Table 5.10

t_i	Exact	Euler $h = 0.025$	Modified Euler $h = 0.05$	Runge-Kutta Order Four $h = 0.1$
0.0	0.5000000	0.5000000	0.5000000	0.5000000
0.1	0.6574145	0.6554982	0.6573085	0.6574144
0.2	0.8292986	0.8253385	0.8290778	0.8292983
0.3	1.0150706	1.0089334	1.0147254	1.0150701
0.4	1.2140877	1.2056345	1.2136079	1.2140869
0.5	1.4256394	1.4147264	1.4250141	1.4256384

EXERCISE SET 5.4

1. Use the Modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.

a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0,$ with $h = 0.5$; actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.

b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ with $h = 0.5$; actual solution $y(t) = t + \frac{1}{1-t}$.

c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.25$; actual solution $y(t) = t\ln t + 2t$.

d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.25$; actual solution $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.

2. Use the Modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.

a. $y' = e^{t-y}, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.5$; actual solution $y(t) = \ln(e^t + e - 1)$.

b. $y' = \dfrac{1+t}{1+y}, \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.5$; actual solution $y(t) = \sqrt{t^2 + 2t + 6} - 1$.

c. $y' = -y + ty^{1/2}, \quad 2 \le t \le 3, \quad y(2) = 2,$ with $h = 0.25$; actual solution $y(t) = \left(t - 2 + \sqrt{2}ee^{-t/2}\right)^2$.

d. $y' = t^{-2}(\sin 2t - 2ty), \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.25$; actual solution $y(t) = \frac{1}{2}t^{-2}(4 + \cos 2 - \cos 2t)$.

3. Use the Modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.

a. $y' = y/t - (y/t)^2, \quad 1 \le t \le 2, \quad y(1) = 1,$ with $h = 0.1$; actual solution $y(t) = t/(1 + \ln t)$.

b. $y' = 1 + y/t + (y/t)^2, \quad 1 \le t \le 3, \quad y(1) = 0,$ with $h = 0.2$; actual solution $y(t) = t\tan(\ln t)$.

c. $y' = -(y + 1)(y + 3), \quad 0 \le t \le 2, \quad y(0) = -2,$ with $h = 0.2$; actual solution $y(t) = -3 + 2(1 + e^{-2t})^{-1}$.

d. $y' = -5y + 5t^2 + 2t, \quad 0 \le t \le 1, \quad y(0) = \frac{1}{3},$ with $h = 0.1$; actual solution $y(t) = t^2 + \frac{1}{3}e^{-5t}$.

4. Use the Modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.

a. $y' = \dfrac{2 - 2ty}{t^2 + 1}, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.1$; actual solution $y(t) = \dfrac{2t + 1}{t^2 + 1}$.

b. $y' = \dfrac{y^2}{1 + t}, \quad 1 \le t \le 2, \quad y(1) = -(\ln 2)^{-1},$ with $h = 0.1$; actual solution $y(t) = \dfrac{-1}{\ln(t + 1)}$.

c. $y' = (y^2 + y)/t, \quad 1 \le t \le 3, \quad y(1) = -2,$ with $h = 0.2$; actual solution $y(t) = \dfrac{2t}{1 - 2t}$.

d. $y' = -ty + 4t/y, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.1$; actual solution $y(t) = \sqrt{4 - 3e^{-t^2}}$.

5. Repeat Exercise 1 using the Midpoint method.
6. Repeat Exercise 2 using the Midpoint method.
7. Repeat Exercise 3 using the Midpoint method.
8. Repeat Exercise 4 using the Midpoint method.
9. Repeat Exercise 1 using Heun's method.
10. Repeat Exercise 2 using Heun's method.
11. Repeat Exercise 3 using Heun's method.
12. Repeat Exercise 4 using Heun's method.
13. Repeat Exercise 1 using the Runge-Kutta method of order four.
14. Repeat Exercise 2 using the Runge-Kutta method of order four.
15. Repeat Exercise 3 using the Runge-Kutta method of order four.
16. Repeat Exercise 4 using the Runge-Kutta method of order four.
17. Use the results of Exercise 3 and linear interpolation to approximate values of $y(t)$, and compare the results to the actual values.
    a. $y(1.25)$ and $y(1.93)$ **b.** $y(2.1)$ and $y(2.75)$
    c. $y(1.3)$ and $y(1.93)$ **d.** $y(0.54)$ and $y(0.94)$
18. Use the results of Exercise 4 and linear interpolation to approximate values of $y(t)$, and compare the results to the actual values.
    a. $y(0.54)$ and $y(0.94)$ **b.** $y(1.25)$ and $y(1.93)$
    c. $y(1.3)$ and $y(2.93)$ **d.** $y(0.54)$ and $y(0.94)$
19. Repeat Exercise 17 using the results of Exercise 7.
20. Repeat Exercise 18 using the results of Exercise 8.
21. Repeat Exercise 17 using the results of Exercise 11.
22. Repeat Exercise 18 using the results of Exercise 12.
23. Repeat Exercise 17 using the results of Exercise 15.
24. Repeat Exercise 18 using the results of Exercise 16.
25. Use the results of Exercise 15 and Cubic Hermite interpolation to approximate values of $y(t)$, and compare the approximations to the actual values.
    a. $y(1.25)$ and $y(1.93)$ **b.** $y(2.1)$ and $y(2.75)$
    c. $y(1.3)$ and $y(1.93)$ **d.** $y(0.54)$ and $y(0.94)$
26. Use the results of Exercise 16 and Cubic Hermite interpolation to approximate values of $y(t)$, and compare the approximations to the actual values.
    a. $y(0.54)$ and $y(0.94)$ **b.** $y(1.25)$ and $y(1.93)$
    c. $y(1.3)$ and $y(2.93)$ **d.** $y(0.54)$ and $y(0.94)$
27. Show that the Midpoint method and the Modified Euler method give the same approximations to the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 1, \quad y(0) = 1,$$

for any choice of h. Why is this true?

28. Water flows from an inverted conical tank with circular orifice at the rate

$$\frac{dx}{dt} = -0.6\pi r^2\sqrt{2g}\,\frac{\sqrt{x}}{A(x)},$$

where r is the radius of the orifice, x is the height of the liquid level from the vertex of the cone, and $A(x)$ is the area of the cross section of the tank x units above the orifice. Suppose $r = 0.1$ ft, $g = 32.1$ ft/s^2, and the tank has an initial water level of 8 ft and initial volume of $512(\pi/3)$ ft^3. Use the Runge-Kutta method of order four to find the following.
    a. The water level after 10 min with $h = 20$ s
    b. When the tank will be empty, to within 1 min.

29. The irreversible chemical reaction in which two molecules of solid potassium dichromate ($K_2Cr_2O_7$), two molecules of water (H_2O), and three atoms of solid sulfur (S) combine to yield three molecules of the gas sulfur dioxide (SO_2), four molecules of solid potassium hydroxide (KOH), and two molecules of solid chromic oxide (Cr_2O_3) can be represented symbolically by the *stoichiometric equation*:

$$2K_2Cr_2O_7 + 2H_2O + 3S \longrightarrow 4KOH + 2Cr_2O_3 + 3SO_2.$$

If n_1 molecules of $K_2Cr_2O_7$, n_2 molecules of H_2O, and n_3 molecules of S are originally available, the following differential equation describes the amount $x(t)$ of KOH after time t:

$$\frac{dx}{dt} = k\left(n_1 - \frac{x}{2}\right)^2\left(n_2 - \frac{x}{2}\right)^2\left(n_3 - \frac{3x}{4}\right)^3,$$

where k is the velocity constant of the reaction. If $k = 6.22 \times 10^{-19}, n_1 = n_2 = 2 \times 10^3$, and $n_3 = 3 \times 10^3$, use the Runge-Kutta method of order four to determine how many units of potassium hydroxide will have been formed after 0.2 s?

30. Show that the difference method

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + a_1 f(t_i, w_i) + a_2 f(t_i + \alpha_2, w_1 + \delta_2 f(t_i, w_i)), \end{aligned}$$

for each $i = 0, 1, \ldots, N-1$, cannot have local truncation error $O(h^3)$ for any choice of constants a_1, a_2, α_2, and δ_2.

31. Show that Heun's method can be expressed in difference form, similar to that of the Runge-Kutta method of order four, as

$$\begin{aligned} w_0 &= \alpha, \\ k_1 &= hf(t_i, w_i), \\ k_2 &= hf\left(t_i + \frac{h}{3}, w_i + \frac{1}{3}k_1\right), \\ k_3 &= hf\left(t_i + \frac{2h}{3}, w_i + \frac{2}{3}k_2\right), \\ w_{i+1} &= w_i + \frac{1}{4}(k_1 + 3k_3), \end{aligned}$$

for each $i = 0, 1, \ldots, N-1$.

32. The Runge-Kutta method of order four can be written in the form

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + \frac{h}{6}f(t_i, w_i) + \frac{h}{3}f(t_i + \alpha_1 h, w_i + \delta_1 hf(t_i, w_i)) \\ &\quad + \frac{h}{3}f(t_i + \alpha_2 h, w_i + \delta_2 hf(t_i + \gamma_2 h, w_i + \gamma_3 hf(t_i, w_i))) \\ &\quad + \frac{h}{6}f(t_i + \alpha_3 h, w_i + \delta_3 hf(t_i + \gamma_4 h, w_i + \gamma_5 hf(t_i + \gamma_6 h, w_i + \gamma_7 hf(t_i, w_i)))). \end{aligned}$$

Find the values of the constants

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \delta_1,\ \delta_2,\ \delta_3,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_6,\ \text{and}\ \gamma_7.$$

5.5 Error Control and the Runge-Kutta-Fehlberg Method

In Section 4.6 we saw that the appropriate use of varying step sizes for integral approximations produced efficient methods. In itself, this might not be sufficient to favor these methods due to the increased complication of applying them. However, they have another feature