# Bài tập Bài 7.3a Phương pháp Euler

**1.** Use Euler's method to approximate the solutions for each of the following initial-value problems.

**a.** $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0, \text{ with } h = 0.5$

**b.** $y' = 1 + (t-y)^2, \quad 2 \le t \le 3, \quad y(2) = 1, \text{ with } h = 0.5$

**c.** $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2, \text{ with } h = 0.25$

**d.** $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1, \text{ with } h = 0.25$

**2.** Use Euler's method to approximate the solutions for each of the following initial-value problems.

**a.** $y' = e^{t-y}, \quad 0 \le t \le 1, \quad y(0) = 1, \text{ with } h = 0.5$

**b.** $y' = \dfrac{1+t}{1+y}, \quad 1 \le t \le 2, \quad y(1) = 2, \text{ with } h = 0.5$

**c.** $y' = -y + ty^{1/2}, \quad 2 \le t \le 3, \quad y(2) = 2, \text{ with } h = 0.25$

**d.** $y' = t^{-2}(\sin 2t - 2ty), \quad 1 \le t \le 2, \quad y(1) = 2, \text{ with } h = 0.25$

**3.** The actual solutions to the initial-value problems in Exercise 1 are given here. Compare the actual error at each step to the error bound.

**a.** $y(t) = \dfrac{1}{5}te^{3t} - \dfrac{1}{25}e^{3t} + \dfrac{1}{25}e^{-2t}$

**b.** $y(t) = t + \dfrac{1}{1-t}$

**c.** $y(t) = t \ln t + 2t$

**d.** $y(t) = \dfrac{1}{2}\sin 2t - \dfrac{1}{3}\cos 3t + \dfrac{4}{3}$

**4.** The actual solutions to the initial-value problems in Exercise 2 are given here. Compute the actual error and compare this to the error bound if Theorem 5.9 can be applied.

**a.** $y(t) = \ln(e^t + e - 1)$

**b.** $y(t) = \sqrt{t^2 + 2t + 6} - 1$

**c.** $y(t) = \left(t - 2 + \sqrt{2}ee^{-t/2}\right)^2$

**d.** $y(t) = \dfrac{4 + \cos 2 - \cos 2t}{2t^2}$

**5.** Use Euler's method to approximate the solutions for each of the following initial-value problems.

**a.** $y' = y/t - (y/t)^2, \quad 1 \le t \le 2, \quad y(1) = 1, \text{ with } h = 0.1$

**b.** $y' = 1 + y/t + (y/t)^2, \quad 1 \le t \le 3, \quad y(1) = 0, \text{ with } h = 0.2$

**c.** $y' = -(y+1)(y+3), \quad 0 \le t \le 2, \quad y(0) = -2, \text{ with } h = 0.2$

**d.** $y' = -5y + 5t^2 + 2t, \quad 0 \le t \le 1, \quad y(0) = \frac{1}{3}, \text{ with } h = 0.1$

**6.** Use Euler's method to approximate the solutions for each of the following initial-value problems.

**a.** $y' = \dfrac{2-2ty}{t^2+1}, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.1$

**b.** $y' = \dfrac{y^2}{1+t}, \quad 1 \le t \le 2, \quad y(1) = -(\ln 2)^{-1}$, with $h = 0.1$

**c.** $y' = (y^2+y)/t, \quad 1 \le t \le 3, \quad y(1) = -2$, with $h = 0.2$

**d.** $y' = -ty + 4ty^{-1}, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.1$

**7.** The actual solutions to the initial-value problems in Exercise 5 are given here. Compute the actual error in the approximations of Exercise 5.

**a.** $y(t) = \dfrac{t}{1+\ln t}$

**b.** $y(t) = t\tan(\ln t)$

**c.** $y(t) = -3 + \dfrac{2}{1+e^{-2t}}$

**d.** $y(t) = t^2 + \dfrac{1}{3}e^{-5t}$

**8.** The actual solutions to the initial-value problems in Exercise 6 are given here. Compute the actual error in the approximations of Exercise 6.

**a.** $y(t) = \dfrac{2t+1}{t^2+1}$

**b.** $y(t) = \dfrac{-1}{\ln(t+1)}$

**c.** $y(t) = \dfrac{2t}{1-2t}$

**d.** $y(t) = \sqrt{4-3e^{-t^2}}$

**9.** Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$ :

**a.** Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of $y$.

**b.** Use the answers generated in part (a) and linear interpolation to approximate the following values of $y$, and compare them to the actual values.

**i.** $y(1.04)$ **ii.** $y(1.55)$ **iii.** $y(1.97)$

**c.** Compute the value of $h$ necessary for $|y(t_i) - w_i| \le 0.1$, using Eq. (5.10).

10. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2, \quad y(1) = -1,$$

with exact solution $y(t) = -1/t$:

a. Use Euler's method with $h = 0.05$ to approximate the solution, and compare it with the actual values of $y$.

b. Use the answers generated in part (a) and linear interpolation to approximate the following values of $y$, and compare them to the actual values.

   i. $y(1.052)$ ii. $y(1.555)$ iii. $y(1.978)$

c. Compute the value of $h$ necessary for $|y(t_i) - w_i| \le 0.05$ using Eq. (5.10).

11. Given the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 5, \quad y(0) = 1,$$

with exact solution $y(t) = e^{-t} + t$:

a. Approximate $y(5)$ using Euler's method with $h = 0.2$, $h = 0.1$, and $h = 0.05$.

b. Determine the optimal value of $h$ to use in computing $y(5)$, assuming $\delta = 10^{-6}$ and that Eq. (5.14) is valid.

12. Consider the initial-value problem

$$y' = -10y, \quad 0 \le t \le 2, \quad y(0) = 1,$$

which has solution $y(t) = e^{-10t}$. What happens when Euler's method is applied to this problem with $h = 0.1$? Does this behavior violate Theorem 5.9?

13. Use the results of Exercise 5 and linear interpolation to approximate the following values of $y(t)$. Compare the approximations obtained to the actual values obtained using the functions given in Exercise 7.

a. $y(1.25)$ and $y(1.93)$

b. $y(2.1)$ and $y(2.75)$

c. $y(1.3)$ and $y(1.93)$

d. $y(0.54)$ and $y(0.94)$

14. Use the results of Exercise 6 and linear interpolation to approximate the following values of $y(t)$. Compare the approximations obtained to the actual values obtained using the functions given in Exercise 8.

a. $y(0.25)$ and $y(0.93)$

b. $y(1.25)$ and $y(1.93)$

c. $y(2.10)$ and $y(2.75)$

d. $y(0.54)$ and $y(0.94)$

**15.** Let $E(h) = \dfrac{hM}{2} + \dfrac{\delta}{h}$.

**a.** For the initial-value problem

$$y' = -y + 1, \quad 0 \le t \le 1, \quad y(0) = 0,$$

compute the value of $h$ to minimize $E(h)$. Assume $\delta = 5 \times 10^{-(n+1)}$ if you will be using $n$-digit arithmetic in part (c).

**b.** For the optimal $h$ computed in part (a), use Eq. (5.13) to compute the minimal error obtainable.

**c.** Compare the actual error obtained using $h = 0.1$ and $h = 0.01$ to the minimal error in part (b). Can you explain the results?

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**16.** In a circuit with impressed voltage $\mathcal{E}$ having resistance $R$, inductance $L$, and capacitance $C$ in parallel, the current $i$ satisfies the differential equation

$$\frac{di}{dt} = C\frac{d^2\mathcal{E}}{dt^2} + \frac{1}{R}\frac{d\mathcal{E}}{dt} + \frac{1}{L}\mathcal{E}.$$

Suppose $C = 0.3$ farads, $R = 1.4$ ohms, $L = 1.7$ henries, and the voltage is given by

$$\mathcal{E}(t) = e^{-0.06\pi t}\sin(2t - \pi).$$

If $i(0) = 0$, find the current $i$ for the values $t = 0.1j$, where $j = 0, 1, \ldots, 100$.

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**17.** In a book entitled *Looking at History Through Mathematics*, Rashevsky [Ra], pp. 103–110, considers a model for a problem involving the production of nonconformists in society. Suppose that a society has a population of $x(t)$ individuals at time $t$, in years, and that all nonconformists who mate with other nonconformists have offspring who are also nonconformists, while a fixed proportion $r$ of all other offspring are also nonconformist. If the birth and death rates for all individuals are assumed to be the constants $b$ and $d$, respectively, and if conformists and nonconformists mate at random, the problem can be expressed by the differential equations

$$\frac{dx(t)}{dt} = (b-d)x(t) \quad \text{and} \quad \frac{dx_n(t)}{dt} = (b-d)x_n(t) + rb(x(t) - x_n(t)),$$

where $x_n(t)$ denotes the number of nonconformists in the population at time $t$.

**a.** Suppose the variable $p(t) = x_n(t)/x(t)$ is introduced to represent the proportion of nonconformists in the society at time $t$. Show that these equations can be combined and simplified to the single differential equation

$$\frac{dp(t)}{dt} = rb(1 - p(t)).$$

**b.** Assuming that $p(0) = 0.01$, $b = 0.02$, $d = 0.015$, and $r = 0.1$, approximate the solution $p(t)$ from $t = 0$ to $t = 50$ when the step size is $h = 1$ year.

**c.** Solve the differential equation for $p(t)$ exactly, and compare your result in part (b) when $t = 50$ with the exact value at that time.

## Modified Euler Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))], \quad \text{for} \quad i = 0, 1, \ldots, N-1.$$