PES University, Bangalore

(Established Under Karnataka Act 16 of 2013)

Department of Science and Humanities

Engineering Mathematics - I - UE21MA101

Unit - 1: Sequences & Series: Class Work Problems (10 Hours)

1. Examine the following sequences for convergence:

   i) $a_n = \frac{n^2-2n}{3n^2+n}$

   ii) $a_n = 2^n$

   iii) $a_n = 3 + (-1)^n$.

2. Show that the sequence $x_n = \frac{3n+4}{2n+1}$ is i) monotonic decreasing; ii) bounded; and iii) tends to limit $\frac{3}{2}$.

3. Discuss the convergence of the series $1 - \frac{1}{2} + \frac{1}{4} + \ldots + (-\frac{1}{2})^{n-1} + \ldots$.

4. Test the convergence of the following series of positive terms:

   (i) $\frac{1}{4 \cdot 7 \cdot 10} + \frac{4}{7 \cdot 10 \cdot 13} + \frac{9}{10 \cdot 13 \cdot 16} + \ldots \infty$

   (ii) $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \ldots \infty$

   (iii) $\displaystyle\sum_{n=1}^{\infty} \left[\sqrt{n^4+1} - \sqrt{n^4-1}\right]$

   (iv) $\displaystyle\sum_{n=1}^{\infty} \left[\sqrt[3]{n^3+1} - n\right]$

   (v) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$

   (vi) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n+1}}$

5. Apply the integral test to determine the convergence of the p-series: $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$

6. Test the convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(logn)^p}$

7. Test the following series for convergence:

   (i) $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^3} + \ldots$

(ii) $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}+\ldots$

(iii) $\frac{1^2}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}+\ldots$

8. Discuss the nature of the series $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{(n^n)^2}$

9. Test the series for convergence $\frac{1^2}{4^2}+\frac{1^2\cdot5^2}{4^2\cdot8^2}+\frac{1^2\cdot5^2\cdot9^2}{4^2\cdot8^2\cdot12^2}+\ldots$

10. Test the series for convergence $x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\ldots$

11. Test the series for convergence $\displaystyle\sum_{n=1}^{\infty}\frac{4\cdot7\cdot10\ldots(3n+1)}{1\cdot2\cdot3\ldots n}x^n$

12. Discuss the nature of the following series:

    (i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$

    (ii) $\displaystyle\sum_{n=1}^{\infty}a^{n^2}x^n$, $a<1$

    (iii) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{n^2}\cdot\frac{1}{3^n}$

    (iv) $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

13. Find the nature of the series $1+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\ldots$, $x>0$

Unit - 2: Partial Differentiation: Class Work Problems (12 Hours)

1. If $u = x^2tan^{-1}(\frac{y}{x}) - y^2tan^{-1}(\frac{x}{y})$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}$

2. Find the value of n so that $v = r^n(3cos^2\theta - 1)$ satisfies the equation:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial v}{\partial \theta}\right) = 0.$$

3. Find $\frac{df}{dt}$ at $t = 0$, where

   (i) $f(x, y) = xcosy + e^x siny, x = t^2 + 1, y = t^3 + t.$

   (ii) $f(x, y, z) = x^3 + xz^2 + y^3 + xyz, x = e^t, y = cost, z = t^3.$
   Answer: (i) $\frac{df}{dt} = e$**; (ii)** $\frac{df}{dt} = 3$

4. If $u = f(x - y, y - z, z - x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

5. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

6. If $u = x^2y$ and $x^2 + xy + y^2 = 1$, find $\frac{du}{dx}$
   Answer: $\frac{du}{dx} = 2xy + x^2\left(\frac{-2x-y}{x+2y}\right)$

7. If $x^y + y^x = c$, where c is a constant, find $\frac{dy}{dx}$
   Answer: $\frac{dy}{dx} = \frac{-(yx^{y-1}+y^x logy)}{(x^y logx+xy^{x-1})}$

8. If $x^2 + y^2 + z^2 = a^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial y}{\partial x}$ at $(1, -1, 2)$.
   Answer: $\frac{\partial z}{\partial x} = -\frac{1}{2}; \frac{\partial y}{\partial x} = 1$

9. Verify Euler's theorem for the following functions:

   (i) $u = y^n log\left(\frac{x}{y}\right)$

   (ii) $u = cos^{-1}\left(\frac{x}{y}\right) + tan^{-1}\left(\frac{y}{x}\right)$

10. If $u = \left(\sqrt{x^4 + y^4}\right) tan^{-1}(\frac{y}{x})$, prove the following:

    i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

    ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u$

11. If $u = x^3sin^{-1}(\frac{y}{x}) + x^4tan^{-1}(\frac{y}{x})$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ at $x = 1, y = 1$. **Answer:** $\frac{17\pi}{2}$

12. If $u = e^{\left(\frac{x^3+y^3}{3x+4y}\right)}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2ulogu.$

13. Find the Taylor's series expansion of $e^x cosy$ about the point $x = 1, y = \frac{\pi}{4}$.
Answer: $\frac{e}{\sqrt{2}}\left[1 + (x-1) - (y - \frac{\pi}{4}) + \frac{(x-1)^2}{2!} - (x-1)(y-\frac{\pi}{4}) - \frac{(y-\frac{\pi}{4})^2}{2!} + \ldots\right]$

14. Expand $e^{ax} sinby$ about origin upto 3^{rd} degree terms.
Answer: $(by + abxy) + \frac{1}{6}(3a^2bx^2y - b^3y^3) + \ldots$

15. Expand $log(1 + x - y)$ upto third degree terms about the origin.
Answer: $log(1 + x - y) = (x - y) - \frac{1}{2}(x-y)^2 + \frac{1}{3}(x-y)^3 + \ldots$

16. Compute $tan^{-1}\left(\frac{0.9}{1.1}\right)$ approximately.
Answer: $tan^{-1}\left(\frac{0.9}{1.1}\right) = 0.685$

17. Find the maximum and minimum values of the function $u(x, y) = x^3 + y^3 - 3x - 12y + 20$
Answer: u is maximum at $(-1, -2)$ and the maximum value is 38.
u is minimum at $(1, 2)$ and the minimum value is 2;
and $(-1, 2)$ & $(1, -2)$ are the saddle points.

18. Find the shortest distance from origin to the surface $xyz^2 = 2$.
Answer: 2

19. A scope probe in the shape of ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earth atmosphere and its surface begins to heat. After one hour the temperature at the point (x, y, z) is on the surface is $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the hottest point on the probe surface.
Answer: $\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}$

20. Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $xy + yz + zx = 3a^2$.
Answer: Minimum value of $f(x, y, z) = 3a^2$.

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Engineering Mathematics - I - UE21MA141A

Unit - 3: Ordinary Differential Equations: Class Work Problems (12 Hours)

1. Solve the equation $y^4dx = (x^{-\frac{3}{4}} - y^3x)dy$
   Answer: $(xy)^{\frac{7}{4}} = -\frac{7}{5}y^{-\frac{5}{4}} + c.$

2. Solve the differential equation $y' + 4xy + xy^3 = 0.$
   Answer: $y = (ce^{4x^2} - \frac{1}{4})^{-\frac{1}{2}}$

3. Solve the differential equation $\frac{dy}{dx} - y = y^2(sinx + cosx)$
   Answer: $y = \frac{1}{ce^{-x} - sinx}$

4. Check the equation $(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$ for exactness. If it is exact, find the solution.
   Answer: The given equation is exact and the solution is $x^3 + 2xe^y + y3 = c.$

5. Solve $(2xycosx^2 - 2xy + 1)dx + (sinx^2 - x^2 + 3)dy = 0.$
   Answer: $ysinx^2 - x^2y + x + 3y = c.$

6. Determine for what values of a and b, the following differential equation is exact and obtain the general solution of the exact equation $(y + x^3)dx + (ax + by^3)dy = 0.$
   Answer: $a = 1$ and the solution is $xy + \frac{x^4}{4} + \frac{by^4}{4} = c$ for all b; and c is the arbitrary constant.

7. Solve the differential equation $(5x^3 + 12x^2 + 6y^2)dx + 6xydy = 0.$
   Answer: $x^5 + 3x^4 + 3x^2y^2 = c.$

8. Solve the differential equation $(xy + y^2)dx + (x + 2y - 1)dy = 0.$
   Answer: $e^x(xy - y + y^2) = c.$

9. Solve the differential equation $(3x^2y^3e^y + y^3 + y^2)dx + (x^3y^3e^y - xy)dy = 0.$
   Answer: $x^3e^y + x + \frac{x}{y} = c.$

10. Solve the differential equation $(\frac{y}{x} \cdot secy - tany)dx + (secy \cdot logx - x)dy = 0.$
    Answer: $ylogx - xsiny = c.$

11. Solve the differential equation $(2xy + x^2)y' = 3y^2 + 2xy.$
    Answer: $\frac{x^3}{y(x+y)} = c.$

12. Solve the differential equation $(xysinxy + cosxy)ydx + (xysinxy - cosxy)xdy = 0.$
    Answer: $\frac{xsecxy}{y} = c.$

13. Find the orthogonal trajectories of the hyperbolas $x^2 - y^2 = c.$
    Answer: $xy = c.$

14. Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is a parameter.
Answer: $x^2 + y^2 - 2a^2 logx = c$.

15. Show that the one parameter family of curves $y^2 = 4c(c + x)$ are self orthogonal.

16. Find the orthogonal trajectories of the family of curves:
(i) $r^2 = csin(2\theta)$ and (ii) $r = c(sec\theta + tan\theta)$.
Answer: (i) $r^2 = c^* cos(2\theta)$ and (ii) $r = c^* e^{-sin\theta}$.

17. Solve $p^3 + 2xp^2 - p^2y^2 - 2xy^2p = 0$.
Answer: $(y - c)(y + x^2 - c)(\frac{1}{y} + x + c) = 0$.

18. Solve $p(p + y) = x(x + y)$.
Answer: $(y - \frac{x^2}{2} - c)(e^x(x + y - 1) - c) = 0$.

19. Solve $x^2(\frac{dy}{dx})^4 + 2x\frac{dy}{dx} - y = 0$.
Answer: $y = c^4 + 2c\sqrt{x}$.

20. Solve $y = xp^2 + p$.
Answer: $x = \frac{1}{(1-p)^2}(logp - p + c)$ and $y = \frac{p^2}{(1-p)^2}(logp - p + c)$

21. Solve $y = 2px + y^2p^3$.
Answer: $y^2 = 2cx + c^3$.

22. Solve $y = p^2y + 2px$.
Answer: $y^2 = c^2 + 2cx$.

23. If the temperature of the air is 30^0C and a metal ball cools from 100^0C to 70^0C in 15 minutes, find how long will it take for the metal ball to reach a temperature of 40^0C.
Answer: $k \approx 0.0373$ and $t = 52.17 \approx 52.2$. Thus, we conclude that it will take 52.2 minutes for the metal ball to reach a temperature of 40^0C.

24. A bottle of mineral water at a room temperature of 72^0F is kept in a refrigerator where the temperature is 44^0F. After half an hour, water cooled to 61^0F.
(i) What is the temperature of the mineral water in another half an hour?
(ii) How long will it take to cool to 50^0F?
Answer: (i) $k \approx 0.0166$ and $(T)_{t=60} \approx 54.3$. Thus the temperature of the mineral water after another half an hour is 54.3^0F.
(ii) We have to find t when $T = 50$.
$t \approx 92.8$. Thus we conclude that it will take 93 minutes (about one and half hours) for the cooling of the mineral water to 50^0F.

Unit - 4: Higher Order Differential Equations: Class Work Problems (12 Hours)

1. Solve the differential equation $4y'''' - 12y''' - y'' + 27y' - 18y = 0$.
   Answer: $y(x) = c_1e^x + c_2e^{2x} + c_3e^{\frac{-3x}{2}} + c_4e^{\frac{3x}{2}}$.

2. Solve the initial value problem:
   $y''' - 6y'' + 11y' - 6y = 0; y(0) = 0, y'(0) = -4, y''(0) = -18$.
   Answer: The constants $c_1 = 1; c_2 = 2; c_3 = -3$ and $y(x) = e^x + 2e^{2x} - 3e^{3x}$.

3. Solve $y'' + 2y' + y = 2e^{3x}$.
   Answer: $y(x) = (c_1 + c_2x)e^{-x} + \frac{e^{3x}}{8}$.

4. Solve $y''' - 2y'' - 5y' + 6y = 2e^x + 4e^{3x} + 7e^{-2x} + 8e^{2x} + 15$.
   Answer: $y(x) = c_1e^x + c_2e^{3x} + c_3e^{-2x} - \frac{1}{3}xe^x + \frac{2}{5}xe^{3x} + \frac{7}{15}xe^{-2x} - 2e^{2x} + \frac{15}{6}$.

5. Solve $y'' + 4y = sin3x + cos2x$.
   Answer: $y(x) = c_1cos2x + c_2sin2x - \frac{1}{5}sin3x + \frac{x}{4}sin2x$.

6. Solve $y'' + 5y' - 6y = sin4x \cdot sinx$.
   Answer: $y(x) = c_1e^x + c_2e^{-6x} + \frac{1}{2}\left[\frac{sin3x - cos3x}{30} + \frac{31cos5x - 25sin5x}{1586}\right]$.

7. Solve $y'' - y = 2x^4 - 3x + 1$.
   Answer: $y(x) = c_1e^x + c_2e^{-x} - [2x^4 + 24x^2 - 3x + 49]$.

8. Solve $y''' - y = x^5 + 3x^4 - 2x^3$.
   Answer: $y(x) = c_1e^x + e^{-\frac{x}{2}}\left[c_2cos\frac{\sqrt{3}}{2}x + c_3sin\frac{\sqrt{3}}{2}x\right] - [x^5 + 3x^4 - 2x^3 + 60x^2 + 72x + 12]$.

9. Solve $y'''' - y = cosx \cdot coshx$.
   Answer: $y(x) = c_1e^x + c_2e^{-x} + c_3cosx + c_4sinx - \frac{1}{5}cosx \cdot coshx$.

10. Solve $y''' - 7y' - 6y = e^{2x}(1 + x)$.
    Answer: $y(x) = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x} - \frac{e^{2x}}{12}\left[x + \frac{17}{12}\right]$.

11. Solve $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + logx$.
    Answer: $y(x) = \frac{c_1}{x} + \sqrt{x}\left[c_2cos\frac{\sqrt{3}}{2}logx + c_3sin\frac{\sqrt{3}}{2}logx\right] + \frac{x}{2} + logx$.

12. $2x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - \frac{y}{x} = 5 - \frac{sin(logx)}{x^2}$.
    Answer: $y(x) = \frac{c_1}{x} + c_2\sqrt{x} + \frac{5}{2}x - \frac{1}{13}\left[\frac{3cos(logx) - 2sin(logx)}{x}\right]$.

13. Solve $(2x + 5)^2\frac{d^2y}{dx^2} - 6(2x + 5)\frac{dy}{dx} + 8y = 6x$.
    Answer: $y(x) = c_1(2x + 5)^{2+\sqrt{2}} + c_2(2x + 5)^{2-\sqrt{2}} - \frac{3}{4}(2x + 5) - \frac{15}{8}$.

14. Solve $(3x - 2)^2\frac{d^2y}{dx^2} - 3(3x - 2)\frac{dy}{dx} = 9(3x - 2)sin(log(3x - 2))$.
    Answer: $y(x) = c_1 + c_2(3x - 2)^2 - \frac{1}{2}(3x - 2)sin(log(3x - 2))$.

15. Find the general solution of the equation $y'' + 3y' + 2y = 2e^x$ using the method of variation of parameters.
    Answer: $y(x) = c_1e^{-x} + c_2e^{-2x} + \frac{1}{3}e^x$.

16. Find the general solution of the equation $y^{''} + 16y = 32sec2x$ using the method of variation of parameters.
Answer: $y(x) = c_1cos4x + c_2sin4x + 8cos2x - 4sin4x \cdot log(sec2x + tan2x)$.

17. Find the general solution of the equation $y^{''} - y = 2(1 - e^{-2x})^{-\frac{1}{2}}$ using the method of variation of parameters.
Answer: $y(x) = c_1e^x + c_2e^{-x} - e^x \cdot sin^{-1}(e^{-x}) - (e^{2x} - 1)^{\frac{1}{2}}e^{-x}$.

18. A voltage $E = E_0e^{-at}$, where E_0 and a are constants is applied at time $t > 0$ to an LR circuit of inductance L and resistance R. Find the charge and current at time $t > 0$, given that the circuit carries no charge and no current at time $t = 0$.
Answer: $q = \frac{E_0}{a-\frac{R}{L}}\left[\frac{1}{R}(1 - e^{-\left(\frac{R}{L}\right)t}) - \frac{1}{aL}(1 - e^{-at})\right]$, which is the charge at time $t > 0$; and $i = \frac{E_0}{L\left(a-\frac{R}{L}\right)}\left[e^{-(\frac{R}{L})t} - e^{-at}\right]$, which is the current at time $t > 0$.

19. At time $t > 0$, an e.m.f of voltage $E = E_0(1 - cost)$, where E_0 is a constant is applied to an LRC circuit for which $L = R = C = 1$. Initially, there is no charge or current in the circuit. Find the charge and current at time $t > 0$.
Answer: $q = E_0\left[e^{-\frac{t}{2}}\left[\frac{1}{\sqrt{3}}sin\frac{\sqrt{3}}{2}t - cos\frac{\sqrt{3}}{2}t\right] + (1 - sint)\right]$, which is the charge at time $t > 0$; and $i = E_0\left[e^{-\frac{t}{2}}\left[cos\frac{\sqrt{3}}{2}t + \frac{1}{2}(\sqrt{3} - \frac{1}{\sqrt{3}})sin\frac{\sqrt{3}}{2}t\right] - cost\right]$, which is the current at time $t > 0$.

20. A body weighing 4.9 kg is hung from a spring. A pull of 10 kg weight will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement (x) of the body from its equilibrium position at time t seconds; the maximum velocity; and the period of oscillation.
Answer: Displacement $x = 0.06\,cos20t$; maximum velocity is 1.2 m/sec; and the period of oscillation is 0.314 sec.

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Unit - 5: Special Functions: Class Work Problems (10 Hours)

Beta and Gamma functions:

Prove the following standard results.

1. $\beta(m, n) = \beta(n, m)$
2. $\beta(m, n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta \cdot cos^{2n-1}\theta d\theta = 2\int_0^{\frac{\pi}{2}} sin^{2n-1}\theta \cdot cos^{2m-1}\theta d\theta$
3. $\int_0^{\frac{\pi}{2}} sin^m\theta \cdot cos^n\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right), m > -1, n > -1.$
4. $\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$
5. Relationship between Beta and Gamma functions: $\beta(m, n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}, m > 0; n > 0.$
6. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
7. Given that $\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{sinp\pi}$, show that $\Gamma(p) \cdot \Gamma(1-p) = \frac{\pi}{sinp\pi}, 0 < p < 1.$

Problems on Beta and Gamma functions:

1. Evaluate $\int_0^{\infty} \sqrt{x} \cdot e^{-x^2} dx$. **Answer**: $\frac{1}{2}\Gamma\left(\frac{3}{4}\right)$
2. Evaluate $\int_0^{\infty} e^{-x^3} dx$. **Answer**: $\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$
3. Using Beta and Gamma functions, evaluate the integral $\int_{-1}^{1} (1 - x^2)^n dx$, where n is a positive integer. **Answer**: $\frac{2^{2n+1}(n!)^2}{(2n+1)!}$
4. Evaluate $\int_0^{\infty} (x^2 + 4)e^{-2x^2} dx$. **Answer**: $\frac{17\sqrt{\pi}}{8\sqrt{2}}$
5. Evaluate $\int_0^{\infty} 2^{-9x^2} dx$ using the Gamma function. **Answer**: $\frac{1}{6}\sqrt{\frac{\pi}{ln2}}$
6. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{sin^8 x}}{\sqrt{cosx}} dx$. **Answer**: $\frac{60}{13}\frac{\Gamma(\frac{5}{6})\cdot\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{12})}$

7. $\int_0^2 (8-x^3)^{-\frac{1}{3}}dx$. **Answer**: $\frac{2\pi}{3\sqrt{3}}$

8. $\int_0^a x^4\sqrt{a^2-x^2}dx$. **Answer**: $\frac{\pi a^6}{32}$

9. Evaluate $\int_0^3 \frac{x^{\frac{3}{2}}}{\sqrt{3-x}} \times \int_0^1 \frac{dx}{\sqrt{1-x^{\frac{1}{4}}}}$. **Answer**: $\frac{432\pi}{35}$

10. Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (cos\theta + sin\theta)^{\frac{1}{3}}$. **Answer**: $\frac{6\sqrt{\pi}}{2^{\frac{5}{6}}}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})}$

Special Functions:

1. Bessel's Differential Equation and Bessel Functions: Obtain the series solution of Bessel's differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)y = 0$$

of order v, where v is a non-negative real number.

2. Prove that $J_{-n}(x) = (-1)^n J_n(x)$, where n is a positive integer.

Derivatives of Bessel functions:

3. Prove that $[x^v J_v(x)]' = x^v J_{v-1}(x)$

4. Prove that $[x^{-v} J_v(x)]' = -x^{-v} J_{v+1}(x)$

Recurrence relations:
Bessel's function of the first kind satisfies the following recurrence relations:

5. Prove that $xJ_v'(x) = xJ_{v-1}(x) - vJ_v(x)$

6. Prove that $xJ_v'(x) = vJ_v(x) - xJ_{v+1}(x)$

7. Prove that $2J_v'(x) = J_{v-1}(x) - J_{v+1}(x)$

8. Prove that $2vJ_v(x) = x\left[J_{v-1}(x) + J_{v+1}(x)\right]$
Note: Prove the recurrence relations {Problem.3 - Problem.8} using the definition of Bessel function.

Problems:

9. Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.
Answer: $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$.

10. Express $J_{\frac{5}{2}}(x)$ and $J_{-\frac{5}{2}}(x)$ in terms sine and cosine functions, where $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} sinx$ and $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} cosx$.
Answer: $J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[sinx \left(\frac{3-x^2}{x^2} \right) - \frac{3cosx}{x} \right]$; $J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3sinx}{x} + \left(\frac{3-x^2}{x^2} \right) cosx \right]$.

11. Evaluate (i) $\displaystyle\int J_3(x)dx$; (ii) $\displaystyle\int x^4 J_1(x)dx$.
Answer: (i) $c - J_2(x) - \frac{2}{x}J_1(x)$; (ii) $(8x^2 - x^4)J_0(x) + (4x^3 - 16x)J_1(x)$.

12. **Generating function for Bessel function** $J_n(x)$: Obtain the generating function for Bessel's function as $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$.

13. **Orthogonality of Bessel function**: Prove that $\displaystyle\int_0^a xJ_n(\alpha x)J_n(\beta x) := \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{a^2}{2}J_{n+1}^2(a\alpha) & \text{if } \alpha = \beta \end{cases}$, where α and β are the roots of $J_n(ax) = 0$.

14. Establish the **Jacobi series**:
i) $cos(xcos\theta) = J_0 - 2J_2cos2\theta + 2J_4cos4\theta - \ldots$
ii) $sin(xcos\theta) = 2\left[J_1cos\theta - J_3cos3\theta + J_5cos5\theta - \ldots\right]$; and hence prove that
$J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \ldots = 1$.

15. **Bessel's integral formula**: Prove that $J_n(x) = \frac{1}{\pi}\displaystyle\int_0^{\pi} cos(n\theta - xsin\theta)d\theta$, where n is a positive integer.