# BÀI TẬP BÀI 7.6 PHƯƠNG TRÌNH VI PHÂN BẬC CAO VÀ HỆ PHƯƠNG TRÌNH VI PHÂN

**1.** Use the Runge-Kutta method for systems to approximate the solutions of the following systems of first-order differential equations, and compare the results to the actual solutions.

**a.** $u_1' = 3u_1 + 2u_2 - (2t^2+1)e^{2t}, \quad u_1(0) = 1;$
$u_2' = 4u_1 + u_2 + (t^2+2t-4)e^{2t}, \quad u_2(0) = 1; \quad 0 \le t \le 1; \quad h = 0.2;$
actual solutions $u_1(t) = \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} + e^{2t}$ and $u_2(t) = \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} + t^2e^{2t}$.

**b.** $u_1' = -4u_1 - 2u_2 + \cos t + 4\sin t, \quad u_1(0) = 0;$
$u_2' = 3u_1 + u_2 - 3\sin t, \quad u_2(0) = -1; \quad 0 \le t \le 2; \quad h = 0.1;$
actual solutions $u_1(t) = 2e^{-t} - 2e^{-2t} + \sin t$ and $u_2(t) = -3e^{-t} + 2e^{-2t}$.

**c.** $u_1' = u_2, \quad u_1(0) = 1;$
$u_2' = -u_1 - 2e^t + 1, \quad u_2(0) = 0;$
$u_3' = -u_1 - e^t + 1, \quad u_3(0) = 1; \quad 0 \le t \le 2; \quad h = 0.5;$
actual solutions $u_1(t) = \cos t + \sin t - e^t + 1$, $u_2(t) = -\sin t + \cos t - e^t$, and $u_3(t) = -\sin t + \cos t$.

**d.** $u_1' = u_2 - u_3 + t, \quad u_1(0) = 1;$
$u_2' = 3t^2, \quad u_2(0) = 1;$
$u_3' = u_2 + e^{-t}, \quad u_3(0) = -1; \quad 0 \le t \le 1; \quad h = 0.1;$
actual solutions $u_1(t) = -0.05t^5 + 0.25t^4 + t + 2 - e^{-t}$, $u_2(t) = t^3 + 1$, and $u_3(t) = 0.25t^4 + t - e^{-t}$.

**2.** Use the Runge-Kutta method for systems to approximate the solutions of the following systems of first-order differential equations, and compare the results to the actual solutions.

**a.** $u_1' = u_1 - u_2 + 2, \quad u_1(0) = -1;$
$u_2' = -u_1 + u_2 + 4t, \quad u_2(0) = 0; \quad 0 \le t \le 1; \quad h = 0.1;$
actual solutions $u_1(t) = -\frac{1}{2}e^{2t} + t^2 + 2t - \frac{1}{2}$ and $u_2(t) = \frac{1}{2}e^{2t} + t^2 - \frac{1}{2}$.

**b.** $u_1' = \frac{1}{9}u_1 - \frac{2}{3}u_2 - \frac{1}{9}t^2 + \frac{2}{3}, \quad u_1(0) = -3;$
$u_2' = u_2 + 3t - 4, \quad u_2(0) = 5; \quad 0 \le t \le 2; \quad h = 0.2;$
actual solutions $u_1(t) = -3e^t + t^2$ and $u_2(t) = 4e^t - 3t + 1$.

**c.** $u_1' = u_1 + 2u_2 - 2u_3 + e^{-t}, \quad u_1(0) = 3;$
$u_2' = u_2 + u_3 - 2e^{-t}, \quad u_2(0) = -1;$
$u_3' = u_1 + 2u_2 + e^{-t}, \quad u_3(0) = 1; \quad 0 \le t \le 1; \quad h = 0.1;$
actual solutions $u_1(t) = -3e^{-t} - 3\sin t + 6\cos t$, $u_2(t) = \frac{3}{2}e^{-t} + \frac{3}{10}\sin t - \frac{21}{10}\cos t - \frac{2}{5}e^{2t}$,
and $u_3(t) = -e^{-t} + \frac{12}{5}\cos t + \frac{9}{5}\sin t - \frac{2}{5}e^{2t}$.

**d.** $u_1' = 3u_1 + 2u_2 - u_3 - 1 - 3t - 2\sin t, \quad u_1(0) = 5;$
$u_2' = u_1 - 2u_2 + 3u_3 + 6 - t + 2\sin t + \cos t, \quad u_2(0) = -9;$
$u_3' = 2u_1 + 4u_3 + 8 - 2t, \quad u_3(0) = -5; \quad 0 \le t \le 2; \quad h = 0.2;$
actual solutions $u_1(t) = 2e^{3t} + 3e^{-2t} + 1$, $u_2(t) = -8e^{-2t} + e^{4t} - 2e^{3t} + \sin t$, and $u_3(t) = 2e^{4t} - 4e^{3t} - e^{-2t} - 2$.

**3.** Use the Runge-Kutta for Systems Algorithm to approximate the solutions of the following higher-order differential equations, and compare the results to the actual solutions.

**a.** $y'' - 2y' + y = te^t - t, \quad 0 \le t \le 1, \quad y(0) = y'(0) = 0$, with $h = 0.1$;
actual solution $y(t) = \frac{1}{6}t^3e^t - te^t + 2e^t - t - 2$.

**b.** $t^2y'' - 2ty' + 2y = t^3 \ln t, \quad 1 \le t \le 2, \quad y(1) = 1, \quad y'(1) = 0$, with $h = 0.1$;
actual solution $y(t) = \frac{7}{4}t + \frac{1}{2}t^3 \ln t - \frac{3}{4}t^3$.

**c.** $y''' + 2y'' - y' - 2y = e^t, \quad 0 \le t \le 3, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 0$, with $h = 0.2$;
actual solution $y(t) = \frac{43}{36}e^t + \frac{1}{4}e^{-t} - \frac{4}{9}e^{-2t} + \frac{1}{6}te^t$.

**d.** $t^3y''' - t^2y'' + 3ty' - 4y = 5t^3 \ln t + 9t^3, \quad 1 \le t \le 2, \quad y(1) = 0, \quad y'(1) = 1, \quad y''(1) = 3$, with $h = 0.1$; actual solution $y(t) = -t^2 + t\cos(\ln t) + t\sin(\ln t) + t^3 \ln t$.

**4.** Use the Runge-Kutta for Systems Algorithm to approximate the solutions of the following higher-order differential equations, and compare the results to the actual solutions.

**a.** $y'' - 3y' + 2y = 6e^{-t}, \quad 0 \le t \le 1, \quad y(0) = y'(0) = 2$, with $h = 0.1$;
actual solution $y(t) = 2e^{2t} - e^t + e^{-t}$.

**b.** $t^2y'' + ty' - 4y = -3t, \quad 1 \le t \le 3, \quad y(1) = 4, \quad y'(1) = 3$, with $h = 0.2$;
actual solution $y(t) = 2t^2 + t + t^{-2}$.

**c.** $y''' + y'' - 4y' - 4y = 0, \quad 0 \le t \le 2, \quad y(0) = 3, \quad y'(0) = -1, \quad y''(0) = 9$, with $h = 0.2$;
actual solution $y(t) = e^{-t} + e^{2t} + e^{-2t}$.

**d.** $t^3y''' + t^2y'' - 2ty' + 2y = 8t^3 - 2, \quad 1 \le t \le 2, \quad y(1) = 2, \quad y'(1) = 8, \quad y''(1) = 6$, with $h = 0.1$; actual solution $y(t) = 2t - t^{-1} + t^2 + t^3 - 1$.

**5.** Change the Adams Fourth-Order Predictor-Corrector Algorithm to obtain approximate solutions to systems of first-order equations.

**6.** Repeat Exercise 2 using the algorithm developed in Exercise 5.

**7.** Repeat Exercise 1 using the algorithm developed in Exercise 5.

**8.** Suppose the swinging pendulum described in the lead example of this chapter is 2 ft long and that $g = 32.17$ ft/s$^2$. With $h = 0.1$ s, compare the angle $\theta$ obtained for the following two initial-value problems at $t = 0$, 1, and 2 s.

**a.** $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\sin\theta = 0, \quad \theta(0) = \dfrac{\pi}{6}, \quad \theta'(0) = 0,$

**b.** $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0, \quad \theta(0) = \dfrac{\pi}{6}, \quad \theta'(0) = 0,$

**9.** The study of mathematical models for predicting the population dynamics of competing species has its origin in independent works published in the early part of the 20th century by A. J. Lotka and V. Volterra (see, for example, [Lo1], [Lo2], and [Vo]).

Consider the problem of predicting the population of two species, one of which is a predator, whose population at time $t$ is $x_2(t)$, feeding on the other, which is the prey, whose population is $x_1(t)$. We will assume that the prey always has an adequate food supply and that its birth rate at any time is proportional to the number of prey alive at that time; that is, birth rate (prey) is $k_1x_1(t)$. The death rate of the prey depends on both the number of prey and predators alive at that time. For simplicity, we assume death rate (prey) = $k_2x_1(t)x_2(t)$. The birth rate of the predator, on the other hand, depends on its food supply, $x_1(t)$, as well as on the number of predators available for reproduction purposes. For this reason, we assume that the birth rate (predator) is $k_3x_1(t)x_2(t)$. The death rate of the predator will be taken as simply proportional to the number of predators alive at the time; that is, death rate (predator) = $k_4x_2(t)$.

Since $x_1'(t)$ and $x_2'(t)$ represent the change in the prey and predator populations, respectively, with respect to time, the problem is expressed by the system of nonlinear differential equations

$$x_1'(t) = k_1x_1(t) - k_2x_1(t)x_2(t) \quad \text{and} \quad x_2'(t) = k_3x_1(t)x_2(t) - k_4x_2(t).$$

Solve this system for $0 \le t \le 4$, assuming that the initial population of the prey is 1000 and of the predators is 500 and that the constants are $k_1 = 3$, $k_2 = 0.002$, $k_3 = 0.0006$, and $k_4 = 0.5$. Sketch a graph of the solutions to this problem, plotting both populations with time, and describe the physical phenomena represented. Is there a stable solution to this population model? If so, for what values $x_1$ and $x_2$ is the solution stable?

**10.** In Exercise 9 we considered the problem of predicting the population in a predator-prey model. Another problem of this type is concerned with two species competing for the same food supply. If the numbers of species alive at time $t$ are denoted by $x_1(t)$ and $x_2(t)$, it is often assumed that, although the birth rate of each of the species is simply proportional to the number of species alive at that time, the death rate of each species depends on the population of both species. We will assume that the population of a particular pair of species is described by the equations

$$\frac{dx_1(t)}{dt} = x_1(t)[4 - 0.0003x_1(t) - 0.0004x_2(t)] \quad \text{and} \quad \frac{dx_2(t)}{dt} = x_2(t)[2 - 0.0002x_1(t) - 0.0001x_2(t)].$$

If it is known that the initial population of each species is 10,000, find the solution to this system for $0 \le t \le 4$. Is there a stable solution to this population model? If so, for what values of $x_1$ and $x_2$ is the solution stable?