Week2 Module 5: The Quadratic Equation

CSCI E-5a: Programming in R

Let's clear the environment:

```
rm( list = ls() )
```

Module Overview and Learning Objectives

In this module, we'll see how to use R to calculate the roots of a quadratic polynomial.

- In Section 1, we'll review the concept of a quadratic polynomial from high-school algebra.
- In Section 2, we'll see how to use the discriminant to determine the number of distinct roots of a quadratic polynomial.
- In Section 3, we'll see how to use the quadratic formula to solve for the roots of a quadratic polynomial.
- In Section 4, we'll look at some additional examples.

When you've completed this module, you should be able to:

- Define a quadratic polynomial.
- Determine the number of roots of a polynomial by using the discriminant.
- Calculate the roots of a quadratic polynomial.

There are no new functions in this module.

OK! Let's get started with a quick review of high-school algebra.

Section 1: Algebra Review

> **Main Idea:** We can define the concept of a quadratic polynomial.

In this section, we'll review the concept of a quadratic polynomial from high-school algebra.

> **Definition:** A *quadratic polynomial* is a polynomial where the highest power is 2.

For instance, this is a quadratic polynomial, because the highest power of the variable x is 2:

$$f(x) \; = \; x^2 - 6x + 5$$

We can express this concept algebraically like this:

$$f(x) \; = \; ax^2 + bx + c$$

The a, b, and c symbols represent constants, and in any practical situation we would always have specific values for these constants.

For instance, for the example above we have $a = 1$, $b = -6$, and $c = 5$.

A particular value of x that makes the polynomial $f(x)$ equal to 0 is called a *root* of that polynomial.

Let's consider the particular value $x = 5$.

Then:

$$f(5) \; = \; 5^2 - (6 \times 5) + 5 \; = \; 25 - 30 + 5 \; = \; 0$$

Let's do this in R.

First, we'll define variables:

```
# Example 1: Define variables

a <- 1
b <- -6
c <- 5
```

Let's check the value of $f(5)$ to see if it's a root:

```
# Example 2: Evaluate f(5)

x <- 5

value.of.f.at.x.equals.5 <-
    a * x^2 + b * x + c

cat(
    "Value of f(x) at x = 5:",
    formatC(
        value.of.f.at.x.equals.5,
        format = "f",
        digits = 2
    )
)
```

```
## Value of f(x) at x = 5: 0.00
```

So the particular value $x = 5$ is a root of the quadratic polynomial $f(x) = x^2 - 5x + 4$.

So that's what a quadratic polynomial is.

Now let's see how to determine the number of roots of a quadratic polynomial.

Section 2: The Discriminant

> **Main Idea:** We can determine the number of roots of a quadratic polynomial.

In this section, we'll see how to use the discriminant to determine the number of distinct roots of a quadratic polynomial.

In order to calculate the roots of a quadratic polynomial, we first of all have to figure out how many distinct roots there are.

A quadratic polynomial can have zero roots, one root, or two roots.

To determine the number of roots of a quadratic polynomial, we first calculate the *discriminant*, which is denoted Δ:

$$\Delta \ = \ b^2 - 4ac$$

Let's calculate the discriminant for our previous example:

```r
# Example 3: Calculating the discriminant

discriminant <-
  b^2 - 4 * a * c

cat(
  "Discriminant:",
  formatC(
    discriminant,
    format = "f",
    digits = 2
  )
)
```

```
## Discriminant: 16.00
```

So the value of the discriminant for that quadratic polynomial is 16.

Then, using the discriminant, we can determine the number of roots of the quadratic polynomial:

- If the discriminant is strictly less than 0, then there are no roots.
- If the discriminant is equal to 0, then there is exactly one root x.
- If the discriminant is strictly greater than 0, then the quadratic polynomial has exactly two distinct roots.

So that's how to use the discriminant to determine the number of distinct roots of a quadratic polynomial.

Now let's see how to actually calculate the values of any roots.

Section 3: Calculating the Roots

> **Main Idea:** *We can calculate the roots of a quadratic polynomial.*

In section, we'll see how to use the quadratic formula to solve for the roots of a quadratic polynomial.

Before we try to calculate the values of any roots, we first of all have to be sure how many roots there are.

If there are no real roots, then there's nothing to calculate.

If the discriminant is equal to 0, then there is exactly one root x, and the formula for x is:

$$x \;=\; -\frac{b}{2a}$$

If the discriminant is strictly greater than 0, then the quadratic polynomial has exactly two distinct roots.

The first root x_1 is:

$$x_1 \;=\; \frac{-b-\sqrt{b^2-4ac}}{2a}$$

The second root is:

$$x_2 \;=\; \frac{-b+\sqrt{b^2-4ac}}{2a}$$

Let's recall our original example of a quadratic polynomial:

$$f(x) \;=\; x^2-6x+5$$

Recall from Example 3 that the discriminant for this quadratic polynomial is $\Delta = 16$.

Since the discriminant is strictly greater than 0, there are exactly two roots for this function.

Let's calculate the first root:

```
# Example 4: Calculate the first root

first.root <-
    (-b - sqrt( b^2 - 4 * a * c ) ) / (2 * a)

cat(
    "The first root of the function is:",
    formatC(
        first.root,
        format = "f",
        digits = 2
    )
)
```

```
## The first root of the function is: 1.00
```

Let's check this:

```
# Example 5: Checking the first root

a * (first.root^2) + (b * first.root) + c
```

```
## [1] 0
```

Now let's calculate the second root, reporting our result using a `cat()` statement displaying 2 decimal places:

```r
# Example 6: Calculating the second root

second.root <-
    (-b + sqrt( b^2 - 4 * a * c ) ) / (2 * a)

cat(
    "The second root of the equation is:",
    formatC(
        second.root,
        format = "f",
        digits = 2
    )
)
```

```
## The second root of the equation is: 5.00
```

Let's check this:

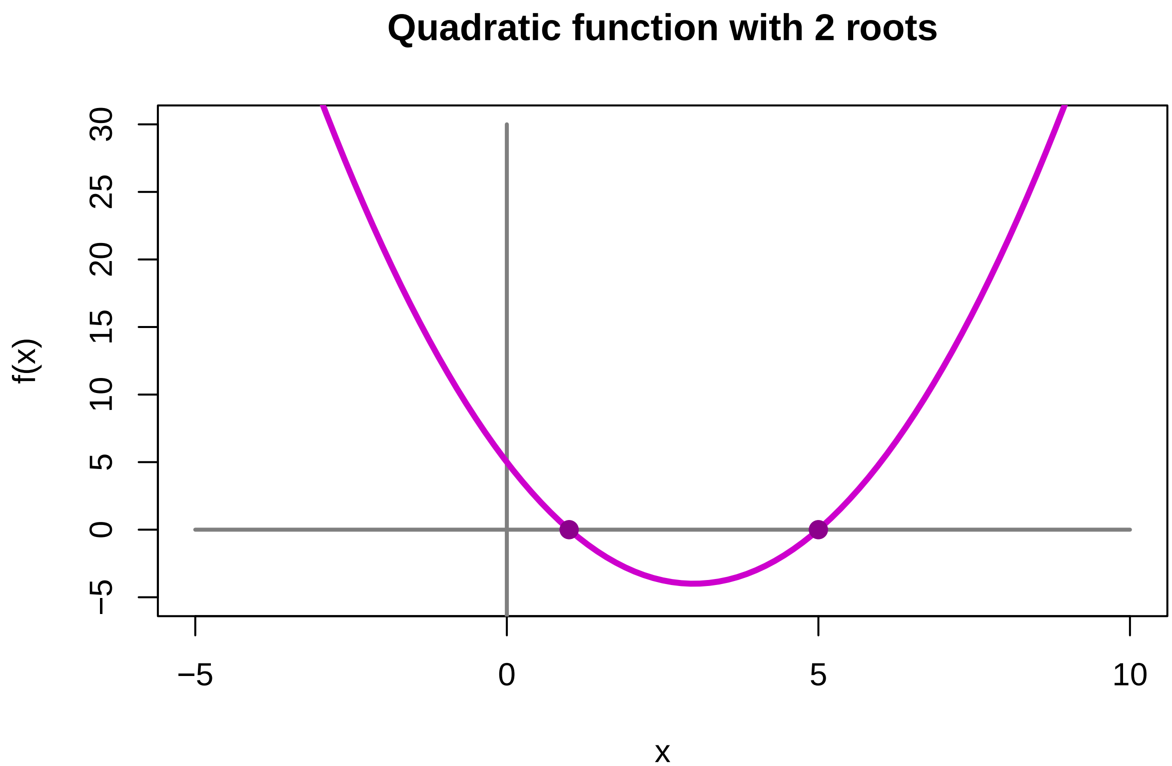
```r
# Example 7: Checking the second root

a * (second.root^2) + (b * second.root) + c
```

```
## [1] 0
```

Now let's make a graph:

```r
# Example 8: Creating a graph

# First, let's create an empty plot:

plot(
    x = NULL,
    xlim = c(-5, 10),
    ylim = c(-5, 30),
    main = "Quadratic function with 2 roots",
    xlab = "x",
    ylab = "f(x)"
)


# Next, let's draw in the x- and y-axes:

segments(
    x0 = -5,
    y0 = 0,
    x1 = 10,
    y1 = 0,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)
```

```r
# Here's the vertical reference line:

segments(
    x0 = 0,
    y0 = -20,
    x1 = 0,
    y1 = 30,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Now we can graph the quadratic function:

curve(
    expr = a*x^2 + b*x + c,
    lty = "solid",
    lwd = 3,
    col = "magenta3",
    add = TRUE
)


# Finally, we'll put points at the locations of the roots:

points(
    x = first.root,
    y = 0,
    pch = 19,
    cex = 1.2,
    col = "magenta4"
)

points(
    x = second.root,
    y = 0,
    pch = 19,
    cex = 1.2,
    col = "magenta4"
)
```



So that's how to use the quadratic formula to solve for the roots of a quadratic polynomial.

Let's look at some more examples.

Section 4: More Examples

Main Idea: *We explore two more examples.*

In this section, we'll analyze two more examples.

First, let's consider this quadratic polynomial:

$$x^2 - 6x + 9 \ = \ 0$$

First, let's define some variables:

```
# Example 9: Defining variables

a <- 1
b <- -6
c <- 9
```

Then the discriminant is:

```r
# Example 10: Calculating the discriminant

b^2 - 4 * a * c
```

```
## [1] 0
```

Since the discriminant is equal to 0, there is exactly one root for this quadratic polynomial:

```r
# Example 11: Calculating the single root

only.root <- - b / (2 * a)

cat(
    "The only root of the polynomial is:",
    formatC(
        only.root,
        format = "f",
        digits = 2
    )
)
```

```
## The only root of the polynomial is: 3.00
```

Let's check this:

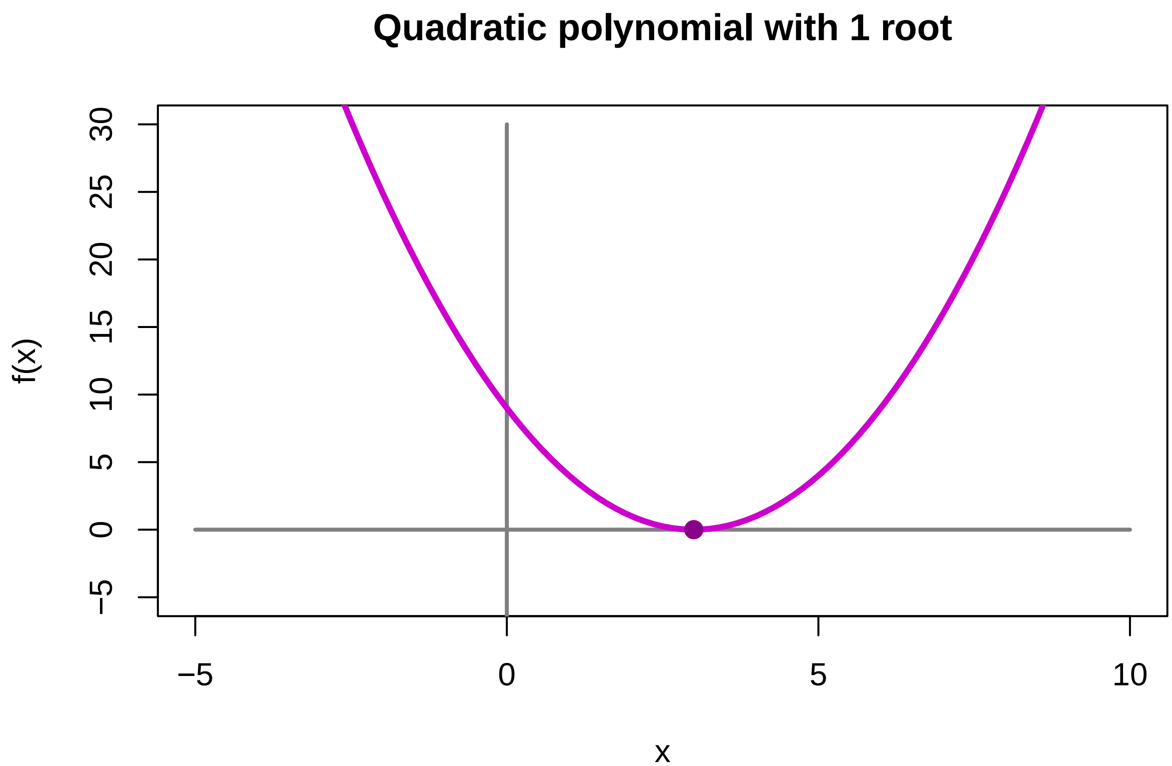
```r
# Example 12: Checking the single root

a * only.root^2 + b * only.root + c
```

```
## [1] 0
```

Now let's draw the graph:

```r
# Example 13: Drawing the graph

# First, let's create an empty plot:

plot(
    x = NULL,
    xlim = c(-5, 10),
    ylim = c(-5, 30),
    main = "Quadratic polynomial with 1 root",
    xlab = "x",
    ylab = "f(x)"
)


# Next, let's draw in the x- and y-axes:

segments(
    x0 = -5,
```

```r
    y0 = 0,
    x1 = 10,
    y1 = 0,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Here's the vertical reference line:

segments(
    x0 = 0,
    y0 = -20,
    x1 = 0,
    y1 = 30,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Now we can graph the quadratic function:

curve(
    a*x^2 + b*x + c,
    lty = "solid",
    lwd = 3,
    col = "magenta3",
    add = TRUE
)


# Finally, we'll put a point at the locations of the only root:

points(
    x = only.root,
    y = 0,
    pch = 19,
    cex = 1.2,
    col = "magenta4"
)
```



Now for the second example.

Let's consider this equation:

$$x^2 - 6x + 14 \;=\; 0$$

First, let's define some variables:

```
# Example 14: Defining variables

a <- 1
b <- -6
c <- 14
```

Then the discriminant is:

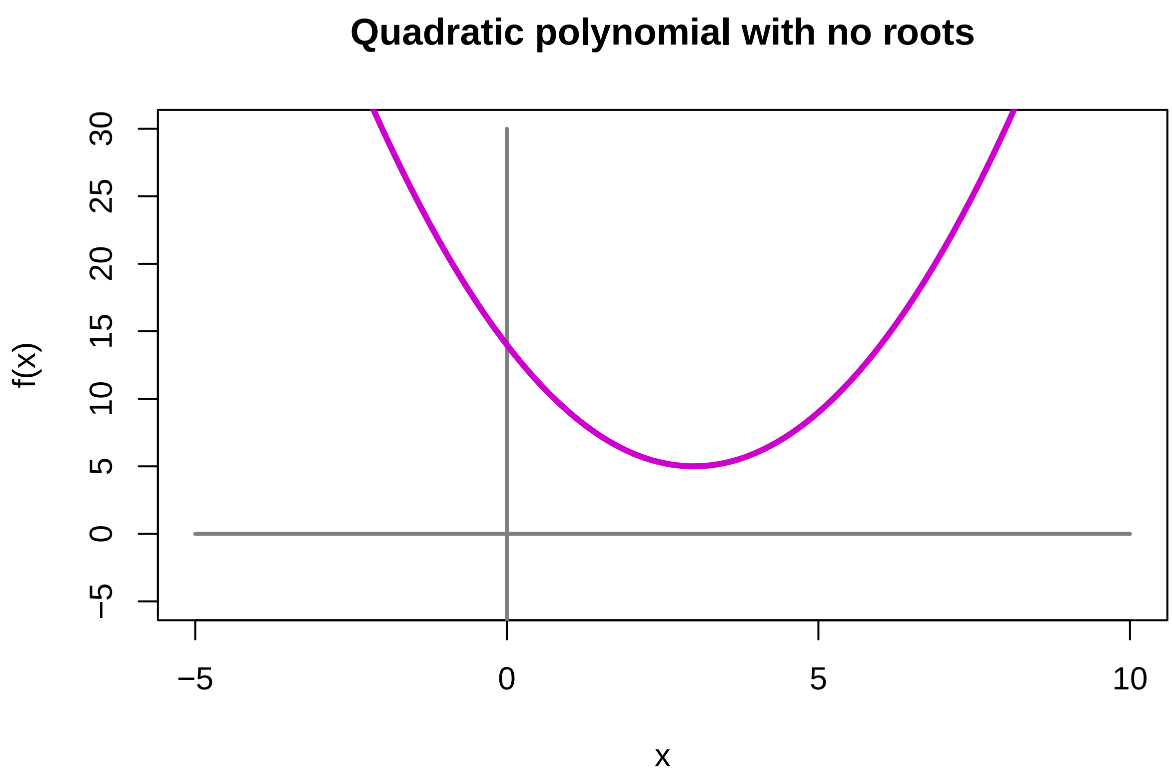
```
# Example 15: Calculating the discriminant

b^2 - 4 * a * c
```

```
## [1] -20
```

Since the discriminant is less than 0, this quadratic function has no roots.

Now let's draw the graph:

```r
# Example 16: Drawing the graph

# First, let's create an empty plot:

plot(
    x = NULL,
    xlim = c(-5, 10),
    ylim = c(-5, 30),
    main = "Quadratic polynomial with no roots",
    xlab = "x",
    ylab = "f(x)"
)


# Next, let's draw in the x- and y-axes:

segments(
    x0 = -5,
    y0 = 0,
    x1 = 10,
    y1 = 0,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Here's the vertical reference line:

segments(
    x0 = 0,
    y0 = -20,
    x1 = 0,
    y1 = 30,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Now we can graph the quadratic function:

curve(
    a*x^2 + b*x + c,
    lty = "solid",
    lwd = 3,
    col = "magenta3",
    add = TRUE
)
```




```
# Since this quadratic polynomial has no roots, there are no points to plot!
```

So those are two more examples of quadratic polynomials and their roots.

Now let's review what we've learned in this module.

Exercise 2.1: Quadratic Polynomials

Part (a)

Consider the quadratic function $f(x)$:

$$f(x) \;=\; 2x^2 - 4x - 16$$

First, determine the number of roots of this polynomial. Then, if there are any roots, calculate their value using the appropriate formula. Finally, create a graph so we can visualize the function and its roots.

Part (b)

Consider the quadratic function $g(x)$:

$$g(x) \;=\; 2x^2 - 4x + 8$$

First, determine the number of roots of this polynomial. Then, if there are any roots, calculate their value using the appropriate formula. Finally, create a graph so we can visualize the function and its roots.

Part (c)

Consider the quadratic function $h(x)$:

$$h(x) \; = \; 2x^2 - 4x + 2$$

First, determine the number of roots of this polynomial. Then, if there are any roots, calculate their value using the appropriate formula. Finally, create a graph so we can visualize the function and its roots.

Module Review

In this module, we saw how to use R to calculate the roots of a quadratic polynomial.

- In Section 1, we reviewed the concept of a quadratic polynomial from high-school algebra.
- In Section 2, we saw how to use the discriminant to determine the number of roots of a quadratic polynomial.
- In Section 3, we saw how to use the quadratic formula to solve for the distinct roots of a quadratic polynomial.
- In Section 4, we looked at some additional examples.

Now that you've completed this module, you should be able to:

- Define a quadratic polynomial.
- Determine the number of roots of a polynomial by using the discriminant.
- Calculate the roots of a quadratic polynomial.

There were no new functions in this module.

Solutions to the Exercises

Exercise 2.1: Quadratic Polynomials

Part (a)

Consider the quadratic function $f(x)$:

$$f(x) \; = \; 2x^2 - 4x - 16$$

First, determine the number of roots of this polynomial. Then, if there are any roots, calculate their value using the appropriate formula. Finally, create a graph so we can visualize the function and its roots.

Solution

First, let's define variables for the coefficients of the quadratic function:

```r
a <- 2

b <- -4

c <- -16
```

Then the discriminant is:

```
b^2 - 4 * a * c
```

```
## [1] 144
```

Since the discriminant is strictly greater than 0, there are exactly two roots for this polynomial

Let's calculate the first root:

```
first.root <- (-b - sqrt( b^2 - 4 * a * c ) ) / (2 * a)

cat(
    "The first root of the function is:",
    formatC(
        first.root,
        format = "f",
        digits = 2
    )
)
```

```
## The first root of the function is: -2.00
```

Let's check this:

```
a * first.root^2 + b * first.root + c
```

```
## [1] 0
```

Now let's calculate the second root, reporting our result using a `cat()` statement displaying 2 decimal places:

```
second.root <- (-b + sqrt( b^2 - 4 * a * c ) ) / (2 * a)

cat(
    "The second root of the equation is:",
    formatC(
        second.root,
        format = "f",
        digits = 2
    )
)
```

```
## The second root of the equation is: 4.00
```

Let's check this:

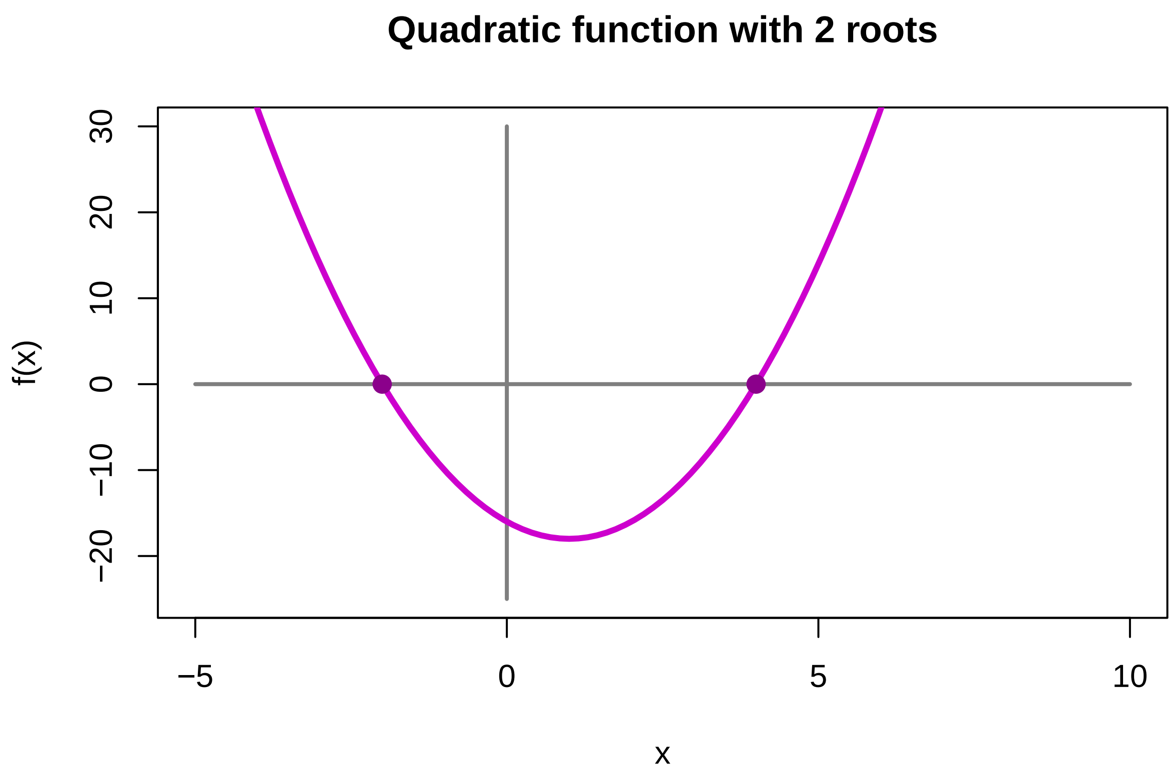
```
a * second.root^2 + b * second.root + c
```

```
## [1] 0
```

Now let's make a graph:

```r
# First, let's create an empty plot:

plot(
    x = NULL,
    xlim = c(-5, 10),
    ylim = c(-25, 30),
    main = "Quadratic function with 2 roots",
    xlab = "x",
    ylab = "f(x)"
)


# Next, let's draw in the x- and y-axes:

segments(
    x0 = -5,
    y0 = 0,
    x1 = 10,
    y1 = 0,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Here's the vertical reference line:

segments(
    x0 = 0,
    y0 = -25,
    x1 = 0,
    y1 = 30,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Now we can graph the quadratic function:

curve(
    a*x^2 + b*x + c,
    lty = "solid",
    lwd = 3,
    col = "magenta3",
    add = TRUE
)


# Finally, we'll put points at the locations of the roots:

points(
    x = first.root,
```

```
    y = 0,
    pch = 19,
    cex = 1.2,
    col = "magenta4"
)

points(
    x = second.root,
    y = 0,
    pch = 19,
    cex = 1.2,
    col = "magenta4"
)
```



Part (b)

Consider the quadratic function $g(x)$:

$$g(x) \ = \ 2x^2 - 4x + 8$$

First, determine the number of roots of this polynomial. Then, if there are any roots, calculate their value using the appropriate formula. Finally, create a graph so we can visualize the function and its roots.

Solution

First, let's define variables for the coefficients of the quadratic function:

```r
a <- 2

b <- -4

c <- 8
```

Then the discriminant is:

```r
b^2 - 4 * a * c
```

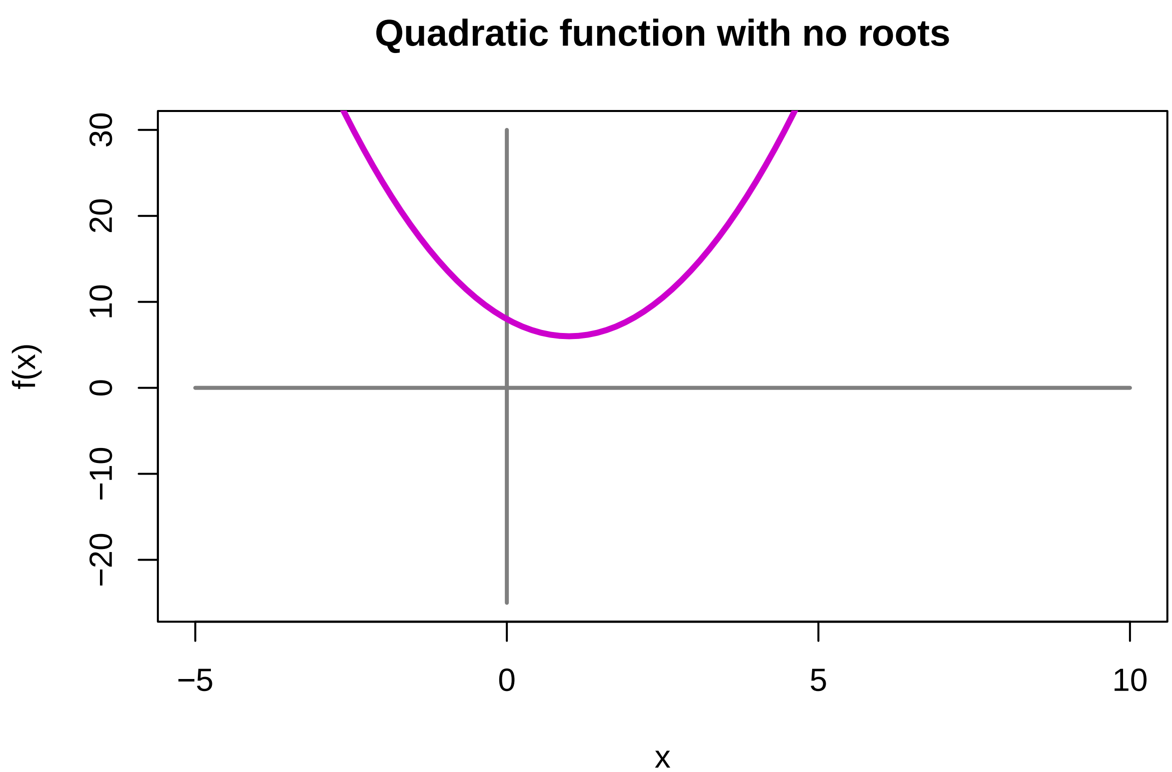
```
## [1] -48
```

Since the discriminant is negative, there are no roots for this quadratic polynomial.

Now let's make a graph:

```r
# First, let's create an empty plot:

plot(
    x = NULL,
    xlim = c(-5, 10),
    ylim = c(-25, 30),
    main = "Quadratic function with no roots",
    xlab = "x",
    ylab = "f(x)"
)


# Next, let's draw in the x- and y-axes:

segments(
    x0 = -5,
    y0 = 0,
    x1 = 10,
    y1 = 0,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Here's the vertical reference line:

segments(
    x0 = 0,
    y0 = -25,
    x1 = 0,
    y1 = 30,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)
```

```r
# Now we can graph the quadratic function:

curve(
    a*x^2 + b*x + c,
    lty = "solid",
    lwd = 3,
    col = "magenta3",
    add = TRUE
)
```




```r
# Finally, there are no roots for this quadratic polynomial,
# so we're done!
```

Part (c)

Consider the quadratic function $h(x)$:

$$h(x) \;=\; 2x^2 - 4x + 2$$

First, determine the number of roots of this polynomial. Then, if there are any roots, calculate their value using the appropriate formula. Finally, create a graph so we can visualize the function and its roots.

Solution

First, let's define variables for the coefficients of the quadratic function:

```r
a <- 2

b <- -4

c <- 2
```

Then the discriminant is:

```r
b^2 - 4 * a * c
```

```
## [1] 0
```

Since the discriminant is zero, there is exactly one root for this quadratic polynomial.

```r
only.root <-
    - b / (2 * a)

cat(
    "Only root of h(x):",
    formatC(
        only.root,
        format = "f",
        digits = 2
    )
)
```

```
## Only root of h(x): 1.00
```

Let's check this:

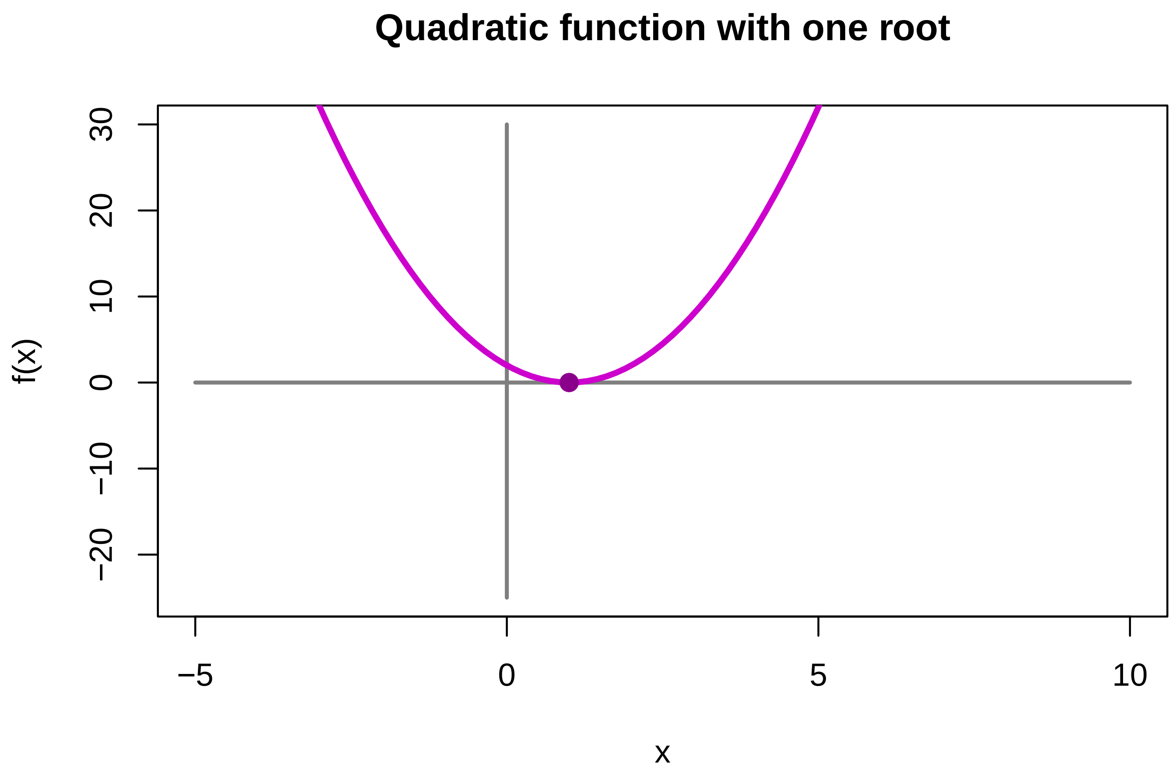
```r
a * only.root^2 + b * only.root + c
```

```
## [1] 0
```

Now let's make a graph:

```r
# First, let's create an empty plot:

plot(
    x = NULL,
    xlim = c(-5, 10),
    ylim = c(-25, 30),
    main = "Quadratic function with one root",
    xlab = "x",
    ylab = "f(x)"
)


# Next, let's draw in the x- and y-axes:

segments(
```

```r
    x0 = -5,
    y0 = 0,
    x1 = 10,
    y1 = 0,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Here's the vertical reference line:

segments(
    x0 = 0,
    y0 = -25,
    x1 = 0,
    y1 = 30,
    lty = "solid",
    lwd = 2,
    col = "gray50"
)


# Now we can graph the quadratic function:

curve(
    a*x^2 + b*x + c,
    lty = "solid",
    lwd = 3,
    col = "magenta3",
    add = TRUE
)


# Finally, there is only one root for this polynomial:

points(
    x = only.root,
    y = 0,
    pch = 19,
    cex = 1.2,
    col = "magenta4"
)
```



Exercise 2.2 Newton's Method

So far, we've found one root of the quadratic polynomial $f(x) = x^2 - 6x + 5$. Now let's find the other, again by using Newton's method.

This time, start the procedure with an initial guess of 6. Run the process for 5 steps, reporting each intermediate result using a `cat()` statement, displaying the values with 6 decimal places.

Solution

First, let's create some variables to hold the values of the coefficients:

```
a <- 1

b <- -6

c <- 5
```

Now let's make our initial guess:

```
x.1 <- 6
```

Now we'll calculate the second guess, and the report it:

```
x.2 <-
    (a * x.1^2 - c) /
    (2 * a * x.1 + b)

cat(
    "x.2 =",
    formatC(
        x.2,
        format = "f",
        digits = 6
    )
)
```

```
## x.2 = 5.166667
```

Now let's calculate and report our third guess:

```
x.3 <-
    (a * x.2^2 - c) /
    (2 * a * x.2 + b)

cat(
    "x.3 =",
    formatC(
        x.3,
        format = "f",
        digits = 6
    )
)
```

```
## x.3 = 5.006410
```

Now for the fourth estimate:

```
x.4 <-
    (a * x.3^2 - c) /
    (2 * a * x.3 + b)

cat(
    "x.4 =",
    formatC(
        x.4,
        format = "f",
        digits = 6
    )
)
```

```
## x.4 = 5.000010
```

And finally the fifth estimate:

```r
x.5 <-
    (a * x.4^2 - c) /
    (2 * a * x.4 + b)

cat(
    "x.5 =",
    formatC(
        x.5,
        format = "f",
        digits = 6
    )
)
```

```
## x.5 = 5.000000
```