Probability Analysis of Monopoly

May 16, 2018

Throughout this document the following Jupyter notebook style and packages are used.

```
In [1]: %%html
        <link href="https://fonts.googleapis.com/css?family=Open+Sans" rel="stylesheet">
        <style>#notebook-container{font-size: 13pt;font-family:'Open Sans', sans-serif;} div.tex
```

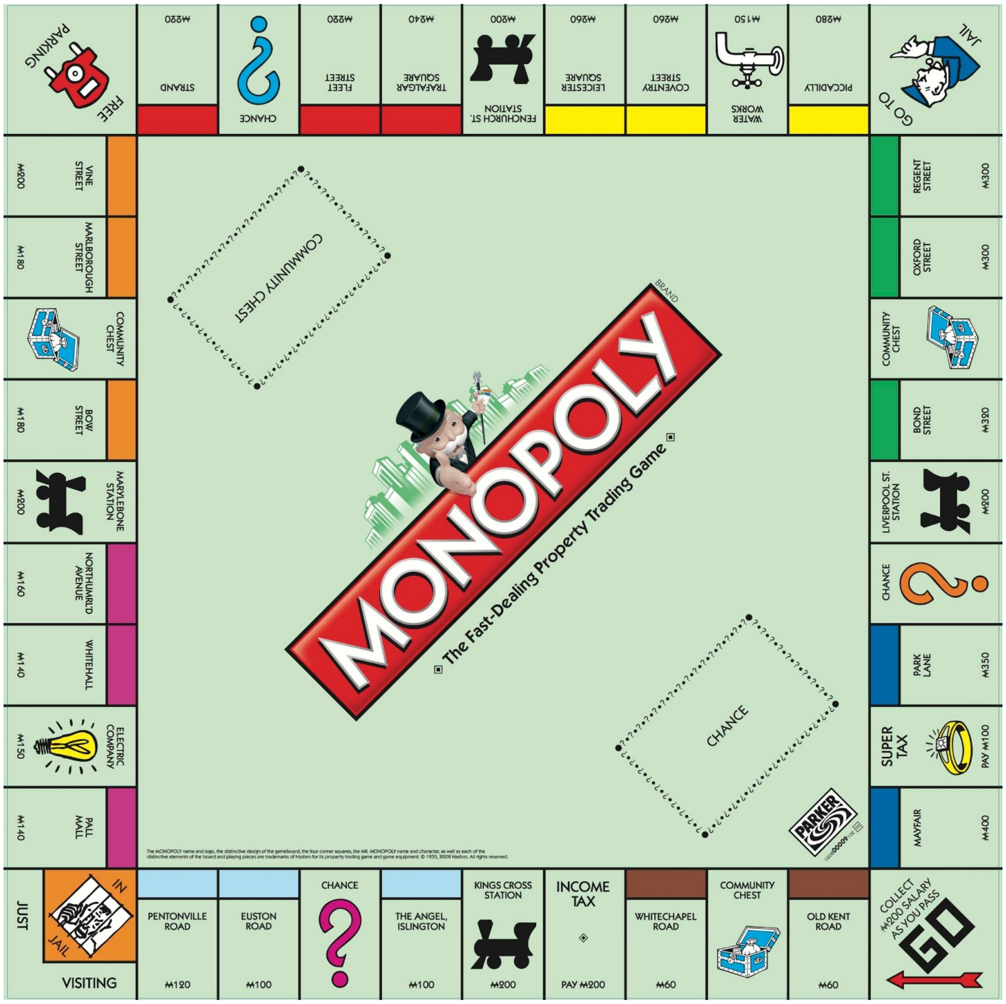
```
<IPython.core.display.HTML object>
```

```
In [2]: %pylab inline
```

```
Populating the interactive namespace from numpy and matplotlib
```

```
In [3]: import pandas as pd
```

Contents



monopoly

1 Goal

In this document we are going to analyze the probabilities in Monopoly to answer the question: which are the best houses to buy?

To answer this question we will create a simulated version of Monopoly and determine the probabilities to land on each square. Then we calculate the expected value for each square. The squares with the highest expected values are the best squares to have.

Finally we will create an exact version of our model with a Markov matrix. We use this to do an error analysis.

2 Game rules

2.1 Squares

Each edge has 9 positions and there are 4 edges. There are 4 corners. Which gives a total of 40 positions where a player can land. The labels are numbered starting at GO.

2.1.1 Labels

```
In [4]: squares_labels = ['start', 'b1', 'cc1', 'b2', 'it', 't1',
                          'lb1', 'c1', 'lb2', 'lb3', 'jail', 'p1',
                          'ec', 'p2', 'p3', 'ts2', 'o1', 'cc2',
                          'o2', 'o3', 'p', 'r1', 'c2', 'r2', 'r3',
                          'ts3', 'y1', 'y2', 'ww', 'y3', 'gtj',
                          'g1', 'g2', 'cc3', 'g3', 'ts4', 'c3',
                          'db1', 'st', 'db2']

        squares_total = len(squares_labels)
        print('There are {} squares.'.format(squares_total))
```

```
There are 40 squares.
```

2.1.2 Descriptions

We also want to know the proper names, so we don't have to look up the labels.

```
In [5]: squares_description = ['Start', 'Brown 1', 'Community Chest 1', 'Brown 2',
                               'Income Tax', 'Train Station 1', 'Light Blue 1',
                               'Chance 1', 'Light Blue 2', 'Light Blue 3', 'Jail',
                               'Purple 1', 'Electric Company', 'Purple 2',
                               'Purple 3', 'Train Station 2', 'Orange 1',
                               'Community Chest 2', 'Orange 2', 'Orange 3',
                               'Free Parking', 'Red 1', 'Chance 2', 'Red 2',
                               'Red 3', 'Train Station 3', 'Yellow 1', 'Yellow 2',
                               'Water Works', 'Yellow 3', 'Go to Jail', 'Green 1',
                               'Green 2', 'Community Chest 3', 'Green 3',
                               'Train Station 4', 'Chance 3', 'Dark Blue 1',
                               'Super Tax', 'Dark Blue 2']
```

```
In [6]: print('There are {} descriptions.'.format(len(squares_description)))
```

```
There are 40 descriptions.
```

2.1.3 Purchasable

We want to know if they are purchasable so we can sort on that later.

```
In [7]: squares_purchasable = [False, True, False, True, False,
                               True, True, False, True, True,
                               False, True, True, True, True,
                               True, True, False, True, True,
                               False, True, False, True, True,
                               True, True, True, True, True,
                               False, True, True, False, True,
                               True, False, True, False, True]
```

2.1.4 Rent

We want to use the rent paid at each square to calculate the expected value. The utility company charge 4 times roll if one is owned, and 10 times roll if both owned. For one railway we charge $25, two $50, three $100, and all four $200.

To find the rent for a utility company, we find the expected value for throwing a dices times 4.

$$4 \cdot E(\bar{k}) = 4 \cdot \frac{1}{6} \cdot (1+2+3+4+5+6)$$

```
In [8]: E_u = 4 * (1+2+3+4+5+6) / 6
```

We pick the value for one railway which is 25.

```
In [9]: E_r = 25
```

```
In [10]: squares_rent = [0, 2, 0, 4, 0, E_r, 6, 0, 6, 8, 0, 10, E_u,
                         10, 12, E_r, 14, 0, 14, 16, 0, 18, 0, 18,
                         20, E_r, 22, 22, E_u, 24, 0, 26, 26, 0, 28,
                         E_r, 0, 35, 0, 50]
```

2.1.5 Grouping

We want to know in what group they are so we can aggregate our data later.

```
In [11]: squares_aggregate = ['Start', 'Brown', 'Community Chest', 'Brown',
                              'Income Tax', 'Train Station', 'Light Blue',
                              'Chance', 'Light Blue', 'Light Blue', 'Jail',
                              'Purple', 'Utilities', 'Purple', 'Purple',
                              'Train Station', 'Orange', 'Community Chest',
                              'Orange', 'Orange', 'Free Parking', 'Red',
                              'Chance', 'Red', 'Red', 'Train Station', 'Yellow',
                              'Yellow', 'Utilities', 'Yellow', 'Go to Jail',
                              'Green', 'Green', 'Community Chest', 'Green',
                              'Train Station', 'Chance', 'Dark Blue',
                              'Super Tax', 'Dark Blue']
```

2.2 Cards

There are two decks of cards.

- Community Cards
- Chance Cards

Each deck contains 16 cards.

2.2.1 Community Cards

Monopoly has 16 community cards.

cc

Because we are determining the probabilities, we are only interested in the following cards:

- advance to go
- go to jail
- get out of jail, free
- go back 2 spaces

2.2.2 Community deck implementation

We implement the community deck in a class. The class keeps track of a list with 16 cards. An index points to the next card. When we are out of cards, we reset the index and reshuffle the cards.

```
In [12]: from random import shuffle

         class CommunityDeck():
             def __init__(self):
                 self.deck = [0] * 16
                 self.deck[0] = 'gtg' # go to go
                 self.deck[1] = 'gtj' # go to jail
                 self.deck[2] = 'goj' # get out of jail
                 self.deck[3] = 'gb2' # go back 2 steps
                 self.index = 16

             def draw_card(self):
                 if self.index >= len(self.deck):
                     self.index = 0
```

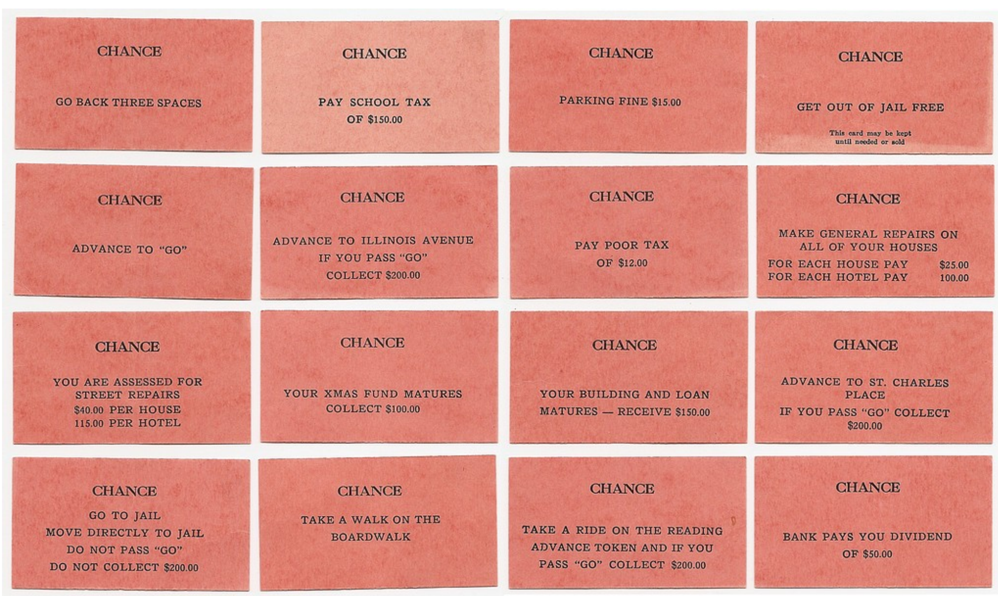
```
            shuffle(self.deck)
        card = self.deck[self.index]
        self.index += 1
        return card
```

Now we test it:

```
In [13]: deck = CommunityDeck()
         deck.deck

Out[13]: ['gtg', 'gtj', 'goj', 'gb2', 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]

In [14]: deck.draw_card()

Out[14]: 0

In [15]: deck.deck

Out[15]: [0, 0, 0, 'gb2', 0, 'goj', 0, 'gtg', 0, 0, 0, 0, 'gtj', 0, 0, 0]

In [16]: deck.index

Out[16]: 1
```

2.2.3 Chance Cards

Monopoly has 16 chance cards.



chance

Because we are determining the probabilities, we are only interested in the following cards:

- go back three spaces
- get out of jail free
- advance to go
- advance to illinois avenue (R3)
- go to jail

2.2.4 Chance deck implementation

We implement the chance deck in a class. The class keeps track of a list with 16 cards. An index points to the next card. When we are out of cards, we reset the index and reshuffle the cards.

```
In [17]: from random import shuffle

         class ChanceDeck():
             def __init__(self):
                 self.deck = [0] * 16
                 self.deck[0] = 'gtg' # go to go
                 self.deck[1] = 'gtj' # go to jail
                 self.deck[2] = 'goj' # get out of jail
                 #self.deck[3] = 'gb3' # go back 3
                 self.deck[4] = 'r3'  # go to red 3 (r3)
                 self.index = 16

             def draw_card(self):
                 if self.index >= len(self.deck):
                     self.index = 0
                     shuffle(self.deck)
                 card = self.deck[self.index]
                 self.index += 1
                 return card
```

Now we test it:

```
In [18]: deck = ChanceDeck()
         deck.deck

Out[18]: ['gtg', 'gtj', 'goj', 0, 'r3', 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]

In [19]: deck.draw_card()

Out[19]: 0

In [20]: deck.deck

Out[20]: [0, 0, 'gtj', 'gtg', 'goj', 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 'r3']

In [21]: deck.index

Out[21]: 1
```

2.3 Dice

We will be implementing the dice as a class. This allows us to encapsulate how the result are determined. It makes it easier to implements other scenarios such as throwing with multiple dices.

```
In [22]: from random import randint

         class Dice():
             def __init__(self, dices = 1, sides = 6):
                 self.dices = dices
                 self.sides = sides

             def throw(self):
                 return sum([randint(1, self.sides) for _ in range(self.dices)])
```

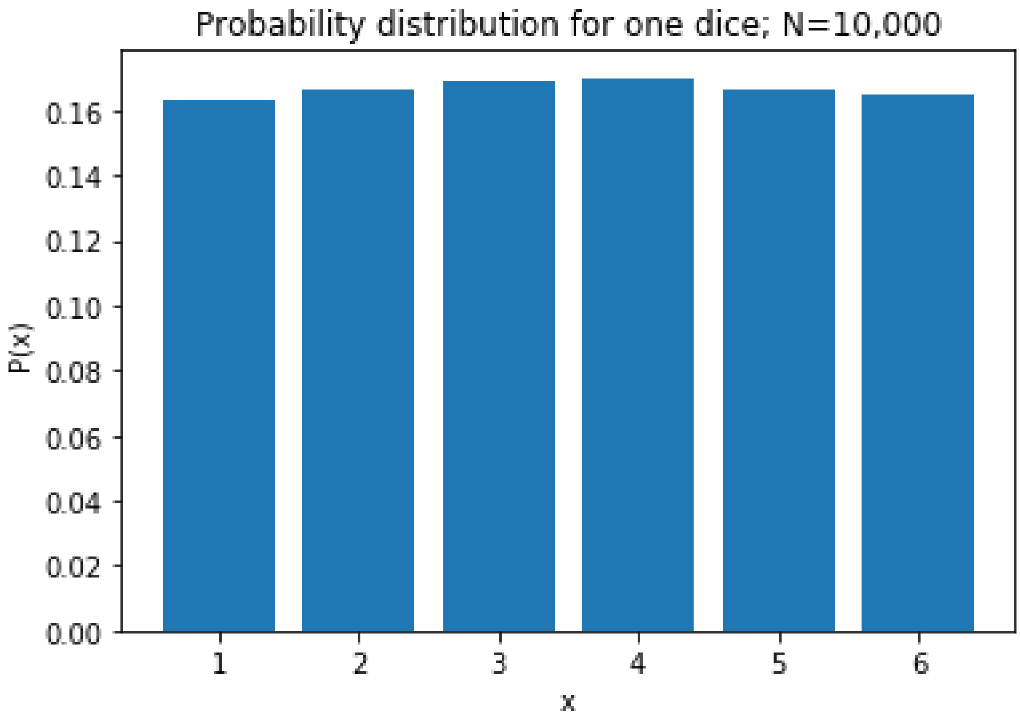
Rolling one time:

```
In [23]: dice = Dice()
         dice.throw()

Out[23]: 5
```

2.3.1 Simple: one dice with six sides

A simple setup would be one dice with six sides. This will give uniformly distributed probabilities.

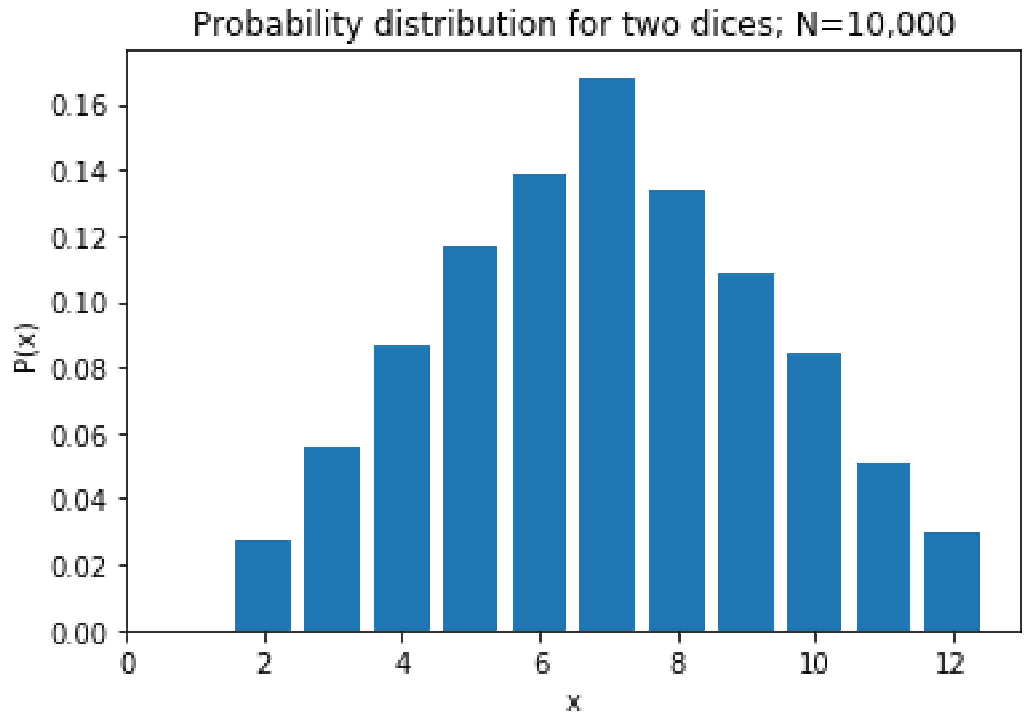
```
In [24]: dice = Dice()
         sides = [0] * dice.dices * dice.sides
         N = 10000
         for i in range(N): sides[dice.throw()-1] += 1
         sides = np.array(sides) / N
         bar(range(1,len(sides)+1), sides);
         ylabel('P(x)')
         xlabel('x')
         title('Probability distribution for one dice; N=10,000');
```



2.3.2 Advanced: two dice with six sides

Monopoly is played with two dices. This will give the following probability distribution:

```
In [25]: dice = Dice(2)
         sides = [0] * dice.dices * dice.sides
         N = 10000
         for i in range(N): sides[dice.throw()-1] += 1
         sides = np.array(sides) / N
         bar(range(1,len(sides)+1), sides);
         ylabel('P(x)')
         xlabel('x')
         title('Probability distribution for two dices; N=10,000');
```



3 Simulated model

We are playing a simplified version of Monopoly. We will not keep track of money. There will only be one player. If a player goes to jail, the player can continue immediately on the next turn. We will keep track of the card decks. The game will be played with 2 dices. We only count when we land on a square. If we are moved to jail for example, the next round will continue from that new position.

3.1 Algorithm

Here we are going to simulate a game for N amount of rounds. The game algorithm is simple:

1. Roll the dices (there are 2 dices with 6 squares)
2. Move to the new position
3. Increment the square counter for that position
4. Check and handle go to jail
5. Check and handle community chest
6. Check and handle chance

3.2 Modulo arithmetic for position tracking

We can easily keep track of our position with modulo arithemetic. Let C be our position (or `index`), d the result from throwing the dice, and n the current round. To determine our new position we calculate:

$$C_{n+1} \equiv C_n + d \pmod{40}$$

The modulo is 40 because that are the total amount of squares.

3.3 Implementation

Below is the implementation for the Monopoly simulation.

In [26]:
```
dice = Dice(2)
community_deck = CommunityDeck()
chance_deck = ChanceDeck()

index = 0 # position
total_squares = len(squares_labels)
squares = [0] * total_squares

rounds = 1000000 # N

for i in range(rounds):

    # Throw the dice and move our position on the board.
    steps = dice.throw()
    index = (index + steps) % total_squares

    # We landed on go to jail.
    if squares_labels[index] is 'gtj':
        index = squares_labels.index('jail')

    # We landed on the community card.
    if squares_labels[index] in ['cc1', 'cc2', 'cc3']:
        card = community_deck.draw_card()
        if card is 'gtg': index = squares_labels.index('start')
        if card is 'gtj': index = squares_labels.index('jail')
        if card is 'gb2':
            if index >= 2: index -= 2
            if index < 2: index = total_squares-abs(index-2)-1

    # We landed on the chance card.
    if squares_labels[index] in ['c1', 'c2', 'c3']:
        card = chance_deck.draw_card()
        if card is 'gtg': index = squares_labels.index('start')
        if card is 'gtj': index = squares_labels.index('jail')
        if card is 'r3': index = squares_labels.index('r3')
        if card is 'gb3':
            if index >= 3: index -= 3
            if index < 3: index = total_squares-abs(index-3)-1
```

```
        # Update the counter
        squares[index] += 1
```

It takes around 2.7 seconds to run a game when $N = 1,000,000$. Because there is only one loop the algorithm will scale linearly.

4 Probability statistics

Now we can proceed to analyze our results.

4.1 Determining probabilities

With the number of times that each square is visited, and the total rounds N we can calculate the probabilities. The probability that a square is visited is:

$$P(\bar{x} = x) = \frac{\text{Times visited}}{\text{N}}$$

We can calculate the expected value for each square in terms of money with:

$$E(\bar{k}) = P(\bar{x} = x) \cdot \text{Rent}$$

We also want to create a `DataFrame` in Python to easily keep track of everything.

```
In [27]: df = pd.DataFrame(index=range(total_squares))
         df['Square'] = squares_labels
         df['Description'] = squares_description
         df['Purchasable'] = squares_purchasable
         df['Rent'] = squares_rent
         df['Visited'] = squares
         df['Probability'] = df['Visited'] / rounds
         df['Aggregate'] = squares_aggregate
         df['Expected Value'] = df['Probability'] * df['Rent']
```

We can calculate a quick summary about the data:

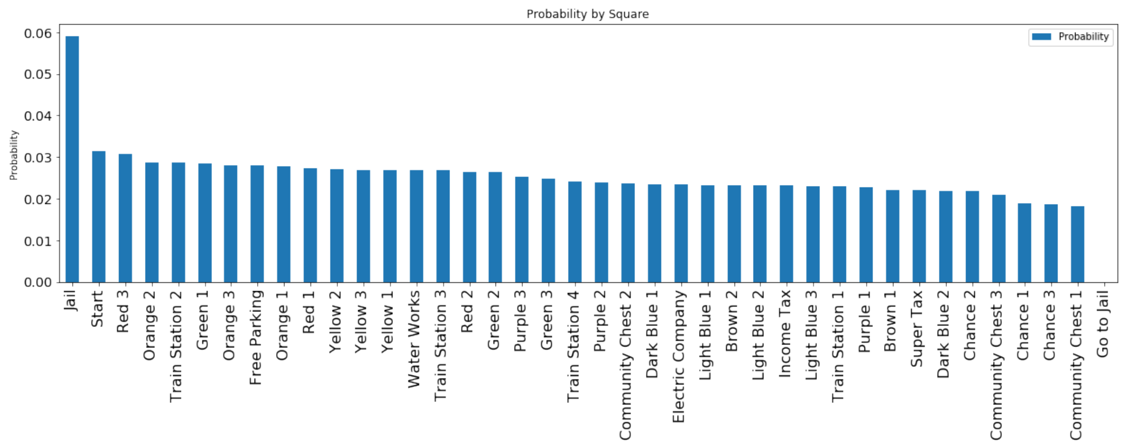
```
In [28]: print('Total rounds: {}'.format(rounds))
         print('Visited avg:  {}'.format(df['Visited'].mean()))
         print('Visited min:  {}'.format(df['Visited'].min()))
         print('Visited max:  {}'.format(df['Visited'].max()))
         print('Visited std:  {:.2f}'.format(df['Visited'].std()))
```

```
Total rounds: 1000000
Visited avg:  25000.0
Visited min:  0
Visited max:  59078
Visited std:  7443.10
```

4.2 Plot of probabilities by square

If we sort these values descending on the probability, we can easily see which squares have the highest probability to be visited.

```
In [29]: plt.rc('xtick', labelsize=16)
         plt.rc('ytick', labelsize=14)
         df[['Description', 'Probability']].sort_values(by='Probability', ascending=False)\
             .plot(kind='bar', figsize=(20,5))
         plt.xticks(range(total_squares), df[['Description', 'Probability']]
             .sort_values(by='Probability', ascending=False)['Description'])
         plt.ylabel('Probability')
         plt.title('Probability by Square');
```



Here we can conclude that `Orange 1` is the most visited square. Also notice that `Orange 2` and `Orange 3` are pretty high. It seems that `Orange` is the best street to have.

4.3 Table of probabilities by square

Below is the full table with all the squares and their corresponding values.

```
In [30]: df.loc[:, df.columns.isin(['Square', 'Description', 'Probability'])].sort_values(by='Pr

Out[30]:     Square       Description  Probability
         10    jail              Jail     0.059078
         0    start             Start     0.031421
         24      r3             Red 3     0.030681
         18      o2          Orange 2     0.028676
         15     ts2   Train Station 2     0.028584
         31      g1           Green 1     0.028344
         19      o3          Orange 3     0.027934
         20       p      Free Parking     0.027895
         16      o1          Orange 1     0.027742
         21      r1             Red 1     0.027343
```

27	y2	Yellow 2	0.027192
29	y3	Yellow 3	0.026966
26	y1	Yellow 1	0.026950
28	ww	Water Works	0.026825
25	ts3	Train Station 3	0.026798
23	r2	Red 2	0.026371
32	g2	Green 2	0.026300
14	p3	Purple 3	0.025249
34	g3	Green 3	0.024771
35	ts4	Train Station 4	0.024023
13	p2	Purple 2	0.023990
17	cc2	Community Chest 2	0.023722
37	db1	Dark Blue 1	0.023417
12	ec	Electric Company	0.023359
6	lb1	Light Blue 1	0.023246
3	b2	Brown 2	0.023238
8	lb2	Light Blue 2	0.023200
4	it	Income Tax	0.023191
9	lb3	Light Blue 3	0.023079
5	t1	Train Station 1	0.023079
11	p1	Purple 1	0.022665
1	b1	Brown 1	0.022156
38	st	Super Tax	0.022138
39	db2	Dark Blue 2	0.021942
22	c2	Chance 2	0.021792
33	cc3	Community Chest 3	0.020822
7	c1	Chance 1	0.018835
36	c3	Chance 3	0.018737
2	cc1	Community Chest 1	0.018249
30	gtj	Go to Jail	0.000000

4.4 Top 10 highest probability squares

The top 10 squares that have the highest probability for a player to land on are:

```
In [31]: df.loc[df['Purchasable'] == True, df.columns.isin(['Square', 'Description', 'Probabilit
             .sort_values('Probability', ascending=False).head(10)
```

Out[31]:

	Square	Description	Probability
24	r3	Red 3	0.030681
18	o2	Orange 2	0.028676
15	ts2	Train Station 2	0.028584
31	g1	Green 1	0.028344
19	o3	Orange 3	0.027934
16	o1	Orange 1	0.027742
21	r1	Red 1	0.027343
27	y2	Yellow 2	0.027192
29	y3	Yellow 3	0.026966
26	y1	Yellow 1	0.026950

The total probability for all 10 squares is:

```
In [32]: df.loc[df['Purchasable'] == True].sort_values('Probability', ascending=False)\
             .head(10)['Probability'].sum()
```

```
Out[32]: 0.28041199999999999
```

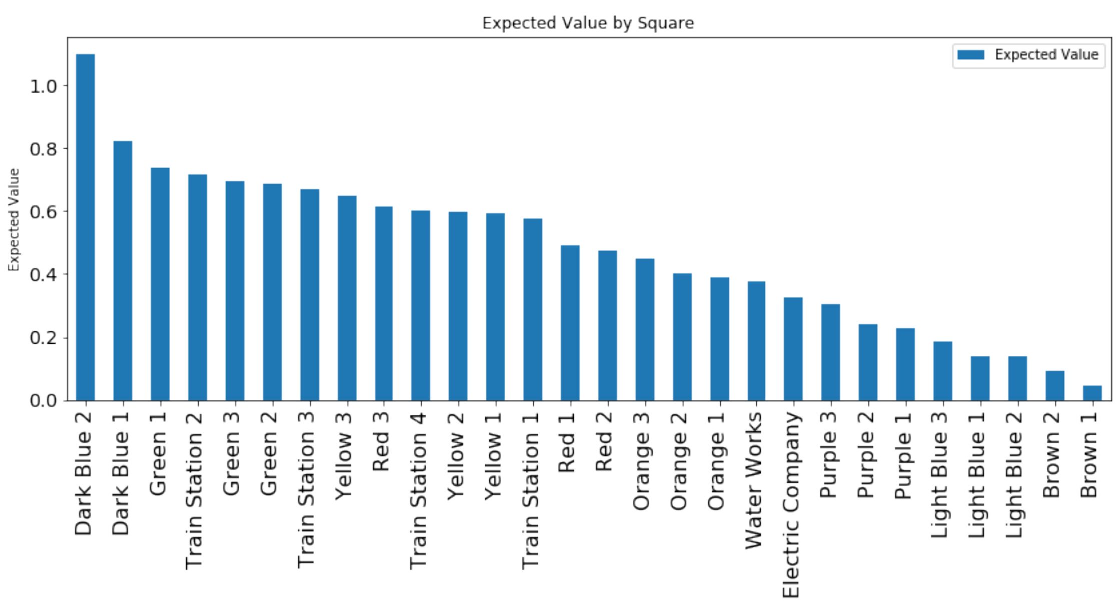
4.5 Plot of expected value per turn

Now we want to know how much each square generates per turn with the found probabilities and the rent the player needs to pay when we land on it. First we make a selection to only get the purchasable squares.

```
In [33]: rentable_df = pd.DataFrame(df.loc[df['Purchasable'] == True]).reindex()
```

Plotting this gives:

```
In [34]: plt.rc('xtick', labelsize=16)
         plt.rc('ytick', labelsize=14)
         rentable_df[['Description', 'Expected Value']].sort_values(by='Expected Value', ascendi
             .plot(kind='bar', figsize=(14,5))
         plt.xticks(range(len(rentable_df.index)), rentable_df[['Description', 'Expected Value']
             .sort_values(by='Expected Value', ascending=False)['Description'])
         plt.ylabel('Expected Value')
         plt.title('Expected Value by Square');
```



4.6 Table of expected value per turn

The full table of expected values is below.

```
In [35]: rentable_df.loc[:, rentable_df.columns.isin(['Square', 'Description', 'Rent', 'Probabil
         .sort_values('Expected Value', ascending=False)
```

Out[35]:

	Square	Description	Rent	Probability	Expected Value
39	db2	Dark Blue 2	50.0	0.021942	1.097100
37	db1	Dark Blue 1	35.0	0.023417	0.819595
31	g1	Green 1	26.0	0.028344	0.736944
15	ts2	Train Station 2	25.0	0.028584	0.714600
34	g3	Green 3	28.0	0.024771	0.693588
32	g2	Green 2	26.0	0.026300	0.683800
25	ts3	Train Station 3	25.0	0.026798	0.669950
29	y3	Yellow 3	24.0	0.026966	0.647184
24	r3	Red 3	20.0	0.030681	0.613620
35	ts4	Train Station 4	25.0	0.024023	0.600575
27	y2	Yellow 2	22.0	0.027192	0.598224
26	y1	Yellow 1	22.0	0.026950	0.592900
5	t1	Train Station 1	25.0	0.023079	0.576975
21	r1	Red 1	18.0	0.027343	0.492174
23	r2	Red 2	18.0	0.026371	0.474678
19	o3	Orange 3	16.0	0.027934	0.446944
18	o2	Orange 2	14.0	0.028676	0.401464
16	o1	Orange 1	14.0	0.027742	0.388388
28	ww	Water Works	14.0	0.026825	0.375550
12	ec	Electric Company	14.0	0.023359	0.327026
14	p3	Purple 3	12.0	0.025249	0.302988
13	p2	Purple 2	10.0	0.023990	0.239900
11	p1	Purple 1	10.0	0.022665	0.226650
9	lb3	Light Blue 3	8.0	0.023079	0.184632
6	lb1	Light Blue 1	6.0	0.023246	0.139476
8	lb2	Light Blue 2	6.0	0.023200	0.139200
3	b2	Brown 2	4.0	0.023238	0.092952
1	b1	Brown 1	2.0	0.022156	0.044312

5 Grouped probability statistics

We want to answer the following questions:

1. What are the best streets to have?
2. What is the probability to be in jail?
3. What is the probability to draw a card?

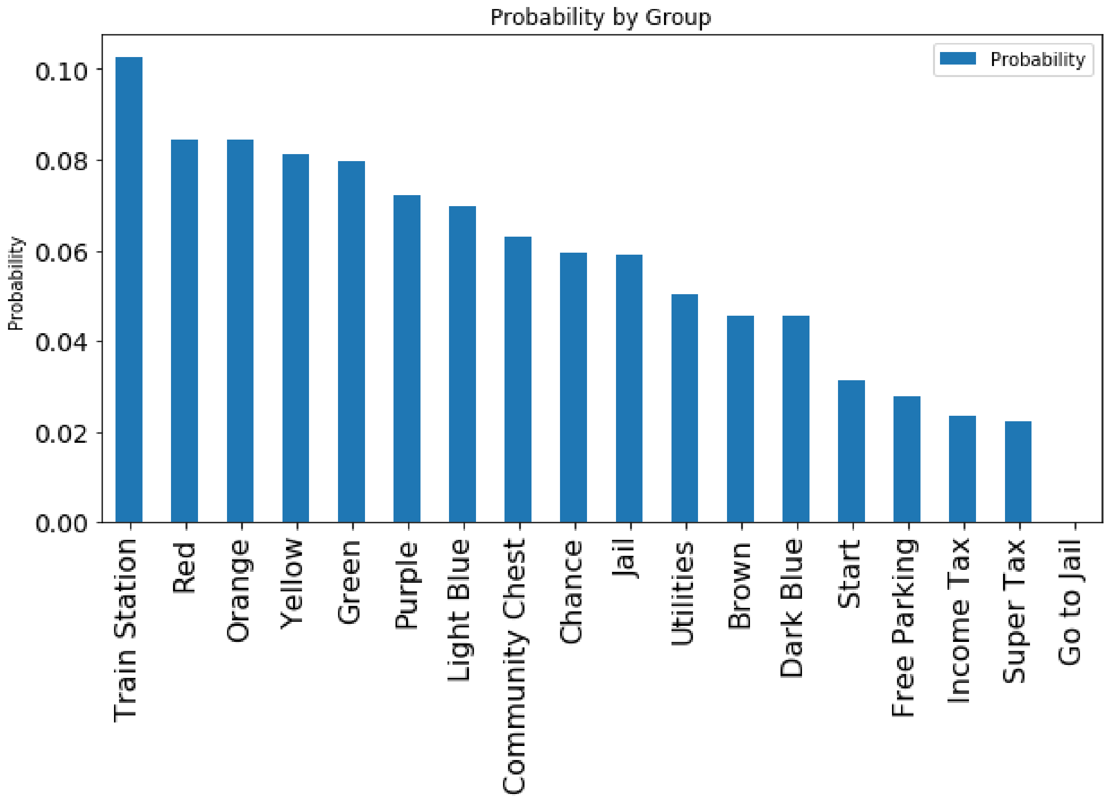
5.1 Plot of probabilities by group

To find what the probabilities are per street, chance, community chest, etc., we are going to aggregate the possibilities.

```
In [36]: aggregated_df = pd.DataFrame(df.groupby(['Aggregate'])['Probability'].sum()).reset_inde
```

Now we plot the aggregated probabilities.

```
In [37]: plt.rc('xtick', labelsize=16)
         plt.rc('ytick', labelsize=14)
         aggregated_df[['Aggregate', 'Probability']].sort_values(by='Probability', ascending=Fal
             .plot(kind='bar', figsize=(10,5))
         plt.xticks(range(len(aggregated_df.index)), aggregated_df[['Aggregate', 'Probability']]
             .sort_values(by='Probability', ascending=False)['Aggregate'])
         plt.ylabel('Probability')
         plt.title('Probability by Group');
```



5.2 Table of probabilities by group

A total overview of all the probabilities can be found in the table below:

```
In [38]: aggregated_df.sort_values('Probability', ascending=False)

Out[38]:          Aggregate  Probability
         15   Train Station     0.102484
         12             Red     0.084395
```

```
10           Orange     0.084352
17           Yellow     0.081108
6             Green     0.079415
11           Purple     0.071904
9        Light Blue     0.069525
2   Community Chest     0.062793
1            Chance     0.059364
8              Jail     0.059078
16        Utilities     0.050184
0             Brown     0.045394
3         Dark Blue     0.045359
13            Start     0.031421
4      Free Parking     0.027895
7        Income Tax     0.023191
14        Super Tax     0.022138
5        Go to Jail     0.000000
```

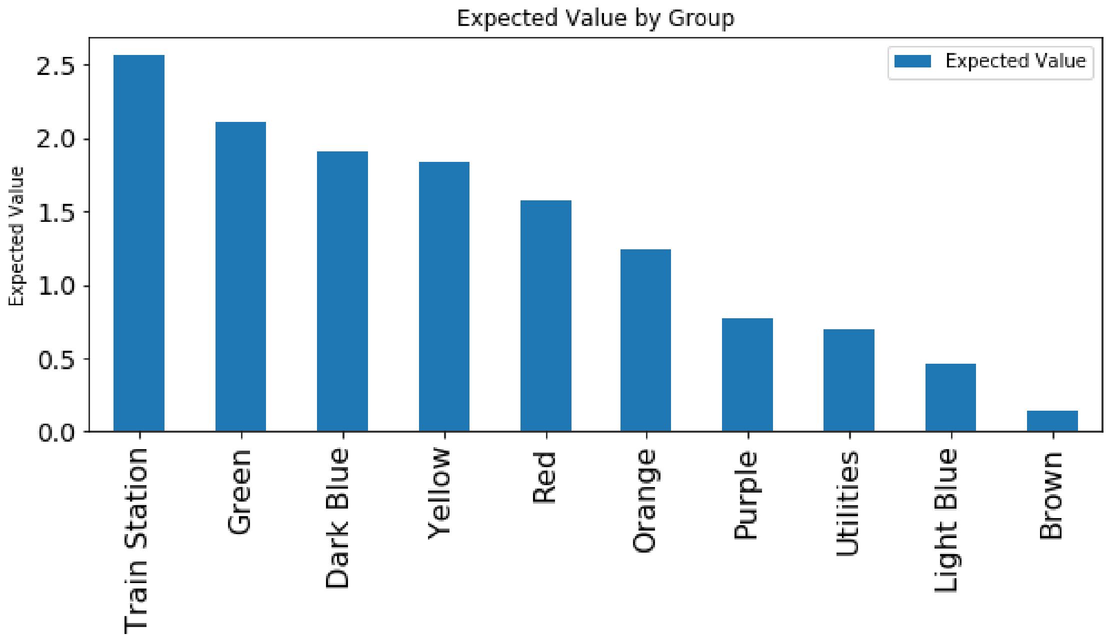
5.3 Plot of expected values by group

If we find the expected values by each aggregate we can find out which group generated the most money per turn.

```
In [39]: aggregated_ev_df = pd.DataFrame(df.loc[df['Purchasable'] == True]\
             .groupby(['Aggregate'])['Expected Value'].sum()).reset_index()
```

Plotting this gives:

```
In [40]: plt.rc('xtick', labelsize=16)
         plt.rc('ytick', labelsize=14)
         aggregated_ev_df[['Aggregate', 'Expected Value']].sort_values(by='Expected Value', asce
             .plot(kind='bar', figsize=(10,4))
         plt.xticks(range(len(aggregated_ev_df.index)), aggregated_ev_df[['Aggregate', 'Expected
             .sort_values(by='Expected Value', ascending=False)['Aggregate'])
         plt.ylabel('Expected Value')
         plt.title('Expected Value by Group');
```



We can conclude that the `Train Station` yields the most. This are however 4 squares. The best street to have is `Green`.

5.4 Table of expected values by group

A full table of expected values can be found below

```
In [41]: aggregated_ev_df.sort_values('Expected Value', ascending=False)
```

Out[41]:

	Aggregate	Expected Value
7	Train Station	2.562100
2	Green	2.114332
1	Dark Blue	1.916695
9	Yellow	1.838308
6	Red	1.580472
4	Orange	1.236796
5	Purple	0.769538
8	Utilities	0.702576
3	Light Blue	0.463308
0	Brown	0.137264

6 Other probabilities

6.1 Train station probabilities

We can conclude that `Train Station` has the highest probability to land on. However, we need to take into account that there are four squares to land on.

```
In [42]: df.loc[df['Aggregate'] == 'Train Station', df.columns.isin(['Square', 'Description', 'P
             .sort_values('Probability', ascending=False)
```

```
Out[42]:    Square      Description  Probability  Expected Value
         15     ts2  Train Station 2     0.028584        0.714600
         25     ts3  Train Station 3     0.026798        0.669950
         35     ts4  Train Station 4     0.024023        0.600575
         5       t1  Train Station 1     0.023079        0.576975
```

6.2 Probability to be in jail

To find the total probability to be in jail, we need to take into account that:

- We can land on jail.
- We can land on go to jail.
- There is one community card which sends you to jail.
- There is one chance card which sends you to jail.

Each deck has 16 cards, therefore the probability to draw go to jail is $P(\bar{x} = \text{go to jail}) = \frac{1}{16}$.

$$P(\bar{x} = \text{in jail}) = P(\bar{x} = \text{jail}) + P(\bar{x} = \text{go to jail}) + \frac{1}{16}\left[P(\bar{x} = \text{community chest}) + P(\bar{x} = \text{chance})\right]$$

```
In [43]: P_jail           = sum(aggregated_df.loc[aggregated_df['Aggregate'] == 'Jail'
         P_go_to_jail     = sum(aggregated_df.loc[aggregated_df['Aggregate'] == 'Go to Jail'
         P_community_card = sum(aggregated_df.loc[aggregated_df['Aggregate'] == 'Community Chest
         P_chance_card    = sum(aggregated_df.loc[aggregated_df['Aggregate'] == 'Chance'
```

```
In [44]: P_jail + P_go_to_jail + 1/16 * (P_community_card + P_chance_card)
```

```
Out[44]: 0.06671281249999996
```

6.3 Probability to draw a card

To find the probability to draw a card, we simply calculate:

$$P(\bar{x} = \text{draw a card}) = P(\bar{x} = \text{community chest}) + P(\bar{x} = \text{chance})$$

```
In [45]: P_community_card + P_chance_card
```

```
Out[45]: 0.122157
```

Where the probabilities for the community chest square are:

```
In [46]: df.loc[df['Aggregate'] == 'Community Chest', df.columns.isin(['Square', 'Description',
             .sort_values('Probability', ascending=False)
```

```
Out[46]:    Square        Description  Probability
         17     cc2  Community Chest 2     0.023722
         33     cc3  Community Chest 3     0.020822
         2      cc1  Community Chest 1     0.018249
```

And the probabilities for the chance square are:

```
In [47]: df.loc[df['Aggregate'] == 'Chance', df.columns.isin(['Square', 'Description', 'Probabil
             .sort_values('Probability', ascending=False)
```

```
Out[47]:     Square Description  Probability
         22      c2     Chance 2     0.021792
         7       c1     Chance 1     0.018835
         36      c3     Chance 3     0.018737
```

7 Markov matrix model

Another way to find the probabilities, would be to create a Markov matrix with the probability to go another square, for each individual square. This will be a 40×40 square matrix M.

```
In [56]: M = np.eye(40)
```

```
In [57]: M
```

```
Out[57]: array([[ 1.,  0.,  0., ...,  0.,  0.,  0.],
                [ 0.,  1.,  0., ...,  0.,  0.,  0.],
                [ 0.,  0.,  1., ...,  0.,  0.,  0.],
                ...,
                [ 0.,  0.,  0., ...,  1.,  0.,  0.],
                [ 0.,  0.,  0., ...,  0.,  1.,  0.],
                [ 0.,  0.,  0., ...,  0.,  0.,  1.]])
```

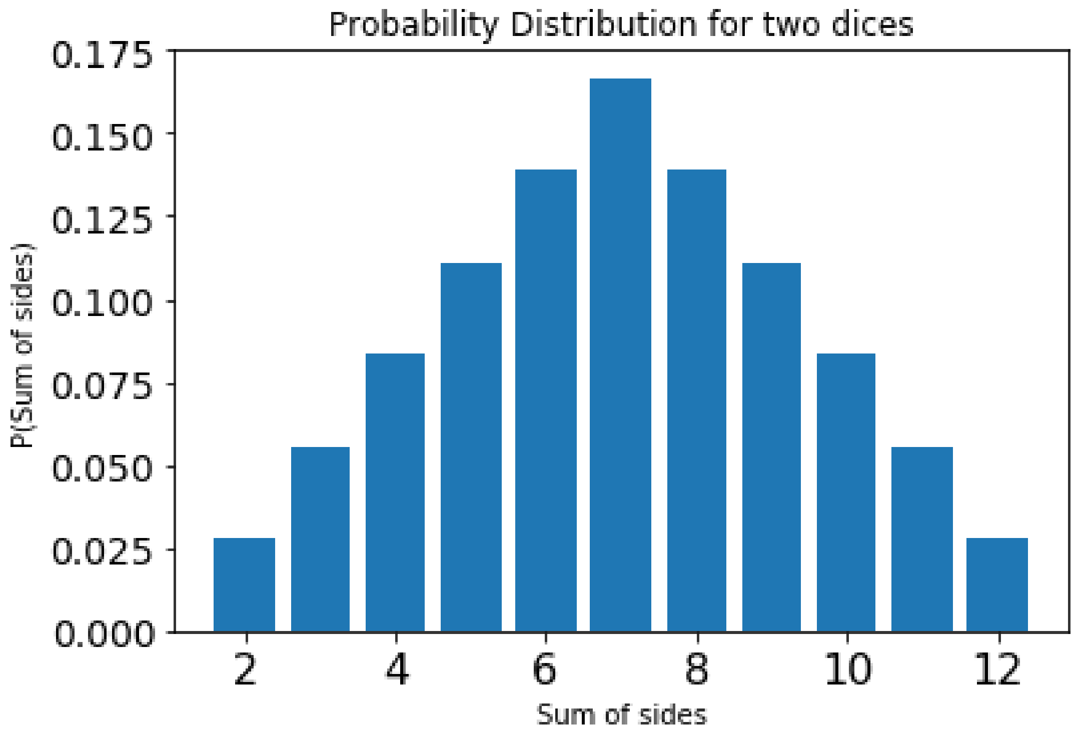
Then we need to fill in the matrix. For each square we are going to calculate the probabilities to get on any other square. For the most the probability will be 0 because we can't throw high enough numbers. We also need to take into the account that there are probabilities that are generated by the cards and add those to the respecting elements.

When we have filled the matrix with all the values we let the matrix converge with $\lim_{n\to\infty} M^n$. A simple way to approximate this would be to square the matrix a few times, like M^{64}. The resulting matrix will have the probabilities for every square.

7.1 Probabilities for throwing two dices

To fill our matrix M we first need to find the probability distribution for two dices.

```
In [58]: outcomes = [x+y for x in range(1,7) for y in range(1,7)]
         probabilities = {}
         for o in set(outcomes): probabilities[o] = outcomes.count(o) / len(outcomes)
         side, p = probabilities.keys(), probabilities.values()
         bar(side, p)
         xlabel('Sum of sides')
         ylabel('P(Sum of sides)')
         title('Probability Distribution for two dices');
```



The range of values for two dices is:

```
In [59]: throw_min = min(side)
         throw_max = max(side)
         throw_range = throw_max - throw_min
         print('Outcomes are min: {}, max: {} and range: {}.'.format(throw_min, throw_max, throw
         prob_df = pd.DataFrame()
         prob_df['Sum of sides'] = side
         prob_df['P(Sum of sides)'] = p
```

```
Outcomes are min: 2, max: 12 and range: 10.
```

7.2 Probabilities for any position

We want to create a matrix row for each tile. We want to calculate the probability to land on another square, calculated for all squares. We also want to use the modulo arithmetic here to wrap our position around the board.

```
In [60]: def get_probability_for_throw(n):
             if n < 2 or n > 12: return 0
             return probabilities[n]

         def get_probabilities_for_position(pos):
             squares = [0] * 40
```

```
for i in range(40):
    squares[(pos+i)%40] = get_probability_for_throw(i)

# Go to jail
squares[10] += squares[2] * 1/16  # Go to jail community card 1
squares[10] += squares[7] * 1/16  # Go to jail chance card 1
squares[10] += squares[17] * 1/16 # Go to jail community card 2
squares[10] += squares[22] * 1/16 # Go to jail chance card 2
squares[10] += squares[33] * 1/16 # Go to jail community card 3
squares[10] += squares[36] * 1/16 # Go to jail chance card 3

# Go to start
squares[0] += squares[2] * 1/16   # Go to start community card 1
squares[0] += squares[7] * 1/16   # Go to start chance card 1
squares[0] += squares[17] * 1/16  # Go to start community card 2
squares[0] += squares[22] * 1/16  # Go to start chance card 2
squares[0] += squares[33] * 1/16  # Go to start community card 3
squares[0] += squares[36] * 1/16  # Go to start chance card 3

# Go back 3 (this creates a loophole because you go back to the community
# chest, and those probabilities need be recalculated again, for which I am too laz
#squares[4]  += squares[7] * 1/16   # Go back 3 chance card 1
#squares[19] += squares[22] * 1/16 # Go back 3 chance card 2
#squares[33] += squares[36] * 1/16 # Go back 3 chance card 3

# Go back 2
squares[0]  += squares[2] * 1/16   # Go back 2 community card 1
squares[15] += squares[17] * 1/16 # Go back 2 community card 2
squares[31] += squares[33] * 1/16 # Go back 2 community card 3

# Go to r3
squares[24] += squares[7] * 1/16   # Go back 3 chance card 1
squares[24] += squares[22] * 1/16 # Go back 3 chance card 2
squares[24] += squares[36] * 1/16 # Go back 3 chance card 3

# Community
squares[2]  *= 13/16
squares[17] *= 13/16
squares[33] *= 13/16

# Chance
squares[7]  *= 13/16
squares[22] *= 13/16
squares[36] *= 13/16

return squares
```

Now we test to see that we indeed get the probabilities, starting from any square n.

```
In [61]: get_probabilities_for_position(24)
```

```
Out[61]: [0.008680555555555556,
          0,
          0.0,
          0,
          0,
          0,
          0,
          0.0,
          0,
          0,
          0.008680555555555556,
          0,
          0,
          0,
          0,
          0.0,
          0,
          0.0,
          0,
          0,
          0,
          0,
          0.0,
          0,
          0.001736111111111111,
          0,
          0.027777777777777776,
          0.05555555555555555,
          0.08333333333333333,
          0.1111111111111111,
          0.1388888888888889,
          0.1736111111111111,
          0.1388888888888889,
          0.09027777777777778,
          0.08333333333333333,
          0.05555555555555555,
          0.022569444444444444,
          0,
          0,
          0]
```

The sum of the column should equal 1.

```
In [62]: sum(get_probabilities_for_position(24))
```

```
Out[62]: 0.9999999999999989
```

Take into account that our floating point arithmetic has a few rounding errors, the results seems to be correct. Now we are going to generate the probabilities for all the squares.

```
In [63]: chances = []
         for p in range(40): chances += get_probabilities_for_position(p)
```

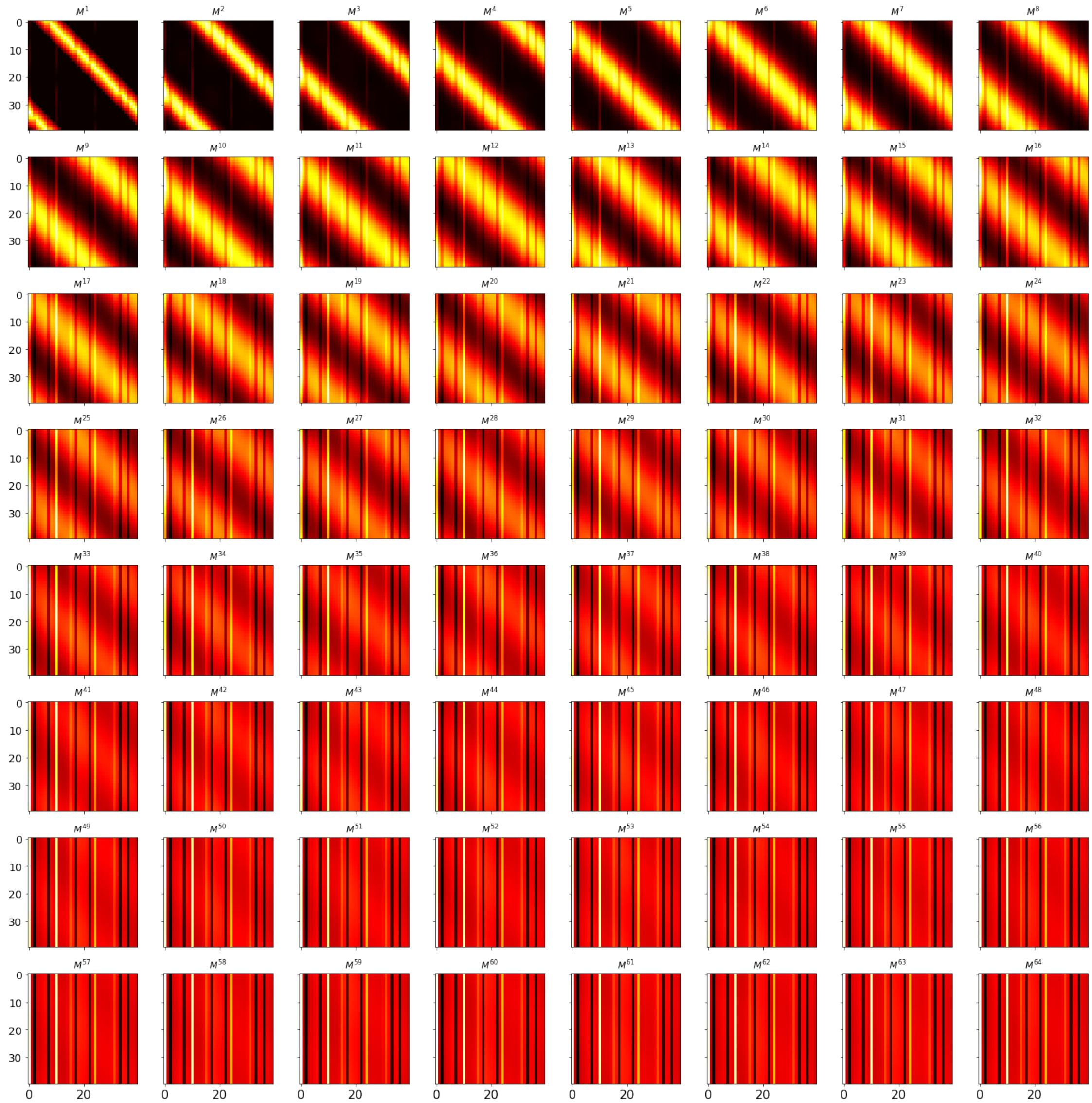
For a 40×40 matrix, we should have 1600 elements.

```
In [64]: len(chances)

Out[64]: 1600
```

7.3 Stable situation of M

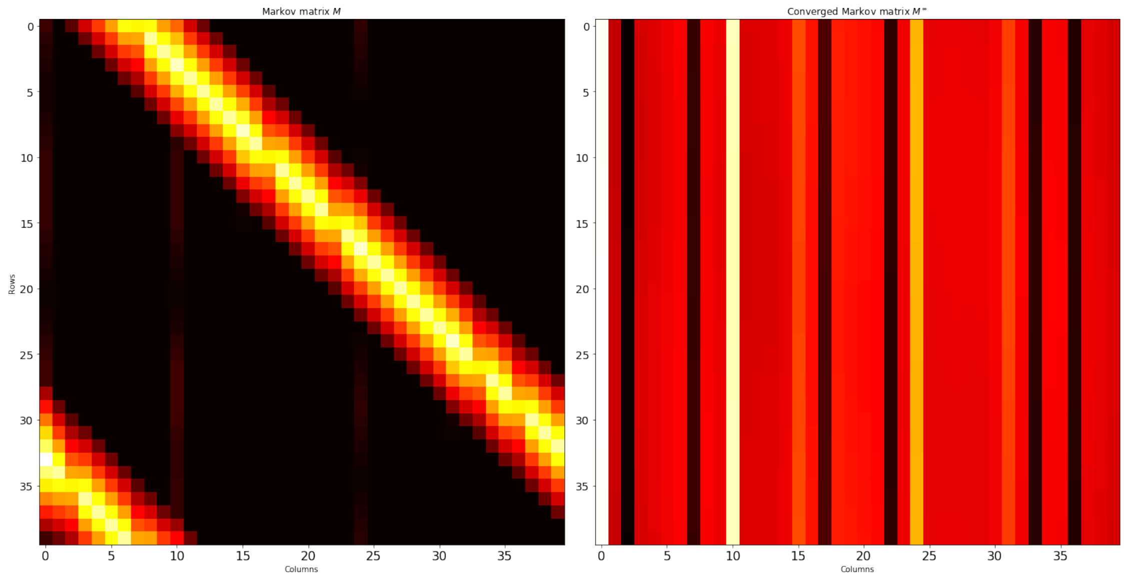
Next we apply $\lim_{n\to\infty} M^n$, to converge to the stable situation of that matrix. Because of floating point arithmetic we keep iterating until we run out of our bit space, which is around M^{64}. If we plot a heat map for every M^n, where n is between $[1, 64]$, we get:

```
In [65]: from mpl_toolkits.axes_grid1 import Grid

In [66]: grid = 8
         N = grid**2
         M = Mn = np.array(chances).reshape(40,40)
         plt.figure(figsize=(20,20))

         for i in range(N):
             ax = plt.subplot(sqrt(N),sqrt(N),i+1)
             if i < N-sqrt(N): plt.setp(ax.get_xticklabels(), visible=False)
             if i % sqrt(N) != 0: plt.setp(ax.get_yticklabels(), visible=False)
             plt.imshow(Mn, cmap='hot')
             plt.title('$M^{' + str(i+1) + '}$')
             Mn = np.dot(M, Mn)

         plt.tight_layout()
```



Which looks quite cool. You can really see how the probabilities spread through the matrix to their exact values. If we look closer at M^1 and M^{64}:

```
In [67]: plt.figure(figsize=(20,10))
         plt.subplot(1,2,1)
         plt.imshow(M, cmap='hot')
         plt.title('Markov matrix $M$')
         plt.ylabel('Rows')
         plt.xlabel('Columns')
         plt.subplot(1,2,2)
         plt.imshow(Mn, cmap='hot')
         plt.title('Converged Markov matrix $M^\infty$')
         plt.xlabel('Columns')
         plt.tight_layout()
```



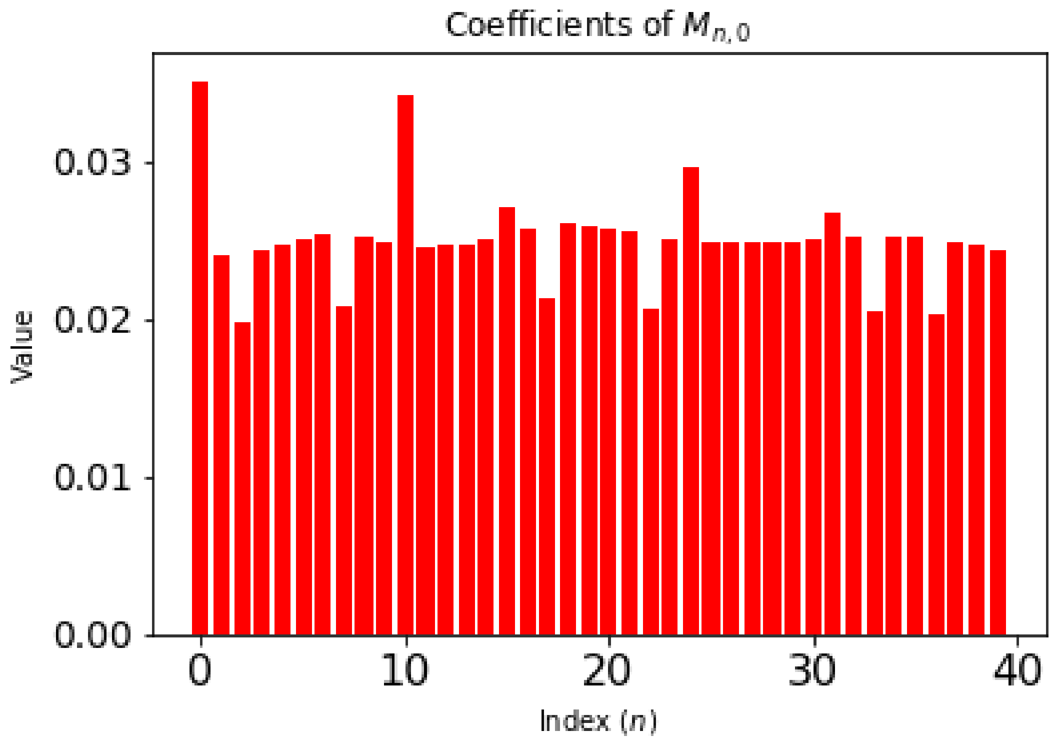
Any row from M has the same probability values.

In [68]: Mn[0,]

```
Out[68]: array([ 0.03504868,  0.02401996,  0.01968376,  0.0243919 ,  0.0246727 ,
                 0.0249952 ,  0.02533011,  0.02077604,  0.02526376,  0.02489455,
                 0.03415223,  0.02451821,  0.02460155,  0.02465195,  0.02495032,
                 0.02707982,  0.02577487,  0.02124669,  0.02602765,  0.02582802,
                 0.02568831,  0.02547366,  0.02054732,  0.02500047,  0.02953957,
                 0.02479888,  0.02485625,  0.02485469,  0.024812  ,  0.02479916,
                 0.02505142,  0.02676756,  0.02512208,  0.02042048,  0.0251762 ,
                 0.02511513,  0.02027001,  0.02479571,  0.02463601,  0.02436713])
```

If we plot the coefficicients of the first row of M:

```
In [69]: plt.bar(range(len(Mn[0,])), Mn[0,], facecolor='r')
         plt.title('Coefficients of $M_{n,0}$')
         plt.xlabel('Index ($n$)')
         plt.ylabel('Value');
```



7.4 Error analysis

Finally we add them to a `DataFrame`, and compare our previous results with our new exact results.

- In our simulation we added the probabilities from `Go to Jail` to `Jail`, so we should also do that for M.
- We removed the `Go back 3` chance card probabilities.

```
In [70]: df = pd.DataFrame()
         df['Label'] = squares_labels
         df['Description'] = squares_description
         df['Simulated probabilities'] = squares
         df['Simulated probabilities'] /= rounds
         df['Converged probabilities'] = Mn[0,]
         df.loc[10, 'Converged probabilities'] += df.loc[30, 'Converged probabilities']
         df.loc[30, 'Converged probabilities'] = 0
         df['Error'] = df['Simulated probabilities'] - df['Converged probabilities']
         df['Absolute Error'] = df['Error'].abs()
         df['Error percentage'] = df['Absolute Error'] / df['Simulated probabilities']
         df
```

```
Out[70]:     Label         Description  Simulated probabilities  \
         0   start               Start                  0.031421
```

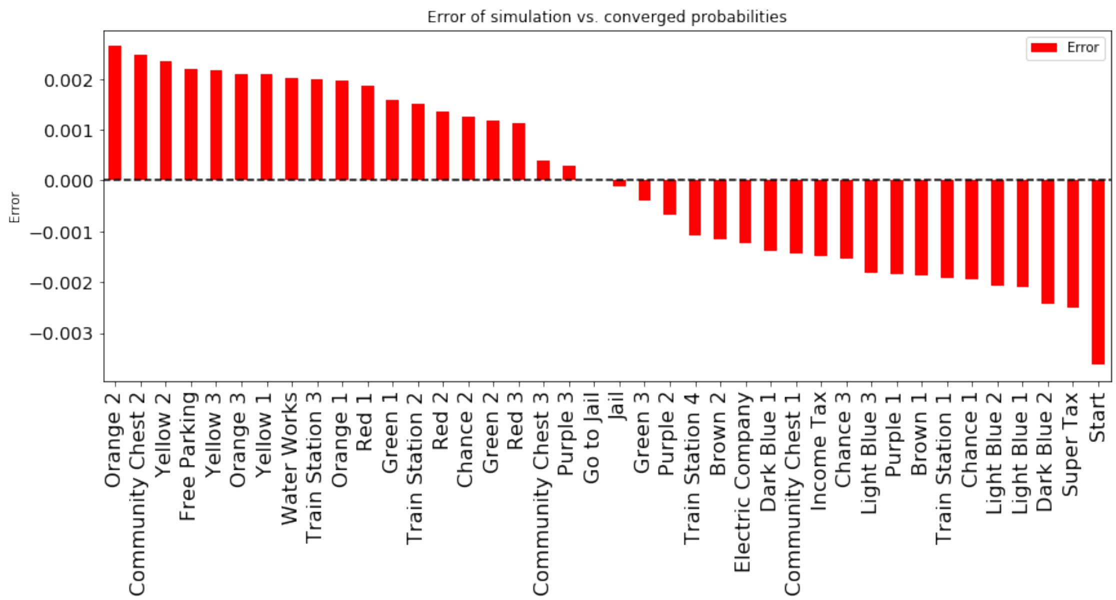
1	b1	Brown 1	0.022156
2	cc1	Community Chest 1	0.018249
3	b2	Brown 2	0.023238
4	it	Income Tax	0.023191
5	t1	Train Station 1	0.023079
6	lb1	Light Blue 1	0.023246
7	c1	Chance 1	0.018835
8	lb2	Light Blue 2	0.023200
9	lb3	Light Blue 3	0.023079
10	jail	Jail	0.059078
11	p1	Purple 1	0.022665
12	ec	Electric Company	0.023359
13	p2	Purple 2	0.023990
14	p3	Purple 3	0.025249
15	ts2	Train Station 2	0.028584
16	o1	Orange 1	0.027742
17	cc2	Community Chest 2	0.023722
18	o2	Orange 2	0.028676
19	o3	Orange 3	0.027934
20	p	Free Parking	0.027895
21	r1	Red 1	0.027343
22	c2	Chance 2	0.021792
23	r2	Red 2	0.026371
24	r3	Red 3	0.030681
25	ts3	Train Station 3	0.026798
26	y1	Yellow 1	0.026950
27	y2	Yellow 2	0.027192
28	ww	Water Works	0.026825
29	y3	Yellow 3	0.026966
30	gtj	Go to Jail	0.000000
31	g1	Green 1	0.028344
32	g2	Green 2	0.026300
33	cc3	Community Chest 3	0.020822
34	g3	Green 3	0.024771
35	ts4	Train Station 4	0.024023
36	c3	Chance 3	0.018737
37	db1	Dark Blue 1	0.023417
38	st	Super Tax	0.022138
39	db2	Dark Blue 2	0.021942

	Converged probabilities	Error	Absolute Error	Error percentage
0	0.035049	-0.003628	0.003628	0.115454
1	0.024020	-0.001864	0.001864	0.084129
2	0.019684	-0.001435	0.001435	0.078621
3	0.024392	-0.001154	0.001154	0.049656
4	0.024673	-0.001482	0.001482	0.063891
5	0.024995	-0.001916	0.001916	0.083028
6	0.025330	-0.002084	0.002084	0.089655

7	0.020776	-0.001941	0.001941	0.103055
8	0.025264	-0.002064	0.002064	0.088955
9	0.024895	-0.001816	0.001816	0.078667
10	0.059204	-0.000126	0.000126	0.002127
11	0.024518	-0.001853	0.001853	0.081765
12	0.024602	-0.001243	0.001243	0.053194
13	0.024652	-0.000662	0.000662	0.027593
14	0.024950	0.000299	0.000299	0.011829
15	0.027080	0.001504	0.001504	0.052623
16	0.025775	0.001967	0.001967	0.070908
17	0.021247	0.002475	0.002475	0.104347
18	0.026028	0.002648	0.002648	0.092354
19	0.025828	0.002106	0.002106	0.075391
20	0.025688	0.002207	0.002207	0.079107
21	0.025474	0.001869	0.001869	0.068366
22	0.020547	0.001245	0.001245	0.057116
23	0.025000	0.001371	0.001371	0.051971
24	0.029540	0.001141	0.001141	0.037203
25	0.024799	0.001999	0.001999	0.074600
26	0.024856	0.002094	0.002094	0.077690
27	0.024855	0.002337	0.002337	0.085956
28	0.024812	0.002013	0.002013	0.075042
29	0.024799	0.002167	0.002167	0.080355
30	0.000000	0.000000	0.000000	NaN
31	0.026768	0.001576	0.001576	0.055618
32	0.025122	0.001178	0.001178	0.044788
33	0.020420	0.000402	0.000402	0.019283
34	0.025176	-0.000405	0.000405	0.016358
35	0.025115	-0.001092	0.001092	0.045462
36	0.020270	-0.001533	0.001533	0.081817
37	0.024796	-0.001379	0.001379	0.058876
38	0.024636	-0.002498	0.002498	0.112838
39	0.024367	-0.002425	0.002425	0.110525

If we plot a bar chart of the error between our simulated probabilities and our exact probabilities, we get:

```
In [71]: ax = df[['Description', 'Error']].sort_values(by='Error', ascending=False)\
             .plot(kind='bar', figsize=(14,5), facecolor='r')
         plt.xticks(range(len(df.index)), df[['Description', 'Error']]
             .sort_values(by='Error', ascending=False)['Description'])
         plt.ylabel('Error')
         plt.title('Error of simulation vs. converged probabilities')
         ax.axhline(0, c='black', ls='dashed');
```



Calculating the accuracy for the simulated model by subtracting the average error yields:

```
In [72]: print('Average accuracy: {:.2f}%'.format((1-df['Error percentage'].mean())*100))
```

```
Average accuracy: 93.23%
```

We should take into consideration that we removed the `Go back 3` from the chance card in both models.