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Title: An almost perfectly predictable Δ^0_2- sequence with no optimal predictor

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Abstract

In [2] it was shown that there exists a sequence, recursive in $0''$, which can be predicted with arbitrarily small asymptotic error but for which no predictor is optimal. Here, we consider the question of how complex such a sequence might be. We demonstrate that the complexity can be lifted to $0'$. Further questions are posed, in particular: can such a sequence be c.e.?

Let $A \subseteq \mathbb{N}$. Define $\rho_n(A) := \frac{|A\lceil n|}{n+1}$, where $A\lceil n = \chi_A \restriction [0, n]$, $\overline{\rho}(A) = \limsup_{n\to\infty} \rho_n(A)$, $\underline{\rho}(A) = \liminf_{n\to\infty} \rho_n(A)$ and $\rho(A) := \lim_{n\to\infty} \rho_n(A)$, if this limit exists. We say that sets $A, B \subseteq \mathbb{N}$ are generically similar if $\rho(A \triangle B) = 0$. A proper predictor is a total computable function from $2^{<\omega}$ to $\{0, 1\}$. If f is a predictor and $A \subseteq \mathbb{N}$, let $fA := \{n : f(A\lceil n - 1) = 1\}$ (assuming that $A\lceil -1 = \chi_A \restriction [0, -1] = \chi_A \restriction \emptyset = \emptyset$). We say that a proper predictor f is optimal for A if $\rho(A \triangle fA)$ exists and for every proper predictor g, if $\rho(A \triangle gA)$ exists then $\rho(A \triangle fA) \leq \rho(A \triangle gA)$.

Theorem 1 ([2])**.** *There exists a set $A \subseteq \mathbb{N}$ such that:*

1. *For every proper predictor f, it is not the case that $\rho(A \triangle fA) = 0$.*
2. *There exists an infinite sequence of proper predictors $f_0, f_1, \ldots$ such that $\lim_t \rho(A \triangle f_t A) = 0$.*

The construction used in [2] requires $0''$ as an oracle (one reason for this is that the enumeration of proper predictors is of such complexity). Here we show that the construction can be lifted to $0'$ which is much more interesting because sets $\leq 0'$ posses recursive approximations (in particular, the famous Chaitin constant belongs to $0'$).

Theorem 2. *There exists a set $A \subseteq \mathbb{N}$ such that:*

1. *For every proper predictor f, it is not the case that $\rho(A \triangle fA) = 0$.*
2. *There exists an infinite sequence of proper predictors $f_0, f_1, \ldots$ such that $\lim_t \rho(A \triangle f_t A) = 0$.*
3. *$A \leq 0'$.*

Proof. **Construction.** Let $\phi_0, \phi_1, \ldots$ be a canonical enumeration of all partial recursive functions from $2^{<\omega}$ to $\{0, 1\}$. Without loss of generality, we can assume that ϕ_0 is a proper predictor. Let $R_n = \{m : 2^k | \wedge 2^{k+1} \not| m\}$ [1]. The construction proceeds in stages $s = 0, 1, \ldots$.

Stage 0. $\alpha_0 = 0$. Let $i_0 : \mathbb{N} \to \mathbb{N}$ be the identity function.

Stage $s+1$. Let n be such that $s+1 \in R_n$. If $\phi_{i_s(n)}(\beta)\downarrow$ for all β of length $\leq s+1$, set $\alpha_{s+1} = \alpha_s(1-\phi_{i_s(n)}(\alpha_s))$. Otherwise (i.e., if $\phi_{i_s(n)}(\beta)\uparrow$ for some β of length $\leq s+1$), set $\alpha_{s+1} := \alpha_s(1-\phi_0(\alpha_s))$ and $i_{s+1}(n) := 0$.

We set $A = \bigcup_{s=0}^{\infty} \alpha_s$.

Verification. The idea of the construction is similar to the one from [2]. Each ϕ_k is assigned numbers from R_k which is a set of density $2^{-(k+1)}$. If ϕ_k is a proper predictor, then the construction makes ϕ_k to fail on R_k, just as it was in [2].

The main difference lies in cases where ϕ_k is not a proper predictor. Then at some stage $t+1$ we have $t+1 \in R_k$, $i(n) = k$ and $\phi_k(\beta)\uparrow$ for some β of length $\leq t+1$ (simply put: we discover that ϕ_k is not total). In that case ϕ_k is not assigned to elements from R_k anymore. Instead, ϕ_0 is. Since up to stage $t+1$ at most finitely many elements from R_k have been handled, the construction makes ϕ_0 to fail on the rest of the elements from R_k. Hence $\rho(A \cap R_k \triangle \phi_0 A \cap R_k) = \rho(R_k)$.

The contents of the proof from [2] remain virtually unchanged, except that h_0, X are replaced by ϕ_0, A, respectively. Moreover, instead of finding appropriate h_i's, ϕ_i's are to be found. Let us briefly recall that we show that there exists a proper predictor g such that $\rho(A\triangle gA) = \frac{1}{2}$. For each predictor f such that $\rho(A\triangle fA)$ is defined, we show that there exists a proper predictor f' such that $\rho(A\triangle fa) > \rho(A\triangle f'A)$.

Finally, we show that there exist predictors $f_0, f_1, \ldots$ such that $\lim_n \rho(A\triangle f_n A) = 0$. This is done as follows. We set $f_0 = g$ (see paragraph above). Suppose f_n is defined. Let $e = \lim_s i_s(n)$ (note that $e = n$ or $e = 0$). Define $f_{n+1}(x) = f_n(x)$ if $|x| \notin R_n$ and $f_{n+1}(x) = 1 - \phi_e(x)$ if $|x| \in R_n$. This makes $\rho(A\triangle f_{n+1}) = \rho(A\triangle f_n A) - \rho(R_n)$. Since $\rho(A\triangle f_0 A) = \frac{1}{2}$ and $\Sigma_{i=1}^{\infty}\rho(R_i) = \frac{1}{2}$, we clearly have $\lim_n \rho(A\triangle f_n A) = 0$. □

By the limit lemma [4, 3], the set A from the above theorem is Δ_2^0.

It is not known whether there exists a computably enumerable sequence satisfying all the properties listed in Theorem 2.

References

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