = \frac{-1}{(Z-2)^2}$$

$$f'(z) = \frac{1}{(Z-2)^2}$$

$\therefore$ the function is analytic in any region not including $Z = 2$

$$\lim_{\Delta Z \to 0} \frac{\dfrac{1}{Z + \Delta Z - 2} - \dfrac{1}{Z-2}}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{\dfrac{\cancel{Z} - \cancel{2} - \cancel{Z} - \Delta Z + \cancel{2}}{(Z+\Delta Z - 2)(Z-2)}}{\Delta Z}$$

$$\lim_{\Delta Z \to 0} \frac{\dfrac{-\Delta Z}{(Z+\Delta Z - 2)(Z-2)}}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{-1}{(Z+\Delta Z - 2)(Z-2)}$$

$$= \frac{-1}{(Z+0-2)(Z-2)} = \frac{-1}{(Z-}$$

التاريخ / /

if the imaginary part of an $\boxed{\text{analytic}}$ function is $2x(1-y)$ determine (a) the real part, (b) the function.

$V = 2x(1-y)$ Imaginary Part

the function is analytic therefore Cauchy Riemann Equation satisfied

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad , \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$\frac{\partial V}{\partial y} = -2x \quad , \quad \frac{\partial V}{\partial x} = 2 - 2y$$

$$\frac{\partial U}{\partial x} = -2x \rightarrow U = -x^2 + f(y)$$

$$\frac{\partial U}{\partial y} = f'(y)$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \rightarrow f'(y) = -2 + 2y \rightarrow f(y) = -2y + y^2 + C$$

$$\therefore U = -x^2 - 2y + y^2 + C$$

to find the function, $w = f(z) = u(x,y) + iv(x,y)$

$$w = -x^2 - 2y + y^2 + C + 2x(1-y)i$$

putting $y = 0$, $x = z$

$$w = -z^2 + C + 2zi$$

$$w = -z^2 + 2zi + C$$

Z.W

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$-xe^{-x}\sin y + e^{-x}\sin y + ye^{-x}\cos y = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y - f'($$

$$f'(x) = 0 \rightarrow f(x) = c$$

$$V = -xe^{-x}\sin y + ye^{-x}\cos y + c$$

$$u = e^{-x}x\cos y + e^{-x}y\sin y$$

$$w = f(z) = u(x,y) + iv(x,y)$$

$$w = e^{-x}x\cos y + e^{-x}y\sin y + i(-xe^{-x}\sin y + ye^{-x}\cos y + c)$$

Putting $x = z$, $y = 0$

$$w = ze^{-z} + ic$$

~~since~~ $\cancel{f(0) = 1 \Rightarrow 1 = 0 + ic \Rightarrow ic = 1}$

$$\cancel{w = ze^{-z} + 1}$$

Z.W

Construct an analytic function $f(z)$ whose real part is $e^{-x}(x\cos y + y\sin y)$ and for which $f(0)=1$

Because the function is analytic, the Cauchy Riemann Equation is Satisfied.

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad , \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$U = e^{-x} x\cos y + e^{-x} y\sin y \quad \text{real Part.}$$

$$\frac{\partial U}{\partial x} = [e^{-x} - xe^{-x}]\cos y - e^{-x} y\sin y$$

$$\frac{\partial U}{\partial x} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y$$

$$\frac{\partial U}{\partial y} = -xe^{-x}\sin y + e^{-x}[\sin y + y\cos y]$$

$$\frac{\partial V}{\partial y} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y$$

$$V = e^{-x}\sin y - xe^{-x}\sin y - e^{-x}\int y\sin y\,dy + f(x)$$

$$udv = uv - \int v\,du$$

$$\int y\sin y\,dy \ , \ u = y \ , \ du = dy \ , \ dv = \sin y \ , \ v = -\cos y$$

$$\int y\sin y = -y\cos y + \int \cos y\,dy = -y\cos y + \sin y$$

$$V = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y - e^{-x}\sin y + f(x)$$

$$V = -xe^{-x}\sin y + ye^{-x}\cos y + f(x)$$

$$\frac{\partial V}{\partial x} = -[e^{-x}\sin y - xe^{-x}\sin y] - ye^{-x}\cos y + f'(x)$$

Z.W

Prove that $f(z) = Z|Z|$ is not analytic $\boxed{\text{any Where}}$

$$f'(z) = \lim_{\Delta Z \to 0} \frac{f(Z+\Delta Z) - f(z)}{\Delta Z}$$

$$f'(z) = \lim_{\Delta Z \to 0} \frac{\cancel{Z|Z|} + \Delta Z|\Delta Z| - \cancel{Z|Z|}}{\Delta Z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)|z+\Delta z| - z|z|}{\Delta z}$$

$$f'(Z) = \lim_{\Delta Z \to 0} \frac{\cancel{\Delta Z}|\Delta Z|}{\cancel{\Delta Z}}$$

$$= \lim_{\Delta z \to 0} \frac{z|z| + z|\Delta z| + \Delta z|z| - \Delta z|z|}{\Delta z}$$

$$f'(z) = \lim_{\Delta Z \to 0} |\Delta Z| = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} |\Delta X + i\Delta y| = 0$$

$$= \frac{z|\Delta z| + \Delta z|z| - \Delta z|\Delta z|}{\Delta z}$$

$$f'(Z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sqrt{\Delta x^2 + \Delta y^2}$$

e^{-x} $x = u$ $dx = du$ $dv = e^{-x}$ $v = -e^{-x}$

if $\Delta y = 0 \rightarrow f'(z) = \lim_{\Delta x \to 0} \sqrt{\Delta X^2} = \lim_{\Delta X \to 0} \Delta X = \cancel{\Delta X}$ 0

$uv - \int v\,du$ $= \int (\cos)$ $= \sin$ dv $v = \cos$ $\int e^{-7}$ 0

if $\Delta x = 0 \rightarrow f(z) = \lim_{\Delta y \to 0} \sqrt{\Delta y^2} = \lim_{\Delta y \to 0} \Delta y = \Delta y$

$$\lim_{\Delta z \to 0} \frac{(z+\Delta z)|z+\Delta z| - z|z|}{\Delta z}$$

$\therefore \Delta x \neq \Delta y$

$\boxed{\text{the function is not analytic any Where.}}$

$$\lim_{\Delta z \to 0} \frac{z|\cancel{z}| + z|\Delta z| + \Delta z|z| + \Delta z|\Delta z| - z|\cancel{z}|}{\Delta z}$$

$$= \frac{0}{0} = 0$$

Z.W

(a) If $w = f(z) = z + \frac{1}{z}$, find $\frac{dw}{dz}$ directly from the definition

(b) For What finite Values of z is $f(z)$ non-analytic.

a-

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\cancel{z} + \Delta z - \cancel{z}}{\Delta z} + \lim_{\Delta z \to 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z}$$

$$f'(z) = 1 + \lim_{\Delta z \to 0} \frac{\frac{\cancel{z} - (\cancel{z} + \Delta z)}{z(z+\Delta z)}}{\Delta z}$$

أكو إشارة + تفصل الحدين

عكس — أفصل الحد بكرة.

$$f'(z) = 1 + \lim_{\Delta z \to 0} \frac{-\cancel{\Delta z}}{z(z+\Delta z)\cancel{\Delta z}}$$

$$f'(z) = 1 + \lim_{\Delta z \to 0} \frac{-1}{z^2 + z\Delta z}$$

$$f'(z) = 1 - \frac{1}{z^2}$$

b- for $z = 0$ the function is not analytic and the derivative does not exist.

Z. W

الموضوع : التاريخ / / ٢٠٠

if $Z = \rho e^{i\phi}$ and $f(z) = u(\rho,\phi) + i v(\rho,\phi)$, where ρ and ϕ are polar Coordinates, Show that the Cauchy-Riemann equation are

$$\boxed{\frac{\partial U}{\partial \rho} = \frac{1}{\rho}\frac{\partial V}{\partial \phi}, \quad \frac{\partial V}{\partial \rho} = -\frac{1}{\rho}\frac{\partial u}{\partial \phi}}$$

from Euler formula, $e^{i\theta} = \cos\theta + i\sin\theta$

$$Z = \rho e^{i\phi} = \rho(\cos\phi + i\sin\phi)$$

$$Z = \rho\cos\phi + i\rho\sin\phi$$

$$U = \rho\cos\phi, \quad V = \rho\sin\phi$$

$$\frac{\partial U}{\partial \rho} = \cos\phi, \quad \frac{\partial V}{\partial \phi} = \rho\cos\phi$$

$$\frac{\partial U}{\partial \rho} = \frac{1}{\rho}\frac{\partial V}{\partial \phi} = \frac{1}{\rho}\rho\cos\phi = \cos\phi$$

$$\therefore \frac{\partial U}{\partial \rho} = \frac{1}{\rho}\frac{\partial V}{\partial \phi}$$

$$\frac{\partial V}{\partial \rho} = \sin\phi, \quad \frac{\partial u}{\partial \phi} = -\rho\sin\phi$$

$$\frac{\partial V}{\partial \rho} = -\frac{1}{\rho}(-\rho\sin\phi) = \sin\phi$$

$$\therefore \frac{\partial V}{\partial \rho} = -\frac{1}{\rho}\frac{\partial u}{\partial \phi}$$

7. W

Prove that there is no analytic function whose imaginary Part is $x^2 - 2y$

$V = x^2 - 2y$

Because there is no analytic function, this mean Cauchy Rieman equation are not Satisfied.

to find U

$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \rightarrow \frac{\partial U}{\partial x} = -2 \rightarrow U_1 = -2x + f(y)$

$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \rightarrow \frac{\partial U}{\partial y} = -2x \rightarrow U_2 = -2xy + f(x)$

$U = U_1 + U_2 = -2x - 2xy + f(y) + f(x)$

$U = -2x - 2xy + C$

$\frac{\partial U}{\partial x} = -2 - 2y \quad , \quad \frac{\partial V}{\partial y} = -2$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore \frac{\partial U}{\partial x} \neq \frac{\partial V}{\partial y}$

$\therefore$ Cauchy Riemann are not Satisfy

$V = x^2 - 2y \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\frac{\partial v}{\partial x} = 2x = -\frac{\partial u}{\partial y} \rightarrow \frac{\partial u}{\partial y} = -2x \rightarrow \boxed{u = -2xy + f(x)}$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial x} = -2y + f'(x) = -2 \quad y: f'(x) = 0$

$\therefore f(x) = c$

$u = -2xy \quad \therefore z = -2xy + i(x^2 - 2)$

$\frac{\partial u}{\partial x} = -2y \quad , \frac{\partial v}{\partial y} = -2$

Z.W

analytic function

$$2u\left(\cancel{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}}\right)+2v\left(\cancel{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}}\right)+2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial y}\right)^2+2$$

By Cauchy Riemann Equation

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

صارت

$$=2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2+2\left(-\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial x}\right)^2$$

$$=4\left(\frac{\partial u}{\partial x}\right)^2+4\left(\frac{\partial v}{\partial x}\right)^2=4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]=4|f'(z$$

B-

$$\boxed{\left(\frac{\partial}{\partial x}|f(z)|\right)^2}+\left(\frac{\partial}{\partial y}|f(z)|\right)^2=|f'(z)|^2$$

$(ABS)^2$

A^2B^2

$f(z)=u+iv \longrightarrow |f(z)|=\sqrt{u^2+v^2}$

لايجوز تربيع المشتقة

$\frac{\partial^2}{\partial x^2}|f(z)|^2$ X

$$\left(\frac{\partial}{\partial x}|f(z)|^2\right)^2=\left(\frac{\partial}{\partial x}\sqrt{u^2+v^2}\right)^2=\frac{\partial^2}{\partial x^2}(u^2+v^2)$$

$$=\frac{\partial^2}{\partial x^2}u^2+\frac{\partial^2}{\partial x^2}v^2$$

لهذا راح نستخدم قاعدة السلسلة محلول في الملزمة الوصفية.

$$=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u^2\right)+\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}v^2\right)$$

$$=\frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right)+$$

Z.W

$w = f(z) = u(x,y) + iv(x,y)$

فاكر $\Rightarrow |a+ib| = \sqrt{a^2+b^2}$

if $f(z)$ is an $\boxed{\text{analytic function}}$; show that

A- $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

$f(z) = u + iv \rightarrow |f(z)|^2 = \boxed{u^2 + v^2}$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = \frac{\partial^2}{\partial x^2}(u^2+v^2) + \frac{\partial^2}{\partial y^2}(u^2+v^2)$$

$$\frac{\partial^2}{\partial x^2}(u^2+v^2) = \frac{\partial^2}{\partial x^2}u^2 + \frac{\partial^2}{\partial x^2}v^2 = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u^2\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}v^2\right)$$

$$= \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(2v\frac{\partial v}{\partial x}\right)$$

لاحظ شوية المشتقة الحاصلة ضرب

$$= 2u\frac{\partial^2 u}{\partial x^2} + 2\boxed{\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}} + 2v\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}$$

$$= 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2v\frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\right)^2$$

$$\frac{\partial^2}{\partial y^2}(u^2+v^2) = \frac{\partial^2}{\partial y^2}u^2 + \frac{\partial^2}{\partial y^2}v^2 = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}u^2\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}v^2\right)$$

$$= \frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(2v\frac{\partial v}{\partial y}\right)$$

$$= 2u\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y}\frac{\partial u}{\partial y} + 2v\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}$$

$$= 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2v\frac{\partial^2 v}{\partial y^2} + 2\left(\frac{\partial v}{\partial y}\right)^2$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2v\frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$$
$$+ 2v\frac{\partial^2 v}{\partial y^2} + 2\left(\frac{\partial v}{\partial y}\right)^2$$

7..W

معناها كل, ثابتة

Determine Whether $|Z|^2$ has aderivative $\boxed{\text{any Where}}$

By Using definition

$$f'(z) = \lim_{\Delta Z \to 0} \frac{f(Z+\Delta Z) - f(z)}{\Delta Z}$$

$$f'(z) = \lim_{\Delta Z \to 0} \frac{\boxed{|Z+\Delta Z|^2} - |Z|^2}{\Delta Z}$$

$$f'(z) = \lim_{\Delta Z \to 0} \frac{\cancel{|Z|^2} + 2|Z||\Delta Z| + |\Delta Z|^2 - \cancel{|Z|^2}}{\Delta Z}$$

$$f'(z) = \lim_{\Delta Z \to 0} \frac{2|Z||\Delta Z| + |\Delta Z|^2}{\Delta Z}$$

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{2|x+iy||\Delta x+i\Delta y| + |\Delta x+i\Delta y|^2}{\Delta x + i\Delta y}$$

if $\Delta y = 0$

$$f'(z) = \lim_{\Delta x \to 0} \frac{2|x+iy||\Delta x| + |\Delta x|^2}{\Delta x}$$

$$f'(z) = \lim_{\Delta x \to 0} 2|x+iy| + |\Delta x| = 2|x+iy| = 2Z$$

if $\Delta x = 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{2|x+iy||i\Delta y| + |i\Delta y|^2}{i\Delta y}$$

$$f'(z) = \lim_{\Delta y \to 0} 2|x+iy| + |i\Delta y| = 2|x+iy| = 2Z$$

$\therefore$ has derivative any Where.

Z.W

[illegible]

Prove that $\frac{d}{dz}(Z^2\bar{Z})$ does not exist $\boxed{\text{any Where}}$

By Using definition $\qquad \bar{Z} \Rightarrow$ Complex Conjugate

$$f'(z) = \lim_{\Delta Z \to 0} \frac{f(Z + \Delta Z) - f(Z)}{\Delta Z}$$

$$f'(z) = \lim_{\Delta Z \to 0} \frac{(Z^2\bar{Z} + \Delta Z^2 \Delta \bar{Z}) - Z^2\bar{Z}}{\Delta Z}$$

$$f'(z) = \lim_{\Delta Z \to 0} \frac{\cancel{Z^2\bar{Z}} + \Delta Z^2 \Delta \bar{Z} - \cancel{Z^2\bar{Z}}}{\Delta Z}$$

$$f'(z) = \lim_{\Delta Z \to 0} \frac{\Delta Z^2 \, \Delta \bar{Z}}{\Delta Z} = \lim_{\Delta Z \to 0} \Delta Z \, \Delta \bar{Z}$$

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Delta x + i\Delta y)(\Delta x - i\Delta y)$$

if $\Delta y = 0$

$$f'(z) = \lim_{\Delta x \to 0} (\Delta x)(\Delta x) = \lim_{\Delta x \to 0} (\Delta x)^2 = 0$$

if $\Delta x = 0$

[illegible]

$$f'(z) = \lim_{\Delta y \to 0} (i\Delta y)(i\Delta y) = \lim_{\Delta y \to 0} (\Delta y)^2 = 0$$

the derivative does not exist any Where

the function non-analytic.

Z.W

(b) $\because \phi + i\psi$ is analytic

$\therefore$ Satisfy Cauchy Riemann Eq

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \phi}{\partial x} = \frac{2(y-2)}{[(x-1)^2+(y-2)^2]} = \frac{2\,\cancel{(y-2)}}{(y-2)^2\left[(x-1)^2/(y-2)^2+1\right]}$$

$u = \frac{x-1}{y-2}$

$\frac{\partial u}{\partial x} = \frac{1}{y-2}$

$$\phi = 2\tan^{-1}((x-1)/(y-2)) + f(y)$$

$$\frac{\partial \phi}{\partial y} = \frac{2(x-1)}{(y-2)^2\left[\frac{(x-1)^2}{(y-2)^2}+1\right]} + f'(y) = \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \frac{2(x-1)}{(y-2)^2\left[\frac{(x-1)^2}{(y-2)^2}+1\right]} + f'(y) = \frac{2(x-1)}{[(x-1)^2+(y-2)^2]}$$

$\Rightarrow f'(y) = 0$

$\therefore f(y) = C$

$\therefore \phi = 2\tan^{-1}(x-1/y-2) + C$

(c) $f(z) = \phi + i\psi$

$= 2\tan^{-1}(x-1/y-2) + C + i\ln((x-1)^2+(y-2)^2)$

Putting $y=0$, $x=z$

$\Rightarrow f(z) = 2\tan^{-1}(1-z/2) + i\ln((z-1)^2+4)$

Z.W

(a) Prove that $\psi = \ln[(x-1)^2 + (y-2)^2]$ is harmonic in every region which does not include the point (1,2). (b) Find a function ϕ such that $\phi + i\psi$ is analytic. (c) Express $\phi + i\psi$ as a function of z.

(a) $\psi = \ln[(x-1)^2 + (y-2)^2]$

$$\frac{\partial \psi}{\partial x} = \frac{2(x-1)}{[(x-1)^2 + (y-2)^2]} , \quad \frac{\partial \psi}{\partial y} = \frac{2(y-2)}{[(x-1)^2 + (y-2)^2]}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{[(x-1)^2 + (y-2)^2](2) - 2(x-1)\,2(x-1)}{[(x-1)^2 + (y-2)^2]^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2(x-1)^2 + 2(y-2)^2 - 4(x-1)^2}{[(x-1)^2 + (y-2)^2]^2} = \frac{-2(x-1)^2 + 2(y-2)^2}{[(x-1)^2 + (y-2)^2]^2}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{[(x-1)^2 + (y-2)^2](2) - 2(y-2)\,2(y-2)}{[(x-1)^2 + (y-2)^2]^2}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{2(x-1)^2 + \boxed{2(y-2)^2 - 4(y-2)^2}}{[(x-1)^2 + (y-2)^2]^2} = \frac{2(x-1)^2 - 2(y-2)^2}{[(x-1)^2 + (y-2)^2]^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{-2(x-1)^2 + 2(y-2)^2 + 2(x-1)^2 - 2(y-2)^2}{[(x-1)^2 + (y-2)^2]^2} = 0$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$\therefore$ ψ Satisfy Laplace eq $\rightarrow$ $\therefore$ ψ is harmonic.

Z. W

(b)

Note: $\cos\theta i = \cosh\theta$

$\sin\theta i = i\sinh\theta$

$$\cos 2z = \cos 2(x+iy) = \cos(2x+2yi)$$
$$= \cos 2x \cos 2yi - \sin 2x \sin 2yi$$
$$= \cos 2x \cosh 2y - i\sin 2x \sinh 2y$$

$$u = \cos 2x \cosh 2y \quad , \quad v = -\sin 2x \sinh 2y$$

$$\frac{\partial u}{\partial x} = -2\sin 2x \cosh 2y \quad , \quad \frac{\partial v}{\partial x} = -2\cos 2x \sinh 2y$$

$$\frac{\partial u}{\partial y} = 2\cos 2x \sinh 2y \quad , \quad \frac{\partial v}{\partial y} = -2\sin 2x \cosh 2y$$

$$\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ Cauchy Riemann Eq. Satisfy.

(c) $$\sinh 4z = \sinh 4(x+iy) = \sinh(4x+4yi)$$
$$= \sinh 4x \cosh 4yi + \cosh 4x \sinh 4yi$$
$$= \sinh 4x \cos 4y + i\cosh 4x \sin 4y$$

$$u = \sinh 4x \cos 4y \quad , \quad v = \cosh 4x \sin 4y$$

$$\frac{\partial u}{\partial x} = 4\cosh 4x \cos 4y \quad , \quad \frac{\partial v}{\partial x} = 4\sinh 4x \sin 4y$$

$$\frac{\partial u}{\partial y} = -4\sinh 4x \sin 4y \quad , \quad \frac{\partial v}{\partial y} = 4\cosh 4x \cos 4y$$

$$\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ Cauchy Riemann Equation Satisfy.

sin = cos

sinh = cosh

sin sinh

sin

i sinh

cos

cosh

sinh

i sin

Mujtaba

1/2/2016

14/8/2016

Z.W

Verify that the Cauchy-Riemann equations are Satisfy for the functions

$$(a)\ e^{Z^2},\quad (b)\ \cos 2Z \quad (c)\ \sinh 4Z.$$

(a)

$$e^{Z^2},\quad Z^2 = (X+iy)^2 = x^2+2xyi-y^2$$

$$\omega = e^{x^2+2xyi-y^2} = e^{x^2+2xyi}/e^{y^2} = e^{x^2}\,e^{2xyi}/e^{y^2}$$

$e^{i\theta}$

$$\omega = e^{x^2}(\cos 2xy + i\sin 2xy)/e^{y^2}$$

$$\omega = e^{x^2}\cos 2xy/e^{y^2} + i\,e^{x^2}\sin 2xy/e^{y^2}$$

$$u = \frac{e^{x^2}\cos 2xy}{e^{y^2}},\quad v = \frac{e^{x^2}\sin 2xy}{e^{y^2}}$$

$$\frac{\partial u}{\partial x} = \frac{1}{e^{y^2}}\left[-e^{x^2}\sin 2xy\,(2y) + \cos 2xy\, e^{x^2}(2x)\right]$$

$$\frac{\partial u}{\partial x} = \frac{1}{e^{y^2}}\left[-2y\,e^{x^2}\sin 2xy + 2x\,e^{x^2}\cos 2xy\right]$$

$$\frac{\partial u}{\partial y} = e^{x^2}\left[-e^{-y^2}\sin 2xy\,(2x) - \cos 2xy\, e^{-y^2}\,2y\right]$$

$$\frac{\partial u}{\partial y} = \frac{1}{e^{y^2}}\left[-2x\,e^{x^2}\sin 2xy - 2y\,e^{x^2}\cos 2xy\right]$$

$$\frac{\partial v}{\partial x} = \frac{1}{e^{y^2}}\left[e^{x^2}\cos 2xy\,(2y) + \sin 2xy\, e^{x^2}\,2x\right]$$

$$\frac{\partial v}{\partial x} = \frac{1}{e^{y^2}}\left[2y\,e^{x^2}\cos 2xy + 2x\,e^{x^2}\sin 2xy\right]$$

$$\frac{\partial v}{\partial y} = e^{x^2}\left[e^{-y^2}\cos 2xy\,(2x) - \sin 2xy\, e^{-y^2}(2y)\right] = \frac{1}{e^{y^2}}\left[2x\,e^{x^2}\cos 2xy - 2y\,e^{x^2}\sin 2xy\right]$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

∴ Cauchy Riemann Equation Satisfy

7. W

2/

Show that the function $x^2 + iy^3$ is not analytic any where. Reconcile this with the fact that the Cauchy - Riemann equations are Satisfied at $x = 0, y = 0$

اعلان
ترافق

$u = x^2$
$v = y^3$
$\frac{\partial u}{\partial x} = 2x$
$\frac{\partial v}{\partial y} = 3y^2$

$w = f(z) = u(x,y) + iv(x,y)$

$u = x^2 \quad , \quad v = y^3$

from Cauchy Riemann Eq.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial v}{\partial y} = 3y^2 \quad , \quad \frac{\partial u}{\partial y} = 0 \quad , \quad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$\ln r e^{i\theta}$ $\ln r + \ln e^{i\theta}$

the function is not analytic any where because Cauchy Riemann Equation are not Satisfy.

Z.W

Complex.

Provethatthe function $u=2x(1-y)$ is harmonic (b) Find v such that $f(z)=u+iv$ is analytic (c) Express $f(z)$ interm

$$u=2x(1-y)=2x-2xy$$

$2(1-y)$

$2x(1)$

$$\frac{\partial u}{\partial x}=2-2y \rightarrow \frac{\partial^2 u}{\partial x^2}=0$$

$2x-2xy$

$$\frac{\partial u}{\partial y}=-2x \rightarrow \frac{\partial^2 u}{\partial y^2}=0$$

$2-2y$

$-y$

from Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

the function u Satisfy laplace eq $\rightarrow$ u is harmonic.

$\because$ $f(z)$ is analytic

$\therefore$ Satisfy Cauchy Riemann Equation.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y}=2-2y \rightarrow v=2y-y^2+f(x)$$

$$\frac{\partial v}{\partial x}=f'(x)$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \rightarrow -2x=-f'(x) \rightarrow f'(x)=2x \rightarrow f(x)=x^2+c$$

$$v=2y-y^2+x^2$$

$$w=f(z)=u(x,y)+iv(x,y)$$

$$w=2x(1-y)+i(2y-y^2+x^2)$$

putting $y=0$; $x=z$

$$w=2z+iz^2$$

Z.W

$$= \int_{-4}^{4} (4y+4)\,dy - \int_{-4}^{4} \left(\frac{y^5}{64} + \frac{y^4}{64}\right) dy$$

$$= (2y^2+4y)\Big|_{-4}^{4} - \left(\frac{y^6}{64\times6} + \frac{y^5}{64\times5}\right)\Big|_{-4}^{4}$$

$$= 32 - \left\{\left(\frac{64}{6} + \frac{16}{5}\right) - \left(\frac{64}{6} - \frac{16}{5}\right)\right\}$$

$$= 32 - \left(\frac{32}{5}\right) = 32 - \frac{32}{5} = \frac{128}{5}$$

$$u = 2x - 2xy$$

$$\frac{\partial u}{\partial x} = 2 - 2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \therefore \frac{\partial v}{\partial y} = 2 - 2y \Rightarrow v = 2y - y^2 + f(x)$$

$$\frac{\partial u}{\partial y} = -2x \Rightarrow 2x = f'(x) \quad \therefore f(x) = \frac{2}{2}x^2$$

$$\therefore v = 2y - y^2 + x^2$$

$$f(z) =$$

Z.W

Evaluate $\oint (x^2 - 2xy)\,dx + (x^2y+3)\,dy$ around the boundary of the regio[n] defined by $y^2 = 8x$ and $x=2$ (a) directly (b) By Using green theorem

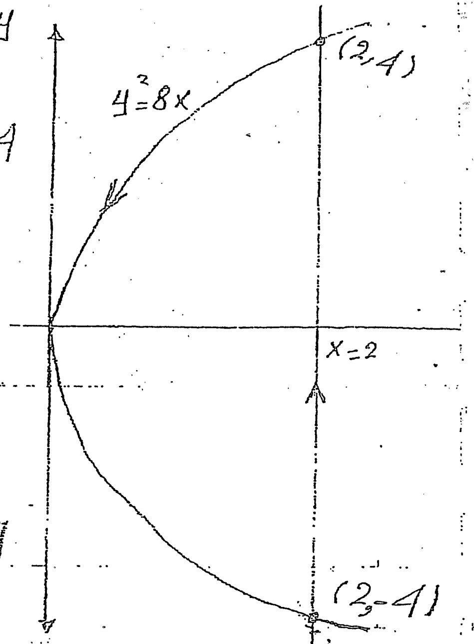
(a) direct method.

$(2,4)$

From $(2,-4)$ to $(2,4)$

$x=2$, $dx=0$ while y varies from $y=-4$ to $y=4$

$$\int_{-4}^{4} (4y+3)\,dy = 2y^2+3y\Big|_{-4}^{4} = \underline{24}$$



From $(2,4)$ to $(2,-4)$ along $y^2=8x$

$x=\frac{1}{8}y^2$, $dx = \frac{1}{4}y\,dy$

$$\int_{4}^{-4}\left[\left(\frac{1}{64}y^4-\frac{1}{4}y^3\right)\frac{1}{4}y\,dy+\left(\frac{1}{64}y^5+3\right)dy\right]$$

$$\int_{4}^{-4}\left[\frac{1}{256}y^5dy-\frac{1}{16}y^4dy+\frac{1}{64}y^5dy+3dy\right]$$

$$\int_{4}^{-4}\left[\frac{5}{256}y^5dy-\frac{1}{16}y^4dy+3dy\right]=\frac{5}{256\times6}y^6\Big|_{4}^{-4}-\frac{1}{16\times5}y^5\Big|_{4}^{-4}+3$$

$$=\frac{2048}{16\times5}-24=\frac{128}{5}-24$$

the total Integral $= 24+\frac{128}{5}-24=\frac{128}{5}$

By Using green theorem

$$\int[P\,dx+Q\,dy]=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$$=\iint_R\left(\frac{\partial}{\partial x}(x^2y+3)-\frac{\partial}{\partial y}(x^2-2xy)\right)dx\,dy=\iint_R(2xy+2x)\,d[x\,dy]$$

$$=\int_{-4}^{4}\int_{\frac{y^2}{8}}^{2}(2xy+2x)\,dx\,dy=\int_{-4}^{4}(x^2y+x^2)\Big|_{\frac{y^2}{8}}^{2}dy$$

Z.W

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

To show that the line integral is dependent on the path although $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ by solve the integral on the path (a) ~~and on the path (b)~~ where the result is different. Then the integral is depending on the path

These results because P and Q do not have continuous derivatives throughout any region including (0,0) and this was assumed in the necessary condition of "Independent path".

Z.W

$$\int_{1}^{0} \frac{-dy}{1+y^2}$$ let $y = \tan\theta \rightarrow dy = \sec^2\theta\, d\theta$

$\theta = \tan^{-1} y$

$\theta_1 = \tan^{-1}(1) = \frac{\pi}{4}$, $\theta_2 = \tan^{-1}(0) = 0$

$$-\int_{\pi/4}^{0} \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta} = -\int_{\pi/4}^{0} \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = -\int_{\pi/4}^{0} d\theta = -\theta\Big|_{\pi/4}^{0} = \frac{\pi}{4}$$

the total Integral $= \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4} = \pi$.

b- along the Straightline from $(1,0)$ to $(1,-1)$, $x=1$, $dx=0$ While y Varies from 0 to -1

$$\int_{0}^{-1} \frac{dy}{1+y^2} = \int_{0}^{-\pi/4} \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta} = \int_{0}^{-\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = \int_{0}^{-\pi/4} d\theta = \theta\Big|_{0}^{-\pi/4} = -\frac{\pi}{4}$$

from $(1,-1)$ to $(-1,-1)$, $y=-1$, $dy=0$, While x Varies from 1 to -1

$$\int_{1}^{-1} \frac{dx}{x^2+1} = \int_{\pi/4}^{-\pi/4} \frac{\sec^2\theta\, d\theta}{\tan^2\theta+1} = \int_{\pi/4}^{-\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = \int_{\pi/4}^{-\pi/4} d\theta = \theta\Big|_{\pi/4}^{-\pi/4} = -\frac{\pi}{2}$$

from $(-1,-1)$ to $(-1,0)$, $x=-1$, $dx=0$, While y Varies from -1 to 0

$$\int_{-1}^{0} \frac{-dy}{1+y^2} = \int_{-\pi/4}^{0} \frac{-\sec^2\theta\, d\theta}{1+\tan^2\theta} = \int_{-\pi/4}^{0} \frac{-\sec^2\theta\, d\theta}{\sec^2\theta} = -\int_{-\pi/4}^{0} d\theta = -\theta\Big|_{-\pi/4}^{0} = -\frac{\pi}{4}$$

the total Integral $= -\frac{\pi}{4} - \frac{\pi}{2} - \frac{\pi}{4} = -\pi$

c) $P = \frac{y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2}$

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2) - y\,(2y)}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Z.W

Evaluate $\displaystyle\int_{(1,0)}^{(-1,0)} \frac{-y\,dx + x\,dy}{x^2+y^2}$ along the following path

a_ Straight line Segment from (1,0) to (1,1) then to (-1,1) then to (-1,0)

b_ Straight line Segment from (1,0) to (1,-1) then to (-1,-1) then to (-1,0)

C_ Show that although $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the line Integral is dependent on path joining (1,0) to (-1,0) and explain.

$\frac{dy}{x^2+y^2} =$

(a) $\displaystyle\int \left[-\frac{y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} \right]$

along the Straight line from (1,0) to (1,1), $x = 1$, $dx = 0$ While y Varies from 0 to 1

let $y = \tan\theta \rightarrow dy = \sec^2\theta\, d\theta$

$\displaystyle\int_0^1 \frac{dy}{1+y^2} = \tan^{-1} y \Big|_0^1 = \frac{\pi}{4}$ $\quad \theta = \tan^{-1} y$

$\theta_1 = \tan^{-1} 0 = 0$, $\theta_2 = \tan^{-1} 1 = \frac{\pi}{4}$

$$\int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = \int_0^{\pi/4} d\theta = \theta \Big|_0^{\pi/4} = \frac{\pi}{4}$$

from (1,1) to (-1,1), $y = 1$, $dy = 0$, While x Varies from 1 to -1

$\displaystyle\int_1^{-1} -\frac{dx}{x^2+1}$

let $x = \tan\theta \rightarrow dx = \sec^2\theta\, d\theta$

$\theta = \tan^{-1} x$

$\theta_1 = \tan^{-1}(1) = \frac{\pi}{4}$, $\theta_2 = \tan^{-1}(-1) = -\frac{\pi}{4}$

$$-\int_{\pi/4}^{-\pi/4} \frac{\sec^2\theta\, d\theta}{\tan^2\theta+1} = -\int_{\pi/4}^{-\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= -\int_{\pi/4}^{-\pi/4} d\theta = -\theta \Big|_{\pi/4}^{-\pi/4} = \frac{\pi}{2}$$

from (-1,1) to (-1,0), $x = -1$, $dx = 0$, While y Varies from 1 to 0

Z.W

1- Independent of the path
2- An exact differential
3- Conservative

$\Big\} \rightarrow \nabla \times F = 0$

Show that $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\,dz$ is an exact differential. Hence solve the differential equation

$$\nabla \times F = \begin{vmatrix} \vec{i} & & \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (2x\cos y + z\sin y) & (xz\cos y - x^2\sin y) & x\sin y \end{vmatrix}$$

$$\nabla \times F = \left[\frac{\partial}{\partial y} x\sin y - \frac{\partial}{\partial z}(xz\cos y - x^2\sin y)\right]\vec{i} + \left[\frac{\partial}{\partial z}(2x\cos y + z\sin y) - \frac{\partial}{\partial x}\right.$$

$$+ \left[\frac{\partial}{\partial x}(xz\cos y - x^2\sin y) - \frac{\partial}{\partial y}(2x\cos y + z\sin y)\right]\vec{k}$$

$\nabla \times F = 0$ $\quad \therefore$ exact D.Eq.

$$F.dr = \nabla\phi . dr = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$\phi = (2x\cos y\,dx - x^2\sin y\,dy) + (z\sin y\,dx + xz\cos y\,dy + x\sin y\,dz)$$

$$\phi = x^2\cos y + xz\sin y + \text{Constant.}$$

$\int F \cdot dr$

if a particle is attracted toward the origin by a force prop... nth power of the distance from the origin, Show that the work done ... this force in moving the particle from the point (x_0, y_0) to the point (... independent of the path along which the particle is moved. What is the ... of work done?

if $F = \nabla \phi$ where ϕ is Single Valued and has Continuous Partial derivative

$$\text{Work done} = \int_{P_0}^{P_1} F \cdot dr = \int_{P_0}^{P_1} \nabla \phi \cdot dr$$

$$= \int_{P_0}^{P_1} \left(\frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k} \right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_{P_0}^{P_1} \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) = \int_{P_0}^{P_1} d\phi = \phi \Big|_{P_0}^{P_1}$$

$$= \phi(P_1) - \phi(P_0)$$

$$\text{Work done} = \int F \cdot dr = \int_{P_0}^{P_1} k r^n dr = k \frac{r^{n+1}}{n+1} \Big|_{P_0}^{P_1}$$

$$= \frac{k}{n+1} \left\{ (x_1^2 + y_1^2)^{(n+1)/2} - (x_0^2 + y_0^2)^{(n+1)/2} \right\}$$

$F \propto r^n$

$F = k r^n$

Z.W

(a) Is there a function ϕ such that $E = \nabla\phi$? If so find it. (b) Evaluate $\oint E \cdot dr$ around C, C is any simple closed curve.

$$E = -\nabla\phi$$

$$rx\vec{i} + ry\vec{j} + rz\vec{k} = -\frac{\partial\phi}{\partial x}\vec{i} - \frac{\partial\phi}{\partial y}\vec{j} - \frac{\partial\phi}{\partial z}\vec{k}$$

$$\frac{\partial\phi}{\partial x} = -rx = -(x^2+y^2+z^2)^{1/2}x \rightarrow \phi = -\frac{1}{2}\frac{(x^2+y^2+z^2)^{3/2}}{3/2} + C$$

$$\phi = -\frac{1}{3}r^3 + C$$

$$\frac{\partial\phi}{\partial y} = -ry = -(x^2+y^2+z^2)^{1/2}y = -\frac{1}{2}\frac{(x^2+y^2+z^2)^{3/2}}{3/2} + C$$

$$\phi = -\frac{1}{3}r^3 + C$$

$$\frac{\partial\phi}{\partial z} = -rz = -(x^2+y^2+z^2)^{1/2}z = -\frac{1}{2}\frac{(x^2+y^2+z^2)^{3/2}}{3/2} + C$$

$$\phi = -\frac{1}{3}r^3 + C$$

$$\therefore \quad \phi = -\frac{r^3}{3} + C$$

(b)

$$\int E \cdot dr = \int \nabla\phi \cdot dr = \int\left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right)\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = \int d\phi = \phi$$

a)

$$\nabla \times E = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xr & yr & zr \end{vmatrix}$$

$$\nabla \times E = \left(\frac{\partial}{\partial y}zr - \frac{\partial}{\partial z}yr\right)\vec{i} + \left(\frac{\partial}{\partial z}xr - \frac{\partial}{\partial x}zr\right)\vec{j} + \left(\frac{\partial}{\partial x}yr - \frac{\partial}{\partial y}xr\right)$$

$\nabla \times E = 0$

Z.W $\quad$ $\therefore$ there is a function ϕ.

Verify Green's lemma for the Integral $\int (x^2+y)\,dx - xy^2\,dy$ taken a[round] the boundary of the Square whose Vertices are $(0,0), (1,0), (1,1)$ ar[e]

direct method.

along the Straightline from $(0,0)$ to $(1,0)$, $y=0$, $dy=0$, While X Varies from

$$\int_0^1 \left[(x^2+0)\,dx - x(0)^2(0)\right] = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

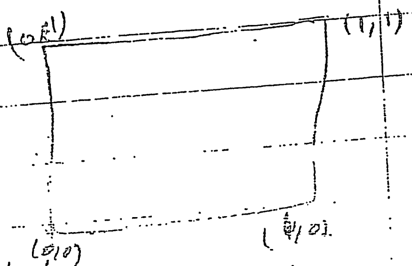
along the Straightline from $(1,0)$ to $(1,1)$, $X=1$, $dX=0$ While y Varies from 0

$$\int_0^1 \left[(1+y)(0) - (1)y^2\,dy\right] = \int_0^1 -y^2\,dy = -\frac{1}{3}y^3\Big|_0^1 = -\frac{1}{3}$$

along the Straightline from $(1,1)$ to $(0,1)$, $y=1$, $dy=0$ While X Varies from 0 to 1

$$\int_1^0 \left[(x^2+1)\,dx - x^2(1)^2(0)\right] = \int_1^0 (x^2+1)\,dx = \frac{1}{3}x^3 + x\Big|_1^0 = -\left(\frac{1}{3}+1\right) = -\frac{4}{3}$$

along the Straight line from $(0,1)$ to $(0,0)$, $X=0$, $dX=0$ While y Varies fro[m]



$$\int_1^0 \left[(0^2+y)(0) - (0)y^2\,dy\right] = 0$$

$\therefore$ the total Integral $= -\frac{4}{3}$

Green theorem

$$\int \left[P\,dx + Q\,dy\right] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2+y)\right) dx\,dy = \iint_R (-y^2-1)\,dx\,dy$$

$$= \int_0^1\int_0^1 (-y^2-1)\,dx\,dy = \int_0^1 (-y^2x - x)\Big|_0^1 dy = \int_0^1 (-y^2-1)\,dy$$

$$= -\frac{1}{3}y^3 - y\Big|_0^1 = -\frac{1}{3} - 1 = -\frac{4}{3}$$

$\therefore$ Green theorem Verif[ied]

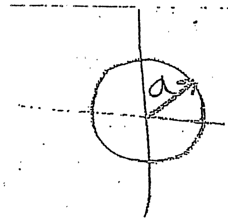
Z.W

Verify Green lemma for the Integral $\int (x-2y)dx + x\,dy$ taken around the Circle $x^2+y^2=a^2$

direct method.

$r=a$
$x = a\cos\theta$
$y = a\sin\theta$

$$\int (x-2y)dx + x\,dy$$



$$x = r\cos\theta \rightarrow dx = -r\sin\theta\,d\theta$$
$$y = r\sin\theta \rightarrow dy = r\cos\theta\,d\theta$$

$$\int_0^{2\pi} (a\cos\theta - 2a\sin\theta)(-a\sin\theta\,d\theta) + \int_0^{2\pi} a\cos\theta(a\cos\theta\,d\theta)$$

$$\int_0^{2\pi} -a^2\sin\theta\cos\theta\,d\theta + \int_0^{2\pi} 2a^2\sin^2\theta\,d\theta + \int_0^{2\pi} a^2\cos^2\theta\,d\theta$$

$$\int_0^{2\pi} -a^2\sin\theta\cos\theta\,d\theta + \int_0^{2\pi} a^2(1-\cos 2\theta)\,d\theta + \int_0^{2\pi} a^2\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$a^2\frac{\cos^3\theta}{3}\Big|_0^{2\pi} + a^2\theta\Big|_0^{2\pi} - \frac{a^2}{2}\sin 2\theta\Big|_0^{2\pi} + \frac{1}{2}a^2\theta\Big|_0^{2\pi} + \frac{a^2}{2}\sin 2\theta\Big|_0^{2\pi}$$

$$0 + 2\pi a^2 - 0 + \pi a^2 + 0 = 3\pi a^2$$

By Using Green theorem

$$\int [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad 1 + \frac{\partial}{\partial y} \quad 3\int_0\int_0^a 3\,r\,dr\,d\theta$$

$$= \iint_R \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(x-2y)\right) dx\,dy = \iint_R (1+2)\,dx\,dy$$

$$= 3\int_0^{\pi/2}\int_0^{\sqrt{a^2-y^2}} dx\,dy = 3\int_0^{\pi/2} x\Big|_0^{\sqrt{a^2-y^2}} dy = 3\int_0^{\pi/2}\sqrt{a^2-y^2}\,dy$$

$$= 3\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta\,d\theta) = 3a^2\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= 3a^2\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 3a^2\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} = \frac{3a^2\pi}{4}\times 4$$

$$= 3a^2\pi$$

Margin working:

$$= 3\int_0^{\frac{\pi}{2}}\int_0^a r\,dr\,d\theta$$
$$= 12\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_0^a d\theta$$
$$= 12\int_0^{\frac{\pi}{2}}\frac{a^2}{2}\,d\theta$$
$$= 6a^2\frac{\pi}{2}$$
$$= 3a^2\pi$$

Z.W

Along What Curve of the family $y = Kx(1-x)$ does the Integral

$$\int_{0,0}^{1,0} y(x-y)\,dx$$ attain its largest Value?

(0,0) (1,0)

$$\int_0^1 Kx(1-x)(x-Kx+Kx^2)\,dx = \int_0^1 (Kx-Kx^2)(x-Kx+Kx^2)\,dx$$

$$= \int_0^1 (Kx^2 - K^2x^2 + K^2x^3 - Kx^3 + K^2x^3 - K^2x^4)\,dx$$

$$= \frac{1}{3}Kx^3\Big|_0^1 - \frac{1}{3}K^2x^3\Big|_0^1 + \frac{1}{4}K^2x^4 - \frac{1}{4}Kx^4\Big|_0^1 + \frac{1}{4}K^2x^4\Big|_0^1 - \frac{1}{5}K^2x^5\Big|_0^1$$

$$= \frac{1}{3}K - \frac{1}{3}K^2 + \frac{1}{4}K^2 - \frac{1}{4}K + \frac{1}{4}K^2 - \frac{1}{5}K^2$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right)K + \left(-\frac{1}{3} + \frac{1}{4} + \frac{1}{4} - \frac{1}{5}\right)K^2$$

$$K = \frac{1}{12}K - \frac{1}{30}K^2$$

$$dK = \frac{1}{12} - \frac{1}{15}K \qquad dK = 0$$

$$\frac{1}{12} - \frac{1}{15}K = 0 \rightarrow \frac{1}{15}K = \frac{1}{12} \rightarrow K = \frac{15}{12} \rightarrow K = \frac{5}{4}$$

اعظم قيمة عندما المشتقة تساوي صفر

هو معطى في السؤال التكامل نحلها اولا ثم نشتقها

ونجعلها تساوي الصفر نعوض عن K

Z.W

Evaluate $\int_c [(2xy^3 - y^2\cos x)dx + (1 - 2y\sin x + 3x^2y^2)dy]$ along the parabola $2x = \pi y^2$ from $(0,0)$ to $(\pi/2, 1)$.

$2x = \pi y^2 \rightarrow x = \frac{\pi}{2}y^2 \rightarrow dx = \pi y\,dy$

$\int_0^1 [(2x\frac{\pi}{2}y^2y^3 - y^2\cos\frac{\pi}{2}y^2)\pi y\,dy + (1 - 2y\sin\frac{\pi}{2}y^2 + 3\frac{\pi^2}{4}y^4y^2)dy]$

$\int_0^1 [(\pi y^5 - y^2\cos\frac{\pi}{2}y^2)\pi y\,dy + (1 - 2y\sin\frac{\pi}{2}y^2 + \frac{3\pi^2}{4}y^6)dy]$

$\int_0^1 [\pi^2y^6dy - \pi y^3\cos\frac{\pi}{2}y^2dy + dy - 2y\sin\frac{\pi}{2}y^2dy + \frac{3}{4}\pi^2y^6dy]$

$\int_0^1 [\frac{7}{4}\pi^2y^6dy - \pi y^3\cos\frac{\pi}{2}y^2dy + dy - 2y\sin\frac{\pi}{2}y^2dy]$

$\int_0^1 \frac{7}{4}\pi^2y^6dy - \int_0^1 \pi y^3\cos\frac{\pi}{2}y^2dy + \int_0^1 dy - \int_0^1 2y\sin\frac{\pi}{2}y^2dy$

$\frac{\pi^2}{4}y^7\Big|_0^1 - \int_0^1 \pi y^3\cos\frac{\pi}{2}y^2dy + y\Big|_0^1 + \frac{2}{\pi}\cos\frac{\pi}{2}y^2\Big|_0^1$

$Udv = UV - \int V\,du$

$\int \pi y^3\cos\frac{\pi}{2}y^2dy$ let $U = y^2 \rightarrow du = 2y\,dy$

$y^2 \quad \pi y$

$dv = \pi y\cos\frac{\pi}{2}y^2dy \rightarrow V = \sin\frac{\pi}{2}y^2$

$\frac{\pi}{2}\cdot 2y$

$Udv = y^2\sin\frac{\pi}{2}y^2 - \int 2y\sin\frac{\pi}{2}y^2dy = y^2\sin\frac{\pi}{2}y^2 + \frac{2}{\pi}\cos\frac{\pi}{2}y^2$

$\frac{\pi^2}{4}y^7\Big|_0^1 - y^2\sin\frac{\pi}{2}y^2\Big|_0^1 - \frac{2}{\pi}\cos\frac{\pi}{2}y^2\Big|_0^1 + y\Big|_0^1 + \frac{2}{\pi}\cos\frac{\pi}{2}y^2\Big|_0^1$

$\frac{\pi^2}{4}y^7\Big|_0^1 - y^2\sin\frac{\pi}{2}y^2\Big|_0^1 + y\Big|_0^1$

$\frac{\pi^2}{4} - 1 + 1 = \frac{\pi^2}{4}$

$\int u\,dv = uv - \int v\,du$

H.W

$\int x^2\cos x \cdot dx$

Let $u = x^2$ → مقابل الاشتقاق $du = 2x \cdot dx$

$dv = \cos x$ → مقابل التكامل $v = \sin x$

$$\left[y^5 - 6y^4 + 9y^3 + y - y^2\sin(y+3)\right]_0^2 = 201.65$$

- along the Straight line from (5, 2) to (2, 2), $y = 2$, $dy = 0$, While x varies from 2

$$\int_5^2 \left[(2x(2)^3 - (2)^2\cos x)dx + (1 - 2(2)\sin x + 3x^2(2)^2)(0)\right]$$

$$\int_5^2 [16x - 4\cos x]\,dx = 8x^2\Big|_5^2 - 4\sin x\Big|_5^2 = -167.79$$

$\frac{16x^2}{2} - 4\sin x$

$8x^2 - 4\sin x$

$(8\times 4 - 4\sin 2)$

$-(8\times 25 - 4\sin 5)$

31.86

$= 167.79$

along the Straight line from (2, 2) to (0,0)

$$\frac{x-2}{-2} = \frac{y-2}{-2} \longrightarrow x = y \longrightarrow dx = dy$$

$$\int_2^0 \left[(2y^4 - y^2\cos y)dy + (1 - 2y\sin y + 3y^4)dy\right]$$

$$\int_2^0 \left[2y^4 dy - y^2\cos y\,dy + dy - 2y\sin y\,dy + 3y^4 dy\right]$$

$$\int_2^0 \left[5y^4 dy + dy - y^2\cos y\,dy - 2y\sin y\,dy\right]$$

$Udv = UV - \int v\,du$

$\int y^2\cos y\,dy \longrightarrow$ let $U = y^2 \longrightarrow du = 2y\,dy$

$dv = \cos y\,dy \longrightarrow V = \sin y$

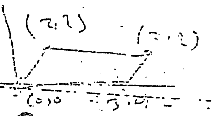
$Udv = y^2\sin y - \int 2y\sin y\,dy$

$$\int_2^0 5y^4 dy + \int_2^0 dy - y^2\sin y\Big|_2^0 + \int_2^0 2y\sin y\,dy - \int_2^0 2y\sin y\,dy$$

$$y^5\Big|_2^0 + y\Big|_2^0 - y^2\sin y = -33.86$$

$\therefore$ total Integral $= 0 + 201.65 - 167.79 - 33.86 = 0$

Z.W



Evaluate the line Integral $\int_c [(2xy^3 - y^2\cos x)dx + (1-2y\sin x + 3x^2y^2)dy]$ around parallelgram with Vertices at $(0,0), (3,0), (5,2), (2,2)$

along the Straight line from $(0,0)$ to $(3,0)$, $y=0, dy=0$, While x Varies from 0 to 3

$$\int_0^3 [(2x(0)^3 - (0)^2\cos x)dx + (1-2(0)\sin x + 3x^2(0)^2)(0)] = 0$$

along the Straight line from $(3,0)$ to $(5,2)$

$$\frac{x-3}{2} = \frac{y}{2} \rightarrow 2y = 2(x-3) \rightarrow y = x-3 \rightarrow x = y+3 \rightarrow dx = dy$$

$$\int_0^2 [((2(y+3)y^3 - y^2\cos(y+3))dy + (1-2y\sin(y+3) + 3(y+3)^2y^2)dy]$$

$$\int_0^2 [(2y^4+6y^3-y^2\cos(y+3))dy + (1-2y\sin(y+3)+3y^2(y^2+6y+9))dy]$$

$$\int_0^2 [2y^4dy + 6y^3dy - y^2\cos(y+3)dy + dy - 2y\sin(y+3)dy + 3y^4dy + 18y^3dy + 27y^2dy]$$

$$\int_0^2 [5y^4dy + 24y^3dy + 27y^2dy + dy - y^2\cos(y+3)dy - 2y\sin(y+3)dy]$$

$$\int_0^2 5y^4dy + \int_0^2 24y^3dy + \int_0^2 27y^2dy + \int_0^2 dy - \int_0^2 y^2\cos(y+3)dy - \int_0^2 2y\sin(y+3)\,dy$$

$$y^5\Big|_0^2 + 6y^4\Big|_0^2 + 9y^3\Big|_0^2 + y\Big|_0^2 - \int_0^2 y^2\cos(y+3)dy - \int_0^2 2y\sin(y+3)dy$$

$$U\,dV = U\,V - \int V\,du$$

$$\int y^2\cos(y+3)dy \quad \text{let } U = y^2 \rightarrow du = 2y\,dy$$

$$dv = \cos(y+3)dy \rightarrow V = \sin(y+3)$$

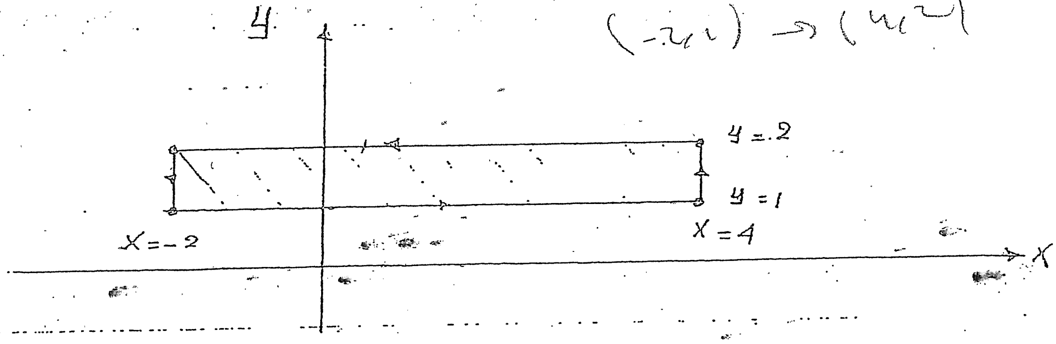
$$U\,dV = y^2\sin(y+3) - \int_0^2 2y\sin(y+3)dy$$

$$y^5\Big|_0^2 + 6y^4\Big|_0^2 + 9y^3\Big|_0^2 + y\Big|_0^2 \quad y^2\sin(y+3) + \int_0^2 2y\sin(y+3)dy - \int_0^2 2y\sin(y+3)dy$$

Z.W

Evaluate the line Integral Using Green theorem and Check the answer by evaluate it directly $\oint_C 3xy\,dx + 2xy\,dy$ Where C is the rectangular bounded by $X=-2$, $X=4$, $y=1$ and $y=2$

$(-2,1) \to (4,2)$



By Using Green theorem

$$\int_C [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial}{\partial x} 2xy - \frac{\partial}{\partial y} 3xy\right) dx\,dy$$

$$= \iint_R (2y - 3x)\,dx\,dy = \int_{-2}^{4}\int_{1}^{2} (2y - 3x)\,dy\,dx$$

$$= \int_{-2}^{4} \left[y^2 - 3yx\right]_1^2 dx = \int_{-2}^{4} (4 - 6x - 1 + 3x)\,dx$$

$$= \int_{-2}^{4} (3 - 3x)\,dx = 3x - \frac{3}{2}x^2 \Big|_{-2}^{4}$$

$$= (12 - 24) - (-6 - 6) = 0$$

By direct method

along the line from $(-2,1)$ to $(4,1)$, $y=1$, $dy=0$

Showthat

$$\oint_C 4x^3y\,dx + x^4dy = 0$$

for any Closed Curve C to Which Green theorem applies.

By Using Green Theorem

$$\int [P dx + Q dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$P = 4x^3y \quad , \quad Q = x^4$

$\frac{\partial P}{\partial y} = 4x^3 \quad , \quad \frac{\partial Q}{\partial x} = 4x^3$

$$\int [P dx + Q dy] = \iint_R (4x^3 - 4x^3) = 0$$

$Pdx \quad Qdy$

$\frac{\partial P}{\partial y} = 0$

$4x^3 \quad 4x^3$

Z.W

For $F = \frac{1}{x^2+y^2}(-y, x)$ and C any Circle of radius $r > 0$ not Contai[ning]

the origin, Show that $\oint_c F \cdot dr = 0$

$$F = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$

$$\int F \cdot dr = \int \frac{-y\,dx + x\,dy}{x^2+y^2}$$

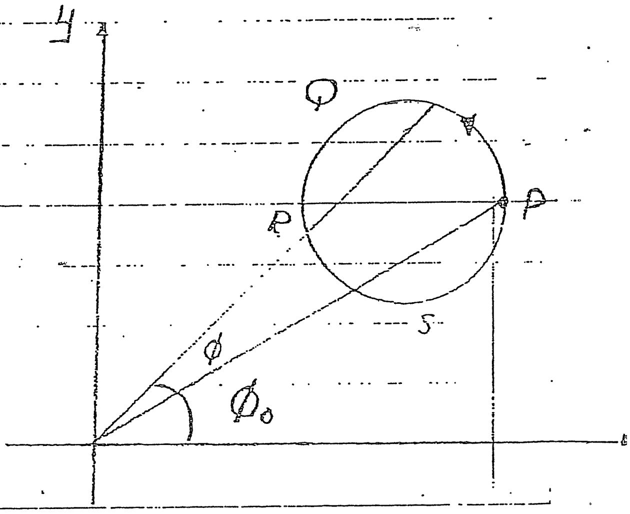
let $x = r\cos\theta \rightarrow dx = -r\sin\theta\, d\theta$

$y = r\sin\theta \rightarrow dy = r\cos\theta\, d\theta$

$$\int F \cdot dr = \int \frac{-r\sin\theta(-r\sin\theta\, d\theta) + r\cos\theta(r\cos\theta\, d\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$\int F \cdot dr = \int \frac{\cancel{r^2}(\sin^2\theta + \cos^2\theta)\,d\theta}{\cancel{r^2}(\cos^2\theta + \sin^2\theta)} = \int d\theta$$

for Closed Curve PQRSP not Surrounding origin $\phi = \phi_0$ at P and $\phi = \phi_0$ after Complete Circuit back to P

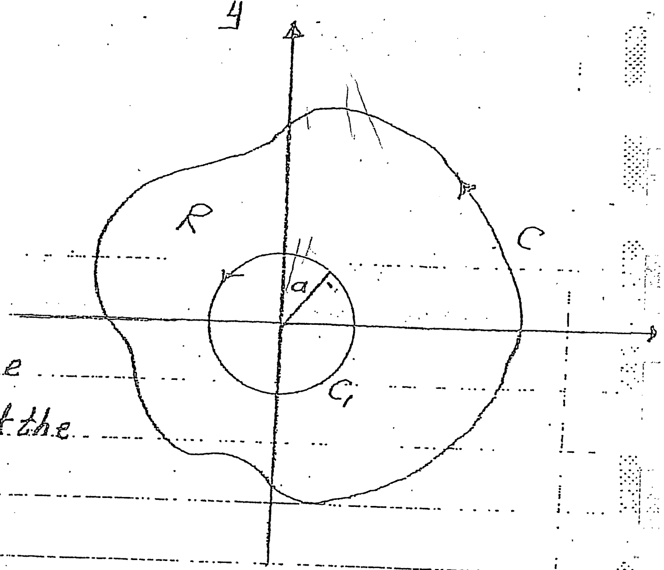


$$\int F \cdot dr = \int_{\phi_0}^{\phi_0} d\phi = \phi\Big|_{\phi_0}^{\phi_0} = 0$$

Z.W

For $F(x,y)=\frac{1}{x^2+y^2}(-y,x)$, Show that $\oint_C F(x,y)\cdot dr = 2\pi$

for every simple Closed Curve C enclosing the origin.



$$F = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$

$$\int F\cdot dr = \int \frac{-y\,dx + x\,dy}{x^2+y^2}$$

let C be any Simple Closed Curve enclosed the origin and let C_1 be the Circle of Small radius a, Centered at the Origin and positively oriented.

$$\int F\cdot dr = \int_C F\cdot dr - \int_{C_1} F\cdot dr$$

Green

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy - \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R \left[\frac{(x^2+y^2)(-1)+y(2y)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(-1)+y(2y)}{(x^2+y^2)^2}\right] dx\,dy$$

$= 0$ path Independent mean

This gives us

$$\int_C F\cdot dr = \int_{C_1} F\cdot dr \qquad \int F\cdot dr = \int d\theta$$

for Circle, $x = a\cos\theta$, $dx = -a\sin\theta\, d\theta$

$y = a\sin\theta$, $dy = a\cos\theta\, d\theta$

$$\int F\cdot dr = \int_0^{2\pi} \frac{-a\sin\theta(-a\sin\theta\, d\theta) + a\cos\theta(a\cos\theta\, d\theta)}{a^2\cos^2\theta + a^2\sin^2\theta} = 2\pi$$

Z.W

Evaluate $\oint_C F\cdot dr$, Where $F = \left(\frac{x}{x^2+y^2}\vec{i} + \frac{y}{x^2+y^2}\vec{j}\right)$ and C i any positively oriented simple Closed Curve Containing the origin.

$$\int F\cdot dr = \frac{x\,dx}{x^2+y^2} + \frac{y\,dy}{x^2+y^2}$$

let

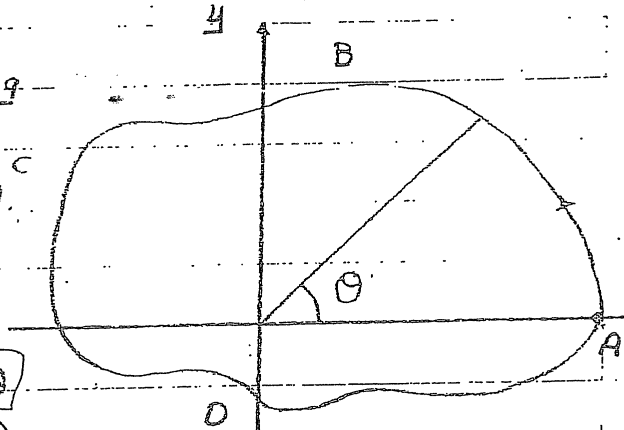
$$x = r\cos\theta \longrightarrow dx = -r\sin\theta\, d\theta$$
$$y = r\sin\theta \longrightarrow dy = r\cos\theta\, d\theta$$

$$\int F\cdot dr = \int \frac{-r^2\cos\theta\sin\theta\, d\theta}{r^2\cos^2\theta + r^2\sin^2\theta} + \int \frac{r^2\sin\theta\cos\theta\, d\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$\int F\cdot dr = -\int \frac{r^2\cos\theta\sin\theta\, d\theta}{r^2(\cos^2\theta + \sin^2\theta)} + \int \frac{r^2\sin\theta\cos\theta\, d\theta}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= -\int \cos\theta\sin\theta\, d\theta + \int \sin\theta\cos\theta\, d\theta$$

for Closed Curve ABCDA Surrounding the origin $\theta = 0$ at A and $\theta = 2\pi$ after Completely Circuit back to A.



$$\int F\cdot dr = -\int_0^{2\pi} \cos\theta\sin\theta\, d\theta + \int_0^{2\pi} \boxed{\sin\theta\cos\theta\, d\theta}$$

$$\int F\cdot dr = -\frac{1}{2}\int_0^{2\pi} \sin 2\theta\, d\theta + \frac{1}{2}\int_0^{2\pi} \sin 2\theta\, d\theta$$

$$\int F\cdot dr = \frac{1}{4}\cos 2\theta\Big|_0^{2\pi} - \frac{1}{4}\cos 2\theta\Big|_0^{2\pi} = 0$$

7. W

$$= \int \frac{\cancel{\rho^2}\,\cancel{(\cos^2\phi + \sin^2\phi)}\, d\phi}{\cancel{\rho^2}\,\cancel{(\cos^2\phi + \sin^2\phi)}} = \int d\phi$$

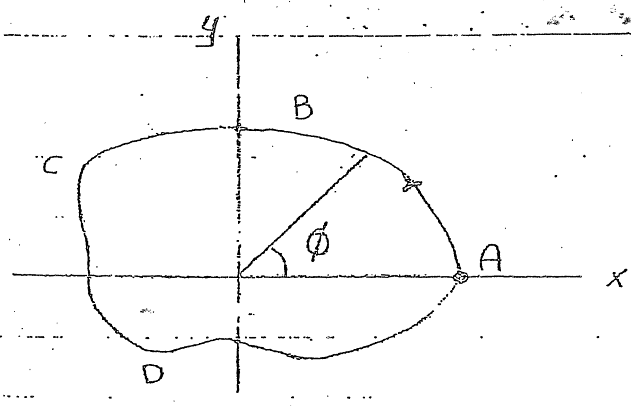


Fig (a)

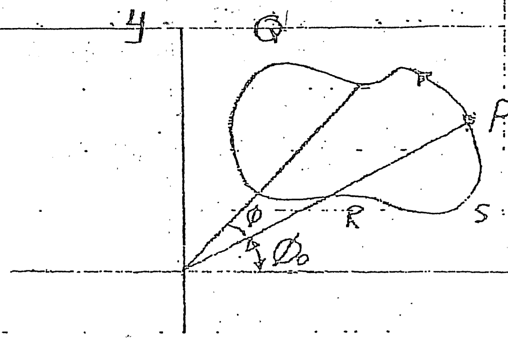


Fig.(b)

for a Closed Curve ABCDA (fig a) Surrounding the origin.
$\phi = 0$ at A and $\phi = 2\pi$ after Complete Circuit back to A.
In this Case Integral equal to

$$\int_0^{2\pi} d\phi = \phi\Big|_0^{2\pi} = 2\pi$$

for a Closed Curve PQRSP (fig b) not Surrounding the Origin
$\phi = \phi_0$ at P and $\phi = \phi_0$ after Complete Circuit back to P

$$\int_{\phi_0}^{\phi_0} d\phi = \phi\Big|_{\phi_0}^{\phi_0} = 0$$

explain the result :

يناقض / تعارض　　ناقض

the result would seem to contradict the property that "the line Integral is zero around every closed Curve"
However, No contradiction because of P & Q are not continuous derivative
have

Z.W

contradiction
لأن P و Q لهم مشتقات غير مستمرة ← مشتقة لها

Mar. 2009

Let $F = \frac{-yi + xj}{x^2 + y^2}$ (a) Calculate $\nabla \times F$. (b) Evaluate

around any Closed Path and explain the result.

$$F = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$

(a)

$$\nabla \times F = \begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$\nabla \times F = \left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}\frac{x}{x^2+y^2}\right)\vec{i} + \left(\frac{\partial}{\partial z}\frac{-y}{x^2+y^2} - \frac{\partial}{\partial x}0\right)\vec{j}$$

$$+\left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right)\vec{K}$$

$$\nabla \times F = 0 + 0 + \left(\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right)\vec{K}$$

$$\nabla \times F = \left(\frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2}\right)\vec{k} = 0$$

$$\nabla \times F = 0$$

(b)

$$\int F \cdot dr = \int \frac{-y\,dx + x\,dy}{x^2+y^2}$$

$x = \rho\cos\phi \rightarrow dx = -\rho\sin\phi\, d\phi$

$y = \rho\sin\phi \rightarrow dy = \rho\cos\phi\, d\phi$

Where (ρ, ϕ) are polar Coordinates.

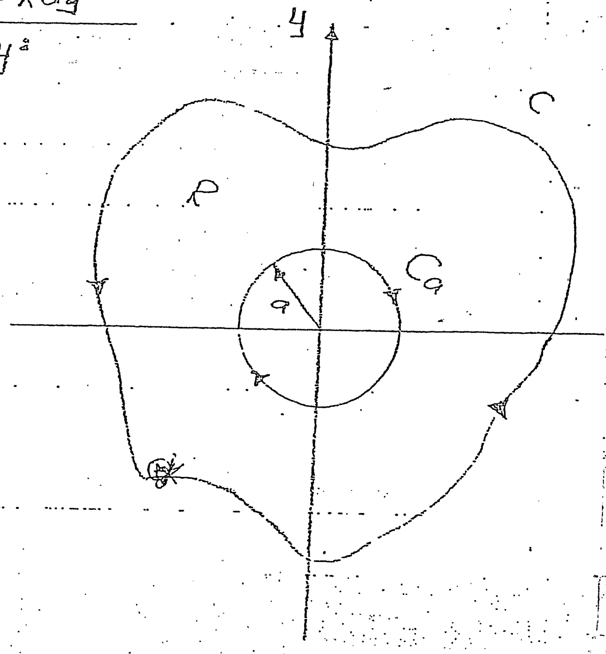
$$\int \frac{-y\,dx + x\,dy}{x^2+y^2} = \int \frac{-\rho\sin\phi(-\rho\sin\phi\,d\phi) + \rho\cos\phi(\rho\cos\phi\,d\phi)}{\rho^2\cos^2\phi + \rho^2\sin^2\phi}$$

Z.W

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \oint_{C_a} \frac{-y\,dx + x\,dy}{x^2+y^2}$$

we can rewrite as

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \oint_{-C_a} \frac{-y\,dx + x\,dy}{x^2+y^2}$$



we shown that the original Integral Can be evaluated by Integrating Counter Clockwise around a Circle of radius (a) that is Center at the origin and lies with in the region enclosed by C

for Circle

$x = r\cos\theta \rightarrow dx = -r\sin\theta\, d\theta$

$y = r\sin\theta \rightarrow dy = r\cos\theta\, d\theta$

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{-r\sin\theta(-r\sin\theta\, d\theta) + r\cos\theta\,(r\cos\theta\, d\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \int_0^{2\pi} \frac{r^2(\sin^2\theta+\cos^2\theta)\,d\theta}{r^2(\sin^2\theta+\cos^2\theta)} = \int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} = 2\pi$$

$\int_{\Gamma} - \int_{C_a} = 0 \qquad \int_{C_a} = (\quad)$

$\int_{\Gamma} - \int_{C_a} = \cancel{(\quad)} \int_{C_a}$

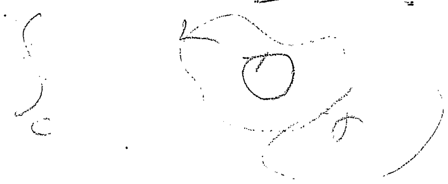
$\int_{\Gamma} \qquad \int_{total} - \int_{C_a} \int_{a} = 0$

Z.W

Evaluate the integral $\oint_C \dfrac{-y\,dx + x\,dy}{x^2 + y^2}$

if C is a piece wise smooth Simple Closed Curve oriented Clockwise Such that (a) C dose not enclosed the origin and (b) C

(b) enclosed the origin.



(a)

$$P = -\frac{y}{x^2+y^2} \quad , \quad Q = \frac{x}{x^2+y^2}$$

$$\frac{\partial P}{\partial y} = -\frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

[illegible] Green [illegible]

if x and y are not both zero, if C dose not enclosed the origin we have.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

on the Simply Connected reigon enclosed by C, the given Integral is Zero by Green theorem.

(b) for this Purpose we Construct a Circle C_a with Clockwise orientation Centered at the origin and has a Small radius (a).

it lies inside the region enclosed by C. This Creats a multiply Connected region (R) Whose boundary Curves C and C_a have orientation as Shown in figure

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} + \oint_{C_a} \frac{-y\,dx + x\,dy}{x^2+y^2} = \iint_R 0\,dA = 0$$

Z.W

سؤال 10

Show that Green lemma fails to hold for the function

$$P = \frac{y}{x^2+y^2} \quad \text{and} \quad Q = \frac{x}{x^2+y^2}$$

if R is the interior of the Circle $C: x^2+y^2=1$. Explain

$$\int_C [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\int_C [P\,dx + Q\,dy] = \iint_R \left(\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right) dx\,dy$$

$$= \iint_R 0\,dx\,dy = 0$$

$\therefore$ Green theorem fails

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

اذا كان التكامل path independent كان نظرية

fails to hold ~~[illegible]~~ green

Sep 2010 هذا السؤال

Let $\vec{F}$ be a force field such that $\vec{F} = -\nabla\phi$. if $\vec{F}$ is given $\vec{F} = 2xy\,\vec{i} + (x^2-1)\,\vec{j}$:

i- Find the Scalar potential ϕ of the force $\vec{F}$

ii- Find the work done in moving an object in this field along the Spiral $r = 2\theta$ from $\theta = 0$ to $\theta = \frac{5\pi}{2}$.

i: $F = -\nabla\phi$

$$2xy\,\vec{i} + (x^2-1)\,\vec{j} = -\left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j}\right)$$

$$\frac{\partial\phi}{\partial x} = -2xy \quad , \quad \frac{\partial\phi}{\partial y} = -(x^2-1) = -x^2+1$$

$$\phi = -x^2y + f(y)$$

$$\phi = -x^2y + y + g(x)$$

These agree if we choose $f(y) = y$, $g(x) = 0$

$$\phi = -x^2y + y + C$$

ii - work done

$$-\int F\cdot dr = \int \nabla\phi\cdot dr = \int\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy\right) = \int d\phi = \phi$$

$x = r\cos\theta = 2\theta\cos\theta$ $\qquad$ $y = r\sin\theta = 2\theta\sin\theta$

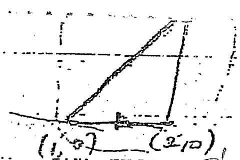
at $\theta = 0 \rightarrow x = 0$ $\qquad$ at $\theta = 0 \rightarrow y = 0$

at $\theta = \frac{5\pi}{2} \rightarrow x = 0$ $\qquad$ at $\theta = \frac{5\pi}{2} \rightarrow y = 5\pi$

$$\int F\cdot dr = \left[\phi\right]_{(0,0)}^{(0,5\pi)} = \left[-x^2y + y + C\right]_{(0,0)}^{(0,5\pi)} = 5\pi$$

$\therefore$ Work done $= 5\pi$

Z.W



a) Prove that $\int_{(1,0)}^{(2,1)} \left[(2xy - y^4 + 3)\,dx + (x^2 - 4xy^3)\,dy\right]$ is indepen[dent] of the Path joining (1,0) and (2,1) (b) Evaluate the Integral in (a)

(a)

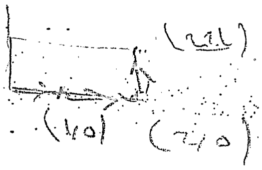
$P = 2xy - y^4 + 3 \qquad Q = x^2 - 4xy^3$

$\frac{\partial P}{\partial y} = 2x - 4y^3 \qquad \frac{\partial Q}{\partial x} = 2x - 4y^3$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\therefore$ Independent of path.

(b) Integral from (1,0) to (2,0) and then from (2,0) to (2,1)

$\int_c = \int_1^2 3\,dx + \int_0^1 (4 - 8y^3)\,dy$



$y = 0 \quad dy = 0$

$= 3x\Big|_1^2 + 4y\Big|_0^1 - 2y^4\Big|_0^1 = 5$

or $\frac{\partial \phi}{\partial x} = 2xy - y^4 + 3 \rightarrow \phi = x^2y - y^4x + 3x + f(y)$

$\frac{\partial \phi}{\partial y} = x^2 - 4xy^3 \rightarrow \phi = x^2y - xy^4 + g(x)$

if $f(y) = 0$, $g(x) = 3x \rightarrow \phi = x^2y - y^4x + 3x + C$

$\int_c = \int_{(1,0)}^{(2,1)} d\left[x^2y - xy^4 + 3x + C\right] = \left[x^2y - y^4x + 3x + C\right]_{(1,0)}^{(2,1)}$

$\int_c = 5$

Using Green lemma, Show that the area bounded by a simple closed Curve is given by the formula $A = \frac{1}{2}\int_c x\,dy - y\,dx$. is this formula Correct for regions bounded by more than one simple Closed Curve ?

$$\int_c [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Putting $P = -y$, $Q = x$

$$\int_c [x\,dy - y\,dx] = \iint_R \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(-y)\right) dx\,dy$$

$$\int_c [x\,dy - y\,dx] = \iint_R (2)\, dx\,dy$$

$$\int_c [x\,dy - y\,dx] = 2\iint_R A$$

$$A = \frac{1}{2}\int_c [x\,dy - y\,dx]$$

yes

Z.W

Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^2$

(Hint: Parametric equations are $x = a\cos^3 t$, $y = a\sin^3 t$, $(0 \leq t \leq$

$x = a\cos^3 t \longrightarrow dx = 3a\cos^2 t(-\sin t)dt = -3a\cos^2 t\sin t\, dt$

$y = a\sin^3 t \longrightarrow dy = 3a\sin^2 t\cos t\, dt$

$$A = \frac{1}{2}\int [x\,dy - y\,dx]$$

$$A = \frac{1}{2}\int_0^{2\pi}\left[a\cos^3 t\, 3a\sin^2 t\cos t\, dt + a\sin^3 t\, 3a\cos^2 t\sin t\, dt\right]$$

$$A = \frac{1}{2}\int_0^{2\pi}\left[3a^2\cos^4 t\sin^2 t\, dt + 3a^2\sin^4 t\cos^2 t\, dt\right]$$

$$A = \frac{3a^2}{2}\int_0^{2\pi}\left[\cos^2 t\cos^2 t\sin^2 t\, dt + \sin^2 t\sin^2 t\cos^2 t\, dt\right]$$

$$A = \frac{3a^2}{8}\int_0^{2\pi}\left[(1+\cos 2t)(1+\cos 2t)\sin^2 t\, dt + (1-\cos 2t)(1-\cos 2t)\cos^2 t\, dt\right]$$

$$A = \frac{3a^2}{8}\int_0^{2\pi}\left[(1+2\cos 2t+\cos^2 2t)\frac{1}{2}(1-\cos 2t)dt + (1-2\cos 2t+\cos^2 2t)\frac{1}{2}(1+\right.$$

$$A = \frac{3a^2}{16}\int_0^{2\pi}\left[1+2\cos 2t+\cos^2 2t-\cos 2t-2\cos^2 2t-\cos^2 2t\cos 2t\right.$$

$$\left. +1-2\cos 2t+\cos^2 2t+\cos 2t-2\cos^2 2t+\cos^2 2t\cos 2t\right]dt$$

$$A = \frac{3a^2}{16}\int_0^{2\pi}\left[2+2\cos^2 2t-4\cos^2 2t\right]dt = \frac{3a^2}{16}\int_0^{2\pi}\left[2-2\cos^2 2t\right]dt$$

$$A = \frac{3a^2}{8}\int_0^{2\pi}\sin^2 2t\, dt = \frac{3a^2}{8}\int_0^{2\pi}\left(\frac{1-\cos 4t}{2}\right)dt$$

$$A = \frac{3a^2}{16}\left[t - \frac{1}{4}\sin 4t\right]_0^{2\pi} = \frac{3a^2}{16}(2\pi - 0)$$

$$A = \frac{3a^2\pi}{8}$$

Z.W

$$= \iint_R (y^2 - x^2)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x^{1/2}} (y^2 - x^2)\,dy\,dx = \int_0^1 \left(\frac{1}{3}y^3 - x^2 y\right)\Big|_{x^2}^{x^{1/2}} dx$$

$$= \int_0^1 \left[\frac{1}{3}x^{3/2} - x^{5/2} - \frac{1}{3}x^6 + x^4\right]$$

$$= \frac{2}{15}x^{5/2} - \frac{2}{7}x^{7/2} - \frac{1}{21}x^6 + \frac{1}{5}x^5\Big|_0^1 = 0$$

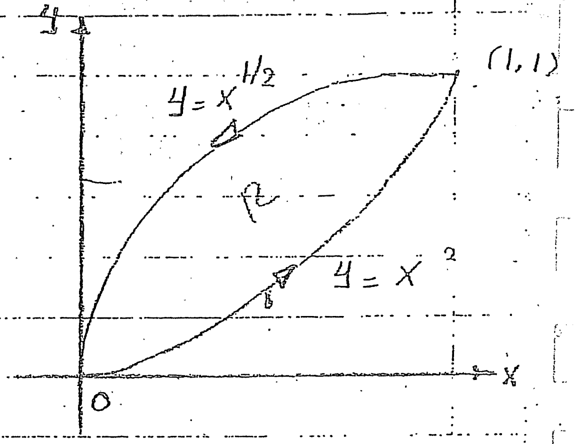
$\therefore$ Green theorem is Verify.

Z.W

Verify Green theorem in the Plane for $\oint_C x^2 y\,dx + (y + xy^2)\,dy$

Where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$.

direct Method



along the Curve $y = x^2$ from $(0,0)$ to $(1,1)$

$$y = x^2 \rightarrow dy = 2x\,dx$$

$$\int_0^1 \left[x^4 dx + (x^2 + x^5)2x\,dx\right]$$

$$\int_0^1 \left[x^4 dx + 2x^3 dx + 2x^6 dx\right]$$

$$\frac{1}{5}x^5\Big|_0^1 + \frac{1}{2}x^4\Big|_0^1 + \frac{2}{7}x^7\Big| = \frac{1}{5} + \frac{1}{2} + \frac{2}{7}$$

along the Curve $y = x^{1/2}$ from $(1,1)$ to $(0,0)$

$$y = x^{1/2} \rightarrow dy = \frac{1}{2}x^{-1/2}dx$$

$$\int_1^0 \left[x^{5/2} dx + (x^{1/2} + x^2)\frac{1}{2}x^{-1/2}dx\right]$$

$$\int_1^0 \left[x^{5/2} dx + \frac{1}{2}dx + \frac{1}{2}x^{3/2}dx\right] = \left[\frac{2}{7}x^{7/2} + \frac{1}{2}x + \frac{1}{5}x^{5/2}\right]_1^0$$

$$= -\frac{2}{7} - \frac{1}{2} - \frac{1}{5}$$

$\therefore$ total Integral $= \frac{1}{5} + \frac{1}{2} + \frac{2}{7} - \frac{2}{7} - \frac{1}{2} - \frac{1}{5} = 0$

By Using Green theorem

$$\int \left[P\,dx + Q\,dy\right] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

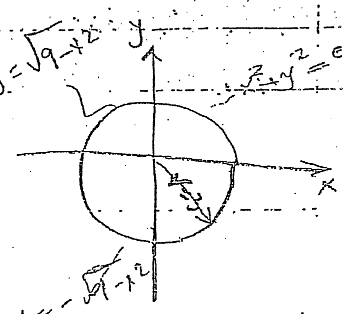
$$= \iint_R \left(\frac{\partial}{\partial x}(y + xy^2) - \frac{\partial}{\partial y}x^2 y\right)dx\,dy$$

Z.W $\quad y^2 - x^2$

$$= \int_0^{2\pi}\int_0^{\sqrt{9-x^2}} (1+2y)\,dx\,dy = \int_0^{2\pi} (y+y^2)\Big|_0^{\sqrt{9-x^2}} dx\,dy$$

$-\sqrt{9-x^2}$

$$= \int_0^{2\pi} \left(\sqrt{9-x^2} + (9-x^2)\right) dx$$



$x = r\cos\theta \ , \ dx = -r\sin\theta\, d\theta$

$$= \int_0^{2\pi} \left(\sqrt{9-9\cos^2\theta} + 9 - 9\cos^2\theta\right)(-3\sin\theta\, d\theta)$$

$$= \int_0^{2\pi} \left(3\sin\theta + 9 - 9\cos^2\theta\right)(-3\sin\theta\, d\theta)$$

$$= -9\int_0^{2\pi} \sin^2\theta\, d\theta - 27\int_0^{2\pi} \sin\theta\, d\theta + 27\int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta$$

$$= -\frac{9}{2}\int_0^{2\pi} (1-\cos 2\theta)\, d\theta - 27\int_0^{2\pi} \sin\theta\, d\theta + 27\int_0^{2\pi} \cos^2\theta\sin\theta\, d$$

$$= -\frac{9}{2}\theta + \frac{9}{4}\sin 2\theta + 27\cos\theta - 9\cos^3\theta \Big|_0^{2\pi} = -9\pi$$

$\therefore$ Green theorem is not verify

~~فيه خطأ تكون حدود التكامل dy أخذت من 0 ← $\sqrt{9-x^2}$ والصحيح أن تأخذ من~~
~~$-\sqrt{9-x^2}$ ← $\sqrt{9-x^2}$ ثم أكمل بطريقة الحل [illegible]~~

[illegible]

Z.W

Verify Green theorem in the Plane for $\oint_C (x^2 - y^2)dx + x\,dy$ When C is the Circle $x^2 + y^2 = 9$.

① direct Method.

$x^2 + y^2 = 9$, $r = 3$

$x = r\cos\theta \longrightarrow dx = -r\sin\theta\, d\theta$

$y = r\sin\theta \longrightarrow dy = r\cos\theta\, d\theta$

$$\int_0^{2\pi} \left[(9\cos^2\theta - 9\sin^2\theta)(-3\sin\theta\, d\theta) + 9\cos^2\theta\, d\theta \right]$$

$$\int_0^{2\pi} \left[-27\cos^2\theta\sin\theta\, d\theta + 27\sin^2\theta\sin\theta\, d\theta + 9\cos^2\theta\, d\theta \right]$$

$$\int_0^{2\pi} \left[-27\cos^2\theta\sin\theta\, d\theta + 27(1-\cos^2\theta)\sin\theta\, d\theta + 9\cos^2\theta\, d\theta \right]$$

$$\int_0^{2\pi} \left[-27\cos^2\theta\sin\theta\, d\theta + 27\sin\theta\, d\theta - 27\cos^2\theta\sin\theta\, d\theta + 9\cos^2\theta\, d\theta \right]$$

$$\int_0^{2\pi} \left[-54\cos^2\theta\sin\theta\, d\theta + 27\sin\theta\, d\theta + \frac{9}{2}(1+\cos 2\theta)\, d\theta \right]$$

$$\int_0^{2\pi} \left[-54\cos^2\theta\sin\theta\, d\theta + 27\sin\theta\, d\theta + \frac{9}{2}\, d\theta + \frac{9}{2}\cos 2\theta\, d\theta \right]$$

$$\left[\frac{54}{3}\cos^3\theta - 27\cos\theta + \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta \right]_0^{2\pi}$$

$(0 - 0 + 9\pi + 0) = 9\pi$

② By Using Green theorem

$$\int \left[P\,dx + Q\,dy \right] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \iint_R \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(x^2 - y^2) \right) dx\,dy$$

$$= \iint_R (1 + 2y)\, dx\,dy = \int_0^{2\pi}\int_0^3 (1 + 2r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (r + 2r^2\sin\theta)\, dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + \frac{2}{3}r^3\sin\theta \right]_0^3 d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} + 18\sin\theta \right) d\theta = \left. \frac{9}{2}\theta - 18\cos\theta \right]_0^{2\pi} = 9\pi$$

2) Z.W $(1 - 2r\sin\theta)\, r\,dr\,d\theta$

$= (0 + 64\pi + 0 + 0 + 0) = \underline{64\pi}$

By Using Green theorem.

$$\int_c [P dx + Q dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}(5y + 3x - 6) - \frac{\partial}{\partial y}(2x - y + 4)\right) dx\, dy$$

$$= \iint_R (3+1)\, dx\, dy = 4 \int_0^{\pi/2} \int_0^{\sqrt{16-y^2}} \cancel{dx}\, dy = 4 \int_0^{\pi/2} x \Big|_0^{\sqrt{16-y^2}} dy$$

$$= 4 \int_0^{\pi/2} \sqrt{16 - y^2}\, dy$$

$4 \iint_R r\, dr\, d\theta$

$= 4 \int_0^{2\pi} \frac{r^2}{2} \Big]_0^4 d\theta$

$y = r \sin\theta \longrightarrow dy = r \cos\theta\, d\theta$

$$= 4 \int_0^{\pi/2} \sqrt{16 - 16\sin^2\theta}\, (4\cos\theta\, d\theta) = 4 \int_0^{\pi/2} (16 \cos^2\theta)\, d\theta$$

$= 4 \times 8 [\theta]_0^{2\pi}$

$$= 64 \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\, d\theta = 32\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$= 2 \times 2\pi$

$= \underline{64\pi}$

$$= 32\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = 16\pi$$

the total Integral $= 16\pi \times 4 = \underline{64\pi}$

Z.W

Evaluate $\oint[(2x-y+4)dx+(5y+3x-6)dy]$ around a Circle of radius 4 with Centre at $(0,0)$.

direct Method,

the equation of Circle, $x^2+y^2=16$ $(r=4)$

$x=r\cos\theta \rightarrow dx=-r\sin\theta d\theta$

$y=r\sin\theta \rightarrow dy=r\cos\theta d\theta$

$$\int_0^{2\pi}[(8\cos\theta-4\sin\theta+4)(-4\sin\theta d\theta)+(20\sin\theta+12\cos\theta-6)(4\cos\theta$$

$$=\int_0^{2\pi}[-32\cos\theta\sin\theta d\theta+16\sin^2\theta d\theta-16\sin\theta d\theta+80\sin\theta\cos\theta d\theta$$

$$+48\cos^2\theta d\theta-24\cos\theta d\theta]$$

$$=\int_0^{2\pi}[48\sin\theta\cos\theta d\theta+16\sin^2\theta d\theta-16\sin\theta d\theta+48\cos^2\theta d\theta-24\cos\theta$$

$$=\int_0^{2\pi}[\frac{48}{2}\sin 2\theta d\theta+16(1-\cos^2\theta)d\theta-16\sin\theta d\theta+48\cos^2\theta d\theta$$

$$-24\cos\theta d\theta]$$

$$=\int_0^{2\pi}[24\sin 2\theta d\theta+16d\theta-16\sin\theta d\theta+32\cos^2\theta d\theta-24\cos\theta d\theta]$$

$$=\int_0^{2\pi}[24\sin 2\theta d\theta+16d\theta-16\sin\theta d\theta+32(\frac{1+\cos 2\theta}{2})d\theta-24\cos\theta d\theta]$$

$$=\int_0^{2\pi}[24\sin 2\theta d\theta+16d\theta-16\sin\theta d\theta+16d\theta+16\cos 2\theta d\theta-24\cos\theta d\theta$$

$$=\int_0^{2\pi}[24\sin 2\theta d\theta+32d\theta-16\sin\theta d\theta+16\cos 2\theta d\theta-24\cos\theta d\theta]$$

$$=-12\cos 2\theta+32\theta+16\cos\theta+8\sin 2\theta-24\sin\theta\Big|_0^{2\pi}$$

Z.W

Work the Previous Problem for the line Integral

$$\oint_C (x^2+y^2)\,dx + 3xy^2\,dy$$

the equation of circle $x^2+y^2=4$, $r=2$

$x = r\cos\theta \rightarrow dx = -r\sin\theta\, d\theta$

$y = r\sin\theta \rightarrow dy = r\cos\theta\, d\theta$

$$\int_0^{2\pi}\left[(4\cos^2\theta + 4\sin^2\theta)(-2\sin\theta\, d\theta) + 3(2\cos\theta)(2\sin\theta)^2(2\cos\theta\, d\theta)\right]$$

$4(1)$

$$\int_0^{2\pi}\left[\boxed{-8\sin\theta\, d\theta} + 48\sin^2\theta\cos^2\theta\, d\theta\right]$$

$$\int_0^{2\pi}\left[-8\sin\theta\, d\theta + \frac{48}{4}\sin^2 2\theta\, d\theta\right]$$

$$\int_0^{2\pi}\left[-8\sin\theta\, d\theta + 12\sin^2 2\theta\, d\theta\right]$$

$\rho = r\sin\theta$

$$\int_0^{2\pi}\left[-8\sin\theta\, d\theta + 12\left(\frac{1-\cos 4\theta}{2}\right)d\theta\right]$$

12

$$\int_0^{2\pi}\left[-8\sin\theta\, d\theta + 6\, d\theta - 6\cos 4\theta\, d\theta\right]$$

$$8\cos\theta\Big|_0^{2\pi} + 6\theta\Big|_0^{2\pi} - \frac{6}{4}\sin 4\theta\Big|_0^{2\pi} = (0 + 12\pi - 0) = 12\pi$$

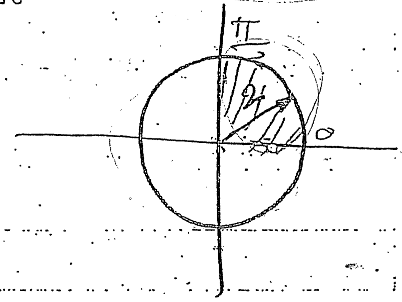
Z.W

Evaluate $\oint_c (3x+4y)dx+(2x-3y)dy$ where C, a circle of rad two with centre at the origin of the xy-plane is traversed in the positive sense.

the Equation of the Circle $(x-x_0)^2+(y-y_0)^2=r^2$ $(r=2)$

$x^2+y^2=4$

By Using Green theorem.



$$\int_c [Pdx+Qdy] = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$$=\iint_R \left[\frac{\partial}{\partial x}(2x-3y)-\frac{\partial}{\partial y}(3x+4y)\right]dx\,dy$$

$$\int_0^2\int_0^{\pi} r\,dr\,d\theta$$

$$\iint -2\,dx\,dy \quad = \iint_R (2-4)\,dx\,dy = \iint_R (-2)\,dx\,dy$$

$$=\int_0^{\pi/2}\int_0^2 -2r\,dr\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{2,\ x=\sqrt{4-y^2}} (-2)\,dx\,dy = -2\int_0^{\pi/2} x\Big|_0^{\sqrt{4-y^2}}dy$$

$$=\int_0^{\pi/2} -4\,d\theta$$

$$=-2\pi$$

$$=-2\int_0^{\pi/2}\sqrt{4-y^2}\,dy$$

$$=4\times -2\pi$$

$$=8\pi$$

let $y=r\sin\theta \to dy = r\cos\theta\,d\theta$ $(r=2)$

$$=-2\int_0^{\pi/2}\sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta = -8\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=-8\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = -4\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$=-4\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = -4\left(\frac{\pi}{2}-0\right) = -2\pi$$

the total Integral $= -2\pi\times 4 = -8\pi$

Z.W

Show that in the polar Coordinates (ρ, ϕ) the expression $x\,dy - y\,dx = \rho^2 d\phi$. Interpret $\frac{1}{2}\int x\,dy - y\,dx$

$x = \rho\cos\phi \rightarrow dx = -\rho\sin\phi\,d\phi$

$y = \rho\sin\phi \rightarrow dy = \rho\cos\phi\,d\phi$

$x\,dy - y\,dx = \rho\cos\phi\,\rho\cos\phi\,d\phi + \rho\sin\phi\,\rho\sin\phi\,d\phi$

$= \rho^2\cos^2\phi\,d\phi + \rho^2\sin^2\phi\,d\phi$

$= \rho^2 d\phi\,(\cos^2\phi + \sin^2\phi) = \rho^2 d\phi$

But

$A = \frac{1}{2}\int_c [x\,dy - y\,dx]$

$A = \frac{1}{2}\rho^2 d\phi$

Z.W

if $x = \rho \cos\phi$, $y = \rho \sin\phi$, prove that

$$\frac{1}{2} \oint [x\,dy - y\,dx] = \frac{1}{2} \int \rho^2 \, d\phi$$ and Interpret

$$x = \rho \cos\phi \rightarrow dx = -\rho \sin\phi \, d\phi$$
$$y = \rho \sin\phi \rightarrow dy = \rho \cos\phi \, d\phi$$

$$\frac{1}{2} \int [x\,dy - y\,dx]$$

$$\frac{1}{2} \int [\rho\cos\phi \, \rho\cos\phi \, d\phi + \rho\sin\phi \, \rho\sin\phi \, d\phi]$$

$$\frac{1}{2} \int [\rho^2 \cos^2\phi \, d\phi + \rho^2 \sin^2\phi \, d\phi]$$

$$\frac{1}{2} \int \rho^2 \, d\phi \, (\cos^2\phi + \sin^2\phi) = \frac{1}{2} \int \rho^2 \, d\phi$$

But

$$A = \frac{1}{2} \int_c [x\,dy - y\,dx]$$

$$\therefore A = \frac{1}{2} \int \rho^2 \, d\phi$$

Mujtaba Noori Saleh.
M.Sc. Structure

$$\int_1^0 4y\,dy = 2y^2\Big|_1^0 = -2$$

the total Integral $= 1 + \frac{8}{3} - 2 = \frac{8}{3} - 1 = \frac{5}{3}$

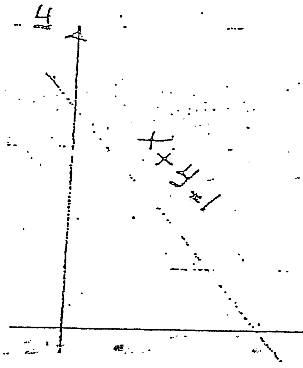
By Using Green Theorem

$-8y^2 - xy$

$$\int [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right) dx\,dy$$

$$= \iint_R (-6y + 16y)\,dx\,dy = \iint_R (10y)\,dx\,dy$$



$$= \int_0^1 \int_{y=0}^{y=1-x} 10y\,dy\,dx = \int_0^1 [5y^2]_0^{1-x}\,dx$$

$$= \int_0^1 \left[5(1-x)^2\right] dx = -\frac{5}{3}(1-x)^3\Big|_0^1 = \frac{5}{3}$$

Green theorem is Verify.

Z.W

$$= \iint_R [-6y + 16y]\,dx\,dy = \int_0^1 \int_{x^2}^{x^{1/2}} [-6y+16y]\,\overset{dy\,dx}{dx\,dy}$$

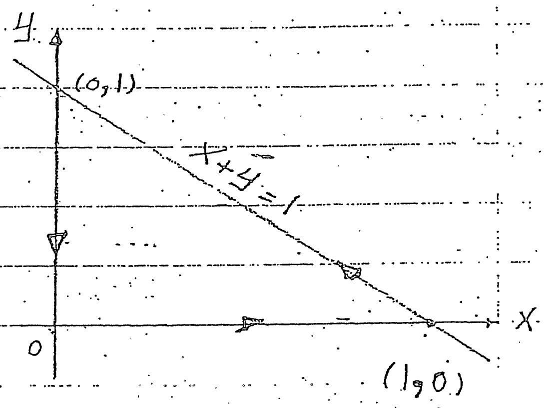
$$= \int_0^1 \left[-3y^2+8y^2\right]_{x^2}^{x^{1/2}} dx = \int_0^1 \left[5y^2\right]_{x^2}^{x^{1/2}} dx$$

$$= \int_0^1 [5x - 5x^4]\,dx = \frac{5}{2}x^2\Big|_0^1 - x^5\Big|_0^1 = \frac{5}{2} - 1 = \frac{3}{2}$$

(b.) $x=0$, $y=0$, $x+y=1$

first method (direct method).

along the straight line from $(0,0)$ to $(1,0)$, $y=0$, $dy=0$ while x varies from 0 to 1.



$$\int \left[(3x^2-8y^2)dx + (4y-6xy)dy\right]$$

$$\int_0^1 3x^2\,dx = x^3\Big|_0^1 = 1 ,$$

along the line $(x+y=1)$ from $(1,0)$ to $(0,1)$

$y = 1 - x \rightarrow dy = -dx$

$$\int_1^0 \left[(3x^2 - 8(1-x)^2 dx - (4(1-x) - 6x(1-x))dx\right]$$

$$\int_1^0 \left[3x^2 - 8dx + 16x\,dx - 8x^2dx - 4dx + 4x\,dx + 6x\,dx - 6x^2dx\right]$$

$$x^3\Big|_1^0 - 8x\Big|_1^0 + 8x^2\Big|_1^0 - \frac{8}{3}x^3\Big|_1^0 - 4x\Big|_1^0 + 2x^2\Big|_1^0 + 3x^2\Big|_1^0 - 2x^3\Big|_1^0 = \frac{8}{3}$$

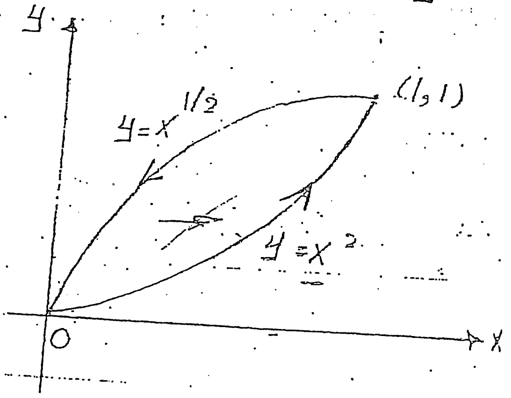
along the straight line from $(0,1)$ to $(0,0)$, $x=0$, $dx=0$

while y varies from 1 to 0

Verify Green's theorem in the plane for $\oint_C (3x^2-8y^2)dx + (4y-6$
where C is the boundary of the region defined by (a) $y=\sqrt{x}$, $y=$
(b) $x=0$, $y=0$, $x+y=1$

(a) $y=x^{1/2}$, $y=x^2$

along the Curve $y=x^2$ from $(0,0)$ to $(1,1)$.



(1) $y=x^2 \to dy=2x\,dx$

$$\int_0^1 \left[(3x^2-8x^4)dx + (4x^2-6x^3)2x\,dx\right]$$

$$\int_0^1 \left[3x^2dx - 8x^4dx + 8x^3dx - 12x^4dx\right]$$

$$x^3\Big|_0^1 - \frac{8}{5}x^5\Big|_0^1 + 2x^4\Big|_0^1 - \frac{12}{5}x^5\Big|_0^1 = 1-\frac{8}{5}+2-\frac{12}{5} = -1$$

مهم

along the Curve $y=x^{1/2}$ from $(1,1)$ to $(0,0)$.

(2) $y=x^{1/2} \to dy=\frac{1}{2}x^{-1/2}dx$

لاحظ حدود التكامل

$$\int_1^0 \left[(3x^2-8x)dx + (4x^{1/2}-6x^{3/2})\frac{1}{2}x^{-1/2}dx\right]$$

$$\int_1^0 \left[3x^2dx - 8x\,dx + 2dx - 3x\,dx\right] = x^3\Big|_1^0 - 4x^2\Big|_1^0 + 2x\Big|_1^0 - \frac{3}{2}x^2\Big|_1^0 = \frac{5}{2}$$

$\therefore$ total Integral $= -1+\frac{5}{2} = \frac{3}{2}$

By Using Green Theorem.

$\sqrt{x}-x^2=0$

$$\int[P\,dx+Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$x^{} - x^4 = 0$

$x(1-x^3)=0$

$x=0$ or $x^3=1$

$$= \iint_R \left(\frac{\partial}{\partial x}(4y-6xy) - \frac{\partial}{\partial y}(3x^2-8y^2)\right)dx\,dy$$

$x=1$

$y=1$

$$= \int_0^{\pi/2} 60\left(\frac{1-\cos 4\theta}{2}\right) d\theta + \int_0^{\pi/2} 80\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 30\int_0^{\pi/2} d\theta - 30\int_0^{\pi/2} \cos 4\theta \, d\theta + 20\int_0^{\pi/2} (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= 30\int_0^{\pi/2} d\theta - 30\int_0^{\pi/2} \cos 4\theta \, d\theta + 20\int_0^{\pi/2} d\theta + 40\int_0^{\pi/2} \cos 2\theta + 20\int_0^{\pi/2} \left(\frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= 30\int_0^{\pi/2} d\theta - 30\int_0^{\pi/2} \cos 4\theta \, d\theta + 20\int_0^{\pi/2} d\theta + 40\int_0^{\pi/2} \cos 2\theta + 10\int_0^{\pi/2} d\theta + 10\int_0^{\pi/2} \cos 4\theta \, d\theta$$

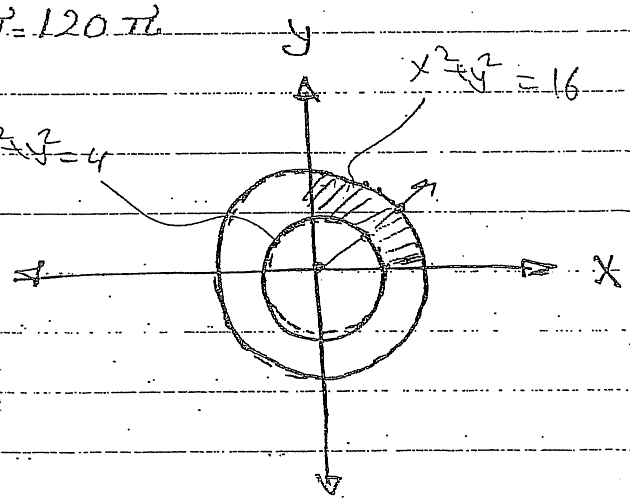
$$= 30\theta\Big|_0^{\pi/2} - \frac{30}{4}\sin 4\theta\Big|_0^{\pi/2} + 20\theta\Big|_0^{\pi/2} + 20\sin 2\theta\Big|_0^{\pi/2} + 10\theta\Big|_0^{\pi/2} + \frac{10}{4}\sin 4\theta\Big|_0^{\pi/2}$$

$$= 15\pi - 0 + 10\pi + 0 + 5\pi + 0 = 30\pi$$

$\therefore$ total Integral $= 4 \times 30\pi = 120\pi$

$dx = r\,dr$

$$\iint_R \boxed{(y^2 + x^2)}\, dx\, dy$$



$$= 4 \times \int_0^{\pi/2} \int_2^4 r^2\, r\, dr\, d\theta$$

$$= 4 \int_0^{\pi/2} r^3\, dr\, d\theta$$

$$= 4 \int_0^{\pi/2} 60\, d\theta$$

$$= 4 * 30\pi = 120\pi$$

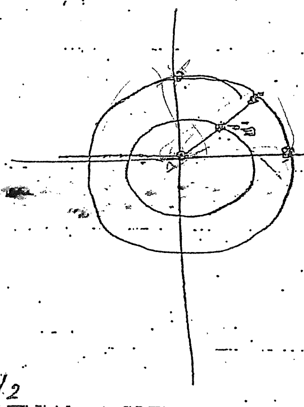
Z.W

$r\,dr\,d\theta$

$\iint_R \underbrace{[y^2+x^2]}_{r^2}\,dx\,dy \qquad x^2+y^2=4 \rightarrow x=\sqrt{4-y^2}$

$x^2+y^2=16 \rightarrow x=\sqrt{16-y^2}$

$$\int_0^{\pi/2}\int_{x=\sqrt{4-y^2}}^{x=\sqrt{16-y^2}} (y^2+x^2)\,dx\,dy \qquad r^2 \quad r\,dr\,d\theta$$



$$= \int_0^{\pi/2}\left[y^2x+\frac{1}{3}x^3\right]_{\sqrt{4-y^2}}^{\sqrt{16-y^2}} dy$$

$$= \int_0^{\pi/2}\left[y^2\sqrt{16-y^2}+\frac{1}{3}\left(\sqrt{16-y^2}\right)^3\right]dy-\int_0^{\pi/2}\left[y^2\sqrt{4-y^2}+\frac{1}{3}\left(\sqrt{4-y^2}\right)^3\right]dy$$

$y=r\sin\theta \rightarrow dy=r\cos\theta\,d\theta$

$4\sin\theta$

$$=\int_0^{\pi/2}\left[16\sin^2\theta\sqrt{16-16\sin^2\theta}+\frac{1}{3}\left(\sqrt{16-16\sin^2\theta}\right)^3\right](4\cos\theta\,d\theta)$$

$$-\int_0^{\pi/2}\left[4\sin^2\theta\boxed{\sqrt{4-4\sin^2\theta}}+\frac{1}{3}\left(\sqrt{4-4\sin^2\theta}\right)^3\right](2\cos\theta\,d\theta)$$

$=\sqrt{4(1-\sin^2\theta)}$
$=\sqrt{4\cos^2\theta}=2\cos\theta$

$$=\int_0^{\pi/2}\left(64\sin^2\theta\cos\theta+\frac{1}{3}(4\cos\theta)^3\right)(4\cos\theta\,d\theta)-\int_0^{\pi/2}\left(8\sin^2\theta\cos\theta+\frac{1}{3}(2\cos\theta)^3\right)(2$$

$$=\int_0^{\pi/2}\left(256\sin^2\theta\cos^2\theta\,d\theta+\frac{256}{3}\cos^4\theta\,d\theta\right)-\int_0^{\pi/2}\left(16\sin^2\theta\cos^2\theta\,d\theta+\frac{16}{3}\cos^4\theta\,d\theta\right.$$

$$=\int_0^{\pi/2}240\sin^2\theta\cos^2\theta\,d\theta+\int_0^{\pi/2}80\cos^4\theta\,d\theta$$

$$=\int_0^{\pi/2}\frac{240}{4}\sin^2 2\theta\,d\theta+\int_0^{\pi/2}80\cos^2\theta\cos^2\theta\,d\theta$$

Z.W

$$\int_C \left[(x^3 - x^2 y)\,dx + x y^2\,dy\right]$$

$$\int_0^{2\pi} \left[(64\cos^3\theta - 64\cos^2\theta\sin\theta)(-4\sin\theta\,d\theta) + 64\cos\theta\sin^2\theta\,(4\cos\theta\,d\theta)\right]$$

$$\int_0^{2\pi} \left[(-256\cos^3\theta\sin\theta\,d\theta + 256\cos^2\theta\sin^2\theta\,d\theta + 256\cos^2\theta\sin^2\theta\,d\theta\right]$$

$$= 256\int_0^{2\pi} \left[-\cos^3\theta\sin\theta\,d\theta + 2\cos^2\theta\sin^2\theta\,d\theta\right]$$

$$256\int_0^{2\pi} \left[-\cos^3\theta\sin\theta\,d\theta + \frac{2}{4}\sin^2 2\theta\,d\theta\right]$$

$$256\int_0^{2\pi} \left[-\cos^3\theta\sin\theta\,d\theta + \frac{1}{2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta\right]$$

$$256\int_0^{2\pi} \left[-\cos^3\theta\sin\theta\,d\theta + \frac{1}{4}\,d\theta - \frac{1}{4}\cos 4\theta\,d\theta\right]$$

$$256\left[\frac{1}{4}\cos^4\theta + \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta\right]_0^{2\pi} = 256\left[(1-1) + \frac{\pi}{2} - (0-0)\right] = \ldots$$

$\therefore$ total Integral $= 128\pi - 8\pi = 120\pi$

By Using Green theorem

$$\int_C \left[P\,dx + Q\,dy\right] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$= \iint_R \left(\frac{\partial}{\partial x} x y^2 - \frac{\partial}{\partial y}(x^3 - x^2 y)\right)dx\,dy$$

$$= \iint_R (y^2 + x^2)\,dx\,dy$$

Verify Green theorem in the plane for $\oint_c [(x^3 - x^2 y)\,dx + x y^2\,dy]$ where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$

⇒ first Solution

along the Circle $x^2 + y^2 = 4$

$x = r\cos\theta \rightarrow dx = -r\sin\theta\,d\theta$, $x = 2\cos\theta \rightarrow dx = -2\sin\theta\,d\theta$

$y = r\sin\theta \rightarrow dy = r\cos\theta\,d\theta$, $y = 2\sin\theta \rightarrow dy = 2\cos\theta\,d\theta$

$$\int_0^{2\pi}\left[(8\cos^3\theta - 8\cos^2\theta\sin\theta)(-2\sin\theta\,d\theta) + 8\cos\theta\sin^2\theta\,(2\cos\theta\,d\theta)\right]$$

$$\int_0^{2\pi}\left[-16\cos^3\theta\sin\theta\,d\theta + 16\cos^2\theta\sin^2\theta\,d\theta + 16\cos^2\theta\sin^2\theta\,d\theta\right]$$

$$16\int_0^{2\pi}\left[-\cos^3\theta\sin\theta\,d\theta + \frac{1}{4}\sin^2 2\theta\,d\theta + \frac{1}{4}\sin^2 2\theta\,d\theta\right]$$

$$16\int_0^{2\pi}\left[-\cos^3\theta\sin\theta\,d\theta + \frac{1}{2}\sin^2 2\theta\,d\theta\right]$$

$$16\int_0^{2\pi}\left[-\cos^3\theta\sin\theta\,d\theta + \frac{1}{2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta\right]$$

$$16\int_0^{2\pi}\left[-\cos^3\theta\sin\theta\,d\theta + \frac{1}{4}d\theta - \frac{1}{4}\cos 4\theta\,d\theta\right]$$

$$16\left[\frac{1}{4}\cos^4\theta + \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta\right]_0^{2\pi} = 16\left(0 + \frac{\pi}{2} - 0\right) = 8\pi$$

$\cos^2\theta\cdot\sin^2\theta\,d\theta = \frac{1}{4}\sin^2 2\theta\,d\theta$

$\sin^2 2\theta = \left(\frac{1-\cos 4\theta}{2}\right)d\theta$

along the Circle $x^2 + y^2 = 16$

$x = 4\cos\theta \rightarrow dx = -4\sin\theta\,d\theta$

$y = 4\sin\theta \rightarrow dy = 4\cos\theta\,d\theta$

$\sqrt{4 - 4\sin^2\theta}$

$\sqrt{4(1-\sin^2\theta)} = 2\cos\theta$

$\sqrt{4\cos^2\theta}$

$x = 4\sin$

$\therefore (1 - \sin^2\theta) = \cos^2\theta$

(a) let C be any simple Closed Curve bounding a region having area A prove that if $a_1, a_2, a_3, b_1, b_2, b_3$ are Constant.

$$\oint_c \left[\underbrace{(a_1x+a_2y+a_3)}_{P}dx+\underbrace{(b_1x+b_2y+b_3)}_{Q}dy\right]=(b_1-a_2)A$$

(b) Under What Conditions will the line Integral around any path C be zero.

$(b_1 - a_2)$

A) By Using Green Theorem.

$$\int_C [P dx + Q dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}(b_1x+b_2y+b_3) - \frac{\partial}{\partial y}(a_1x+a_2y+a_3)\right) dx\,dy$$

$b_1 - a_2$

$$= \iint_R (b_1-a_2)\,dx\,dy = (b_1-a_2)A$$

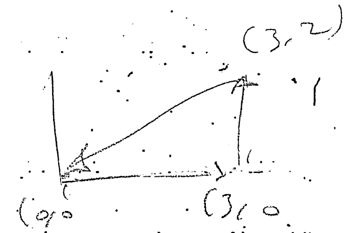
تمثل مساحة الشريحة = $dx\,dy$

$$\iint_R dx\, dy = A$$

B)

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Verify Green's theorem in the Plane for $\oint [(2x-y+4)dx + (5y+3x-6)dy]$ around a triangle in the xy plane with Vertices at $(0,0)$, $(3,0)$, $(3,2)$ traversed in a Counterwise direction.



By Using Green theorem.

$$\int [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

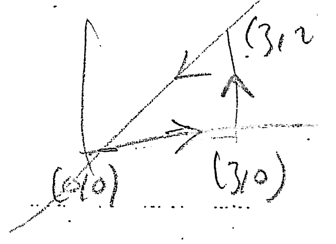
$$= \iint_R \left(\frac{\partial}{\partial x}(5y+3x-6) - \frac{\partial}{\partial y}(2x-y+4)\right) dy\,dx$$

$$= \iint_R (3+1)\,dy\,dx$$

$$= 4\int_0^3 \int_0^{\frac{2}{3}x} dy\,dx = 4\int_0^3 y\Big|_0^{\frac{2}{3}x} dx \qquad \frac{x_1 - x_2}{y_1 - y_2} =$$

$$= 4\int_0^3 \frac{2}{3}x\,dx = \frac{4}{3}x^2\Big|_0^3 = 12$$

$$\frac{x-0}{-3} = \frac{y-0}{-2} \Rightarrow 2x = 3y \qquad \therefore y = \frac{2}{3}x$$



$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

$y=0 \quad dy=0$

$x=3 \quad dy=0$

$$\int_0^3\int_0^2 (3-1)\,dx\,dy \qquad \int_0^3 y\,dy$$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2}$$

$$= \int_0^2\int_0^{2x} \qquad 12 = \int y$$

$$\frac{x-3}{0-3} = \frac{y-2}{0-2}$$

$x - 3 = +\frac{3}{2}y$

$x = \frac{3}{2}y$

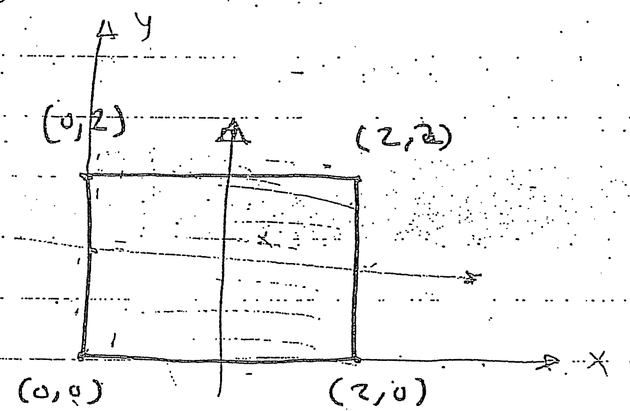
$$x - 3 = \frac{-3y+6}{-2}$$

$$= \iint \left(\frac{\partial}{\partial x}(y^2 - 2xy) - \frac{\partial}{\partial y}(x^2 - xy^3) \right) dx\,dy$$

$$= \int_0^2 \int_0^2 (-2y + 3xy^2)\, dx\, dy$$

$$= \int_0^2 \left(-2xy + \frac{3}{2}x^2y^2 \right) \Big|_0^2 dy$$

$$= \int_0^2 (-4y + 6y^2)\, dy = -2y^2 + 2y^3 \Big|_0^2 = 8$$



$$\int P\,dx + Q\,dy = \int_0^2 \int_0^2 (-2y + 3xy^2)\, dx\, dy$$

$$-2yx + 3\frac{x^2}{2}y^2$$

$$-2y(2) + 3\frac{(2)^2}{2}y^2$$

$$= -4y + 6y^2 \Big]_0^2$$

$$-\frac{4}{2}y^2 + \frac{6}{3}y^3$$

$$-2y^2 + 2y^3$$

$$-2(4) + 2(2)^3$$

$$= -8 + 16 = 8$$

Z.W

$$\frac{x-x_0}{x_0-x_1} = \frac{y-y_0}{y_0-y_1}$$

Verify Green's theorem in the plane for $\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ Where C is a Square with Vertices $(0,0), (2,0), (2,2), (0,2)$.

first Solution

along the Straight line from $(0,0)$ to $(2,0)$, $y=0$, $dy=0$ While x Varies from 0 to 2.

$$\int_0^2 [(x^2 - x(0))dx + (0^2 - 2x(0))0] = \int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3}$$

along the Straight line from $(2,0)$ to $(2,2)$, $x=2$, $dx=0$ While y Varies from 0 to 2.

$$\int_0^2 [(2^2 - 2y^3)(0) + (y^2 - 2(2)y)\,dy] = \int_0^2 (y^2 - 4y)\,dy = \frac{1}{3}y^3 - 2y^2\Big|_0^2 = -\frac{16}{3}$$

along the Straight line from $(2,2)$ to $(0,2)$, $y=2$, $dy=0$ While x Varies from 2 to 0

$$\int_2^0 [(x^2 - x(2)^3)dx + (2^2 - 2x(2))(0)] = \int_2^0 (x^2 - 8x)\,dx = \frac{1}{3}x^3 - 4x^2\Big|_2^0 = \frac{40}{8}$$

along the Straight line from $(0,2)$ to $(0,0)$, $x=0$, $dx=0$ While y Varies from 2 to 0

$$\int_2^0 [(0^2 - 0y^3)(0) + (y^2 - 2(0)y)\,dy] = \int_2^0 y^2\,dy = \frac{1}{3}y^3\Big|_2^0 = -\frac{8}{3}$$

the total Integral $= \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8$

By Using Green theorem

$$\int_C [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Z.W

Determine Whether the force field $F = 2xz\,i + (x^2 - y)\,j + (2z - x^2)\,k$ is Conservative or non Conservative.

[illegible] للتفكير

$$F = 2xz\,\vec{i} + (x^2 - y)\,\vec{j} + (2z - x^2)\,\vec{k}$$

$$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz & (x^2 - y) & (2z - x^2) \end{vmatrix}$$

$$\nabla \times F = \left(\frac{\partial}{\partial y}(2z - x^2) - \frac{\partial}{\partial z}(x^2 - y)\right)\vec{i} + \left(\frac{\partial}{\partial z}2xz - \frac{\partial}{\partial x}(2z - x^2)\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}(x^2 - y) - \frac{\partial}{\partial y}2xz\right)\vec{k}$$

$$\nabla \times F = (2x + 2x)\,\vec{j} + 2x\,\vec{k}$$

$$\nabla \times F = 4x\,\vec{j} + 2x\,\vec{k}$$

$\therefore$ non Conservative.

Z.W

$$d\phi = (x^3 + y^2x + z^2x)\,dx + (x^2y + y^3 + z^2y)\,dy + (x^2z + y^2z + z^3)\,dz$$

$$\phi = \frac{x^4}{4} + \frac{y^4}{4} + \frac{z^4}{4} + C$$

$$\phi = \frac{1}{4}(x^4 + y^4 + z^4) + C$$

$$\phi = \frac{1}{4}(x^2 + y^2 + z^2)^2 + C$$

$$\phi = \frac{1}{4}(r^2)^2 + C \longrightarrow \phi = \frac{r^4}{4} + C$$

or

$$\boxed{\nabla\phi = F} \quad (F \text{ Conservative field})$$

$$\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = r^2x\,\vec{i} + r^2y\,\vec{j} + r^2z\,\vec{k}$$

$$\frac{\partial\phi}{\partial x} = r^2x = (x^2 + y^2 + z^2)\,x \longrightarrow \phi = \frac{(x^2+y^2+z^2)^2}{4} + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = r^2y = (x^2 + y^2 + z^2)\,y \longrightarrow \phi = \frac{(x^2+y^2+z^2)^2}{4} + g(x,z)$$

$$\frac{\partial\phi}{\partial z} = r^2z = (x^2 + y^2 + z^2)\,z \longrightarrow \phi = \frac{(x^2+y^2+z^2)^2}{4} + h(x,y)$$

$$f(y,z) = g(x,z) = h(x,y) = C \qquad \nabla\phi = F$$

$$\therefore \phi = \frac{(x^2+y^2+z^2)^2}{4} + C$$

$$\phi = \frac{r^4}{4} + C$$

Z.W

$dQ = \int f\,dr$

Prove that $F = r^2 \vec{r}$ is Conservative and find the Scalar Poten

$F = r^2 r = r^2 x \vec{i} + r^2 y \vec{j} + r^2 z \vec{k}$

$$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^2 x & r^2 y & r^2 z \end{vmatrix}$$

$$\nabla \times F = \left[\frac{\partial}{\partial y} r^2 z - \frac{\partial}{\partial z} r^2 y\right]\vec{i} + \left[\frac{\partial}{\partial z} r^2 x - \frac{\partial}{\partial x} r^2 z\right]\vec{j}$$
$$+\left[\frac{\partial}{\partial x} r^2 y - \frac{\partial}{\partial y} r^2 x\right]\vec{k}$$

$$\nabla \times F = \left(2r\frac{\partial r}{\partial y} z - 2r\frac{\partial r}{\partial z} y\right)\vec{i} + \left(2r\frac{\partial r}{\partial z} x - 2r\frac{\partial r}{\partial x} z\right)\vec{j}$$
$$+\left(2r\frac{\partial r}{\partial x} y - 2r\frac{\partial r}{\partial y} x\right)\vec{k}$$

$\phi = \int f\,dr$

$$\nabla \times F = \left(2r\frac{y}{r} z - 2r\frac{z}{r} y\right)\vec{i} + \left(2r\frac{z}{r} x - 2r\frac{x}{r} z\right)\vec{j}$$
$$+\left(2r\frac{x}{r} y - 2r\frac{y}{r} x\right)\vec{k}$$

$$\nabla \times F = \underbrace{(2yz - 2zy)}_{=0}\vec{i} + \underbrace{(2zx - 2xz)}_{=0}\vec{j} + \underbrace{(2xy - 2yx)}_{=0}\vec{k} = 0$$

$\nabla \times F = 0$ $\therefore$ Conservative

$$F \cdot dr = \nabla\phi \cdot dr = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi$$

$$d\phi = r^2 x\,dx + r^2 y\,dy + r^2 z\,dz$$

$(x^2+y^2+z^2)^x$

$u \quad du$

$u = x \qquad x = 0 + \frac{\pi}{2}t \qquad dx$

Z.W $\quad dv = r^2$

$du = dx \qquad y = 1 + t \qquad dy$

$v = \frac{r^3}{3}$ * $\qquad z = -1 + 2t$

$= \nabla \times F$ ... $= \text{Curl} F$ ... Conservative

$0 = \nabla \times F$

(a) Prove that $F = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2 + 2)\vec{k}$ is a $\boxed{\text{conservative force field}}$.

(b) Find the $\boxed{\text{Scalar Potential}}$ for F. $\to F \cdot dr \Rightarrow \nabla\phi \cdot dr$

(c) Find the work done in moving an object in this field from from $(0,1,-1)$ to $(\pi/2, -1, 2)$.

(a)

$$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y^2\cos x + z^3) & (2y\sin x - 4) & (3xz^2 + 2) \end{vmatrix}$$

$$\nabla \times F = \left[\frac{\partial}{\partial y}(3xz^2+2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right]\vec{i} + \left[\frac{\partial}{\partial z}(y^2\cos x + z^3) - \frac{\partial}{\partial x}(3xz^2+2)\right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]\vec{k}$$

$$\nabla \times F = (0-0)\vec{i} + (3z^2 - 3z^2)\vec{j} + (2y\cos x - 2y\cos x)\vec{k}$$

$d\phi = F \cdot dr$

$\boxed{\nabla \times F = 0}$

$\therefore$ F is Conservative force field. $F = \nabla\phi$

(b) $F \cdot dr = \nabla\phi \cdot dr = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$

$d\phi = F \cdot d$

$d\phi = (y^2\cos x + z^3)dx + (2y\sin x - 4)dy + (3xz^2+2)dz$

$d\phi = (y^2\cos x\,dx + 2y\sin x\,dy) + (z^3dx + 3xz^2dz) - 4dy + 2dz$

$\phi = y^2\sin x + z^3x - 4y + 2z +$ Constant

or $x = 0 + \frac{\pi}{2}t$

$y = 1 - 1t$

$z = -1 + 2t$

(c) Workdone $= \phi(\pi/2, -1, 2) - \phi(0, 1, -1)$

$= (4\pi + 4 + 4) - (-1 - 4 - 2) = 4\pi + 15$

$dx =$

$dy =$

$dz =$

Z.W $\int F \cdot dr - \int F \cdot dr \quad \int d\phi = \phi$

$d\, d\phi\, d\phi$

$$\phi = 2x^2y - x^3z^2 + C$$

Anther method;

$$A = \nabla\phi$$

$$A \cdot dr = \nabla\phi \cdot dr = \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$A \cdot dr = d\phi$$

$$d\phi = (4xy - 3x^2z^2)\,dx + 2x^2dy - 2x^3z\,dz$$

$$d\phi = (4xy\,dx + 2x^2dy) + (-3x^2z^2dx - 2x^3z\,dz)$$

$$\phi = 2x^2y - x^3z^2 + C$$

$$A = \nabla\phi$$

$$A = \frac{\partial}{\partial x}\phi i + \frac{\partial}{\partial y}\phi j + \frac{\partial\phi}{\partial z}k$$

$$\frac{\partial\phi}{\partial x} = 4xy - 3x^2z^2$$

$$= \frac{4}{2}x^2y - \frac{3}{3}x^3z^2$$

$\nabla \times A = 0$
$\nabla \cdot A = 0$

(a) if $A = (4xy - 3x^2z^2)i + 2x^2j - 2x^3zK$, prove that $\int_C A \cdot dr$ is independent of path the Curve C joining two given point
(b) Show that there is a differentiable function ϕ such that $A = \nabla\phi$ and find it.

(a)

$$\nabla \times A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (4xy - 3x^2z^2) & 2x^2 & -2x^3z \end{vmatrix}$$

$$\nabla \times A = \left(-\frac{\partial}{\partial y} 2x^3z - \frac{\partial}{\partial z} 2x^2\right)\vec{i} + \left(\frac{\partial}{\partial z}(4xy - 3x^2z^2) + \frac{\partial}{\partial x} 2x^3z\right)\vec{j}$$
$$+ \left(\frac{\partial}{\partial x} 2x^2 - \frac{\partial}{\partial y}(4xy - 3x^2z^2)\right)\vec{k}$$

$$\nabla \times A = (0-0)\vec{i} + (-6x^2z + 6x^2z)\vec{j} + (4x - 4x)\vec{k}$$

$\frac{\partial p}{\partial y} = \frac{\partial v}{\partial x}$

$\nabla \times A = 0$ (conservation, independing path)
$\therefore A$ is Conservation field ... conservation
$\therefore \int A \cdot dr$ is independent of path.

(b)
$A = \nabla\phi$
$$(4xy - 3x^2z^2)i + 2x^2j - 2x^3z\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 4xy - 3x^2z^2, \quad \frac{\partial\phi}{\partial y} = 2x^2, \quad \frac{\partial\phi}{\partial z} = -2x^3z$$

$\phi = 2x^2y - x^3z^2 + f(y,z)$
$\phi = 2x^2y + g(x,z)$
$\phi = -x^3z^2 + h(x,y)$
$f(y,z) = 0, \quad g(x,z) = -x^3z^2, \quad h(x,y) = 2x^2y$

Z.W

(a) if $F = \nabla \phi$, where ϕ is single-value and has continu[ous] partial derivatives, show that the work done in moving a particale from one point $P_1 = (x_1, y_1, z_1)$ in this field to another point $P_2 = (x_2, y_2, z_2)$ is independent of the path joining the two po[ints].

(b) Conversely, if $\int F \cdot dr$ is independent of the path C joining any two points, show that there exists a function ϕ such that [$F = \nabla\phi$].

(a.)

$$\text{workdone} = \int_{P_1}^{P_2} F \cdot dr = \int_{P_1}^{P_2} \nabla \phi \cdot dr$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}\right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz\right)$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1) = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

the Integral depend only on points P_1 and P_2 and not the path joining [them].

(b.) $$\phi = \int_{(x_1, y_1, z_1)}^{(x, y, z)} F \cdot dr = \int_{(x_1, y_1, z_1)}^{(x, y, z)} F \cdot \frac{dr}{ds} ds$$

$\nabla \phi \, dr$

By differentation $\frac{d\phi}{ds} = F \cdot \frac{dr}{ds}$, But $\frac{d\phi}{ds} = \nabla\phi \cdot \frac{dr}{ds}$

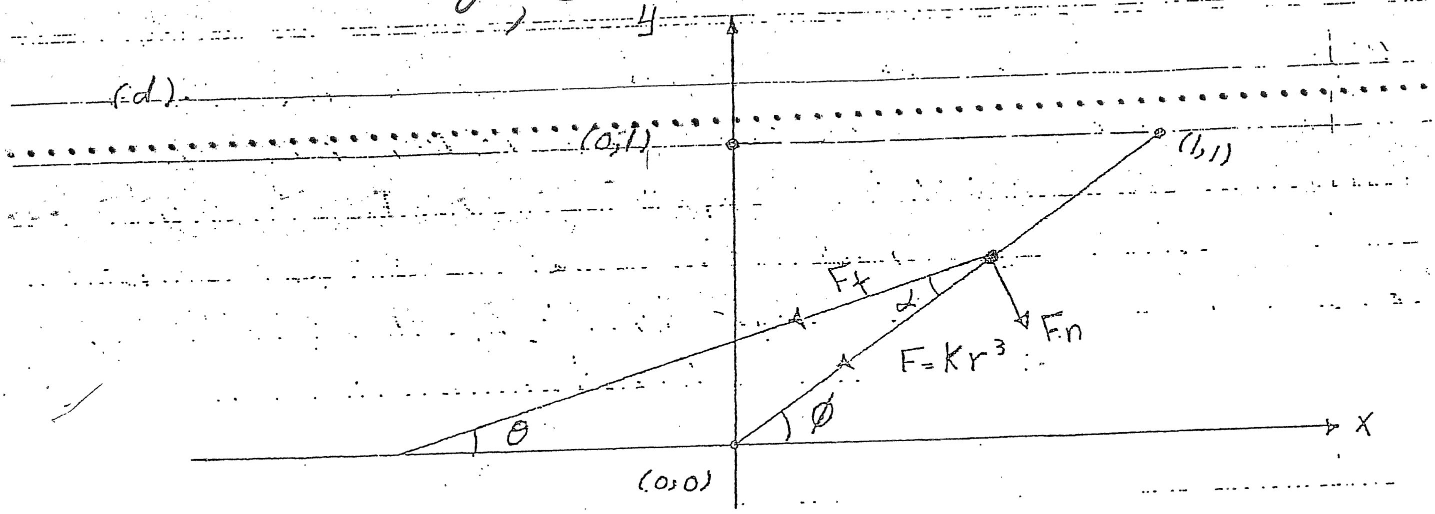
$$\nabla \phi \cdot \frac{dr}{ds} = F \cdot \frac{dr}{ds} \rightarrow (\nabla \phi - F)\frac{dr}{ds} = 0$$

$$\boxed{\nabla \phi = F}$$

$$\phi = \int_{P_1}^{P_2} F \cdot dr$$

Z.W

حل آخر



from (a)

$$w = Kr^2 \int [(x\,dx + y\,dy) + \mu (y\,dx - x\,dy)]$$

from (0,0) to (0,1), $x = 0$, $dx = 0$ While y varies from 0 to 1.

$$w_1 = Kr^2 \int_0^1 [y\,dy] = Kr^2 \frac{1}{2} y^2 \Big|_0^1 = \frac{1}{2} K r^2$$

from (0,1) to (1,1), $y = 1$, $dy = 0$, while x varies from 0 to 1

$$w_2 = Kr^2 \int_0^1 [x\,dx + \mu\,dx] = Kr^2 \left[\frac{1}{2} x^2 + \mu x\right]_0^1 = Kr^2 \left[\frac{1}{2} + \mu\right]$$

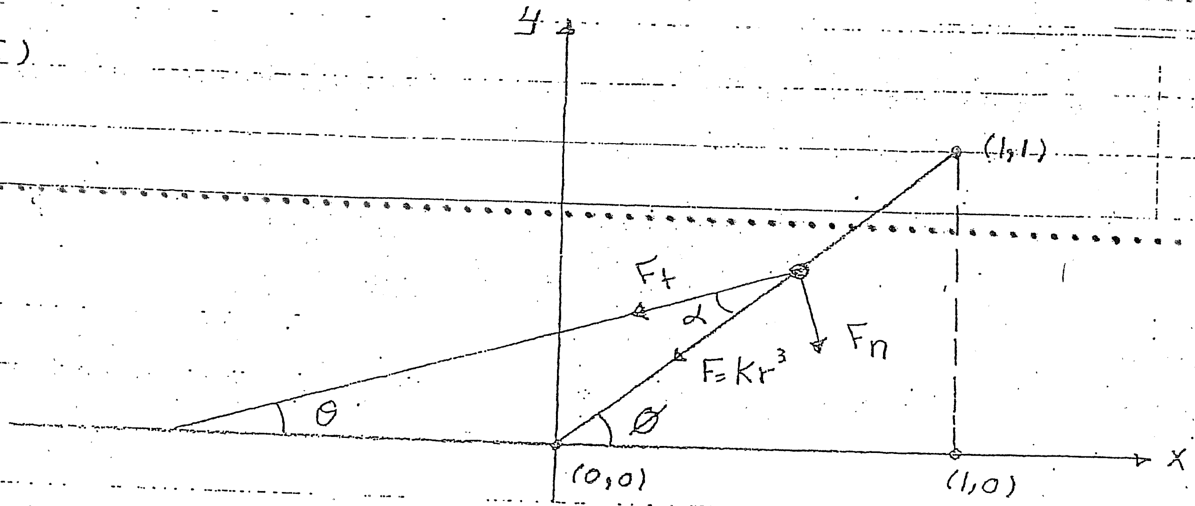
$$= \frac{1}{2} Kr^2 + Kr^2 \mu$$

$$w = \frac{1}{2} Kr^2 + \frac{1}{2} Kr^2 + Kr^2 \mu = Kr^2 + Kr^2 \mu$$

$$w = Kr^2 (1 + \mu)$$

Z.W

(C)



from (a).

$$w = Kr^2 \int [(x\,dx + y\,dy) + \mu(y\,dx - x\,dy)]$$

from (0,0) to (1,0), $y = 0$, $dy = 0$, While x Varies from 0 to 1.

$$w_1 = \int_0^1 Kr^2 (x\,dx) = Kr^2 \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2} K r^2$$

from (1,0) to (1,1), $x = 1$, $dx = 0$ While y Varies from 0 to 1.

$$w_2 = \int_0^1 Kr^2 (y\,dy + \mu(-dy)) = Kr^2 \int_0^1 (y\,dy - \mu\,dy)$$

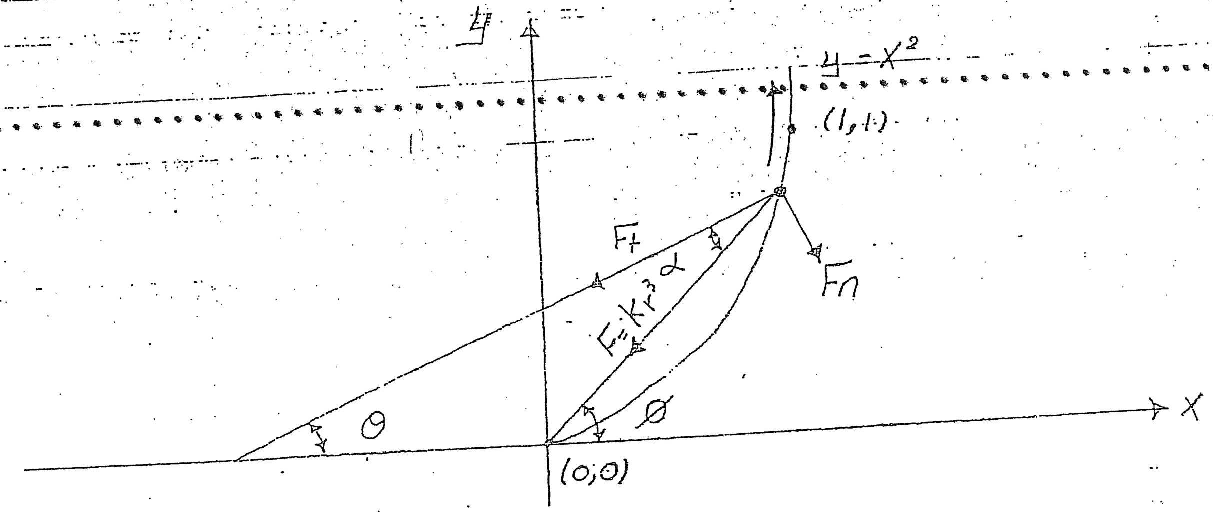
$$= Kr^2 \left[\frac{1}{2} y^2 - \mu y\right]_0^1 = Kr^2 \left[\frac{1}{2} - \mu\right]$$

$$w = \frac{1}{2} Kr^2 + \frac{1}{2} Kr^2 - Kr^2 \mu$$

$$w = Kr^2 - Kr^2 \mu = Kr^2 (1 - \mu)$$

Z.W

(b)



$$F_t = Kr^3\cos\alpha = Kr^3\cos(\phi-\theta)$$

$$F_f = \mu F_n = \mu Kr^3\sin\alpha = \mu Kr^3\sin(\phi-\theta)$$

$$w = Kr^2\int_0^1\left[r\cos(\phi-\theta) + \mu r\sin(\phi-\theta)\right]ds$$

$$w = Kr^2\int_0^1\left[(xdx+ydy)+\mu(ydx-xdy)\right]$$

$$y = x^2 \rightarrow dy = 2xdx$$

$$w_1 = Kr^2\int_0^1(xdx+ydy) = Kr^2\int_0^1(xdx+2x^3dx)$$

$$= Kr^2\left[\frac{1}{2}x^2+\frac{1}{2}x^4\right]_0^1 = Kr^2$$

$$w_2 = Kr^2\mu\int_0^1(ydx-xdy) = Kr^2\mu\int_0^1(x^2dx-2x^2dx)$$

$$= Kr^2\mu\int_0^1(-x^2)dx = -Kr^2\mu\frac{x^3}{3}\Big|_0^1 = -\frac{1}{3}Kr^2\mu$$

$$w = w_1+w_2 = Kr^2 - \frac{1}{3}Kr^2\mu = Kr^2\left(1-\frac{\mu}{3}\right)$$

Z.W

$$= \int K r^2 \left[r \cos(\phi - \theta)\, ds + \mu r \sin(\phi - \theta)\, ds \right]$$

$$= \int K r^2 \left[r(\cos\phi \cos\theta + \sin\phi \sin\theta)\, ds + \mu r(\sin\phi \cos\theta - \sin\theta \cos\phi)\, ds \right]$$

$$= \int K r^2 \left[(r\cos\phi \cos\theta\, ds + r\sin\phi \sin\theta\, ds) + \mu(r\sin\phi \cos\theta\, ds - r\cos\phi \sin\theta\, ds) \right]$$

But;

$r\cos\phi = x$, $r\sin\phi = y$

$\cos\theta\, ds = dx$, $\sin\theta\, ds = dy$

$$= \int K r^2 \left[(x\,dx + y\,dy) + \mu(y\,dx - x\,dy) \right]$$

$y = x \longrightarrow dy = dx$

$$\omega = K r^2 \int_0^1 \left[(x\,dx + x\,dx) + \mu \underbrace{\cancel{(x\,dx - x\,dy)}}_{0} \right]$$

$$\omega = K r^2 \int_0^1 2x\,dx = K r^2\, x^2 \Big|_0^1 = K r^2$$

Z.W

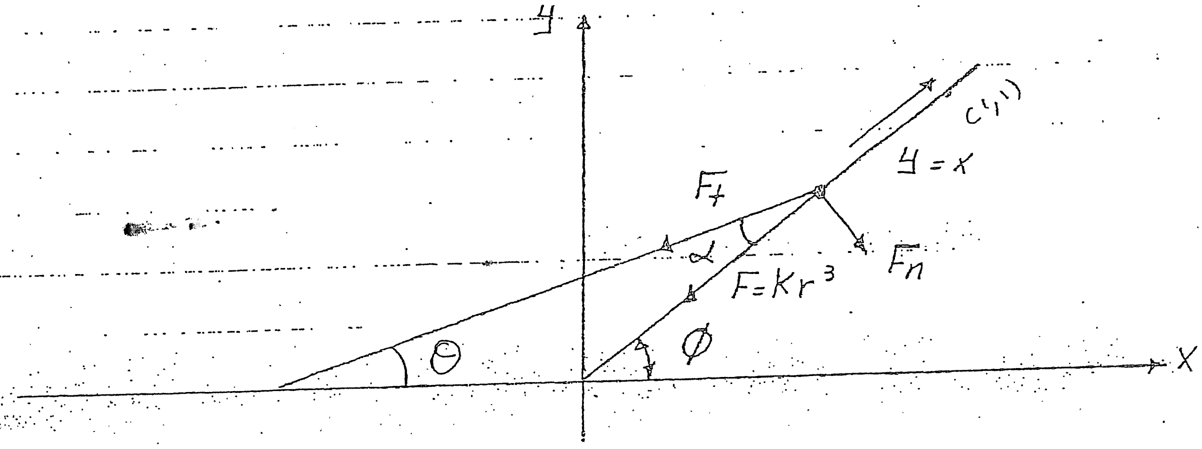
A particle is attracted toward the origin by a force proportional to the Cube of the distance from the origin. How much work is done in moving the particle from the origin to the point (1,1) if in each case the Coefficient of friction between the particle and the Path is μ and if motion takes place:

(a) Along $x = y$ (b) Along the path $y = x^2$

(c) Along the x-axis to (1,0) and then Vertically to (1,1)

(d) Along the y-axis to (0,1) and then horizontally to (1,1).

(a)



let;

- ϕ: the Angle between the force (F) and x-axis
- Θ: the Angle between the tangent force (F_t) and x-axis.
- α: the Angle between the force (F) and tangent force (F_t).

F_t: tangent force

F_n: normal force

F_f: friction force

from the exterior Angle theorem of plane geometery, $\alpha = \phi - \Theta$

$F_t = F\cos\alpha = Kr^3\cos(\phi - \Theta)$

$F_f = \mu F_n = \mu F\sin\alpha = \mu Kr^3\sin(\phi - \Theta)$

let w: Work done by the Particle

$$w = \int Kr^3\cos(\phi - \Theta)\,ds + \int \mu Kr^3\sin(\phi - \Theta)\,ds$$

Z.W

محسن نوري جبار

16/8/2016

$$W = \int K r^2 (x\,dx + y\,dy)$$

- from (0,0) to (1,0), $y = 0$, $dy = 0$, while x varies from 0 to 1

$$W_1 = \int_0^1 K r^2 (x\,dx + (0)(0)) = K r^2 \int_0^1 x\,dx = K r^2 \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} K r^2$$

from (1,0) to (1,1); $x = 1$, $dx = 0$, while y varies from 0 to 1

$$W_2 = \int_0^1 K r^2 ((1)(0) + y\,dy) = K r^2 \int_0^1 y\,dy = K r^2 \frac{y^2}{2}\Big|_0^1 = \frac{1}{2} K r^2$$

$$W = W_1 + W_2 = \frac{1}{2} K r^2 + \frac{1}{2} K r^2 = K r^2$$

$$\cancel{\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta}$$

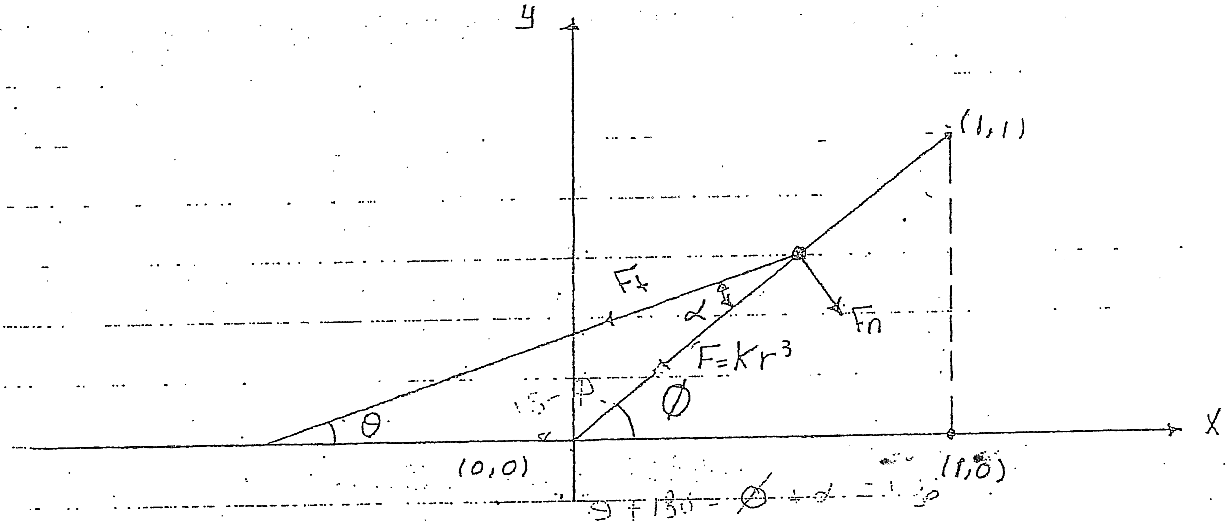
$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$

$$\sin(\phi \mp \theta) = \sin\phi\cos\theta \mp \sin\phi\cos\theta$$

Q Z.W

→ 2009

A particle is attracted toward the origin by a force proportional to Cube of the distance from the origin. What is the amount of work d in moving the particle from the origin to the point (1,1), along the x-axi to (1,0) and then Vertically to (1,1) Neglect friction.



let;

ϕ :- the Angle between the force ($F=kr^3$) and x-axis

θ :- the Angle between the tangent force (F_t) and x-axis

α :- the Angle between the force (F) and the tangent force (F_t).

F_t :- tangent force

F_n :- normal force may neglected because the friction is neglect.

$F_t = F\cos\alpha = kr^3\cos\alpha$

from the exterior Angle theorem in the plane geometry.

$\alpha = \phi - \theta$

let w :- the work done by Particle.

$$w = \int kr^3 \cos\alpha \, ds = \int kr^3 \cos(\phi - \theta)\, ds$$

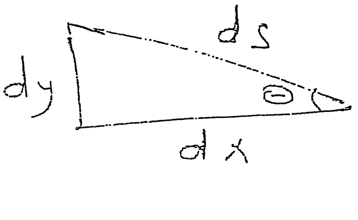
$$= \int kr^3 (\cos\phi \cos\theta + \sin\phi \sin\theta)\, ds$$

$$= \int kr^2 (\overset{x}{r\cos\phi}\, \underbrace{\cos\theta\, ds}_{dx} + \overset{y}{r\sin\phi}\, \underbrace{\sin\theta\, ds}_{dy})$$

But; $x = r\cos\phi$, $y = r\sin\phi$

$dx = \cos\theta\, ds$, $dy = \sin\theta\, ds$

Z.W



But;

$$r\cos\phi = x \quad ; \quad r\sin\phi = y$$

$$\cos\theta\, ds = dx \quad ; \quad \sin\theta\, ds = dy$$

$$w = \int_{0,1}^{1,2} K(x\,dx + y\,dy)$$

$$y = 1 + x^2 \rightarrow dy = 2x\,dx$$

$$w = \int_0^1 K(x\,dx + (1+x^2)2x\,dx)$$

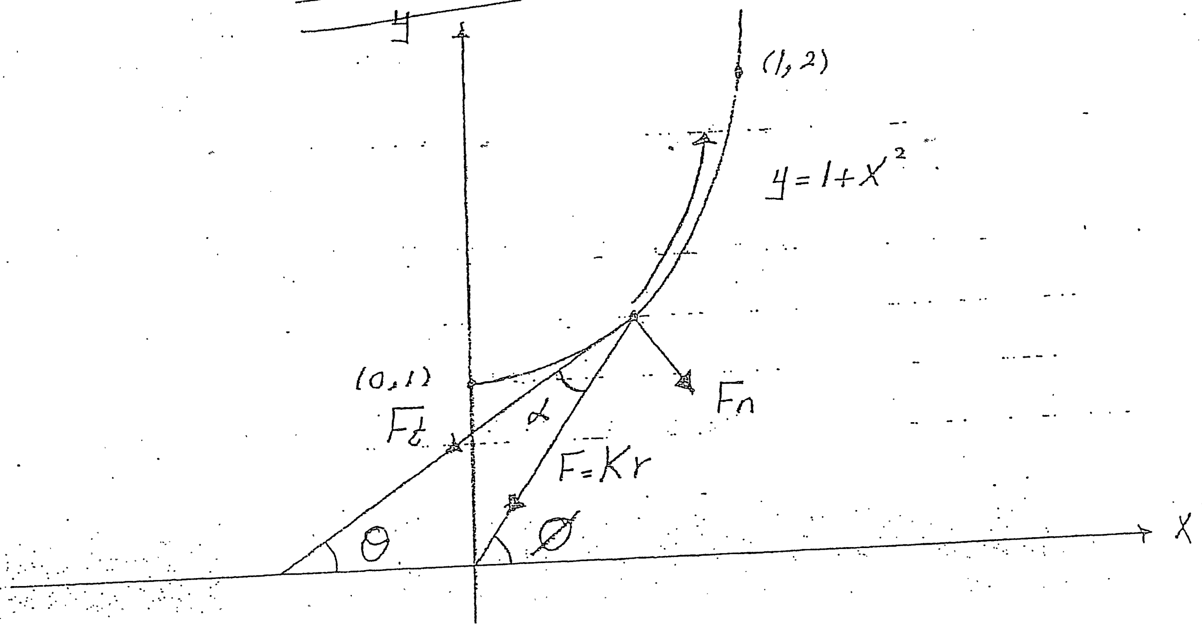
$$w = K\int_0^1 (x\,dx + 2x\,dx + 2x^3\,dx)$$

$$w = K\int_0^1 (3x\,dx + 2x^3\,dx) = K\left[\frac{3}{2}x^2 + \frac{1}{2}x^4\right]_0^1$$

$$= K\left(\frac{3}{2} + \frac{1}{2}\right) = \underline{2K}.$$

ينجذب باتجاه نقطة الأصل

A partical is attracted toward the origin by a force proportional to the distance "r" of the Particle from the origin. What is the work done when the Particle is moved from the Point (0,1) to the Point (1,2) along the Path $y = 1 + x^2$. Neglect friction.



let,

ϕ :- the Angle between the force ($F = Kr$) and x-axis.

θ :- the Angle between the tangent force F_t and x-axis.

α :- the Angle between the force F and tangent force

F_t :- tangent force

F_n :- Normal force may Neglected because the friction is neglected.

$\boxed{F_t = F\cos\alpha = Kr\cos\alpha}$

from the exterior Angle theorem of plane geometry, $\alpha = \phi - \theta$

let, w = work done by Particle

$$w = \int_{0,1}^{1,2} Kr\cos\alpha\, ds = \int_{0,1}^{1,2} Kr\cos(\phi - \theta)\, ds$$

$$= K\int_{0,1}^{1,2} r(\cos\phi\cos\theta + \sin\phi\sin\theta)\, ds$$

$$W = \int_{0,1}^{1,2} Kr\cos\alpha\, ds + \int_{0,1}^{1,2} Kr\mu \sin\alpha\, ds$$

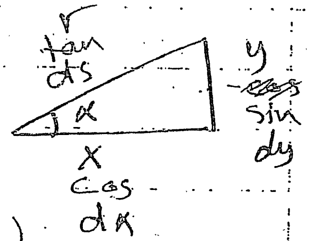
$$W = \int_{0,1}^{1,2} Kr\cos(\phi-\theta)\, ds + \int_{0,1}^{1,2} Kr\mu\sin(\phi-\theta)\, ds$$

$$W = \int_{0,1}^{1,2} Kr\,[\cos\phi\cos\theta + \sin\phi\sin\theta]\, ds \longrightarrow \text{جتا}$$

فرق الزوايا

$$+ \int_{0,1}^{1,2} Kr\mu\,[\underset{y}{\sin\phi}\underset{dx}{\cos\theta} - \cos\phi\sin\theta]\, ds \longrightarrow \text{جا}$$

$$\begin{cases} r\cos\phi = x \quad , \quad r\sin\phi = y \\ \cos\theta\, ds = dx \quad , \quad \sin\theta\, ds = dy \end{cases}$$



$$W = \int_{0,1}^{1,2} K(x\,dx + y\,dy) + \int_{0,1}^{1,2} K\mu(y\,dx - x\,dy)$$

$$y = 1 + x^2 \longrightarrow dy = 2x\,dx$$

$$* \int_{0,1}^{1,2} K(x\,dx + y\,dy) = K\int_0^1 (x\,dx + (1+x^2)\,2x\,dx)$$

$$= K\int_0^1 (x\,dx + 2x\,dx + 2x^3) = K\int_0^1 (3x\,dx + 2x^3\,dx)$$

$$= K\left[\frac{3}{2}x^2 + \frac{1}{2}x^4\right]_0^1 = K\left(\frac{3}{2} + \frac{1}{2}\right) = 2K$$

$$* \int_{0,1}^{1,2} K\mu(y\,dx - x\,dy) = K\mu\int_0^1 (dx + x^2dx - 2x^2dx) = K\mu\left[dx - x^2dx\right]_0^1$$

$$= K\mu\left[x - \frac{1}{3}x^3\right]_0^1 = K\mu\left[1 - \frac{1}{3}\right] = \frac{2}{3}K\mu$$

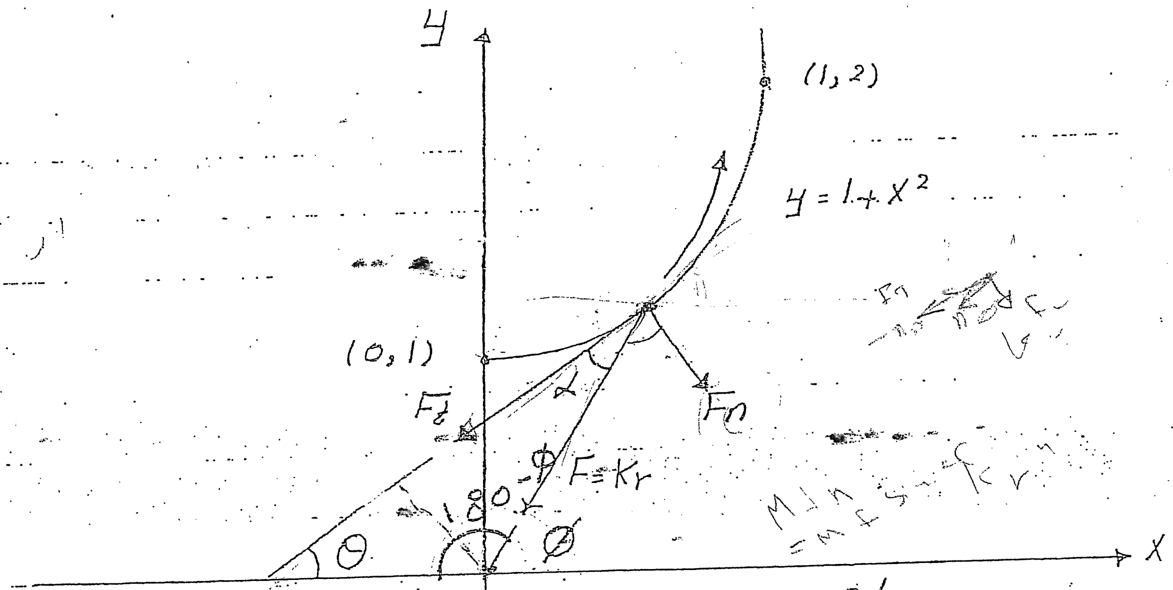
$$W = \underline{2K} + \frac{2}{3}K\mu$$

$$\frac{6}{3}$$

Z.W

$F = kr$

if a particle is attracted toward the origin by a force whose magnitude is proportional to the distance r of the particle from the origin, how much work is done when the particle is moved from the po. (0,1) to the Point (1,2) along the Path $y = 1+x^2$ assuming the Coeffici of friction μ between the Particle and the Path ?



$\theta + \alpha + 180 - \phi = 180$

$-\theta + \phi = \alpha$

let

ϕ :- the Angle between the force ($F = Kr$) and x-axis

θ :- the Angle between the tangent force F_t and x-axis

α :- the Angle between the the force (F) and tangent force (F_t)

F_t :- tangent force

F_n :- Normal force

F_f :- friction force

$F_t = F\cos\alpha = Kr\cos\alpha$

$F_f = \mu F_n = \mu F\sin\alpha = \mu Kr\sin\alpha$

$F_t = F\cos\alpha = Kr\cos\alpha$

$F_f = \mu F\sin\alpha = \mu Kr\sin\alpha$

$F = Kr$

from the exterior angle theorem of plan geometry, $\alpha = \phi - \theta$

let w: work done by the particle

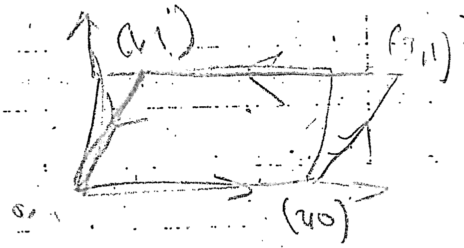
$$w = \int_{0,1}^{1,2} kr(\cos\alpha + \mu\sin\alpha)\,ds$$

$\frac{x-1}{-1} = \frac{y-1}{-1} \rightarrow x = y \rightarrow dx = dy$

$$\int_1^0 \left[3x^2dx + 2xdx - xdx - 3\cos x\,dx\right]$$

$$x^3 + x^2 - \frac{1}{2}x^2 - 3\sin x\Big|_1^0 = -1.5$$

the total Integral $= 8 + 17.5 - 30 - 1.5 = -6$



$y = 0 \quad dy = 0$

$$\int_0^2 3x^2\,dx$$

$$\int \quad \frac{x - x_2}{x_1 - x_2} = \frac{y - y_2}{y_2 - y_1}$$

Z.W

Evaluate $\oint (3x^2+2y)dx - (x+3\cos y)dy$ around the Parallelgram having Vertices at $(0,0)$, $(2,0)$, $(3,1)$ and $(-1,1)$.

along the Straight line from $(0,0)$ to $(2,0)$, $y=0$, $dy=0$ While x Varies from 0 to 2

$$\int_0^2 (3x^2+2(0))dx = \int_0^2 3x^2 dx = x^3\Big|_0^2 = 8$$

from $(2,0)$ to $(3,1)$, the Standard Equation of the line

$$\frac{x-x_0}{A} = \frac{y-y_0}{B} \longrightarrow \frac{x-2}{1} = \frac{y}{1}$$

$$y = x-2 \longrightarrow x = y+2 \longrightarrow dx = dy$$

$$\int_0^1 [3(y+2)^2 dy + 2y\,dy - (y+2)dy - 3\cos y\,dy]$$

$$\int_0^1 [3y^2dy + 12y\,dy + 12dy + 2y\,dy - y\,dy - 2dy - 3\cos y\,dy]$$

$$\int_0^1 [3y^2dy + 13y\,dy + 10dy - 3\cos y\,dy]$$

$$y^3 + \frac{13}{2}y^2 + 10y - 3\sin y\Big|_0^1 = 1 + \frac{13}{2} + 10 - 0 = 17.5$$

along the Straight line from $(3,1)$ to $(1,1)$, $y=1$, $dy=0$ Whil x Varies from 3 to 1

$$\int_3^1 (3x^2+2(1))dx = \int_3^1 (3x^2+2)dx = x^3+2x\Big|_3^1$$

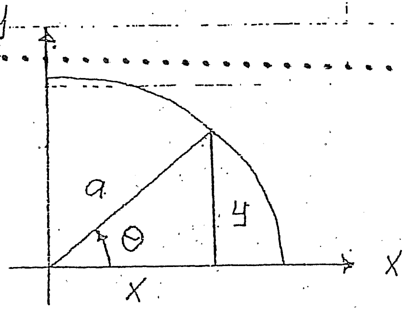
$$= (1-27) + (2-6) = -30$$

from $(1,1)$ to $(0,0)$, the Standard Equation of the line

7. W

Evaluate $\int x^2y^2\,ds$ around the Circle $x^2+y^2=1$

$$\boxed{ds=\sqrt{(dx)^2+(dy)^2}}$$



$$ds=\sqrt{(-a\sin\theta\, d\theta)^2+(a\cos\theta\, d\theta)^2}$$

$$ds=\sqrt{a^2\sin^2\theta\, d\theta^2+a^2\cos^2\theta\, d^2\theta}$$

$$ds=a\,d\theta\ ;\ \cancel{\sqrt{a^2 d\theta^2(\sin^2\theta+\cos^2\theta)}}$$

$x=a\cos\theta$

$y=a\sin\theta$

$dx=-a\sin\theta\, d\theta$

$dy=a\cos\theta\, d\theta$

$$\int_0^{2\pi} a^2\cos^2\theta\, a^2\sin^2\theta\, a\,d\theta$$

$$a^5\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$\frac{a^5}{4}\int_0^{2\pi}(1-\cos^2 2\theta)\,d\theta=\frac{a^5}{4}\left[\int_0^{2\pi}d\theta-\frac{1}{2}\int_0^{2\pi}(1+\cos 4\theta)\,d\theta\right]$$

$$=\frac{a^5}{4}\left[\theta-\frac{1}{2}\theta-\frac{1}{8}\sin 4\theta\right]_0^{2\pi}$$

$$=\frac{a^5}{4}\left[(2\pi-\pi-0)-(0-0-0)\right]=\frac{\pi a^5}{4}$$

$$\int x^2y^2\,ds=\frac{\pi a^5}{4}$$

$x=a\cos\alpha$

$dx=-a\sin\alpha\, d\alpha$

$y=a\sin\alpha$

$dy=a\cos\alpha\, d\alpha$

$$ds=\sqrt{a^2\sin^2\alpha\, d\alpha^2+a^2\cos^2\alpha\, d\alpha^2}$$

$$=\sqrt{a^2 d\alpha^2(1)}=a\,d\alpha$$

$$\int a^2\cos^2\alpha\, a^2\sin^2\alpha\, a\,d\alpha \quad a^5\int\cos^2\alpha\sin^2\alpha\, d\alpha$$

$$a^5\int\left(\frac{1+\cos 2\alpha}{2}\right)\left(\frac{1-\cos 2\alpha}{2}\right)d\alpha=\frac{a^5}{4}(1-\cos^2 2\alpha)$$

Z.W

$$= \frac{1}{2}\int_0^{\pi/2} (d\theta + \cos 2\theta \, d\theta) = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = \frac{\pi}{4}$$

$$\int_0^1 \frac{x^2}{(1-x^2)^{1/2}}\,dx = \int_0^{\pi/2} \frac{\sin^2\theta}{(1-\sin^2\theta)^{1/2}} \cos\theta \, d\theta = \int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta} \cos\theta \, d$$

$$= \int_0^{\pi/2} \sin^2\theta \, d\theta = \int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}-0\right) - (0-0)\right] = \frac{\pi}{4}$$

$$\int_c A \cdot dr = \frac{\pi}{4} + \int_0^1 2x\,dx - \frac{\pi}{4} = \int_0^1 2x\,dx = x^2\Big|_0^1 = 1$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Z. W

معادلة دائرة

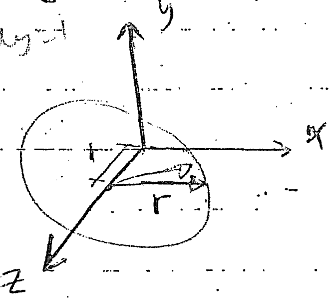
Evaluate $\int_c A\cdot dr$ along the Curve $\boxed{x^2+y^2=1}$, $z=1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$ if $A=(yz+2x)i+xzj+(y+$

$$\vec r = x\vec i + y\vec j + z\vec k \longrightarrow dr = dx\vec i + dy\vec j + dz\vec k$$

$$\int_c A\cdot dr = \left[(yz+2x)\vec i + xz\vec j + (xy+2z)\vec k\right]\cdot(dx\vec i + dy\vec j + dz\vec k)$$

$$\int_c A\cdot dr = \int_c \left[(yz+2x)\,dx + xz\,dy + (xy-2z)\,dz\right]$$

But $z=1$, $dz=0$



$$\int_c \left[(y+2x)\,dx + x\,dy\right]$$

$$x^2+y^2=1 \longrightarrow \boxed{y=(1-x^2)^{1/2}}$$

Polar يحول إلى

$$dy = \frac{1}{2}(1-x^2)^{-1/2}\,2x\,dx = -x(1-x^2)^{-1/2}dx$$

$$\int_0^1 \left[(1-x^2)^{1/2}dx + 2x\,dx \quad \frac{x^2}{(1-x^2)^{1/2}}dx\right]$$

$$\int_0^1 (1-x^2)^{1/2}dx$$

التكامل بطريقة المثلثات

let $x=\sin\theta \longrightarrow dx=\cos\theta\,d\theta$

$\theta=\sin^{-1}x$

$\theta_1=\sin^{-1}0=0$, $\theta_2=\sin^{-1}1=\pi/2$

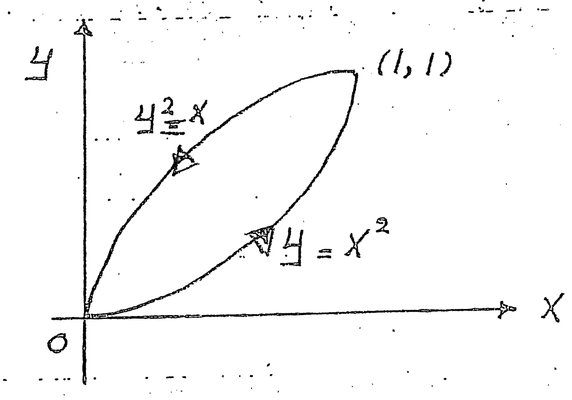
$$\int_0^{\pi/2}(1-\sin^2\theta)^{1/2}\cos\theta\,d\theta = \int_0^{\pi/2}\cos^2\theta\,d\theta = \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

Z.W

2

Evaluate $\oint_C A \cdot dr$ around the Closed Curve C of figure if

$$A = (x-y)i + (x+y)j$$



$$r = x\vec{i} + y\vec{j} \longrightarrow dr = dx\vec{i} + dy\vec{j}$$

$$\int A \cdot dr = \int [(x-y)dx + (x+y)dy]$$

along the Curve $y = x^2 \longrightarrow dy = 2x\,dx$

$$\int_0^1 [(x-x^2)dx + (x+x^2)2x\,dx] = \int_0^1 [x\,dx - x^2dx + 2x^2dx + 2x^3dx]$$

$$\int_0^1 [x\,dx + x^2dx + 2x^3dx] = \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{2}x^4 \Big|_0^1 = \frac{4}{3}$$

along the Curve $y^2 = x$ ($y = x^{1/2}$) from (1,1) to (0,0)

$$dy = \frac{1}{2}x^{-1/2}dx$$

$$\int_1^0 [(x - x^{1/2})dx + (x + x^{1/2})\frac{1}{2}x^{-1/2}dx]$$

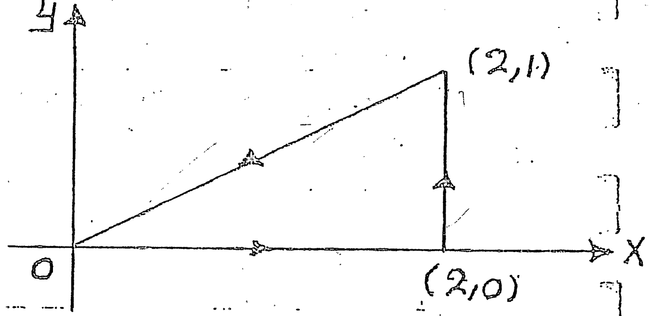
$$\int_1^0 [x\,dx - x^{1/2}dx + \frac{1}{2}x^{1/2}dx + \frac{1}{2}dx]$$

$$\frac{1}{2}x^2 - \frac{2}{3}x^{3/2} + \frac{1}{3}x^{3/2} + \frac{1}{2}x \Big|_1^0 = -\frac{2}{3}$$

$$\int A \cdot dr = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

Z.W

if $F=(2x+y^2)i+(3y-4x)j$ Evaluate $\oint_c F.dr$ around the triangle C of figure (a) in the indicated direction (b) opposite to the indicate direction



$F=(2x+y^2)\vec{i}+(3y-4x)\vec{j}$

$r=x\vec{i}+y\vec{j} \rightarrow dr=dx\vec{i}+dy\vec{j}$

$$\int_c F\cdot dr = \int_c \left[(2x+y^2)dx+(3y-4x)dy\right]$$

along the Straight line from (0,0) to (2,0), $y=0$, $dy=0$
While x Varies from 0 to 2

$$\int_0^2 \left[(2x+0)dx+(3(0)-4x)0\right]=\int_0^2 2x\,dx = x^2\Big|_0^2 = 4$$

along the Straight line from (2,0) to (2,1), $x=2$, $dx=0$
While y Varies from 0 to 1.

$$\int_0^1 \left[(2(2)+y^2)(0)+(3y-4(2))dy\right]=\int_0^1 (3y-8)dy = \frac{3}{2}y^2-8y\Big|_0^1=-\frac{13}{2}$$

along the Straight line from (2,1) to (0,0)
the Standard Eq. of line $\frac{x-x_0}{A}=\frac{y-y_0}{B} \rightarrow \frac{x-2}{-2}=\frac{y-1}{-1}$

$x-2=2(y-1) \rightarrow x-2=2y-2$

$x=2y \rightarrow dx=2dy$

$$\int_1^0 \left[(4y+y^2)2dy+(3y-8y)dy\right]=\int_1^0 \left[8ydy+2y^2dy-5ydy\right]dy$$

$$=\int_1^0 \left[3ydy+2y^2dy\right]=\frac{3}{2}y^2+\frac{2}{3}y^3\Big|_1^0=-\frac{13}{6}$$

$$\therefore \oint F\cdot dr = 4-\frac{13}{2}-\frac{13}{6}=-\frac{14}{3}$$

$$\int_0^1 [48t^5 dt + 16t^4 dt + 28t^3 dt \; 12t^2 dt]$$

$$8t^6 + \frac{16}{5}t^5 + 7t^4 - 4t^3 \Big|_0^1 = 14.2$$

(C.)

$$x^2 = 4y \rightarrow y = \frac{1}{4}x^2 \rightarrow dy = \frac{1}{2}x\,dx$$

$$3x^3 = 8Z \rightarrow Z = \frac{3}{8}x^3 \rightarrow dZ = \frac{9}{8}x^2 dx$$

$$\int_0^2 [3x^2 dx + (2x\frac{3}{8}x^3 - \frac{1}{4}x^2)\frac{1}{2}x\,dx + \frac{3}{8}x^3\frac{9}{8}x^2 dx]$$

$$\int_0^2 [3x^2 dx + \frac{3}{8}x^5 dx - \frac{1}{8}x^3 dx + \frac{27}{64}x^5 dx]$$

$$x^3 + \frac{1}{16}x^6 - \frac{1}{32}x^4 + \frac{27}{384}x^6 \Big|_0^2 = 16$$

Z.W

Find the work done in moving a particle in the force field
$F = 3x^2 i + (2xz - y) j + z k$ along

(a) the Straight line from $(0,0,0)$ to $(2,1,3)$

(b) the Space Curve $x = 2t^2,\ y = t,\ z = 4t^2 - t$ from $t=0$ to $t=1$

(c) the Curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$

$$\text{Work done} = \int_C F \cdot dr$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \longrightarrow d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\int_C F \cdot dr = \int_C \left[3x^2\,dx + (2xz - y)\,dy + z\,dz\right]$$

$x = x_0 + Ax$

work done $= \int F \cdot dr$

(a) the Parametric Equation of line

$$x = x_0 + At \longrightarrow x = 2t \longrightarrow dx = 2dt$$
$$y = y_0 + Bt \longrightarrow y = t \longrightarrow dy = dt$$
$$z = z_0 + Ct \longrightarrow z = 3t \longrightarrow dz = 3dt$$

$= x = x_0 + At$

$x = 0 + 2t$

$$\int_0^1 \left[3(2t)^2 2dt + (2 \times 2t \times 3t - t)\,dt + (3t)3dt\right]$$

$$\int_0^1 \left[24t^2 dt + 12t^2 dt - t\,dt + 9t\,dt\right] = \int_0^1 \left[36t^2 + 8t\right] dt$$

$$\int_0^1 36t^2 dt + \int_0^1 8t\,dt = 12t^3 + 4t^2 \Big|_0^1 = 16$$

(b) $x = 2t^2$, $y = t$, $z = 4t^2 - t$

$dx = 4t\,dt$, $dy = dt$, $dz = (8t - 1)\,dt$

$$\int_0^1 \left[3(2t^2)^2 4t\,dt + (2 \times 2t^2 \times (4t^2 - t) - t)\,dt + (4t^2 - t)(8t - 1)\,dt\right]$$

$$\int_0^1 \left[48t^5 dt + 16t^4 dt - 4t^3 dt - t\,dt + 32t^3 dt - 4t^2 dt - 8t^2 dt + t\,dt\right]$$

if $F = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$, evaluate $\int_C F.dr$ along the Curve C in the xy plane, $y = x^3$ from the point $(1,1)$ to $(2,8)$

$$F = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j} \quad , \quad dr = dx\,\vec{i} + dy\,\vec{j}$$

$$\int F.dr = \int_c \left[(5xy - 6x^2)dx + (2y - 4x)dy\right]$$

$$y = x^3 \rightarrow dy = 3x^2 dx$$

$$\int_1^2 \left[(5x^4 - 6x^2)dx + (2x^3 - 4x)\,3x^2 dx\right]$$

$$\int_1^2 \left[5x^4 dx - 6x^2 dx + 6x^5 dx - 12x^3 dx\right]$$

$$x^5 - 2x^3 + x^6 - 3x^4 \Big|_1^2 = (32) - (-3) = 35$$

along the Straight line from (0,1,1) to (2,1,1), $y=1$, $dy=0$
$z=1$; $dz=0$ while x varies from 0 to 2.

$$\int_0^2 \left[(2(1)+3)\,dx + x(1)(0) + ((1)(1)-x)(0)\right]$$

$$\int_0^2 5\,dx = 5x\Big|_0^2 = 10$$

$$\int A\cdot dr = 10$$

c) the Parametric equation of the line.

$$x = x_0 + At \rightarrow x = 2t \rightarrow dx = 2\,dt$$

$$y = y_0 + Bt \rightarrow y = t \rightarrow dy = dt$$

$$z = z_0 + Ct \rightarrow z = t \rightarrow dz = dt$$

$$\int_0^1 \left[(2t+3)2\,dt + 2t^2\,dt + (t^2-2t)\,dt\right]$$

$$\int_0^1 \left[4t\,dt + 6\,dt + 2t^2\,dt + t^2\,dt - 2t\,dt\right]$$

$$\int_0^1 \left[2t\,dt + 3t^2\,dt + 6\,dt\right] = t^2 + t^3 + 6t\Big|_0^1 = 8$$

اذا وصلنا (0,0,0) الى (2,1,1) كل متغير يتغير شكل $x = 2y$
مختلفة عن الآخر (x: 0 → 2) (y: 0 → 1) (z: 0 → 1)

$(2y+3)\,2\,dy + (2y \cdot y)\,dy + (y^2 - 2y)\,dy$

[illegible] Parametric [illegible]

[illegible]

[illegible]

[illegible] (0,0,0) [illegible] (1,1,1) [illegible]

Z.W

[illegible] (0,1) [illegible]

If $\vec{A} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$, evaluate $\int_C A \cdot dr$ along the following Paths C:

(a) $x = 2t^2$, $y = t$, $z = t^3$ from $t=0$ to $t=1$

(b) the Straight lines from (0,0,0) to (0,0,1) then to (0,1,1) and then to (2,1,1)

(c) the Straightline joining (0,0,0) and (2,1,1).

$$\int F \cdot dr = \int_C [(2y+3)dx + xz\,dy + (yz-x)dz]$$

a- $x = 2t^2$, $y = t$, $z = t^3$

$dx = 4t\,dt$, $dy = dt$, $dz = 3t^2 dt$

$$\int_0^1 [(2t+3)4t\,dt + 2t^5 dt + (t^4 - 2t^2)3t^2 dt]$$

$$\int_0^1 [8t^2 dt + 12t\,dt + 2t^5 dt + 3t^6 dt - 6t^4 dt]$$

$$\frac{8}{3}t^3 + 6t^2 + \frac{1}{3}t^6 + \frac{3}{7}t^7 - \frac{6}{5}t^5 \Big|_0^1 = \frac{288}{35}$$

b. $\int F \cdot dr = \int_C [(2y+3)dx + xz\,dy + (yz-x)dz]$

along the Straight line from (0,0,0) to (0,0,1), $x=0$, $dx=0$, $y=0$, dy

While z varies from 0 to 1.

$$\int_0^1 [(2(0)+3)(0) + (0)z(0) + ((0)z - 0)dz] = 0$$

along the Straightline from (0,0,1) to (0,1,1), $x=0$, $dx=0$, $z=1$, dz

While y varies from 0 to 1.

$$\int_0^1 [(2y+3)(0) + (0)(1)dy + (y(1) - 0)(0)] = 0$$

$$= \frac{6}{4} z^{4} - \frac{8}{5} z^{5/2} + \frac{1}{2} z + \frac{2}{3} z^{3/2} - \frac{1}{5} z^{5} \Big|_{0}^{1} = \frac{13}{15}$$

$$\int F \cdot dr = \frac{13}{15}$$

C along the Straightline from (0,0,0) to (0,1,0)
$x=0, dx=0, z=0, dz=0$, While y Varies from 0 to 1

$$\int_c F.dr = \int_c [(3x-2y)dx + (y+2z)dy - x^2dz]$$

$$\int_0^1 [(3(0)-2y)(0) + (y+2(0))dy - (0)^2(0)]$$

$$\int_0^1 y\,dy = \frac{1}{2}y^2\Big|_0^1 = \boxed{\frac{1}{2}}$$

along the Straightline from (0,1,0) to (0,1,1), $x=0, dx=0$
$y=1, dy=0$ While Z Varies from 0 to 1

$$\int_0^1 [(3(0)+2(1))(0) + (1+2z)(0) - (0)^2dz] = 0$$

along the Straightline from (0,1,1) to (1,1,1), $y=1, dy=0, z=1, dz=0$
While X Varies from 0 to 1

$$\int_0^1 [(3x-2)dx + (1+2(1))(0) - x^2(0)] = \int_0^1 [(3x-2)dx] = \int_0^1 3x\,dx - \int_0^1 2\,dx$$

$$= \frac{3}{2}x^2 - 2x\Big|_0^1 = \boxed{-\frac{1}{2}}$$

$$\boxed{\int F.dr = \frac{1}{2} + 0 - \frac{1}{2} = 0}$$ مجموع المسارات الثلاثة

d. $x = z^2 \rightarrow dx = 2z\,dz$
$z = y^2 \rightarrow y = z^{1/2} \rightarrow dy = \frac{1}{2}z^{-1/2}dz \rightarrow \boxed{dy = \frac{dz}{2\sqrt{z}}}$

$$\int_0^1 [(3z^2 - 2z^{1/2})2z\,dz + (z^{1/2} + 2z)\frac{1}{2}z^{-1/2}dz - z^4dz]$$

$$\int_0^1 [6z^3dz - 4z^{3/2}dz + \frac{1}{2}dz + z^{1/2}dz - z^4dz]$$

$\sqrt{z} \times \sqrt{z} = z$ $\frac{z}{\sqrt{z}}$ $z^{1/2}$

$\sqrt{z} \cdot \sqrt{z}$
$\sqrt{z}$

Z.W

عند ما يكون straight line

مهملة في الإحداثيات

if $F = (3x - 2y)i + (y + 2z)j - x^2 k$ evaluate $\int_c F \cdot dr$ from $(0,0,0)$ to $(1,1,1)$ where C is a path consisting of (a) the curve $x = t$, $y = t^2$, $z = t^3$ (b) a straight line joining these points, (c) the straight lines from $(0,0,0)$ to $(0,1,0)$, then to $(0,1,1)$ and then to $(1,1,1)$ (d) the curve $x = z^2$, $\boxed{z = y^2}$

يكون به $y = \sqrt{z}$

$$F = (3x - 2y)\vec{i} + (y + 2z)\vec{j} - x^2\vec{k} \quad , \quad dr = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\int F \cdot dr = \int [(3x - 2y)dx + (y + 2z)dy - x^2 dz]$$

a- $x = t$, $y = t^2$, $z = t^3$

$dx = dt$, $dy = 2t\,dt$, $dz = 3t^2 dt$

$$\int F \cdot dr = \int_0^1 [(3t - 2t^2)dt + (t^2 + 2t^3)2t\,dt - t^2\, 3t^2 dt]$$

$$= \int_0^1 [3t\,dt - 2t^2 dt + 2t^3 dt + 4t^4 dt - 3t^4 dt]$$

$$= \int_0^1 [3t\,dt - 2t^2 dt + 2t^3 dt + t^4 dt]$$

$$= \frac{3}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{2}t^4 + \frac{1}{5}t^5 \Big|_0^1 = \frac{3}{2} - \frac{2}{3} + \frac{1}{2} + \frac{1}{5} =$$

b- the straight line from $(0,0,0)$ to $(1,1,1)$

the $\boxed{\text{Standard Equation of line}}$, $\dfrac{x - x_0}{A} = \dfrac{y - y_0}{B} = \dfrac{z - z_0}{C}$

متشابهات (تغير واحد في كل متغير)

$x = y = z$, $dx = dy = dz$

$$\int F \cdot dr = \int_0^1 [(3x - 2x)dx + (x + 2x)dx - x^2 dx]$$

$$= \int_0^1 [x\,dx + x\,dx + 2x\,dx - x^2 dx] = \int_0^1 (4x\,dx - x^2 dx)$$

$$= 2x^2 - \frac{1}{3}x^3 \Big|_0^1 = \frac{5}{3}$$

Z.W

Evaluate $\int_c (2x+y)\,ds$, where C is the Curve in the xy-plane given by $x^2+y^2=25$ and ds is the arc length parameter, from the point (3,4) to (4,3) along the Shortest path.

$$* \quad ds=\sqrt{dx^2+dy^2}=\sqrt{dx^2(1+dy^2/dx^2)}=dx\sqrt{1+(dy/dx)^2}$$

$$x^2+y^2=25 \rightarrow 2x+2y\frac{dy}{dx}=0 \rightarrow -2x=-2y\frac{dy}{dx}$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

$$x^2+y^2=25 \rightarrow y^2=25-x^2 \rightarrow y=\sqrt{25-x^2}$$

$$ds=dx\sqrt{1+(-x/y)^2}=dx\sqrt{1+x^2/y^2}$$

$$\int_c (2x+y)\,ds=\int (2x+\sqrt{25-x^2})\sqrt{1+x^2/y^2}\,dx$$

$$=\int_c 2x\sqrt{1+x^2/y^2}\,dx+\int_c \sqrt{25-x^2}\sqrt{1+x^2/y^2}\,dx$$

$y^2=25-x^2$ $\quad \frac{5}{y}$

$$=\int_c 2x\sqrt{(y^2+x^2)/y^2}\,dx+\int_c \sqrt{25-x^2}\sqrt{(y^2+x^2)/y^2}\,dx$$

$x^2+y^2=25 \rightarrow \sqrt{25}=5$

$$=\int_3^4 2x\frac{5}{\sqrt{25-x^2}}\,dx+\int_3^4 \cancel{\sqrt{25-x^2}}\,\frac{5}{\cancel{\sqrt{25-x^2}}}\,dx$$

$$=5\int_3^4 (25-x^2)^{-1/2}(2x)+5\int_3^4 dx=-10(25-x^2)^{1/2}+5x\Big|_3^4$$

Z.W

$\int (x^2 - y^2)\,dx + \int 2xy\,dy$ $\qquad y = x^2 - x$

$dy = (2x-1)\,dx$

(a) if $F = (x^2 - y^2)\vec{i} + 2xy\,\vec{j}$ evaluate $\int F.dr$ along the Curve C in the xy plane given by $y = x^2 - x$ from the point (1, 0) to (2, 2)
(b) Interpret physically the result obtained.

$$F = (x^2 - y^2)\,\vec{i} + 2xy\,\vec{j} \quad , \quad dr = dx\,\vec{i} + dy\,\vec{j}$$

$$\int F.dr = \int \left[(x^2 - y^2)\vec{i} + 2xy\,\vec{j}\right] . (dx\,\vec{i} + dy\,\vec{j})$$

$$= \int \left[(x^2 - y^2)\,dx + 2xy\,dy \right]$$

$$y = x^2 - x \longrightarrow dy = (2x-1)\,dx$$

مربع ثنائي

$$\int F.dr = \int_1^2 \left[x^2 dx - \boxed{(x^2 - x)^2} dx + 2x(x^2 - x)(2x-1)\,dx \right]$$

[illegible]

$$= \int_1^2 \left[\cancel{x^2 dx} - x^4 dx + 2x^3 dx - \cancel{x^2 dx} + (2x^3 - 2x^2)(2x-1)\,dx \right]$$

$$= \int_1^2 \left[2x^3 dx - x^4 dx + 4x^4 dx - 2x^3 dx - 4x^3 dx + 2x^2 dx \right]$$

$$= \int_1^2 \left[2x^2 dx + 3x^4 dx - 4x^3 dx \right]$$

$$= \frac{2}{3}x^3 + \frac{3}{5}x^5 - x^4 \Big|_1^2 = 8\frac{4}{15}$$

~~Torque ... center force~~

$$\int_2^0 \left[(2y+4)\frac{3}{2}dy + (5y + \frac{9}{2}y - 6)dy \right]$$

$$\int_2^0 \left[3y\,dy + 6\,dy + 5y\,dy + 4.5\,dy - 6\,dy \right]$$

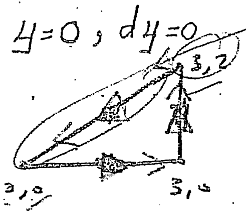
$$\int_2^0 \left[12.5\,y\,dy \right] = 12.5\,\frac{y^2}{2}\Big|_2^0 = -25$$

$$\int_c \left[(2x - y + 4)dx + (5y + 3x - 6)dy \right] = 21 + 16 - 25 = \boxed{12}$$

Z.W

Evaluate $\oint [(2x-y+4)dx + (5y+3x-6)dy]$ around a triangle in the xy-plane with Vertices at $(0,0)$, $(3,0)$, $(3,2)$ traversed in a Counter Clockwise direction.

- along the Straight line from $(0,0)$ to $(3,0)$ $y=0$, $dy=0$ While x Varies from 0 to 3.



$$\int_0^3 [(2x-0+4)dx + (5(0)+3x-6)(0)]$$

$$\int_0^3 [(2x+4)dx] = \int_0^3 2x\,dx + 4\int_0^3 dx = x^2 + 4x\Big|_0^3 = 21$$

- along the Straight line from $(3,0)$ to $(3,2)$, $x=3$, $dx=0$ while y Varies from 0 to 2.

$$\int_0^2 [(2(3)-y+4)(0) + (5y+3(3)-6)dy]$$

$$\int_0^2 [(5y+3)dy] = \frac{5}{2}y^2 + 3y\Big|_0^2 = 10+6 = 16$$

- along the Straight line from $(3,2)$ to $(0,0)$.

the Standard Equation of line $\frac{x-x_0}{A} = \frac{y-y_0}{B}$

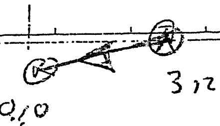
$$\frac{x-3}{-3} = \frac{y-2}{-2}$$

$$2(x-3) = 3(y-2) \rightarrow x-3 = \frac{3}{2}(y-2) \rightarrow x-3 = \frac{3}{2}y - 3$$

$$x = \frac{3}{2}y \rightarrow dx = \frac{3}{2}dy$$

$$\int_2^0 [(2x-y+4)dx + (5y+3x-6)dy]$$

$$\int_2^0 [(3y-y+4)\frac{3}{2}dy + (5y+\frac{9}{2}y-6)dy]$$



$A = 0-3$

$$\int_1^2 (y-1)\,dy = \int_1^2 y\,dy - \int_1^2 dy = \frac{1}{2}y^2 - y\Big|_1^2 = (2-2) - \left(\frac{1}{2} - 1\right) = \frac{1}{2}$$

along the Straight line from $(1,2)$ to $(4,2)$

along the Straight line from $(1,2)$ to $(4,2)$ while x Varies from 1 to 4

$y = 2 \rightarrow dy = 0$ while x Varies from 1 to 4

$$\int_1^4 (x+2)\,dx = \int_1^4 x\,dx + \int_1^4 2\,dx = \frac{1}{2}x^2 + 2x\Big|_1^4 = (8+8) - \left(\frac{1}{2}+2\right) = 13.5$$

هنا يجمع تكاملي الأول والثاني ليحصل على
الكلي للمسار

$$\int_C \left[(x+y)\,dx + (y-x)\,dy\right] = \frac{1}{2} + 13.5 = \boxed{14}$$

d- $x = 2t^2 + t + 1 \rightarrow dx = (4t+1)\,dt$

$y = t^2 + 1 \rightarrow dy = 2t\,dt$

$$\int_C \left[(x+y)\,dx + (y-x)\,dy\right]$$

$$\int_C \left[(2t^2+t+1+t^2+1)(4t+1)\,dt + (t^2+1-2t^2-t-1)\,2t\,dt\right]$$

$$\int_C \left[(3t^2+t+2)(4t+1)\,dt + (-t^2-t)\,2t\,dt\right]$$

$$\int_C \left[12t^3+4t^2+8t+3t^2+t+2-2t^3-2t^2\right]dt$$

$$\int_0^1 \left[10t^3+5t^2+9t+2\right]dt = \frac{10}{4}t^4 + \frac{5}{3}t^3 + \frac{9}{2}t^2 + 2t\Big|_0^1$$

$$= \left(\frac{10}{4} + \frac{5}{3} + \frac{9}{2} + 2\right) = \frac{32}{3}$$

$y = \frac{3}{3}x$

Z.W

Evaluate $\int_{(1,1)}^{(4,2)} [(x+y)dx + (y-x)dy]$ along a- the Parabola $y^2=x$

b- a Straight line c- Straight lines from (1,1) to (1,2) and then to (4,2) d- the Curve $x=2t^2+t+1$, $y=t^2+1$

a- $\frac{x-x_0}{x_0-x_1} = \frac{y-y_0}{y_0-y_1}$

$$\int_C [P(g(y),y)\,g'(y)dy + Q(g(y),y)dy]$$

$$y^2=x \to 2y\frac{dy}{dx}=1 \to dx = 2y\,dy$$

$$\int_1^2 [(y^2+y)2y\,dy + (y-y^2)dy] = \int_1^2 [2y^3dy+2y^2dy+y\,dy-y^2dy]$$

$$\int_1^2 [2y^3dy+y^2dy+y\,dy] = \frac{1}{2}y^4+\frac{1}{3}y^3+\frac{1}{2}y^2\Big|_1^2$$

$$\left(8+\frac{8}{3}+2\right)-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{2}\right) = \frac{34}{3}$$

$t^2+1-2t^2-t-1=0$

$-t^2-t=0$

$-t(t+1)=0$

$t=0$ or $t=-1$

b- the Straight line joining $P_1(1,1)$ and $P_2(4,2)$

the Standard Equation of line

$$\frac{x-x_0}{A}=\frac{y-y_0}{B} \to \frac{x-1}{3}=\frac{y-1}{1}$$

$$(x-1)=3(y-1) \to x-1=3y-3 \to \boxed{x=3y-2} \to 1=3\frac{dy}{dx} \to dx$$

$$\int_1^2[(3y-2+y)3dy+(y-3y+2)dy] = \int_1^2[(4y-2)3dy+(-2y+2)dy]$$

$$=\int_1^2[12y\,dy-6dy-2y\,dy+2dy] = \int_1^2[10y\,dy-4dy]$$

$$=5y^2-4y\Big|_1^2 = (20-8)-(5-4)=11$$

c- along the Straight line from (1,1) to (1,2)

$x=1$, $dx=0$, While y Varies from 1 to 2

Evaluate $\int_{-1,0}^{1,0} y(1+x)\,dy$

(a) along the x-axis and (b) along $y = 1 - x^2$

(a) $y = 0$, $dy = 0$

$$\int_{-1}^{1} (0)(1+x)(0) = 0$$

(b) $y = 1 - x^2 \longrightarrow dy = -2x\,dx$

$$\int_{-1}^{1} (1-x^2)(1+x)(-2x\,dx) = \int_{-1}^{1} (1+x-x^2-x^3)(-2x\,dx)$$

$$= \int_{-1}^{1} (-2x - 2x^2 + 2x^3 + 2x^4)\,dx$$

$$= -x^2 - \frac{2}{3}x^3 + \frac{1}{2}x^4 + \frac{2}{5}x^5 \Big|_{-1}^{1}$$

$$= \left(-1 - \frac{2}{3} + \frac{1}{2} + \frac{2}{5}\right) - \left(-1 + \frac{2}{3} + \frac{1}{2} - \frac{2}{5}\right) = -\frac{8}{15}$$

Z.W

Line Integration

Along What Curve of the family $y = x^n$ does the Integ[ral]

$$\int_{0,0}^{1,1} (25xy - 8y^2)\,dx$$ attain its largest Value?

$$\int_0^1 (25xy - 8y^2)\,dx$$

$$\int_0^1 (25x\,x^n - 8(x^n)^2)\,dx$$

الواحد اتى لاي رقم زاد

$$\int_0^1 (25x^{n+1} - 8x^{2n})\,dx$$

$$25\frac{x^{n+2}}{n+2} - 8\frac{x^{2n+1}}{2n+1}\Bigg|_0^1$$

$$\frac{25}{n+2} - \frac{8}{2n+1} = \frac{25(2n+1) - 8(n+2)}{(n+2)(2n+1)}$$

$$\frac{50n + 25 - 8n - 16}{(2n^2 + n + 4n + 2)} = \frac{42n + 9}{(2n^2 + 5n + 2)}$$

$$dn = \frac{(2n^2+5n+2)(42) - (42n+9)(4n+5)}{(2n^2+5n+2)^2}$$

أكبر قيمة عند المشتقة

$dn = 0$ so that the Integral attain its largest Value.

$$(2n^2+5n+2)(42) - (42n+9)(4n+5) = 0$$

$$84n^2 + 210n + 84 - (168n^2 + 210n + 36n + 45) = 0$$

$$84n^2 + 210n + 84 - 168n^2 - 210n - 36n - 45 = 0$$

$$84n^2 - 36n + 39 = 0$$

$$n = \frac{36 \pm \sqrt{36^2 + 4 \times 84 \times 39}}{-2 \times -84} \quad \therefore n = 0.5$$

if $\nabla\psi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 -$
find ψ.

$$\nabla\psi = \frac{\partial\psi}{\partial x}\vec{i} + \frac{\partial\psi}{\partial y}\vec{j} + \frac{\partial\psi}{\partial z}\vec{k}$$

$$\frac{\partial\psi}{\partial x} = y^2 - 2xyz^3 \rightarrow \psi = y^2x - x^2yz^3 + f(y,z)$$

$$\frac{\partial\psi}{\partial y} = 3 + 2xy - x^2z^3 \rightarrow \psi = 3y + xy^2 - x^2yz^3 + f(x,z)$$

$$\frac{\partial\psi}{\partial z} = 6z^3 - 3x^2yz^2 \rightarrow \psi = \frac{3}{2}z^4 - x^2yz^3 + f(y,z)$$

<u>or</u>

$$\nabla\psi \cdot dr = \vec{V} \cdot dr$$

$$d\psi = (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (6z^3 - 3x^2yz^2)d$$

$$d\psi = (y^2dx + 2xydy) + (-2xyz^3dx - 3x^2yzdz) + 3dy + 6z^3d$$

$$\psi = xy^2 - x^2yz^3 + 3y + (3/2)z^4 + \text{Constant}$$

Human Resource

Z.W

$$\frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k} = -xr^{-2}\vec{i} - yr^{-2}\vec{j} - zr^{-2}\vec{k}$$

$$\frac{\partial \phi}{\partial x} = -xr^{-2} = -\frac{x}{r^2} = -\frac{x}{(x^2+y^2+z^2)}$$

$$\phi = -\frac{1}{2}\ln(x^2+y^2+z^2) + C$$

$$\phi = -\frac{1}{2}\ln r^2 + C$$

$$\frac{\partial \phi}{\partial y} = -yr^{-2} = -\frac{y}{r^2} = -\frac{y}{x^2+y^2+z^2}$$

$$\phi = -\frac{1}{2}\ln(x^2+y^2+z^2) + C \rightarrow \phi = -\frac{1}{2}\ln r^2 + C$$

$$\frac{\partial \phi}{\partial z} = -zr^{-2} = -\frac{z}{r^2} = -\frac{z}{x^2+y^2+z^2}$$

$$\phi = -\frac{1}{2}\ln(x^2+y^2+z^2) + C \rightarrow \phi = -\frac{1}{2}\ln r^2 + C$$

$$\therefore \phi = -\frac{1}{2}\ln r^2 + C$$

$$\phi(a) = 0 \rightarrow -\frac{1}{2}\ln a^2 + C = 0 \rightarrow \boxed{C = \frac{1}{2}\ln a^2}$$

$$\phi = -\frac{1}{2}\ln r^2 + \frac{1}{2}\ln a^2$$

$$\phi = \frac{(1/2)\ln a^2}{(1/2)\ln r^2} = \frac{\ln a^2}{\ln r^2} = \frac{2\ln a}{2\ln r} = \ln(a/r)$$

$$\therefore \phi = \ln(a/r)$$

Z.W

بزاري

Show that $E = \vec{r}/r^2$ is irrotational. Find ϕ such that $E = -\nabla\phi$ and such that $\phi(a) = 0$ where $a > 0$

$$E = \frac{\vec{r}}{r^2} = xr^{-2}\vec{i} + yr^{-2}\vec{j} + zr^{-2}\vec{k}$$

$$\nabla \times E = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xr^{-2} & yr^{-2} & zr^{-2} \end{vmatrix}$$

$$\nabla \times E = \left(\frac{\partial}{\partial y} zr^{-2} - \frac{\partial}{\partial z} yr^{-2}\right)\vec{i} + \left(\frac{\partial}{\partial z} xr^{-2} - \frac{\partial}{\partial x} zr^{-2}\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x} yr^{-2} - \frac{\partial}{\partial y} xr^{-2}\right)\vec{k}$$

$$\frac{\partial}{\partial y} zr^{-2} = -2zr^{-3}\frac{\partial r}{\partial y} = -2zyr^{-4}$$

$$\frac{\partial}{\partial z} yr^{-2} = -2yr^{-3}\frac{\partial r}{\partial z} = -2zyr^{-4}$$

$$\frac{\partial}{\partial z} xr^{-2} = -2xr^{-3}\frac{\partial r}{\partial z} = -2xzr^{-4}$$

$$\frac{\partial}{\partial x} zr^{-2} = -2zr^{-3}\frac{\partial r}{\partial x} = -2xzr^{-4}$$

$\nabla \times E = 0$

$$\frac{\partial}{\partial x} yr^{-2} = -2yr^{-3}\frac{\partial r}{\partial x} = -2yxr^{-4}$$

$$\frac{\partial}{\partial y} xr^{-2} = -2xr^{-3}\frac{\partial r}{\partial y} = -2xyr^{-4}$$

$\nabla \times E = 0$ $\therefore$ irrotational

$E = -\nabla\phi \rightarrow \nabla\phi = -E$

A B

$A \times B$

If A and B are irrotational prove that $A \times B$ is Solenoidal.

$$\left.\begin{array}{l}\nabla \times A = 0 \\ \nabla \times B = 0\end{array}\right\} \text{irrotational}$$

$$A \times B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$A \times B = (A_2B_3 - A_3B_2)\,\vec{i} + (A_3B_1 - A_1B_3)\,\vec{j} + (A_1B_2 - A_2B_1)\,\vec{k}$$

$$\boxed{\nabla \cdot (A \times B)} = \frac{\partial}{\partial x}(A_2B_3 - A_3B_2) + \frac{\partial}{\partial y}(A_3B_1 - A_1B_3) + \frac{\partial}{\partial z}(A_1B_2 - A_2B_1)$$

$$\frac{\partial}{\partial x}A_2B_3 - \frac{\partial}{\partial x}A_3B_2 + \frac{\partial}{\partial y}A_3B_1 - \frac{\partial}{\partial y}A_1B_3 + \frac{\partial}{\partial z}A_1B_2 - \frac{\partial}{\partial z}A_2B_1$$

$$A_2\frac{\partial B_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial B_2}{\partial x} - B_2\frac{\partial A_3}{\partial x} + A_3\frac{\partial B_1}{\partial y} + B_1\frac{\partial A_3}{\partial y} - A_1\frac{\partial B_3}{\partial y} - B_3\frac{\partial A_1}{\partial y}$$

$$+ A_1\frac{\partial B_2}{\partial z} + B_2\frac{\partial A_1}{\partial z} - A_2\frac{\partial B_1}{\partial z} - B_1\frac{\partial A_2}{\partial z}$$

$$\left(B_1\frac{\partial A_3}{\partial y} - B_1\frac{\partial A_2}{\partial z}\right) + \left(B_2\frac{\partial A_1}{\partial z} - B_2\frac{\partial A_3}{\partial x}\right) + \left(B_3\frac{\partial A_2}{\partial x} - B_3\frac{\partial A_1}{\partial y}\right) + \left(-A_1\frac{\partial B_3}{\partial y} + A_1\frac{\partial B_2}{\partial z}\right)$$

$$+ \left(-A_2\frac{\partial B_1}{\partial z} + A_2\frac{\partial B_3}{\partial x}\right) + \left(-A_3\frac{\partial B_2}{\partial x} + A_3\frac{\partial B_1}{\partial y}\right)$$

$$B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + B_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + B_3\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) - A_2\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)$$

$$- A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$

$$(B_1\vec{i} + B_2\vec{j} + B_3\vec{k}) \cdot \left\{\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\vec{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\vec{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\vec{k}\right\}$$

$$- (A_1\vec{i} + A_2\vec{j} + A_3\vec{k}) \cdot \left\{\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\vec{i} + \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\vec{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\vec{k}\right\}$$

$$B \cdot (\nabla \times A) - A \cdot (\nabla \times B) = 0$$

Z.W

$\therefore$ Solenoidal.

if w is a Constant Vector and $V = w \times r$, prove that

$\operatorname{div} V = 0$

w = angular Velocity Vector

$w = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{K}$

$r = x\vec{i} + y\vec{j} + z\vec{K}$

$$V = w \times r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$V = (w_2 z - w_3 y)\vec{i} + (w_3 x - w_1 z)\vec{j} + (w_1 y - w_2 x)\vec{K}$$

$\operatorname{div} V = \nabla \cdot V$

$$= \frac{\partial}{\partial x}(w_2 z - w_3 y) + \frac{\partial}{\partial y}(w_3 x - w_1 z) + \frac{\partial}{\partial z}(w_1 y - w_2 x)$$

$= 0$

$\therefore \operatorname{div} V = 0$

Z.W

* (a) Prove $\nabla^2 f(r) = \dfrac{d^2 f}{dr^2} + \dfrac{2}{r}\dfrac{df}{dr}$

(b) Find $f(r)$ Such that $\nabla^2 f(r) = 0$

$$\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r)$$

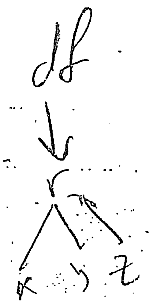
$\dfrac{df}{dr} = f'(r)$ [illegible]

$$= \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r)$$

$$\frac{\partial^2}{\partial x^2} f(r) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(r)\right) \quad \boxed{= \frac{\partial}{\partial x}\left(\frac{dF}{dr}\frac{\partial r}{\partial x}\right)}$$

[illegible]

$$= \frac{\partial}{\partial x}\left(\frac{dF}{dr}\frac{x}{r}\right)$$



$$= \frac{d^2F}{dr^2}\frac{\partial r}{\partial x}\frac{x}{r} + \frac{dF}{dr}\left(r^{-1} - x r^{-2}\frac{\partial r}{\partial x}\right)$$

$$= \frac{d^2F}{dr^2}\frac{x}{r}\frac{x}{r} + \frac{dF}{dr}\left(\frac{1}{r} - x r^{-2}\frac{x}{r}\right)$$

$$= \frac{d^2F}{dr^2}\frac{x^2}{r^2} + \frac{dF}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3}\right)$$

and similar $\dfrac{\partial^2}{\partial y^2} f(r) = \dfrac{d^2F}{dr^2}\dfrac{y^2}{r^2} + \dfrac{dF}{dr}\left(\dfrac{1}{r} - \dfrac{y^2}{r^3}\right)$

$$\frac{\partial^2}{\partial z^2} f(r) = \frac{d^2F}{dr^2}\frac{z^2}{r^2} + \frac{dF}{dr}\left(\frac{1}{r} - \frac{z^2}{r^3}\right)$$

$$\nabla^2 f(r) = \frac{d^2F}{dr^2}\frac{x^2}{r^2} + \frac{d^2F}{dr^2}\frac{y^2}{r^2} + \frac{d^2F}{dr^2}\frac{z^2}{r^2} +$$

$$\frac{dF}{dr}\left(\frac{3}{r} - \frac{x^2+y^2+z^2}{r^3}\right)$$

$$= \frac{d^2F}{dr^2}\frac{x^2+y^2+z^2}{r^2} + \frac{dF}{dr}\left(\frac{3}{r} - \frac{r^2}{r^3}\right)$$

$$= \frac{d^2F}{dr^2}\frac{r^2}{r^2} + \frac{dF}{dr}\left(\frac{3}{r} - \frac{1}{r}\right) = \frac{d^2F}{dr^2} + \frac{2}{r}\frac{dF}{dr}$$

$$\frac{\partial}{\partial z}(-3r^{-4}z) = -3r^{-4} + 12r^{-5}\frac{\partial r}{\partial z}z$$

$$= -3r^{-4} + 12r^{-5}\frac{z}{r}z$$

$$= -3r^{-4} + 12z^2r^{-6}$$

$$\nabla.[r\nabla(1/r^3)] = -3r^{-4} + 12x^2r^{-6} - 3r^{-4} + 12y^2r^{-6}$$

$$-3r^{-4} + 12z^2r^{-6}$$

$$= -9r^{-4} + 12r^{-6}\underbrace{(x^2+y^2+z^2)}_{r^2}$$

$$= -9r^{-4} + 12r^{-6}r^2$$

$$= -9r^{-4} + 12r^{-4}$$

$$= 3r^{-4}$$

Z.W

Evaluate $\nabla \cdot [r \nabla (1/r^3)]$

$$\nabla(1/r^3) = \nabla r^{-3} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) r^{-3}$$

$$= \frac{\partial}{\partial x} r^{-3}\vec{i} + \frac{\partial}{\partial y} r^{-3}\vec{j} + \frac{\partial}{\partial z} r^{-3}\vec{k}$$

$$= -3r^{-4}\frac{\partial r}{\partial x}\vec{i} - 3r^{-4}\frac{\partial r}{\partial y}\vec{j} - 3r^{-4}\frac{\partial r}{\partial z}\vec{k}$$

$$= -3r^{-4}\frac{x}{r}\vec{i} - 3r^{-4}\frac{y}{r}\vec{j} - 3r^{-4}\frac{z}{r}\vec{k}$$

$$= -3x r^{-5}\vec{i} - 3y r^{-5}\vec{j} - 3z r^{-5}\vec{k}$$

$$= -3r^{-5}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\therefore r\nabla(1/r^3) = r\left(-3r^{-5}(x\vec{i} + y\vec{j} + z\vec{k})\right)$$

$$r\nabla(1/r^3) = -3r^{-4}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla \cdot [r\nabla(1/r^3)] = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot(-3r^{-4}x\vec{i} - 3r^{-4}y\vec{j} - 3r^{-4}z\vec{k})$$

$$= \frac{\partial}{\partial x}(-3r^{-4}x) + \frac{\partial}{\partial y}(-3r^{-4}y) + \frac{\partial}{\partial z}(-3r^{-4}z)$$

$$\frac{\partial}{\partial x}(-3r^{-4}x) = -3r^{-4} + 12r^{-5}\frac{\partial r}{\partial x}x = -3r^{-4} + 12r^{-5}\frac{x}{r}x$$

$$= -3r^{-4} + 12x^2r^{-6}$$

$$\frac{\partial}{\partial y}(-3r^{-4}y) = -3r^{-4} + 12r^{-5}\frac{\partial r}{\partial y}y$$

$$= -3r^{-4} + 12r^{-5}\frac{y}{r}y$$

$$= -3r^{-4} + 12y^2r^{-6}$$

Z.W

if A is a Constant Vector, prove $\nabla(r.\vec{A}) = \vec{A}$

$r = x\vec{i} + y\vec{j} + z\vec{k}$

$A = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

$r.A = xA_1 + yA_2 + zA_3$

$\nabla(r.A) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(xA_1 + yA_2 + zA_3)$

$= \frac{\partial}{\partial x}(xA_1 + yA_2 + zA_3)\vec{i} + \frac{\partial}{\partial y}(xA_1 + yA_2 + zA_3)\vec{j}$

$+ \frac{\partial}{\partial z}(xA_1 + yA_2 + zA_3)\vec{k}$

$= A_1\vec{i} + A_2\vec{j} + A_3\vec{k} = \vec{A}$

$\therefore \nabla(r.A) = A$

Z.W

*Find the Values of the Constants a, b, c So that the direction derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in a direction parallel to the Z-axis.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\nabla\phi = (ay^2 + 3cz^2x^2)\vec{i} + (2axy + bz)\vec{j} + (by + 2czx^3)\vec{k}$$

$$\nabla\phi = (4a + 3c)\vec{i} + (4a - b)\vec{j} + (2b - 2c)\vec{k}$$

$$\nabla\phi \cdot \vec{i} = 0 \rightarrow (4a + 3c) = 0 \rightarrow a = -\frac{3}{4}c \text{ —— (1)}$$

$$\nabla\phi \cdot \vec{j} = 0 \rightarrow 4a - b = 0 \rightarrow a = \frac{b}{4} \text{ —— (2)}$$

from equation (1) and (2) we get, $b = -3c$

$$|\nabla\phi| = \sqrt{(4a + 3c)^2 + (4a - b)^2 + (2b - 2c)^2}, \quad |\nabla\phi| = 64$$

$$4096 = (4a + 3c)^2 + (4a - b)^2 + (2b - 2c)^2$$

$$4096 = (-6c - 2c)^2 \rightarrow c = 8$$

$$a = -6, \quad b = -24$$

Z.W

In What direction from the Point (1, 3, 2) is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of this maximum?

max $\nabla\phi$

In the direction of $\nabla\phi$ (grad ϕ)

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\nabla\phi = 2z\vec{i} - 2y\vec{j} + 2x\vec{k}$$

$$\nabla\phi = 4\vec{i} - 6\vec{j} + 2\vec{k}$$

$$|\nabla\phi| = \sqrt{4^2 + (-6)^2 + (2)^2} = 2\sqrt{14}$$

∇Q

Z.W

Find the Constants a and b so that the Surface $ax^2 - byz = (a+2)x$ will be orthogonal to the Surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$

$Q_1 = ax^2 - byz - (a+2)x$

for first Surface

$\phi = ax^2 - byz - (a+2)x$

$\phi = ax^2 - byz - ax - 2x$

$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\vec{i} + \frac{\partial\phi_1}{\partial y}\vec{j} + \frac{\partial\phi_1}{\partial z}\vec{k}$

$\nabla\phi_1 = (2ax - a - 2)\vec{i} + (-bz)\vec{j} + (-by)\vec{k}$

$\nabla\phi_1 = (2a - a - 2)\vec{i} - 2b\vec{j} + b\vec{k}$

$\nabla\phi_1 = (a-2)\vec{i} - 2b\vec{j} + b\vec{k}$

for Second Surface

$\phi = 4x^2y + z^3 - 4$

$\nabla\phi_2 = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$

$\nabla\phi_2 = -8\vec{i} + 4\vec{j} + 12\vec{k}$

$\boxed{\nabla\phi_1 \cdot \nabla\phi_2 = 0}$ (Orthogonal)

$((a-2)\vec{i} - 2b\vec{j} + b\vec{k}) \cdot (-8\vec{i} + 4\vec{j} + 12\vec{k}) = 0$

$-8(a-2) - 8b + 12b = 0$

$-8a + 16 + 4b = 0$

Assume $b = 1 \rightarrow a = 5/2$

Z.W.

Find the unit outward drawn normal to the Surface $(x-1)^2+y^2+(z+2)^2=9$ at the Point $(3,1,-4)$

$$\phi = (x-1)^2+y^2+(z+2)^2-9$$

$$\nabla\phi = 2(x-1)\vec{i}+2y\vec{j}+2(z+2)\vec{k}$$

$$\nabla\phi = 4\vec{i}+2\vec{j}-4\vec{k}$$

$$|\nabla\phi| = \sqrt{4^2+2^2+(-4)^2} = 6$$

the Unit Vector $= \frac{1}{6}(4\vec{i}+2\vec{j}-4\vec{k})$

$$= \frac{1}{3}(2\vec{i}+\vec{j}-2\vec{k})$$

نريد متجه الوحدة العمودي على السطح

متجه الوحدة العمودي

تذكر $\frac{1}{\nabla\phi}$ ...

$$\text{if } A(x,y,z) = A_1\vec{i} + A_2\vec{j} + A_3\vec{K}, \text{ Show that}$$

$$dA = (\nabla A_1 \cdot dr)\vec{i} + (\nabla A_2 \cdot dr)\vec{j} + (\nabla A_3 \cdot dr)\vec{K}$$

$$\boxed{dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz}$$ قانون

$$dA = \left(\frac{\partial A_1}{\partial x}\vec{i} + \frac{\partial A_2}{\partial x}\vec{j} + \frac{\partial A_3}{\partial x}\vec{K}\right)dx + \left(\frac{\partial A_1}{\partial y}\vec{i} + \frac{\partial A_2}{\partial y}\vec{j} + \frac{\partial A_3}{\partial y}\vec{K}\right)dy$$

$$+ \left(\frac{\partial A_1}{\partial z}\vec{i} + \frac{\partial A_2}{\partial z}\vec{j} + \frac{\partial A_3}{\partial z}\vec{K}\right)dz$$

$$= \left(\frac{\partial A_1}{\partial x}dx\vec{i} + \frac{\partial A_1}{\partial y}dy\vec{i} + \frac{\partial A_1}{\partial z}dz\vec{i}\right) + \left(\frac{\partial A_2}{\partial x}dx\vec{j} + \frac{\partial A_2}{\partial y}dy\vec{j} + \frac{\partial A}{\partial z}\right.$$

$$+ \left(\frac{\partial A_3}{\partial x}dx\vec{K} + \frac{\partial A_3}{\partial y}dy\vec{K} + \frac{\partial A_3}{\partial z}dz\vec{K}\right)$$

$$= \left(\frac{\partial A_1}{\partial x}dx + \frac{\partial A_1}{\partial y}dy + \frac{\partial A_1}{\partial z}dz\right)\vec{i} + \left(\frac{\partial A_2}{\partial x}dx + \frac{\partial A_2}{\partial y}dy + \frac{\partial A_2}{\partial z}dz\right)\vec{j}$$

$$+ \left(\frac{\partial A_3}{\partial x}dx + \frac{\partial A_3}{\partial y}dy + \frac{\partial A_3}{\partial z}dz\right)\vec{K}$$

$$= \left\{\left(\frac{\partial A_1}{\partial x}\vec{i} + \frac{\partial A_1}{\partial y}\vec{j} + \frac{\partial A_1}{\partial z}\vec{K}\right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{K})\right\}\vec{i}$$

$$+ \left\{\left(\frac{\partial A_2}{\partial x}\vec{i} + \frac{\partial A_2}{\partial y}\vec{j} + \frac{\partial A_2}{\partial z}\vec{K}\right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{K})\right\}\vec{j}$$

$$+ \left\{\left(\frac{\partial A_3}{\partial x}\vec{i} + \frac{\partial A_3}{\partial y}\vec{j} + \frac{\partial A_3}{\partial z}\vec{K}\right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{K})\right\}\vec{K}$$

$$= (\nabla A_1 \cdot dr)\vec{i} + (\nabla A_2 \cdot dr)\vec{j} + (\nabla A_3 \cdot dr)\vec{K}$$

Z.V

if $\nabla U = 2r^4 \vec{r}$, find U.

$$\nabla U = 2r^4 x \vec{i} + 2r^4 y \vec{j} + 2r^4 z \vec{k}$$

$$\nabla U = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}$$

$$U = \int \frac{\partial U}{\partial x}\, dx$$

$$U = \int 2r^4 x\, dx = 2\int (r^2)^2 x\, dx = 2\int (x^2+y^2+z^2)^2 x\, dx$$

$$= \frac{1}{3}(x^2+y^2+z^2)^3 + C = \frac{1}{3}(r^2)^3 + C = \frac{r^6}{3} + C$$

or

$$U = \int \frac{\partial U}{\partial y}\, dy$$

$$U = \int 2r^4 y\, dy = 2\int (r^2)^2 y\, dy = 2\int (x^2+y^2+z^2)^2 y\, dy$$

$$= \frac{1}{3}(x^2+y^2+z^2)^3 + C = \frac{1}{3}(r^2)^3 + C = \frac{r^6}{3} + C$$

or

$$U = \int \frac{\partial U}{\partial z}\, dz$$

$$U = \int 2r^4 z\, dz = 2\int (r^2)^2 z\, dz = 2\int (x^2+y^2+z^2) z\, dz$$

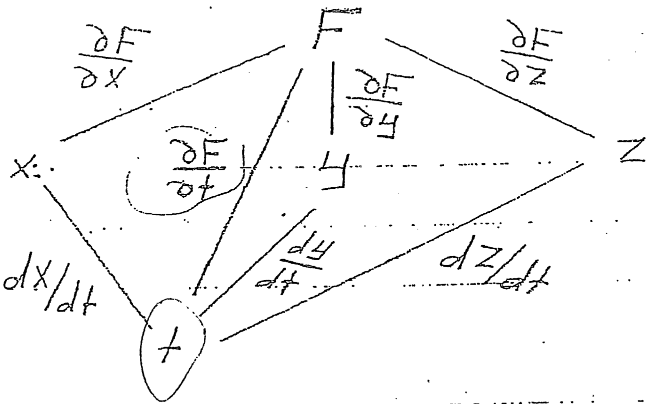
$$= \frac{1}{3}(x^2+y^2+z^2)^3 + C = \frac{1}{3}(r^2)^3 + C = \frac{r^6}{3} + C$$

Z.W

افتراق

if F is a differentiable function of $(x, y, z), t$ where x, y, z are differentiable functions of t prove that

$$\frac{dF}{dt} = \boxed{\frac{\partial F}{\partial t} + \nabla F \cdot \frac{dr}{dt}}$$



$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}\right)\cdot\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right)$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \nabla F \cdot \frac{dr}{dt}$$

Z.W

Prove $\nabla\left(\frac{F}{G}\right) = \frac{G\nabla F - F\nabla G}{G^2}$ if $G \neq 0$

$$\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(\frac{F}{G}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{F}{G}\right)\vec{i} + \frac{\partial}{\partial y}\left(\frac{F}{G}\right)\vec{j} + \frac{\partial}{\partial z}\left(\frac{F}{G}\right)\vec{k}$$

$$\frac{\partial}{\partial x}\left(\frac{F}{G}\right) = \frac{G\frac{\partial F}{\partial x} - F\frac{\partial G}{\partial x}}{G^2}$$

$$\frac{\partial}{\partial y}\left(\frac{F}{G}\right) = \frac{G\frac{\partial F}{\partial y} - F\frac{\partial G}{\partial y}}{G^2}$$

$$\frac{\partial}{\partial z}\left(\frac{F}{G}\right) = \frac{G\frac{\partial F}{\partial z} - F\frac{\partial G}{\partial z}}{G^2}$$

$$\frac{1}{G^2}\left(G\frac{\partial F}{\partial x}\vec{i} - F\frac{\partial G}{\partial x}\vec{i} + G\frac{\partial F}{\partial y}\vec{j} - F\frac{\partial G}{\partial y}\vec{j} + G\frac{\partial F}{\partial z}\vec{k} - F\frac{\partial G}{\partial z}\vec{k}\right)$$

$$\frac{1}{G^2}G\left(\frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}\right) - \frac{1}{G^2}F\left(\frac{\partial G}{\partial x}\vec{i} + \frac{\partial G}{\partial y}\vec{j} + \frac{\partial G}{\partial z}\vec{k}\right)$$

$$\frac{1}{G^2}G\nabla F - \frac{1}{G^2}F\nabla G$$

$$\frac{G\nabla F - F\nabla G}{G^2}$$

$$\therefore \nabla\left(\frac{F}{G}\right) = \frac{G\nabla F - F\nabla G}{G^2}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \quad \frac{\vec{r}}{r^5} = \frac{x\vec{i}}{r^5} + \frac{y\vec{j}}{r^5} + \frac{z\vec{k}}{r^5}$$

* Find $\phi(r)$ Such that $\nabla\phi = \dfrac{\vec{r}}{r^5}$ and $\phi(1) = 0$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\nabla\phi = \frac{x}{r^5}\vec{i} + \frac{y}{r^5}\vec{j} + \frac{z}{r^5}\vec{k}$$

$$\frac{\partial\phi}{\partial x} = \frac{x}{r^5}, \quad \frac{\partial\phi}{\partial y} = \frac{y}{r^5}, \quad \frac{\partial\phi}{\partial z} = \frac{z}{r^5}$$

$$\phi = \int \frac{\partial\phi}{\partial x}\,dx = \int x\,r^{-5}\,dx = \int x\,(x^2+y^2+z^2)^{-5/2}\,dx$$

$$= \frac{1}{2}\,\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + C = -\frac{1}{3}r^{-3} + C \longrightarrow (1)$$

$$\phi = \int \frac{\partial\phi}{\partial y}\,dy = \int y\,r^{-5}\,dy = \int y\,(x^2+y^2+z^2)^{-5/2}\,dy$$

$$= \frac{1}{2}\,\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + C = -\frac{1}{3}r^{-3} + C \longrightarrow (2)$$

$$\phi = \int \frac{\partial\phi}{\partial z}\,dz = \int z\,r^{-5}\,dz = \int z\,(x^2+y^2+z^2)^{-5/2}$$

$$= \frac{1}{2}\,\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + C = -\frac{1}{3}r^{-3} + C \longrightarrow (3)$$

$$\therefore \phi = -\frac{1}{3}r^{-3} + C$$

$$\phi(1) = 0 \longrightarrow 0 = -\frac{1}{3} + C \longrightarrow C = \frac{1}{3}$$

$$\phi = -\frac{1}{3}r^{-3} + \frac{1}{3} = \frac{1}{3}(1 - r^{-3})$$

$$\phi = \frac{1}{3}\left(1 - \frac{1}{r^3}\right)$$

$nr^{n-2} + n(n-2)x^2 r^{n-4} + nr^{n-2} + n(n-2)y^2 r^{n-4}$

$+ nr^{n-2} + n(n-2)z^2 r^{n-4}$

$3nr^{n-2} + n(n-2) r^{n-4} (x^2+y^2+z^2)$

$3nr^{n-2} + n(n-2) r^{n-4}\, r^2$

$3nr^{n-2} + n(n-2)\, r^{n-2}$

$r^{n-2}(3n + n^2 - 2n)$

$r^{n-2}(n^2+n) = n(n+1) r^{n-2}$

$u = (x^2+y^2+z^2)^{-\frac{5}{2}}$

$dv = x\,dx$

$u\,dv = uv - \int v\,du$

$du = -\frac{5}{2}(\ )^{-\frac{7}{2}}$

$du = -\frac{5}{2}(\ )^{-\frac{7}{2}} *$

$u = x$

$du = dx$

$v = (\ )^{-\frac{5}{2}}$

$dv = -\frac{3}{2}(\ )$

Z.W

Prove $\nabla^2 r^n = n(n+1)r^{n-2}$ Where n is a Constant.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ تقست بالوحدة

$$\nabla^2 r^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n$$

$$\frac{\partial^2}{\partial x^2} r^n + \frac{\partial^2}{\partial y^2} r^n + \frac{\partial^2}{\partial z^2} r^n$$

$$\frac{\partial^2}{\partial x^2} r^n = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} r^n\right) = \frac{\partial}{\partial x}\left(n\, r^{n-1} \frac{\partial r}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(n r^{n-1} \frac{x}{r}\right) = \frac{\partial}{\partial x}\left(n r^{n-2} x\right)$$

$$= n r^{n-2} + n(n-2) r^{n-3} \frac{\partial r}{\partial x} x = n r^{n-2} + n(n-2) r^{n-3} \frac{x}{r} x$$

$\Phi \cdot \nabla Q$

$$= \underbrace{n r^{n-2} + n(n-2) x^2 r^{n-4}}$$

$$\frac{\partial^2}{\partial y^2} r^n = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} r^n\right) = \frac{\partial}{\partial y}\left(n r^{n-1} \frac{\partial r}{\partial y}\right) = \frac{\partial}{\partial y}\left(n r^{n-1} \frac{y}{r}\right)$$

$$= \frac{\partial}{\partial y}\left(n r^{n-2} y\right) = n r^{n-2} + n(n-2) r^{n-3} \frac{\partial r}{\partial y} y$$

$$= n r^{n-2} + n(n-2) r^{n-3} \frac{y}{r} y = n r^{n-2} + n(n-2) y^2 r^{n-4}$$

$$\frac{\partial^2}{\partial z^2} r^n = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} r^n\right) = \frac{\partial}{\partial z}\left(n r^{n-1} \frac{\partial r}{\partial z}\right) = \frac{\partial}{\partial z}\left(n r^{n-1} \frac{z}{r}\right)$$

$$= \frac{\partial}{\partial z}\left(n r^{n-2} z\right) = n r^{n-2} + n(n-2) r^{n-3} \frac{\partial r}{\partial z} z$$

$$= n r^{n-2} + n(n-2) r^{n-3} \frac{z}{r} z$$

$$= n r^{n-2} + n(n-2) z^2 r^{n-4}$$

Z.W $\quad n r^{n-2} + n(n-2) x^2 r^{n-4} + n r^{n-2}$

$$3 n r^{n-2} + n(n-2) r^{n-4} (x^2 + y^2 + z^2)$$

$$3 n r^{n-2} + n(n-2) r^{n-2}$$

Prove Curl (ϕ grad ϕ) = 0

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\phi\nabla\phi = \phi\frac{\partial\phi}{\partial x}\vec{i} + \phi\frac{\partial\phi}{\partial y}\vec{j} + \phi\frac{\partial\phi}{\partial z}\vec{k}$$

$$\nabla\times\phi\nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \phi\frac{\partial\phi}{\partial x} & \phi\frac{\partial\phi}{\partial y} & \phi\frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$\nabla\times\phi\nabla\phi = \left(\phi\frac{\partial^2\phi}{\partial z\partial y} - \phi\frac{\partial^2\phi}{\partial z\partial y}\right)\vec{i} - \left(\phi\frac{\partial^2\phi}{\partial z\partial x} - \phi\frac{\partial\phi}{\partial x\partial z}\right)$$
$$+\left(\phi\frac{\partial^2\phi}{\partial x\partial y} - \phi\frac{\partial^2\phi}{\partial y\partial x}\right)\vec{k}$$

$$\nabla\times\phi\nabla\phi = 0$$

if U and V are differentiable Scalar fields prove that $\nabla U \times \nabla V$ is Solenoidal.

$$\nabla U = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}$$

$$\nabla V = \frac{\partial V}{\partial x}\vec{i} + \frac{\partial V}{\partial y}\vec{j} + \frac{\partial V}{\partial z}\vec{k}$$

$$\nabla U \times \nabla V = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \\ \partial V/\partial x & \partial V/\partial y & \partial V/\partial z \end{vmatrix}$$

$$\underline{\nabla U \times \nabla V} = \left(\frac{\partial U}{\partial y}\frac{\partial V}{\partial z} - \frac{\partial U}{\partial z}\frac{\partial V}{\partial y}\right)\vec{i} + \left(\frac{\partial U}{\partial z}\frac{\partial V}{\partial x} - \frac{\partial U}{\partial x}\frac{\partial V}{\partial z}\right)\vec{j} + \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial V}{\partial x}\right)\vec{k}$$

$$\underline{\nabla \cdot (\nabla U \times \nabla V)} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\frac{\partial V}{\partial z} - \frac{\partial U}{\partial z}\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial z}\frac{\partial V}{\partial x} - \frac{\partial U}{\partial x}\frac{\partial V}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial V}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\frac{\partial V}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial z}\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial z}\frac{\partial V}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial y}\frac{\partial V}{\partial x}\right)$$

$$= \frac{\partial U}{\partial y}\frac{\partial^2 V}{\partial z \partial x} + \frac{\partial V}{\partial z}\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial U}{\partial z}\frac{\partial^2 V}{\partial y \partial x} - \frac{\partial V}{\partial y}\frac{\partial^2 U}{\partial x \partial z} + \frac{\partial U}{\partial z}\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial z \partial y} - \frac{\partial U}{\partial x}\frac{\partial^2 V}{\partial z \partial y} - \frac{\partial V}{\partial z}\frac{\partial^2 U}{\partial x \partial y}$$

$$+ \frac{\partial U}{\partial x}\frac{\partial^2 V}{\partial y \partial z} + \frac{\partial V}{\partial y}\frac{\partial^2 U}{\partial x \partial z} - \frac{\partial U}{\partial y}\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x \partial z} = 0$$

$\nabla \cdot (\nabla U \times \nabla V) = 0$ $\quad\therefore$ It is Solenoidal

Z.W

Show that $\vec{A} = (2x^2+8xy^2z)\vec{i} + (3x^3y-3xy)\vec{j} - (4y^2z^2+2x^3z)\vec{k}$ is not Solenoidal but $B = xyz^2A$ is Solenoidal

$$\nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left\{(2x^2+8xy^2z)\vec{i} + (3x^3y-3xy)\vec{j} + (4y^2z^2+2x^3z)\vec{k}\right\}$$

$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}(2x^2+8xy^2z) + \frac{\partial}{\partial y}(3x^3y-3xy) - \frac{\partial}{\partial z}(4y^2z^2+2x^3z)$$

$$= (4x+8y^2z) + (3x^3-3x) - (8y^2z+2x^3)$$

$$= 4x + 8y^2z + 3x^3 - 3x - 8y^2z - 2x^3 = x + x^3$$ Not Solenoidal

$$B = xyz^2A = xyz^2\left\{(2x^2+8xy^2z)\vec{i} + (3x^3y-3xy)\vec{j} - (4y^2z^2+2x^3z)\vec{k}\right\}$$

$$B = (2x^3yz^2+8x^2y^3z^3)\vec{i} + (3x^4y^2z^2-3x^2y^2z^2)\vec{j} - (4xy^3z^4+2x^4yz^3)\vec{k}$$

$$\nabla\cdot B = \frac{\partial}{\partial x}(2x^3yz^2+8x^2y^3z^3) + \frac{\partial}{\partial y}(3x^4y^2z^2-3x^2y^2z^2) - \frac{\partial}{\partial z}(4xy^3z^4+2x^4yz^3)$$

$$\nabla\cdot B = (6x^2yz^2+16xy^3z^3) + (6x^4yz^2-6x^2yz^2) - (16xy^3z^3+6x^4yz^2)$$

$$\nabla\cdot B = 6x^2yz^2+16xy^3z^3+6x^4yz^2-6x^2yz^2-16xy^3z^3-6x^4yz^2$$

$$\nabla\cdot B = 0$$ ∴ Solenoidal

Show that $A = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational. Find ϕ such that $A = \nabla\phi$.

Curl

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (6xy + z^3) & (3x^2 - z) & (3xz^2 - y) \end{vmatrix}$$

$$\nabla \times A = \left\{ \frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z) \right\} \vec{i} + \left\{ \frac{\partial}{\partial z}(6xy + z^3) - \frac{\partial}{\partial x}(3xz^2 - y) \right\} \vec{j} + \left\{ \frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3) \right\} \vec{k}$$

$$\nabla \times A = (-1 + 1)\vec{i} + (3z^2 - 3z^2)\vec{j} + (6x - 6x)\vec{k} = 0$$

$\nabla \times A = 0 \quad \therefore$ irrotational.

$A = \nabla\phi$

$$\frac{\partial \phi}{\partial x} = 6xy + z^3, \quad \frac{\partial \phi}{\partial y} = 3x^2 - z, \quad \frac{\partial \phi}{\partial z} = 3xz^2 - y$$

$$\phi = 3x^2 y + z^3 x + f(y, z)$$

$$\phi = 3x^2 y - zy + f(x, z)$$

$$\phi = xz^3 - zy + f(x, y)$$

if $f(y, z) = -zy$, $f(x, z) = xz^3$, $f(x, y) = 3x^2 y$

$$\phi = 3x^2 y - zy + xz^3$$

$\vec{A} = \nabla\phi$

[illegible]

[illegible]

Prove that the Vector $\vec{A} = 3y^4z^2 i + 4x^3z^2 j - 3x^2y^2 k$ is Solenoidal.

$$div\,\vec{A} = \nabla \cdot \vec{A}$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k})$$

$$= \frac{\partial}{\partial x}3y^4z^2 + \frac{\partial}{\partial y}4x^3z^2 + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0 + 0 + 0 = 0$$

$$div\,\vec{A} = \nabla \cdot \vec{A} = 0 \quad \therefore \text{Solenoidal}$$

$\nabla \cdot A = 0$

∇A Grad,

$= \nabla \cdot A$ Diver.

$= \nabla \times A$ Curl

A

$\nabla \cdot A$

Find $\nabla \psi$ Where $\psi = (x^2+y^2+z^2)\, e^{-\sqrt{x^2+y^2+z^2}}$

$$\nabla \psi = \frac{\partial \psi}{\partial x}\vec{i} + \frac{\partial \psi}{\partial y}\vec{j} + \frac{\partial \psi}{\partial z}\vec{k}$$

$$* \; \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} r^2 e^{-r} = 2r\frac{\partial r}{\partial x} e^{-r} - e^{-r}\frac{\partial r}{\partial x} r^2$$

$$= 2r\frac{x}{r} e^{-r} - e^{-r}\frac{x}{r} r^2$$

$$= 2x e^{-r} - x e^{-r} r = x e^{-r}(2-r)$$

$$* \; \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} r^2 e^{-r} = 2r\frac{\partial r}{\partial y} e^{-r} - e^{-r}\frac{\partial r}{\partial y} r^2$$

$$= 2r\frac{y}{r} e^{-r} - e^{-r}\frac{y}{r} r^2$$

$$= 2y e^{-r} - y e^{-r} r = y e^{-r}(2-r)$$

$$\frac{\partial \psi}{\partial z} = \frac{\partial}{\partial z} r^2 e^{-r} = 2r\frac{\partial r}{\partial z} e^{-r} - e^{-r}\frac{\partial r}{\partial z} r^2$$

$$= 2r\frac{z}{r} e^{-r} - e^{-r}\frac{z}{r} r^2$$

$$= 2z e^{-r} - z e^{-r} r = z e^{-r}(2-r)$$

$$\nabla \psi = x e^{-r}(2-r)\vec{i} + y e^{-r}(2-r)\vec{j} + z e^{-r}(2-r)\vec{k}$$

$$\nabla \psi = (e^{-r}(2-r))(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla \psi = e^{-r}(2-r)\vec{r}$$

if U is a differentiable function of x, y, z

prove $\nabla U . dr = du$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$U = f(x, y, z)$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$\nabla U = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}$$

$$\nabla U . dr = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}\right) . (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du$$

$\therefore$ $\nabla U . dr = du$

$$\frac{\partial}{\partial x} U i + \frac{\partial}{\partial y} U j + \frac{\partial}{\partial z} U k =$$

Z.W

Evaluate $\nabla \cdot (r^3 \vec{r})$

$$\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(x r^3 \vec{i} + y r^3 \vec{j} + z r^3 \vec{k}\right)$$

$$\frac{\partial}{\partial x} x r^3 + \frac{\partial}{\partial y} y r^3 + \frac{\partial}{\partial z} z r^3$$

$$\frac{\partial}{\partial x} x r^3 = r^3 + 3r^2 \frac{\partial r}{\partial x} x = r^3 + 3r^2 \frac{x}{r} x = r^3 + 3x^2 r$$

$$\frac{\partial}{\partial y} y r^3 = r^3 + 3r^2 \frac{\partial r}{\partial y} y = r^3 + 3r^2 \frac{y}{r} y = r^3 + 3y^2 r$$

$$\frac{\partial}{z} z r^3 = r^3 + 3r^2 \frac{\partial r}{\partial z} z = r^3 + 3r^2 \frac{z}{r} z = r^3 + 3z^2 r$$

$$\nabla \cdot (r^3 r) = r^3 + 3x^2 r + r^3 + 3y^2 r + r^3 + 3z^2 r$$

$$= 3r^3 + 3r(x^2 + y^2 + z^2) = 3r^3 + 3r r^2$$

$$= 3r^3 + 3r^3 = 6r^3$$

$$\nabla \cdot (r^3 r) = 6r^3$$

$$= \phi \frac{\partial^2 \psi}{\partial x^2} + \phi \frac{\partial^2 \psi}{\partial y^2} + \phi \frac{\partial^2 \psi}{\partial z^2} + \psi \frac{\partial^2 \phi}{\partial x^2} + \psi \frac{\partial^2 \phi}{\partial y^2}$$

$$+ \psi \frac{\partial^2 \phi}{\partial z^2} + 2\left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial z} \right)$$

$$= \phi \nabla^2 \psi + \psi \nabla^2 \phi + 2 \nabla \phi \cdot \nabla \psi$$

Prove $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\nabla\psi + \psi\nabla^2\phi$

$$\nabla^2(\phi\psi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\phi\psi)$$

$$= \frac{\partial^2}{\partial x^2}\phi\psi + \frac{\partial^2}{\partial y^2}\phi\psi + \frac{\partial^2}{\partial z^2}\phi\psi$$

$$\frac{\partial^2}{\partial x^2}\phi\psi = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\phi\psi\right) = \frac{\partial}{\partial x}\left(\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right)$$

$$= \phi\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \psi\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x}$$

$$\boxed{= \phi\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \psi\frac{\partial^2\phi}{\partial x^2}}$$

$$\frac{\partial^2}{\partial y^2}\phi\psi = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\phi\psi\right) = \frac{\partial}{\partial y}\left(\phi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\phi}{\partial y}\right)$$

$$= \phi\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \psi\frac{\partial^2\phi}{\partial y^2} + \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y}$$

$$= \phi\frac{\partial^2\psi}{\partial y^2} + 2\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \psi\frac{\partial^2\phi}{\partial y^2}$$

$$\frac{\partial^2}{\partial z^2}\phi\psi = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\phi\psi\right) = \frac{\partial}{\partial z}\left(\phi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\frac{\partial^2\psi}{\partial z^2} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z} + \psi\frac{\partial^2\phi}{\partial z^2} + \frac{\partial\psi}{\partial z}\frac{\partial\phi}{\partial z}$$

$$= \phi\frac{\partial^2\psi}{\partial z^2} + 2\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z} + \psi\frac{\partial^2\phi}{\partial z^2}$$

$$\phi\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \psi\frac{\partial^2\phi}{\partial x^2} + \phi\frac{\partial^2\psi}{\partial y^2} + 2\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \psi\frac{\partial^2\phi}{\partial y^2}$$

$$+ \phi\frac{\partial^2\psi}{\partial z^2} + 2\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z} + \psi\frac{\partial^2\phi}{\partial z^2}$$

ϕ

Prove that $\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = 0$

$$\nabla \cdot \left(\frac{r}{r^3}\right) = \nabla \cdot (r^{-3} r) = \nabla \cdot (x r^{-3} \vec{i} + y r^{-3} \vec{j} + z r^{-3} \vec{k})$$

$$\left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}\right) \cdot (x r^{-3} \vec{i} + y r^{-3} \vec{j} + z r^{-3} \vec{k})$$

$$\frac{\partial}{\partial x} x r^{-3} + \frac{\partial}{\partial y} y r^{-3} + \frac{\partial}{\partial z} z r^{-3}$$

$$\frac{\partial}{\partial x} x r^{-3} = r^{-3} - 3 r^{-4} \frac{\partial r}{\partial x} x = r^{-3} - 3 r^{-4} \frac{x}{r} x = r^{-3} - 3x^2 r^{-5}$$

$$\frac{\partial}{\partial y} y r^{-3} = r^{-3} - 3 r^{-4} \frac{\partial r}{\partial y} y = r^{-3} - 3 r^{-4} \frac{y}{r} y = r^{-3} - 3y^2 r^{-5}$$

$$\frac{\partial}{\partial z} z r^{-3} = r^{-3} - 3 r^{-4} \frac{\partial r}{\partial z} z = r^{-3} - 3 r^{-4} \frac{z}{r} z = r^{-3} - 3z^2 r^{-5}$$

$$r^{-3} - 3x^2 r^{-5} + r^{-3} - 3y^2 r^{-5} + r^{-3} - 3z^2 r^{-5}$$

$$3r^{-3} - 3r^{-5}(x^2 + y^2 + z^2) = 3r^{-3} - 3r^{-5} r^2$$

$$= 3r^{-3} - 3r^{-3} = 0$$

$$\therefore \nabla \cdot \left(\frac{r}{r^3}\right) = 0$$

Evaluate $\nabla \times (\vec{r}/r^2)$

$$\frac{\vec{r}}{r^2} = \frac{x}{r^2}\vec{i} + \frac{y}{r^2}\vec{j} + \frac{z}{r^2}\vec{k} = x r^{-2}\vec{i} + y r^{-2}\vec{j} + z r^{-2}\vec{k}$$

$$\nabla \times (r/r^2) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x r^{-2} & y r^{-2} & z r^{-2} \end{vmatrix}$$

$$\nabla \times (\vec{r}/r^2) = \left(\frac{\partial}{\partial y} z r^{-2} - \frac{\partial}{\partial z} y r^{-2}\right)\vec{i} + \left(\frac{\partial}{\partial z} x r^{-2} - \frac{\partial}{\partial x} z r^{-2}\right)\vec{j}$$
$$+ \left(\frac{\partial}{\partial x} y r^{-2} - \frac{\partial}{\partial y} x r^{-2}\right)\vec{k}$$

$$\boxed{\frac{\partial}{\partial y} z r^{-2}} = -2 r^{-3}\frac{\partial r}{\partial y} z = -2 r^{-3}\frac{y}{r} z = -2 z y r^{-4}$$

$$\frac{\partial}{\partial z} y r^{-2} = -2 r^{-3}\frac{\partial r}{\partial z} y = -2 r^{-3}\frac{z}{r} y = -2 z y r^{-4}$$

$$\frac{\partial}{\partial z} x r^{-2} = -2 r^{-3}\frac{\partial r}{\partial z} x = -2 r^{-3}\frac{z}{r} x = -2 z x r^{-4}$$

$$\frac{\partial}{\partial x} z r^{-2} = -2 r^{-3}\frac{\partial r}{\partial x} z = -2 r^{-3}\frac{x}{r} z = -2 x z r^{-4}$$

$$\frac{\partial}{\partial x} y r^{-2} = -2 r^{-3}\frac{\partial r}{\partial x} y = -2 r^{-3}\frac{x}{r} y = -2 x y r^{-4}$$

$$\frac{\partial}{\partial y} x r^{-2} = -2 r^{-3}\frac{\partial r}{\partial y} x = -2 r^{-3}\frac{y}{r} x = -2 y x r^{-4}$$

$$\nabla \times (r/r^2) = (-2zyr^{-4} + 2zyr^{-4})\vec{i} + (-2zxr^{-4} + 2zxr^{-4})\vec{j} + (-2xyr^{-4} + 2xyr^{-4})\vec{k}$$

$$\nabla \times (r/r^2) = 0$$

$$= \frac{\partial}{\partial x}(-2x\,r^{-4}) = -2\left(r^{-4} - 4r^{-5}\frac{\partial r}{\partial x}x\right)$$

$$= -2\left(r^{-4} - 4r^{-5}\frac{x}{r}x\right) = -2(r^{-4} - 4x^2r^{-6})$$

$$\frac{\partial^2}{\partial y^2}r^{-2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}r^{-2}\right) = \frac{\partial}{\partial y}\left(-2r^{-3}\frac{\partial r}{\partial y}\right) = \frac{\partial}{\partial y}\left(-2r^{-3}\frac{y}{r}\right)$$

$$= \frac{\partial}{\partial y}(-2y\,r^{-4}) = -2\left(r^{-4} - 4r^{-5}\frac{\partial r}{\partial y}y\right)$$

$$= -2\left(r^{-4} - 4r^{-5}\frac{y}{r}y\right) = -2(r^{-4} - 4y^2r^{-6})$$

$$\frac{\partial^2}{\partial z^2}r^{-2} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}r^{-2}\right) = \frac{\partial}{\partial z}\left(-2r^{-3}\frac{\partial r}{\partial z}\right) = \frac{\partial}{\partial z}\left(-2r^{-3}\frac{z}{r}\right)$$

$$= \frac{\partial}{\partial z}(-2z\,r^{-4}) = -2\left(r^{-4} - 4r^{-5}\frac{\partial r}{\partial z}z\right)$$

$$= -2\left(r^{-4} - 4r^{-5}\frac{z}{r}z\right) = -2(r^{-4} - 4z^2r^{-6})$$

$$\nabla^2[\nabla\cdot(\mathbf{r}/r^2)] = -2(r^{-4} - 4x^2r^{-6} + r^{-4} - 4y^2r^{-6} + r^{-4} - 4z^2r^{-6})$$

$$= -2(3r^{-4} - 4r^{-6}(x^2+y^2+z^2))$$

$$= -2(3r^{-4} - 4r^{-6}r^2) = -2(3r^{-4} - 4r^{-4})$$

$$= -2(-r^{-4}) = 2r^{-4}$$

$$\nabla^2[\nabla\cdot(\mathbf{r}/r^2)] = 2r^{-4}$$

2

نفسة بالوحدة

Evaluate $\nabla^2[\nabla\cdot(\vec{r}/r^2)]$ 2012 [illegible]

$$\nabla\cdot(\vec{r}/r^2) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(\frac{x\vec{i}+y\vec{j}+z\vec{k}}{r^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^2}\right)$$

$$= \frac{\partial}{\partial x} x r^{-2} + \frac{\partial}{\partial y} y r^{-2} + \frac{\partial}{\partial z} z r^{-2}$$

$$\frac{\partial}{\partial x} x r^{-2} = r^{-2} - 2r^{-3}\frac{\partial r}{\partial x} x = r^{-2} - 2r^{-3}\frac{x}{r} x$$

$$= r^{-2} - 2x^2 r^{-4}$$

$$\frac{\partial}{\partial y} y r^{-2} = r^{-2} - 2r^{-3}\frac{\partial r}{\partial y} y = r^{-2} - 2r^{-3}\frac{y}{r} y$$

$$= r^{-2} - 2y^2 r^{-4}$$

$$\frac{\partial}{\partial z} z r^{-2} = r^{-2} - 2r^{-3}\frac{\partial r}{\partial z} z = r^{-2} - 2r^{-3}\frac{z}{r} z$$

$$= r^{-2} - 2z^2 r^{-4}$$

$$\nabla\cdot(r/r^2) = r^{-2} - 2x^2 r^{-4} + r^{-2} - 2y^2 r^{-4} + r^{-2} - 2z^2 r^{-4}$$

$$= 3r^{-2} - 2r^{-4}(x^2+y^2+z^2)$$

$$= 3r^{-2} - 2r^{-4} r^2 = 3r^{-2} - 2r^{-2} = r^{-2}$$

$$\nabla\cdot(r/r^2) = r^{-2}$$

$$\nabla^2[\nabla\cdot(r/r^2)] = \nabla^2 r^{-2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^{-2}$$

$$= \frac{\partial^2}{\partial x^2} r^{-2} + \frac{\partial^2}{\partial y^2} r^{-2} + \frac{\partial^2}{\partial z^2} r^{-2}$$

$$\frac{\partial^2}{\partial x^2} r^{-2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} r^{-2}\right) = \frac{\partial}{\partial x}\left(-2r^{-3}\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(-2r^{-3}\frac{x}{r}\right)$$

if $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find $\phi(x,y,z)$, if $\phi(1,-2,2) = 4$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3, \quad \frac{\partial\phi}{\partial y} = x^2z^3, \quad \frac{\partial\phi}{\partial z} = 3x^2yz^2$$

$$\phi = \int\frac{\partial\phi}{\partial x}dx, \quad \phi = \int\frac{\partial\phi}{\partial y}dy, \quad \phi = \int\frac{\partial\phi}{\partial z}dz$$

$$\phi = \int 2xyz^3 dx = x^2yz^3 + C$$

$$\phi = \int x^2z^3 dy = x^2yz^3 + C$$

$$\phi = \int 3x^2yz^2 dz = x^2yz^3 + C$$

$$\therefore \quad \phi = x^2yz^3 + C$$

$$\phi(1,-2,2) = 4 \rightarrow 4 = -16 + C \rightarrow C = 20$$

$$\phi = x^2yz^3 + 20$$

(يد) O.K

* Prove $\boxed{\nabla f(r) = \frac{f'(r)\,\vec{r}}{|r|}}$

$$\nabla f(r) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) f(r)$$

$$= \frac{\partial}{\partial x} f(r)\,\vec{i} + \frac{\partial}{\partial y} f(r)\,\vec{j} + \frac{\partial}{\partial z} f(r)\,\vec{k}$$

$$\frac{\partial}{\partial x} f(r) = f'(r)\frac{\partial r}{\partial x} = f'(r)\,\frac{x}{r}$$

$$\frac{\partial}{\partial y} f(r) = f'(r)\frac{\partial r}{\partial y} = f'(r)\,\frac{y}{r}$$

$$\frac{\partial}{\partial z} f(r) = f'(r)\frac{\partial r}{\partial z} = f'(r)\,\frac{z}{r}$$

$$\nabla f(r) = f'(r)\frac{x}{r}\vec{i} + f'(r)\frac{y}{r}\vec{j} + f'(r)\frac{z}{r}\vec{k}$$

$$\nabla f(r) = \frac{f'(r)}{r}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla f(r) = \frac{f'(r)\,\vec{r}}{r}$$

$$\nabla f(r) = \frac{f'(r)\,\vec{r}}{|r|}$$

$$\left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot f(r)$$

$$= \frac{\partial f(r)}{\partial x} i + \frac{\partial f r}{\partial y} j + \frac{\partial f r}{\partial z} k$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = f'(r) \cdot \frac{x}{r}$$

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) \cdot (xi + yj + zk)$$

$\sqrt{x^2 + y^2 + z^2}$ r r^2

الموضوع :

Find $\nabla |r|^3$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$r = (x^2+y^2+z^2)^{1/2} \quad , \quad r^3 = (x^2+y^2+z^2)^{3/2}$$

$$\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(x^2+y^2+z^2)^{3/2}$$

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2}\vec{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{3/2}\vec{j} + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{3/2}$$

$$\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2}\cancel{2}x\vec{i} + \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2}\cancel{2}y\vec{j} + \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2}\cancel{2}z\vec{k}$$

$$3rx\vec{i} + 3ry\vec{j} + 3rz\vec{k}$$

$$3r(x\vec{i} + y\vec{j} + z\vec{k}) = 3r\bar{r}$$

Evaluate $\nabla^2 (\ln r)$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$r = (x^2 + y^2 + z^2)^{1/2}$$

$$\nabla^2(\ln r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\ln r)$$

$$= \frac{\partial^2}{\partial x^2}\ln r + \frac{\partial^2}{\partial y^2}\ln r + \frac{\partial^2}{\partial z^2}\ln r$$

$$\frac{\partial^2}{\partial x^2}\ln r = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\ln r\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\frac{x}{r}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right)$$

$$= \frac{\partial}{\partial x} x r^{-2} = r^{-2} - 2r^{-3}\frac{\partial r}{\partial x}x = r^{-2} - 2r^{-3}\frac{x}{r}x$$

$$= r^{-2} - 2x^2 r^{-4}$$

$$\frac{\partial^2}{\partial y^2}\ln r = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\ln r\right) = \frac{\partial}{\partial y}\left(\frac{1}{r}\frac{\partial r}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{r}\frac{y}{r}\right) = \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right)$$

$$= \frac{\partial}{\partial y}(y r^{-2}) = r^{-2} - 2r^{-3}\frac{\partial r}{\partial y}y = r^{-2} - 2r^{-3}\frac{y}{r}y$$

$$= r^{-2} - 2y^2 r^{-4}$$

$$\frac{\partial^2}{\partial z^2}\ln r = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\ln r\right) = \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial r}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{z}{r}\right) = \frac{\partial}{\partial z}\left(\frac{z}{r^2}\right)$$

$$= \frac{\partial}{\partial z} z r^{-2} = r^{-2} - 2r^{-3}\frac{\partial r}{\partial z}z = r^{-2} - 2r^{-3}\frac{z}{r}z$$

$$= r^{-2} - 2z^2 r^{-4}$$

$$\nabla^2(\ln r) = r^{-2} - 2x^2 r^{-4} + r^{-2} - 2y^2 r^{-4} + r^{-2} - 2z^2 r^{-4}$$

$$= 3r^{-2} - 2r^{-4}(x^2 + y^2 + z^2) = 3r^{-2} - 2r^{-4}r^2 = 3r^{-2} - 2r^{-2}$$

$$= r^{-2} = \frac{1}{r^2}$$

$$-r^{-3}+3x^2r^{-5}-r^{-3}+3y^2r^{-5}-r^{-3}+3z^2r^{-5}=0$$

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$$-3r^{-3}+\boxed{3r^{-5}}(x^2+y^2+z^2)=0$$

$$-3r^{-3}+3r^{-5}\,r^{2}=0$$

$$-3r^{-3}+3r^{-3}=0$$

$$\therefore\quad \nabla^2\left(\frac{1}{r}\right)=0$$

Prove that $\nabla^2\left(\frac{1}{|r|}\right) = 0$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$r = (x^2 + y^2 + z^2)^{1/2}$$

$$\nabla^2\left(\frac{1}{r}\right) = 0 \longrightarrow \nabla^2 r^{-1} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^{-1} = 0$$

$$\frac{\partial^2}{\partial x^2} r^{-1} + \frac{\partial^2}{\partial y^2} r^{-1} + \frac{\partial^2}{\partial z^2} r^{-1} = 0$$

$$\frac{\partial^2}{\partial x^2} r^{-1} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} r^{-1}\right) = \frac{\partial}{\partial x}\left(-r^{-2}\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(-r^{-2}\frac{x}{r}\right)$$

$$= \frac{\partial}{\partial x}(-r^{-3}x) = -\left(r^{-3} - 3r^{-4}\frac{\partial r}{\partial x}x\right) = -\left(r^{-3} - 3r^{-4}\frac{x}{r}x\right)$$

$$= -(r^{-3} - 3r^{-5}x^2) = -r^{-3} + 3x^2 r^{-5}$$

$$\frac{\partial^2}{\partial y^2} r^{-1} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} r^{-1}\right) = \frac{\partial}{\partial y}\left(-r^{-2}\frac{\partial r}{\partial y}\right) = \frac{\partial}{\partial y}\left(-r^{-2}\frac{y}{r}\right)$$

$$= \frac{\partial}{\partial y}(-y r^{-3}) = -\left(r^{-3} - 3r^{-4}\frac{\partial r}{\partial y}y\right)$$

$$= -\left(r^{-3} - 3r^{-4}\frac{y}{r}y\right) = -(r^{-3} - 3r^{-5}y^2) = -r^{-3} + 3y^2 r^{-5}$$

$$\frac{\partial^2}{\partial z^2} r^{-1} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} r^{-1}\right) = \frac{\partial}{\partial z}\left(-r^{-2}\frac{\partial r}{\partial z}\right) = \frac{\partial}{\partial z}\left(-r^{-2}\frac{z}{r}\right)$$

$$= \frac{\partial}{\partial z}(-z r^{-3}) = -\left(r^{-3} - 3r^{-4}\frac{\partial r}{\partial z}z\right) = -\left(r^{-3} - 3r^{-4}\frac{z}{r}z\right)$$

$$= -(r^{-3} - 3z^2 r^{-5}) = -r^{-3} + 3z^2 r^{-5}$$

Find the acute angle between the Surface $xy^2z = 3x$ and $3x^2 - y^2 + 2z = 1$ at the Point $(1, -2, 1)$

for first Surface

$$\phi_1 = xy^2z - 3x - z^2$$

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\vec{i} + \frac{\partial\phi_1}{\partial y}\vec{j} + \frac{\partial\phi_1}{\partial z}\vec{k}$$

$$\nabla\phi_1 = (y^2z - 3)\vec{i} + (2xyz)\vec{j} + (xy^2 - 2z)\vec{k}$$

$$\boxed{\nabla\phi_1 = \vec{i} - 4\vec{j} + 2\vec{k}}$$ at the Point $(1, -2, 1)$

$$|\nabla\phi_1| = \sqrt{1^2 + (4)^2 + 2^2} = \sqrt{21}$$

for Second Surface

$$\phi_2 = 3x^2 - y^2 + 2z - 1$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\vec{i} + \frac{\partial\phi_2}{\partial y}\vec{j} + \frac{\partial\phi_2}{\partial z}\vec{k}$$

$$\nabla\phi_2 = 6x\vec{i} - 2y\vec{j} + 2\vec{k}$$

$$\nabla\phi_2 = 6\vec{i} + 4\vec{j} + 2\vec{k}$$ at the Point $(1, -2, 1)$

$$|\nabla\phi_2| = \sqrt{6^2 + 4^2 + 2^2} = \sqrt{56}$$

$$\nabla\phi_1 \cdot \nabla\phi_2 = 6 - 16 + 4 = -6$$

$$\boxed{\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1|\,|\nabla\phi_2| \cos\theta}$$

$$-6 = \sqrt{21}\sqrt{56}\cos\theta \rightarrow \theta =$$

$\nabla f_1 \cdot \nabla f_2 = |\nabla f_1| \cdot |\nabla f_2|$

Q: What Value of the Constant a will the Vector $A = (axy - z^3)\vec{i} + (a-2)x^2\vec{j} + (1-a)xz^2 K$ have its Curl identically equal to zero ?

$$\nabla \times A = 0$$

$$\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{K}\right) \times \left\{(axy - z^3)\vec{i} + (a-2)x^2\vec{j} + (1-a)xz^2\vec{K}\right\}$$

$$\nabla \times A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$$\nabla \times A = \left\{\frac{\partial}{\partial y}(1-a)\cancel{xz^2} - \frac{\partial}{\partial z}(a-2)x^2\right\}\vec{i} + \left\{\frac{\partial}{\partial z}(axy - z^3) - \frac{\partial}{\partial x}(1-a)xz^2\right\}\vec{j} + \left\{\frac{\partial}{\partial x}(a-2)x^2 - \frac{\partial}{\partial y}(axy - z^3)\right\}\vec{K} = 0$$

$$(-3z^2 - (1-a)z^2)\vec{j} + (2(a-2)x - ax)\vec{K} = 0$$

$$(-3z^2 - z^2 + az^2)\vec{j} + (2ax - 4x - ax)\vec{K} = 0$$

$$(-4z^2 + az^2)\vec{j} + (ax - 4x)\vec{K} = 0$$

$$-4z^2 + az^2 = 0 \rightarrow az^2 = 4z^2 \rightarrow a = 4$$

$$ax - 4x = 0 \rightarrow ax = 4x \rightarrow a = 4$$

صغر
علي
مكمل

Find equations for the tangent plane and normal line to the Surface $Z = X^2 + y^2$ at the point $(2, -1, 5)$

$$\phi = X^2 + y^2 - Z$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi = 4\vec{i} - 2\vec{j} - \vec{k}$$

$\nabla\phi = \frac{\partial\phi}{\partial x} \quad \frac{\partial\phi}{\partial y}$

$2x\vec{i} + 2y\vec{j} - 1\vec{k}$

$4 \quad -2 \quad -1$

$(\vec{r} - \vec{r_0}) \cdot \vec{N} = 0$

$[(x-2)\vec{i} + (y+1)\vec{j} + (z-5)\vec{k}] \cdot [4\vec{i} - 2\vec{j} - \vec{k}]$

$4(x-2) - 2(y+1) - (z-5) = 0$

$4x - 8 - 2y - 2 - z + 5 = 0$

$\boxed{4x - 2y - z = 5}$

the Equation of tangent Plane.

$$f_x(X - X_0) + f_y(y - y_0) + f_z(Z - Z_0) = 0$$

$$4(X - 2) - 2(y + 1) - (Z - 5) = 0$$

$$4X - 8 - 2y - 2 - Z + 5 = 0$$

$$4X - 2y - Z = 5$$

$f_x(X - X_0) + f_y(y - y_0) + f_z(z$

$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y}$

the Equation of the Normal line

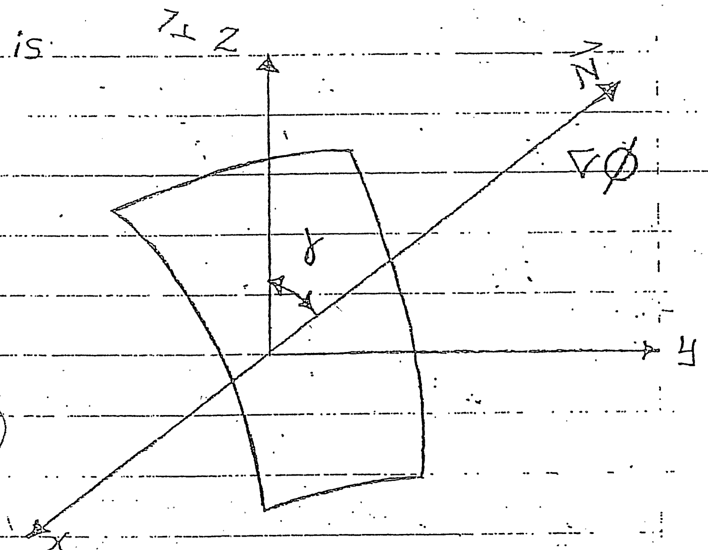
$$\frac{X - X_0}{f_x} = \frac{y - y_0}{f_y} = \frac{Z - Z_0}{f_z}$$

$$\frac{X - 2}{4} = \frac{y + 1}{-2} = \frac{Z - 5}{-1}$$

Prove that the acute angle γ between the Z axis and the normal to the Surface $F(x,y,z)=0$ at any point is given by

$$\sec\gamma = \sqrt{F_x^2 + F_y^2 + F_z^2} \Big/ |F_z|$$

the Normal to the Surface is the gradient ∇F



$F(x,y,z)=0$

$$\nabla F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{K}$$

$$\nabla F = f_x\vec{i} + f_y\vec{j} + f_z\vec{K}$$

$\vec{r} = z\vec{K}$, $\vec{r}$ position vector from origin to any point $(0,0,z)$

$$d\vec{r} = dz\vec{K}$$

$$\boxed{\nabla\phi \cdot d\vec{r} = |\nabla\phi|\,|dr|\cos\gamma}$$

$$f_z\,dz = \sqrt{f_x^2 + f_y^2 + f_z^2}\,\sqrt{(dz)^2}\,\cos\gamma$$

$$f_z = \sqrt{f_x^2 + f_y^2 + f_z^2}\,\cos\gamma$$

$$\sec\gamma = \sqrt{f_x^2 + f_y^2 + f_z^2} \Big/ f_z$$

الموضوع :   التاريخ   /   / ٢٠٠

Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ in the direction $2i - 3j + 6k$

$$\phi = 4xz^3 - 3x^2y^2z$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\nabla\phi = (4z^3 - 6xy^2z)\vec{i} - 6x^2yz\vec{j} + (12xz^2 - 3x^2y^2)\vec{k}$$

$$\nabla\phi = 8\vec{i} + 48\vec{j} + 84\vec{k}$$

$$\vec{V} = 2\vec{i} - 3\vec{j} + 6\vec{k}$$

$$|\vec{V}| = \sqrt{2^2 + (-3)^2 + 6^2} = 7$$

$$\text{direction } \vec{v} = \frac{\vec{V}}{|\vec{V}|} = \frac{1}{7}(2\vec{i} - 3\vec{j} + 6\vec{k}) = \frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}$$

normal

the directional derivative $= \nabla\phi \cdot \vec{V} \rightarrow$ unit vector

$$= \frac{16}{7} - \frac{144}{7} + \frac{504}{7} = \frac{376}{7}$$

a) Find the directional derivative of $U = 2xy - z^2$ at $(2,-1,1)$ in direction toward $(3,1,-1)$

(b) In What direction is the directional derivative a maximum

(c) What is the Value of this maximum?

(a) $U = 2xy - z^2$ directional deriv $= \frac{d\phi}{ds} = \nabla\phi \cdot u$ unit vector

$$\nabla U = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}$$

$$\nabla U = 2y\vec{i} + 2x\vec{j} - 2z\vec{k}$$

$$\nabla U = -2\vec{i} + 4\vec{j} - 2\vec{k} \quad \text{at Point } (2,-1,1)$$

$$\vec{V} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

$$\vec{V} = (3-2)\vec{i} + (1+1)\vec{j} + (-1-1)\vec{k}$$

$$\vec{V} = \vec{i} + 2\vec{j} - 2\vec{k}$$

$$|\vec{V}| = \sqrt{1^2 + 2^2 + (-2)^2} = 3$$

$$\vec{V} = \frac{1}{3}(\vec{i} + 2\vec{j} - 2\vec{k}) = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k} \text{ (unit vector)}$$

$$\nabla U \cdot \vec{V} = \frac{2}{3} + \frac{8}{3} + \frac{4}{3} = \frac{10}{3}$$

(b) In the direction of the gradient (grade U) ∇U

(c) $|\nabla U| = \sqrt{(-2)^2 + (4)^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$

[illegible] $= \nabla\phi \cdot U$ unit vector

[illegible] $|\nabla U|$

Find the equations of (a) tangent plane and (b) normal line to the Surface $x^2+y^2=4z$ at $(2,-4,5)$.

$\phi = x^2+y^2-4z$

to find the Normal to the Surface

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

لان نعوض في نقطة لو اتخلص من المجاهيل

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} - 4\vec{k}$$

متجه

$$\nabla\phi = 4\vec{i} - 8\vec{j} - 4\vec{k} \quad \text{at Point } (2,-4,5)$$

Equation of tangent Plan.

$f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$

$4(x-2) - 8(y+4) - 4(z-5) = 0$

$4x - 8 - 8y - 32 - 4z + 20 = 0$

$4x - 8y - 4z = 20$

$x - 2y - z = 5$

المعادلة بدلالة المتغيرات

المتجه العمودي الذي وجدناه

Equation of Normal line

$$\frac{x-x_0}{f_x} = \frac{y-y_0}{f_y} = \frac{z-z_0}{f_z}$$

$$\frac{x-2}{1} = \frac{y+4}{-2} = \frac{z-5}{-1}$$

$(r-r_0)\cdot N$

$x-x_0 \quad y-y_0 = z-z_0$

$x-x_1 \quad y-y_1 \quad z-z_1$

..W

Showthat $\nabla \times (r^2 \vec{r}) = 0$ where $r = xi + yj + zk$ and $r = |r|$

$$r^2 = x^2 + y^2 + z^2$$

$$r^2 r = (x^2+y^2+z^2)(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= (x^2+y^2+z^2)x\vec{i} + (x^2+y^2+z^2)y\vec{j} + (x^2+y^2+z^2)z\vec{k}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla \times (r^2 r) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2+y^2+z^2)x & (x^2+y^2+z^2)y & (x^2+y^2+z^2)z \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(x^2+y^2+z^2)z - \frac{\partial}{\partial z}(x^2+y^2+z^2)y\right\}\vec{i} + \left\{\frac{\partial}{\partial z}(x^2+y^2+z^2)x - \frac{\partial}{\partial x}(x^2+y^2+z^2)z\right\}\vec{j} + \left\{\frac{\partial}{\partial x}(x^2+y^2+z^2)y - \frac{\partial}{\partial y}(x^2+y^2+z^2)x\right\}\vec{k}$$

$$= (2yz - 2zy)\vec{i} + (2zx - 2xz)\vec{j} + (2xy - 2yx)\vec{k} = 0$$

$$\therefore \nabla \times (r^2 r) = 0$$

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)A_1\vec{i}-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)A_2\vec{j}$$

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)A_3\vec{k}+\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}\right)\vec{i}+\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}\right)\vec{i}$$

$$+\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial z}\right)\vec{i}+\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial y}\right)\vec{j}+\frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial z}\right)\vec{j}+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x}\right)\vec{j}$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial z}\right)\vec{k}+\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x}\right)\vec{k}+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial y}\right)\vec{k}$$

$$=-\nabla^2 A_1\vec{i}-\nabla^2 A_2\vec{j}-\nabla^2 A_3\vec{k}+\frac{\partial}{\partial x}\vec{i}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$

$$+\frac{\partial}{\partial y}\vec{j}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)+\frac{\partial}{\partial z}\vec{k}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$

$$=-\nabla^2(A_1\vec{i}+A_2\vec{j}+A_3\vec{k})+\frac{\partial}{\partial x}\vec{i}(\nabla\cdot A)+\frac{\partial}{\partial y}\vec{j}(\nabla\cdot A)$$

$$+\frac{\partial}{\partial z}\vec{k}(\nabla\cdot A)$$

$$=-\nabla^2 A+\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)(\nabla\cdot A)$$

$$=-\nabla^2 A+\nabla(\nabla\cdot A)$$

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* Prove that $\nabla \times (\nabla \times \vec{A}) = -\nabla^2 \vec{A} + \nabla(\nabla \cdot \vec{A})$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

$$\nabla \times (\nabla \times A) = \nabla \times \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla \times \left\{ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\vec{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\vec{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\vec{k} \right\}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) & \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) & \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \end{vmatrix}$$

$$= \left\{ \frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) \right\}\vec{i} + \left\{ \frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \right\}\vec{j} + \left\{ \frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) \right\}\vec{k}$$

نضيف ، لها $\frac{\partial^2 A_1}{\partial x^2}$

$$= \frac{\partial^2 A_2}{\partial y \partial x}\vec{i} - \frac{\partial^2 A_1}{\partial y^2}\vec{i} - \frac{\partial^2 A_1}{\partial z^2}\vec{i} + \frac{\partial^2 A_3}{\partial x \partial z}\vec{i} + \frac{\partial^2 A_3}{\partial z \partial y}\vec{j} - \frac{\partial^2 A_2}{\partial z^2}\vec{j}$$

لها $\frac{\partial^2 A_2}{\partial y^2}$

$$- \frac{\partial^2 A_2}{\partial x^2}\vec{j} + \frac{\partial^2 A_1}{\partial y \partial x}\vec{j} + \frac{\partial^2 A_1}{\partial x \partial z}\vec{k} - \frac{\partial^2 A_3}{\partial x^2}\vec{k} - \frac{\partial^2 A_3}{\partial y^2}\vec{k} + \frac{\partial^2 A_2}{\partial y \partial z}\vec{k}$$

لها $\frac{\partial^2 A_3}{\partial z^2}$

$$= \frac{\partial^2 A_1}{\partial x^2}\vec{i} - \frac{\partial^2 A_1}{\partial y^2}\vec{i} - \frac{\partial^2 A_1}{\partial z^2}\vec{i} - \cancel{\frac{\partial^2 A_1}{\partial x^2}\vec{i}} + \frac{\partial^2 A_2}{\partial y \partial x}\vec{i} + \frac{\partial^2 A_3}{\partial x \partial z}\vec{i} - \frac{\partial^2 A_2}{\partial x^2}\vec{j}$$

$$+ \frac{\partial^2 A_2}{\partial y^2}\vec{j} - \frac{\partial^2 A_2}{\partial z^2}\vec{j} - \cancel{\frac{\partial^2 A_2}{\partial y^2}\vec{j}} + \frac{\partial^2 A_3}{\partial z \partial y}\vec{j} + \frac{\partial^2 A_1}{\partial y \partial x}\vec{j} - \frac{\partial^2 A_3}{\partial x^2}\vec{k} - \frac{\partial^2 A_3}{\partial y^2}\vec{k}$$

$$- \frac{\partial^2 A_3}{\partial z^2}\vec{k} + \cancel{\frac{\partial^2 A_3}{\partial z^2}\vec{k}} + \frac{\partial^2 A_1}{\partial x \partial z}\vec{k} + \frac{\partial^2 A_2}{\partial y \partial z}\vec{k}$$

Prove that

$$\nabla(FG) = F\nabla G + G\nabla F$$

$$\nabla(FG) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{K}\right)(FG)$$

$$= \frac{\partial}{\partial x}(FG)\vec{i} + \frac{\partial}{\partial y}(FG)\vec{j} + \frac{\partial}{\partial z}(FG)\vec{K}$$

$$= F\frac{\partial G}{\partial x}\vec{i} + G\frac{\partial F}{\partial x}\vec{i} + F\frac{\partial G}{\partial y}\vec{j} + G\frac{\partial F}{\partial y}\vec{j}$$

$$+ F\frac{\partial G}{\partial z}\vec{K} + G\frac{\partial F}{\partial z}\vec{K}$$

$$= F\frac{\partial G}{\partial x}\vec{i} + F\frac{\partial G}{\partial y}\vec{j} + F\frac{\partial G}{\partial z}\vec{K} + G\frac{\partial F}{\partial x}\vec{i}$$

$$+ G\frac{\partial F}{\partial y}\vec{j} + G\frac{\partial F}{\partial z}\vec{K}$$

$$= F\left(\frac{\partial G}{\partial x}\vec{i} + \frac{\partial G}{\partial y}\vec{j} + \frac{\partial G}{\partial z}\vec{K}\right) + G\left(\frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{K}\right)$$

$$= F\nabla G + G\nabla F$$

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$$= B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + B_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + B_3\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$-\left[A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) + A_2\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right) + A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\right]$$

$$= (B_1\vec{i} + B_2\vec{j} + B_3\vec{k}) \cdot \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\vec{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\vec{j}\right.$$

$$\left. + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\vec{k}\right] - (A_1\vec{i} + A_2\vec{j} + A_3\vec{k}) \cdot \left[\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\vec{i}\right.$$

$$\left. + \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\vec{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\vec{k}\right]$$

$$= B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

$$\therefore \nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

Provethat $\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{K}$$

$$\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{K}$$
$$\vec{B} = B_1\vec{i} + B_2\vec{j} + B_3\vec{K}$$

$$A\times B = \begin{vmatrix} i & j & K \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$A\times B = (A_2B_3 - A_3B_2)\vec{i} + (A_3B_1 - A_1B_3)\vec{j} + (A_1B_2 - A_2B_1)\vec{K}$$

$$\nabla\cdot(A\times B) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{K}\right)\cdot\left[(A_2B_3 - A_3B_2)\vec{i} + (A_3B_1 - A_1B_3)\vec{j} + (A_1B_2 - A_2B_1)\vec{K}\right]$$

$$= \frac{\partial}{\partial x}(A_2B_3 - A_3B_2) + \frac{\partial}{\partial y}(A_3B_1 - A_1B_3) + \frac{\partial}{\partial z}(A_1B_2 - A_2B_1)$$

$$= \frac{\partial}{\partial x}A_2B_3 - \frac{\partial}{\partial x}A_3B_2 + \frac{\partial}{\partial y}A_3B_1 - \frac{\partial}{\partial y}A_1B_3 + \frac{\partial}{\partial z}A_1B_2 - \frac{\partial}{\partial z}A_2B_1$$

$$= A_2\frac{\partial B_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial B_2}{\partial x} - B_2\frac{\partial A_3}{\partial x} + A_3\frac{\partial B_1}{\partial y} + B_1\frac{\partial A_3}{\partial y}$$
$$- A_1\frac{\partial B_3}{\partial y} - B_3\frac{\partial A_1}{\partial y} + A_1\frac{\partial B_2}{\partial z} + B_2\frac{\partial A_1}{\partial z} - A_2\frac{\partial B_1}{\partial z} - B_1\frac{\partial A_2}{\partial z}$$

$$= B_1\frac{\partial A_3}{\partial y} - B_1\frac{\partial A_2}{\partial z} + B_2\frac{\partial A_1}{\partial z} - B_2\frac{\partial A_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} - B_3\frac{\partial A_1}{\partial y}$$
$$- A_1\frac{\partial B_3}{\partial y} + A_1\frac{\partial B_2}{\partial z} - A_2\frac{\partial B_1}{\partial z} + A_2\frac{\partial B_3}{\partial x} - A_3\frac{\partial B_2}{\partial x} + A_3\frac{\partial B_1}{\partial y}$$

Prove that $\nabla \times (U\vec{A}) = (\nabla U)\times \vec{A} + U(\nabla \times \vec{A})$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}, \quad A = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

$$UA = UA_1\vec{i} + UA_2\vec{j} + UA_3\vec{k}$$

$$\nabla \times (UA) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\times\left(UA_1\vec{i} + UA_2\vec{j} + UA_3\vec{k}\right)$$

$$\nabla \times (UA) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ UA_1 & UA_2 & UA_3 \end{vmatrix}$$

$$\nabla \times (UA) = \left(\frac{\partial}{\partial y}UA_3 - \frac{\partial}{\partial z}UA_2\right)\vec{i} + \left(\frac{\partial}{\partial z}UA_1 - \frac{\partial}{\partial x}UA_3\right)\vec{j} + \left(\frac{\partial}{\partial x}UA_2 - \frac{\partial}{\partial y}UA_1\right)\vec{k}$$

هنفاضل U و A

scalar function نعتبر الاتجاه

$$= U\frac{\partial A_3}{\partial y}\vec{i} + A_3\frac{\partial U}{\partial y}\vec{i} - U\frac{\partial A_2}{\partial z}\vec{i} - A_2\frac{\partial U}{\partial z}\vec{i} + U\frac{\partial A_1}{\partial z}\vec{j} + A_1\frac{\partial U}{\partial z}\vec{j}$$

$$- U\frac{\partial A_3}{\partial x}\vec{j} - A_3\frac{\partial U}{\partial x}\vec{j} + U\frac{\partial A_2}{\partial x}\vec{k} + A_2\frac{\partial U}{\partial x}\vec{k} - U\frac{\partial A_1}{\partial y}\vec{k} - A_1\frac{\partial U}{\partial y}\vec{k}$$

$$= A_3\frac{\partial U}{\partial y}\vec{i} - A_2\frac{\partial U}{\partial z}\vec{i} + A_1\frac{\partial U}{\partial z}\vec{j} - A_3\frac{\partial U}{\partial x}\vec{j} + A_2\frac{\partial U}{\partial x}\vec{k} - A_1\frac{\partial U}{\partial y}\vec{k}$$

$$+ U\frac{\partial A_3}{\partial y}\vec{i} - U\frac{\partial A_2}{\partial z}\vec{i} + U\frac{\partial A_1}{\partial z}\vec{j} - U\frac{\partial A_3}{\partial x}\vec{j} + U\frac{\partial A_2}{\partial x}\vec{k} - U\frac{\partial A_1}{\partial y}\vec{k}$$

$$= \left(\frac{\partial U}{\partial y}A_3 - \frac{\partial U}{\partial z}A_2\right)\vec{i} + \left(\frac{\partial U}{\partial z}A_1 - \frac{\partial U}{\partial x}A_3\right)\vec{j} + \left(\frac{\partial U}{\partial x}A_2 - \frac{\partial U}{\partial y}A_1\right)\vec{k}$$

$$+ U\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\vec{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\vec{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\vec{k}\right]$$

$$= (\nabla U)\times\vec{A} + U(\nabla\times\vec{A})$$

التاريخ / /

Prove that $\nabla\cdot(\phi A) = (\nabla\phi)\cdot A + \phi(\nabla\cdot A)$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{K}$$

$$A = A_1\vec{i} + A_2\vec{j} + A_3\vec{K}$$

$$\phi A = \phi A_1\vec{i} + \phi A_2\vec{j} + \phi A_3\vec{K}$$

$$\nabla\cdot(\phi A) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{K}\right)\cdot\left(\phi A_1\vec{i} + \phi A_2\vec{j} + \phi A_3\vec{K}\right)$$

$$= \frac{\partial}{\partial x}\phi A_1 + \frac{\partial}{\partial y}\phi A_2 + \frac{\partial}{\partial z}\phi A_3$$

$$= \phi\frac{\partial A_1}{\partial x} + A_1\frac{\partial\phi}{\partial x} + \phi\frac{\partial A_2}{\partial y} + A_2\frac{\partial\phi}{\partial y} + \phi\frac{\partial A_3}{\partial z} + A_3\frac{\partial\phi}{\partial z}$$

$$= A_1\frac{\partial\phi}{\partial x} + A_2\frac{\partial\phi}{\partial y} + A_3\frac{\partial\phi}{\partial z} + \phi\frac{\partial A_1}{\partial x} + \phi\frac{\partial A_2}{\partial y} + \phi\frac{\partial A_3}{\partial z}$$

$$= \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{K}\right)\cdot\left(A_1\vec{i} + A_2\vec{j} + A_3\vec{K}\right) +$$

$$\phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= (\nabla\phi)\cdot A + \phi\left[\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{K}\right)\cdot\left(A_1\vec{i} + A_2\vec{j} + A_3\vec{k}\right)\right]$$

$$= (\nabla\phi)\cdot A + \phi(\nabla\cdot A)$$

$$\therefore\ \nabla\cdot(\phi A) = (\nabla\phi)\cdot A + \phi(\nabla\cdot A)$$

زينل محمد رشيد

Prove that $\nabla \times (A+B) = \nabla \times A + \nabla \times B$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$A = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$
$$B = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$$
$$A+B = (A_1+B_1)\vec{i} + (A_2+B_2)\vec{j} + (A_3+B_3)\vec{k}$$

$$\nabla \times (A+B) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (A_1+B_1) & (A_2+B_2) & (A_3+B_3) \end{vmatrix}$$

$$\nabla \times (A+B) = \left[\frac{\partial}{\partial y}(A_3+B_3) - \frac{\partial}{\partial z}(A_2+B_2)\right]\vec{i} + \left[\frac{\partial}{\partial z}(A_1+B_1) - \frac{\partial}{\partial x}(A_3+B_3)\right]\vec{j}$$
$$+\left[\frac{\partial}{\partial x}(A_2+B_2) - \frac{\partial}{\partial y}(A_1+B_1)\right]\vec{k}$$

$$= \frac{\partial A_3}{\partial y}\vec{i} + \frac{\partial B_3}{\partial y}\vec{i} - \frac{\partial A_2}{\partial z}\vec{i} - \frac{\partial B_2}{\partial z}\vec{i} + \frac{\partial A_1}{\partial z}\vec{j} + \frac{\partial B_1}{\partial z}\vec{j} - \frac{\partial A_3}{\partial x}\vec{j} - \frac{\partial B_3}{\partial x}\vec{j}$$
$$+ \frac{\partial A_2}{\partial x}\vec{k} + \frac{\partial B_2}{\partial x}\vec{k} - \frac{\partial A_1}{\partial y}\vec{k} - \frac{\partial B_1}{\partial y}\vec{k}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\vec{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\vec{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\vec{k} +$$
$$\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\vec{i} + \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\vec{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\vec{k}$$

$$= \nabla \times A + \nabla \times B$$

$$\therefore \nabla \times (A+B) = \nabla \times A + \nabla \times B$$