UEMA151: CLASS WORK PROBLEMS

UNIT 2: VECTOR CALCULUS

1	Introduction to vector differentiation and Gradient of a scalar function.
2	Directional derivatives, Angle between the surfaces
3	Divergence,
4	Curl and related properties
5	Application problems on Gradient, divergence and curl.
6	Introduction to line, surface and volume integrals
7	Problems on line Surface and Volume integral
8	Green's theorem
9	Problems on Green's theorem
10	Stoke's theorem
11	Problems on Stoke's theorem
12	Gauss's Divergence theorem

VECTOR IDENTITIES:

1	$div\ grad\ f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
2	$curl\ grad\ f = \nabla \times \nabla f = \mathbf{0}$
3	$div\ curl\ \mathbf{F} = \nabla . \nabla \times \mathbf{F} = 0$
4	$div\ (\mathbf{F} \times \mathbf{G}) = \mathbf{G} . (curl\ \mathbf{F}) - \mathbf{F} . (curl\ \mathbf{G})$
5	$curl\ curl\ \mathbf{F} = grad\ div\ \mathbf{F} - \nabla^2\mathbf{F}$,
6	$grad\ div\ \mathbf{F} = curl\ curl\ \mathbf{F} + \nabla^2\mathbf{F}$
7	$div\ (f\,\mathbf{G}) = (\mathbf{grad}\ f) . \mathbf{G} + f\,(\mathbf{div}\ \mathbf{G})$
8	$curl\ (f\,\mathbf{G}) = (grad\ f) \times \mathbf{G} + f\,(curl\ \mathbf{G})$
9	$curl\ (\mathbf{F} \times \mathbf{G}) = \mathbf{F}\,(div\ \mathbf{G}) - \mathbf{G}\,(div\ \mathbf{F}) + (\mathbf{G} . \nabla)\,\mathbf{F} - (\mathbf{F} . \nabla)\,\mathbf{G}$

PROBLEMS ON VECTOR DIFFERENTIATION

1. If $\emptyset = x^3y + xy^2 + 3y$ find
   a. $\nabla\emptyset$ b) $\nabla\emptyset$ at (0,0,0) c) $\nabla\emptyset$ at (1,1,1)

   Ans: $(\boldsymbol{3x^2y + y^2})\boldsymbol{i} + (\boldsymbol{x^3 + 2xy + 3})\boldsymbol{j} + \boldsymbol{0k},\ \boldsymbol{3j},\ \sqrt{\mathbf{52}}$

2. Find the gradient of the following scalar point function $f(x, y, z) = 3x^2y - y^3z^2$ at (1, -2, -1).
   Ans: -(12i+9j+16k)

3. If $\vec{r} = xi + yj + zk, |\vec{r}| = r$, Find $\nabla\emptyset$ if $\emptyset = log|r|$.
   Ans: $\frac{\vec{r}}{r^2}$
4. The electrostatic potential V in a region is given by $v = \frac{y}{(x^2+y^2+z^2)^{3/2}}$.
   Suppose a unit charge is in the region at the point with coordinates (2,1,0). Find the direction at this point, in which the rate of decrease in potential is greatest.
   Ans: 0.107i-0.036j-0k

5. Find the normal vector and the unit normal vector to the given curve/surface$x^3 + y^3 + 3xyz = 3$ at the point (1,2, -1).
   Ans: $(-\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}), \frac{(-\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k})}{\sqrt{14}}$

6. What is the angle between the normal to the surface $xy = z^2$at the points (1,4,2) and (-3, -3,3)?
   Ans: $\boldsymbol{cos^{-1}}(\frac{1}{\sqrt{22}})$

7. Find the values of constants λ and $\boldsymbol{\mu}$ so that the surfaces $\lambda x^2 - \boldsymbol{\mu} yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ intersect orthogonally at (1, -1,2).
   Ans: $\boldsymbol{\mu} = \mathbf{1}, \boldsymbol{\lambda} = \mathbf{5/2}$

8. Find the directional derivative of $\vec{V}^2$ where $\vec{V} = xy^2i + zy^2j + xz^2k$ at the point (2,0,3) in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point (3,2,1).

   Ans: $\frac{\mathbf{702}\sqrt{\mathbf{14}}}{\mathbf{7}}$
9. Find the directional derivative of $f(x, y, z) = 4xz^3 - 3x^2y^2z$ at (2,-1,2) in the direction 2i-3j+6k.
   Ans: $\frac{\mathbf{376}}{\mathbf{7}}$
10. In what direction from the point (1,1, -1) is the directional derivative of $\emptyset = x^2 - 2y^2 + 4z^2$ a maximum. Also find the value of this maximum directional derivative.
   Ans: 2i-4j-8k; 2$\sqrt{\mathbf{21}}$

11. Show that the field force given by $F(x, y, z) = (y^2 cosx + z^3)i + (2ysinx - 4)j + (3xz^2 + 2)k$ is conservative and find its scalar potential.

Ans: $\emptyset(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{y^2 sinx + xz^3 - 4y + 2z + k_1}$

12. If $\vec{V} = zx^2 i + 2y^3 z^2 j + xyz^2 k$ =find div V.

Ans: $\boldsymbol{div\ v = \nabla . v = 2xz + 6y^2 z^2 + 2xyz}$

13. Show that the vector field $\vec{V} = x\ siny\ i + y\ sinx\ j - z(sinx + siny)k$ is solenoidal.

14. Find the curl of the vector field $\vec{V} = xyzi + 2x^2 yj + (xz^2 - zy^2)k$ and verify that div (curl $\vec{V}$)=0.

Ans: curl $\boldsymbol{\vec{V} = -[2yzi + (z^2 - xy)j + (xz - 4xy)k]}$

15. Find the constants 'a', 'b' and 'c' so that $\vec{V} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrotational and hence find its scalar potential $\emptyset(x, y, z)$.

Ans: a=4,b=2,c=-1. $\emptyset(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{\frac{x^2}{2} + 2xy + 4xz - 3\frac{y^2}{2} - yz + z^2 + c}$

16. Find the value of constant 'a' so that the vector $\vec{A} = (x + 3y)\ i + (y - 2z)j + (x + az)k$ is solenoidal.

Ans: -2

17. A fluid motion is given by $\vec{v} = (ysinz - sinx)i + (x\ sinz + 2yz)j + (xycosz + y^2)k$. Is the motion irrotational? If so, find the velocity potential.

Ans: Yes, $\emptyset = \boldsymbol{xysinz + cosx + zy^2 + c}$

18. If $f = 3x^2 y, \emptyset = xz^2 - 2y$ evaluate $grad[grad(f) . grad(\emptyset)]$.

Ans: $\boldsymbol{(6yz^2 - 12x)i + 6xz^2 j + 12xyzk}$

VECTOR INTEGRATION

1. Find the work done in moving a particle in the force field $F(x,y,z) = 3x^2i + (2xz - y)j + zk$ along the space curve $x = 2t^2, y = t, z = 4t^2 - t$ from t=0 to t=1.
   Ans: 14.2

2. Find the constant 'a' so that the vector field $v = (axy - z^3)i + (a - 2)x^2j + (1 - a)xz^2k$ is conservative. Calculate its scalar potential and the work done in moving a particle form P: (1,2,-3) to Q: (1,-4,2) in the field.
   Ans: a=4.

GREEN'S THEOREM

1. Verify the Green's theorem in the plane for $\oint_C (xy + y^2)dx + x^2dy$ where, C is the closed curve of the region bounded by $y = x \text{ and } y = x^2$.
2. Find the work done by the force $F(x,y,z) = (-16y + sinx^2)i + (4e^y + 3x^2)j$ acting along the simple closed curve 'C' bounded by the curves $y = x, y = -x \text{ and } x^2 + y^2 = 16$.
   Ans: 64 π
3. Apply the Green's theorem to show that the area bounded by a simple closed curve 'C' is given by $\frac{1}{2}\oint x\,dy - ydx$. Hence, find the area of an ellipse whose semi-major and minor axes are of lengths 'a' and 'b'.
4. Evaluate $\oint(2x - y^3)\,dx - xydy$ over the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.
   Ans: **60 π**

Surface Integrals

1. Evaluate the surface integral $\iint_S F.n\,dA$ where $F(x,y,z) = (x + y^2)i - 2xj + 2yzk$ and S is the portion of the plane 2x+y+2z=6 which is in the first octant.
   Ans: 81.

2. Evaluate the surface integral $\iint_S F.n\,dA$, where $F = zi + xj - 3y^2z\,k$ and S is the portion of the surface of the cylinder $\boldsymbol{x^2 + y^2 = 16}$ included in the first octant between z=0 and z=5.
Ans: **90**.

Stokes's Theorem

1. Verify Stokes's theorem for the vector field $F = (2x - y)i - yz^2j - y^2z\,k$ over the upper half of the surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane.
2. Apply Stokes's theorem to prove that $\oint_C ydx + z\,dy + xdz = -2\sqrt{2}\,\pi\,a^2$, where C is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0, x + y = 2a$ and begins at the point (2a,0,0) and goes at first below the z-plane.
3. Evaluate $\oint(2y^3\,i + x^3j + zk)$.(dxi + dy j+ dz k) over the surface of the cone $z = \sqrt{x^2 + y^2}$ below $z = 4$.

Gauss's Divergence Theorem

1. Use divergence theorem to evaluate $\iint_S F.n\,dA$, where $F = 2x^2y\,i - y^2j + 4xz^2k$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and x=2.
   Ans: **180**.
2. Use divergence theorem to evaluate $\iint_S v.n\,dA$ where $v = x^2zi + yj - xz^2k$ and 'S' is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane z=4y.
   Ans: $\boldsymbol{8\pi}$
3. By transforming to tiple integral, evaluate $I = \iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dxdy$ where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular discs z=0 and z=b.
   Ans:$\boldsymbol{\frac{5}{4}\pi a^4 b}$

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UNIT 2 : VECTOR CALCULUS

CLASS WORK PROBLEMS.

Problems on Vector Differentiation:

1) If $\phi = x^3y + xy^2 + 3y$ find

a) $\nabla\phi$ b) $\nabla\phi$ at $(0,0,0)$ c) $\nabla\phi$ at $(1,1,1)$.

$$\phi = x^3y + xy^2 + 3y$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= (3x^2y + y^2)\hat{i} + (x^3 + 2xy + 3)\hat{j} + 0.$$

$$\nabla\phi_{\text{at }(0,0,0)} = 3\hat{j}$$

$$\nabla\phi \text{ at } (1,1,1) = 4\hat{i} + 6\hat{j}.$$

2) Find the gradient of the following scalar point function $f(x,y,z) = 3x^2y - y^3z^2$ at $(1,-2,-1)$.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\,\hat{k}.$$

$$\nabla f = 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\,\hat{j} - 2y^3z\,\hat{k}.$$

$(\nabla f)_{at}\ (1,-2,-1)$

$$= 6(1)(-2)\,\hat{i} + (3-12)\,\hat{j} - 2(-8)(-1)\,\hat{k}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

3) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $|\vec{r}| = r$, find $\nabla\phi$ if $\phi = \log r$.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$. $\Rightarrow r^2 = x^2+y^2+z^2$

$\therefore \phi = \log\sqrt{x^2+y^2+z^2} = \log r.$

$2r \cdot \frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}.$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial\phi}(\log r)\,\hat{i} + \frac{\partial}{\partial y}(\log r)\,\hat{j} + \frac{\partial}{\partial z}(\log r)\,\hat{k}$$

$$= \frac{1}{r}\cdot\frac{\partial r}{\partial \cancel{x}}\hat{i} + \frac{1}{r}\cdot\frac{\partial r}{\partial y}\hat{j} + \frac{1}{r}\cdot\frac{\partial r}{\partial z}\hat{k}.$$

$$= \frac{1}{r}\cdot\frac{x}{r}\hat{i} + \frac{1}{r}\cdot\frac{y}{r}\hat{j} + \frac{1}{r}\frac{z}{r}\cdot\hat{k}$$

$$= \frac{1}{r^2}(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{\vec{r}}{r^2}.$$

4). The electro static potential V in a region is given by $V = \dfrac{y}{(x^2+y^2+z^2)^{3/2}}$. Suppose a unit charge is in the region at the pt with coordinates (2, 1, 0). Find the direction at this point, in which the rate of decrease in potential is greatest.

$$V = y(x^2+y^2+z^2)^{-3/2}$$

$$\nabla V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}$$

$$= \left(y\cdot\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x)\right)\hat{i} +$$

$$\left[(x^2+y^2+z^2)^{-3/2} + y\left(-3/2\right)(x^2+y^2+z^2)^{-5/2}(2y)\right]\hat{j}$$

$$+\left(y\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}\,2z\right)\hat{k}.$$

$$(\nabla V)_{at\ (2,1,0)} = \frac{-6}{(5)^{5/2}}\hat{i} + \left(\frac{1}{(5)^{3/2}} - \frac{3}{(5)^{5/2}}\right)\hat{j}$$

$$\text{Rate of decrease} = -\nabla V$$

$$= \frac{6}{(5)^{5/2}}\hat{i} + \left(\frac{3}{(5)^{5/2}} - \frac{1}{(5)^{3/2}}\right)\hat{j}.$$

$$= 0.107\hat{i} - 0.036\hat{j}.$$

This vector points in the direction of greatest rate of decrease in potential.

5) Find the normal vector and the unit normal vector to the given surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$

$$\phi = x^3 + y^3 + 3xyz - 3.$$

We know that $\nabla\phi$ is a vector normal to the surface.

We have $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$$\nabla\phi = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + 3xy\hat{k}$$

$$(\nabla\phi)_{at\ (1,2,-1)} = -3\hat{i} + 9\hat{j} + 6\hat{k}.$$

The required unit vector normal

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{(-3)^2 + (9)^2 + (6)^2}} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}}.$$

6). What is the angle between the normal to the surface $xy = z^2$ at the points $(1, 4, 2)$ and $(-3, -3, 3)$.

Let $f = xy - z^2$.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= y\hat{i} + x\hat{j} - 2z\hat{k}$$

$$(\nabla f)_{at\ (1,4,2)} = 4\hat{i} + \hat{j} - 4\hat{k} = \nabla f_1$$

$$(\nabla f)\ \text{at}\ (-3,-3,3) = -3\hat{i} - 3\hat{j} - 6\hat{k} = \nabla f_2$$

$$\therefore \cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

$$= \frac{-12 - 3 + 24}{\sqrt{16+1+16}\ \sqrt{9+9+36}} = \frac{9}{\sqrt{33}\sqrt{54}}.$$

$$\therefore \theta = \cos^{-1}\left(\frac{9}{\sqrt{33}\sqrt{54}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right).$$

7) Find the Values of constants λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda+2)x$ and $4x^2y + z^3 = 4$ intersect orthogonally at $(1,-1,2)$.

First we have to ensure that the given point lies on both the surfaces.

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Substituting $(1,-1,2)$ onto the equation

$\lambda x^2 - \mu yz = (\lambda+2)x$ we obtain

$$\lambda + 2\mu = \lambda + 2$$

$$\boxed{\mu = 1}$$

Also if $(1,-1,2)$ is substituted onto the LHS of the equation $4x^2y + z^3 = 4$ we get 4 which is equal to the R.H.S.

$\Rightarrow$ the given point lies on both the surfaces when $\mu = 1$.

In order to find λ we have to use the orthogonality condition $\nabla\phi_1 \cdot \nabla\phi_2 = 0$.

where

$$\phi_1 = \lambda x^2 - \mu yz - (\lambda+2)x.$$

$$\nabla\phi_1 = [2\lambda x - (\lambda+2)]\hat{i} + (-\mu z)\hat{j} + (-\mu y)\hat{k}.$$

$$\phi_2 = 4x^2y + z^3$$

$$\nabla\phi_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$$

$$\nabla\phi_1 \text{ at } (1,-1,2) = (\lambda-2)\hat{i} - 2\mu\hat{j} + \mu\hat{k}.$$

$$\nabla\phi_2 \text{ at } (1,-1,2) = -8\hat{i} + 4\hat{j} + 12\hat{k}.$$

$\nabla\phi_1 \cdot \nabla\phi_2 = 0$

gives $\quad -8(\lambda-2) - 8\mu + 12\mu = 0.$

$-8\lambda + 4\mu + 16 = 0.$

But $\mu = 1, \quad \therefore \lambda = 5/2.$

Thus $\lambda = 5/2$ and $\mu = 1$ are the required Values.

8) Find the directional derivative of V^2 where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point $(2,0,3)$ in the direction of the outward normal to the sphere $x^2+y^2+z^2=14$ at the point $(3,2,1)$

$$\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$$

$$V = |\vec{V}| = \sqrt{x^2y^4 + z^2y^4 + x^2z^4}$$

$$\Rightarrow V^2 = x^2y^4 + z^2y^4 + x^2z^4.$$

$$\nabla V^2 = (2xy^4 + 2xz^4)\hat{i} + (4x^2y^3 + 4y^3z^2)\hat{j} + (2zy^4 + 4x^2z^3)\hat{k}.$$

$$(\nabla V^2)_{at\ (2,0,3)} = 324\hat{i} + 432\hat{k}$$

Let

$$\phi = x^2+y^2+z^2-14$$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$(\nabla\phi)_{at\ (3,2,1)} = 6\hat{i} + 4\hat{j} + 2\hat{k}.$$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{36+16+4}} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}.$$

Directional derivative of V^2 along $\nabla\phi$

$$= \nabla V^2 \cdot \hat{n} \quad \text{where } \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= (324\hat{i} + 432\hat{k}) \cdot \left(\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right)$$

$$= \frac{1404}{\sqrt{14}} = \frac{702\sqrt{14}}{7}.$$

9) Find the directional derivative of $f(x,y,z) = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Directional derivative of f at $(2,-1,2)$ in the direction $\vec{u} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$$= \nabla f \cdot \hat{u} \quad \text{where} \quad \hat{u} = \frac{\vec{u}}{|\vec{u}|}.$$

$$f = 4xz^3 - 3x^2y^2z$$

$$\nabla f = (4z^3 - 6xy^2z)\hat{i} - 6x^2yz\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

$$(\nabla f)_{at\ (2,-1,2)} = 8\hat{i} + 48\hat{j} + 84\hat{k}$$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

Directional Derivative of f along $\vec{u}$

$$= \nabla f \cdot \hat{u}$$

$$= (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

$$= \frac{16 - 144 + 504}{7}$$

$$= \frac{376}{7}.$$

10) In what direction from the point $(1, 1, -1)$ is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ a maximum. Also find the value of this maximum directional derivative.

$$\phi = x^2 - 2y^2 + 4z^2$$

$$\nabla\phi = 2x\hat{i} - 4y\hat{j} + 8z\hat{k}$$

$$(\nabla\phi)_{at\ (1,1,-1)} = 2\hat{i} - 4\hat{j} - 8\hat{k}$$

The value of maximum directional derivative is given by

$$|\nabla\phi| = \sqrt{4+16+64} = \sqrt{84} = 2\sqrt{21}$$

STOKE'S THEOREM :

Let $\vec{F}$ be a vector function, having continuous first partial derivatives in a domain in space containing an open two sided surface S bounded by a simple closed curve C then

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = \oint_C \vec{F} \cdot d\vec{r}.$$

where $\hat{n}$ is a unit normal of S and C is traversed in the positive direction.

Problems on Stokes Theorem :

1) Verify Stoke's theorem for the vector field $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over the upper half of the surface of $x^2+y^2+z^2=1$ bounded by its projection on the XY-plane.

By Stoke's Theorem, we have.

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = \oint_C \vec{F} \cdot dr.$$

L.H.S

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds$$

$$\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}\left[-2yz - (-2yz)\right] - \hat{j}\left[0-0\right] + \hat{k}\left[0 - (-1)\right]$$

$$= \hat{k}.$$

Normal vector $\nabla\phi = \nabla(x^2+y^2+z^2-1)$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}.$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}.$$

$$= x\hat{i}+y\hat{j}+z\hat{k}$$

Projecting on XY plane.

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{z}.$$

$$\therefore \iint_S (\nabla\times\vec{F})\,\hat{n}\,ds = \iint_R \hat{k}\cdot(x\hat{i}+y\hat{j}+z\hat{k})\cdot\frac{dx\,dy}{z}$$

$$= \iint_R \cancel{z}\cdot\frac{dx\,dy}{\cancel{z}}$$ = Area enclosed by the circle $x^2+y^2=1$. $z=0$.

$$= \pi.$$

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R.H.S

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \left((2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}\right)\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \oint_C (2x-y)\,dx - yz^2dy - y^2z.dz.$$

$C : x^2+y^2 = 1$ where $z=0$

Put $x = \cos\theta$
$y = \sin\theta$

$$= \int_C (2x-y)\,dx.$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\,d\theta$$

$$= \int_0^{2\pi} \left(-\sin 2\theta + \frac{1}{2} - \frac{\cos 2\theta}{2}\right)d\theta$$

$$= \left[\frac{\cos 2\theta}{2} + \frac{1}{2}\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0) = \pi.$$

L.H.S = R.H.S

Hence Stokes Theorem is Verified.

2) Apply Stoke's Theorem to prove that

$\oint_C y\,dx + z\,dy + x\,dz = -2\sqrt{2}\pi a^2$, where C is the curve given by $x^2+y^2+z^2-2ax-2ay=0$, $x+y=2a$ and begins at the point $(2a, 0, 0)$ and goes at first below the z-plane.

Consider

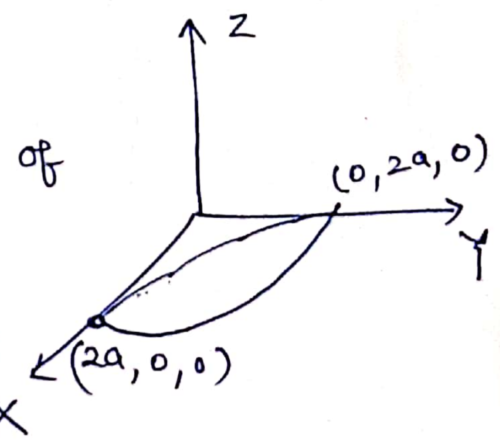
$$x^2+y^2+z^2-2ax-2ay=0$$

$$\Rightarrow (x-a)^2+(y-a)^2+z^2 = 2a^2 \quad \text{—(1)}.$$

This represents a sphere with centre at $(a, a, 0)$ and radius equal to $\sqrt{2}\,a$.

The equation $x+y=2a$ represents a plane.

Thus C is the intersection of the sphere (1) and the plane $x+y=2a$.



Further, the centre $(a, a, 0)$ of the sphere lies on the plane $x+y=2a$. Thus C is the great circle of the sphere passing through $(2a,0,0)$ and $(0, 2a, 0)$.

Hence C is a circle lying on the plane $x+y=2a$, with centre $(a, a, 0)$ and radius equal to $\sqrt{2}\,a$. Let S be the region bounded by this circle C.

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Now,

$$y\,dx + z\,dy + x\,dz = (y\hat{i} + z\hat{j} + x\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}).$$

$$y\,dx + z\,dy + x\,dz = \vec{F} \cdot d\vec{r}.$$

where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}.$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$$= \hat{i}[0-1] - \hat{j}[1-0] + \hat{k}[0-1]$$

$$= -\hat{i} - \hat{j} - \hat{k}.$$

Unit normal drawn to the plane $x+y=2a$ is

$$\phi : x+y-2a$$

$$\nabla\phi = \hat{i} + \hat{j}$$

$$|\nabla\phi| = \sqrt{1+1} = \sqrt{2}$$

$$\hat{n} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}.$$

$$\nabla \times \vec{F} \cdot \hat{n} = (-\hat{i} - \hat{j} - \hat{k}) \cdot \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$$

$$= -1/\sqrt{2} - 1/\sqrt{2} = -\sqrt{2}.$$

By Stoke's Theorem we have

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\cdot ds$$

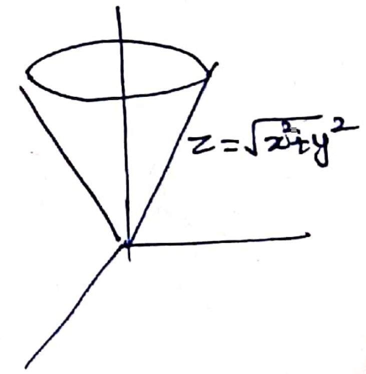
$$= \iint_S -\sqrt{2}\cdot ds.$$

$$= -\sqrt{2}.(\text{Area of the } \odot^{le}\ c)$$

$$= -\sqrt{2}\,\pi(a\sqrt{2})^2$$

$$= -2\sqrt{2}\,\pi a^2.$$

3) Evaluate $\oint(2y^3\hat{i}+x^3\hat{j}+z\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$ over the surface of the cone $z=\sqrt{x^2+y^2}$ below $z=4$.



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds.$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z. \\ 2y^3 & x^3 & z \end{vmatrix}$$

$$= \hat{i}[0-0]-\hat{j}[0-0]+\hat{k}[3x^2-6y^2)$$

$$\phi = x^2+y^2-z^2$$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}-2z\hat{k}}{2\sqrt{x^2+y^2+z^2}.}$$

$$= \frac{2x\hat{i}+2y\hat{j}-2z\hat{k}}{2\sqrt{2}\,z} = \frac{x\hat{i}+y\hat{j}-z\hat{k}}{\sqrt{2}\,z}.$$

Projecting on XY-plane

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{\left|{-z}/{\sqrt{2}z}\right|} = \sqrt{2}\,dx\,dy.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R (3x^2-6y^2)\hat{k}\cdot\left(\frac{x\hat{i}+y\hat{j}-z\hat{k}}{\sqrt{2}\,z}\right)\sqrt{2}\,dx\,dy$$

$$= \iint_R \frac{-\cancel{z}(3x^2-6y^2)\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\,\cancel{z}}dx\,dy$$

$$= -\iint_R (3x^2-6y^2)\,dx\,dy.$$

R is the circular region bounded by $x^2+y^2=16$.

$$= -\int_0^{2\pi}\int_{r=0}^{4} (3r^2\cos^2\theta - 6r^2\sin^2\theta)\,r\,dr\,d\theta.$$

$$= -\int_0^{2\pi}\left(3\cos^2\theta \cdot \left[\frac{r^4}{4}\right]_0^4 - 6\sin^2\theta\left[\frac{r^4}{4}\right]_0^4\right)d\theta$$

$$= -\int_0^{2\pi} \frac{3}{4}(256)\cos^2\theta - \frac{6\times 256}{4}\sin^2\theta \, d\theta$$

$$= -\left[192 \cdot \int_0^{2\pi}\cos^2\theta \, d\theta - 384\int_0^{2\pi}\sin^2\theta \, d\theta\right]$$

$$= -\left[192\pi - 384\pi\right]$$

$$= 192\pi$$

Gauss - Divergence Theorem.

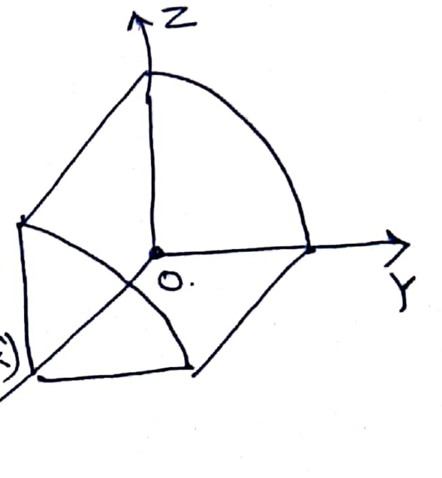
Let $\vec{F}$ be a vector function having continuous derivatives in a Volume V bounded by a closed Surface S then

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

Where $\hat{n}$ is the outward drawn (positive) normal to S.

Problems

1) Use divergence theorem to evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = 2x^2y\,\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2+z^2=9$ and $x=2$



By divergence thm we have

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv.$$

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(2x^2y\,\hat{i} - y^2\hat{j} + 4xz^2\hat{k})$$

$$= 4xy - 2y + 8xz.$$

$$\therefore \iiint_V \nabla\cdot\vec{F}\, dv = \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}}\int_{x=0}^{2} (4xy - 2y + 8xz)\, dx\cdot dy\, dz.$$

$$= \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}} \left[4y\cdot\frac{x^2}{2} - 2yx + 8z\cdot\frac{x^2}{2}\right]_0^2 dy\, dz$$

$$= \int_{z=0}^{3} \int_{y=0}^{\sqrt{9-z^2}} 2y(4) - 4y + 4z\cdot(4) \; dy \, dz.$$

$$= \int_{0}^{3} \int_{0}^{\sqrt{9-z^2}} (4y + 16z) \, dy \, dz$$

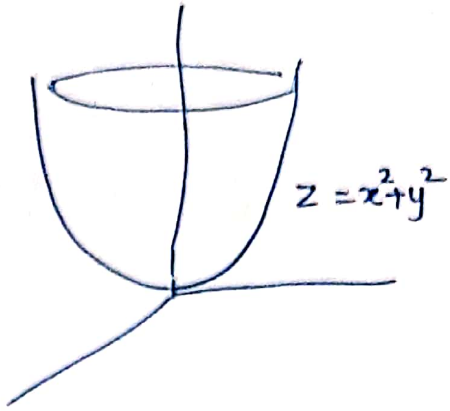
$$= \int_{0}^{3} \left[4 \cdot \frac{y^2}{2} + 16z \cdot y \right]_{0}^{\sqrt{9-z^2}} \cdot dz$$

$$= \int_{0}^{3} 2(9-z^2) + 16z \cdot \sqrt{9-z^2} \, . \, dz.$$

$= 180$ (by calculator).

2). Use divergence theorem to evaluate $\iint_S \vec{V} \cdot \hat{n} \, ds$ where $\vec{V} = x^2 z \hat{\imath} + y \hat{\jmath} - xz^2 \hat{k}$ and S is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4y$.

By Gauss divergence Theorem we have



$$\iint_S \vec{V}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{V}\, dV.$$

$$\vec{V} = x^2 z\hat{i} + y\hat{j} - xz^2\hat{k}$$

$$\nabla\cdot\vec{V} = 2xz + 1 - 2xz = 1.$$

$$\therefore \iint_S \vec{V}\cdot\hat{n}\, ds = \iiint_V 1 \cdot dx\,dy\,dz.$$

Using cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dx\,dy\,dz = r\,dz\,dr\,d\theta$

we have

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{4\sin\theta}\int_{z=r^2}^{4r\sin\theta} r\cdot dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{4\sin\theta} (4r\sin\theta - r^2)r\,dr\,d\theta$$

$$= \int_0^{\pi} 4\sin\theta\cdot\left[\frac{r^3}{3}\right]_0^{4\sin\theta} - \left[\frac{r^4}{4}\right]_0^{4\sin\theta} \cdot d\theta$$

$$= \frac{64}{3}\cdot\int_0^{\pi}\sin^4\theta\cdot d\theta = \frac{64}{3}\times\frac{3\pi}{8} = 8\pi.$$

3) By transforming to triple integral, evaluate

$I = \iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where

S is the closed surface consisting of the cylinder $x^2+y^2=a^2$ $(0 \le z \le b)$. and the circular discs $z=0$ and $z=b$.

We know that from projection formula.

$ds = \dfrac{dx\,dy}{(\hat{n}\cdot\hat{k})}$ | $ds = \dfrac{dy\,dz}{\hat{n}\cdot\hat{i}}$ | $ds = \dfrac{dx\,dz}{\hat{n}\cdot\hat{j}}$

$(\hat{n}\cdot\hat{k})ds = dx\,dy$ | $(\hat{n}\cdot\hat{i})ds = dy\,dz$ | $(\hat{n}\cdot\hat{j})ds = dx\,dz$.

Given integral. $\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$.

$$= \iint_S x^3.(\hat{n}\cdot\hat{i})ds + (x^2y)(\hat{n}\cdot\hat{j})ds + x^2z\,(\hat{n}\cdot\hat{k})ds.$$

$$= \iint_S x^3\cdot(\hat{i}\cdot\hat{n})ds + (x^2y)(\hat{j}\cdot\hat{n})ds + x^2z(\hat{k}\cdot\hat{n})ds$$

$$= \iint_S (x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k})\cdot\hat{n}\cdot ds.$$

$\therefore\ \vec{F} = x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k}$

By Gauss - Divergence Thm we have

$$\oiint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(x^2z)$$

$$= 3x^2 + x^2 + x^2 = 5x^2$$

$$\oiint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V 5x^2\,dv = \iiint_V 5x^2\,dx\,dy\,dz$$

Converting to cylindrical coordinates.

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$dx\,dy\,dz = r\,dz\,dr\,d\theta$.

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{b} 5r^2\cos^2\theta \cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a} 5r^3\cos^2\theta\,[z]_0^b\,.\,dr\,d\theta.$$

$$= 5b\cdot\int_0^{2\pi}\int_0^{a} \cos^2\theta\cdot\left[\frac{r^4}{4}\right]_0^a\cdot d\theta$$

$$= \frac{5ba^4}{4}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \frac{5ba^4}{4}\times\pi = \frac{5\pi a^4 b}{4}$$

Another method for first problem.

$$\iiint_V \nabla\cdot\vec{F}\, dV = \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}}\int_{x=0}^{2} (4xy - 2y + 8xz)\cdot dx\, dy\, dz$$

$$= \int_0^3\int_0^{\sqrt{9-z^2}} \left[4y\cdot\frac{x^2}{2} - 2yx + 8z\cdot\frac{x^2}{2}\right]_0^2 dy\, dz.$$

$$= \int_0^3\int_0^{\sqrt{9-z^2}} 2y(4) - 4y + 4z(4)\, dy\, dz$$

$$= \int_0^3\int_0^{\sqrt{9-z^2}} (4y + 16z)\, dy\, dz$$

Using polar coordinates $y = r\cos\theta$
$z = r\sin\theta.$
$dy\,dz = r\,dr\,d\theta.$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{3} [4\cdot(r\cos\theta) + 16\cdot r\sin\theta]\, r\, dr\, d\theta.$$

$$= \int_0^{\pi/2} 4\cos\theta\cdot\left[\frac{r^3}{3}\right]_0^3 + 16\sin\theta\left[\frac{r^3}{3}\right]_0^3 \cdot d\theta$$

$$= \frac{4\times 27}{3}\int_0^{\pi/2}\cos\theta\, d\theta + \frac{16\cdot\times \cancel{27}^{9}}{\cancel{3}}\int_0^{\pi/2}\sin\theta\, d\theta$$

$$= 36.\left[\sin\theta\right]_0^{\pi/2} + 144\left[-\cos\theta\right]_0^{\pi/2} = 180.$$