UE20MA151

ENGINEERING MATHEMATICS

CLASS WORK PROBLEMS

UNIT 5 : FOURIER SERIES

1) Obtain a Fourier Series for $f(x) = x^3$ for $-\pi < x < \pi$.

$f(x) = x^3$ is an odd function.

since $f(-x) = (-x)^3 = -x^3 = -f(x)$.

Therefore for an odd function, the fourier expansion is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \left(\text{Here } a_0 = 0 \text{ and } a_n = 0\right).$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$.

calculation of Fourier Coefficient:

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^3 \cdot \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right) - 3x^2\left(\frac{-1}{n}\right)\frac{\sin nx}{n} + 6x \cdot \left(\frac{-1}{n^2}\right)\cdot\left(\frac{-\cos nx}{n}\right) - 6\cdot\left(\frac{1}{n^3}\right)\frac{\sin nx}{n}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[-\frac{x^3\cos nx}{n} + \frac{6x\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi^3\cos n\pi}{n} + \frac{6\pi\cos n\pi}{n^3} - 0\right]$$

$$= \frac{2}{\pi}\times\pi\left[\frac{-\pi^2(-1)^n}{n} + \frac{6(-1)^n}{n^3}\right]$$

$$b_n = 2(-1)^n\left[\frac{-\pi^2}{n} + \frac{6}{n^3}\right]$$

$\therefore$ The fourier expansion is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\boxed{x^3 = \sum_{n=1}^{\infty} 2(-1)^n \cdot \left[\frac{-\pi^2}{n} + \frac{6}{n^3}\right] \sin nx.}$$

2) Find the Fourier series of $f(x) = x + x^2$ for $-\pi < x < \pi$. and hence deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$$

Solution:

The Fourier expansion of $f(x)$ over $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin nx\,dx.$$

Calculation of Fourier coefficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \left(\frac{\pi^2}{2} + \frac{(-\pi)^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} \cancel{-\frac{\pi^2}{2}} + \frac{\pi^3}{3}\right]$$

$$a_0 = \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right] = \frac{2\pi^2}{3}. \Rightarrow \boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx \cdot dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cdot \cos nx \, dx.$$

$$= \frac{1}{\pi}\Big[(x+x^2)\cdot \frac{\sin nx}{n} - (1+2x)\cdot\left(\frac{1}{n}\right)\cdot\left(\frac{-\cos nx}{n}\right)$$

$$+ 2\cdot\left(\frac{-1}{n^2}\right)\cdot \frac{\sin nx}{n}\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[+(1+2x)\left(\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2}\Big((1+2\pi)\cos n\pi - (1-2\pi)\underbrace{\cos n(-\pi)}_{\cos n\pi}\Big]$$

$$= \frac{1}{\pi n^2}\left(\cancel{\cos n\pi} + 2\pi \cos n\pi - \cancel{\cos n\pi} + 2\pi \cos n\pi\right)$$

$$= \frac{4\pi \cos n\pi}{n^2 \pi} = \frac{4(-1)^n}{n^2}.$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin nx \cdot dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cdot \sin nx \, dx.$$

$$= \frac{1}{\pi}\Big[(x+x^2)\cdot\left(-\frac{\cos nx}{n}\right) - (1+2x)\cdot\left(\frac{-1}{n}\right)\cdot\left(\frac{+\sin nx}{n}\right)$$

$$+ 2\cdot\left(\frac{-1}{n^2}\right)\left(\frac{-\cos nx}{n}\right)\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) + \frac{2}{n^3}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\Big(\left(\pi+\pi^2\right)\left(-\frac{\cos n\pi}{n}\right) + \frac{2}{n^3}\cos n\pi$$

$$- \left[\left(-\pi+\pi^2\right)\left(-\frac{\cos n(-\pi)}{n}\right) + \frac{2}{n^3}\cos n(-\pi)\right]\Big).$$

$$= \frac{1}{\pi}\Big[-\frac{\pi\cos n\pi}{n} \cancel{-\frac{\pi^2}{n}\cos n\pi} \cancel{+\frac{2}{n^3}\cos n\pi} - \frac{\pi\cos n\pi}{n}.$$

$$\cancel{+\frac{\pi^2}{n}\cos n\pi} \cancel{-\frac{2}{n^3}\cos n\pi}\Big]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n}\right] = \frac{-2}{n}\cos n\pi = \frac{-2}{n}(-1)^n$$

$$\boxed{b_n = \frac{2}{n}(-1)^{n+1}}$$

The required Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$\boxed{x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx + \frac{2}{n}(-1)^{n+1} \sin nx} \quad (1)$$

Deduction

Put $x = \pi$ in (1)

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx + \frac{2}{n}(-1)^{n+1} \sin nx$$

since $f(x) = x + x^2$ is defined in $-\pi < x < \pi$

The value of $f(x)$ at $x = \pi$ be given by $\frac{1}{2}[f(-\pi) + f(\pi)]$.

$$\therefore \frac{1}{2}\{f(-\pi) + f(\pi)\} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos n\pi$$

$$\frac{1}{2}\{(\cancel{-\pi}) + \pi^2 + \cancel{\pi} + \pi^2\} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^{2n}.$$

$$\frac{\cancel{2}\pi^2}{\cancel{2}} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\frac{2}{3}\pi^2 = 4\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3 \times 4}$$

$$\boxed{\sum \frac{1}{n^2} = \frac{\pi^2}{6}.}$$

3) Obtain the Fourier series to represent e^{-ax} from $x=-\pi$ to $x=\pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$

The Fourier expansion of $f(x)$ over $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx.$$

<u>Calculation of Fourier coefficients</u>:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax} \cdot dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$a_0 = \frac{-1}{a\pi}\left[e^{-a\pi} - e^{a\pi}\right] = \frac{2\sinh a\pi}{a\pi}.$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx \, dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cdot \cos nx \, dx.$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}\left(-a\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$\int e^{ax}\cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right).$$

$$= \frac{-a}{\pi}\left[\frac{e^{-ax}\cos nx}{a^2+n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{-a}{\pi(a^2+n^2)}\cdot\left[e^{-a\pi}\cos n\pi - e^{-a(-\pi)}\cos n(-\pi)\right].$$

$$= -\frac{a\cos n\pi}{\pi(a^2+n^2)}\left[e^{-a\pi} - e^{a\pi}\right]$$

$$\boxed{a_n = \frac{2(-1)^n . a . \sinh a\pi}{\pi(a^2+n^2)}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cdot \sin nx\, dx.$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}(-a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}.$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-a\pi}\cos n\pi - e^{-a(-\pi)}\cos n(-\pi)\right]$$

$$b_n = \frac{-n\cos n\pi}{\pi(a^2+n^2)}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$\boxed{b_n = \frac{2n(-1)^n\cdot\sinh a\pi}{\pi(a^2+n^2)}}.$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx).$$

The required Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$e^{-ax} = \frac{1}{2} \cdot \frac{2 \sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n \cdot \sinh a\pi}{\pi(a^2+n^2)} \cos nx$$

$$+ \sum_{n=1}^{\infty} \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)} \sin nx.$$

$$\boxed{e^{-ax} = \frac{\sinh a\pi}{a\pi} \left\{ 1 + \sum_{n=1}^{\infty} \frac{2a^2(-1)^n \cos nx}{\pi^2+n^2} + \frac{2an(-1)^n \sin nx}{a^2+n^2} \right\}}$$

To deduce the series we shall put $a=1$, $x=0$ in the Fourier series expansion.

$$e^{-a(0)} = \frac{\sinh \pi}{\pi} \left\{ 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} \cos n(0) \right\}$$

$$1 = \frac{\sinh \pi}{\pi} \left\{ 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} \right\}.$$

$$\frac{\pi}{\sinh \pi} = 1 + 2\left[\frac{-1}{1+1^2} + \frac{1}{1+2^2} + \frac{-1}{1+3^2} + \frac{1}{1+4^2} + \cdots \right]$$

$$\frac{\pi}{\sinh \pi} = 2\left(\frac{1}{5} - \frac{1}{10} + \frac{1}{17} + \cdots \right)$$

4) Find the Fourier Series expansion of

$f(x) = x(1-x)(2-x)$ in $(0,2)$.

The fourier expansion of $f(x)$ over $(0, 2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}.$$

The period of $f(x) = 2$.

$$2l = 2 \Rightarrow \boxed{l=1}.$$

The relevant Fourier series is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x$$

where $a_0 = \int_0^2 f(x)\,dx.$

$$a_n = \int_0^2 f(x) \cos n\pi x \, dx$$

$$b_n = \int_0^2 f(x) \sin n\pi x \, dx.$$

Calculation of Fourier Coefficients.

$$a_0 = \int_0^2 f(x)\,dx$$

$$= \int_0^2 (2x - 3x^2 + x^3)\,dx$$

$$= \left[2\cdot\frac{x^2}{2} - 3\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = 4 - 8 + \frac{16}{4}$$

$$\boxed{a_0 = 0}.$$

$$a_n = \int_0^2 f(x) \cos n\pi x \, dx.$$

$$= \int_0^2 (2x - 3x^2 + x^3) \cos n\pi x \, dx.$$

$$= \Big[(2x - 3x^2 + x^3) \frac{\sin n\pi x}{n\pi} - (2 - 6x + 3x^2) \frac{-\cos n\pi x}{(n\pi)^2}$$

$$+ (-6 + 6x) \left(\frac{-1}{n^2\pi^2}\right) \frac{\sin n\pi x}{n\pi} - 6 \left(\frac{-1}{n^3\pi^3}\right)\left(\frac{-\cos n\pi x}{n\pi}\right) \Big]_0^2$$

$$= \left[(2 - 6x + 3x^2) \cdot \frac{\cos n\pi x}{(n\pi)^2} - \frac{6}{(n\pi)^4} \cos n\pi x \right]_0^2.$$

$$= \left[2 \cdot \frac{\cos 2n\pi}{(n\pi)^2} - \frac{6}{(n\pi)^4} \cos 2n\pi - \left(\frac{2}{(n\pi)^2} - \frac{6}{(n\pi)^4} \right) \right]$$

$$= \left[\frac{2}{(n\pi)^2} - \frac{6}{(n\pi)^4} - \frac{2}{(n\pi)^2} + \frac{6}{(n\pi)^4} \right]$$

$$= 0$$

$$\boxed{a_n = 0.}$$

$$b_n = \int_0^2 f(x) \sin n\pi x \, dx$$

$$= \int_0^2 (2x - 3x^2 + x^3) \sin n\pi x \, dx.$$

$$= \left[(2x - 3x^2 + x^3)\left(\frac{-\cos n\pi x}{n\pi}\right) - (2 - 6x + 3x^2)\left(\frac{-1}{n\pi}\right)\frac{\sin n\pi x}{n\pi} \right.$$

$$\left. + (-6 + 6x)\left(\frac{-1}{(n\pi)^2}\right)\left(\frac{-\cos n\pi x}{n\pi}\right) - (6)\frac{}{(n\pi)^3} \cdot \frac{\sin n\pi x}{n\pi} \right]_0^2$$

$$= \left[-(2x - 3x^2 + x^3)\frac{\cos n\pi x}{n\pi} + (-6 + 6x) \cdot \frac{\cos n\pi x}{(n\pi)^3} \right]_0^2$$

$$= \left[0 + 6 \cdot \frac{\cos 2n\pi}{(n\pi)^3} - \frac{(-6) \cdot}{(n\pi)^3} \right]$$

$$= \frac{6}{(n\pi)^3} + \frac{6}{(n\pi)^3}$$

$$\boxed{b_n = \frac{12}{n^3\pi^3}.}$$

$$b_n = \frac{12}{n^3\pi^3}.$$

The required Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x$$

$$f(x) = 0 + \sum_{n=1}^{\infty} 0 \cdot \cos n\pi x + \frac{12}{n^3\pi^3} \sin n\pi x.$$

$$\boxed{\therefore f(x) = \sum_{n=1}^{\infty} \frac{12}{n^3\pi^3} \sin n\pi x.}$$

5) A periodic function of period 4 is defined as $f(x) = |x|$, $-2 < x < 2$. Find its Fourier series expansion.

The Fourier expansion of $f(x)$ over $(0, 2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}.$$

Here $l = 2$.

The relevant Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2}$$

where

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx$$

$$a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cdot \cos \frac{n\pi x}{2} \cdot dx$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\cdot \sin\frac{n\pi x}{2}\,dx.$$

<u>Calculation of Fourier Coefficients</u>

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx.$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} |x|\cdot dx.$$

$f(x) = |x|$ is an even function.

$\Rightarrow b_n = 0.$

$$a_0 = \frac{1}{\cancel{2}}\times\cancel{2}\int_{0}^{2} x\cdot dx = \left[\frac{x^2}{2}\right]_0^2$$

$$\boxed{a_0 = 2} \qquad = \frac{1}{2}(4) = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2}{2}\int_{0}^{2} x\cdot\cos\frac{n\pi x}{2}\,.dx.$$

$$= \left[x\cdot\frac{\sin\frac{n\pi x}{2}}{n\pi/2} - (1)\cdot\frac{1}{\left(\frac{n\pi}{2}\right)}\cdot\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}\right]_0^2$$

$$a_n = \frac{4}{n^2\pi^2}\left[\cos n\pi - 1\right] = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

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The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$$

$$\boxed{|x| = \frac{2}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \cos\frac{n\pi x}{2}.}$$

6) Find the Fourier series of

$$f(x) = \begin{cases} 2 & -2 < x < 0 \\ x & 0 < x < 2. \end{cases}$$

The Fourier expansion of $f(x)$ over $(-l, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}.$$

Here $l = 2$.

The relevant Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}$$

where

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x) \cos\frac{n\pi x}{2}\,dx$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x) \sin\frac{n\pi x}{2}\,dx$$

Calculation of Fourier Coefficients

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\, dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2 \cdot dx + \int_{0}^{2} x\, dx\right]$$

$$= \frac{1}{2}\left\{ 2\left[x\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_0^2 \right\}$$

$$a_0 = \frac{1}{2}\left\{4 + 2\right\} = \underline{\underline{3}}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x) \cdot \cos\frac{n\pi x}{2}\, dx.$$

$$a_n = \frac{1}{2}\left\{\int_{-2}^{0} 2 \cdot \cos\frac{n\pi x}{2}\, dx + \int_0^2 x \cdot \cos\frac{n\pi x}{2}\, dx\right\}$$

$$= \frac{1}{2}\left\{\left[2 \cdot \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_{-2}^{0} + \left[x \cdot \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (1)\cdot\left(\frac{1}{\frac{n\pi}{2}}\right)\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2\right\}$$

$$= \frac{1}{2}\left\{ + \frac{4}{n^2\pi^2} \cdot \cos\frac{n\pi x}{2}\right\}_0^2 = \frac{2}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$\boxed{a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]}$$

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$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\cdot \sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left\{ \int_{-2}^{0} 2\cdot \sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cdot \sin\frac{n\pi x}{2}\,dx \right\}.$$

$$= \frac{1}{2}\left[(2)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_{-2}^{0} + \left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - \frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2 \right]$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\left[-1+(-1)^n\right] + \frac{2}{n\pi}\left(2\cdot(-1)^n\right)\right]$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}(-1) + (-1)^n\frac{4}{n\pi} - \frac{4}{n\pi}(-1)^n\right].$$

$$\boxed{b_n = \frac{-2}{n\pi}}$$

Fourier series is given by

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\frac{n\pi x}{2}$$

$$- \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi x}{2}$$

Half-Range Sine and Cosine Series

1) Find the Fourier Sine Series of the function $f(x) = \cos x$ defined on the interval $[0, \pi]$.

We require half range Fourier sine series for $f(x) = \cos x$ in $[0, \pi]$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin nx \, dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos x \cdot \sin nx \, dx.$$

$$\boxed{2 \sin A \cdot \cos B = \sin(A+B) + \sin(A-B)}$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(nx + x) + \sin(nx - x) \, dx.$$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x + \sin(n-1)x] \, dx.$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} - \left[\frac{-1}{n+1} - \frac{1}{n-1} \right] \right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n \left[\frac{1}{n+1} + \frac{1}{n-1}\right] + \frac{n-\cancel{1}+n+\cancel{1}}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n \left[\frac{n-1+n+1}{n^2-1}\right] + \frac{2n}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2n(-1)^n}{n^2-1} + \frac{2n}{n^2-1}\right]$$

$$\boxed{b_n = \frac{2n}{\pi}\left[\frac{(-1)^n+1}{n^2-1}\right] \quad \text{where } n \neq 1.}$$

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Calculation of b_1 :

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \cdot \sin x \, dx$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(2x) + \sin(0) \cdot dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$b_1 = \frac{-1}{2\pi}\left[\cos 2\pi - \cos 0\right] = 0 .$$

$\therefore$ The half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= 0 + \sum_{n=2}^{\infty} \frac{2n}{\pi}\left[\frac{(-1)^n+1}{n^2-1}\right] \sin nx$$

$$\boxed{\cos x = \sum_{n=2}^{\infty} \frac{2n}{\pi}\left[\frac{(-1)^n+1}{n^2-1}\right] \sin nx .}$$

2) If $f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$ then

Show that

$$f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} \cdots\right]$$

We require to find Half range Fourier sine series for $f(x)$.

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cdot \sin nx \, dx.$$

<u>Calculation of b_n:</u>

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \cdot \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\left[x \cdot \left(-\frac{\cos nx}{n}\right) - (1) \cdot \left(\frac{-1}{n}\right) \frac{\sin nx}{n}\right]_0^{\pi/2}\right.$$

$$\left. + \left[(\pi - x) \cdot \left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-1}{n}\right) \frac{\sin nx}{n}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cdot\left(\pi/_2 \cdot \cos n\pi/_2 - 0\right) + \frac{1}{n^2}\left(\sin n\pi/_2 - \sin 0\right)\right.$$

$$-\frac{1}{n}\left[\left(\pi - \pi\right)\cdot \cos n\pi - \left(\pi - \pi/_2\right)\cos n\pi/_2\right]$$

$$\left. -\frac{1}{n^2}\left[\sin n\pi - \sin n\pi/_2\right]\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{-\pi}{2n}\cdot\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \cancel{\frac{\pi}{2n}\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right]$$

$$\boxed{b_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}.}$$

The required sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi n^2}\sin\frac{n\pi}{2}\cdot \sin nx.$$

$$\boxed{f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots\right\}}$$

3) Obtain the Fourier expansion of $x \sin x$ as a cosine series in $(0, \pi)$.

The Half range Fourier Cosine Series is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Where
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

Calculation of Fourier Coefficients:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cdot dx$$

$$= \frac{2}{\pi}\left[x\cdot(-\cos x) - (1)\cdot(-1)\cdot \sin x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi + 0\right]$$

$$= +\frac{2\pi}{\pi} = 2 \Rightarrow \boxed{a_0 = 2.}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x \cdot \cos nx \, dx.$$

$$2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B).$$

$$= \frac{1}{\pi}\int_0^{\pi} x \cdot \left[\sin(n+1)x - \sin(n-1)x\right] dx.$$

$$= \frac{1}{\pi}\Bigg[x \cdot \left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - (1)\cdot\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\Bigg]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - 0\right]\right]$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = \frac{(-1)^n}{n+1} + \frac{(-1)^{n-1}}{n-1}$$

$$\Rightarrow (-1)^n\left[\frac{+1}{n+1} - \frac{1}{n-1}\right] = (-1)^n\left[\frac{n-1-n-1}{n^2-1}\right]$$

$$\boxed{\text{NOTE}: (-1)^{-1} = \frac{1}{-1} = -1.}$$

$$\boxed{a_n = \frac{2(-1)^{n+1}}{n^2-1} \quad \text{where } n \neq 1}$$

Calculation of a_1 :

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \cdot \sin x \cdot \cos x \, dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cdot \frac{\sin 2x}{2} \, dx.$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-1}{2}\right)\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-1}{2}\left[\pi \cos 2\pi - 0\right]\right]$$

$$\Rightarrow \frac{-1}{2\cancel{\pi}} \times \cancel{\pi} \cos 2\pi = \frac{-1}{2}.$$

$$\boxed{a_1 = -\tfrac{1}{2}.}$$

The Half range Fourier Cosine Series. is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx.$$

$$\boxed{x \sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} a_n \cos nx.}$$

4) obtain the half range sine series for the function $f(x) = x^2$ in the interval $0 < x < 3$.

Half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx.$$

Here $l = 3$.

$\therefore$ The relevant Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3}$$

Where

$$b_n = \frac{2}{3} \int_0^3 f(x) \cdot \sin \frac{n\pi x}{3} dx.$$

Calculation of Fourier Coefficients

$$b_n = \frac{2}{3} \int_0^3 f(x) \sin \frac{n\pi x}{3} dx.$$

$$= \frac{2}{3} \int_0^3 x^2 \cdot \sin \frac{n\pi x}{3} dx.$$

$$b_n = \frac{2}{3}\left[x^2 \cdot \left(\frac{-\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}} \right) - (2x)\cdot\left(\frac{-1}{\frac{n\pi}{3}}\right) \frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}} \right.$$

$$\left. + 2\cdot\left(\frac{-1}{\left(\frac{n\pi}{3}\right)^2}\right)\cdot \frac{\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}} \right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{n\pi}\left[3^2\cdot\cos n\pi - 0\right] - \frac{27\times 2}{n^3\pi^3}\cdot\left[\cos n\pi - 1\right]\right]$$

$$= \frac{-2\times 9\,(-1)^n}{n\pi} - \frac{\cancel{27}^{9}\times 4}{\cancel{3}\cdot n^3\pi^3}(-1)^n + \frac{\cancel{27}^{9}\times 4}{\cancel{3}\,n^3\pi^3}.$$

$$= \frac{36}{n^3\pi^3} \cdot \frac{-18(-1)^n}{n\pi} - \frac{36}{n^3\pi^3}(-1)^n.$$

$$b_n = \frac{36}{n^3\pi^3}\left(1-(-1)^n\right) - \frac{18}{n\pi}(-1)^n.$$

$\therefore$ Half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3}$$

$$\boxed{x^2 = \sum_{n=1}^{\infty}\left(\frac{36}{n^3\pi^3}\left(1-(-1)^n\right) - \frac{18}{n\pi}(-1)^n\right)\sin\frac{n\pi x}{3}}$$

5) Find the half-range cosine series for

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \end{cases}$$

Cosine Series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where

$$a_0 = \frac{2}{l} \int_0^l f(x) \cdot dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \cos \frac{n\pi x}{l} \, dx .$$

Here function is defined in half-range, i.e. $(0, l) \Rightarrow \boxed{l = 2}$

Calculation of Fourier Coefficients :

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$l = 2$.

$$a_0 = \frac{\cancel{2}}{\cancel{2}} \int_0^2 f(x) \, dx.$$

$$= \int_0^2 f(x) \, dx = \int_0^1 x \, dx + \int_1^2 (2-x) \, dx.$$

$$a_0 = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$a_0 = 1.$$

$$a_n = \frac{2}{l}\int_0^l f(x).\ \cos\frac{n\pi x}{l}\ .dx.$$

$$= \frac{2}{2}\int_0^2 f(x).\ \cos\frac{n\pi x}{2}\ dx$$

$$= \int_0^1 x.\ \cos\frac{n\pi x}{2}\ dx + \int_1^2 (2-x).\ \cos\frac{n\pi x}{2}\ dx.$$

$$= \left[x.\ \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]\right]_0^1$$

$$+ \left[(2-x).\ \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]_1^2$$

$$= \cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right) - \cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}}$$

$$- \frac{4}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$\boxed{a_n = \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left[1 + (-1)^n\right]}$$

Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}.$$

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left(1+(-1)^n\right) \cos \frac{n\pi x}{2}}$$

COMPLEX FORM OF THE FOURIER SERIES

1) Find the Complex Fourier Series of $f(x) = \cos ax$, $-\pi < x < \pi$, where 'a' is not an integer.

The Complex Fourier Series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

Where

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot e^{-inx}\,dx$$

<u>Calculation of Fourier Coefficients</u>:

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax \cdot e^{-inx}\,dx.$$

$$\boxed{\int e^{ax} \cdot \cos bx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]}$$

Here $a = -in$, $b = a$.

$$\therefore C_n = \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2+a^2}\left(-in\cos ax + a\sin ax\right)\right]_{-\pi}^{\pi}$$

$$C_n = \frac{1}{2\pi}\left\{ \frac{e^{-inx}}{a^2-n^2}(-in\cos ax + a\sin ax)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{a^2-n^2}(-in\cos a\pi + a\sin a\pi) - \frac{e^{in\pi}}{a^2-n^2}(-in\cos a(-\pi) + a\sin a(-\pi))\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[in\cos a\pi\left(-e^{-in\pi} + e^{in\pi}\right) + a\sin a\pi\left(e^{-in\pi} + e^{in\pi}\right)\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[in\cos a\pi\,(2i\sin n\pi) + a\sin a\pi\,(2\cos n\pi)\right]$$

$$C_n = \frac{1}{2\pi(a^2-n^2)}(2a\sin a\pi\cos n\pi)$$

$$\boxed{C_n = \frac{(-1)^n \cdot a\sin a\pi}{\pi(a^2-n^2)}}$$

The Complex Fourier Series is

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \cdot a\sin a\pi}{\pi(a^2-n^2)}\, e^{inx}}$$

2) Find the complex form of the Fourier series of the $f(x) = e^{-ax}$ in $-1 \leq x \leq 1$.

Solution:

The complex form of Fourier series is given by.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{in\pi x}{l}}$$

where
$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) \cdot e^{-\frac{in\pi x}{l}} \, dx.$$

Calculation of Fourier Coefficients:

$$c_n = \frac{1}{2} \int_{-1}^{1} f(x) \cdot e^{-in\pi x} \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-ax} \cdot e^{-in\pi x} \, dx.$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-(a+in\pi) \cdot x} \, dx$$

$$= \frac{1}{2} \left[\frac{e^{-(a+in\pi)x}}{-(a+in\pi)} \right]_{-1}^{1}$$

$$= \frac{-1}{2(a+in\pi)}\left[e^{-(a+in\pi)} - e^{(a+in\pi)}\right]$$

$$= \frac{e^{(a+in\pi)} - e^{-(a+in\pi)}}{2(a+in\pi)}$$

$e^{in\pi} = \cos n\pi + i\sin n\pi$

$$= \frac{e^{a}\cdot e^{in\pi} - e^{-a}\cdot e^{-in\pi}}{2(a+in\pi)}$$

$e^{-in\pi} = \cos n\pi - i\sin n\pi$

$$C_n = \frac{(-1)^n}{a+in\pi}\left(\frac{e^{a}-e^{-a}}{2}\right)$$

$$C_n = \frac{(-1)^n \cdot \sinh a}{a+in\pi}.$$

Therefore the complex Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \cdot \sinh a}{a+in\pi} \cdot e^{inx}.$$

$$\boxed{f(x) = e^{-ax} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \cdot (a-in\pi)\cdot \sinh a}{a^2+n^2\pi^2}\, e^{inx}}$$

3) Find the Complex form of the Fourier Series of the periodic function

$f(x) = \sin x$, $0 < x < \pi$, $f(x+\pi) = f(x)$.

Solution.

$$f(x) = \sin x \qquad 0 < x < \pi$$

Here $2l = \pi$, $\Rightarrow l = \pi/2$.

Complex form of Fourier Series is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{i2nx}$$

where

$$C_n = \frac{1}{2l} \int_0^{2l} f(x) \cdot e^{-\frac{in\pi x}{l}} dx.$$

$$C_n = \frac{1}{\pi} \int_0^{\pi} \sin x \cdot e^{-i2nx} dx.$$

$$\boxed{\int e^{ax} \cdot \sin bx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)}$$

$a = -2in$, $b = 1$.

$$C_n = \frac{1}{\pi}\left[\frac{e^{-i2nx}}{(-i2n)^2+1^2}(-i2n\sin x - \cos x)\right]_0^{\pi}$$

$$= \frac{1}{\pi(4n^2-1)}\left[e^{-i2nx}(2in\sin x + \cos x)\right]_0^{\pi}$$

$$= \frac{1}{\pi(4n^2-1)}\left[e^{-i2n\pi}(2in\sin\pi + \cos\pi) - e^{0}(2in\sin 0 + \cos 0)\right]$$

$$= \frac{1}{\pi(4n^2-1)}\left[1(0+(-1)) - 1(+1)\right].$$

$$\boxed{C_n = \frac{-2}{\pi(4n^2-1)}}$$

$\therefore$ The complex form of Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{-2}{\pi(4n^2-1)} e^{i2nx}.$$

$$\boxed{f(x) = \sin x = \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)} e^{i2nx}.}$$