Fast Coordination of Distributed Energy Resources Over Time-Varying Communication Networks

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***Abstract*—In this article, we consider the problem of optimally coordinating the response of a group of distributed energy resources (DERs) controlled by distributed agents so they collectively meet the electric power demanded by a collection of loads while minimizing the total generation cost and respecting the DER capacity limits. This problem can be cast as a convex optimization problem, where the global objective is to minimize a sum of convex functions corresponding to individual DER generation cost while satisfying 1) linear inequality constraints corresponding to the DER capacity limits and 2) a linear equality constraint corresponding to the total power generated by the DERs being equal to the total power demand. We develop distributed algorithms to solve the DER coordination problem over time-varying communication networks with either bidirectional or unidirectional communication links. The proposed algorithms can be seen as distributed versions of a centralized primal–dual algorithm. One of the algorithms proposed for directed communication graphs has a geometric convergence rate even when communication out-degrees are unknown to agents. We showcase the proposed algorithms using the standard IEEE 39-bus test system and compare their performance against other ones proposed in the literature.**

***Index Terms*—Consensus control, directed graphs, distributed algorithms, distributed power generation.**

I. Introduction

IT IS envisioned that present-day power grids, which are dependent on centralized power generation stations, will transition toward more decentralized power generation mostly based on distributed energy resources (DERs). One of the obstacles in making this shift happen is to find effective control strategies for coordinating DERs. In this regard, and partly due to high variability introduced by renewable-based generation resources, DERs will need to more frequently adjust their setpoints, which entails the development of fast control strategies. Also, because of the communication overhead, it may not be feasible to use a centralized approach to coordinate a large number of DERs over a large geographic area. This necessitates DER coordination using distributed control strategies that scale well to power networks of large size.

In this work, we consider a group of DERs and electrical loads, which are interconnected by an electric power network, and can exchange information among themselves via some communication network. Each DER is endowed with a power generation cost function, which is unknown to other DERs, and its power output is upper- and lower-limited by some (possibly time-varying) capacity constraints. A computing device, referred to as an agent, attached to each DER is able to communicate with the computing devices (agents) of other DERs located within its communication range. Then, the objective is to determine, in a distributed manner, DER optimal power outputs so as to satisfy total electric power demand while minimizing the total generation cost and respecting DER capacity limits. This DER coordination problem can be cast as a convex optimization problem (see, e.g., [1], [2], [3], [4], [5], [6], [7]), where the global objective is to minimize a sum of convex functions corresponding to the costs of generating power from the DERs while satisfying linear inequality constraints on the power produced by each DER, and a linear equality constraint corresponding to the total generated power being equal to the total power consumed by the electrical loads.

We consider two types of communication networks described by either an undirected or directed graph based on various modes of communication between communication devices: simplex mode, half-duplex mode, and full-duplex mode (see, e.g., [8, Ch. 11]). In simplex mode, transmission occurs only in one direction. In half-duplex mode, transmission occurs in both directions, but only in one direction at a time. In full-duplex mode, the end communication devices transmit and receive information simultaneously. In our work, undirected graphs are used to represent networks where communication devices operate in full-duplex mode allowing us to use a significantly simpler distributed algorithm than the one proposed for directed graphs. Directed graphs are used to represent a more general class of networks where communication devices operate in any of the

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three aforementioned modes but require a more complicated algorithm to coordinate DERs.

As packets of data transmitted by communication devices are not guaranteed to reach their destination, communication links of the corresponding communication graph will only be present with some probability making the communication graph time-varying. Typical causes of packet losses are a link congestion due to limited bandwidth, interference from other signals, hardware faults, and, in the case of wireless communication networks, weak wireless signal, and a receiver being able to receive and process only one signal at a time [8]. Not handling unreliable communication in a proper way will prevent us from finding an optimal solution to the DER coordination problem. This serves as a motivation for designing algorithms that solve the DER coordination problem by properly handling unreliable communication.

To this end, we focus on the challenges that arise due to the time-varying nature of the underlying communication network and address the DER coordination problem via distributed algorithms that are capable of operating over time-varying communication graphs with either 1) bidirectional or 2) unidirectional communication links. These algorithms also have a geometric convergence rate, which is a desirable feature for ensuring fast performance. We believe that the proposed algorithms can be extended to solve more complex DER coordination problems with additional constraints, e.g., line flow constraints, voltage constraints, or reactive power balance constraints, as long as these are linear or have a separable structure, i.e., each constraint is local or involves only a pair of neighboring agents.

A vast body of work has focused on solving the DER coordination problem in a distributed way (see, e.g., [1], [2], [3], [4], [5], [6], [7], [9], [10], [11], [12]). Earlier works focused on time-invariant communication networks (see, e.g., [1], [2], [9], [10]). In one of the earliest works, Zhang and Chow [1] proposed a distributed approach in which agents' local estimates are driven to the optimal incremental cost via the leader–follower consensus algorithm. Domínguez-García et al. [2] utilize the so-called *ratio-consensus algorithm* (see, e.g., [13], [14]) to distributively compute the solution to the dual formulation of the DER coordination problem. Later works focused on time-varying communication networks (see, e.g., [3], [4], [5], [6], [7]). For example, in [3], assuming that generation cost functions are quadratic, Kar and Hug propose a distributed algorithm that uses a consensus term to converge to a common incremental cost, and a subgradient term to satisfy the total load demand. In [7], Wu et al. propose a robustified version of the so-called subgradient-push method (see, e.g., [15]) that utilizes the so-called push-sum protocol (see, e.g., [16], [17], [18]) to converge to a consensual solution over time-varying directed communication networks. However, the convergence of the algorithms proposed in [3] and [7] is not guaranteed to be geometrically fast and might be slow due to the fact that the algorithms use a diminishing stepsize. In [11] and [12], the authors propose distributed algorithms based on the dual-ascent method that have a geometric convergence rate but require the agents to know their communication out-degrees.

Our starting point in the design of our algorithms is a primal–dual algorithm (first-order Lagrangian method), where the dual variable associated with the power balance constraint depends on the total power imbalance (supply–demand mismatch). We then develop distributed versions of this primal–dual algorithm by having DER agents closely emulate the iterations of the primal–dual algorithm. To this end, each agent maintains an estimate of the dual variable and updates it using a local estimate of the total power imbalance and the neighbors' estimates of the total power imbalance. The update of the total power imbalance estimate is based on the gradient tracking idea that appeared in [19]. To enable agents to operate over time-varying directed communication graphs when their communication out-degrees are unknown to them, we propose a robust distributed primal–dual algorithm that converges geometrically fast.

Each proposed algorithm is viewed as a feedback interconnection of the (centralized) primal–dual algorithm representing the nominal system and the error dynamics due to the nature of the distributed implementation. The key ingredient for establishing the convergence results is to show that both systems are finite-gain stable, which then allows us to use the small-gain theorem (see, e.g., [20]) to establish the convergence of the feedback interconnected system. The small-gain-theorem-based analysis first appeared in [19] in the context of distributed algorithms for solving an unconstrained consensus optimization problem.

Contributions: We note that the distributed algorithms in [19], which were proposed to solve an unconstrained distributed optimization problem, cannot be easily applied to solve the DER coordination problem because of the additional inequality and equality constraints that appear in our problem formulation. The presence of these inequality and equality constraints poses major challenges in establishing the convergence of the distributed algorithms we propose and in quantifying an upper bound for stepsizes where this convergence occurs. Showing the finite-gain stability of the two systems, the nominal system and the error dynamics, which comprise the distributed algorithm, requires a nontrivial analysis. Although the distributed algorithms in [11] and [12] can be used to solve the DER coordination problem over directed graphs, they require each agent to know its communication out-degree, i.e., the number of neighbors who received its information, at every iteration. Obtaining such information at every iteration is a very challenging task when communication links are directed and not reliable. To overcome these issues, we propose a distributed algorithm, which attains a geometric convergence rate and where each agent is only required to know its nominal out-degree, i.e., the number of neighbors who can receive its information, which can be learned or precomputed before executing the proposed algorithm.

Additionally, we propose a different algorithm for undirected graphs used to represent the communication networks where communication devices operate in the so-called full-duplex mode (see, e.g., [8, Ch. 11]) that allows communication devices to transmit and receive information simultaneously. This distributed algorithm is significantly simpler than the one proposed for directed graphs and does not require any additional information about the nominal out-degrees. We also show that other variants of the primal–dual algorithm (see, e.g., [21], [22]) can be leveraged as a base for designing the distributed counterparts as long as these variants under additive disturbance are finite-gain stable maps from the disturbance to their output, and the outputs of the individual systems in the feedback system representation enter as additive disturbances to each other.

II. Preliminaries

In this section, we formulate the DER coordination problem and give an overview of the small-gain theorem for discrete-time systems.

A. DER Coordination Problem

We consider a collection of DERs and electrical loads interconnected by a power network. Let p_i denote the power output of the DER at bus i, $1 \leq i \leq n$. Let ℓ_i denote the amount of power to be consumed by the load at bus i, $1 \leq i \leq n$, which is locally measured and known to the DER at the same bus, and assumed to be fixed while the DER coordination problem is solved. Let $\underline{p}_i$ and $\overline{p}_i$ denote the lower and upper limits, respectively, on the power that the DER at bus i can generate. Also, let $f_i(\cdot)$ denote the cost function associated with the electric power generated by the DER at bus i. Then, our main objective is to determine the power that needs to be generated by the DERs in order to collectively satisfy the total electric power demand, $\sum_{i=1}^n \ell_i$, while minimizing the total generation cost, $\sum_{i=1}^n f_i(p_i)$.

More formally, we consider the following DER coordination problem that has been studied in [1], [2], [3], [4], [6], and [7]:

$$\underset{p\in\mathbb{R}^n}{\text{minimize}} \sum_{i=1}^{n} f_i(p_i) \tag{1a}$$

$$\text{subject to } \mathbf{1}^\top p = \mathbf{1}^\top \ell \tag{1b}$$

$$\underline{p} \leq p \leq \overline{p} \tag{1c}$$

where $p = [p_1, \ldots, p_n]^\top$, $\ell = [\ell_1, \ldots, \ell_n]^\top$, $\underline{p} = [\underline{p}_1, \ldots, \underline{p}_n]^\top$, $\overline{p} = [\overline{p}_1, \ldots, \overline{p}_n]^\top$, and $\mathbf{1}$ is the all-ones vector (its size should be clear from the context). We assume that $\mathbf{1}^\top \underline{p} \leq \mathbf{1}^\top \ell \leq \mathbf{1}^\top \overline{p}$, which makes (1) feasible. A widely used function for representing the generation cost of electric power generator units is a quadratic function (see, e.g., [23, Ch. 11], [24, Ch. 5]), which belongs to the class of twice differentiable and strongly convex functions as stated in the following assumption.

Assumption 1: Each cost function $f_i(\cdot)$ is twice differentiable and strongly convex with parameter $m > 0$, i.e., $f_i''(x) \geq m$, $\forall x \in [\underline{p}_i, \overline{p}_i]$, $\forall i \in \mathcal{V}$.

Let $\mathcal{A}(x)$ denote the set of active inequality constraints for a feasible point $x \in \mathbb{R}^n$ given by

$$\mathcal{A}(x) := \{j \mid x_j = \overline{p}_j \text{ or } x_j = \underline{p}_j\}.$$

Let e_j denote an n-dimensional vector whose jth entry is equal to 1 and all other entries are equal to zero. We make the following standard regularity assumption on the solution of (1) (see, e.g., [25, Ch. 3.3]).

Assumption 2: The optimal solution of (1), denoted by $p^\star$, is regular, namely, the equality constraint gradient, given by $\mathbf{1}$, and the active inequality constraint gradients, given by e_j, $j \in \mathcal{A}(p^\star)$, are linearly independent.

The main objective of our work in this article is to design a distributed algorithm for solving (1) geometrically fast over time-varying communication networks.

B. Small-Gain Theorem

In the following, we give a brief overview of the main analysis tool used in later developments—the small-gain theorem (see, e.g., [20, Th. 5.6]) for discrete-time systems. For the forthcoming developments, we adopt the appropriate metric for measuring the energy content of signals of interest. For a given sequence of iterates, $\{x[k]\}_{k=0}^\infty$, where $x[k] \in \mathbb{R}^n$, consider the following norm (previously used in [19]):

$$\|x\|_2^{a,K} := \max_{0\leq k\leq K} a^{-k}\|x[k]\|_2$$

for some $a \in (0,1)$, where $\|\cdot\|_2$ is the Euclidean norm. If $\|x\|_2^{a,K}$ is bounded for all $K \geq 0$, then, $a^{-k}\|x[k]\|_2$ is bounded for all $k \geq 0$, and, thus, it follows that $x[k]$ converges to zero at a geometric rate $\mathcal{O}(a^k)$.

Now, consider a feedback connection of two discrete-time systems, $\mathcal{H}_1$ and $\mathcal{H}_2$, such that

$$e_2[k+1] = \mathcal{H}_1(e_1[k])$$
$$e_1[k+1] = \mathcal{H}_2(e_2[k]).$$

We assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite-gain stable in the sense of the norm $\|\cdot\|_2^{a,K}$, namely, the following relations hold:

$$\|e_2\|_2^{a,K} \leq \gamma_1\|e_1\|_2^{a,K} + \beta_1 \tag{2a}$$

$$\|e_1\|_2^{a,K} \leq \gamma_2\|e_2\|_2^{a,K} + \beta_2 \tag{2b}$$

for some nonnegative constants β_1, β_2, γ_1, and γ_2. From (1), we have that

$$\|e_2\|_2^{a,K} \leq \gamma_1\|e_1\|_2^{a,K} + \beta_1 \leq \gamma_1\gamma_2\|e_2\|_2^{a,K} + \gamma_1\beta_2 + \beta_1$$

which by rearranging yields $\|e_2\|_2^{a,K} \leq \frac{\gamma_1\beta_2+\beta_1}{1-\gamma_1\gamma_2}$. Similarly, $\|e_1\|_2^{a,K} \leq \frac{\gamma_2\beta_1+\beta_2}{1-\gamma_1\gamma_2}$. Then, if $\gamma_1\gamma_2 < 1$, $\|e_1\|_2^{a,K}$ and $\|e_2\|_2^{a,K}$ are bounded, and $e_1[k]$ and $e_2[k]$ converge to zero at a geometric rate $\mathcal{O}(a^k)$.

III. DER Coordination Over Time-Varying Undirected Graphs

In this section, we present a distributed algorithm for solving the DER coordination problem (1) over time-varying undirected communication graphs.

A. Communication Network Model

Here, we introduce the model describing the communication network that enables the bidirectional exchange of information between DER agents. Let $\mathcal{G}^{(0)} = (\mathcal{V}, \mathcal{E}^{(0)})$ denote an undirected graph, where each agent in the vertex set $\mathcal{V} := \{1, 2, \ldots, n\}$ corresponds to a DER, and $\{i, j\} \in \mathcal{E}^{(0)}$ if there is a communication link between DER agents i and j that allows them to exchange information. It is assumed that $\mathcal{G}^{(0)}$ does not contain self-loops. During any bounded time interval (t_k, t_{k+1}), successful data transmissions among the DER agents can be captured by the undirected graph $\mathcal{G}^{(c)}[k] = (\mathcal{V}, \mathcal{E}^{(c)}[k])$, where $\mathcal{E}^{(c)}[k] \subseteq \mathcal{E}^{(0)}$ is the set of active communication links, with $\{i, j\} \in \mathcal{E}^{(c)}[k]$ if agents i and j simultaneously (and successfully) exchange information with each other during time interval (t_k, t_{k+1}). Let $\mathcal{N}_i := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}^{(0)}\}$ denote the set of nominal neighbors of agent i, and define its nominal degree, which includes itself, as $d_i^+ := |\mathcal{N}_i^+| + 1$. Let $\mathcal{N}_i[k]$ denote the set of neighbors of agent i during time interval (t_k, t_{k+1}), i.e., $\mathcal{N}_i[k] := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}^{(c)}[k]\}$. We make the following standard assumption regarding the connectivity of the network (see, e.g., [15], [19]).

Assumption 3: There exists some positive integer B such that the graph with vertex set $\mathcal{V}$ and edge set $\bigcup_{l=kB}^{(k+1)B-1} \mathcal{E}^{(c)}[l]$ is connected for $k = 0, 1, \ldots$.

To satisfy Assumption 3, active communication links are not required to form a connected graph during every time interval (t_k, t_{k+1}), $k = 1, 2, \ldots$, but only during some longer time intervals of bounded length. Note that Assumption 3 necessarily implies that the nominal graph $G^{(0)}$ is connected.

B. Distributed Primal–Dual Algorithm

Our starting point to solve (1) is the following primal–dual algorithm [25, Ch. 4.4] with the additional projection:

$$p_i[k+1] = \left[p_i[k] - s f_i'(p_i[k]) + s\xi\overline{\lambda}[k]\right]_{\underline{p}_i}^{\overline{p}_i} \tag{3a}$$

$$\overline{\lambda}[k+1] = \overline{\lambda}[k] - s\mathbf{1}^\top(p[k] - \ell) \tag{3b}$$

where $p[k] = [p_1[k], \ldots, p_n[k]]^\top$, $[\cdot]_{\underline{p}_i}^{\overline{p}_i}$ denotes the projection onto the interval $[\underline{p}_i, \overline{p}_i]$, i.e.,

$$[x]_{\underline{p}_i}^{\overline{p}_i} = \begin{cases} \underline{p}_i & \text{if } x < \underline{p}_i \\ x & \text{if } x \in [\underline{p}_i, \overline{p}_i] \\ \overline{p}_i & \text{if } x > \overline{p}_i \end{cases}$$

$s > 0$ is a constant stepsize, $\xi > 0$ is a constant parameter, and $\overline{\lambda}[k]$ is the estimate of the Lagrange multiplier at time k associated with the power balance constraint, $\mathbf{1}^\top p = \mathbf{1}^\top \ell$. Algorithm (3) does not conform to the general communication model described in Section III-A because in order to execute it, the total power imbalance, $\mathbf{1}^\top(p[k] - \ell)$, at time k is needed to update $\overline{\lambda}[k]$.

To design a distributed version of (3), each agent i needs to have a local estimate of $\overline{\lambda}[k]$, denoted by $\lambda_i[k]$. To update $\lambda_i[k]$, it should also have an estimate of $\mathbf{1}^\top(p[k] - \ell)$. One such estimate that can be constructed purely based on the local power imbalance is $\hat{n}(p_i[k] - \ell_i)$, where $\hat{n}$ is some estimate of n that every agent has,[1] e.g., $\hat{n}$ can be one, which leads us to the following distributed algorithm:

$$p_i[k+1] = [p_i[k] - s f_i'(p_i[k]) + s\xi\lambda_i[k]]_{\underline{p}_i}^{\overline{p}_i} \tag{4a}$$

$$\lambda_i[k+1] = \left(1 - \sum_j w_{ij}[k]\right)\lambda_i[k] + \sum_j w_{ij}[k]\lambda_j[k] - s\hat{n}(p_i[k] - \ell_i) \tag{4b}$$

where $w_{ij}[k] = w_{ji}[k] \geq \eta$ if $\{i, j\} \in \mathcal{E}^{(c)}[k]$, $w_{ij}[k] = 0$ if $\{i, j\} \notin \mathcal{E}^{(c)}[k]$, and the constant $\eta > 0$ is chosen so that $1 - \sum_j w_{ij}[k] \geq \eta$. Even if $\hat{n}$ is an accurate estimate of n, $\hat{n}(p_i[k] - \ell_i)$ is a very crude estimate of $\mathbf{1}^\top(p[k] - \ell)$, and results in poor performance as will be demonstrated later via numerical simulations.

A better approach is to let each agent estimate the total power imbalance by using its local power imbalance and the estimates of its neighbors. To elaborate on this further, we let y_i denote agent i's estimate of the total power imbalance. Then, one way to update y_i is as follows:

$$y_i[k+1] = \left(1 - \sum_j w_{ij}[k]\right) y_i[k] + \sum_j w_{ij}[k] y_j[k] + \hat{n}(p_i[k+1] - p_i[k]) \tag{5}$$

where $y_i[0] = \hat{n}(p_i[0] - \ell_i)$. In (5), agent i first computes the average of its estimate and the estimates of its neighbors, and then adds $\hat{n}(p_i[k+1] - p_i[k])$ to ensure that the average of all total power imbalance estimates is always equal to $(\hat{n}/n)\mathbf{1}^\top(p[k] - \ell)$, which is equal to the total power imbalance, $\mathbf{1}^\top(p[k] - \ell)$ if $\hat{n} = n$. This second step allows local estimates to remain close to the total power imbalance. In the following, we provide the complete update formulae for the primal and dual variables:

$$p_i[k+1] = [p_i[k] - s f_i'(p_i[k]) + s\xi\lambda_i[k]]_{\underline{p}_i}^{\overline{p}_i} \tag{6a}$$

$$\lambda_i[k+1] = \left(1 - \sum_j w_{ij}[k]\right)\lambda_i[k] + \sum_j w_{ij}[k]\lambda_j[k] - s y_i[k] \tag{6b}$$

$$y_i[k+1] = \left(1 - \sum_j w_{ij}[k]\right) y_i[k] + \sum_j w_{ij}[k] y_j[k] + \hat{n}(p_i[k+1] - p_i[k]). \tag{6c}$$

Note that in (6b), agent i computes the (weighted) average of its estimate and the estimates of its neighbors, which yields a good estimate of $\overline{\lambda}$. The iterations in (6) are initialized with $\lambda_i[0] = 0$, and $y_i[0] = \hat{n}(p_i[0] - \ell_i)$.

C. Feedback Interconnection Representation of the Distributed Primal–Dual Algorithm

In the following, we represent (6) as a feedback interconnection of a nominal system, denoted by $\mathcal{H}_1$, and a disturbance system, denoted by $\mathcal{H}_2$, which allows us to utilize the small-gain theorem for convergence analysis purposes. To this end, let $e[k] := \lambda[k] - (\frac{1}{n}\mathbf{1}^\top\lambda[k])\mathbf{1}$, $\lambda[k] = [\lambda_1[k], \ldots, \lambda_n[k]]^\top$, and $\hat{\lambda}[k] := (1/\hat{n})\mathbf{1}^\top\lambda[k]$; then, we define the nominal system, $\mathcal{H}_1$, as follows:

$$\mathcal{H}_1 : \begin{cases} p[k+1] = \left[p[k] - s\nabla f(p[k]) + s\xi\frac{\hat{n}}{n}\mathbf{1}\hat{\lambda}[k] + s\xi e[k]\right]_{\underline{p}}^{\overline{p}} & \text{(7a)} \\ \hat{\lambda}[k+1] = \hat{\lambda}[k] - s\mathbf{1}^\top(p[k] - \ell) & \text{(7b)} \end{cases}$$

where $\nabla f(p[k]) = [f_1'(p_1[k]), f_2'(p_2[k]), \ldots, f_n'(p_n[k])]^\top$, and $[\cdot]_{\underline{p}}^{\overline{p}}$ denotes the componentwise projection onto the box $[\underline{p}, \overline{p}]$, i.e., $[x]_{\underline{p}}^{\overline{p}} = [[x_1]_{\underline{p}_1}^{\overline{p}_1}, [x_2]_{\underline{p}_2}^{\overline{p}_2}, \ldots, [x_n]_{\underline{p}_n}^{\overline{p}_n}]^\top$. Note that in order to obtain (7a), we substituted $e[k] + (\hat{n}/n)\mathbf{1}\hat{\lambda}[k]$ for $\lambda[k]$ in (6a), and summed (6b) over all i and divided the result by $\hat{n}$ to obtain (7b). We note that $e[k]$ is the vector of deviations of the local estimates of the Lagrange multiplier from their average at time instant k; without $e[k]$, the nominal system $\mathcal{H}_1$ has almost the same form as (3). Now, we define the disturbance system,

[1]We note that $\hat{n}$ must be precomputed before running the distributed algorithms we propose. The problem of estimating distributively the number of nodes in various types of networks has been extensively discussed and tackled in a number of works (see, e.g., [26], [27], [28] and references therein) under various assumptions on the underlying communication graph.

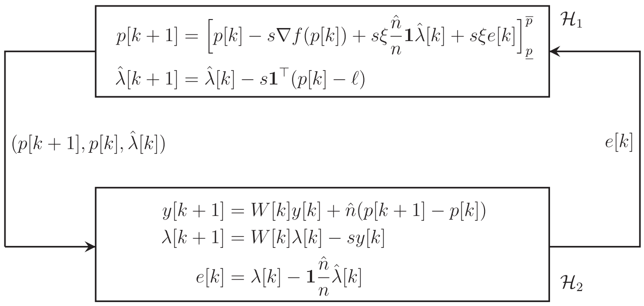

Fig. 1. Algorithm (2) as a feedback system.

$\mathcal{H}_2$, as follows:

$$\mathcal{H}_2 : \begin{cases} y[k+1] = W[k]y[k] + \hat{n}(p[k+1] - p[k]) & \text{(8a)} \\ \lambda[k+1] = W[k]\lambda[k] - sy[k] & \text{(8b)} \\ e[k] = \lambda[k] - (\hat{n}/n)\mathbf{1}\hat{\lambda}[k] & \text{(8c)} \end{cases}$$

where $W[k] = [w_{ij}[k]] \in \mathbb{R}^{n\times n}$ is a weight matrix at time instant k. Then, as illustrated in Fig. 1, algorithm (6) can be viewed as a feedback interconnection of $\mathcal{H}_1$ and $\mathcal{H}_2$, where (p^*, λ^*) is the equilibrium of (7) when $e[k] \equiv 0$, for all $k \geq 0$. Finding the relationship between the loop gain of the feedback system and the stepsize s allows us to quantify the effect of the feedback system on the convergence error in terms of the stepsize. We later show that the loop gain can be decreased by decreasing s. As a matter of fact, if the loop gain is sufficiently small, then, the feedback loop does not amplify the energy of the convergence error, and, on the contrary, the error eventually decays to zero, which follows from the small-gain theorem.

Remark 1: We note that the feedback system representation in Fig. 1 allows one to decouple the convergence analysis of the distributed algorithm into two separate parts. In the first part, we analyze the convergence properties of the basic primal–dual algorithm (1) under disturbance, whereas in the second part, we analyze the convergence properties of the underlying consensus algorithm under disturbance. Then, the small-gain theorem merges these two parts to show the convergence properties of the entire system.

D. Convergence Analysis

In order to invoke the small-gain theorem, we must first show that the following relations between the energy of the convergence error and that of the disturbance hold:

R1. $\|z\|_2^{a,K} \leq \alpha_1\|e\|_2^{a,K} + \beta_1$ for some positive α_1 and β_1

R2. $\|e\|_2^{a,K} \leq s\alpha_2\|z\|_2^{a,K} + \beta_2$ for some positive α_2 and β_2

for some $a \in (0, 1)$, small enough $s > 0$, and $\forall\xi > 0$, where

$$z[k] := \begin{bmatrix} p[k] - p^* \\ \hat{\lambda}[k] - \lambda^* \end{bmatrix}$$

denotes the convergence error. The results R1 and R2 are equivalent to ensuring that the systems $\mathcal{H}_1$ and $\mathcal{H}_2$ in Fig. 1 are finite-gain stable. From R1 and R2, it can be determined that the loop gain is $s\alpha_1\alpha_2$. Noticing that the gain $s\alpha_1\alpha_2$ becomes strictly smaller than 1 for sufficiently small s, we later show that $\|z\|_2^{a,K}$ becomes bounded for all $K > 0$, and that $z[k]$ converges to zero at a geometric rate $\mathcal{O}(a^k)$. In the following, we show that the relations R1 and R2 hold and present the convergence results for algorithm (6).

We first establish that $p^* = p^\star$, namely, p^* is the solution of (1); its proof can be found in the Appendix.

Lemma 1: Consider (p^*, λ^*), namely, the equilibrium of the nominal system $\mathcal{H}_1$ with $e[k] \equiv 0$, $\forall k$. Then, p^* is the solution of (1).

To provide an upper bound on the stepsize s, we introduce some notation. It follows from the proof of [19, Lemma 3.10] that $\|e[k]\|_2$ is always bounded, i.e., $\|e[k]\|_2 \leq M$, for some $M > 0$. Define $\psi := \max(\mathbf{1}^\top(\ell - \underline{p}), \mathbf{1}^\top(\overline{p} - \ell))/\sqrt{\xi\hat{n}}$

$$\begin{aligned} \Delta\varphi_i^{\max} := \max\Bigg(& \max_{j\in\mathcal{V}}\Big(\overline{p}_j - \underline{p}_j + f_j'(\overline{p}_j)/\sqrt{\xi\hat{n}}\Big) \\ & + 2\sqrt{\xi\hat{n}}/n\psi - f_i'(\underline{p}_i)/\sqrt{\xi\hat{n}} + 2M\sqrt{\xi/\hat{n}} \\ & f_i'(\overline{p}_i)/\sqrt{\xi\hat{n}} + 2M\sqrt{\xi/\hat{n}} + 2\sqrt{\xi\hat{n}}/n\psi \\ & - \min_{j\in\mathcal{V}}\Big(\underline{p}_j - \overline{p}_j + f_j'(\underline{p}_j)/\sqrt{\xi\hat{n}}\Big)\Bigg) \end{aligned} \tag{9}$$

$$\zeta_i := \min(\overline{p}_i - p_i^*, p_i^* - \underline{p}_i)/\Delta\varphi_i^{\max} \tag{10}$$

and let constants a_1, a_2, and δ be such that $a_1 > 0$, and

$$\delta := (\xi\hat{n}/n)^2\sum_{i=1}^{n}\frac{a_1 s^2}{1 + s^2 + 2\zeta_i} \tag{11}$$

$$a_2 := (\xi\hat{n}/n)a_1 + (\xi\hat{n}/n)^2 s^2 a_1 n + (\xi\hat{n}/n)a_1 s^2 - \delta. \tag{12}$$

Note that since p^* is regular, there exists l such that $\underline{p}_l < p_l^* < \overline{p}_l$, yielding $\zeta_l > 0$. Let $E := \mathbf{1}\mathbf{1}^\top$, $\underline{D} \in \mathbb{R}^{n\times n}$ and $\overline{D} \in \mathbb{R}^{n\times n}$ denote diagonal matrices with $\underline{D}_{ii} = \min_{\underline{p}_i \leq p_i \leq \overline{p}_i} f_i''(p_i)$ and $\overline{D}_{ii} = \max_{\underline{p}_i \leq p_i \leq \overline{p}_i} f_i''(p_i)$, respectively. Define

$$F(s) := \begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} \tag{13}$$

where

$$\begin{aligned} F_{11}(s) = {} & 2sa_1\underline{D} - s^2 a_1(I + (\overline{D})^2 + 2\xi\hat{n}/nE) \\ & - s^2 a_2 E + s^3 a_1(-2\xi\hat{n}I + 2\underline{D} - I) \\ & - s^2\delta I - s^4 a_1(\overline{D})^2 \end{aligned}$$

$$F_{12}(s) = \xi\hat{n}/ns^4 a_1(\overline{D}\mathbf{1} + \mathbf{1}),\ F_{21}(s) = F_{12}(s)^\top$$

$$F_{22}(s) = (\xi\hat{n}/n)^2 a_1 s^2\left(\sum_{i=1}^{n}\frac{s^2 + 2\zeta_i}{1 + s^2 + 2\zeta_i} - s^2 n\right) - a_2 s^3.$$

Since $\frac{s^2+2\zeta_i}{1+s^2+2\zeta_i} \geq \frac{2\zeta_i}{1+2\zeta_i}$, we have that, for sufficiently small $s > 0$

$$F_{22}(s) > (\xi\hat{n}/n)^2 a_1 s^2\left(\sum_{i=1}^{n}\frac{2\zeta_i}{1 + 2\zeta_i} - s^2 n\right) - a_2 s^3 > 0$$

where the last inequality follows since the quadratic term dominates the higher-order terms for sufficiently small $s > 0$. Let $S^{(F)}(s)$ denote the Schur complement of $F_{22}(s)$ in $F(s)$, namely, $S^{(F)}(s) := F_{11}(s) - F_{12}(s)F_{22}(s)^{-1}F_{21}(s)$. It is easy to see that, for sufficiently small $s > 0$, $S^{(F)}(s)$ is positive definite, since the linear term dominates the higher-order terms.

Define

$$\overline{s} := \min\{s > 0 : S^{(F)}(s) \text{ is not positive definite or } F_{22}(s) = 0\}. \tag{14}$$

Then, by [29, Th. 7.7.6], since $S^{(F)}(s)$ is positive definite and $F_{22}(s) > 0$, $\forall s \in (0, \overline{s})$, $F(s)$ is positive definite, $\forall s \in (0, \overline{s})$. In the next result, we establish that $\mathcal{H}_1$ is finite-gain stable.

Proposition 1: Let Assumptions 1 and 2 hold. Then, under (7), we have that

$$\text{R1. } \|z\|_2^{a,K} \le \alpha_1 \|e\|_2^{a,K} + \beta_1 \tag{15}$$

for some positive α_1 and β_1, $a \in (0,1)$, $\forall s \in (0, \min(\overline{s}, 1/\sqrt{\xi \hat{n}}))$, and $\forall \xi > 0$.

Proof: Define

$$P := \begin{bmatrix} a_1 I & -s a_1 \xi \frac{\hat{n}}{n} \mathbf{1} \\ -s a_1 \xi \frac{\hat{n}}{n} \mathbf{1}^\top & a_2 \end{bmatrix}.$$

Then, the Schur complement of a_2 in P, denoted by S, is given by

$$S := a_1 I - (1/a_2)(s a_1 \xi \hat{n}/n)^2 \mathbf{1}\mathbf{1}^\top.$$

From (11), we have that $\delta < (\xi \hat{n}/n)^2 \, s^2 a_1 n$; then, it follows from (12) that

$$a_2 > (\xi \hat{n}/n) a_1. \tag{16}$$

By using (16), we have that[2]

$$S/a_1 > I - s^2 \xi \hat{n}/n \mathbf{1}\mathbf{1}^\top.$$

Since $s < 1/\sqrt{\xi \hat{n}}$, it follows that S is positive definite. Then, by [29, Th. 7.7.6], since S is positive definite and $a_2 > 0$, P is positive definite. Define a vector norm $\|x\|_P := \sqrt{x^\top P x}$, and

$$\varphi[k] := p[k] - s\nabla f(p[k]) + s\xi \frac{\hat{n}}{n} \mathbf{1} \hat{\lambda}[k] \tag{17a}$$

$$\varphi^* := p^* - s\nabla f(p^*) + s\xi \frac{\hat{n}}{n} \mathbf{1} \lambda^* \tag{17b}$$

$$\epsilon[k] := s\xi e[k] \tag{17c}$$

$$\Delta\varphi[k] := \varphi[k] + \epsilon[k] - [\varphi[k] + \epsilon[k]]_{\underline{p}}^{\overline{p}} \tag{17d}$$

$$\Delta\varphi^* := \varphi^* - [\varphi^*]_{\underline{p}}^{\overline{p}}. \tag{17e}$$

The result of the following lemma is needed for the subsequent development; its proof can be found in the Appendix.

Lemma 2:

$$\left\| \begin{bmatrix} [\varphi[k] + \epsilon[k]]_{\underline{p}}^{\overline{p}} - [\varphi^*]_{\underline{p}}^{\overline{p}} \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} \right\|_P \le \left\| \begin{bmatrix} \varphi[k] + \epsilon[k] - \varphi^* \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} \right\|_{P+T}$$

where

$$T := \begin{bmatrix} a_1 \, s^2 I & \mathbf{0}_n \\ \mathbf{0}_n^\top & \delta \end{bmatrix}.$$

■

[2]We write $A > B$ ($A \ge B$) to denote that a matrix $A - B$ is positive definite (positive semidefinite).

Next, it follows from the mean value theorem [30, Th. 5.1] applied to each component in $\nabla f(p[k]) - \nabla f(p^*)$ that

$$\nabla f(p[k]) - \nabla f(p^*) = \nabla^2 f(\upsilon[k])(p[k] - p^*) \tag{18}$$

where $\upsilon[k] := [\upsilon_1[k], \upsilon_2[k], \ldots, \upsilon_n[k]]^\top$, with $\upsilon_i[k]$ lying on the line segment connecting $p_i[k]$ and p_i^*, and $\nabla^2 f(\upsilon[k])$ is the Hessian of $f(\mathrm{x})$ at $\mathrm{x} = \upsilon[k]$. Define

$$A[k] := \begin{bmatrix} I - s\nabla^2 f(\upsilon[k]) & s\xi \frac{\hat{n}}{n} \mathbf{1} \\ -s\mathbf{1}^\top & 1 \end{bmatrix}$$

then, note that

$$\begin{bmatrix} \varphi[k] - \varphi^* \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} = A[k] z[k]. \tag{19}$$

It follows from the triangle inequality that

$$\begin{aligned} &\left\| \begin{bmatrix} \varphi[k] + \epsilon[k] - \varphi^* \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} \right\|_{P+T} \\ &\le \left\| \begin{bmatrix} \varphi[k] - \varphi^* \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} \right\|_{P+T} + \sqrt{a_1(1+s^2)} \|\epsilon[k]\|_2. \end{aligned} \tag{20}$$

By noting that

$$z[k+1] = \begin{bmatrix} [\varphi[k] + \epsilon[k]]_{\underline{p}}^{\overline{p}} - [\varphi^*]_{\underline{p}}^{\overline{p}} \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix}$$

and putting together the results in Lemma 2, (19), and (20), we obtain that

$$\begin{aligned} \|z[k+1]\|_P \le &\sqrt{z[k]^\top A[k]^\top (P+T) A[k] z[k]} \\ &+ \sqrt{a_1(1+s^2)} \|\epsilon[k]\|_2. \end{aligned} \tag{21}$$

Define $D_k := \nabla^2 f(\upsilon[k])$, $Q[k] := P - A[k]^\top (P+T) A[k]$; then, we show that $Q[k] \ge s^3 P$. Define $R[k] := Q[k] - s^3 P$ given by

$$R[k] = \begin{bmatrix} R_{11}[k] & R_{12}[k] \\ R_{21}[k] & R_{22}[k] \end{bmatrix}$$

where

$$\begin{aligned} R_{11}[k] = \, & 2 s a_1 D_k - s^2 a_1 (I + D_k^2 + 2\xi \hat{n}/n E) - s^2 a_2 E \\ & + s^3 a_1 (\xi \hat{n}/n E D_k + \xi \hat{n}/n D_k E + 2 D_k - I) \\ & - s^2 \delta I - s^4 a_1 D_k^2 \\ R_{12}[k] = \, & \xi \hat{n}/n s^4 a_1 (D_k \mathbf{1} + \mathbf{1}), \; R_{21}[k] = R_{12}[k]^\top \\ R_{22}[k] = \, & (\xi \hat{n}/n)^2 a_1 \, s^2 \left(\sum_{i=1}^{n} \frac{s^2 + 2\zeta_i}{1 + s^2 + 2\zeta_i} - s^2 n \right) - a_2 \, s^3. \end{aligned}$$

Let $S^{(R)}[k]$ denote the Schur complement of $R_{22}[k]$ in $R[k]$, namely, $S^{(R)}[k] := R_{11}[k] - R_{12}[k] R_{22}[k]^{-1} R_{21}[k]$. We recall that $S^{(F)}(s) > 0$ and $F_{22}(s) > 0$, $\forall s \in (0, \overline{s})$. Since $S^{(R)}[k] \ge S^{(F)}(s)$ and $R_{22}[k] = F_{22}(s)$, for all k, it follows that $S^{(R)}[k] > 0$ and $R_{22}[k] > 0$. Then, by [29, Th. 7.7.6], $R[k]$ is positive definite for all k. Hence, $Q[k] \ge s^3 P$, yielding the following relation from (21):

$$\|z[k+1]\|_P \le \sqrt{1-s^3}\|z[k]\|_P + \sqrt{a_1(1+s^2)}\|\epsilon[k]\|_2. \tag{22}$$

Recalling that $\epsilon[k] = s\xi e[k]$, and letting $\gamma := \sqrt{1-s^3}$, and $b_1 := \sqrt{a_1(1+s^2)}s\xi$, we rewrite (22) as follows:

$$\|z[k+1]\|_P \le \gamma\|z[k]\|_P + b_1\|e[k]\|_2. \tag{23}$$

Now, by choosing $a > \gamma$, and multiplying both sides of (6) by $a^{-(k+1)}$, we obtain

$$a^{-(k+1)}\|z[k+1]\|_P \le \frac{\gamma}{a}a^{-k}\|z[k]\|_P + \frac{b_1}{a}a^{-k}\|e[k]\|_2. \tag{24}$$

Then, by taking $\max_{0\le k\le K}(\cdot)$ on both sides of (24), we obtain

$$\max_{0\le k\le K} a^{-(k+1)}\|z[k+1]\|_P \le \frac{\gamma}{a}\max_{0\le k\le K} a^{-k}\|z[k]\|_P + \frac{b_1}{a}\max_{0\le k\le K} a^{-k}\|e[k]\|_2. \tag{25}$$

Since

$$\max_{0\le k\le K} a^{-(k+1)}\|z[k+1]\|_P = \max_{0\le k\le K+1} a^{-k}\|z[k]\|_P - \|z[0]\|_P$$

the relation (25) can be written as

$$\|z\|_P^{a,K+1} \le \frac{\gamma}{a}\|z\|_P^{a,K} + \frac{b_1}{a}\|e\|_2^{a,K} + \|z[0]\|_P \tag{26}$$

where $\|x\|_P^{a,K} := \max_{0\le k\le K} a^{-k}\|x[k]\|_P$ for a sequence $\{x[k]\}_{k=0}^{\infty}$. Since $\|z\|_P^{a,K+1} \ge \|z\|_P^{a,K}$, it follows from (26) that

$$\|z\|_P^{a,K} \le \frac{\gamma}{a}\|z\|_P^{a,K} + \frac{b_1}{a}\|e\|_2^{a,K} + \|z[0]\|_P. \tag{27}$$

Then, after rearranging (27), we obtain

$$\|z\|_P^{a,K} \le \frac{b_1}{a-\gamma}\|e\|_2^{a,K} + \frac{a}{a-\gamma}\|z[0]\|_P.$$

Because $\|\cdot\|_2 \le \alpha\|\cdot\|_P$ for some $\alpha > 0$, we have that $\|z\|_P^{a,K} \ge \|z\|_2^{a,K}/\alpha$. Hence

$$\frac{\|z\|_2^{a,K}}{\alpha} \le \frac{b_1}{a-\gamma}\|e\|_2^{a,K} + \frac{a}{a-\gamma}\|z[0]\|_P$$

which can be rewritten as

$$\|z\|_2^{a,K} \le \alpha_1\|e\|_2^{a,K} + \beta_1$$

where $\alpha_1 = \frac{b_1\alpha}{a-\gamma}$, $\beta_1 = \frac{a\alpha}{a-\gamma}\|z[0]\|_P$, yielding (15). □

We omit the proof of the next result, where we show that system $\mathcal{H}_2$ is finite-gain stable, since it is analogous to that of a similar result proposed for directed communication graphs in Section IV.

Proposition 2: Let Assumption 3 hold. Then, under (8), we have that

$$\text{R2. } \|e\|_2^{a,K} \le s\alpha_2\|z\|_2^{a,K} + \beta_2$$

for some positive α_2 and β_2, $a \in (0,1)$.

Now, we show the convergence of algorithm (6) by applying the small-gain theorem to the results in Propositions 1–2.

Proposition 3: Let Assumptions 1, 2, and 3 hold. Then, under algorithm (6)

$$\|z\|_2^{a,K} \le \beta \tag{28}$$

for some $a \in (0,1)$, $\beta > 0$, $\forall s \in (0, \min(\bar{s}, 1/\sqrt{\xi\hat{n}}, 1/(\alpha_1\alpha_2)))$, and $\forall\xi > 0$. In particular, $(p_i[k], \lambda_i[k])$ converges to (p_i^*, λ^*), $\forall i$, at a geometric rate $\mathcal{O}(a^k)$.

Proof: By using Propositions 1 and 2, it follows that

$$\|z\|_2^{a,K} \le \alpha_1\|e\|_2^{a,K} + \beta_1 \le \alpha_1(s\alpha_2\|z\|_2^{a,K} + \beta_2) + \beta_1$$

which, after rearranging, results in

$$\|z\|_2^{a,K} \le \frac{\alpha_1\beta_2 + \beta_1}{1 - s\alpha_1\alpha_2} =: \beta$$

yielding (28). Hence, for $s < 1/(\alpha_1\alpha_2)$, we have that $s\alpha_1\alpha_2 < 1$, which ensures that β is finite. ■

Remark 2: Propositions 1–3 imply that instead of (3), other variants of the primal–dual algorithm (see, e.g., [21], [22]) can be used as a base for designing the distributed counterparts as long as these variants satisfy the relation R1, and the outputs of $\mathcal{H}_1$ and $\mathcal{H}_2$ enter as additive disturbances to each other.

IV. DER Coordination Over Time-Varying Directed Graphs

In this section, we present a distributed algorithm for solving the DER coordination problem (1) over time-varying directed communication graphs.

A. Communication Network Model

Here, we introduce the model describing the communication network that enables the unidirectional exchange of information between DER agents. Let $\mathcal{G}^{(0)} = (\mathcal{V}, \mathcal{E}^{(0)})$ denote a directed graph, where each agent in the vertex set $\mathcal{V} := \{1, 2, \ldots, n\}$ corresponds to a DER, and $(i,j) \in \mathcal{E}^{(0)}$ if there is a communication link that allows agent i to send information to agent j (but not vice versa). We assume that $\mathcal{G}^{(0)}$ does not have self-loops. During any time interval (t_k, t_{k+1}), successful data transmissions among the DER agents can be captured by the directed graph $\mathcal{G}^{(c)}[k] = (\mathcal{V}, \mathcal{E}^{(c)}[k])$, where $\mathcal{E}^{(c)}[k] \subseteq \mathcal{E}^{(0)}$ is the set of active communication links, with $(i,j) \in \mathcal{E}^{(c)}[k]$ if agent j receives information from agent i during time interval (t_k, t_{k+1}), but not necessarily vice versa. Let $\mathcal{N}_i^+ := \{j \in \mathcal{V} : (i,j) \in \mathcal{E}^{(0)}\}$ denote the set of nominal out-neighbors of agent i, and define its nominal out-degree, which includes itself, as $d_i := |\mathcal{N}_i^+| + 1$. Let $\mathcal{N}_i^+[k]$ and $\mathcal{N}_i^-[k]$ denote the sets of out-neighbors and in-neighbors of agent i, respectively, during time interval (t_k, t_{k+1}), i.e., $\mathcal{N}_i^+[k] := \{j \in \mathcal{V} : (i,j) \in \mathcal{E}^{(c)}[k]\}$ and $\mathcal{N}_i^-[k] := \{\ell \in \mathcal{V} : (\ell, i) \in \mathcal{E}^{(c)}[k]\}$. We define agent i's instantaneous (communication) out-degree (including itself) to be $D_i^+[k] := |\mathcal{N}_i^+[k]| + 1$. We make the following standard and reasonable assumptions (see, e.g., [31], [32]).

Assumption 4: The nominal communication graph $\mathcal{G}^{(0)} = (\mathcal{V}, \mathcal{E}^{(0)})$ is strongly connected.

Assumption 5: There exists some positive integer B such that $(i,j) \in \bigcup_{l=k}^{k+B-1} \mathcal{E}^{(c)}[l]$ for every $(i,j) \in \mathcal{E}^{(0)}$, $k = 0, 1, \ldots$.

Assumption 5 is satisfied if a communication link in the nominal communication graph becomes active at least once over every time interval of some bounded length. We also assume that each agent knows its nominal out-degree, as stated next.

Assumption 6: The value of d_i^+ is known to agent i, $i = 1, 2, \ldots, n$.

Notice that Assumption 6 is weaker than the standard assumption (see, e.g., [12], [15], [19]) that the instantaneous out-degree

$D_i^+[k]$ is known to agent i at every time instant k, which is hard to obtain in practice due to unreliable communication.

B. Ratio Consensus Algorithm

We provide a brief overview of the ratio consensus algorithm (see, e.g., [13], [33]) utilized to estimate the average power imbalance (or total power imbalance if n is known) and to update the Lagrange multipliers, $\lambda_i[k]$, in the distributed algorithms proposed later.

Consider a group of agents indexed by the set $\mathcal{V}$, each with some real initial value, i.e., v_i at agent i. Each agent aims to obtain the average of the initial values via exchange of information over the graph $\mathcal{G}^{(c)}[k]$. To this end, we let agent i maintain two variables, $\mu_i[k]$ and $\nu_i[k]$ such that $\mu_i[0] = v_i$ and $\nu_i[0] = 1$. We first consider the following updates performed by agent i:

$$\mu_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} \frac{\mu_j[k]}{D_j^+[k]} \tag{29a}$$

$$\nu_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} \frac{\nu_j[k]}{D_j^+[k]} \tag{29b}$$

$$r_i[k+1] = \frac{\mu_i[k+1]}{\nu_i[k+1]}. \tag{29c}$$

We write (29a)–(29b) in a matrix-vector form as follows:

$$\mu[k+1] = P[k]\mu[k] \tag{30a}$$

$$\nu[k+1] = P[k]\nu[k] \tag{30b}$$

where $P[k]$ is an $n \times n$ matrix with $P_{ij}[k] = 1/D_j^+[k]$, $j \in \mathcal{N}_i^-[k] \cup \{i\}$, and $P_{ij}[k] = 0$, otherwise. We note that $P[k]$ is column stochastic, and it can be shown that $r_i[k]$ converges to the average of the initial values, namely, $\lim_{k\to\infty} r_i[k] = \frac{\sum_i v_i}{n}$ [13], as long as Assumptions 4 and 5 hold.

C. Preliminary Distributed Primal–Dual Algorithm

We begin with the following preliminary distributed primal–dual algorithm, where the averaging step is based on the ratio consensus algorithm (29), executed by each agent i as follows:

$$p_i[k+1] = [p_i[k] - s f_i'(p_i[k]) + s\xi x_i[k]]_{\underline{p}_i}^{\overline{p}_i} \tag{31a}$$

$$\lambda_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} \frac{\lambda_j[k] - s y_j[k]}{D_j^+[k]} \tag{31b}$$

$$v_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} \frac{v_j[k]}{D_j^+[k]} \tag{31c}$$

$$x_i[k+1] = \frac{\lambda_i[k+1]}{v_i[k+1]} \tag{31d}$$

$$y_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} \frac{y_j[k]}{D_j^+[k]} + \hat{n}(p_i[k+1] - p_i[k]) \tag{31e}$$

where $y_i[k]$ is the estimate at instant k of the total power imbalance, $\mathbf{1}^\top(p[k] - \ell)$, at agent i. The iterations in (31) are initialized with $x_i[0] = 0$, $\lambda_i[0] = 0$, $v_i[0] = 1$, and $y_i[0] = \hat{n}(p_i[0] - \ell_i)$. We note that the iterations used to update $\lambda_i[k]$, $v_i[k]$, and $y_i[k]$ are similar to the so-called Push-DIGing algorithm proposed in [19] for solving an unconstrained consensus optimization problem.

As in the case of undirected graphs, algorithm (31) can be represented as a feedback interconnection of a nominal system, denoted by $\vec{\mathcal{H}}_1$, and a disturbance system, denoted by $\vec{\mathcal{H}}_2$. To this end, let $e[k] := x[k] - (\frac{1}{n}\mathbf{1}^\top\lambda[k])\mathbf{1}$, and $\hat{\lambda}[k] := \frac{1}{\hat{n}}\mathbf{1}^\top\lambda[k]$; then, we define the nominal system, $\vec{\mathcal{H}}_1$, as follows:

$$\vec{\mathcal{H}}_1 : \begin{cases} p[k+1] = \big[p[k] - s\nabla f(p[k]) + s\xi\frac{\hat{n}}{n}\mathbf{1}\hat{\lambda}[k] \\ \qquad\qquad + s\xi e[k]\big]_{\underline{p}}^{\overline{p}} & \text{(32a)} \\ \hat{\lambda}[k+1] = \hat{\lambda}[k] - s\mathbf{1}^\top(p[k] - \ell) & \text{(32b)} \end{cases}$$

where we substituted $e[k] + \frac{\hat{n}}{n}\mathbf{1}\hat{\lambda}[k]$ for $x[k]$ in (31a) to obtain (32a), and we summed (31b) over all i and divided the result by $\hat{n}$ to obtain (32b). We note that $e[k]$ is the vector of deviations from their average at instant k of the local estimates of the Lagrange multiplier; without $e[k]$, the nominal system $\vec{\mathcal{H}}_1$ has almost the same form as (3). In fact, the nominal systems for the undirected and directed cases are the same, namely, $\vec{\mathcal{H}}_1 = \mathcal{H}_1$. Now, we define the disturbance system, $\vec{\mathcal{H}}_2$, as follows:

$$\vec{\mathcal{H}}_2 : \begin{cases} y[k] = P[k-1]y[k-1] \\ \qquad\quad + \hat{n}(p[k] - p[k-1]) & \text{(33a)} \\ \lambda[k+1] = P[k](\lambda[k] - sy[k]) & \text{(33b)} \\ v[k+1] = P[k]v[k], & \text{(33c)} \\ x[k+1] = (V[k+1])^{-1}\lambda[k+1] & \text{(33d)} \\ e[k] = x[k] - \left(\frac{\hat{n}}{n}\hat{\lambda}[k]\right)\mathbf{1} & \text{(33e)} \end{cases}$$

where $V[k] := \mathrm{diag}(v[k])$, $P[k] \in \mathbb{R}^{n\times n}$ with

$$P_{ij}[k] := \begin{cases} \frac{1}{D_j^+[k]}, & \text{if } i = j \text{ or } (j,i) \in \mathcal{E}^{(c)}[k] \\ 0, & \text{else.} \end{cases}$$

Then, algorithm (31) can be viewed as a feedback interconnection of $\vec{\mathcal{H}}_1$ and $\vec{\mathcal{H}}_2$, where (p^*, λ^*) is the equilibrium of (32) when $e[k] \equiv 0$, $\forall k$.

Since it is not assumed that each agent i knows its communication out-degree, $D_i^+[k]$, at every time instant k, the averaging step in Algorithm (31) cannot be executed. However, Algorithm (31) provides some basic ideas behind the main algorithm presented next.

D. Running-Sum Ratio Consensus Algorithms

Without knowledge of the instantaneous out-degrees, the averaging step in Algorithm (31) based on the ratio consensus algorithm (29) cannot be executed. In fact, if d_i^+ is used in (29) instead of $D_i^+[k]$, then, $P[k]$ is not necessarily column stochastic, precluding the DER agents from achieving consensus. However, the loss of column-stochasticity can be fixed by augmenting the original network of agents with additional virtual agents and links such that if agent i does not receive a packet from agent j, we let a virtual agent receive the packet via a virtual link [14]. This allows us to augment (30) with additional states corresponding to the virtual agents so that the augmented system becomes

$$\mu'[k+1] = \tilde{P}[k]\mu'[k] \tag{34a}$$

$$\nu'[k+1] = \tilde{P}[k]\nu'[k] \tag{34b}$$

where $\mu'[k]$ and $\nu'[k]$ are the augmented state vectors that, in addition to the states of the real agents, also contain the states of the virtual agents. The matrix $\tilde{P}[k]$ can be made column stochastic by carefully updating the states of the virtual agents. To explain this, we consider agents i and j connected via a communication link $(j,i) \in \mathcal{E}^{(0)}$, and let μ'_{ji} denote the state of the corresponding virtual agent. If $(j,i) \notin \mathcal{E}^{(c)}[k]$, then, $\mu'_{ji}[k+1] = \mu'_{ji}[k] + \frac{\mu_j[k]}{d_j^+}$, namely, the virtual agent receives the packet from agent j. If $(j,i) \in \mathcal{E}^{(c)}[k]$, then, we consider the following two options for updating the state of the virtual agent.

1) The virtual agent sends the value of its current state, $\mu'_{ji}[k]$, to agent i, and sets the value of the next state to zero, i.e., $\mu'_{ji}[k+1] = 0$. In the meantime, agent j sends $\mu_j[k]/d_j^+$ to agent i.
2) The virtual agent sends a portion of its current state, $\gamma\mu'_{ji}[k]$, to agent i, where γ is strictly positive and less than 1, and retains the other portion by performing the following update:

$$\mu'_{ji}[k+1] = (1-\gamma)\mu'_{ji}[k] + (1-\gamma)\mu_j[k]/d_j^+$$

where we notice that agent j sends $(1-\gamma)\mu_j[k]/d_j^+$ to the virtual agent and the remaining portion, $\gamma\mu_j[k]/d_j^+$, to agent i.

The first option was chosen in the original running-sum ratio consensus algorithm (see, e.g., [14]). In this work, we select the second option, since it allows us to considerably simplify the convergence analysis. To account for the absence of the virtual agents and links in the actual communication network, additional computations must be performed by each transmitting/receiving agent to effectively capture the effect of the updates by the virtual agents on the states of the actual agents. To this end, we let agent j broadcast the running sums $\sum_{t=0}^{k} \mu_j[t]/d_j^+$ and $\sum_{t=0}^{k} \nu_j[t]/d_j^+$. Then, $\mu_i[k]$ and $\nu_i[k]$ are updated by agent i as follows:

$$\mu_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} (\mu_{ij}[k+1] - \mu_{ij}[k]) \tag{35a}$$

$$\nu_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} (\nu_{ij}[k+1] - \nu_{ij}[k]) \tag{35b}$$

$$r_i[k+1] = \frac{\mu_i[k+1]}{\nu_i[k+1]} \tag{35c}$$

where $\mu_{ij}[k]$ and $\nu_{ij}[k]$ are additional variables maintained by agent i and updated using the running sums received from agent j as follows:

$$\mu_{ij}[k+1] = \begin{cases} (1-\gamma)\mu_{ij}[k] + \gamma\sum_{t=0}^{k} \frac{\mu_j[t]}{d_j^+}, & \text{if } j \in \mathcal{N}_i^-[k] \\ \mu_{ij}[k] + \frac{\mu_j[k]}{d_j^+}, & \text{if } j = i \\ \mu_{ij}[k], & \text{otherwise} \end{cases} \tag{36a}$$

$$\nu_{ij}[k+1] = \begin{cases} (1-\gamma)\nu_{ij}[k] + \gamma\sum_{t=0}^{k} \frac{\nu_j[t]}{d_j^+}, & \text{if } j \in \mathcal{N}_i^-[k] \\ \nu_{ij}[k] + \frac{\nu_j[k]}{d_j^+}, & \text{if } j = i \\ \nu_{ij}[k], & \text{otherwise.} \end{cases} \tag{36b}$$

It is straightforward to see that the use of the running sums in the updates has the same effect on the states of the actual agents as the updates by the virtual agents have. Furthermore, by using the results in [14], it can be shown that $r_i[k]$ asymptotically converges with probability one to the average of the initial values, namely

$$\lim_{k\to\infty} r_i[k] = \frac{\sum_i v_i}{n}.$$

E. Robust Distributed Primal–Dual Algorithm

By utilizing the running-sum ratio-consensus algorithm (35)–(36) in the averaging step, we develop a robust extension of algorithm (31). We show that this robust extension is able to solve the DER coordination problem (1) even when every agent i only knows its nominal out-degree, d_i^+, but not its instantaneous out-degree, $D_i^+[k]$.

We let agent j broadcast the running sums $\sum_{t=0}^{k} \frac{\lambda_j[t]}{d_j^+}$, $\sum_{t=0}^{k} \frac{v_j[t]}{d_j^+}$, and $\sum_{t=0}^{k} \frac{y_j[t]}{d_j^+}$ to its neighbors at each $k \geq 0$. Agent i performs the following updates:

$$p_i[k+1] = [p_i[k] - s f_i'(p_i[k]) + s\xi x_i[k]]_{\underline{p}_i}^{\overline{p}_i} \tag{37a}$$

$$\lambda_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} (\lambda_{ij}[k+1] - \lambda_{ij}[k] - s y_{ij}[k+1] + s y_{ij}[k]) \tag{37b}$$

$$v_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} (v_{ij}[k+1] - v_{ij}[k]) \tag{37c}$$

$$x_i[k+1] = \frac{\lambda_i[k+1]}{v_i[k+1]} \tag{37d}$$

$$y_i[k+1] = \sum_{j \in \mathcal{N}_i^-[k] \cup \{i\}} (y_{ij}[k+1] - y_{ij}[k]) + \hat{n}(p_i[k+1] - p_i[k]) \tag{37e}$$

where $\lambda_{ij}[k]$, $v_{ij}[k]$, and $y_{ij}[k]$ are updated using the running sums received by agent i from agent j, and given by

$$\lambda_{ij}[k+1] = \begin{cases} (1-\gamma)\lambda_{ij}[k] + \gamma\sum_{t=0}^{k} \frac{\lambda_j[t]}{d_j^+}, & \text{if } j \in \mathcal{N}_i^-[k] \\ \lambda_{ij}[k] + \frac{\lambda_j[k]}{d_j^+}, & \text{if } j = i \\ \lambda_{ij}[k], & \text{otherwise} \end{cases} \tag{38a}$$

$$v_{ij}[k+1] = \begin{cases} (1-\gamma)v_{ij}[k] + \gamma\sum_{t=0}^{k} \frac{v_j[t]}{d_j^+}, & \text{if } j \in \mathcal{N}_i^-[k] \\ v_{ij}[k] + \frac{v_j[k]}{d_j^+}, & \text{if } j = i \\ v_{ij}[k], & \text{otherwise} \end{cases} \tag{38b}$$

$$y_{ij}[k+1] = \begin{cases} (1-\gamma)y_{ij}[k] + \gamma\sum_{t=0}^{k} \frac{y_j[t]}{d_j^+}, & \text{if } j \in \mathcal{N}_i^-[k] \\ y_{ij}[k] + \frac{y_j[k]}{d_j^+}, & \text{if } j = i \\ y_{ij}[k], & \text{otherwise} \end{cases} \tag{38c}$$

where $0 < \gamma < 1$.

Remark 3: Algorithm (37)–(38) relies on Assumption 6, which can be further relaxed by replacing the running-sum ratio consensus algorithm with its modification in [14, Algorithm 2] for handling imprecise knowledge of the nominal out-degrees. However, the convergence analysis of the resulting distributed approach is nontrivial and left for future work.

F. Feedback Representation of the Robust Distributed Primal–Dual Algorithm in Virtual Domain

To facilitate the understanding of algorithm (37)–(38), we represent it as a feedback interconnection of a nominal system, denoted by $\vec{\mathcal{H}}_1^r$, and a disturbance system, denoted by $\vec{\mathcal{H}}_2^r$. However, unlike the previously described feedback representations, this representation will be given in the virtual domain using the virtual agents and links. Moreover, we show that while executing algorithm (37)–(38), the actual agents, essentially, execute (31) over a virtual network augmented with the virtual agents and links.

Consider a set of virtual agents denoted by $\mathcal{S} = \{n+1, \ldots, n+|\mathcal{E}^{(0)}|\}$, where the virtual agents correspond to the edges in $\mathcal{E}^{(0)}$ through a one-to-one map $\mathbb{I}$ such that $\mathbb{I}(j,i) \in \mathcal{S}$ for $(j,i) \in \mathcal{E}^{(0)}$. Consider neighboring agents i and j, i.e., $(j,i) \in \mathcal{E}^{(0)}$, and a virtual agent $l \in \mathcal{S}$ corresponding to the link from j to i, i.e., $\mathbb{I}(j,i) = l$. Let $\widetilde{\mathcal{N}}_l^-[k]$ denote the set of in-neighbors of agent l at instant k given by $\widetilde{\mathcal{N}}_l^-[k] = \{j\}$, $\forall k$, implying that agent l always receives a packet from agent j. Let $\widetilde{\mathcal{N}}_i^-[k]$ denote the augmented set of in-neighbors of agent $i \in \mathcal{V}$ at instant k given by

$$\widetilde{\mathcal{N}}_i^-[k] = \mathcal{N}_i^-[k] \cup \{a \in \mathcal{S} : a = \mathbb{I}(j,i), j \in \mathcal{N}_i^-[k]\} \quad (39)$$

which contains the set of in-neighbors $\mathcal{N}_i^-[k]$ and the set of virtual agents, from which agent i receives a packet at instant k. Note that the definition of $\widetilde{\mathcal{N}}_i^-[k]$ in (39) implies that agent i receives a packet from agent l at instant k if agent i receives a packet from agent j at instant k.

If we let agent $l \in \mathcal{S}$ execute the following iterations:

$$\lambda_l[k+1] = \begin{cases} \lambda_l[k] + \frac{\lambda_j[k]}{d_j^+}, & j \notin \widetilde{\mathcal{N}}_i^-[k] \\ (1-\gamma)\lambda_l[k] + (1-\gamma)\frac{\lambda_j[k]}{d_j^+}, & \text{otherwise} \end{cases} \quad (40a)$$

$$v_l[k+1] = \begin{cases} v_l[k] + \frac{v_j[k]}{d_j^+}, & j \notin \widetilde{\mathcal{N}}_i^-[k] \\ (1-\gamma)v_l[k] + (1-\gamma)\frac{v_j[k]}{d_j^+}, & \text{otherwise} \end{cases} \quad (40b)$$

$$y_l[k+1] = \begin{cases} y_l[k] + \frac{y_j[k]}{d_j^+}, & j \notin \widetilde{\mathcal{N}}_i^-[k] \\ (1-\gamma)y_l[k] + (1-\gamma)\frac{y_j[k]}{d_j^+}, & \text{otherwise} \end{cases} \quad (40c)$$

where $\lambda_l[0] = 0$, $v_l[0] = 0$, and $y_l[0] = 0$, then, it is not difficult to see that the agent i's updates in (37) are equivalent to the following iterations in the virtual domain:

$$p_i[k+1] = [p_i[k] - sf_i'(p_i[k]) + s\xi x_i[k]]_{\underline{p}_i}^{\overline{p}_i} \quad (41a)$$

$$\lambda_i[k+1] = \frac{\lambda_i[k] - sy_i[k]}{d_i^+} + \sum_{a \in \widetilde{\mathcal{N}}_i^-[k]} \gamma \frac{\lambda_a[k] - sy_a[k]}{d_a^+} \quad (41b)$$

$$v_i[k+1] = \frac{v_i[k]}{d_i^+} + \sum_{a \in \widetilde{\mathcal{N}}_i^-[k]} \gamma \frac{v_a[k]}{d_a^+} \quad (41c)$$

$$x_i[k+1] = \frac{\lambda_i[k+1]}{v_i[k+1]} \quad (41d)$$

$$y_i[k+1] = \frac{y_i[k]}{d_i^+} + \sum_{a \in \widetilde{\mathcal{N}}_i^-[k]} \gamma \frac{y_a[k]}{d_a^+} + \hat{n}(p_i[k+1] - p_i[k]) \quad (41e)$$

where $d_a^+ := 1$, $a \in \mathcal{S}$. Note that the information sent by the virtual agents is treated by agent i the same way as the information sent by the real agents, and that the iterations in (41) are, in essence, similar to those in (31); i.e., while executing algorithm (37)–(38), the actual agents execute (31) over a virtual network augmented with the virtual agents and links. Now, we define $N := n + |\mathcal{E}^{(0)}|$, and $\tilde{P}[k] \in \mathbb{R}^{N \times N}$ such that

$$\tilde{P}_{ij}[k] := \begin{cases} \frac{\gamma}{d_j^+}, & \text{if } i \in \mathcal{V}, j \in \widetilde{\mathcal{N}}_i^-[k] \\ \frac{1-\gamma}{d_j^+}, & \text{if } i \in \mathcal{S}, \mathbb{I}(j,l) = i, j \in \widetilde{\mathcal{N}}_l^-[k] \\ \frac{1}{d_j^+}, & \text{if } i \in \mathcal{S}, \mathbb{I}(j,l) = i, j \notin \widetilde{\mathcal{N}}_l^-[k] \\ 0, & \text{else} \end{cases}$$

$$\tilde{P}_{ii}[k] := \begin{cases} \frac{1}{d_i^+}, & \text{if } i \in \mathcal{V} \\ \frac{1-\gamma}{d_i^+}, & \text{if } i \in \mathcal{S}, \mathbb{I}(j,l) = i, j \in \widetilde{\mathcal{N}}_l^-[k] \\ \frac{1}{d_i^+}, & \text{if } i \in \mathcal{S}, \mathbb{I}(j,l) = i, j \notin \widetilde{\mathcal{N}}_l^-[k]. \end{cases}$$

Here, $\tilde{P}[k]$ denotes the one-step state transition matrix at time k, and determines the values of $\lambda[k]$, $v[k]$, and $y[k]$ at the next iteration, e.g., $v[k+1] = \tilde{P}[k]v[k]$. Note that $\tilde{P}[k]$ is column stochastic. Furthermore, for $i = 1, \ldots, N$, we have that

$$\begin{aligned} \tilde{P}_{ij}[k] &\geq \min(\gamma, 1-\gamma) \min_{j \in \mathcal{V} \cup \mathcal{S}} \frac{1}{d_j^+} \\ &\geq \min(\gamma, 1-\gamma)/n := \tau, \ j \in \widetilde{\mathcal{N}}_i^-[k] \cup \{i\} \quad \forall k \end{aligned} \quad (42)$$

where we used the fact that $d_j^+ \leq n$, $\forall j$. This, in particular, implies that all diagonal entries in $\tilde{P}[k]$ are always strictly positive. For further development, we establish the following result using the analysis from the proof of [15, Lemma 4]. However, there are some subtle differences due to the fact that $v_i[0] = 0$, for $i \in \mathcal{S}$. We recall that $v_i[0] = 1$, for $i \in \mathcal{V}$.

Lemma 3: For $i = 1, 2, \ldots, N$, we have that

$$v_i[k] \geq \frac{1-\gamma}{n} \tau^{N(2B-1)} \quad \forall k \geq 1. \quad (43)$$

Proof: Since $\tilde{P}_{ii}[k] = 1/d_i^+$, $\forall i \in \mathcal{V}$, and $d_i^+ \leq n$, we have that $\tilde{P}_{ii}[k] \geq 1/n$, $\forall i \in \mathcal{V}$ and $k \geq 0$. Hence

$$(\tilde{P}[k+1] \ldots \tilde{P}[0])_{ii} \geq \frac{1}{n} (\tilde{P}[k] \ldots \tilde{P}[0])_{ii}$$

for $i = 1, \ldots, n$. Because $\tau < 1/n$, it becomes clear that when $1 \leq k \leq N(2B-1)$

$$(\tilde{P}[k-1]\ldots\tilde{P}[0]\mathbf{v}[0])_i \geq \tilde{P}_{ii}[k-1]\ldots\tilde{P}_{ii}[0] \geq 1/n^{N(2B-1)} > \tau^{N(2B-1)} \quad (44)$$

for all $i \in \mathcal{V}$, where $\mathbf{v}[t] := [v_1[t], v_2[t], \ldots, v_N[t]]^\top$. We recall that, by (42), we have that $\tilde{P}_{ij}[k] \geq \tau,\ i = 1, \ldots, N,\ \forall j \in \widetilde{\mathcal{N}}_i^-[k] \cup \{i\},\ \forall k$. Then, as shown in [32, Lemma 2], for $t \geq (N-1)(2B-1)$, we have that

$$(\tilde{P}[t-1]\ldots\tilde{P}[0])_{ij} \geq \tau^{(N-1)(2B-1)} \quad \forall i, j. \quad (45)$$

By combining (44) and (45) and using the fact that $v_i[0] = 1$, for $i = 1, \ldots, n$, we find that

$$v_i[k+1] = (\tilde{P}[k]\ldots\tilde{P}[0]\mathbf{v}[0])_i \geq \tau^{N(2B-1)} \quad \forall k \geq 0. \quad (46)$$

Now, consider a virtual agent $l \in \mathcal{S}$ such that $\mathbb{I}(i,j) = l$ for some $i, j \in \mathcal{V}$. Noticing that $i \in \widetilde{\mathcal{N}}_l^-[k],\ \forall k \geq 0$, and $d_i^+ \leq n$, for $l = n+1, \ldots, N$, we have from (40b) that

$$v_l[k+1] \geq \frac{1-\gamma}{d_i^+}v_i[k] \geq \frac{1-\gamma}{n}\tau^{N(2B-1)} \quad \forall k \geq 0. \quad (47)$$

Combining (46) and (47) yields (43). ■

Next, we define additional virtual variables maintained by the virtual agents. For $i \in \mathcal{S}$, we define

$$x_i[k] := \begin{cases} \frac{\lambda_i[k]}{v_i[k]}, & \text{if } k > 0 \\ 0, & \text{if } k = 0. \end{cases}$$

We let $p_i[k]$ denote the iterate for the produced power at the virtual agent $i \in \mathcal{S}$ at instant k, $f_i(p_i) := mp_i^2$ denote the cost function, $\underline{p}_i = \overline{p}_i = 0$ denote the capacity constraints, and $\ell_i = 0$ denote the consumed power. Since $\underline{p}_i = \overline{p}_i = 0$, we have that $p_i[k] = 0$, for all $k \geq 0$. Since $p_i[k] = 0$, $\forall k$, and $\ell_i = 0$, these virtual variables do not have any effect on the solution of the considered problem, and are only needed for describing the feedback system.

Next, we let

$$\begin{aligned}
\mathbf{x}[k] &= [x_1[k], x_2[k], \ldots, x_N[k]]^\top \\
\boldsymbol{\lambda}[k] &= [\lambda_1[k], \lambda_2[k], \ldots, \lambda_N[k]]^\top \\
\mathbf{y}[k] &= [y_1[k], y_2[k], \ldots, y_N[k]]^\top \\
\mathbf{p}[k] &= [p_1[k], p_2[k], \ldots, p_N[k]]^\top \\
\boldsymbol{\ell} &= [\ell_1, \ell_2, \ldots, \ell_N]^\top \\
\mathbf{f}(\mathbf{p}[k]) &= [f_1(p_1[k]), f_2(p_2[k]), \ldots, f_N(p_N[k])]^\top \\
\underline{\mathbf{p}} &= [\underline{p}_1, \ldots, \underline{p}_N]^\top, \overline{\mathbf{p}} = [\overline{p}_1, \ldots, \overline{p}_N]^\top \\
v[k] &= [v_1[k], v_2[k], \ldots, v_n[k]]^\top \\
V[k] &:= \mathrm{diag}(v[k]), \tilde{V}[k] := \mathrm{diag}(\mathbf{v}[k]).
\end{aligned}$$

Noticing that, by Lemma 3, $\tilde{V}[k]$ is invertible for all $k \geq 1$, we define

$$\mathbf{h}[k] := \begin{cases} (\tilde{V}[k])^{-1}\mathbf{y}[k] & \text{if } k > 0 \\ \begin{bmatrix} (V[k])^{-1}y[k] \\ \mathbf{0}_{N-n} \end{bmatrix} & \text{if } k = 0 \end{cases} \quad (48)$$

where $y[k] = [y_1[k], y_2[k], \ldots, y_n[k]]^\top$. [Instead of defining $\mathbf{h}[k]$ as $(\tilde{V}[k])^{-1}\mathbf{y}[k]$, we adopt the definition in (48), because $\tilde{V}[k]$ is not invertible at $k = 0$.] Let $\tilde{R}[k] := (\tilde{V}[k+1])^{-1}\tilde{P}[k]\tilde{V}[k]$, I_n denote the $n \times n$ identity matrix, $\mathbf{0}_{a\times b}$ denote the $a \times b$ all-zeros matrix, and

$$\tilde{I} = \begin{bmatrix} I_n & \mathbf{0}_{n\times(N-n)} \\ \mathbf{0}_{(N-n)\times n} & \mathbf{0}_{(N-n)\times(N-n)} \end{bmatrix}.$$

Let $\overline{\mathbf{x}}[k] := \mathbf{1}^\top\mathbf{x}[k]/N$, $\tilde{\mathbf{x}}[k] = \mathbf{x}[k] - \mathbf{1}\overline{\mathbf{x}}[k]$, and $\hat{\mathbf{x}}[k] := \mathbf{1}^\top\mathbf{x}[k]/\hat{n}$. Then, by substituting $(\hat{n}/N)\hat{\mathbf{x}}[k] + \tilde{\mathbf{x}}[k]$ for $\mathbf{x}[k]$, $\tilde{V}[k]\mathbf{x}[k]$ for $\boldsymbol{\lambda}[k]$ and $(\tilde{V}[k+1])^{-1}\tilde{P}[k]\tilde{V}[k]$ for $\tilde{R}[k]$, and by using (40)–(41) and (48), we obtain that

$$\mathbf{p}[k+1] = \left[\mathbf{p}[k] - s\nabla\mathbf{f}(\mathbf{p}[k]) + s\xi\frac{\hat{n}}{N}\hat{\mathbf{x}}[k]\mathbf{1} + s\xi\tilde{\mathbf{x}}[k]\right]_{\underline{\mathbf{p}}}^{\overline{\mathbf{p}}} \quad (49a)$$

$$\begin{aligned}
\mathbf{x}[k+1] &= (\tilde{V}[k+1])^{-1}\left(\tilde{P}[k]\tilde{V}[k]\mathbf{x}[k] - s\tilde{I}\tilde{P}[k]\mathbf{y}[k]\right) \\
&= (\tilde{V}[k+1])^{-1}\tilde{P}[k]\tilde{V}[k]\mathbf{x}[k] \\
&\quad - s\tilde{I}(\tilde{V}[k+1])^{-1}\tilde{P}[k]\mathbf{y}[k] \\
&= (\tilde{V}[k+1])^{-1}\tilde{P}[k]\tilde{V}[k]\mathbf{x}[k] \\
&\quad - s\tilde{I}(\tilde{V}[k+1])^{-1}\tilde{P}[k]\tilde{V}[k]\mathbf{h}[k] \\
&= \tilde{R}[k]\mathbf{x}[k] - s\tilde{I}\tilde{R}[k]\mathbf{h}[k] \quad (49b)
\end{aligned}$$

$$\begin{aligned}
\mathbf{h}[k+1] &= (\tilde{V}[k+1])^{-1}\mathbf{y}[k+1] \\
&= (\tilde{V}[k+1])^{-1}\left(\tilde{P}[k]\mathbf{y}[k] + \hat{n}(\mathbf{p}[k+1] - \mathbf{p}[k])\right) \\
&= (\tilde{V}[k+1])^{-1}\left(\tilde{P}[k]\tilde{V}[k]\mathbf{h}[k] + \hat{n}(\mathbf{p}[k+1] - \mathbf{p}[k])\right) \\
&= \tilde{R}[k]\mathbf{h}[k] + \hat{n}(\tilde{V}[k+1])^{-1}(\mathbf{p}[k+1] - \mathbf{p}[k]). \quad (49c)
\end{aligned}$$

Let

$$\overline{\mathbf{h}}[k] := \frac{\mathbf{1}^\top\mathbf{y}[k]}{\mathbf{1}^\top\mathbf{v}[k]} = \frac{\hat{n}}{n}\mathbf{1}^\top(\mathbf{p}[k] - \ell). \quad (50)$$

Since $\tilde{R}[k]$ is row-stochastic [19], the following relations hold:

$$\hat{\mathbf{x}}[k] = \frac{1}{\hat{n}}\mathbf{1}^\top\mathbf{x}[k] = \frac{1}{\hat{n}}\mathbf{1}^\top\tilde{R}[k]\mathbf{1}\overline{\mathbf{x}}[k] \quad (51)$$

$$\overline{\mathbf{h}}[k] = \frac{1}{n}\mathbf{1}^\top\tilde{I}\tilde{R}[k]\mathbf{1}\overline{\mathbf{h}}[k]. \quad (52)$$

By using (50), (51), and (52), we find from (49b) that

$$\begin{aligned}
\hat{\mathbf{x}}[k+1] &= \frac{1}{\hat{n}}\mathbf{1}^\top\mathbf{x}[k+1] = \frac{1}{\hat{n}}\mathbf{1}^\top\tilde{R}[k]\mathbf{x}[k] - s\frac{1}{\hat{n}}\mathbf{1}^\top\tilde{I}\tilde{R}[k]\mathbf{h}[k] \\
&= \hat{\mathbf{x}}[k] - s\mathbf{1}^\top(\mathbf{p}[k] - \ell) + \left(\frac{1}{\hat{n}}\mathbf{1}^\top\tilde{R}[k]\mathbf{x}[k] - \hat{\mathbf{x}}[k]\right) \\
&\quad - s\left(\frac{1}{\hat{n}}\mathbf{1}^\top\tilde{I}\tilde{R}[k]\mathbf{h}[k] - \frac{n}{\hat{n}}\overline{\mathbf{h}}[k]\right) \\
&= \hat{\mathbf{x}}[k] - s\mathbf{1}^\top(\mathbf{p}[k] - \ell) + \frac{1}{\hat{n}}\mathbf{1}^\top\tilde{R}[k]\left(\mathbf{x}[k] - \mathbf{1}\overline{\mathbf{x}}[k]\right)
\end{aligned}$$

$$
\begin{aligned}
&- s\frac{1}{\hat{n}}\mathbf{1}^\top \tilde{I}\tilde{R}[k]\left(\mathbf{h}[k]-\mathbf{1}\overline{\mathbf{h}}[k]\right)\\
&= \hat{\mathbf{x}}[k] - s\mathbf{1}^\top(\mathbf{p}[k]-\ell) + \frac{1}{\hat{n}}\mathbf{1}^\top\tilde{R}[k]\tilde{\mathbf{x}}[k]\\
&- s\frac{1}{\hat{n}}\mathbf{1}^\top\tilde{I}\tilde{R}[k]\tilde{\mathbf{h}}[k], \qquad (53)
\end{aligned}
$$

where $\tilde{\mathbf{h}}[k] := \mathbf{h}[k] - \mathbf{1}\overline{\mathbf{h}}[k]$. Then, we use (49a) and (53) to determine the nominal system, $\vec{\mathcal{H}}_1^r$, as follows:

$$
\vec{\mathcal{H}}_1^r : \begin{cases}
\mathbf{p}[k+1] = \Big[\mathbf{p}[k] - s\nabla\mathbf{f}(\mathbf{p}[k]) \\
\qquad + s\xi\frac{\hat{n}}{N}\hat{\mathbf{x}}[k]\mathbf{1} + s\xi\tilde{\mathbf{x}}[k]\Big]_{\underline{\mathbf{p}}}^{\overline{\mathbf{p}}} & (54a)\\
\hat{\mathbf{x}}[k+1] = \hat{\mathbf{x}}[k] - s\mathbf{1}^\top(\mathbf{p}[k]-\ell)\\
\qquad + \frac{1}{\hat{n}}\mathbf{1}^\top\tilde{R}[k]\tilde{\mathbf{x}}[k]\\
\qquad - s\frac{1}{\hat{n}}\mathbf{1}^\top\tilde{I}\tilde{R}[k]\tilde{\mathbf{h}}[k]. & (54b)
\end{cases}
$$

Now, we use (49b) and (49c) to determine the disturbance system, $\vec{\mathcal{H}}_2$, as follows:

$$
\vec{\mathcal{H}}_2^r : \begin{cases}
\mathbf{h}[k] = \tilde{R}[k-1]\mathbf{h}[k-1]\\
\qquad + \hat{n}(\tilde{V}[k])^{-1}(\mathbf{p}[k]-\mathbf{p}[k-1]) & (55a)\\
\mathbf{x}[k+1] = \tilde{R}[k]\mathbf{x}[k] - s\tilde{I}\tilde{R}[k]\mathbf{h}[k] & (55b)\\
\mathbf{e}[k] = [\tilde{\mathbf{x}}[k]^\top, s\tilde{\mathbf{h}}[k]^\top]^\top. & (55c)
\end{cases}
$$

Then, algorithm (37)–(38) can be viewed as a feedback interconnection of $\vec{\mathcal{H}}_1^r$ and $\vec{\mathcal{H}}_2^r$, where $(\mathbf{p}^*, \mathbf{x}^*)$ is the equilibrium of (5) when $\mathbf{e}[k] \equiv 0$, $\forall k$.

G. Convergence Analysis

To establish the convergence results for algorithm (37)–(38), we show that $\vec{\mathcal{H}}_1^r$ and $\vec{\mathcal{H}}_2^r$ are finite-gain stable. This allows us to apply the small-gain theorem to prove that algorithm (37)–(38) converges to an optimal solution geometrically fast.

For our analysis, we need the following result, where we recall that $\tilde{R}[k] = (\tilde{V}[k+1])^{-1}\tilde{P}[k]\tilde{V}[k]$; its proof can be found in the Appendix.

Lemma 4: For $i = 1, \ldots, N$, we have that

$$
\tilde{R}_{ij}[k] \geq \frac{1-\gamma}{n^2}\tau^{N(2B-1)+1} \quad \forall j \in \widetilde{\mathcal{N}}_i^-[k] \cup \{i\}, k \geq 1. \qquad (56)
$$

The proof of the next result, where we show that $\vec{\mathcal{H}}_1^r$ is finite-gain stable, is omitted because it is identical to that of Proposition 1. The value of the parameter $\overline{s}$ is found using the same expressions in (9)–(14) by substituting N for n, and using a different value for ψ given by

$$
\psi := \max\left(\mathbf{1}^\top(\ell - \underline{p}), \mathbf{1}^\top(\overline{p} - \ell)\right)/\sqrt{\xi\hat{n}} + MN/\hat{n}. \qquad (57)
$$

Proposition 4: Let Assumptions 1 and 2 hold. Then, under (5), we have that

$$
\text{R1. } \|\mathbf{z}\|_2^{a,K} \leq \alpha_1\|\mathbf{e}\|_2^{a,K} + \beta_1
$$

for some positive α_1 and β_1, $a \in (0,1)$, $\forall s \in (0, \min(\overline{s}, 1/\sqrt{\xi\hat{n}}))$, and $\forall \xi > 0$, where

$$
\mathbf{z}[k] := \begin{bmatrix}\mathbf{p}[k]-\mathbf{p}^*\\ \hat{\mathbf{x}}[k]-\mathbf{x}^*\end{bmatrix}.
$$

In the next result, we show that $\vec{\mathcal{H}}_2^r$ is finite-gain stable.

Proposition 5: Let Assumptions 4, 5, and 6 hold. Then, under (55), we have that

$$
\text{R2. } \|\mathbf{e}\|_2^{a,K} \leq s\alpha_2\|\mathbf{z}\|_2^{a,K} + \beta_2 \qquad (58)
$$

for some positive α_2 and β_2, $a \in (0,1)$.

Proof: Letting $\boldsymbol{\delta}[k+1] := \hat{n}(\mathbf{p}[k+1] - \mathbf{p}[k])$, and using the triangle inequality, we obtain that

$$
\begin{aligned}
\|\boldsymbol{\delta}[k+1]\|_2 &= \hat{n}\|\mathbf{p}[k+1] - \mathbf{p}^* - \mathbf{p}[k] + \mathbf{p}^*\|_2\\
&\leq \hat{n}(\|\mathbf{p}[k+1]-\mathbf{p}^*\|_2 + \|\mathbf{p}[k]-\mathbf{p}^*\|_2)\\
&\leq \hat{n}(\|\mathbf{z}[k+1]\|_2 + \|\mathbf{z}[k]\|_2). \qquad (59)
\end{aligned}
$$

Then, by taking $\max_{0\leq k\leq K}(\cdot)$ on both sides of (59), we obtain

$$
\begin{aligned}
\max_{0\leq k\leq K}\|\boldsymbol{\delta}[k+1]\|_2 &\leq \hat{n}\Big(\max_{0\leq k\leq K}\|\mathbf{z}[k+1]\|_2\\
&\quad + \max_{0\leq k\leq K}\|\mathbf{z}[k]\|_2\Big)\\
&\leq 2\hat{n}\max_{0\leq k\leq K+1}\|\mathbf{z}[k]\|_2. \qquad (60)
\end{aligned}
$$

Since $\max_{0\leq k\leq K}\|\boldsymbol{\delta}[k+1]\|_2 \geq \max_{0\leq k\leq K+1}\|\boldsymbol{\delta}[k]\|_2 - \|\boldsymbol{\delta}[0]\|_2$, it follows from (60) that

$$
\|\boldsymbol{\delta}\|_2^{a,K} \leq 2\hat{n}\|\mathbf{z}\|_2^{a,K} + \|\boldsymbol{\delta}[0]\|_2. \qquad (61)
$$

Recall that $\mathbf{e}[k] = [\tilde{\mathbf{x}}[k]^\top, s\tilde{\mathbf{h}}[k]^\top]^\top$; then, by using the fact that $\sqrt{a^2+b^2} \leq a+b, \forall a,b \in \mathbb{R}^+$, we have that

$$
\|\mathbf{e}[k]\|_2 \leq \|\tilde{\mathbf{x}}[k]\|_2 + s\|\tilde{\mathbf{h}}[k]\|_2. \qquad (62)
$$

By using (42) and the result in Lemma 4, the following lemmata can be established by borrowing much of the analysis from the proofs of [19, Lemmas 15–16].

Lemma 5:

$$
\|\tilde{\mathbf{h}}\|_2^{a,K} \leq \gamma_1\|\boldsymbol{\delta}\|_2^{a,K} + \gamma_2 \qquad (63)
$$

for some positive constants γ_1 and γ_2.

Lemma 6:

$$
\|\tilde{\mathbf{x}}\|_2^{a,K} \leq s\gamma_3\|\tilde{\mathbf{h}}\|_2^{a,K} + \gamma_4 \qquad (64)
$$

for some positive constants γ_3 and γ_4.

By using (64), (63), and (61) in (62), we obtain

$$
\begin{aligned}
\|\mathbf{e}\|_2^{a,K} &\leq \|\tilde{\mathbf{x}}\|_2^{a,K} + s\|\tilde{\mathbf{h}}\|_2^{a,K} \leq s(1+\gamma_3)\|\tilde{\mathbf{h}}\|_2^{a,K} + \gamma_4\\
&\leq s\gamma_1(1+\gamma_3)\|\boldsymbol{\delta}\|_2^{a,K} + s\gamma_2(1+\gamma_3) + \gamma_4\\
&\leq 2s\gamma_1(1+\gamma_3)\hat{n}\|\mathbf{z}\|_2^{a,K} + s\gamma_2(1+\gamma_3) + \gamma_4\\
&\quad + s\gamma_1(1+\gamma_3)\|\boldsymbol{\delta}[0]\|_2,
\end{aligned}
$$

which can be rewritten as

$$
\|\mathbf{e}\|_2^{a,K} \leq s\alpha_2\|\mathbf{z}\|_2^{a,K} + \beta_2
$$

where $\alpha_2 = 2\gamma_1(1+\gamma_3)\hat{n}$, and $\beta_2 = s\gamma_2(1+\gamma_3) + \gamma_4 + s\gamma_1(1+\gamma_3)\|\boldsymbol{\delta}[0]\|_2$, yielding (58). ∎

In the following, we state the convergence results for algorithm (37)–(38), which can be shown by applying the small-gain theorem to the results in Propositions 4–5, similar to the analysis in the proof of Proposition 3.

Proposition 6: Let Assumptions 1, 2, 4, 5, and 6 hold. Then, under algorithm (37)–(38)

$$\|\mathbf{z}\|_2^{a,K} \leq \beta$$

for some $\beta > 0$, $a \in (0,1)$, $\forall s \in (0, \min(\overline{s}, 1/\sqrt{\xi \hat{n}}, 1/(\alpha_1 \alpha_2)))$, and $\forall \xi > 0$. In particular, $(p_i[k], x_i[k])$ converges to $(\mathbf{p}_i^*, \mathbf{x}^*)$, $i = 1, \ldots, n$, at a geometric rate $\mathcal{O}(a^k)$.

Finally, in the following, we establish that $p^* := [\mathbf{p}_1^*, \ldots, \mathbf{p}_n^*]$ is the solution of (1). [The proof is similar to the proof of Lemma 1.]

Lemma 7: Consider $(\mathbf{p}^*, \mathbf{x}^*)$, namely, the equilibrium of the nominal system $\vec{\mathcal{H}}_1^r$ with $\mathbf{e}[k] \equiv 0$, $\forall k$. Then, p^* is the solution of (1).

V. Numerical Simulations

In this section, we present numerical results that illustrate the performance of the proposed algorithms using the IEEE 39-bus test system [34].

We randomly pick the load demands and generation capacity constraints of the DERs. For each i, we choose $f_i(p_i) = a_i p_i^2$, where $a_i > 0$ is randomly selected.

A. Time-Varying Undirected Communication Graphs

First, we illustrate the performance of the proposed algorithm (6) over time-varying undirected communication graphs. In the communication model, every pair of agents is connected by a bidirectional communication link if there is an electrical line between their respective DERs. Communication links are assumed to fail with probability 0.2 independently (and independently between different time steps). The weights $a_{ij}[k]$, $\{i,j\} \in \mathcal{E}^{(0)}$, are picked using the Metropolis rule [35], namely

$$a_{ij}[k] = \begin{cases} \frac{1}{\max(d_i, d_j)}, & \{i,j\} \in \mathcal{E}^{(c)}[k] \\ 0, & \text{otherwise.} \end{cases}$$

For convenience, algorithm (6) is referred to as $\mathbf{PD}_1$. We compare its performance with that of algorithm (1), referred to as $\mathbf{PD}_2$. We run $\mathbf{PD}_1$ using a constant stepsize $s = 0.01$, $\xi = 0.06$, and different values for $\hat{n}$, namely, $\hat{n} = 10$ and $\hat{n} = 30$. We execute $\mathbf{PD}_2$ using a constant stepsize $s = 0.01$, and, also, using a diminishing stepsize of the form $s[k] = a/(k+b)$, where $a > 0$ and $b > 0$, with $\xi = 0.06$ and $\hat{n} = 30$. Both algorithms are initialized with $p[0] = 0$. In Fig. 2, we provide the convergence error, namely, the Euclidean distance between the exact and iterative solutions, $\|p[k] - p^*\|_2$, for both algorithms. It can be seen that $\mathbf{PD}_1$ significantly outperforms $\mathbf{PD}_2$ when $\hat{n} = 30$ is used and has geometric convergence speed. We also note that using $\hat{n} = 10$ yields significantly slower performance than when $\hat{n} = 30$ is used. This further implies that if the power network size tends to vary substantially over time, e.g., due to the connection of new DERs, it needs to be estimated on a regular basis to maintain fast performance of $\mathbf{PD}_1$.

B. Time-Varying Directed Communication Graphs

Next, we illustrate the performance of the proposed algorithm (37)–(38) over time-varying directed communication graphs. In the communication model, every pair of agents are connected by a single or two opposite unidirectional communication links if there is an electrical line between their respective DERs. We assign the orientations of the communication links such that the nominal communication graph $\mathcal{G}^{(0)}$ is strongly connected. Communication links fail with probability 0.2 independently (and independently between different time steps). We also assume that out-degrees, $D_i^+[k]$, are unknown to DER agents.

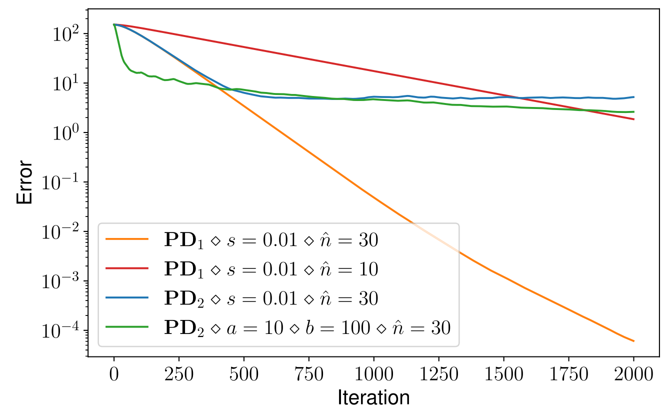

Fig. 2. Trajectories of $\|p[k] - p^*\|_2$ for algorithms $\mathbf{PD}_1$, $\mathbf{PD}_2$.

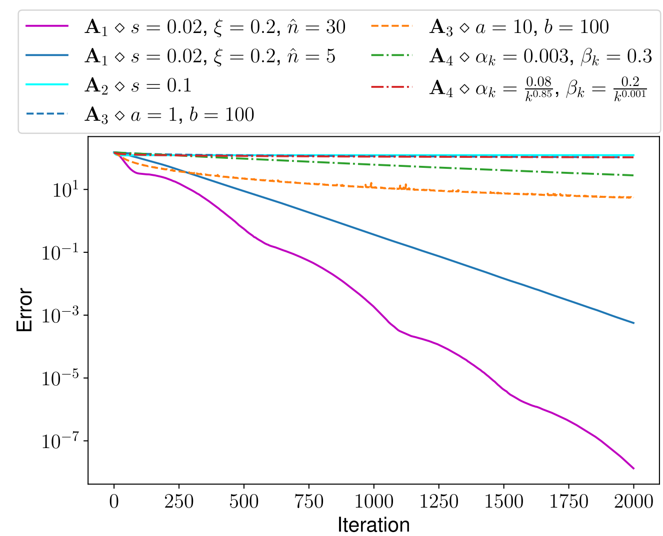

Fig. 3. Trajectories of $\|p[k] - p^*\|_2$ for algorithms $\mathbf{A}_1$–$\mathbf{A}_4$.

We compare the performance of algorithm (37)–(38), for convenience referred to as $\mathbf{A}_1$, against that of the distributed algorithms proposed in [3], [7], and [11], referred to as $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{A}_4$, respectively. Algorithms $\mathbf{A}_1$ and $\mathbf{A}_2$ use a constant stepsize s. In contrast, $\mathbf{A}_3$ and $\mathbf{A}_4$ need to use a diminishing stepsize in order to guarantee convergence. However, if the stepsize is constant and sufficiently small, $\mathbf{A}_3$ and $\mathbf{A}_4$ can still achieve convergence within a small error. We run $\mathbf{A}_1$ using $\gamma = 0.9$, $s = 0.02$, $\xi = 0.2$, and different values for $\hat{n}$, namely, $\hat{n} = 5$ and $\hat{n} = 30$. We executed $\mathbf{A}_3$ using different diminishing stepsizes of the form $s[k] = a/(k+b)$, where $a > 0$ and $b > 0$. We run $\mathbf{A}_4$ using the following parameter values: $(\alpha_k, \beta_k) = (0.003, 0.3)$ and $(\alpha_k, \beta_k) = (\frac{0.08}{k^{0.85}}, \frac{0.2}{k^{0.001}})$, which are also used in the numerical simulations in [3].

In Fig. 3, we provide the convergence error, namely, the Euclidean distance between the exact and iterative solutions,

$\|p[k]-p^*\|_2$, for all algorithms. Effects of unreliable communication can be observed in Fig. 3; in particular, $\mathbf{A}_2$ fails to converge because time-varying out-degrees, $D_i^+[k]$, are unknown to DER agents, whereas $\mathbf{A}_3$ and $\mathbf{A}_4$ exhibit very slow performance. By contrast, $\mathbf{A}_1$ manages to find an optimal solution with high accuracy, when $\hat{n}=30$ is used, and has geometric convergence speed despite the effects of unreliable communication. We note that using $\hat{n}=5$ results in noticeably slower performance. Similar to the undirected case, it can then be inferred that the power network size must be regularly estimated if it can vary substantially over time in order to maintain fast performance of $\mathbf{A}_1$. Through numerical simulations, we also observed that it is in general difficult to choose the right values for a and b in order for $\mathbf{A}_3$ to operate well. In fact, if the ratio a/b is large, $\mathbf{A}_3$ might exhibit oscillatory behavior. But setting a/b to a small value results in a slow convergence.

Numerical results shown in Figs. 2 and 3 demonstrate that the error between exact and computed solutions becomes negligibly small after 2000 iterations. Therefore, when we dispatch computed DER setpoints, namely, the amount of power each DER needs to produce, the error will have a negligible impact on the operation of the considered power system.

VI. Conclusion

We presented distributed algorithms for solving the DER coordination problem over time-varying communication graphs. The algorithms have a geometric convergence rate. One important future direction is to extend the proposed algorithms to solve more complex DER coordination problems with additional constraints, e.g., line flow constraints, voltage constraints, or reactive power balance constraints. Another interesting endeavor for future work is to leverage the methods based on the notion of Integral Quadratic Constraints (see, e.g., [36]) for refining the convergence results or even redesigning some of the proposed algorithms.

Appendix

A. Proof of Lemma 1

Proof: At the equilibrium, we have that

$$p^* = [p^* - s\nabla f(p^*) + s\xi(\hat{n}/n)\mathbf{1}\lambda^*]_{\underline{p}}^{\overline{p}}$$

$$\lambda^* = \lambda^* - s\mathbf{1}^\top(p^*-\ell).$$

Then, the following relations hold:

$$0 = \nabla f(p^*) - \xi(\hat{n}/n)\mathbf{1}\lambda^* + \mu^* - \nu^* \quad (65a)$$

$$0 = \mathbf{1}^\top(p^*-\ell) \quad (65b)$$

$$0 = \mu_i^*(p_i^* - \overline{p}_i) \quad (65c)$$

$$0 = \nu_i^*(\underline{p}_i - p_i^*), i=1,\dots,n \quad (65d)$$

where $\mu^* = [\mu_1^*,\dots,\mu_n^*]^\top$, with $\mu_i^* \geq 0, i=1,\dots,n$, and $\nu^* = [\nu_1^*,\dots,\nu_n^*]^\top$, with $\nu_i^* \geq 0$, $i=1,\dots,n$. Noticing that (65) represents the Karush–Kuhn–Tucker (KKT) conditions for (1), it follows from [25, Prop. 3.3.1] that p^* is the solution of (1). ■

B. Proof of Lemma 2

Proof: For our analysis, we need the results in the following lemmas.

Lemma 8: Let $v, w, \underline{v}, \overline{v} \in \mathbb{R}$, where $\underline{v} \leq \overline{v}$, and define $\Delta v := v - [v]_{\underline{v}}^{\overline{v}}$ and $\Delta w := w - [w]_{\underline{v}}^{\overline{v}}$. Then, we have that

$$(\Delta v - \Delta w)([v]_{\underline{v}}^{\overline{v}} - [w]_{\underline{v}}^{\overline{v}}) \geq 0. \quad (66)$$

Proof: Suppose $[v]_{\underline{v}}^{\overline{v}} > [w]_{\underline{v}}^{\overline{v}}$, then, it follows that $\Delta v \geq 0$, and $\Delta w \leq 0$. Hence, $\Delta v - \Delta w \geq 0$, and the result (66) holds. ■

Lemma 9: The following relation holds:

$$\min(\overline{p}_i - p_i^*, p_i^* - \underline{p}_i) \geq \zeta_i |\Delta\varphi_i[k] - \Delta\varphi_i^*|,\ i=1,2,\dots,n \quad (67)$$

for any s such that $0 < s \leq 1/\sqrt{\xi\hat{n}}$, and all $k \geq 0$.

Proof: Note that if $p_i^* = \overline{p}_i$ or $p_i^* = \underline{p}_i$, then, $\zeta_i = 0$, and the result (67) holds trivially. Suppose $\underline{p}_i < p_i^* < \overline{p}_i$. We first show that $\hat{\lambda}[k]$ is always bounded. We recall that $\psi := \max(\mathbf{1}^\top(\ell - \underline{p}), \mathbf{1}^\top(\overline{p}-\ell))/\sqrt{\xi\hat{n}}$ and define

$$\lambda^{\min} := \frac{n}{s\xi\hat{n}}\left(\min_{i\in\mathcal{V}}\left(\underline{p}_i - \overline{p}_i + sf_i'(\underline{p}_i)\right) - s\xi M\right)$$

$$\lambda^{\max} := \frac{n}{s\xi\hat{n}}\left(\max_{i\in\mathcal{V}}\left(\overline{p}_i - \underline{p}_i + sf_i'(\overline{p}_i)\right) + s\xi M\right).$$

We show by contradiction that

$$\lambda^{\min} - 2\psi \leq \hat{\lambda}[k] \leq \lambda^{\max} + 2\psi \quad \forall k. \quad (68)$$

Suppose $\hat{\lambda}[k_0] > \lambda^{\max} + 2\psi$ for some $k_0 > 0$. Since

$$|\hat{\lambda}[k+1] - \hat{\lambda}[k]| \leq \psi \quad \forall k$$

there exists $t \geq 0$ such that

$$\hat{\lambda}[t] \in [\lambda^{\max}, \lambda^{\max}+\psi].$$

Then, by using (7a) and the fact that $\|e[k]\|_2 \leq M$, we have that $p_j[t+1] = \overline{p}_j$, $j=1,2,\dots,n$, and

$$\hat{\lambda}[t+1] \leq \hat{\lambda}[t] + \psi \leq \lambda^{\max} + 2\psi.$$

In the next iterations, for $\tau \geq t+1$, we have that $p_j[\tau] = \overline{p}_j$, $j=1,2,\dots,n$, as long as $\hat{\lambda}[\tau] \geq \lambda^{\max}$, and that

$$\hat{\lambda}[\tau+1] = \hat{\lambda}[\tau] - s\mathbf{1}^\top(\overline{p}-\ell) < \hat{\lambda}[\tau].$$

Hence,

$$\hat{\lambda}[k] \leq \lambda^{\max} + 2\psi \quad \forall k.$$

Similarly, it can be shown that

$$\hat{\lambda}[k] \geq \lambda^{\min} - 2\psi \quad \forall k.$$

Therefore, the result (68) holds. Next, we show that $|\Delta\varphi_i[k] - \Delta\varphi_i^*| \leq \Delta\varphi_i^{\max}$. Since $\underline{p}_i < p_i^* < \overline{p}_i$, and (p^*, λ^*) is the equilibrium of (7) when $e[k] \equiv 0$, for all $k \geq 0$, it follows that $\Delta\varphi_i^* = 0$. By using (17), and the facts that $p_i[k]$, $\hat{\lambda}[k]$, and $e_i[k]$ are always bounded and $s \leq 1/\sqrt{\xi\hat{n}}$, it is straightforward to derive the upper bound $\Delta\varphi_i^{\max}$, which is given by

$$\Delta\varphi_i^{max} := \max\left(\max_{j\in\mathcal{V}}\left(\overline{p}_j - \underline{p}_j + f_j'(\overline{p}_j)/\sqrt{\xi\hat{n}}\right)\right.$$

$$+ 2\sqrt{\xi\hat{n}}/n\psi - f_i'(\underline{p}_i)/\sqrt{\xi\hat{n}} + 2M\sqrt{\xi/\hat{n}},$$
$$f_i'(\overline{p}_i)/\sqrt{\xi\hat{n}} + 2M\sqrt{\xi/\hat{n}} + 2\sqrt{\xi\hat{n}}/n\psi$$
$$- \min_{j\in\mathcal{V}} \left(\underline{p}_j - \overline{p}_j + f_j'(\underline{p}_j)/\sqrt{\xi\hat{n}}\right)\bigg).$$

Then, by using the definition of ζ_i in (10), we obtain (67). ■

It follows from Lemma 8 that, for $i = 1, 2, \ldots, n$

$$(\Delta\varphi_i[k] - \Delta\varphi_i^*)([\varphi_i[k] + \epsilon_i[k]]_{\underline{p}_i}^{\overline{p}_i} - [\varphi_i^*]_{\underline{p}_i}^{\overline{p}_i}) \geq 0. \quad (69)$$

Moreover, by Lemma 9, we have that

$$(\Delta\varphi_i[k] - \Delta\varphi_i^*)([\varphi_i[k] + \epsilon_i[k]]_{\underline{p}_i}^{\overline{p}_i} - [\varphi_i^*]_{\underline{p}_i}^{\overline{p}_i})$$
$$\geq \zeta_i(\Delta\varphi_i[k] - \Delta\varphi_i^*)^2, \ i = 1, 2, \ldots, n. \quad (70)$$

Then, we obtain that

$$\left\| \begin{bmatrix} \varphi[k] + \epsilon[k] - \varphi^* \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} \right\|_{P+T}^2$$
$$= \left\| \begin{bmatrix} [\varphi[k] + \epsilon[k]]_{\underline{p}}^{\overline{p}} - [\varphi^*]_{\underline{p}}^{\overline{p}} \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} \right\|_P^2$$
$$- 2sa_1\xi\frac{\hat{n}}{n}(\Delta\varphi[k] - \Delta\varphi^*)^\top \mathbf{1}(\hat{\lambda}[k+1] - \lambda^*)$$
$$+ 2a_1(1+s^2)(\Delta\varphi[k] - \Delta\varphi^*)^\top([\varphi[k] + \epsilon[k]]_{\underline{p}}^{\overline{p}} - [\varphi^*]_{\underline{p}}^{\overline{p}})$$
$$+ a_1(1+s^2)\left\|\Delta\varphi[k] - \Delta\varphi^*\right\|_2^2 + \delta(\hat{\lambda}[k+1] - \lambda^*)^2$$
$$+ a_1 s^2 \left\|[\varphi[k] + \epsilon[k]]_{\underline{p}}^{\overline{p}} - [\varphi^*]_{\underline{p}}^{\overline{p}}\right\|_2^2$$
$$\geq \left\| \begin{bmatrix} [\varphi[k] + \epsilon[k]]_{\underline{p}}^{\overline{p}} - [\varphi^*]_{\underline{p}}^{\overline{p}} \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} \right\|_P^2$$
$$- 2sa_1\xi\frac{\hat{n}}{n}(\Delta\varphi[k] - \Delta\varphi^*)^\top \mathbf{1}(\hat{\lambda}[k+1] - \lambda^*)$$
$$+ 2a_1\sum_{i=1}^n \zeta_i(\Delta\varphi_i[k] - \Delta\varphi_i^*)^2$$
$$+ a_1(1+s^2)\left\|\Delta\varphi[k] - \Delta\varphi^*\right\|_2^2 + \delta(\hat{\lambda}[k+1] - \lambda^*)^2 \quad (71)$$
$$= \left\| \begin{bmatrix} [\varphi[k] + \epsilon[k]]_{\underline{p}}^{\overline{p}} - [\varphi^*]_{\underline{p}}^{\overline{p}} \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} \right\|_P^2$$
$$+ \sum_{i=1}^n \Bigg(\sqrt{a_1(1+s^2+2\zeta_i)}(\Delta\varphi_i[k] - \Delta\varphi_i^*)$$
$$- \sqrt{\frac{a_1 s^2}{1+s^2+2\zeta_i}}\xi\frac{\hat{n}}{n}(\hat{\lambda}_i[k+1] - \lambda_i^*)\Bigg)^2 \quad (72)$$
$$\geq \left\| \begin{bmatrix} [\varphi[k] + \epsilon[k]]_{\underline{p}}^{\overline{p}} - [\varphi^*]_{\underline{p}}^{\overline{p}} \\ \hat{\lambda}[k+1] - \lambda^* \end{bmatrix} \right\|_P^2 \quad (73)$$

where in (71) we used (69) and (70), and in (72), we used the definition of δ in (11). ■

C. Proof of Lemma 4

Proof: From the definition of $\tilde{R}[k]$, we have that $\tilde{R}_{ij}[k] = \tilde{P}_{ij}[k]v_j[k]/v_i[k+1]$. From Lemma 3, we have that

$$v_j[k] \geq \frac{1-\gamma}{n}\tau^{N(2B-1)}, k \geq 1.$$

Since $\mathbf{1}^\top v[t] = n, \forall t \geq 0$, it follows that $v_i[k+1] \leq n$. We recall that, by (42), $\tilde{P}_{ij}[k] \geq \tau, i = 1, \ldots, N, \forall j \in \widetilde{\mathcal{N}}_i^-[k] \cup \{i\}$, $\forall k$. Hence

$$\tilde{R}_{ij}[k] \geq \frac{1-\gamma}{n^2}\tau^{N(2B-1)+1}$$

$i = 1, \ldots, N, \forall j \in \widetilde{\mathcal{N}}_i^-[k] \cup \{i\}, \forall k$, yielding (56). ■

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