

# Set-Based Control Barrier Functions and Safety Filters

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**Abstract**—High performance and formal safety guarantees are common requirements for industrial control applications. Control barrier function (CBF) methods provide a systematic approach to the modularization of safety and performance. However, the design of such CBFs can be challenging, which limits their applicability to large-scale or data-driven systems. This paper introduces the concept of a set-based CBF for linear systems with convex constraints. By leveraging control invariant sets from reachability analysis and predictive control, the set-based CBF is defined implicitly through the minimal scaling of such a set to contain the current system state. This approach enables the development of implicit, data-driven, and high-dimensional CBF representations. The paper demonstrates the design of a safety filter using set-based CBFs, which is suitable for real-time implementations and learning-based approximations to reduce online computational demands. The effectiveness of the method is illustrated through comprehensive simulations on a high-dimensional mass-spring-damper system and a motion control task, and it is validated experimentally using an electric drive application with short sampling times, highlighting its practical benefits for safety-critical control.

**Index Terms**—Control barrier functions, Safety-critical control, Invariant sets, NL Predictive control

## I. INTRODUCTION

Modern industrial control systems are increasingly required to deliver high performance, such as energy efficiency, comfort, and precise reference tracking while simultaneously guaranteeing safety under stringent operational constraints. Achieving this balance is particularly challenging, as safety requirements often necessitate conservative designs that limit achievable performance. Safety filters have emerged as a modular and permissive supervisory layer that can be integrated with high-performance controllers, human operators, or learning-based policies, as illustrated, e.g., in [1]. These filters act as real-time certifiers, ensuring that safety specifications are maintained even in the presence of potentially unsafe control actions. Various research approaches have been developed for the design of safety filters, all of which build on the concept of set invariance, see, e.g., [1] for an overview. Reachability-based methods [2] involve set-based propagation of all potential system trajectories, control barrier functions (CBF) [3] are based on concepts from Lyapunov theory to identify inputs for set invariance, and predictive control techniques [4] use a receding horizon optimal control problem to ensure recursive constraint satisfaction at each control sampling time step.

Recent research has highlighted the potential of combining these approaches to leverage their complementary

strengths [1]. However, a unified framework that systematically integrates reachability analysis, CBFs, and predictive control remains an open problem, primarily due to the differing assumptions and mathematical structures inherent to each method.

**Contributions:** This work introduces a unifying set-based CBF framework for safety filter design in linear systems with convex state and input constraints. The proposed approach constructs CBFs via the minimal scaling of a control invariant set that contains the current state, thereby enabling the use of scalable, implicitly defined, or data-driven safe sets such as zonotopes, polytopes, convex hulls of trajectories, or feasible sets of predictive controllers. This generalization overcomes the limitations of classical CBFs, which typically require explicit (analytic) representations of the safe set. The resulting set-based CBF safety filter inherits the favorable properties of conventional CBF-based designs, including the ability to shape the closed-loop behavior near the safe set boundary through the selection of class  $\mathcal{K}$  functions [5]–[7] (Theorem 1). Moreover, continuity of the set-based CBF ensures asymptotic stability of the safe set under the safety filter, provided the underlying set is robust control invariant with respect to small perturbations (Theorem 2).

We present a systematic procedure for incorporating set-based CBF constraints into safety filter optimization problems (Section IV), covering a range of invariant set representations, including polytopes in both vertex and half-space forms, zonotopes, and predictive safe sets. The resulting safety filter problems are amenable to efficient real-time implementation using standard optimization tools. Furthermore, we demonstrate that the set-based CBF can be efficiently approximated using learning-based methods, thereby reducing online computational requirements without compromising safety guarantees (Section IV-E). By leveraging techniques from stochastic MPC, we also provide a straightforward extension toward stochastic disturbances (Section IV-F). The effectiveness and scalability of the proposed framework are validated through comprehensive simulations (Section V), including a high-dimensional mass-spring-damper system and a motion control task, as well as experimental results on an electric drive system with short sampling times (Section VI).

**Related Work:** Scalable safe set computations for safety filter design have been addressed through reachability-based methods [8], system-level synthesis [9], and predictive control techniques [4], [10]. While these methods enable the efficient construction of safe sets for complex and high-dimensional systems, they do not inherently provide CBF-based mechanisms for shaping the closed-loop behavior near the boundary of the safe set. In contrast, the set-based CBF framework

introduced in this work combines the advantages of scalable safe set computation with the ability to systematically influence closed-loop dynamics, and can be efficiently designed to guarantee both safety and asymptotic stability of the safe set.

Recent advances in integrating CBFs with predictive control have been reported in [1], where the sum of slack variables from a soft-constrained predictive control problem is utilized as a CBF. Although this approach guarantees asymptotic stability of the safe set, it does not provide a systematic means to tune the closed-loop behavior near the safe set boundary, as is standard in the CBF literature [7]. In contrast, the set-based CBF framework introduced in this paper generalizes predictive control methods by uniformly scaling the constraints of the underlying problem for any given state, yielding a CBF that is continuously differentiable in the vicinity of the safe set boundary. This smoothness not only enables more precise shaping of the closed-loop response but also facilitates efficient and accurate learning-based approximations, compared to, e.g., [11]. While a related mechanism has been explored in [12] for predictive safety filters, that approach relies on iterative constraint tightening and the availability of a classical CBF for design. In contrast, our method enables the direct transformation of any nominal or robust recursively feasible MPC formulation into a set-based CBF, broadening its applicability and simplifying the design process for linear systems with convex constraints.

Connections between reachability-based safety frameworks and CBFs have been explored in [13] to enhance closed-loop performance with less conservative safety interventions. While this approach is capable of handling uncertain nonlinear system dynamics, it relies on Hamilton-Jacobi reachability analysis, which requires solving partial differential equations and is therefore computationally tractable only for systems with low-dimensional state spaces.

In summary, while previous research has explored various combinations of safety filter design methods and control barrier functions - often in a fragmented manner (see also [1] for a comprehensive overview) - a unified framework has remained elusive. By focusing on linear systems, this paper combines the strengths of these approaches into a cohesive methodology, providing efficient design procedures for set-based CBFs that guarantee asymptotic stability of the safe set.

*Outline of this article:* Section II-A formalizes the system class, constraints, and safety filter problem considered in this article, and Section II-B recalls the concept of CBFs and their discrete-time counterparts, which are used to design a safety filter based on CBFs. We then introduce set-based CBFs in Section III. Section IV provides a general procedure for incorporating set-based CBF constraints into safety filter problems, and shows how to design set-based CBFs using common control invariant set computations. Section IV-E introduces an efficient learning procedure to approximate set-based CBFs with robustness guarantees to reduce online computational requirements. Finally, we demonstrate the effectiveness of the proposed framework through simulations (Section V) and experiments (Section VI).

*Notation:* Let  $c\mathcal{A} \triangleq \{ca | a \in \mathcal{A}\}$  for some set  $\mathcal{A} \subseteq \mathbb{R}^n$  and scalar  $c \in \mathbb{R}$ . A continuous function  $\alpha : [0, a) \rightarrow \mathbb{R}$  for some  $a > 0$  is said to be class  $\mathcal{K}$  ( $\alpha \in \mathcal{K}$ ) if it is strictly increasing and  $\alpha(0) = 0$  and is said to be extended class  $\mathcal{K}$  ( $\alpha \in \mathcal{K}^e$ ) if it is a class  $\mathcal{K}$  function defined on  $(-a, b)$  with  $a, b > 0$ . A ball of radius  $r > 0$  defined by the norm  $\|\cdot\|$  is denoted by  $\mathcal{B}_n(r) \triangleq \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ . The Minkowski sum of two sets  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$  is defined as  $\mathcal{A} \oplus \mathcal{B} \triangleq \{a+b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ . The Pontryagin difference of two sets  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$  is defined as  $\mathcal{A} \ominus \mathcal{B} \triangleq \{x \in \mathbb{R}^n \mid x \oplus \mathcal{B} \subseteq \mathcal{A}\}$ . The interior of a set  $\mathcal{A} \subseteq \mathbb{R}^n$  is denoted by  $\text{int}(\mathcal{A})$ .

## II. PRELIMINARIES

### A. Problem Setting

We consider linear, time-invariant systems of the form

$$x(k+1) = Ax(k) + Bu(k), \quad (1)$$

with states  $x(k) \in \mathbb{R}^{n_x}$  and inputs  $u(k) \in \mathbb{R}^{n_u}$  at time step  $k$ . System (1) is subject to state and input constraints

$$x(k) \in \mathcal{X}, \quad u(k) \in \mathcal{U}, \quad (2)$$

with compact and convex sets  $\mathcal{X} \subset \mathbb{R}^{n_x}$  and  $\mathcal{U} \subset \mathbb{R}^{n_u}$  containing the origin. An extension to the case of additive stochastic disturbances is outlined in Section IV-E and omitted for brevity in the following.

Safety filters  $\kappa_f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u}$  are concerned with ensuring that the safety specifications (2) are satisfied when the system is presented with potentially unsafe desired control inputs  $u_{\text{des}}(k)$ . Instead of applying  $u_{\text{des}}(k)$  directly to the system, the safety filter  $\kappa_f(x(k), u_{\text{des}}(k))$  monitors the current system state  $x(k)$  and the desired control input signals  $u_{\text{des}}(k)$  and only applies them if safety can be certified. Otherwise, the safety filter overrides  $u_{\text{des}}(k)$  to produce a safe control signal with minimal intervention so that the constraints will be satisfied according to (2) for all future times.

Two key properties are desirable for practical safety filters [1] and will be the goal of this article. First, the filter should be permissive, that is, it should interfere minimally with the desired control input while providing safety. Second, it should provide a way to tune how ‘fast’ a system can approach its safety limits. In the following sections, we introduce CBF-based safety filters and define different kinds of set invariance used throughout this article. Then, we show how to leverage safe sets from scalable reachability analysis and predictive control to construct the underlying CBFs efficiently.

### B. Safety Filters based on Control Barrier Functions

Safety filters using CBFs have, e.g., been proposed in [10], [11], [13] and are given by the optimal input resulting from the following optimization problem:

$$\min_{u \in \mathcal{U}} G(u, u_{\text{des}}) \quad (3a)$$

$$\text{s.t. } h(Ax + Bu) - h(x) \geq -\Delta h(x), \quad (3b)$$

that is,

$$\kappa_f(x, u_{\text{des}}) \triangleq u^*(x, u_{\text{des}}), \quad (4)$$

with  $u^*(x, u_{\text{des}})$  denoting the optimal solution to (3), given a state  $x$ . A commonly applied filtering objective is given by  $G(u, u_{\text{des}}) = \|u - u_{\text{des}}\|$ . Other objective functions in literature account for different norms and include future desired inputs, see, e.g., [1]. Safety with respect to the system constraints (2) is encoded in the so-called CBF constraint (3b), based on a discrete-time CBF  $h$  [6, Definition III.1] defined as follows.

**Definition 1** (Control Barrier Function). *A function  $h : \mathcal{D} \rightarrow \mathbb{R}$  with compact domain  $\mathcal{D} \subset \mathbb{R}^{n_x}$  is a control barrier function (CBF) with a corresponding safe set  $\mathcal{S} \triangleq \{x \in \mathbb{R}^{n_x} : h(x) \geq 0\} \subseteq \mathcal{D}$  for (1) and (2), if  $\mathcal{S} \subseteq \mathcal{X}$  and  $\mathcal{D}$  are non-empty and compact,  $h(x)$  is continuous on  $\mathcal{D}$ , and if there exists a function  $\alpha \in \mathcal{K}^e$  such that for all  $x \in \mathcal{D}$  there exists an input  $u \in \mathcal{U}$  satisfying*

$$h(Ax + Bu) - h(x) \geq -\Delta h(x), \quad (5)$$

with

$$\Delta h(x) \triangleq \begin{cases} \min(\alpha(h(x)), h(x)), & \text{if } h(x) \geq 0, \\ \max(\alpha(h(x)), h(x)), & \text{else.} \end{cases} \quad (6)$$

The set of safe control inputs at  $x \in \mathcal{S}$  w.r.t.  $h$  is given by

$$K_{\text{CBF}}(x) \triangleq \{u \in \mathcal{U} \mid (5)\}. \quad (7)$$

Based on Definition 1, the CBF constraint (3b) ensures that the value of the CBF along closed-loop trajectories does not decrease ‘too’ rapidly, when starting within the safe set, i.e.,  $x \in \mathcal{S}$ . Specifically, the first case in (6) permits a maximal decrease of  $h(x)$  for  $h(x) \geq 0$ , thereby guaranteeing that  $h(Ax + B\kappa_{\text{cbf}}(x, u_{\text{des}})) \geq 0$ , which implies  $Ax + B\kappa_{\text{cbf}}(x, u_{\text{des}}) \in \mathcal{S}$ . If the system state is outside the safe set, i.e.,  $x \notin \mathcal{S}$  due to disturbances or unsafe initial conditions, the second case in (6) enforces a monotonic increase of the CBF value  $h(x)$  along trajectories, thereby ensuring asymptotic stability of the safe set.

Compared to [6, Definition III.1], we adopt a more restrictive CBF decrease bound in (6). Rather than requiring a decrease for some continuous function  $\Delta h(x)$  only for  $x \in \mathcal{D} \setminus \mathcal{S}$ , we enforce (5) for all  $x \in \mathcal{D}$  and impose a specific structure on  $\Delta h$  as given in (6). These additional requirements align Definition 1 more closely with the original definitions of CBFs for continuous-time systems [3] and their discrete-time counterparts [5]. As a result, Definition 1 enables systematic shaping of the closed-loop behavior by selecting an extended class  $\mathcal{K}$  function  $\alpha$ , which determines the set of feasible control inputs (7). Illustrative examples of how different choices of  $\alpha$  influence the closed-loop behavior can be found in [7]. The properties discussed above follow from [6, Thm. III.4] and rely on the notion of control invariant sets.

**Definition 2** (Set Invariance). *A set  $\Omega \subseteq \mathbb{R}^{n_x}$  is called control invariant for system (1) subject to (2) if for all  $x \in \Omega$  there exists an input  $u \in \mathcal{U}$  such that  $Ax + Bu \in \Omega$ . The set is called invariant for autonomous systems if for all  $x \in \Omega$  it holds that  $Ax + B\pi(x) \in \Omega$  for some controller  $\pi : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$ .*

**Proposition 1.** *Let  $\mathcal{D} \subset \mathbb{R}^{n_x}$  be a non-empty and compact set. Consider a CBF  $h : \mathcal{D} \rightarrow \mathbb{R}$  with safe set  $\mathcal{S}$  according to Definition 1. If  $\mathcal{S} \subset \mathcal{D}$  and  $\mathcal{D}$  is an invariant set for system (1)*

*under  $u(k) = \kappa(x(k))$  for some  $\kappa : \mathcal{D} \rightarrow \mathcal{U}$  with  $\kappa(x) \in K_{\text{CBF}}(x)$ , then it holds that*

- 1)  $\mathcal{S}$  is an invariant set,
- 2)  $\mathcal{S}$  is asymptotically stable in  $\mathcal{D}$ .

*Proof.* The proof follows directly from the fact that (5) is more restrictive than the original definition [6, Definition III.1].  $\square$

The properties established in Definition 1 and Proposition 1 are fundamental for CBF-based safety filter design. Although constructing such CBFs is generally challenging, the next section demonstrates how set-based CBFs can be systematically designed to satisfy these requirements using invariant sets.

### III. SET-BASED CONTROL BARRIER FUNCTIONS

This section presents a systematic approach for constructing a CBF based on a control invariant set  $\Omega$ . The resulting CBF satisfies Definition 1 and defines the safe set  $\mathcal{S} = \Omega$ . Specifically, for all  $x \in \Omega$  and any  $\alpha \in \mathcal{K}^e$ , the CBF satisfies the first case in (6), establishing the invariance property in Proposition 1, 1). Section III-A extends this framework to guarantee asymptotic stability of the safe set by addressing the second case in (6) according to Proposition 1, 2).

Consider a control invariant set  $\Omega$ , and note that scaling of the form  $\gamma\Omega$  preserves invariance (see Lemma 3). For a given state  $x$ , we define

$$\gamma_{\Omega}(x) \triangleq \min_{\gamma \in \mathbb{R}, \gamma \geq 0} \gamma \text{ s.t. } x \in \gamma\Omega. \quad (8)$$

The set-based CBF is given by

$$h(x) \triangleq 1 - \gamma_{\Omega}(x), \quad (9)$$

for which we can verify basic properties as detailed in the following as an intermediate result.

**Theorem 1.** *Let  $\Omega$  be a convex control invariant set satisfying  $0 \in \text{int}(\Omega)$  and  $\Omega \subseteq \mathcal{X}$ . The corresponding set-based CBF (9) satisfies Definition 1 for any compact  $\mathcal{D} \supseteq \mathcal{S}$  with  $\mathcal{S} = \Omega$ , except for the second case in (6). Under application of  $u(k) = \kappa(x(k))$  for some  $\kappa : \mathcal{D} \rightarrow \mathcal{U}$  with  $\kappa(x) \in K_{\text{CBF}}(x)$ , it holds that  $\mathcal{S}$  is a forward invariant set.*

The proof of Theorem 1 is provided in the appendix. It establishes that applying the CBF safety filter (3) with the set-based CBF (9) constructed from a control invariant set  $\Omega$  guarantees safety for all initial conditions  $x(0) \in \Omega$ , while permitting arbitrary choices of  $\alpha \in \mathcal{K}^e$  to shape the decrease  $\Delta h$ . The impact of this design flexibility is demonstrated in the numerical simulations and experimental results presented in Sections V-B and VI.

#### A. Robust Invariance ensures Asymptotic Stability of $\mathcal{S}$

To ensure that the set-based CBF (9) also satisfies the second case of (6), and thus guarantees asymptotic stability of the safe set  $\mathcal{S}$ , we require the underlying set  $\Omega$  to be robust control invariant with respect to an artificial disturbance set  $\mathcal{W}$  containing the origin in its interior. Intuitively, this means that for every state in  $\Omega$ , there exists a control input that keeps the system within  $\Omega$  despite any disturbance  $w \in \mathcal{W}$ .

Although the actual system (1) is not affected by this auxiliary disturbance, incorporating  $\mathcal{W}$  in the design introduces a margin of robustness. Consequently, a contracted set  $\mathcal{S} = \nu\Omega$  with  $\nu < 1$  can be shown to be asymptotically stable under the set-based CBF, as formalized in the following.

**Definition 3** (Robust Control Invariant Set). *A set  $\tilde{\Omega} \subseteq \mathbb{R}^{n_x}$  is called robust control invariant for system (1) subject to (2) with respect to an additive disturbance set  $\mathcal{W} \subseteq \mathbb{R}^n$  if for all  $x \in \tilde{\Omega}$  there exists an input  $u \in \mathcal{U}$  such that  $Ax + Bu + w \in \tilde{\Omega}$  for all  $w \in \mathcal{W}$ . The set is called robust invariant for autonomous systems if for all  $x \in \tilde{\Omega}$  and  $w \in \mathcal{W}$  it holds that  $Ax + B\pi(x) + w \in \tilde{\Omega}$  for some controller  $\pi : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$ .*

**Theorem 2.** *Let  $\tilde{\Omega} \subset \mathbb{R}^{n_x}$  be a convex robust control invariant set with  $0 \in \text{int}(\Omega)$  and  $\mathcal{S} \subseteq \mathcal{X}$  for some disturbance  $\mathcal{W}$  with  $0 \in \text{int}(\mathcal{W})$ . Let  $\Omega \triangleq \nu\tilde{\Omega}$  for some  $\nu < 1$  such that  $\tilde{\Omega} \ominus \mathcal{W} \subseteq \Omega$  holds. Then, the set-based CBF (9) satisfies Definition 1 with safe set  $\mathcal{S} = \Omega$  and domain  $\mathcal{D} = \tilde{\Omega}$ .*

*Proof.* The missing part in addition to Theorem 1 is to show that the second case of the bound (6) holds. A sufficient condition is to show that for all  $x \in \mathcal{D} \setminus \mathcal{S}$  there exists an  $u \in \mathcal{U}$  such that  $h(Ax + Bu) - h(x) \geq -h(x)$ , which implies  $h(Ax + Bu) \geq 0$ . Since  $\mathcal{D}$  is a robust control invariant set, there exists an input  $u \in \mathcal{U}$  such that  $Ax + Bu + w \in \mathcal{D}$  for all  $w \in \mathcal{W}$ . By definition of the Pontryagin difference, it follows that  $Ax + Bu \in \mathcal{D} \ominus \mathcal{W}$ , which is a subset of  $\mathcal{S}$  by assumption, completing the proof.  $\square$

The choice of  $\mathcal{W}$  and  $\nu$  reflects a trade-off between the size of the region from which the system can recover to the safe set and the conservativeness of the safety filter. Larger perturbations  $\mathcal{W}$  allow for smaller values  $\nu$ , increasing the region of attraction  $\tilde{\Omega} \setminus \nu\tilde{\Omega}$ . This enables recovery from a broader set of unsafe states, but results in a smaller safe set  $\mathcal{S} = \nu\Omega$  and thus a more restrictive safety filter. Conversely, if recovery from outside  $\mathcal{S}$  is not required, robustness can be omitted and the nominal result of Theorem 1 applies. A numerical simulation of Theorem 2 can be found in Section V-A.

#### IV. IMPLEMENTATION AND DESIGN OF SET-BASED CBF SAFETY FILTERS

A variety of methods exist for computing (robust) control invariant sets for linear systems, including classical set invariance [14], viability theory [15], reachability analysis [8], and model predictive control design techniques [16]. However, directly enforcing the CBF constraint (3b) in a safety filter using a set-based CBF (9) leads to a bi-level optimization problem. To address this, we introduce a sufficient reformulation that guarantees satisfaction of the CBF condition (3b) through the following safety filter optimization problem:

$$\min_{u \in \mathcal{U}, \gamma^+ \geq 0} G(u, u_{\text{des}}) \quad (10a)$$

$$\text{s.t. } 1 - \gamma^+ - h(x) \geq -\Delta h(x), \quad (10b)$$

$$Ax + Bu \in \gamma^+ \Omega. \quad (10c)$$

Here, the constraint (10c) ensures that  $1 - \gamma^+$  serves as an upper bound on the set-based CBF, which is non-conservative

since the optimizer can always select  $\gamma^+$  and  $u$  such that  $1 - \gamma^+ = h(Ax + Bu)$  holds. Consequently, the core challenge in designing a CBF safety filter of the form (10) reduces to formulating a suitable, and preferably convex, set-scaling constraint (10c) for a given set  $\Omega$ . Sections IV-A - IV-D illustrate this reformulation for several common classes of robust positive invariant sets. Furthermore, Section IV-E introduces learning-based approximations of set-based CBFs to reduce online computational complexity and Section IV-F concludes with an extension towards stochastic uncertainties.

**Remark 1.** *Evaluating  $h(x) = 1 - \gamma_\Omega(x)$  in the set-based CBF safety filter decrease condition (10b) can require solving an optimization problem prior to each filter evaluation. This can pose challenges for real-time implementation, particularly in high-dimensional systems or when  $\Omega$  is defined implicitly, such as by a convex hull or generator representation. A practical simplification is to augment the objective (10a) with a small penalty term  $\rho(\gamma^+)^2$ , which encourages the optimizer to select  $\gamma^{+,*} = \gamma_\Omega(Ax + Bu)$ , effectively yielding  $h(Ax + Bu)$ , which can be used directly at the next step. However, larger values of  $\rho$  may introduce conservative interventions, and exact equivalence to (10) is only achieved as  $\rho \rightarrow 0$ .*

##### A. Polytopes in H-Representation

Most standard algorithms for computing (maximal) robust controlled invariant sets for linear systems subject to polytopic state and input constraints result in polytopes in half-space representation (H-representation) to define  $\Omega$ , see, e.g., [17]. Such polytopes containing the origin can be expressed as:  $\Omega = \{x \in \mathbb{R}^{n_x} \mid Hx \leq \mathbf{1}\}$ , with  $H \in \mathbb{R}^{N_h \times n_x}$ . Since  $\gamma\Omega = \{x \in \mathbb{R}^{n_x} \mid Hx \leq \gamma\mathbf{1}\}$  it follows that (10c) can be efficiently implemented as

$$H(Ax + Bu) \leq \gamma^+ \mathbf{1}. \quad (11)$$

##### B. Data-Driven Safe Sets: Polytopes in V-Representation

Data-driven (robust) control invariant sets, such as those described in [18] for repetitive tasks or [19] for finite trajectories ending at the origin, are defined as polytopes in vertex representation (V-representation), that is,  $\Omega = \left\{x = \sum_{i=0}^{N_v} \lambda_i v_i \mid \sum_{i=0}^{N_v} \lambda_i = 1, \lambda_i \geq 0\right\}$ . To ensure that  $0 \in \Omega$  we require  $v_0 = \{0\}$ . From [19, Lemma 2] it follows that

$$\gamma_\Omega(x) = \min_{\{\lambda_i\}_{i=0}^{N_v}} 1 - \lambda_0 \text{ s.t. } x = \sum_{i=0}^{N_v} \lambda_i v_i, \sum_{i=0}^{N_v} \lambda_i = 1, \lambda_i \geq 0.$$

This allows to implement constraint (10c) with  $\lambda_i \geq 0$  as

$$Ax + Bu = \sum_{i=0}^{N_v} \lambda_i v_i, \quad \sum_{i=0}^{N_v} \lambda_i = 1, \quad 1 - \lambda_0 = \gamma^+.$$

##### C. Scalable Safe Set Computations: Zonotopes

Zonotopes are a special case of polytopes and provide preferable scalability properties for high-dimensional systems. Zonotopes are centrally symmetric and defined as

$$\Omega = \{x = c + G\lambda \mid \|\lambda\|_\infty \leq 1\},$$

where  $c \in \mathbb{R}^{n_x}$  is the center and  $G \in \mathbb{R}^{n_x \times N_g}$  is the generator matrix having  $N_g$  generators. Based on the definition of zonotopes, it follows that (8) is equivalent to

$$\gamma_\Omega(x) = \min_{\gamma \in \mathbb{R}, \gamma \geq 0} \gamma \text{ s.t. } x = \gamma c + G\lambda, \|\lambda\|_\infty \leq \gamma;$$

which allows a reformulation of (10c) as:

$$Ax + Bu = \gamma^+ c + G\lambda, \|\lambda\|_\infty \leq \gamma^+,$$

with optimization variables  $\lambda \in \mathbb{R}^{N_g}, \gamma^+ \geq 0$ .

#### D. Predictive Safe Sets: Projected Invariant Sets

This section utilizes the fact that the feasible set of a robust predictive control problem is robust positive control invariant [1], [16] and can therefore be used to construct a set-based CBF. To this end, we show that a simple scaling of the predictive control constraints in state and input sequence space equivalently scales the corresponding set of feasible initial states.

Consider the following robust predictive control problem, which is widely used in model predictive control [16] and in the context of predictive safety filters without CBF mechanisms [1]:

$$\min_{\{u_i\}_{i=0}^{N-1}, \{x_i\}_{i=0}^N} G(u_0, u_{\text{des}}) \quad (14a)$$

$$\text{s.t.} \quad x_0 = x, \quad (14b)$$

$$x_N \in \gamma \mathcal{X}_f, \quad (14c)$$

$$\text{for all } i = 0, \dots, N-1 : \quad (14d)$$

$$x_{i+1} = Ax_i + Bu_i, \quad (14e)$$

$$x_i \in \gamma \mathcal{X}_i, \quad (14f)$$

$$u_i \in \gamma \mathcal{U}_i. \quad (14g)$$

For  $\gamma = 1$ , this formulation recovers the standard robust predictive safety filter as in [1, Eq. (54)] for  $\mathcal{E}(x, u) = \mathcal{W}$ . At each time step, we solve (14) for a given state  $x$  and obtain an optimal input and state sequence  $\{u_i^*, \{x_i^*\}$ , from which  $u(k) = u_0^*$  is applied to the system. The objective (14a) approximates the desired safety filter objective (3a), subject to the nominal state and input constraints (14f) and (14g), along the predicted system dynamics (14e). To guarantee robust positive invariance under the optimal input sequence, the state and input constraints  $\mathcal{X}_i$  and  $\mathcal{U}_i$  are iteratively tightened along the prediction horizon, and the terminal set  $\mathcal{X}_f$  in (14c) is chosen to be robust positively invariant, see, e.g., [20, Sec. 15]. Under suitable choices it follows from [20, Thm. 15.1, 2.] that the feasible set

$$\mathcal{X}_N^\gamma \triangleq \{x \mid \exists \{u_i\}_{i=0}^{N-1} \text{ s.t. } (x, \{u_i\}_{i=0}^{N-1}) \in \mathcal{Z}_N^\gamma\}. \quad (15)$$

with

$$\mathcal{Z}_N^\gamma \triangleq \{(x_0, \{u_i\}_{i=0}^{N-1}) \mid (14c)-(14g)\}, \quad (16)$$

is a convex and compact robust control invariant according to Definition 3.

To derive a set-based CBF according to Theorem 2, we need to find an efficient scaling operation of the feasible set (15), which is only defined implicitly via a projection of the high-dimensional set (16) onto the set of feasible initial states (15).

While this projection is generally intractable to compute, we can still implement (10c) using (14c)-(14g) by scaling the set  $\mathcal{X}_N^\gamma$  through  $\gamma$  since  $\gamma \mathcal{X}_N^1 = \mathcal{X}_N^\gamma$ , see Lemma 4. The CBF constraint (10c) can therefore be enforced as follows:

$$\begin{aligned} x_0 &= Ax + Bu, \\ x_N &\in \gamma^+ \mathcal{X}_f, \\ \text{for all } i &= 0, \dots, N-1 : \\ x_{i+1} &= Ax_i + Bu_i, \\ x_i &\in \gamma^+ \mathcal{X}_i, \\ u_i &\in \gamma^+ \mathcal{U}_i. \end{aligned}$$

The resulting CBF-based safety filter (10) can thus be interpreted as a predictive safety filter [1] augmented with a CBF mechanism, enabling systematic shaping of safety interventions near the boundary of the feasible safe set (15). Notably, this approach recovers the predictive CBF formulation in [6] for linear systems, providing a principled design procedure and eliminating the need for heuristic tuning of iterative constraint tightening or large soft constraint penalties.

#### E. Approximate Set-based CBF

To reduce the online computational burden of (10), the set-based CBF introduced can be efficiently approximated offline via imitation learning, following the approach in [11]. Specifically, the mapping  $\gamma_\Omega(x)$  is approximated by a learned function  $\bar{\gamma}_\Omega(x)$ , trained on samples  $\{x_j, \gamma_\Omega(x_j)\}$  generated offline, for example by uniform sampling within a relevant subset of  $\mathcal{D}$ . The resulting approximation  $\bar{h}(x) = 1 - \bar{\gamma}_\Omega(x)$  enables direct implementation of (3) and the resulting safety filter is robust against small approximation errors, if  $h(x)$  is a CBF according to Definition 1 [11, Thm. 2]. The key advantage of the proposed set-based CBF approximation over predictive CBF approximations [11] lies in the favorable Lipschitz continuity of  $\gamma_\Omega(x)$ , as established in the proof of Theorem 2. Furthermore, the set-based CBF yields a smooth transition across the safe set boundary, enabling the use of efficient, smooth function approximators and improving data efficiency compared to non-smooth predictive CBF structures.

#### F. Stochastic Disturbances and Indirect Feedback

While set-based CBF safety filters guarantee asymptotic stability of the safe set under nominal conditions, practical systems are often subject to significant stochastic disturbances. To address this, we extend the set-based CBF framework to handle linear systems with additive stochastic disturbances,

$$x(k+1) = Ax(k) + Bu(k) + Gw_s(k), \quad (18)$$

where  $w_s(k)$  is a random process with known distribution  $\mathcal{Q}_W$ . The objective is to ensure probabilistic satisfaction of state and input constraints, i.e.,

$$\mathbb{P}\{x(k) \in \mathcal{X}^j\} \geq p_x^j, \quad \mathbb{P}\{u(k) \in \mathcal{U}^j\} \geq p_u^j, \quad (19)$$

for prescribed probability levels  $p_x^j, p_u^j \in (0, 1]$ .

Following the stochastic MPC literature [21], we decompose the system into nominal and error dynamics,  $x(k) =$

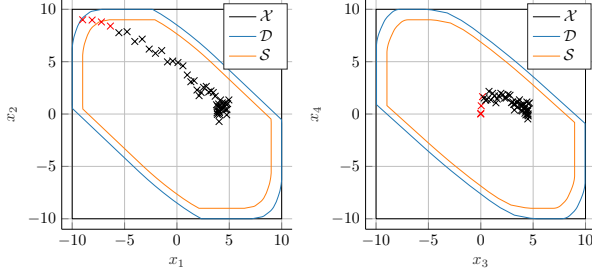


Fig. 1. Simulation of state trajectory and sets. The black/red marks correspond to times where the zonotope-based CBF  $h$  is positive/negative.

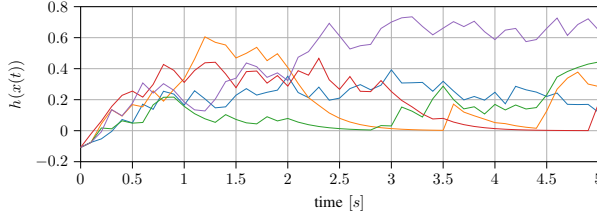


Fig. 2. Evolution of the zonotope-based CBFs  $h$  that asymptotically stabilize  $S$  for different  $u_{\text{des}}(k)$ .

$z(k) + e(k)$ , with  $z(k+1) = Az(k) + Bv(k)$  and  $e(k+1) = Ae(k) + B\pi(e(k)) + Gw_s(k)$ . The controller  $\pi(e(k))$  is designed to keep the error  $e(k)$  small, and probabilistic reachable sets  $\mathcal{R}_k^{j,x}, \mathcal{R}_k^{j,u}$  satisfying  $\Pr\{e(k) \in \mathcal{R}_k^{j,x}\} \geq p_x^j$ ,  $\Pr\{\pi(e(k)) \in \mathcal{R}_k^{j,u}\} \geq p_u^j$  are used to tighten the constraints [21]:  $\bar{\mathcal{X}} \triangleq \bigcap_{j,k} \mathcal{X}_j \ominus \mathcal{R}_k^{j,x}, \bar{\mathcal{U}} \triangleq \bigcap_{j,k} \mathcal{U}_j \ominus \pi(\mathcal{R}_k^{j,u})$ .

A set-based CBF is then constructed for the nominal dynamics using the tightened constraints, and the safety filter optimization is formulated as

$$\min_v G(v + \pi(x - z(k)), u_{\text{des}}) \quad (20a)$$

$$\text{s.t. } v \in \bar{\mathcal{U}}, \quad (20b)$$

$$h(Az(k) + Bv) - h(z(k)) \geq -\Delta h(z(k)), \quad (20c)$$

with  $z(0) = x(0)$ . The resulting control law  $u(k) = v^* + \pi(x(k) - z(k))$  ensures chance constraint satisfaction using standard arguments from stochastic control and this concept can be tailored to various disturbance models and feedback structures [21].

## V. NUMERICAL SIMULATIONS

### A. Chain of Mass-Spring-Damper Systems

To demonstrate the scalability of our approach, we use a chain of 5 mass-spring-damper systems from [22, Sec. 5], i.e.,  $n_x = 10$  and  $n_u = 5$ . Instead of discretizing the continuous-time system using the Euler method, as done in [22], we use the exact discretization to obtain a more accurate discrete-time model. To apply Theorem 2, we choose the additive disturbance set  $\mathcal{W} \subset \mathbb{R}^{10}$  as unit box and use the modeling framework CVXPY [23] to solve the convex optimization problem. By combining [8] and [24], the computation of a zonotopic under-approximation of the maximal robust control invariant set  $\mathcal{D}$  takes 1 s, where the set has 50 generators. In contrast, computing the maximal (non-robust) control invariant set using [17] is aborted prematurely after 24 h.

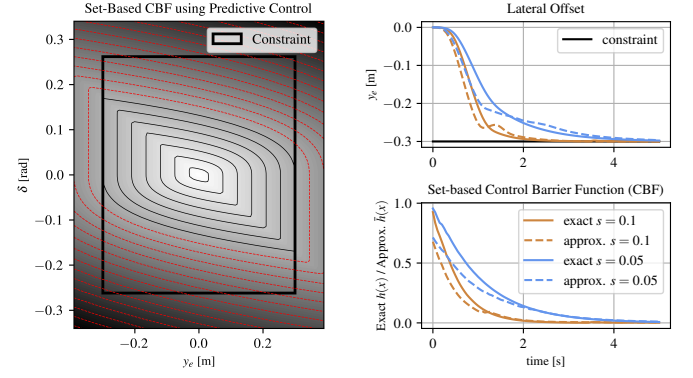


Fig. 3. Lateral motion control example. **Left:** Set-based control barrier function using predictive safe sets, plotted over steering angle  $\delta$  and lateral offset  $y_e$ . The color indicates values ranging from 1 (white) to smaller values (dark). Black contour lines represent safe areas with  $h(x) \geq 0$ , while red contour lines display  $h(x) < 0$ . The remaining states are set to zero. **Right:** Closed-loop trajectories under the safety filter (10) for a constant unsafe desired steering angle using the exact set-based CBF function and its approximation as described in Section IV-E for different damping functions  $\alpha(r) = s \cdot r$ .

In Fig. 1, we show the simulation results when choosing  $S = 0.9\mathcal{D}$ ,  $x_0 = [-9 \ 9 \ 0 \ \dots \ 0]^T \notin S$ , and sampling  $u_{\text{des}}(k) \in \mathbb{R}^5$  randomly from the input constraint set  $\mathcal{U}$ . In Fig. 2, we present the evolution of the zonotope-based CBFs  $h$  for different  $u_{\text{des}}(k)$  randomly sampled from  $\mathcal{U}$ . The system always converges to the safe set, and once inside, the desired damping behavior is evident when approaching  $h(x) = 0$ .

### B. Motion Control

To demonstrate the practical applicability of the set-based CBF framework, we consider a lateral motion control problem for an autonomous vehicle. The system is modeled as a discrete-time linearized single-track vehicle with six states and one input, subject to polytopic state and input constraints reflecting physical and safety requirements. The set-based CBF is defined by the feasible set of a nominal predictive controller, as described in Section IV-D, and compared to its neural network-based approximation (Section IV-E).

Figure 3 (left) shows the set-based CBF values across the relevant state space, illustrating the smooth transition at the safe set boundary. The right figure presents closed-loop trajectories under the safety filter (10) for a constant unsafe steering input, comparing the exact set-based CBF with a neural network approximation (two hidden layers, 2048 neurons each, trained on one million samples using PyTorch [25]). The results confirm that the proposed approach guarantees constraint satisfaction and enables systematic shaping of intervention behavior near the safety boundary. Moreover, the neural network approximation eliminates the need for expensive online optimization, providing real-time safety guarantees with minimal computational overhead. A video of this application can be found at <https://www.youtube.com/watch?v=zRYmDozDStU>.

## VI. EXPERIMENT: ELECTRIC MOTOR

This section illustrates a real-world application of the set-based CBF safety filter designed using a maximal control



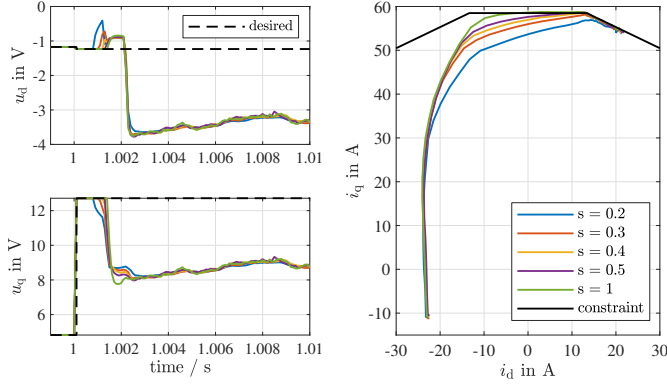


Fig. 4. Test bench results of an electric machine with a change of the desired voltages at time  $t = 1$  s for different CBF decrease bound parameters  $s \in [0.2, 1]$  defining  $\alpha(h(x)) = s \cdot h(x)$  in (6). While the safety does not interfere with the desired input initially, a smaller bound results in an earlier intervention with a stronger ‘damping’ effect near the safe boundary.

invariant set (Theorem 1). To this end, we consider the safe control of an electric motor on a test bench, requiring high sampling rates. The electric motor is a permanent magnet synchronous motor (PMSM) which, at fixed speeds, can be described by a discrete-time LTI model of the form (1) with states  $x = [i_d, i_q]^T \in \mathbb{R}^2$  and  $u = [u_d, u_q, 1]^T \in \mathbb{R}^3$  where  $i_d, i_q, u_d, u_q$  are the dq-components of the electric currents and voltages, respectively. The system is subject to polytope constraints on state and input  $\mathcal{X} = \{x | H_x x \leq h_x\}$  and  $\mathcal{U} = \{u | H_u u \leq h_u\}$ , respectively. For more information on modeling the system and constraints, and potential applications of safety filters for PMSMs, see [26].

The set-based CBF is constructed using the maximal positive control invariant set in H-representation (Section IV-A), yielding the set-based CBF  $h(x) = 1 - \gamma_\Omega(x)$  with  $\gamma_\Omega(x)$  as in (11). The CBF decrease bound is chosen as  $\alpha(h(x)) = s \cdot h(x)$  for  $s \in [0.2, 1]$  to illustrate its damping effect on closed-loop trajectories. The safety filter is implemented as a parametric quadratic program (QP) using (11), with parameters  $\theta = [x^T, u_{\text{des}}^T, h(x)]^T$ , and solved explicitly via the MPT toolbox [17], resulting in a piecewise affine control law with 127 regions.

On dSPACE1007 hardware, the average evaluation time is  $34.2 \mu\text{s}$ , well below the system’s sampling time of  $150 \mu\text{s}$ . Experimental results (Fig. 4) show that smaller  $s$  values yield earlier and more damped interventions near the safety boundary, while  $s = 1$  corresponds to a standard safety filter. This highlights the flexibility of the set-based CBF framework for tuning intervention behavior while ensuring real-time constraint satisfaction.

## VII. CONCLUSION

This article introduced a unifying set-based control barrier function (CBF) framework for safety filter design in linear systems with convex constraints. By leveraging scalable invariant sets, including those defined implicitly or via predictive control, the proposed approach enables systematic and efficient construction of CBFs for high-dimensional and data-driven systems. The framework supports real-time implementation

and learning-based approximations, while providing formal safety and stability guarantees. Numerical simulations on a high-dimensional mass-spring-damper system and a motion control task, along with experimental validation on an electric drive with strict timing requirements, demonstrate the method’s effectiveness, flexibility, and practical applicability to safety-critical control problems.

## APPENDIX A PROOF OF THEOREM 1

Parts a), b), and c) show basic properties of the set-based CBF, leading to invariance in part d):

a) *Safe set  $\mathcal{S}$  and  $h(x)$* : We first show that  $\Omega = \mathcal{S}$  is a safe set according to Definition 1, i.e.,  $x \in \Omega \Leftrightarrow h(x) \geq 0$  on the domain  $\mathbb{R}^{n_x}$ . Case  $x \in \Omega \Rightarrow h(x) \geq 0$ : For all  $x \in \Omega$  we note that  $\bar{\gamma} = 1$  is a feasible but suboptimal solution to (8) implying  $\gamma_\Omega(x) \leq 1$  and therefore  $h(x) = 1 - \gamma_\Omega(x) \geq 0$ . Case  $x \in \Omega \Leftarrow h(x) \geq 0$ : From  $h(x) \geq 0$  and (8) it follows that  $0 \leq \gamma_\Omega(x) \leq 1$  and  $x \in \gamma_\Omega(x)\Omega$ . Lemma 2 implies  $\gamma_\Omega(x)\Omega \subseteq \Omega$  and therefore  $x \in \Omega$ . To establish that  $\gamma_\Omega(x)$  is defined for all  $x \in \mathbb{R}^{n_x}$ , i.e., there exists a solution to (8), consider the case  $x \notin \Omega$ . Since  $0 \in \text{int}(\Omega)$ , there exists a ball of radius  $c > 0$  with respect to some norm  $\|\cdot\|$  around 0 that is contained in  $\Omega$ . Selecting  $\gamma = \|x\|c^{-1}$  implies  $x \in \gamma\Omega$  providing a candidate solution to (8) for all  $x \in \mathbb{R}^{n_x}$ , which implies feasibility for any domain  $\mathcal{D} \subseteq \mathbb{R}^{n_x}$ .

b) *Continuity of  $h(x)$* : Define  $\bar{r} \triangleq 2 \max_{x \in \mathcal{D}} \|x\|$ , which exists due to continuity of  $\|\cdot\|$  and compactness of  $\mathcal{D}$ . There exists a corresponding  $\bar{\gamma} > 0$  such that  $\mathcal{B}_n(\bar{r}) \subseteq \bar{\gamma}\Omega$  since  $0 \in \text{int}(\Omega)$ . It holds for all  $x \in \mathcal{D}$  that  $x \in \mathcal{B}_n(\|x\|) = \|x\|\bar{r}^{-1}\mathcal{B}(\bar{r})$  and due to Lemma 1 it follows that

$$x \in \|x\|\bar{r}^{-1}\bar{\gamma}\Omega.$$

This allows us to derive a continuous upper bound on  $\gamma_\Omega(x)$  around any  $x_0 \in \mathcal{D}$  by constructing a suboptimal solution  $\gamma_c(x, x_0)$  to (8) as follows. Let  $\gamma_c(x, x_0) = \gamma_c^{(1)}(x, x_0) + \gamma_c^{(2)}(x_0)$  resulting due to convexity of  $\Omega$  in  $\gamma_c(x, x_0)\Omega = \gamma_c^{(1)}(x, x_0)\Omega + \gamma_c^{(2)}(x_0)\Omega$ . Select  $\gamma_c^{(1)}(x, x_0) \triangleq \bar{\gamma}\bar{r}^{-1}\|x - x_0\|$  and  $\gamma_c^{(2)}(x_0) \triangleq \gamma_\Omega(x_0)$ . For any  $x, x_0 \in \mathcal{D}$  we verify that  $\gamma_c(x, x_0)$  is a feasible solution to (8) as follows:

$$x = \underbrace{x - x_0}_{\in \gamma_c^{(1)}(x, x_0)\Omega} + \underbrace{x_0}_{\in \gamma_c^{(2)}(x_0)\Omega} \in \gamma_c(x, x_0)\Omega.$$

Finally, given a  $x_0 \in \mathcal{D}$ , it follows for all  $x \in \mathcal{D}$  that

$$\|\gamma_\Omega(x) - \gamma_\Omega(\bar{x})\| \underset{\text{suboptimality}}{\leq} \|\gamma_c(x, \bar{x}) - \gamma_\Omega(\bar{x})\| = \bar{\gamma}\bar{r}^{-1}\|x - \bar{x}\|$$

showing local Lipschitz continuity of  $\gamma_\Omega(x)$  in  $\mathcal{D}$ .

c) *For all  $x \in \mathcal{S}$  the bound (6) holds*: Since  $\mathcal{S} = \Omega$  is control invariant, for any  $x \in \Omega$  we have  $x \in \gamma_\Omega(x)\Omega$ . From Lemma 3 it follows that there exists an input  $u \in \mathcal{U}$  such that  $Ax + Bu \in \gamma_\Omega(x)\Omega$ , implying  $\gamma_\Omega(Ax + Bu) \leq \gamma_\Omega(x)$  and therefore  $h(Ax + Bu) \geq h(x)$ . Since  $-\Delta h(x) \leq 0$  for all  $x \in \Omega$ , it follows that (5) holds.

d) *Forward invariance of  $\Omega$* : Follows directly from [6, Appendix B].

## APPENDIX B TECHNICALITIES

**Lemma 1.** Consider a set  $\Omega \subseteq \mathbb{R}^n$  and a ball  $\mathcal{B}_n(r)$  of radius  $r > 0$  such that  $\mathcal{B}_n(r) \subseteq \Omega$ . It holds that  $\lambda\mathcal{B}_n(r) \subseteq \lambda\Omega$  for all  $\lambda \geq 0$ .

*Proof.* For every  $y \in \lambda\mathcal{B}_n(r)$  we have by definition  $y = \lambda x$ ,  $x \in \mathcal{B}_n(r)$  and  $x \in \Omega$ . Since  $x \in \Omega$  it holds  $\lambda x \in \lambda\Omega$  by definition, i.e.,  $\lambda\Omega = \{\lambda z | z \in \Omega\}$ , completing the proof.  $\square$

**Lemma 2.** Consider a compact and convex set  $\mathcal{C} \subset \mathbb{R}^n$  with  $0 \in \mathcal{C}$ . For all  $\gamma \in [0, 1]$  it holds that  $\gamma\mathcal{C} \subseteq \mathcal{C}$ .

*Proof.* For every  $x \in \mathcal{C}$  we can verify  $\gamma x \in \mathcal{C}$  using convexity, i.e.,  $\gamma x = \gamma \cdot x + (1 - \gamma) \cdot 0$ , since  $0 \in \mathcal{C}$ .  $\square$

**Lemma 3.** Let  $\Omega \subset \mathbb{R}^n$  be a convex control invariant set with  $0 \in \Omega$  satisfying Definition 2. For all  $\gamma \in [0, 1]$  it follows that  $\gamma\Omega$  is a control invariant set according to Definition 2.

*Proof.* For every  $x \in \gamma\Omega$  there exists an input  $u$  such that  $Ax + Bu \in \Omega$ . For every  $y = \gamma x$  select the input  $v = \gamma u$ , yielding  $\gamma(Ax + Bu) \in \gamma\Omega$ , showing control invariance of  $\gamma\Omega$ .  $\square$

**Lemma 4.** For all  $\gamma \geq 0$  it holds that  $\gamma\mathcal{X}_N^1 = \mathcal{X}_N^\gamma$ .

*Proof.* In the case that  $\gamma = 0$  it follows from (14b) and (14f) that  $\mathcal{X}_N^0 = \{0\}$ , which equals  $0\mathcal{X}_N^1$  by definition. For  $\gamma > 0$  we have  $\gamma\mathcal{Z}_N^1 = \{(\gamma z, \{\gamma v_i\}) \mid (z, \{v_i\}) \in \mathcal{Z}_N^1\}$ , where we substitute  $(\gamma z, \{\gamma v_i\}) \triangleq (x, \{u_i\})$  to obtain

$$\gamma\mathcal{Z}_N^1 = \{(x, u_i) \mid (\gamma^{-1}x, \gamma^{-1}\{u_i\}) \in \mathcal{Z}_N^1\}. \quad (21)$$

Next, we rewrite (21) as

$$\left\{ (x, \{u_i\}) \mid \begin{array}{l} (A^N \gamma^{-1}x + \sum_{l=0}^{N-1} A^{N-l-1} B \gamma^{-1}u_l) \in \mathcal{X}_f, \\ \forall i \in \{0, \dots, N-1\}: \\ (A^i \gamma^{-1}x + \sum_{l=0}^{i-1} A^{i-l-1} B \gamma^{-1}u_l) \in \mathcal{X}_i, \\ \gamma^{-1}u_i \in \mathcal{U}_i \end{array} \right\} \\ = \mathcal{Z}_N^\gamma$$

using  $x_0 = x$  and by inserting the dynamics (14e). Through the relation  $\mathcal{Z}_N^\gamma = \gamma\mathcal{Z}_N^1$  we finally obtain

$$\begin{aligned} \mathcal{X}_N^\gamma &= \{x \mid (x, \{u_i\}) \in \gamma\mathcal{Z}_N^1\} \\ &= \{x \mid (\gamma^{-1}x, \{\gamma^{-1}u_i\}) \in \mathcal{Z}_N^1\} \\ &= \{\gamma z \mid (z, \{v_i\}) \in \mathcal{Z}_N^1\} = \gamma\mathcal{X}_N^1, \end{aligned}$$

showing the desired result.  $\square$

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