The local truncation error involved with a predictor-corrector method of the Milne-Simpson type is generally smaller than that of the Adams-Bashforth-Moulton method. But the technique has limited use because of round-off error problems, which do not occur with the Adams procedure. Elaboration on this difficulty is given in Section 5.10.

EXERCISE SET 5.6

1. Use all the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case use exact starting values, and compare the results to the actual values.
   a. $y' = te^{3t} - 2y$, $0 \le t \le 1$, $y(0) = 0$, with $h = 0.2$; actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.
   b. $y' = 1 + (t - y)^2$, $2 \le t \le 3$, $y(2) = 1$, with $h = 0.2$; actual solution $y(t) = t + \frac{1}{1-t}$.
   c. $y' = 1 + y/t$, $1 \le t \le 2$, $y(1) = 2$, with $h = 0.2$; actual solution $y(t) = t \ln t + 2t$.
   d. $y' = \cos 2t + \sin 3t$, $0 \le t \le 1$, $y(0) = 1$, with $h = 0.2$; actual solution $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.
2. Use each of the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case use starting values obtained from the Runge-Kutta method of order four. Compare the results to the actual values.
   a. $y' = \dfrac{2 - 2ty}{t^2 + 1}$, $0 \le t \le 1$, $y(0) = 1$, with $h = 0.1$ actual solution $y(t) = \dfrac{2t + 1}{t^2 + 2}$.
   b. $y' = \dfrac{y^2}{1 + t}$, $1 \le t \le 2$, $y(1) = -(\ln 2)^{-1}$, with $h = 0.1$ actual solution $y(t) = \dfrac{-1}{\ln(t + 1)}$.
   c. $y' = (y^2 + y)/t$, $1 \le t \le 3$, $y(1) = -2$, with $h = 0.2$ actual solution $y(t) = \dfrac{2t}{1 - t}$.
   d. $y' = -ty + 4t/y$, $0 \le t \le 1$, $y(0) = 1$, with $h = 0.1$ actual solution $y(t) = \sqrt{4 - 3e^{-t^2}}$.
3. Use each of the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case use starting values obtained from the Runge-Kutta method of order four. Compare the results to the actual values.
   a. $y' = y/t - (y/t)^2$, $1 \le t \le 2$, $y(1) = 1$, with $h = 0.1$; actual solution $y(t) = \dfrac{t}{1 + \ln t}$.
   b. $y' = 1 + y/t + (y/t)^2$, $1 \le t \le 3$, $y(1) = 0$, with $h = 0.2$; actual solution $y(t) = t\tan(\ln t)$.
   c. $y' = -(y + 1)(y + 3)$, $0 \le t \le 2$, $y(0) = -2$, with $h = 0.1$; actual solution $y(t) = -3 + 2/(1 + e^{-2t})$.
   d. $y' = -5y + 5t^2 + 2t$, $0 \le t \le 1$, $y(0) = 1/3$, with $h = 0.1$; actual solution $y(t) = t^2 + \frac{1}{3}e^{-5t}$.
4. Use all the Adams-Moulton methods to approximate the solutions to the Exercises 1(a), 1(c), and 1(d). In each case use exact starting values, and explicitly solve for w_{i+1}. Compare the results to the actual values.
5. Use Algorithm 5.4 to approximate the solutions to the initial-value problems in Exercise 1.
6. Use Algorithm 5.4 to approximate the solutions to the initial-value problems in Exercise 2.
7. Use Algorithm 5.4 to approximate the solutions to the initial-value problems in Exercise 3.
8. Change Algorithm 5.4 so that the corrector can be iterated for a given number p iterations. Repeat Exercise 7 with $p = 2, 3$, and 4 iterations. Which choice of p gives the best answer for each initial-value problem?
9. The initial-value problem

$$y' = e^y, \quad 0 \le t \le 0.20, \quad y(0) = 1$$

has solution

$$y(t) = 1 - \ln(1 - et).$$

Applying the three-step Adams-Moulton method to this problem is equivalent to finding the fixed point w_{i+1} of

$$g(w) = w_i + \frac{h}{24}\left(9e^{w} + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}\right).$$

a. With $h = 0.01$, obtain w_{i+1} by functional iteration for $i = 2, \ldots, 19$ using exact starting values w_0, w_1, and w_2. At each step use w_i to initially approximate w_{i+1}.

b. Will Newton's method speed the convergence over functional iteration?

10. Use the Milne-Simpson Predictor-Corrector method to approximate the solutions to the initial-value problems in Exercise 3.

11. **a.** Derive the Adams-Bashforth Two-Step method by using the Lagrange form of the interpolating polynomial.

b. Derive the Adams-Bashforth Four-Step method by using Newton's backward-difference form of the interpolating polynomial.

12. Derive the Adams-Bashforth Three-Step method by the following method. Set

$$y(t_{i+1}) = y(t_i) + ahf(t_i, y(t_i)) + bhf(t_{i-1}, y(t_{i-1})) + chf(t_{i-2}, y(t_{i-2})).$$

Expand $y(t_{i+1})$, $f(t_{i-2}, y(t_{i-2}))$, and $f(t_{i-1}, y(t_{i-1}))$ in Taylor series about $(t_i, y(t_i))$, and equate the coefficients of h, h^2 and h^3 to obtain a, b, and c.

13. Derive the Adams-Moulton Two-Step method and its local truncation error by using an appropriate form of an interpolating polynomial.

14. Derive Simpson's method by applying Simpson's rule to the integral

$$y(t_{i+1}) - y(t_{i-1}) = \int_{t_{i-1}}^{t_{i+1}} f(t, y(t))\, dt.$$

15. Derive Milne's method by applying the open Newton-Cotes formula (4.29) to the integral

$$y(t_{i+1}) - y(t_{i-3}) = \int_{t_{i-3}}^{t_{i+1}} f(t, y(t))\, dt.$$

16. Verify the entries in Table 5.12 on page 305.

5.7 Variable Step-Size Multistep Methods

The Runge-Kutta-Fehlberg method is used for error control because at each step it provides, at little additional cost, *two* approximations that can be compared and related to the local truncation error. Predictor-corrector techniques always generate two approximations at each step, so they are natural candidates for error-control adaptation.

To demonstrate the error-control procedure, we construct a variable step-size predictor-corrector method using the four-step explicit Adams-Bashforth method as predictor and the three-step implicit Adams-Moulton method as corrector.

The Adams-Bashforth four-step method comes from the relation

$$\begin{aligned} y(t_{i+1}) = y(t_i) + \frac{h}{24}[&55f(t_i, y(t_i)) - 59f(t_{i-1}, y(t_{i-1})) \\ &+ 37f(t_{i-2}, y(t_{i-2})) - 9f(t_{i-3}, y(t_{i-3}))] + \frac{251}{720}y^{(5)}(\hat{\mu}_i)h^5, \end{aligned}$$

for some $\hat{\mu}_i \in (t_{i-3}, t_{i+1})$. The assumption that the approximations $w_0, w_1, \ldots, w_i$ are all exact implies that the Adams-Bashforth local truncation error is

$$\frac{y(t_{i+1}) - w_{i+1,p}}{h} = \frac{251}{720}y^{(5)}(\hat{\mu}_i)h^4. \qquad (5.40)$$