Bài tập Chương 4 bài 4. Nội suy Spline bậc 3

Phần I

PROBLEM SET 19.4

1. **WRITING PROJECT. Splines.** In your own words, and using as few formulas as possible, write a short report on spline interpolation, its motivation, a comparison with polynomial interpolation, and its applications.

2–9 VERIFICATIONS. DERIVATIONS. COMPARISONS

2. **Individual polynomial q_j.** Show that $q_j(x)$ in (6) satisfies the interpolation condition (4) as well as the derivative condition (5).

3. Verify the differentiations that give (7) and (8) from (6).

4. **System for derivatives.** Derive the basic linear system (9) for $k_1, \cdots, k_{n-1}$ as indicated in the text.

5. **Equidistant nodes.** Derive (14) from (9).

6. **Coefficients.** Give the details of the derivation of a_{j2} and a_{j3} in (13).

7. Verify the computations in Example 1.

8. **Comparison.** Compare the spline g in Example 1 with the quadratic interpolation polynomial over the whole interval. Find the maximum deviations of g and p_2 from f. Comment.

9. **Natural spline condition.** Using the given coefficients, verify that the spline in Example 2 satisfies $g''(x) = 0$ at the ends.

10–16 DETERMINATION OF SPLINES

Find the cubic spline $g(x)$ for the given data with k_0 and k_n as given.

10. $f(-2) = f(-1) = f(1) = f(2) = 0, \quad f(0) = 1,$
$k_0 = k_4 = 0$

Sử dụng hệ pt (14):

$$(14)\ k_{j-1} + 4k_j + k_{j+1} = \frac{3}{h}(f_{j+1} - f_{j-1}),\ j = 1, \ldots, n-1$$

$4k_1 + k_2 = 3$

$k_1 + 4k_2 + k_3 = 0$

$k_2 + 4k_3 = -3$

$k_1 = 0.75,\ k_2 = 0,\ k_3 = -0.75$

$a_{00} = f_0 = 0,\ a_{01} = k_0 = 0,\ a_{02} = -0.75$

$a_{03} = 0.75$

where $j = 0, \cdots, n-1$. Using Taylor's formula, we obtain

$$\begin{aligned}
a_{j0} &= q_j(x_j) = f_j && \text{by (2),} \\
a_{j1} &= q_j'(x_j) = k_j && \text{by (5),} \\
a_{j2} &= \frac{1}{2} q_j''(x_j) = \frac{3}{h_j^2}(f_{j+1} - f_j) - \frac{1}{h_j}(k_{j+1} + 2k_j) && \text{by (7),} \\
a_{j3} &= \frac{1}{6} q_j'''(x_j) = \frac{2}{h_j^3}(f_j - f_{j+1}) + \frac{1}{h_j^2}(k_{j+1} + k_j)
\end{aligned} \tag{13}$$

j	0	1	2	3	4
x_j	-2	-1	0	1	2
f_j	0	0	1	0	0

with a_{j3} obtained by calculating $q_j''(x_{j+1})$ from (12) and equating the result to (8), that is,

$$q_j''(x_{j+1}) = 2a_{j2} + 6a_{j3}h_j = \frac{6}{h_j^2}(f_j - f_{j+1}) + \frac{2}{h_j}(k_j + 2k_{j+1}),$$

and now subtracting from this $2a_{j2}$ as given in (13) and simplifying.

Note that for ***equidistant nodes*** of distance $h_j = h$ we can write $c_j = c = 1/h$ in (6*) and have from (9) simply

(14) $$k_{j-1} + 4k_j + k_{j+1} = \frac{3}{h}(f_{j+1} - f_{j-1}) \qquad (j = 1, \cdots, n-1).$$

where $j = 0, \cdots, n-1$. Using Taylor's formula, we obtain

(13)
$$\begin{aligned}
a_{j0} &= q_j(x_j) = f_j && \text{by (2),}\\
a_{j1} &= q_j'(x_j) = k_j && \text{by (5),}\\
a_{j2} &= \frac{1}{2}q_j''(x_j) = \frac{3}{h_j^2}(f_{j+1} - f_j) - \frac{1}{h_j}(k_{j+1} + 2k_j) && \text{by (7),}\\
a_{j3} &= \frac{1}{6}q_j'''(x_j) = \frac{2}{h_j^3}(f_j - f_{j+1}) + \frac{1}{h_j^2}(k_{j+1} + k_j)
\end{aligned}$$

11. If we started from the piecewise linear function in Fig. 438, we would obtain $g(x)$ in Prob. 10 as the spline satisfying $g'(-2) = f'(-2) = 0$, $g'(2) = f'(2) = 0$. Find and sketch or graph the corresponding interpolation polynomial of 4th degree and compare it with the spline. Comment.

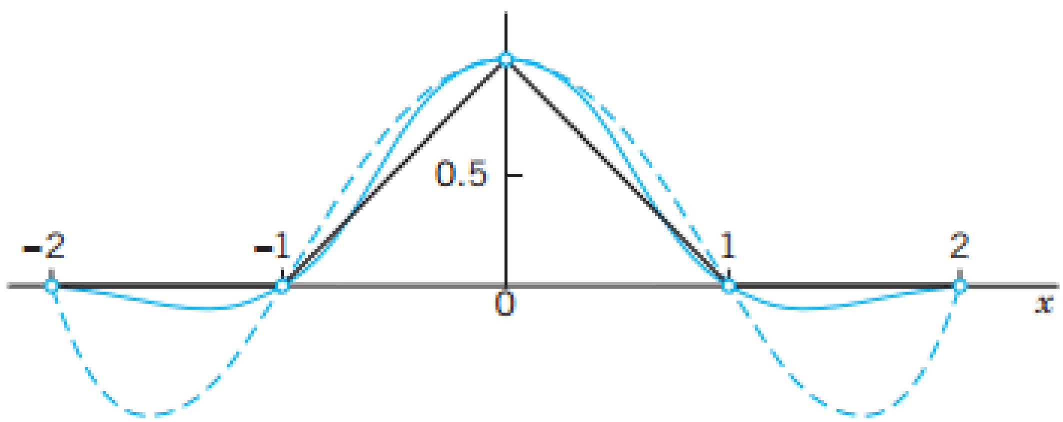


Fig. 438. Spline and interpolation polynomial in Probs. 10 and 11

12. $f_0 = f(0) = 1$, $f_1 = f(2) = 9$, $f_2 = f(4) = 41$, $f_3 = f(6) = 41$, $k_0 = 0$, $k_3 = -12$

13. $f_0 = f(0) = 1$, $f_1 = f(1) = 0$, $f_2 = f(2) = -1$, $f_3 = f(3) = 0$, $k_0 = 0$, $k_3 = -6$

14. $f_0 = f(0) = 2$, $f_1 = f(1) = 3$, $f_2 = f(2) = 8$, $f_3 = f(3) = 12$, $k_0 = k_3 = 0$

15. $f_0 = f(0) = 4, \quad f_1 = f(2) = 0, \quad f_2 = f(4) = 4,$
$f_3 = f(6) = 80, \quad k_0 = k_3 = 0$

16. $f_0 = f(0) = 2, \quad f_1 = f(2) = -2, \quad f_2 = f(4) = 2,$
$f_3 = f(6) = 78, \quad k_0 = k_3 = 0$. Can you obtain the answer from that of Prob. 15?

17. If a cubic spline is three times continuously differentiable (that is, it has continuous first, second, and third derivatives), show that it must be a single polynomial.

18. CAS EXPERIMENT. Spline versus Polynomial. If your CAS gives natural splines, find the natural splines when x is integer from $-m$ to m, and $y(0) = 1$ and all other y equal to 0. Graph each such spline along with the interpolation polynomial p_{2m}. Do this for $m = 2$ to 10 (or more). What happens with increasing m?

19. Natural conditions. Explain the remark after (11).

20. TEAM PROJECT. Hermite Interpolation and Bezier Curves. In **Hermite interpolation** we are looking for a polynomial $p(x)$ (of degree $2n + 1$ or less) such that $p(x)$ and its derivative $p'(x)$ have given values at $n + 1$ nodes. (More generally, $p(x), p'(x), p''(x), \cdots$ may be required to have given values at the nodes.)

(a) Curves with given endpoints and tangents. Let C be a curve in the xy-plane parametrically represented by $\mathbf{r}(t) = [x(t), y(t)]$, $0 \leqq t \leqq 1$ (see Sec. 9.5). Show that for given initial and terminal points of a curve and given initial and terminal tangents, say,

$$A: \quad \mathbf{r}_0 = [x(0), y(0)]$$

$$= [x_0, y_0],$$

$$
\begin{aligned}
B: \quad \mathbf{r}_1 &= [x(1), y(1)] \\
&= [x_1, y_1] \\
\mathbf{v}_0 &= [x'(0), y'(0)] \\
&= [x'_0, y'_0], \\
\mathbf{v}_1 &= [x'(1), y'(1)] \\
&= [x'_1, y'_1]
\end{aligned}
$$

we can find a curve C, namely,

$$
\begin{aligned}
\mathbf{r}(t) = \mathbf{r}_0 &+ \mathbf{v}_0 t \\
&+ (3(\mathbf{r}_1 - \mathbf{r}_0) - (2\mathbf{v}_0 + \mathbf{v}_1))t^2 \\
&+ (2(\mathbf{r}_0 - \mathbf{r}_1) + \mathbf{v}_0 + \mathbf{v}_1)t^3;
\end{aligned} \tag{15}
$$

in components,

$$x(t) = x_0 + x_0't + (3(x_1 - x_0) - (2x_0' + x_1'))t^2$$
$$+ (2(x_0 - x_1) + x_0' + x_1')t^3$$

$$y(t) = y_0 + y_0't + (3(y_1 - y_0) - (2y_0' + y_1'))t^2$$
$$+ (2(y_0 - y_1) + y_0' + y_1')t^3.$$

Note that this is a cubic Hermite interpolation polynomial, and $n = 1$ because we have two nodes (the endpoints of C). (This has nothing to do with the Hermite polynomials in Sec. 5.8.) The two points

$$G_A\colon\ \mathbf{g}_0 = \mathbf{r}_0 + \mathbf{v}_0$$
$$= [x_0 + x_0', y_0 + y_0']$$

and

$$G_B\colon\ \mathbf{g}_1 = \mathbf{r}_1 - \mathbf{v}_1$$
$$= [x_1 - x_1', y_1 - y_1']$$

are called **guidepoints** because the segments AG_A and BG_B specify the tangents graphically. A, B, G_A, G_B determine C, and C can be changed quickly by moving the points. A curve consisting of such Hermite interpolation polynomials is called a **Bezier curve**, after the French engineer P. Bezier of the Renault

Automobile Company, who introduced them in the early 1960s in designing car bodies. Bezier curves (and surfaces) are used in computer-aided design (CAD) and computer-aided manufacturing (CAM). (For more details, see Ref. [E21] in App. 1.)

(b) Find and graph the Bezier curve and its guidepoints if A: [0, 0], B: [1, 0], $\mathbf{v}_0 = [\frac{1}{2}, \frac{1}{2}]$, $\mathbf{v}_1 = [-\frac{1}{2}, -\frac{1}{4}\sqrt{3}]$.

(c) Changing guidepoints changes C. Moving guidepoints farther away results in C "staying near the tangents for a longer time." Confirm this by changing $\mathbf{v}_0$ and $\mathbf{v}_1$ in (b) to $2\mathbf{v}_0$ and $2\mathbf{v}_1$ (see Fig. 439).

(d) Make experiments of your own. What happens if you change $\mathbf{v}_1$ in (b) to $-\mathbf{v}_1$. If you rotate the tangents? If you multiply $\mathbf{v}_0$ and $\mathbf{v}_1$ by positive factors less than 1?

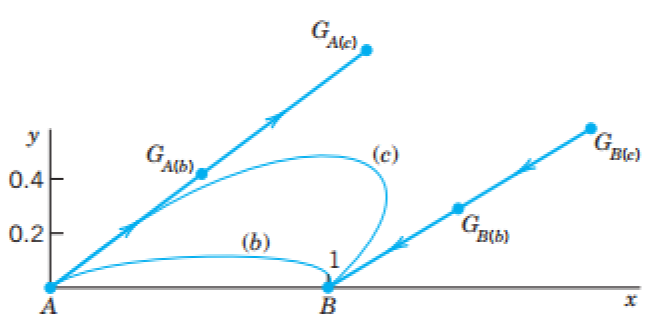


Fig. 439. Team Project 20(b) and (c): Bezier curves

Phần II

1. Determine the natural cubic spline S that interpolates the data $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$.
2. Determine the clamped cubic spline s that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and satisfies $s'(0) = s'(2) = 1$.
3. Construct the natural cubic spline for the following data.

a.

x	$f(x)$
8.3	17.56492
8.6	18.50515

b.

x	$f(x)$
0.8	0.22363362
1.0	0.65809197

c.

x	$f(x)$
−0.5	−0.0247500
−0.25	0.3349375
0	1.1010000

d.

x	$f(x)$
0.1	−0.62049958
0.2	−0.28398668
0.3	0.00660095
0.4	0.24842440

4. Construct the natural cubic spline for the following data.

a.

x	$f(x)$
0	1.00000
0.5	2.71828

b.

x	$f(x)$
−0.25	1.33203
0.25	0.800781

c.

x	$f(x)$
0.1	−0.29004996
0.2	−0.56079734
0.3	−0.81401972

d.

x	$f(x)$
−1	0.86199480
−0.5	0.95802009
0	1.0986123
0.5	1.2943767

5. The data in Exercise 3 were generated using the following functions. Use the cubic splines constructed in Exercise 3 for the given value of x to approximate $f(x)$ and $f'(x)$, and calculate the actual error.
   a. $f(x) = x \ln x$; approximate $f(8.4)$ and $f'(8.4)$.
   b. $f(x) = \sin(e^x - 2)$; approximate $f(0.9)$ and $f'(0.9)$.
   c. $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$; approximate $f(-\frac{1}{3})$ and $f'(-\frac{1}{3})$.
   d. $f(x) = x \cos x - 2x^2 + 3x - 1$; approximate $f(0.25)$ and $f'(0.25)$.

6. The data in Exercise 4 were generated using the following functions. Use the cubic splines constructed in Exercise 4 for the given value of x to approximate $f(x)$ and $f'(x)$, and calculate the actual error.
   a. $f(x) = e^{2x}$; approximate $f(0.43)$ and $f'(0.43)$.
   b. $f(x) = x^4 - x^3 + x^2 - x + 1$; approximate $f(0)$ and $f'(0)$.
   c. $f(x) = x^2 \cos x - 3x$; approximate $f(0.18)$ and $f'(0.18)$.
   d. $f(x) = \ln(e^x + 2)$; approximate $f(0.25)$ and $f'(0.25)$.
7. Construct the clamped cubic spline using the data of Exercise 3 and the fact that
   a. $f'(8.3) = 3.116256$ and $f'(8.6) = 3.151762$
   b. $f'(0.8) = 2.1691753$ and $f'(1.0) = 2.0466965$
   c. $f'(-0.5) = 0.7510000$ and $f'(0) = 4.0020000$
   d. $f'(0.1) = 3.58502082$ and $f'(0.4) = 2.16529366$

8. Construct the clamped cubic spline using the data of Exercise 4 and the fact that
   a. $f'(0) = 2$ and $f'(0.5) = 5.43656$
   b. $f'(-0.25) = 0.437500$ and $f'(0.25) = -0.625000$
   c. $f'(0.1) = -2.8004996$ and $f'(0) = -2.9734038$
   d. $f'(-1) = 0.15536240$ and $f'(0.5) = 0.45186276$

9. Repeat Exercise 5 using the clamped cubic splines constructed in Exercise 7.
10. Repeat Exercise 6 using the clamped cubic splines constructed in Exercise 8.
11. A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \le x < 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

    Find b, c, and d.
12. A clamped cubic spline s for a function f is defined on $[1, 3]$ by

$$s(x) = \begin{cases} s_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{if } 1 \le x < 2, \\ s_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

    Given $f'(1) = f'(3)$, find a, b, c, and d.
13. A natural cubic spline S is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + B(x-1) - D(x-1)^3, & \text{if } 1 \le x < 2, \\ S_1(x) = 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

    If S interpolates the data $(1, 1)$, $(2, 1)$, and $(3, 0)$, find B, D, b, and d.

14. A clamped cubic spline s for a function f is defined by

$$s(x) = \begin{cases} s_0(x) = 1 + Bx + 2x^2 - 2x^3, & \text{if } 0 \le x < 1, \\ s_1(x) = 1 + b(x-1) - 4(x-1)^2 + 7(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

Find $f'(0)$ and $f'(2)$.

15. Construct a natural cubic spline to approximate $f(x) = \cos \pi x$ by using the values given by $f(x)$ at $x = 0, 0.25, 0.5, 0.75$, and 1.0. Integrate the spline over $[0, 1]$, and compare the result to $\int_0^1 \cos \pi x \, dx = 0$. Use the derivatives of the spline to approximate $f'(0.5)$ and $f''(0.5)$. Compare these approximations to the actual values.

16. Construct a natural cubic spline to approximate $f(x) = e^{-x}$ by using the values given by $f(x)$ at $x = 0$, 0.25, 0.75, and 1.0. Integrate the spline over $[0, 1]$, and compare the result to $\int_0^1 e^{-x} \, dx = 1 - 1/e$. Use the derivatives of the spline to approximate $f'(0.5)$ and $f''(0.5)$. Compare the approximations to the actual values.

17. Repeat Exercise 15, constructing instead the clamped cubic spline with $f'(0) = f'(1) = 0$.

18. Repeat Exercise 16, constructing instead the clamped cubic spline with $f'(0) = -1$, $f'(1) = -e^{-1}$.

19. Suppose that $f(x)$ is a polynomial of degree 3. Show that $f(x)$ is its own clamped cubic spline, but that it cannot be its own natural cubic spline.

20. Suppose the data $\{x_i, f(x_i)\}_{i=1}^n$ lie on a straight line. What can be said about the natural and clamped cubic splines for the function f? [*Hint:* Take a cue from the results of Exercises 1 and 2.]

21. Given the partition $x_0 = 0$, $x_1 = 0.05$, and $x_2 = 0.1$ of $[0, 0.1]$, find the piecewise linear interpolating function F for $f(x) = e^{2x}$. Approximate $\int_0^{0.1} e^{2x} \, dx$ with $\int_0^{0.1} F(x) \, dx$, and compare the results to the actual value.

22. Let $f \in C^2[a, b]$, and let the nodes $a = x_0 < x_1 < \cdots < x_n = b$ be given. Derive an error estimate similar to that in Theorem 3.13 for the piecewise linear interpolating function F. Use this estimate to derive error bounds for Exercise 21.

23. Extend Algorithms 3.4 and 3.5 to include as output the first and second derivatives of the spline at the nodes.

24. Extend Algorithms 3.4 and 3.5 to include as output the integral of the spline over the interval $[x_0, x_n]$.

25. Given the partition $x_0 = 0$, $x_1 = 0.05$, $x_2 = 0.1$ of $[0, 0.1]$ and $f(x) = e^{2x}$:

a. Find the cubic spline s with clamped boundary conditions that interpolates f.

b. Find an approximation for $\int_0^{0.1} e^{2x} \, dx$ by evaluating $\int_0^{0.1} s(x) \, dx$.

c. Use Theorem 3.13 to estimate $\max_{0 \le x \le 0.1} |f(x) - s(x)|$ and

$$\left| \int_0^{0.1} f(x) \, dx - \int_0^{0.1} s(x) \, dx \right|.$$

d. Determine the cubic spline S with natural boundary conditions, and compare $S(0.02)$, $s(0.02)$, and $e^{0.04} = 1.04081077$.

26. Let f be defined on $[a, b]$, and let the nodes $a = x_0 < x_1 < x_2 = b$ be given. A quadratic spline interpolating function S consists of the quadratic polynomial

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 \quad \text{on } [x_0, x_1]$$

and the quadratic polynomial

$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 \quad \text{on } [x_1, x_2],$$

such that

i. $S(x_0) = f(x_0), S(x_1) = f(x_1)$, and $S(x_2) = f(x_2)$,

ii. $S \in C^1[x_0, x_2]$.

Show that conditions (i) and (ii) lead to five equations in the six unknowns a_0, b_0, c_0, a_1, b_1, and c_1. The problem is to decide what additional condition to impose to make the solution unique. Does the condition $S \in C^2[x_0, x_2]$ lead to a meaningful solution?

27. Determine a quadratic spline s that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and satisfies $s'(0) = 2$.

28. **a.** The introduction to this chapter included a table listing the population of the United States from 1950 to 2000. Use natural cubic spline interpolation to approximate the population in the years 1940, 1975, and 2020.

b. The population in 1940 was approximately 132,165,000. How accurate do you think your 1975 and 2020 figures are?

29. A car traveling along a straight road is clocked at a number of points. The data from the observations are given in the following table, where the time is in seconds, the distance is in feet, and the speed is in feet per second.

Time	0	3	5	8	13
Distance	0	225	383	623	993
Speed	75	77	80	74	72

a. Use a clamped cubic spline to predict the position of the car and its speed when $t = 10$ s.

b. Use the derivative of the spline to determine whether the car ever exceeds a 55-mi/h speed limit on the road; if so, what is the first time the car exceeds this speed?

c. What is the predicted maximum speed for the car?

30. The 2009 Kentucky Derby was won by a horse named Mine That Bird (at more than 50:1 odds) in a time of 2:02.66 (2 minutes and 2.66 seconds) for the $1\frac{1}{4}$-mile race. Times at the quarter-mile, half-mile, and mile poles were 0:22.98, 0:47.23, and 1:37.49.

a. Use these values together with the starting time to construct a natural cubic spline for Mine That Bird's race.

b. Use the spline to predict the time at the three-quarter-mile pole, and compare this to the actual time of 1:12.09.

c. Use the spline to approximate Mine That Bird's starting speed and speed at the finish line.

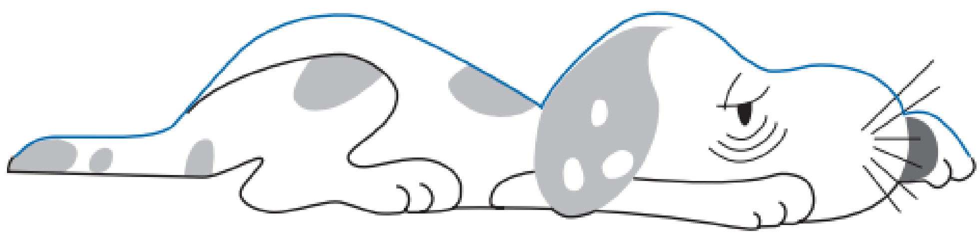
31. It is suspected that the high amounts of tannin in mature oak leaves inhibit the growth of the winter moth (*Operophtera bromata L., Geometridae*) larvae that extensively damage these trees in certain years. The following table lists the average weight of two samples of larvae at times in the first 28 days after birth. The first sample was reared on young oak leaves, whereas the second sample was reared on mature leaves from the same tree.

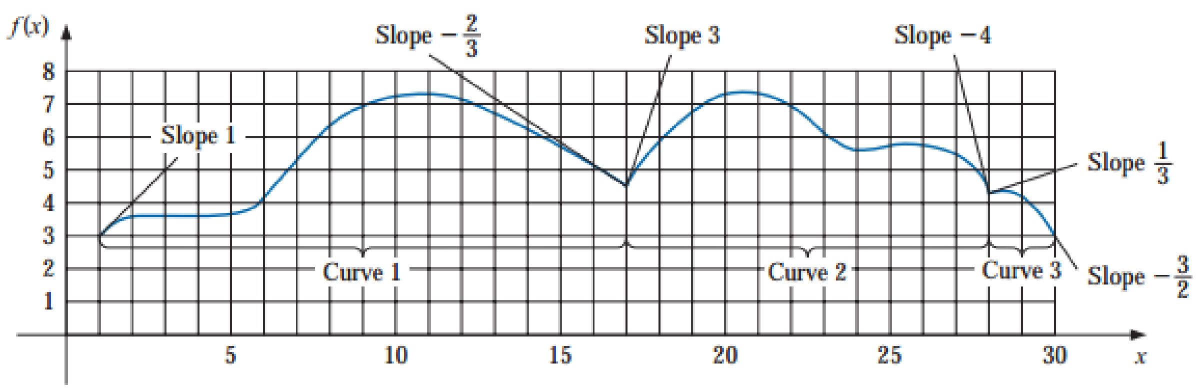
a. Use a natural cubic spline to approximate the average weight curve for each sample.

b. Find an approximate maximum average weight for each sample by determining the maximum of the spline.

Day	0	6	10	13	17	20	28
Sample 1 average weight (mg)	6.67	17.33	42.67	37.33	30.10	29.31	28.74
Sample 2 average weight (mg)	6.67	16.11	18.89	15.00	10.56	9.44	8.89

32. The upper portion of this noble beast is to be approximated using clamped cubic spline interpolants. The curve is drawn on a grid from which the table is constructed. Use Algorithm 3.5 to construct the three clamped cubic splines.





Curve 1				Curve 2				Curve 3			
i	x_i	$f(x_i)$	$f'(x_i)$	i	x_i	$f(x_i)$	$f'(x_i)$	i	x_i	$f(x_i)$	$f'(x_i)$
0	1	3.0	1.0	0	17	4.5	3.0	0	27.7	4.1	0.33
1	2	3.7		1	20	7.0		1	28	4.3	
2	5	3.9		2	23	6.1		2	29	4.1	
3	6	4.2		3	24	5.6		3	30	3.0	−1.5
4	7	5.7		4	25	5.8					
5	8	6.6		5	27	5.2					
6	10	7.1		6	27.7	4.1	−4.0				
7	13	6.7									
8	17	4.5	−0.67								

33. Repeat Exercise 32, constructing three natural splines using Algorithm 3.4.