Bài tập bài 7.1 Giới thiệu phương trình vi phân

Sử dụng Định lý 7.4 để chứng minh các bài toán giá trị ban đầu sau có duy nhất nghiệm và giải chúng, h=0.01, y_1, y_2, y_3 .

$$a)\ \ y' = y\cos t,\ \ 0 \le t \le 1, y(0) = 1$$

$y_1 = y_0 + hf(t_0, y_0)$

$= 1 + 0.01(1*\cos 0) = 1.01$

$y_2 = y_1 + hf(t_1, y_1)$

= 1.01 +
0.01*(1.01*cos(0.01))

= 1.0200995

$y_3 = y_2 + hf(t_2, y_2)$

= 1.02009951 +
0.01*(1.0200995*cos(0.02))

= 1.030298465

$$b)\ \ y' = \frac{2}{t}y + t^2 e^t,\ \ 1 \le t \le 2, y(1) = 0$$

$$c)\ \ y' = -\frac{2}{t}y + t^2 e^t,\ \ 1 \le t \le 2, y(1) = \sqrt{2}e$$

$$d)\ \ y' = \frac{4t^3 y}{1+t^4},\ \ 0 \le t \le 1, y(0) = 1$$

1. Use Theorem 5.4 to show that each of the following initial-value problems has a unique s
find the solution.

a. $y' = y\cos t, \quad 0 \le t \le 1, \quad y(0) = 1.$

b. $y' = \dfrac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0.$

c. $y' = -\dfrac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = \sqrt{2}e.$

d. $y' = \dfrac{4t^3y}{1+t^4}, \quad 0 \le t \le 1, \quad y(0) = 1.$

2. Show that each of the following initial-value problems has a unique solution and find the solution. Can Theorem 5.4 be applied in each case?

a. $y' = e^{t-y}, \quad 0 \le t \le 1, \quad y(0) = 1.$

b. $y' = t^{-2}(\sin 2t - 2ty), \quad 1 \le t \le 2, \quad y(1) = 2.$

c. $y' = -y + ty^{1/2}, \quad 2 \le t \le 3, \quad y(2) = 2.$

d. $y' = \dfrac{ty+y}{ty+t}, \quad 2 \le t \le 4, \quad y(2) = 4.$

3. For each choice of $f(t, y)$ given in parts (a)–(d):

i. Does f satisfy a Lipschitz condition on $D = \{(t, y) \mid 0 \le t \le 1, -\infty < y < \infty\}$?

ii. Can Theorem 5.6 be used to show that the initial-value problem

$$y' = f(t, y), \quad 0 \le t \le 1, \quad y(0) = 1,$$

is well-posed?

a. $f(t, y) = t^2y + 1$ **b.** $f(t, y) = ty$ **c.** $f(t, y) = 1 - y$ **d.** $f(t, y) = -ty + \dfrac{4t}{y}$

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ii. Can Theorem 5.6 be used to show that the initial-value problem

$$y' = f(t, y), \quad 0 \le t \le 1, \quad y(0) = 1,$$

is well-posed?

a. $f(t, y) = e^{t-y}$ **b.** $f(t, y) = \dfrac{1+y}{1+t}$ **c.** $f(t, y) = \cos(yt)$ **d.** $f(t, y) = \dfrac{y^2}{1+t}$

5. For the following initial-value problems, show that the given equation implicitly defines a solution. Approximate $y(2)$ using Newton's method.

a. $y' = -\dfrac{y^3 + y}{(3y^2 + 1)t}, \quad 1 \le t \le 2, \quad y(1) = 1; \quad y^3 t + yt = 2$

b. $y' = -\dfrac{y \cos t + 2te^y}{\sin t + t^2 e^y + 2}, \quad 1 \le t \le 2, \quad y(1) = 0; \quad y \sin t + t^2 e^y + 2y = 1$

6. Prove Theorem 5.3 by applying the Mean Value Theorem 1,8 to $f(t, y)$, holding t fixed.

7. Show that, for any constants a and b, the set $D = \{(t, y) \mid a \le t \le b, -\infty < y < \infty\}$ is convex.

8. Suppose the perturbation $\delta(t)$ is proportional to t, that is, $\delta(t) = \delta t$ for some constant δ. Show directl that the following initial-value problems are well-posed.

a. $y' = 1 - y, \quad 0 \le t \le 2, \quad y(0) = 0$

b. $y' = t + y, \quad 0 \le t \le 2, \quad y(0) = -1$

c. $y' = \dfrac{2}{t} y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0$

d. $y' = -\dfrac{2}{t} y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = \sqrt{2}e$

9. *Picard's method* for solving the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is described as follows: Let $y_0(t) = \alpha$ for each t in $[a, b]$. Define a sequence $\{y_k(t)\}$ of functions by

$$y_k(t) = \alpha + \int_a^t f(\tau, y_{k-1}(\tau))\, d\tau, \quad k = 1, 2, \ldots.$$

a. Integrate $y' = f(t, y(t))$, and use the initial condition to derive Picard's method.

b. Generate $y_0(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$ for the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 1, \quad y(0) = 1.$$

c. Compare the result in part (b) to the Maclaurin series of the actual solution $y(t) = t + e^{-t}$.