Robust Multi-Agent Safety via Tube-Based Tightened Exponential Barrier Functions

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Abstract: This paper presents a constructive framework for synthesizing provably safe controllers for nonlinear multi-agent systems subject to bounded disturbances. The methodology applies to systems representable in Brunovsky canonical form, accommodating arbitrary-order dynamics in multi-dimensional spaces. The central contribution is a method of constraint tightening that formally couples robust error feedback with nominal trajectory planning. The key insight is that the design of an ancillary feedback law, which confines state errors to a robust positively invariant (RPI) tube, simultaneously provides the exact information needed to ensure the safety of the nominal plan. Specifically, the geometry of the resulting RPI tube is leveraged via its support function to derive state-dependent safety margins. These margins are then used to systematically tighten the high relative-degree exponential control barrier function (eCBF) constraints imposed on the nominal planner. This integrated synthesis guarantees that any nominal trajectory satisfying the tightened constraints corresponds to a provably safe trajectory for the true, disturbed system. We demonstrate the practical utility of this formal synthesis method by implementing the planner within a distributed Model Predictive Control (MPC) scheme, which optimizes performance while inheriting the robust safety guarantees.

Keywords: multi-agent systems, cooperative systems, distributed tube model predictive control, support function, exponential control barrier function.

1. INTRODUCTION

To operate successfully in applications like vehicle convoys and aerial swarms, multi-agent systems (MAS) must satisfy strict coordination and safety demands (Barnes et al., 2007). This challenge is amplified for systems with high relative degrees, where the chain of integrators between the control input and the safety output introduces significant response latency, heightening the risk of collisions in safety-critical settings (Nguyen and Sreenath, 2016; Garg et al., 2024). Model Predictive Control (MPC) offers a powerful paradigm for constrained control by optimizing over future predictions (Mayne et al., 2000). In parallel, higher-order and exponential control barrier functions (HOCBFs and eCBFs) have emerged as effective tools for enforcing safety in systems with complex dynamics (Xiao and Belta, 2022, 2019; Tan et al., 2022; Nguyen and Sreenath, 2016; Wang et al., 2021). However, enforcing robust safety for multi-agent systems in the presence of model uncertainty and external disturbances remains a significant challenge, especially when these advanced CBF techniques are applied in a distributed context.

This work introduces a robust safety framework for leader-follower formation control in continuous-time, nonlinear multi-agent systems subject to bounded matched disturbances. The central contribution is a method for formally coupling robust error feedback with nominal trajectory planning to enforce safety specifications. Our approach addresses a critical deficiency in the literature: while methods combining distributed MPC with HOCBFs exist (Wang et al., 2025), their safety analysis is often confined to nominal, disturbance-free models and lacks formal mechanisms to ensure forward invariance for systems under uncertainty. Similarly, other distributed control schemes for multi-agent systems (Tan and Dimarogonas, 2022; Mestres et al., 2024) are often reactive and lack the foresight of a planning horizon, or, if predictive (Jiang and Guo, 2024), have not been formulated to provide formal robustness guarantees for systems with high relative-degree dynamics. By integrating robust tube constructions (Kolathaya and Ames, 2019) with techniques for high-order constraints (Xiao and Belta, 2019; Ames et al., 2017), the presented framework provides a unified solution that simultaneously offers predictive planning, robust safety, and applicability to high-relative-degree systems.

We present a control synthesis framework that seamlessly integrates robust feedback with nominal planning. Local ancillary feedback controllers first confine the deviations of the true agent states from their nominal counterparts within pre-computed robust positively invariant (RPI) tubes. The key insight is that the geometries of these tubes provide sufficient information to systematically tighten the exponential control barrier function (eCBF) constraints imposed on the nominal planner of each agent. This integrated synthesis ensures that any set of nominal trajectories satisfying the tightened constraints corresponds to a provably safe trajectory for the true, disturbed system.

To leverage this synthesis for high-performance planning, we implement the planner using a distributed Model Predictive Control (DMPC) framework. Unlike reactive controllers that rely only on the current state, the predictive horizon of DMPC allows each agent to proactively shape its trajectory to avoid future hazards and inevitable collisions, a known limitation of single-timestep safety methods. The tightened eCBF inequalities are directly incorporated into each agent's local optimization problem, resulting in a computationally tractable controller that combines the foresight of MPC with formal, end-to-end guarantees of robust safety and recursive feasibility.

The remainder of the paper is structured as follows: Section 2 presents the multi-agent dynamics, derives the nominal control model, constructs per-agent RPI tubes around the nominal trajectories, and defines support functions to tighten the nominal eCBF constraints. Section 3 provides the tube-MPC with tightened eCBF inequality framework over fixed communication topology. Section 4 demonstrates the proposed framework via a numerical example with fixed communication topology. Section 5 summarizes contributions and potential future work.

2. PROBLEM FORMULATION AND METHODOLOGY

We consider a system composed of N follower agents indexed by $i \in \mathcal{V} = \{1, \ldots, N\}$ and a leader agent indexed by 0. The dynamics of the i^{th} follower agent is described by the n-th order nonlinear Brunovsky form as:

$$\begin{aligned} \dot{x}^i_p &= x^i_{p+1}, & p &= 1, \ldots, n-1, \\ \dot{x}^i_n &= f^i(x^i, t) + u^i + w^i, & w^i(t) &\in \mathcal{D}_i. \end{aligned} \tag{1}$$

where $x^i_p \in \mathbb{R}^d$ is the p-th component of the state of agent i with dimension d, $u^i \in \mathbb{R}^d$ is its input, and $w^i \in \mathbb{R}^d$ is a bounded time-varying disturbance for agent i. The dynamics of the leader agent is given as:

$$\begin{aligned} \dot{x}^0_p &= x^0_{p+1}, & p &= 1, \ldots, n-1, \\ \dot{x}^0_n &= f^0(x^0, t). \end{aligned} \tag{2}$$

To manage the disturbance $w^i(t)$ in (1), our strategy is to design the high-level planner for a simplified nominal system and use a local feedback controller to confine the resulting error dynamics within a robust positively invariant (RPI) tube (Rawlings et al., 2017; Mayne and Kerrigan, 2007). We therefore define a nominal trajectory $(\bar{x}^i, \bar{u}^i)$ for (1) and its corresponding nominal system as

$$\begin{aligned} \dot{\bar{x}}^i_p &= \bar{x}^i_{p+1}, & p &= 1, \ldots, n-1, \\ \dot{\bar{x}}^i_n &= f^i(\bar{x}^i, t) + \bar{u}^i, \end{aligned} \tag{3}$$

with $\bar{x}^i := [\bar{x}^i_1; \ldots; \bar{x}^i_n]$, and define the nominal error as $z^i_p := x^i_p - \bar{x}^i_p$, $p = 1, \ldots, n$, which is equivalent to defining $z^i := x^i - \bar{x}^i = [z^i_1; \ldots; z^i_n] \in \mathbb{R}^{nd}$. The nominal error dynamics is therefore

$$\begin{aligned} \dot{z}^i_p &= z^i_{p+1} \quad (p = 1, \ldots, n-1), \\ \dot{z}^i_n &= \Delta f(z^i, \bar{x}^i, t) + (u^i - \bar{u}^i) + w^i, \end{aligned} \tag{4}$$

where $\Delta f^i(z^i, \bar{x}^i, t) := f^i(\bar{x}^i + z^i, t) - f^i(\bar{x}^i, t)$. Defining

$$A^i_0 := \begin{bmatrix} 0 & I_d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_d \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad G^i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_d \end{bmatrix}, \tag{5}$$

we write the error dynamics (4) as

$$\dot{z}^i \; = \; A^i_0 \, z^i \; + \; G^i \big(\Delta f^i + (u^i - \bar{u}^i) + w^i\big). \tag{6}$$

2.1 State Based Ancillary Feedback:

Given the error $z^i = x^i - \bar{x}^i$, the ancillary tube feedback is designed as

$$u^i = \bar{u}^i - \sum_{p=1}^{n} K^i_p \, z^i_p \equiv \bar{u}^i - K^i z^i, \tag{7}$$

where $K^i_p \in \mathbb{R}^{d \times d}$ are constant ancillary gains (chosen to stabilize the chain of integrators) and K^i is the corresponding matrix representation of the gains. Substituting (7) into (6) gives

$$\dot{z}^i \; = \; A^i_K \, z^i \; + \; G^i \big(\Delta f^i(z^i, \bar{x}^i, t) + w^i\big), \tag{8}$$

where $A^i_K := A^i_0 - G^i K^i$, by construction (5), is a constant matrix. The gains K^i_p must be picked so that A^i_K is Hurwitz. This can be achieved noting that for any $Q_i \succ 0$, the continuous-time Lyapunov equation is:

$$(A^i_K)^\top P_i + P_i A^i_K = -Q_i, \qquad P_i \succ 0. \tag{9}$$

We assume a Lipschitz bound on the nonlinearity around the nominal trajectory

$\|\Delta f^i(\bar{x}^i, z^i, t)\| \; = \big\| f^i(\bar{x}^i + z^i, t) - f^i(\bar{x}^i, t) \big\| \le \; L_{f^i} \, \|z^i\|$,

with a known constant $L_{f^i} \ge 0$.

2.2 Robust Positive Invariant (RPI) Tube Construction:

Definition 2.1. (Robust Positive Invariance (RPI)). A set $\mathcal{Z}_i \subset \mathbb{R}^n$ is said to be robustly positively invariant if, for all $z^i(t_0) \in \mathcal{Z}_i$ and any $w^i(t) \in \mathcal{D}_i$, the condition $z^i(t) \in \mathcal{Z}_i$ holds for all $t \ge t_0$ (Blanchini and Miani, 2007). □

Lemma 2.1. (RPI ellipsoid). Let $\|w^i(t)\| \le \bar{w}_i$ for all $t \ge t_0$ and that Q_i and the corresponding P_i in (9) are such that, $\frac{\lambda_{\min}(Q_i)}{2\lambda_{\max}(P_i)} > L_{f^i}$. Then the per-agent ellipsoidal tube

$$\mathcal{Z}_i = \{ \, z^i : \; (z^i)^\top P_i z^i \le \rho_i^2 \, \} \tag{10}$$

is robust positively invariant for all ρ_i satisfying

$$\rho_i \ge \frac{2\bar{w}_i \lambda_{\max}(P_i)}{\lambda_{\min}(Q_i) - 2L_{f^i}\lambda_{\max}(P_i)}. \qquad \square$$

Proof. For the Lyapunov function $V_i(z^i) := \|z^i\|^2_{P_i} \equiv (z^i)^\top P_i z^i$, we have

$$\begin{aligned} &\frac{d}{dt} V_i(z^i(t)) = \frac{d}{dt}\big((z^i)^\top P_i z^i\big) \\ &= \dot{z}^{i\top} P_i z^i + (z^i)^\top P_i \dot{z}^i = \; 2\,(z^i)^\top P_i \, \dot{z}^i \\ &\overset{(8)}{=} 2(z^i)^\top P_i A^i_K z^i + 2(z^i)^\top P_i G^i \, \Delta f^i + 2(z^i)^\top P_i G^i \, w^i \\ &= (z^i)^\top \big(P_i A^i_K + (A^i_K)^\top P_i\big) z^i \\ &\qquad\qquad + 2(z^i)^\top P_i G^i \, \Delta f^i + 2(z^i)^\top P_i G^i \, w^i \\ &\overset{(9)}{=} -(z^i)^\top Q_i z^i + 2(z^i)^\top P_i G^i \Delta f^i + 2(z^i)^\top P_i G^i w^i. \end{aligned} \tag{11}$$

The Rayleigh-quotient bound on Q_i is written as

$$-(z^i)^\top Q_i z^i \le -\lambda_{\min}(Q_i)\|z^i\|^2 \tag{12}$$

Also, by the Cauchy–Schwarz inequality

$$\begin{aligned} 2(z^i)^\top P_i G^i \Delta f^i &\le 2\|z^i\| \cdot \|P_i\| \cdot \|G^i \Delta f^i\| \\ &\le 2L_{f^i}\lambda_{\max}(P_i)\|z^i\|^2, \end{aligned} \tag{13}$$

and, similarly,

$$
\begin{aligned}
2(z^i)^\top P_i G^i w^i &\le 2\|z^i\| \cdot \|P_i\| \cdot \|G^i \Delta w^i\| \\
&\le 2\bar{w}_i \lambda_{\max}(P_i)\|z^i\|. \quad (14)
\end{aligned}
$$

Employing (12), (13) and (14), we obtain from (11) that

$$
\begin{aligned}
\frac{d}{dt} V_i(z^i) \le &-\big(\lambda_{\min}(Q_i) - 2L_{f^i}\lambda_{\max}(P_i)\big)\|z^i\|^2 \\
&+ 2\bar{w}_i \lambda_{\max}(P_i)\|z^i\|. \quad (15)
\end{aligned}
$$

For the set $\mathcal{Z}_i$ to be positively invariant, its is required that $\frac{d}{dt}V_i(z^i) \le 0$ for any state z^i on the boundary $\partial\mathcal{Z}_i$ of the tube, which is identified by $V_i(z^i) = \rho_i^2$. This requires that

$$
-\big(\lambda_{\min}(Q_i) - 2L_{f^i}\lambda_{\max}(P_i)\big)\rho_i^2 + 2\bar{w}_i\lambda_{\max}(P_i)\rho_i \le 0. \quad (16)
$$

Since $\|w^i(t)\| \le \bar{w}_i$, a necessary and sufficient condition is

$$
\rho_i \ge \frac{2\bar{w}_i \lambda_{\max}(P_i)}{\lambda_{\min}(Q_i) - 2L_{f^i}\lambda_{\max}(P_i)} \quad (17)
$$

■

2.3 Tightened Safety and Support Functions

To enforce safety for the true state x^i while optimizing over the nominal state $\bar{x}^i$, we derive a tighten term using the per-agent RPI tubes $\mathcal{Z}i$ from Lemma 2.1. This is accomplished by employing the support function of the ellipsoidal tube (Arcari et al., 2023; Villanueva et al., 2017), which calculates the maximum extent of the tube in a specified direction and projects the RPI tube onto the gradient of the eCBF, converting the set of possible disturbances into a single scalar representing the worst-case deviation along the most critical direction. This scalar offset is later used in Section 3 to tighten the nominal eCBF inequality, ensuring that even under worst-case disturbance, the true state remains safe.

To facilitate the discussion, the control-affine dynamics of agent i from (1) is written

$$
\dot{x}^i = F_x(x^i) + F_u u^i + F_w w^i \quad (18)
$$

where $F_x(x^i) = \left[x_2^{i\top}, \cdots, f^i(x^i,t)^\top\right]^\top$, $F_u = [0, 0, \cdots, I]^\top$, $F_w = [0, 0, \cdots, I]^\top$. Accordingly, the nominal control–affine dynamics of agent i from (3) is written

$$
\dot{\bar{x}}^i = F_x(\bar{x}^i) + F_u \bar{u}^i \quad (19)
$$

Given a scalar function $h_\bullet : \mathbb{R}^n \to \mathbb{R}$, the r-th order Lie derivatives of $h_\bullet$ along F_x are defined recursively as:

$$
\begin{aligned}
L^0_{F_x} h_\bullet &:= h_\bullet, \\
L^1_{F_x} h_\bullet &:= \nabla h_\bullet^\top F_x, \\
L^r_{F_x} h_\bullet &:= \nabla(L^{r-1}_{F_x} h_\bullet)^\top F_x, \quad r \ge 2.
\end{aligned}
$$

Similarly, the Lie derivatives of $h_\bullet$ along F_u is defined as:

$$
L_{F_u} h_\bullet := \nabla h_\bullet^\top F_u.
$$

Definition 2.2. (Relative degree r). The relative degree r of the function $h_\bullet$ with respect to the input whenever $L_{F_u} L^j_{F_x} h = 0$ for $j = 1, \ldots, r-2$ and $L_{F_u} L^{r-1}_{F_x} h \ne 0$ (Nguyen and Sreenath, 2016). □

Definition 2.3. (*eCBF* (Nguyen, 2017, Definition 5.1)) For the system (18), and the safety set

$$
\mathcal{C} := \left\{ \begin{array}{ll} (x^1, \cdots, x^i, \cdots, x^N) \in \mathbb{R}^{N \cdot n \cdot d} : & \\ h_{ij}(x^i, x^j) \ge 0, & \forall (i,j) \in \mathcal{V}^2, \\ h_{iO}(x^i) \ge 0, & \forall (i,O) \in \mathcal{V} \times \mathcal{O}_i \end{array} \right\}, \quad (20)
$$

the functions $h_{ij}(x^i, x^j)$ (similarly $h_{iO}(x^i)$) are exponential control barrier functions (eCBF) of relative degree r, if there exist $K_b^{ij} \in \mathbb{R}^{r \times 1}$ s.t.,

$$
\inf_{u^i \in U_i} \left(L^r_{F_x} h_{ij} + L_{F_u} L^{r-1}_{F_x} h_{ij} \cdot u^i + K_b^{ij} \eta_{ij} \right) \ge 0, \quad (21)
$$

with $\eta_{ij}(x^i(t), x^j(t)) = \left[h_{ij}, L_{F_x} h_{ij}, \ldots, L^{r-1}_{F_x} h_{ij}\right]^\top$, for all $(i,j) \in \mathcal{V}^2$ and $(\cdot, \cdots, x^i, \cdots, x^j, \cdots, \cdot) \in \mathcal{C}$, and

$$
h_{ij}(x^i(t), x^j(t)) \ge C_b\, e^{A_b^{ij} t} \eta_{ij}(x^i(t_0), x^j(t_0)) \ge 0, \quad (22)
$$

whenever $h_{ij}(x^i(t_0), x^j(t_0)) \ge 0$, where the matrix A_b^{ij} depends on the choice of K_b^{ij}, and $C_b = [1, 0, \ldots, 0]$. □

Assumption 2.1. We assume that $h_{ij}(x^i, x^j)$, $h_{ij}(\bar{x}^i, \bar{x}^j)$, $h_{iO}(x^i)$ and $h_{iO}(\bar{x}^i)$ are convex and continuously differentiable in their arguments. □

Definition 2.4. (Tightened Safety Function). Given the decomposition $x^i = \bar{x}^i + z^i$, $x^j = \bar{x}^j + z^j$ with $z^i \in \mathcal{Z}_i$, $z^j \in \mathcal{Z}_j$, and $z^{ij} = z^i - z^j$, we define the inter-agent tightened safety function as the worst-case value (infimum) considering all possible errors in relative tube $\mathcal{Z}_{ij} = \mathcal{Z}_i \oplus (-\mathcal{Z}_j)$ in the form of

$$
\begin{aligned}
h^{\text{tight}}_{ij}(\bar{x}^i, \bar{x}^j) &:= \inf_{z^{ij} \in \mathcal{Z}_{ij}} h_{ij}(x^i, x^j) \\
&:= \inf_{z^{ij} \in \mathcal{Z}_{ij}} h_{ij}(\bar{x}^i + z^i, \bar{x}^j + z^j), \quad (23)
\end{aligned}
$$

which occurs at the tube configuration where inter-agent i–j and agent-obstacle i–O are close to violating safety. Similarly, for i–O, $h^{\text{tight}}_{iO}(\bar{x}^i) := \inf_{z^i \in \mathcal{Z}_i} h_{iO}(\bar{x}^i + z^i)$. □

Definition 2.5. (Support Function of RPI Tubes). The support function of $\mathcal{Z}_i$ is defined (Boyd and Vandenberghe, 2004) as

$$
\sigma_{\mathcal{Z}_i}(g_i) := \sup_{z^i \in \mathcal{Z}_i} g_i^\top z^i, \quad g_i \in \mathbb{R}^n, \quad (24)
$$

which expresses how far $\mathcal{Z}_i$ can grow in the direction g_i as measured by the projection $g_i^\top z^i$. □

Remark 2.1. For the relative tubes $\mathcal{Z}_{ij} = \mathcal{Z}_i \oplus (-\mathcal{Z}_j)$ in Definition 2.4, it follows from Definition 2.5 that $\sigma_{\mathcal{Z}_{ij}}([g_i; g_j]) = \sigma_{\mathcal{Z}_i}(g_i) + \sigma_{\mathcal{Z}_j}(g_j)$.

Lemma 2.2. Under the Assumption 2.1 and the assumptions of Lemmas 2.1,

$$
h_{ij}(x^i, x^j) \ge h_{ij}(\bar{x}^i, \bar{x}^j) - \sigma_{\mathcal{Z}_i}(g_i) - \sigma_{\mathcal{Z}_j}(g_j), \quad (25)
$$

$$
h_{iO}(x^i) \ge h_{iO}(\bar{x}^i) - \sigma_{\mathcal{Z}_i}(g_{iO}). \quad (26)
$$

with $g_i := \nabla_{\bar{x}^i} h_{ij}(\bar{x}^i, \bar{x}^j)$, $g_j := \nabla_{\bar{x}^j} h_{ij}(\bar{x}^i, \bar{x}^j)$ and $g_{iO} := \nabla_{\bar{x}^i} h_{iO}(\bar{x}^i)$. □

Proof. Since the sets $(\mathcal{Z}_i, \mathcal{Z}_j)$ are closed, convex, and origin-symmetric, we have the dualities (Hiriart-Urruty and Lemaréchal, 2001, Section 2)

$$
\inf_{z^i \in \mathcal{Z}_i} g_i^\top z^i = -\sup_{z^i \in \mathcal{Z}_i} (-g_i)^\top z^i \overset{\substack{\text{Def.}\\ 2.5}}{=} -\sigma_{\mathcal{Z}_i}(-g_i) \overset{\substack{\text{origin}\\ \text{sym.}}}{=} -\sigma_{\mathcal{Z}_i}(g_i) \quad (27)
$$

By the supporting-hyperplane property for convex h_{ij} (Boyd and Vandenberghe, 2004, Equation 3.2), we can write the first-order global lower bound as

$$h_{ij}(\bar{x}^i+z^i,\bar{x}^j+z^j) \geq h_{ij}(\bar{x}^i,\bar{x}^j)+\nabla h_{ij}(\bar{x}^i,\bar{x}^j)^\top \left[z^i, z^j\right]^\top$$
$$\geq h_{ij}(\bar{x}^i,\bar{x}^j)+\nabla_{\bar{x}^i} h_{ij}(\bar{x}^i,\bar{x}^j)^\top z^i+\nabla_{\bar{x}^j} h_{ij}(\bar{x}^i,\bar{x}^j)^\top z^j$$
$$\geq h_{ij}(\bar{x}^i,\bar{x}^j)+\inf_{z^i} \nabla_{\bar{x}^i} h_{ij}(\bar{x}^i,\bar{x}^j)^\top z^i+\inf_{z^j} \nabla_{\bar{x}^j} h_{ij}(\bar{x}^i,\bar{x}^j)^\top z^j$$
$$\underset{(27)}{\geq} h_{ij}(\bar{x}^i,\bar{x}^j)-\sigma_{\mathcal{Z}_i}(g_i)-\sigma_{\mathcal{Z}_j}(g_j). \quad (28)$$

for $g_i := \nabla_{\bar{x}^i} h_{ij}(\bar{x}^i,\bar{x}^j), g_j := \nabla_{\bar{x}^j} h_{ij}(\bar{x}^i,\bar{x}^j)$.

Similar arguments are used to obtain (26) for $h_{iO}(x^i)$. ■

Lemma 2.3. (Support Function of Tubes). The support function of the safety tube $\mathcal{Z}_i := \{ z^i : (z^i)^\top P_i z^i \leq \rho_i^2 \}$, with ρ_i and $P_i \succ 0$ satisfying the Lyapunov equation (9) as in Lemma 2.1, is

$$\sigma_{\mathcal{Z}_i}(g_i) \;=\; \rho_i \sqrt{g_i^\top P_i^{-1} g_i}. \quad (29)$$

□

Proof. Applying the definition 2.5 to the tube $\mathcal{Z}_i$, we have

$$\sigma_{\mathcal{Z}_i}(g_i) = \max_{z^i} \; g_i^\top z^i \quad \text{s.t.} \quad (z^i)^\top P_i z^i \leq \rho_i^2$$

Defining the Lagrangian

$$\mathcal{L}(z^i,\lambda) \;=\; g_i^\top z^i \;-\; \lambda\big((z^i)^\top P_i z^i - \rho_i^2\big), \qquad \lambda \geq 0.$$

and seeking its critical points, we obtain

$$\frac{\partial \mathcal{L}}{\partial z^i} = g_i - 2\lambda P_i z^i = 0 \quad \Rightarrow \quad (z^i)^\star = \frac{1}{2\lambda} P_i^{-1} g_i,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \quad \Rightarrow \quad ((z^i)^\star)^\top P_i (z^i)^\star = \rho_i^2.$$

Therefore,

$$\left(\frac{1}{2\lambda} P_i^{-1} g_i\right)^\top P_i \left(\frac{1}{2\lambda} P_i^{-1} g_i\right) \;=\; \frac{1}{4\lambda^2}\, g_i^\top P_i^{-1} g_i \;=\; \rho_i^2.$$

which establishes that $\lambda = \dfrac{1}{2\rho_i}\sqrt{g_i^\top P_i^{-1} g_i}$ as well as $(z^i)^\star = \dfrac{\rho_i}{\sqrt{g_i^\top P_i^{-1} g_i}} P_i^{-1} g_i$, thus giving the support function

$$\sigma_{\mathcal{Z}_i}(g_i) = g_i^\top (z^i)^\star = g_i^\top \left(\frac{1}{2\lambda} P_i^{-1} g_i\right) = \rho_i \sqrt{g_i^T P_i^{-1} g_i}.$$

■

3. MAIN RESULT

Building upon the methodology presented in Section 2, we now detail the tube-MPC with tightened eCBF framework that performs optimization on the nominal trajectories. We enforce the tightened eCBF inequalities by replacing $h_{ij}^{\text{tight}}(\cdot)$ and $h_{iO}^{\text{tight}}(\cdot)$ for the zero-order term $\kappa_0 h_{ij}(\cdot)$, while keeping all higher-order Lie derivatives evaluated on $h_{ij}(\cdot)$.

Following the standard eCBF methodology (Nguyen and Sreenath, 2016), we construct exponential control barrier functions Φ_{ij} and Φ_{iO} from $h_{ij}(\bar{x}^i,\bar{x}^j)$ and $h_{iO}(\bar{x}^i)$,

$$\Phi_{ij}(x^i,x^j,u^i) := L_{F_x}^r h_{ij}(x^i,x^j) + \sum_{q=0}^{r-1} \kappa_q L_{F_x}^q h_{ij}(x^i,x^j) + L_{F_u} L_{F_x}^{r-1} h_{ij}(x^i,x^j) u^i, \quad (30)$$

$$\Phi_{iO}(x^i,u^i) := L_{F_x}^r h_{iO}(x^i) + \sum_{q=0}^{r-1} \kappa_q L_{F_x}^q h_{iO}(x^i) + L_{F_u} L_{F_x}^{r-1} h_{iO}(x^i) u^i. \quad (31)$$

3.1 Proposed Tightened eCBF Functions

We propose the following tightened exponential control barrier functions (tight-eCBF) using the standard eCBF (30)–(31), to impose on the nominal trajectories of agents while ensuring safety of the actual trajectories which are subject to unknown disturbances w^i and corresponding errors z^i.

$$\Phi_{ij}^{\text{tight}}(\bar{x}^i,\bar{x}^j,\bar{u}^i) = L_{F_x}^r h_{ij}(\bar{x}^i,\bar{x}^j) + \sum_{q=1}^{r-1} \kappa_q L_{F_x}^q h_{ij}(\bar{x}^i,\bar{x}^j) + \big(L_{F_u} L_{F_x}^{r-1} h_{ij}(\bar{x}^i,\bar{x}^j)\big)\bar{u}^i + \kappa_0\big(h_{ij}(\bar{x}^i,\bar{x}^j) - \delta_{ij}(\bar{x}^i,\bar{x}^j)\big), \quad (32)$$

$$\Phi_{iO}^{\text{tight}}(\bar{x}^i,\bar{u}^i) = L_{F_x}^r h_{iO}(\bar{x}^i) + \sum_{q=1}^{r-1} \kappa_q L_{F_x}^q h_{iO}(\bar{x}^i) + \big(L_{F_u} L_{F_x}^{r-1} h_{iO}(\bar{x}^i)\big)\bar{u}^i + \kappa_0\big(h_{iO}(\bar{x}^i) - \delta_{iO}(\bar{x}^i)\big), \quad (33)$$

where we use notations $\delta_{ij}(\bar{x}^i,\bar{x}^j) := \sigma_{\mathcal{Z}_i}(g_i) + \sigma_{\mathcal{Z}_j}(g_j) \equiv \rho_i \left\|\nabla_{\bar{x}^i} h_{ij}(\bar{x}^i,\bar{x}^j)\right\|_{P_i^{-1}} + \rho_j \left\|\nabla_{\bar{x}^j} h_{ij}(\bar{x}^i,\bar{x}^j)\right\|_{P_j^{-1}}$, together with $\delta_{iO}(\bar{x}^i) := \sigma_{\mathcal{Z}_i}(g_{iO}) \equiv \rho_i \left\|\nabla_{\bar{x}^i} h_{iO}(\bar{x}^i,\bar{x}^j)\right\|_{P_i^{-1}}$.

The following theorem establishes conditions for forward invariance of the true states set $\mathcal{C}$.

Theorem 3.1. (Forward Invariance of Set). Let $r \geq 1$ be the relative degree of the eCBF $h_{ij}(\cdot,\cdot)$ and $h_{iO}(\cdot)$ as in Definition 2.2. Let the nominal input processes $(\bar{u}^1,\cdots,\bar{u}^N)$ and the corresponding nominal trajectories $(\bar{x}^1,\cdots,\bar{x}^N)$, satisfy the inequalities $\Phi_{ij}^{\text{tight}}(\bar{x}^i,\bar{x}^j,\bar{u}^i) \geq 0$ and $\Phi_{iO}^{\text{tight}}(\bar{x}^i,\bar{u}^i) \geq 0$ for the tightened safety functions (32) and (33), for some positive gains $\kappa_0,\ldots,\kappa_{r-1}$ so the polynomial $p(s) = s^r + \kappa_{r-1}s^{r-1} + \cdots + \kappa_1 s + \kappa_0 \equiv \prod_{q=1}^{r}(s+c_q)$ is Hurwitz (i.e., $c_q > 0$ for all q).

Suppose that at the initial time t_0 the safety constraints $h_{ij}(x^i(t_0),x^j(t_0)) \geq 0$, $h_{iO}(x^i(t_0)) \geq 0$, together with $L_{F_x}^k h_{ij}(x^i(t_0),x^j(t_0)) \geq 0$, $L_{F_x}^k h_{iO}(x^i(t_0),x^j(t_0)) \geq 0$, $k \in \{1,\cdots,r\}$ are satisfied, and the initial error is within the safety tube, i.e., $z^i(t_0) \in \mathcal{Z}_i$. Then, under the Assumptions of Lemmas 2.1–2.3,

$$h_{ij}(x^i(t),x^j(t)) \geq 0, \quad h_{iO}(x^i(t)) \geq 0, \quad (34)$$

for all $t \geq t_0$, i.e., the collision-avoidance and obstacle-avoidance set (20) for the state of all agents are forward invariant. □

Proof. By Lemma 2.2, we obtain from (25) that

$$h_{ij}(x^i,x^j) \;\geq\; h_{ij}(\bar{x}^i,\bar{x}^j) - \delta_{ij}(\bar{x}^i,\bar{x}^j).$$

where the support function (29), provided in Lemma 2.3, are bounded by

$$\sigma_{\mathcal{Z}_i}(g) = \rho_i \sqrt{g^\top P_i^{-1} g} \;\leq\; \rho_i \|P_i^{-1/2}\| \|g\|.$$

for $g := \nabla_{\bar{x}^i} h_{ij}(\bar{x}^i,\bar{x}^j)$. Therefore, for all $t \geq t_0$,

$$\delta_{ij}(\bar{x}^i(t),\bar{x}^j(t)) = \sigma_{\mathcal{Z}_i}(\nabla_{\bar{x}^i} h_{ij}) + \sigma_{\mathcal{Z}_j}(\nabla_{\bar{x}^j} h_{ij}) \leq \underbrace{\rho_i \|P_i^{-1/2}\| G_i^{\max} + \rho_j \|P_j^{-1/2}\| G_j^{\max}}_{:=\,\Delta_{ij}}. \quad (35)$$

where $G_i^{\max}$ and $G_j^{\max}$ are finite bounds on the gradients of h_{ij} with respect to $\bar{x}^i$ and $\bar{x}^i$, i.e.,

$$|\nabla_{\bar{x}^i} h_{ij}(\bar{x}^i,\bar{x}^j)| \leq G_i^{\max}, \qquad |\nabla_{\bar{x}^j} h_{ij}(\bar{x}^i,\bar{x}^j)| \leq G_j^{\max}.$$

Hence

$$\delta_{ij}(\bar{x}^i,\bar{x}^j) \;\le\; \Delta_{ij} \quad \forall t \ge t_0.$$

Accordingly, we obtain the bound on the tight-eCBF (32) as

$$\begin{aligned}
\Phi_{ij}^{\text{tight}}\big(\bar{x}^i,\bar{x}^j,\bar{u}^i\big) &= L_{F_x}^r h_{ij}(\bar{x}^i,\bar{x}^j) + \sum_{q=1}^{r-1}\kappa_q L_{F_x}^q h_{ij}(\bar{x}^i,\bar{x}^j) \\
&+ \big(L_{F_u}L_{F_x}^{r-1}h_{ij}(\bar{x}^i,\bar{x}^j)\big)\bar{u}^i + \kappa_0\big(h_{ij}(\bar{x}^i,\bar{x}^j) - \delta_{ij}(\bar{x}^i,\bar{x}^j)\big) \\
&\ge L_{F_x}^r h_{ij}(\bar{x}^i,\bar{x}^j) + \sum_{q=1}^{r-1}\kappa_q L_{F_x}^q h_{ij}(\bar{x}^i,\bar{x}^j) \\
&+ \big(L_{F_u}L_{F_x}^{r-1}h_{ij}(\bar{x}^i,\bar{x}^j)\big)\bar{u}^i + \kappa_0\big(h_{ij}(\bar{x}^i,\bar{x}^j) - \Delta_{ij}\big). \quad (36)
\end{aligned}$$

Therefore, defining the auxiliary eCBF as $\tilde{h}_{ij} := h_{ij} - \Delta_{ij}$, and invoking (Nguyen and Sreenath, 2016, Theorem 2) we deduce that $\tilde{h}_{ij}^{(k)}(\bar{x}^i(t),\bar{x}^j(t)) \ge 0$ for all $t \ge t_0$, which subsequently establishes that the positivity of Φ_{ij}^{tight} in (36) yields

$$h_{ij}(\bar{x}^i,\bar{x}^j) \ge \Delta_{ij},\ \forall t \ge t_0. \quad (37)$$

Substituting (37) into (25) from Lemma 2.2, gives

$$\begin{aligned}
h_{ij}(x^i(t),x^j(t)) \;&\ge\; h_{ij}(\bar{x}^i,\bar{x}^j) - \delta_{ij}(\bar{x}^i,\bar{x}^j) \\
&\ge \Delta_{ij} - \delta_{ij}(\bar{x}^i,\bar{x}^j) \ge 0, \quad \forall t \ge t_0 \quad (38)
\end{aligned}$$

Repeating similar steps for the obstacle safety functions h_{iO}, establishes that $h_{iO}(x^i) \ge 0$ for all $O \in \mathcal{O}_i$. In other words, the true local safety set $\mathcal{C}$ defined in (20) is forward invariant, i.e.,

$$x^i(t) \in \mathcal{C}$$

for all $t \ge t_0$. ■

3.2 Distributed Tube-MPC eCBF Formation Framework

Having established that the enforcement of the proposed tightened safety functions (32) and (33) on nominal trajectories of the agents ensures that the actual trajectories of the system are safe, we now present a Distributed Tube-MPC with eCBF which generates the nominal trajectories.

In order to reduce the amount of computations at the agent level, and to establish distributed control policies, we consider a communication graph $\mathcal{G} = (\mathcal{V},\mathcal{E},A)$ with the node set $\mathcal{V} = \{1,\ldots,N\}$, edge set $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$, and weighted adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$, where

$$a_{ij} := \begin{cases} w_{ij} & \text{if } (j,i) \in \mathcal{E}, \\ 0 & \text{otherwise,} \end{cases}$$

where $w_{ij} > 0$ for all $(j,i) \in \mathcal{E}$. For the special case of undirected graphs, A is symmetric and $w_{ij} = w_{ji}$. We define the neighboring agents set $\mathcal{N}_i := \{\, j \in \mathcal{V} : a_{ij} > 0 \,\}$. The in-degree matrix is denoted by $D = \operatorname{diag}\{d_i\}$ and defined via $d_i = \sum_{j=1}^N a_{ij}$, and the Laplacian is $L := D - A$.

The leader's interaction matrix is defined as $B_0 = \operatorname{diag}\{b_{i0}\} \in \mathbb{R}^{N\times N}$ with

$$b_{i0} := \begin{cases} w_{i0} & \text{if } (0,i) \in \bar{\mathcal{E}}, \\ 0 & \text{otherwise,} \end{cases} \qquad w_{i0} > 0.$$

We introduce a leader node 0 and the augmented graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}},\bar{\mathcal{E}})$ with $\bar{\mathcal{V}} = \{0,1,\ldots,N\}$. The augmented Laplacian used in leader-follower formation control is defined as $L_{B_0} := L + B_0$. In order to ensure that no clusters of agents are isolated from the leader, we impose the following assumption.

Assumption 3.1. Under the fixed topology $\mathcal{G}$, the augmented graph $\bar{\mathcal{G}}$ contains a spanning tree with the leader as the root, i.e., the leader is either directly or indirectly connected to every agent. In other words, whenever $(v_i, v_0) \notin \bar{\mathcal{E}}$, there exists a sequence of nonzero elements of A of the form $a_{ii_2}, a_{i_2i_3}, \cdots, a_{i_{l-1}i'}$, for some $(i',0) \in \bar{\mathcal{E}}$, with the sequence length l being a finite integer. □

To avoid unsafe lack of information in safety critical situations, we impose the following information architecture.

Assumption 3.2. If $\|x_1^i - x_1^j\| \le \phi$ with $\phi \in \mathbb{R}_{>0}$ denoting a position proximity threshold, then $a_{ij} \ne 0$. □

The above assumption ensures that agents within a ϕ-range are connected and can exchange information.

To accommodate tracking with offsets and ensure coordinated movement while maintaining formation, we denote by $\psi_p^i \in \mathbb{R}^d$ and $\psi_p^0 \in \mathbb{R}^d$ desired state offset (formation geometry) from the leader or other agents.

As presented in (Koulong and Pakniyat, 2025, Sections 3.2–3.4) for the distribution of policies, we define the local weighted stability error of each agent i as

$$\begin{aligned}
r^i = &-\nu_1 \sum_{p=1}^{n}\sum_{j=1}^{N}\lambda_p a_{ij}\Big[(x_p^i - \psi_p^i) - (x_p^j - \psi_p^j)\Big] \\
&- \nu_2 \sum_{p=1}^{n}\lambda_p b_{i0}\Big[(x_p^i - \psi_p^i) - (x_p^0 - \psi_p^0)\Big], \quad (39)
\end{aligned}$$

with arbitrary positive constants $\nu_1,\nu_2 > 0$, and design parameters $\lambda_j,\ j = 1,\ldots,n-1$ with the restriction that the associated characteristic polynomial $s^{n-1} + \lambda_{n-1}s^{n-2} + \cdots + \lambda_1$ is Hurwitz.

Using Theorem 3.1, we formulate the distributed Tube-MPC with eCBF constraints, which can be solved using the continuous-time transcription (multiple shooting with RK4 inside the optimal control problem). At each sampling instant t_k, we optimize the nominal trajectory $(\bar{x}^i, \bar{u}^i)$ with sampling period T_s and horizon length H while enforcing the tightened eCBF inequalities on the nominal states, and apply to the plant the tube feedback. For a given prediction horizon $H \in \mathbb{N}$, we minimize the objective below subject to the dynamics (3), with $k = 0,\ldots,H-1$, as follows

$$\min_{\{\bar{u}_k^i\}_{k=0}^{H-1}} \sum_{k=0}^{H-1}\Big(\|\bar{r}_k^i\|_{Q_r}^2 + \|\bar{u}_k^i\|_R^2 + \|\Delta\bar{u}_k^i\|_{R_\Delta}^2\Big) + \|\bar{r}_H^i\|_{P_r}^2 \quad (40)$$

s.t.

$$\begin{aligned}
&\bar{x}_{k+1}^i = \Phi_{\text{RK4}}\big(\bar{x}_k^i, \bar{u}_k^i, T_s\big), && (41)\\
&\bar{x}_0^i = \bar{x}^i(t), && (42)\\
&\big(\bar{x}_k^i, \bar{u}_k^i\big) \in \big(\mathcal{X}_i \ominus \mathcal{Z}_i\big) \times \big(U_i \ominus K^i\mathcal{Z}_i\big), && (43)\\
&\Phi_{ij}^{\text{tight}}\big(\bar{x}_k^i, \bar{x}_k^j, \bar{u}_k^i\big) \ge 0,\ \forall (i,j) \in \mathcal{E}, && (44)\\
&\Phi_{iO}^{\text{tight}}\big(\bar{x}_k^i, \bar{u}_k^i\big) \ge 0,\ \forall (i,O) \in \mathcal{O}^{\text{act}}, && (45)\\
&\bar{x}_H^i \in \bar{\mathcal{X}}_{f,i}, && (46)
\end{aligned}$$

where $\Delta \bar{u}_k^i := \bar{u}_k^i - \bar{u}_{k-1}^i$ is the control increment that penalizes large control changes between consecutive time steps, $(Q_r, P_r, R, R_\Delta) \succ 0$ are weighting matrices. Φ_{ij}^{tight} is given by (32) and Φ_{iO}^{tight} is given by (33). The terminal safe set is defined as

$$\bar{\mathcal{X}}_{f,i} := \Bigg\{ \bar{x}_f^i \in \bar{\mathcal{X}}_i \text{ s.t. } \\ \Phi_{iO}^{\text{tight}}(\bar{x}_f^i, \bar{u}_f^i) \geq 0,\ \forall (i,O) \in \mathcal{O}^{\text{act}} \\ \Phi_{ij}^{\text{tight}}(\bar{x}_f^i, \bar{x}_f^j, \bar{u}_f^i) \geq 0,\ \forall j \in \mathcal{N}_i, \exists \bar{u}_f^i \in \bar{U}_i \Bigg\} \tag{47}$$

The satisfaction of the constraints within the MPC optimization—particularly the tightened eCBF conditions (44) and (45) and the terminal constraint (46) —provides a recursively feasible nominal control inputs. By virtue of Theorem 3.1, implementing this policy using the composite control law (7) guarantees the forward invariance of the safe set for the true system dynamics, preventing collisions despite external disturbances.

4. NUMERICAL EXAMPLE

We demonstrate the proposed framework by simulating a leader-follower formation scenario in a two-dimensional space ($p = 2$). The system consists of one leader and $N = 5$ followers, each with distinct nonlinear dynamics and subject to external disturbances. Each agent's dynamics are modeled with $n = 3$ integrator blocks. The simulation is executed for a total of 30 seconds (sim$T$ = 30s), corresponding to $K_{\text{steps}} = 300$ iterations with a sampling time of $T_s = 0.1$s. The DMPC planner for each agent uses a prediction horizon of $H = 5$. Two circular obstacles at $[1, 1]$ and $[-1.5, 0.5]$ with radii 0.50 and 0.65 respectively include 0.15 inflation margins. Ancillary feedback gains $K_p = -[15, 4, 15, 8, 6, 8]$ were set via pole placement. Formation offsets are $\psi = [[-9; 2], [-6; 2], [0; 2], [6; 2], [9; 2]]$ and eCBF gains $\kappa_0 = 30, \kappa_1 = 38, \kappa_2 = 3$. The leader starts at $[3, 0, 0, 0, 0, 0]$ while followers begin at $[1.0, -0.5, \text{zeros}(1, 4)]$, $[-0.8, 0.4, \text{zeros}(1, 4)]$, $[0.6, 0.6, \text{zeros}(1, 4)]$, $[-0.5, -0.7, \text{zeros}(1, 4)]$, and $[0.7, 0.0, \text{zeros}(1, 4)]$. Cost weights: $Q_r = 50\, I_{10}$, $P_r = 10\, I_{10}$, $R = 0.01\, I_2$, $R_{du} = 0.001\, I_2$. Formation coupling $\nu_1 = \nu_2 = 1$ and stability parameters $\lambda_1 = 100, \lambda_2 = 50$. Figure 1 shows the information flow structure.

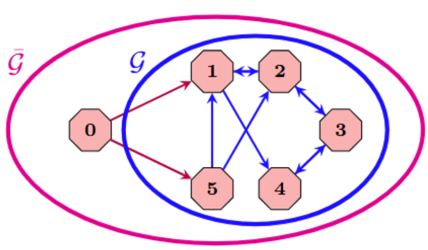
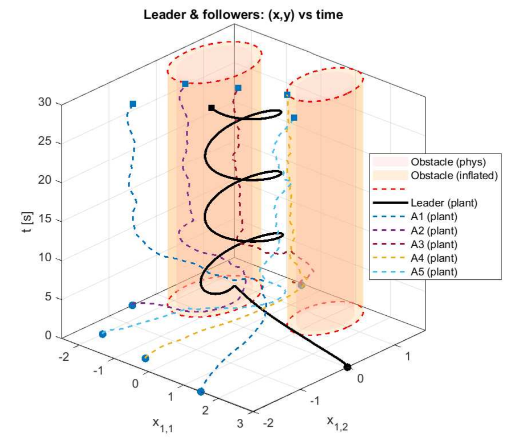


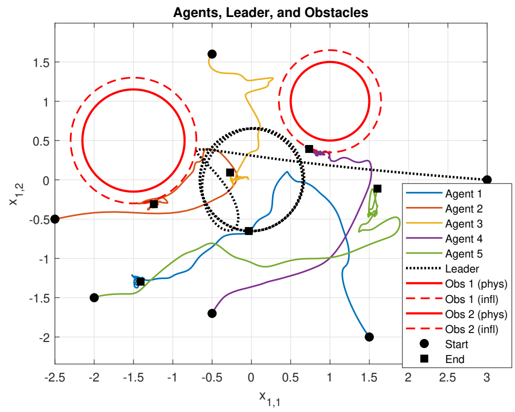
Fig. 1. The considered communication topology $\mathcal{G}$ and the augmented graph $\bar{\mathcal{G}}$ for the example in Section 4.

Leader:

$$\dot{x}_{1,1}^0 = x_{2,1}^0,\ \dot{x}_{1,2}^0 = x_{2,2}^0;\ \dot{x}_{2,1}^0 = x_{3,1}^0,\ \dot{x}_{2,2}^0 = x_{3,2}^0;$$



(a) Leader and Follower Agents $[x_{1,1}^i, x_{1,2}^i, t]$



(b) Leader and Follower Agents $[x_{1,1}^i, x_{1,2}^i]$

Fig. 2. The corresponding evolution of the positional states $x_1^i \in \mathbb{R}^2$, for the leader $i = 0$, and follower agents $i \in \{1, 2, \cdots, 5\}$ with respect to obstacles

$$\begin{aligned}\dot{x}_{3,1}^0 &= -5\,(x_{3,1}^0 + 0.36\,x_{1,1}^0 + 0.84\,x_{2,1}^0 \\ &\quad + 0.15\,(x_{1,1}^0)^3 - 0.4\sin(0.8\,t)), \\ \dot{x}_{3,2}^0 &= -5\,(x_{3,2}^0 + 0.36\,x_{1,2}^0 + 0.84\,x_{2,2}^0 \\ &\quad + 0.15\,(x_{1,2}^0)^3 - 0.4\sin(0.8\,t + \frac{\pi}{2})).\end{aligned}$$

For each follower agent i, the common dynamics are:

$$\dot{x}_{1,1}^i = x_{2,1}^i,\ \dot{x}_{1,2}^i = x_{2,2}^i;\ \ \dot{x}_{2,1}^i = x_{3,1}^i,\ \dot{x}_{2,2}^i = x_{3,2}^i;$$

and the last block (jerk) is:

$$\dot{x}_{3,1}^i = f_1^i(x^i) + u_1^i + w_1^i(t),$$
$$\dot{x}_{3,2}^i = f_2^i(x^i) + u_2^i + w_2^i(t),$$

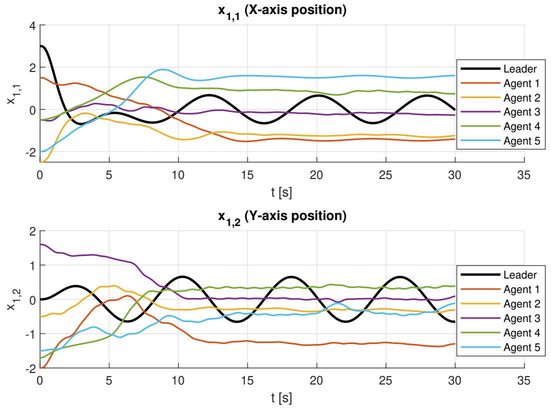
with f_1^i, f_2^i and w^i (disturbance) defined agent-wise below.

Follower $i = 1$

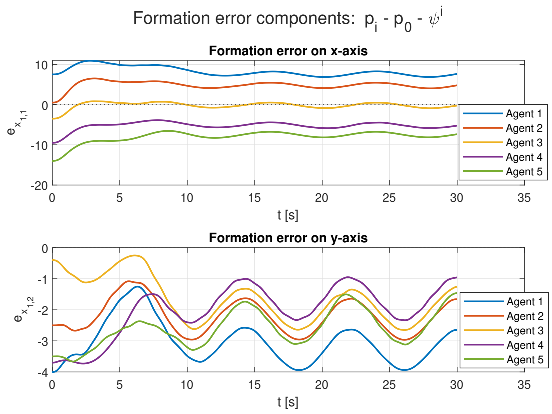
$$f_1^1(x^1) = -k_a^1\Big(x_{3,1}^1 + 0.49\,x_{1,1}^1 + 1.12\,x_{2,1}^1 + 0.12\,(x_{1,1}^1)^3 - 0.25\tanh\big(0.6\,x_{1,1}^1\big)\Big),$$

$$f_2^1(x^1) = -k_a^1\Big(x_{3,2}^1 + 0.49\,x_{1,2}^1 + 1.12\,x_{2,2}^1 + 0.12\,(x_{1,2}^1)^3 - 0.25\tanh\big(0.6\,x_{1,2}^1\big)\Big),$$

$$w_1^1(t) = 0.20\sin(0.9\,t),\ w_2^1(t) = 0.15\sin\Big(1.1\,t + \tfrac{\pi}{7}\Big).$$



(a) Leader and Follower Agents Component States $[x^i_{1,1}, x^i_{1,2}]$



(b) Leader and Follower Agents $[x_{1,1}, x_{1,2}]$

Fig. 3. Relative $[x^i_{1,1}, x^i_{1,2}]$ state Error between Agents and Leader in y-direction

Follower $i = 2$

$$f^2_1(x^2) = -k^2_a\Big(x^2_{3,1} + 0.36\,x^2_{1,1} + 0.84\,x^2_{2,1} + 0.18\,(x^2_{1,1})^3 - 0.15\,(x^2_{1,2})^2\,\tanh(x^2_{1,1})\Big),$$
$$f^2_2(x^2) = -k^2_a\Big(x^2_{3,2} + 0.36\,x^2_{1,2} + 0.84\,x^2_{2,2} + 0.18\,(x^2_{1,2})^3 - 0.15\,(x^2_{1,1})^2\,\tanh(x^2_{1,2})\Big),$$
$$w^2_1(t) = 0.18\,\sin(1.1\,t{+}0.3),\; w^2_2(t) = 0.18\,\sin\Big(0.8\,t{-}\tfrac{\pi}{5}\Big).$$

Follower $i = 3$

$$f^3_1(x^3) = -k^3_a\Big(x^3_{3,1} + 0.25\,\tanh(x^3_{1,1}) + 0.9\,\tanh(x^3_{2,1}) - 0.20\,\sin(0.7\,t)\Big),$$
$$f^3_2(x^3) = -k^3_a\Big(x^3_{3,2} + 0.25\,\tanh(x^3_{1,2}) + 0.9\,\tanh(x^3_{2,2}) - 0.20\,\sin\big(0.9\,t + \tfrac{\pi}{3}\big)\Big),$$
$$w^3_1(t) = 0.18\,\sin(1.1\,t{+}0.3),\; w^3_2(t) = 0.18\,\sin\Big(0.8\,t{-}\tfrac{\pi}{5}\Big).$$

Follower $i = 4$

$$f^4_1(x^4) = -k^4_a\Big(x^4_{3,1} + 0.4225\,x^4_{1,1} + 0.975\,x^4_{2,1} + 0.08\,\big(x^4_{1,1} + x^4_{1,2}\big)^3\Big),$$
$$f^4_2(x^4) = -k^4_a\Big(x^4_{3,2} + 0.4225\,x^4_{1,2} + 0.975\,x^4_{2,2} - 0.08\,\big(x^4_{1,1} - x^4_{1,2}\big)^3\Big),$$
$$w^4_1(t) = 0.12\,\sin(0.6t),\; w^4_2(t) = 0.12\,\sin\big(0.6t + \tfrac{\pi}{2}\big).$$

Follower $i = 5$

$$f^5_1(x^5) = -k^5_a\Big(x^5_{3,1} + 0.3025\,x^5_{1,1} + 0.88\,x^5_{2,1} + 0.15\,(x^5_{1,1})^3 - 0.20\,\tanh\big(0.5\,x^5_{1,1}\big)\Big),$$
$$f^5_2(x^5) = -k^5_a\Big(x^5_{3,2} + 0.3025\,x^5_{1,2} + 0.88\,x^5_{2,2} + 0.15\,(x^5_{1,2})^3 - 0.20\,\tanh\big(0.5\,x^5_{1,2}\big)\Big),$$
$$w^5_1(t) = 0.22\,\sin\Big(0.9\,t + \tfrac{\pi}{8}\Big),\; w^5_2(t) = 0.20\,\sin(1.0\,t).$$

Analysis

The simulation results demonstrate the core promise of the proposed framework: the synthesis of controllers that ensure safe, coordinated multi-agent motion in the presence of disturbances and obstacles.

Figure 2 shows the followers forming up with the leader while remaining outside the physical obstacle discs. Trajectories bend around the inflated safety envelopes (dashed cylinders); the closest passes occur near the right obstacle but stay outside its physical radius, so no violations occur. Pairwise separation is preserved - the eCBFs keep inter-agent distances above d_{min}. These results indicate that the collision- and obstacle-avoidance eCBFs are effective, and the tube tightening keeps the true trajectories within robust envelopes around the nominal plan.

Figure 3 plots the formation error e_i for a moving leader with obstacle detours. As expected under input limits and active eCBF constraints, e_i does not converge to zero; instead it exhibits small, bounded oscillations about zero. After a brief transient, followers settle into leader-synchronous motion with modest, structured deviations. This indicates stable behavior: agents hold their prescribed offsets and preserve obstacle clearance using an affine, solver-friendly eCBF constraint within the tube MPC.

5. CONCLUSION

This paper presented a distributed control framework that formally addresses the critical challenge of safe multi-agent navigation for high-order, nonlinear systems subject to disturbances. We introduced a unified architecture that synergistically combines the predictive capabilities of distributed Model Predictive Control (DMPC) with the formal safety guarantees of exponential Control Barrier Functions (eCBFs).

The core contribution is a method for robustly enforcing safety constraints on a nominal planner. By first constructing robust positively invariant (RPI) tubes to bound state errors, and then using their support functions to systematically tighten the eCBF constraints, the framework provides a formal margin that accounts for worst-case disturbances. This approach ensures that satisfying the tightened constraints on the nominal system guarantees forward invariance of safety sets for the true, disturbed system.

A key feature of this framework is its computational tractability. The synthesis method preserves the affine structure of the eCBF constraints relative to the control inputs, enabling the use of efficient, distributed quadratic programming. The resulting controller is therefore scalable and suitable for real-time implementation. As demonstrated in simulation, the framework achieves robust performance, maintaining bounded formation errors while successfully navigating a complex, obstacle-rich environment.

This research opens several promising avenues for future investigation. The framework could be extended to accommodate more dynamic and realistic communication scenarios, such as time-varying or switched network topologies and the presence of communication delays. Further research could explore the use of data-driven and learning-based methods to estimate disturbance bounds online, which would allow for the adaptive tuning of the RPI tubes to reduce conservatism. Finally, the integration of this provably-safe motion planning layer with higher-level decision-making algorithms would be a key step toward the deployment of fully autonomous and robust multi-agent teams in safety-critical applications.

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