ENGINEERING MATHEMATICS II
UE20MA151
UNIT 1: INTEGRAL CALCULUS

1	Double integrals: Introduction, Evaluation
2	Application of double integrals: Area, Volume of the solid and Average value of the function.
3	Jacobian, Change of variables in Double integral (Polar coordinates)
4	Problems on change of variables in double integral
5	Change of order of integration
6	Triple Integrals: Introduction and Evaluation
7-8	Application of triple integrals: Volume and Average value of the function
9-10	Change of variables in triple integral (Spherical and Cylindrical)
11	Applications of Multiple integrals: Area and volume.
12	Center of Mass and Moment of Inertia.

Problems on double integral

1. Sketch the region 'R' over which we would evaluate the integral $\int_{y=0}^{y=1}\int_{x=0}^{x=2-2y} f(x,y)dx\,dy$.
2. Evaluate $\int_0^4\int_0^4 12\,x^2y^3\;\,dx\,dy$. **Ans: $\mathbf{4^7}$**
3. Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$. **Ans: $\frac{\Pi}{4}\,\mathbf{log}_e(\sqrt{2}+1)$.**
4. Find the volume of the solid bounded above by $f(x,y)\;=x^2$, over the region in the first quadrant bounded by the hyperbola $xy=16$ and the lines $y=x, y=0\;and\;x=8$. **Ans: 448**
5. $\iint_A r^3\,dr\,d\theta$, where A is the area included between the circles r=2sinΘ and r=4sinΘ. **Ans: 22.5 Π**
6. Find the average value of the function $\sqrt{xy-y^2}$, over the triangle with vertex (0,0), (10, 1), (1,1). **Ans: 4/3**
7. Find the smaller of the areas bounded by $y=2-x\;and\;x^2+y^2=4$. Ans: $(\Pi-2)Unit^2$.

CHANGE OF VARIABLES IN DOUBLE INTEGRALS:

1. Transform each of the given integrals to one or more iterated integrals in polar coordinates.

   a) $\int_0^1 dx \int_0^1 f(x,y)dy,$

   b) $\int_0^1 \left[\int_0^{x^2} f(x,y)dy\right] dx.$

   Ans: $\boldsymbol{a)}\ \int_0^{\Pi/4}\left[\int_0^{sec\theta} \boldsymbol{f(rcos\theta, rsin\theta)rdr}\right]\boldsymbol{d}\,\theta +$
   $\int_{\Pi/4}^{\Pi/2}\left[\int_0^{cosec\theta} \boldsymbol{f(rcos\theta, rsin\theta)rdr}\right]\boldsymbol{d}\,\theta$

   b) $\int_0^{\Pi/4}\left[\int_{sec\theta tan\theta}^{sec\theta} \boldsymbol{f(rcos\theta, rsin\theta)rdr}\right]\boldsymbol{d\theta}$

2. Change into polar co-ordinates and evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dy\,dx$.

   Ans: $\boldsymbol{\frac{\Pi}{4}}$ **.**

3. Evaluate double integral $\int_R \int e^{x^2}\,dy\,dx$ where the region R is given by R:$2y \le x \le 2$ and $0 \le y \le 1$. **Ans:** $\boldsymbol{\frac{1}{4}(e^4 - 1)}$ **.**

4. Compute the following integrals by changing to polar coordinates.

   $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx.$ **Ans:** $\boldsymbol{\frac{3}{8}\Pi - 1}$

5. Express $\int_0^{a/\sqrt{2}} \int_0^x x\,dy\,dx + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$, as a single integral and then evaluate it.

   Ans: $\int_0^{\Pi/4}\int_0^{a} \boldsymbol{r^2\, cos\theta dr\, d\theta}, \boldsymbol{\frac{a^3}{3\sqrt{2}}}$

6. Evaluate the following integrals by changing to polar coordinates.

   $\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dy\,dx,$

   Ans: a) $\boldsymbol{\frac{a^5\Pi}{20}}$

7. Find the area inside the circle r=2a cosΘ and outside the circle r=a. Ans: $2a^2\left(\frac{\Pi}{6}+\frac{\sqrt{3}}{4}\right)$ square units.

CHANGING THE ORDER OF INTEGRATION:

1. Show that $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dy\, dx \neq \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dx\, dy$.
2. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy\, dx$ by changing the order of integration.

   Ans: 1
3. Evaluate $\iint (3x^2 + y^2) dA$ over the region bounded by $-2 \leq y \leq 3, y^2 - 3 \leq x \leq y + 3$. **Ans:2375/7**.
4. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ and hence evaluate. **Ans: $\frac{3}{8}$.**
5. Change the order of integration in $\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y} dy\, dx$ and hence evaluate. **Ans: 1.**

TRIPLE INTEGRALS

1. Evaluate $\int_2^3 \int_1^2 \int_2^5 xy^2 dz\, dy\, dx$. Ans: $\frac{35}{2}$
2. Evaluate $I = \int_{x=0}^{x=1} \int_{y=0}^{y=1} \int_{z=0}^{z=1} x + y + z\, dz\, dy\, dx$. Ans: $\frac{3}{2}$.
3. Evaluate the following triple integral.

   $\int_{-1}^1 dz \int_0^z dx \int_{x-z}^{x+z} (x + y + z) dy$ Ans: 0
4. Find the volume of the solid bounded by the surfaces $z = 0, z = 1 - x^2 - y^2, y = 0, y = 1 - x, x = 0$ and $x = 1$. Ans: $\frac{1}{3}$
5. The temperature at a point (x,y,z) of a solid E bounded by the planes $x = 0, y = 0, z = 0$ and the plane $x + y + z = 1$ is $\frac{1}{(1+x+y+z)^3}$ degree Celsius . Find the average temperature over the solid.

   Ans:6 $\left(\frac{\ln 2}{2} - \frac{5}{16}\right)$.
6. Evaluate the triple integral $\iiint_E \sqrt{x^2 + z^2} dxdydz$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. Ans: $\frac{128\pi}{15}$

CHANGE OF VARIABLES IN TRIPLE INTEGRALS: CYLINDRICAL AND SPHERICAL

1. Use cylindrical co-ordinates, to evaluate $\iiint_V (x^2+y^2)dxdydz$ taken over the region V bounded by the paraboloid $z=9-x^2-y^2$ and the plane z=0 Ans: $\frac{243\pi}{2}$
2. By transforming into cylindrical co-ordinates evaluate the integral $\int\int\int(x^2+y^2+z^2)\,dx\,dy\ \,dz$ taken over the region
   $0\le z\le x^2+y^2\le 1.$ Ans: $\frac{5\pi}{12}$
3. Calculate the volume of the solid bounded by the paraboloid $z=2-x^2-y^2$ and the conic surface $z=\sqrt{x^2+y^2}$.

The region of integration is bounded from above by the paraboloid, and from below by the cone (Figure 11).

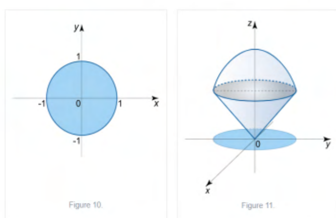


Figure 10. Figure 11.

Ans: $\frac{5\pi}{6}$

4. Evaluate $\iiint_V xyz(x^2+y^2+z^2)^{\frac{n}{2}}dxdydz$ taken through the positive octant of the sphere $x^2+y^2+z^2=b^2$ provided $n+5>0$. Ans: $\frac{b^{n+6}}{8(n+6)}$
5. Find the volume of the portion of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in the first octant using spherical polar coordinates. Ans:πabc/6 cubic units

CENTER OF MASS AND MOMENT OF INERTIA

1. Find the total mass of the region in the cube.
   a. $0\le x\le 1, 0\le y\le 1, 0\le z\le 1$ with density at any point given by xyz. Ans: $\frac{1}{8}$
2. Compute the mass of a sphere of radius b if the density varies inversely as the square of the distance from the center. Ans: 4kπb

3. Compute the moment of inertia of a right circular cylinder of altitude 2h and radius b, relative to the diameter of its median section with density equals to k, a constant. Ans: $k(\frac{2\pi h^3 b^2}{3} + \frac{\pi h b^4}{2})$.

ENGINEERING MATHEMATICS - II

UE20MA151

CLASS WORK PROBLEMS

UNIT 1 : INTEGRAL CALCULUS.

Problems on double integral.

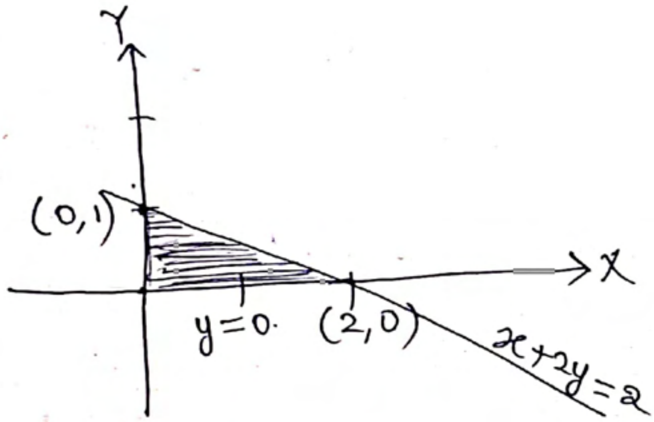
1) Sketch the region R over which we would evaluate the integral

$$\int_{y=0}^{y=1} \int_{x=0}^{x=2-2y} f(x,y)\, dx\, dy.$$

The region R is bounded by $y=0$, $y=1$

$x=0$ and $x = 2-2y \Rightarrow x+2y = 2$

x	0	2
y	1	0



2) Evaluate $\int_0^4 \int_0^4 12x^2y^3\, dx\, dy.$

$$\int_0^4 \int_0^4 12x^2y^3\, dx\, dy = \int_0^4 12y^3 \cdot \left[\frac{x^3}{3}\right]_0^4 dy.$$

$$= \cancel{12}^{4} \cdot \frac{4^3}{\cancel{3}} \cdot \left[\frac{y^4}{4}\right]_0^4 = \frac{4^8}{4} = 4^7$$

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3) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy \cdot dx}{1+x^2+y^2}$

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\ dx}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\ dx}{\left(\sqrt{1+x^2}\right)^2+y^2}$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} . dx.$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx.$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\ dx$$

$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1.$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

NOTE :

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\ dx = \log\left(x+\sqrt{x^2+a^2}\right)$$

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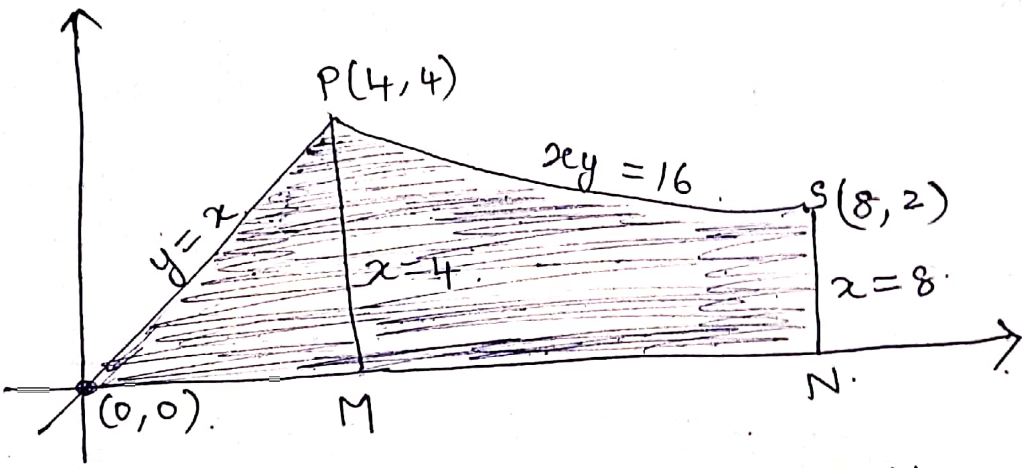
4) Find the Volume of the solid bounded above by $f(x,y) = x^2$ over the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$ and $x = 8$.

The region of integration lies in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$ and $x = 8$.



$$\iint_R x^2\,dx\,dy = \int_0^4\int_0^x x^2\,dy\,dx + \int_4^8\int_0^{16/x} x^2\,dy\,dx.$$

$$= \int_0^4 x^2[y]_0^x\,dx + \int_4^8 x^2\cdot[y]_0^{16/x}\cdot dx$$

$$= \int_0^4 x^3\,dx + \int_4^8 x^2\cdot\frac{16}{x}\,dx.$$

$$= \left.\frac{x^4}{4}\right]_0^4 + 16\cdot\left.\frac{x^2}{2}\right]_4^8 = 64 + 8(8^2-4^2)$$
$$= 64 + 384 = \underline{448}$$

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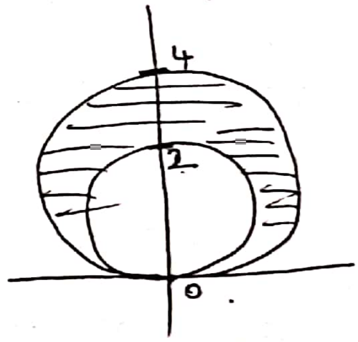
5) $\iint_A r^3 \, dr \, d\theta$ where A is the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Given circles $r = 2\sin\theta$

$r = 4\sin\theta$



$$\iint_A r^3 \, dr \, d\theta = \int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3 . \, dr \, d\theta$$

$$= \int_0^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} . \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi} 4^4 \sin^4\theta - 2^4 \sin^4\theta \, . \, d\theta$$

$$= \frac{1}{4} \times 240 \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= 60 \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= 60 \times \frac{3\pi}{8}$$

$$= \frac{45\pi}{2}.$$

NOTE:

$$\int_0^{\pi} \sin^4\theta \, d\theta$$

$$= 2 \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= 2 \cdot \frac{1}{2} \beta\left(\tfrac{5}{2}, \tfrac{1}{2}\right)$$

$$= \frac{\Gamma\frac{5}{2} \cdot \Gamma\frac{1}{2}}{\Gamma 3} = \frac{\frac{3}{2} \cdot \frac{1}{2} \pi}{2}$$

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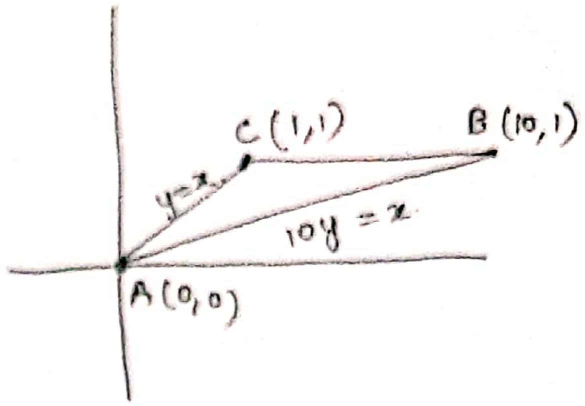
6) Find the average Value of the function $\sqrt{xy-y^2}$ over the triangle with vertex $(0,0)$ $(10,1)$ $(1,1)$



Equ. of AC

$$\frac{y-0}{1-0} = \frac{x-0}{1-0} \Rightarrow y = x.$$

Equ. of AB

$$\frac{y-0}{1-0} = \frac{x-0}{10-0} \Rightarrow 10y = x.$$

Average Value of the fn. $\sqrt{xy-y^2}$ over the $\triangle^{le}$ is

$$\frac{1}{A}\iint_R f(x,y)\cdot dx\,dy = \frac{2}{9}\int_0^1\int_y^{10y}\sqrt{xy-y^2}\,dx\,dy.$$

$$= \frac{2}{9}\cdot\int_0^1 \left[\frac{(xy-y^2)^{3/2}}{3/2\cdot y}\right]_y^{10y} dy.$$

$$= \frac{2}{9}\cdot\frac{2}{3}\int_0^1 \frac{(9y^2)^{3/2}}{y}\cdot dy.$$

$$= \frac{4}{27}\cdot\int_0^1 27y^2\cdot dy$$

$$= \frac{4}{27}\times 27\cdot\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{4}{3}.$$

NOTE:

Area of $\triangle^{le}$ $(x_1 y_1)$ $(x_2 y_2)$ $(x_3 y_3)$

$$= \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1) + x_3(y_1-y_2)\right]$$

$$= \frac{9}{2}$$

Equ. of line joining $(x_1 y_1)$ and $(x_2 y_2)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

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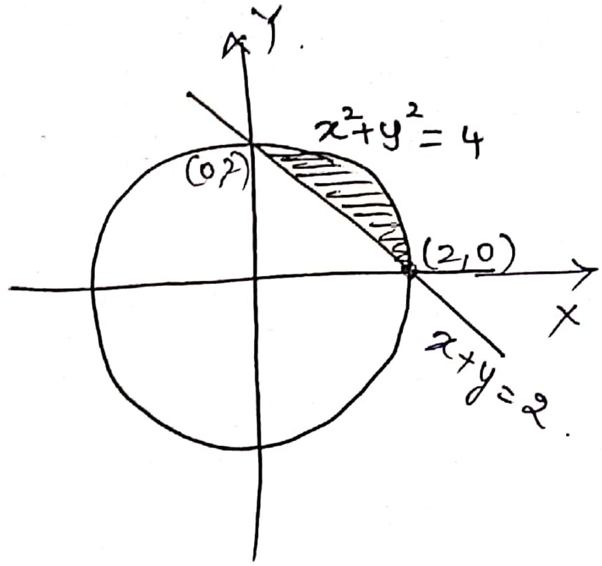
7) Find the smaller of the areas bounded by $y = 2 - x$ and $x^2 + y^2 = 4$.



$$\int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy \cdot dx$$

$$= \int_0^2 \left[y \right]_{2-x}^{\sqrt{4-x^2}} \cdot dx$$

$$= \int_0^2 \sqrt{4-x^2} - (2-x) \cdot dx.$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right]_0^2 - \left[2x - \frac{x^2}{2} \right]_0^2$$

$$= 2 \times \frac{\pi}{2} - (4-2) = \pi - 2 \text{ units}$$

NOTE

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right).$$

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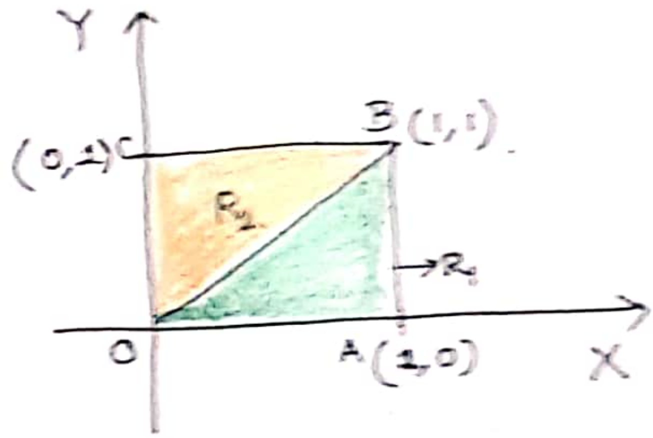
CHANGE OF VARIABLES IN DOUBLE INTEGRALS

1) Transform each of the given integrals to one or more iterated integrals in polar coordinates

a) $\int_0^1 dx \int_0^1 f(x,y)\, dy.$



Region of integration:

Y varies from 0 to 1

X varies from 0 to 1

$\therefore$ Region of integration is square OABC

By changing into polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

In the region R_1, r varies from $r=0$ to $r=\sec\theta$

θ varies from $\theta=0$ to $\theta=\pi/4$.

In the region R_2, r varies from $r=0$ to $r=\operatorname{cosec}\theta$

θ varies from $\theta=\pi/4$ to $\theta=\pi/2$

$$\therefore \int_0^1 dx \int_0^1 f(x,y)\,dy = \int_0^{\pi/4}\int_0^{\sec\theta} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$+ \int_{\pi/4}^{\pi/2}\int_0^{\operatorname{cosec}\theta} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

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b) $\int_0^1 \int_0^{x^2} f(x,y)\, dy \cdot dx$

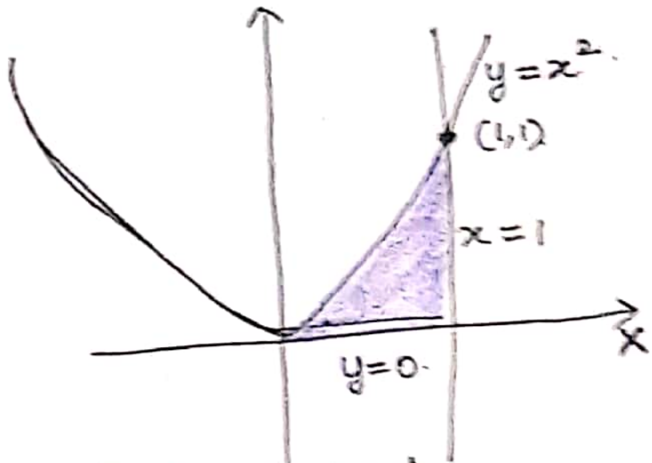
The region of integration is bounded by $y=0,\ y=x^2,\ x=0$ and $x=1$

By changing into polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$



In the region R,

r varies from $\underset{y=x^2}{}$ to $\underset{}{}$ $\sec\theta\tan\theta$ to $\sec\theta$ ($y=x^2 \longleftrightarrow x=1$)

θ varies from 0 to $\pi/4$

$$\boxed{\begin{array}{c} y = x^2 \\ r\sin\theta = r^2\cos^2\theta \\ \sec\theta\tan\theta = r \end{array}}$$

$$\therefore \int_0^1 \int_0^{x^2} f(x,y)\,dy\,dx = \int_0^{\pi/4} \int_{\sec\theta\tan\theta}^{\sec\theta} f(r\cos\theta,\ r\sin\theta)\, r\,dr\,d\theta.$$

2) Change into polar coordinates and evaluate

$$\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\,dy\,dx$$

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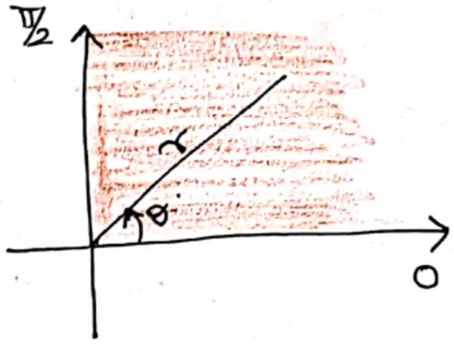
Solution:

Both x & y increase from 0 to ∞. Therefore the region of integration is the first quadrant of the XY plane. In this quadrant

r varies from 0 to ∞

and θ varies from 0 to $\pi/2$.



$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} . r\, dr\, d\theta$$

Put $r^2 = t$

$2r.dr = dt$

$$= \int_0^{\pi/2} \int_0^\infty e^{-t} \frac{dt}{2} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\frac{e^{-t}}{-1} \right]_0^\infty . d\theta$$

$$= \frac{-1}{2} \left[\int_0^{\pi/2} (0-1) . d\theta \right]$$

$$= +\frac{1}{2} \int_0^{\pi/2} d\theta = \frac{1}{2} [\theta]_0^{\pi/2} = \frac{\pi}{4}.$$

NOTE:

It is impossible to solve certain double integrals without changing to polar form.

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3) Evaluate double integral $\iint_R e^{x^2}\,dy\,dx$ where the region R is given by R : $2y \le x \le 2$ and $0 \le y \le 1$

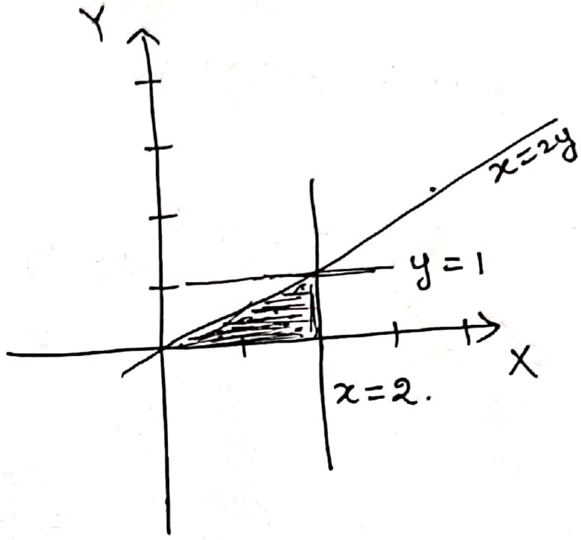
Solution :

The region is bounded by the curves

$x = 2y$, $x = 2$; $y = 0$, $y = 1$.

$x = 2y$

x	0	2	4
y	0	1	2.



$$\iint_R e^{x^2}\,dy\,dx$$

$$= \int_0^2 \int_0^{x/2} e^{x^2}\,dy\,dx.$$

$$= \int_0^2 e^{x^2}\cdot\left[y\right]_0^{x/2}\cdot dx.$$

$$= \int_0^2 e^{x^2}\cdot \frac{x}{2}\cdot dx = \frac{1}{2}\int_0^2 e^{x^2}\cdot x\,dx$$

Put $x^2 = t$

$2x\,dx = dt$.

$$= \frac{1}{2}\int_0^4 e^t\cdot\frac{dt}{2}$$

$$= \frac{1}{4}\left[e^t\right]_0^4 = \frac{e^4-1}{4}.$$

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4) Compute the following integrals by changing to polar coordinates

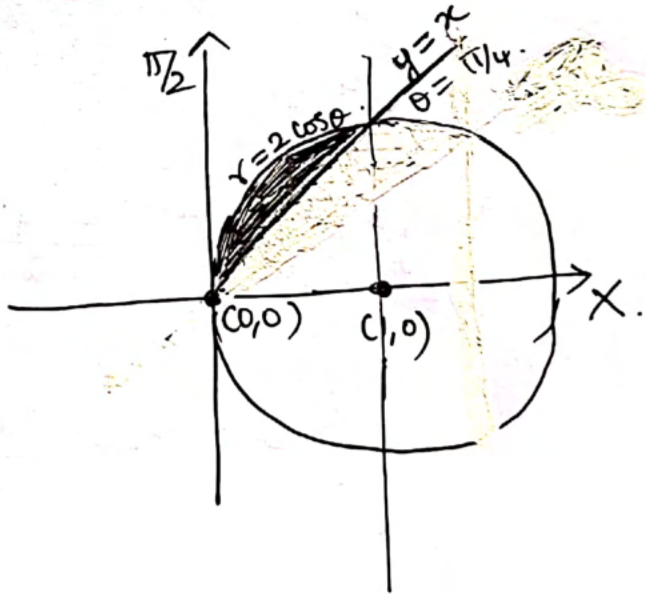
$$\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx.$$

The region is bounded by the curves $y = x$, $y = \sqrt{2x-x^2}$, $x=0$ and $x = 1$.

$$y^2 = 2x - x^2$$

$$x^2 + y^2 = 2x. \longrightarrow \text{A circle with centre at } (1,0).$$

By changing to polar coordinates



sub $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$$

$$= \int_{\pi/4}^{\pi/2} \int_0^{2\cos\theta} r^2 \cdot r\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} \cdot d\theta$$

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$$= \frac{16}{4} \int_{\pi/4}^{\pi/2} \cos^4\theta \, d\theta.$$

$$= 4 \int_{\pi/4}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{4}{4} \int_{\pi/4}^{\pi/2} \left(1 + \cos^2 2\theta + 2\cos 2\theta\right) d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(1 + \frac{1+\cos 4\theta}{2} + 2\cos 2\theta\right) d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{3}{2} + \frac{\cos 4\theta}{2} + 2\cos 2\theta\right) d\theta$$

$$= \left[\frac{3}{2}\theta + \frac{1}{2} \cdot \frac{\sin 4\theta}{4} + \cancel{2} \cdot \frac{\sin 2\theta}{\cancel{2}}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{3}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \frac{1}{8}\left(\sin 4 \cdot \pi/2 - \sin 4 \cdot \pi/4\right)$$

$$+ \sin 2 \cdot \pi/2 - \sin 2 \cdot \pi/4$$

$$= \frac{3\pi}{8} + 0 + (-1)$$

$$= \frac{3\pi}{8} - 1.$$

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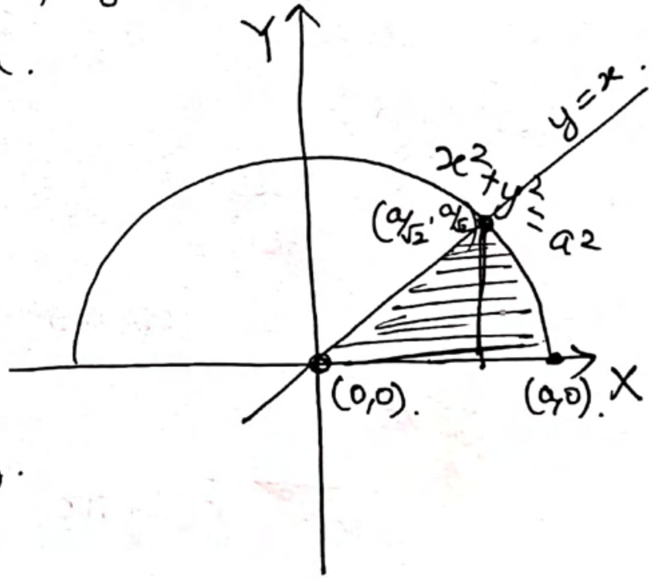
5) Express $\int_0^{a/\sqrt{2}}\int_0^{x} x\,dy\,dx + \int_{a/\sqrt{2}}^{a}\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$ as a single integral and then evaluate it :

The Region of integration is split in to two regions

R_1 : bounded by $y=0,\ y=x$
$x=0,\ x=a/\sqrt{2}$

R_2 : bounded by $y=0,\ y=\sqrt{a^2-x^2}$
$x=a/\sqrt{2},\ x=a.$



By changing in to polar Coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\,dy = r\,dr\,d\theta.$$

$$\int_0^{a/\sqrt{2}}\int_0^{x} x\,dy\,dx + \int_{a/\sqrt{2}}^{a}\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx.$$

$$= \int_0^{\pi/4}\int_0^{a} r\cos\theta \cdot r\,dr\,d\theta.$$

$$= \int_0^{\pi/4} \cos\theta \cdot \left[\frac{r^3}{3}\right]_0^a d\theta = \frac{a^3}{3}\left[\sin\theta\right]_0^{\pi/4}$$

$$= \frac{a^3}{3\sqrt{2}}.$$

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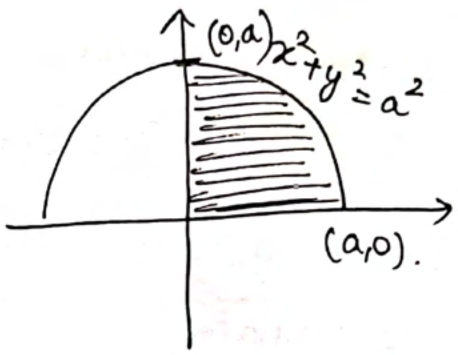
6). Evaluate the following integrals by changing to polar coordinates

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y^2 \cdot \sqrt{x^2+y^2}\, .dy\, dx.$$

The region is bounded by the curves $y=0$, $y=\sqrt{a^2-x^2}$, $x=0$ and $x=a$.



By changing into polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$dx\, dy = r\, dr\, d\theta.$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dy\, dx. = \int_0^{\pi/2}\int_0^a r^2\sin^2\theta \cdot r \cdot r \cdot dr\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cdot \left[\frac{r^5}{5}\right]_0^a d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= \frac{a^5}{5} \cdot \frac{1}{2}\beta\left(3/2, 1/2\right)$$

$$= \frac{a^5}{10} \cdot \frac{\Gamma 3/2 \cdot \Gamma 1/2}{\Gamma 2} = \frac{a^5}{10} \cdot \frac{1/2 \cdot \Gamma 1/2 \cdot \Gamma 1/2}{1}$$

$$= \frac{a^5}{20}\pi$$

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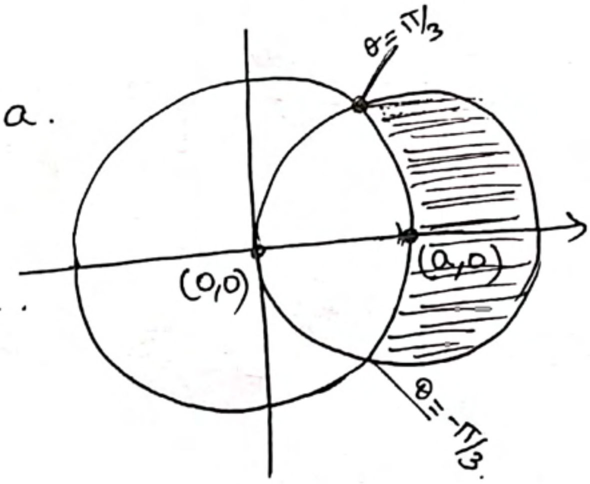
7) Find the area inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$.

$r = 2a\cos\theta$ is a circle with centre $(a,0)$ and radius a.

$r = a$ is a circle with centre $(0,0)$ and radius a.



$$\text{Area} = \int_{-\pi/3}^{\pi/3}\int_{a}^{2a\cos\theta} r\,dr\,d\theta$$

The point of intersection of the circle $r = 2a\cos\theta$ and $r=a$ is

$$2a\cos\theta = a$$

$$\cos\theta = 1/2$$

$$\theta = \pm\pi/3.$$

$$= \int_{-\pi/3}^{\pi/3}\left[\frac{r^2}{2}\right]_{a}^{2a\cos\theta} d\theta.$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4a^2\cos^2\theta - a^2\cdot d\theta$$

$$= \frac{4a^2}{2}\int_{-\pi/3}^{\pi/3}\left(\cos^2\theta\, d\theta\right) - \frac{a^2}{2}\int_{-\pi/3}^{\pi/3} d\theta.$$

$$= 2a^2\times 2\cdot\int_{0}^{\pi/3}\cos^2\theta\,d\theta - \frac{a^2}{\cancel{2}}\times\frac{\cancel{2}\pi}{3}.$$

$$= 4a^2\cdot\int_{0}^{\pi/3}\cos^2\theta\,d\theta - \frac{a^2\pi}{3}.$$

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$$\text{Area} = 4a^2 \cdot \int_0^{\pi/3} \frac{1+\cos 2\theta}{2}\, d\theta - \frac{a^2\pi}{3}.$$

$$= \frac{4a^2}{2} \cdot \left[\theta + \frac{\sin 2\theta}{2.}\right]_0^{\pi/3} - \frac{a^2\pi}{3}$$

$$= 2a^2 \cdot \left[\pi/3 + \frac{1}{2} \cdot \sin 2 \cdot \frac{\pi}{3}\right] - \frac{a^2\pi}{3}.$$

$$= 2a^2 \frac{\pi}{3} + \frac{a^2\sqrt{3}}{2} - \frac{a^2\pi}{3}.$$

$$\text{Area.} = \frac{a^2\pi}{3} + \frac{a^2\sqrt{3}}{2.} \text{ sq. units}.$$

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CHANGING THE ORDER OF INTEGRATION.

1) Show That $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx \neq \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy$.

L.H.S

$$\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,.dy\,.dx.$$

$$=\int_0^1\int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,.dy\,dx.$$

$$=\int_0^1\int_0^1 \frac{2x}{(x+y)^3} - \frac{x+y}{(x+y)^3}\,.dy\,.dx.$$

$$=\int_0^1 \left\{ \cancel{2}x.\left[\frac{1}{-\cancel{2}(x+y)^2}\right] + \frac{1}{x+y} \right\}_0^1 dx$$

$$=\int_0^1 \frac{-x}{(x+1)^2} + \frac{x}{x^2} + \left(\frac{1}{x+1} - \frac{1}{x}\right).dx$$

$$=\int_0^1 \frac{-x}{(x+1)^2} + \cancel{\frac{1}{x}} + \frac{1}{x+1} - \cancel{\frac{1}{x}}$$

$$=\int_0^1 \frac{-x+x+1}{(x+1)^2} = \int_0^1 \frac{1}{(x+1)^2} = \left[\frac{-1}{x+1}\right]_0^1 = \frac{-1}{2}+1$$

$$= \frac{1}{2}$$

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R.H.S

$$\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, dx \cdot dy.$$

$$= \int_0^1 \int_0^1 \frac{x+y-2y}{(x+y)^3}\, dx \cdot dy.$$

$$= \int_0^1 \int_0^1 \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \cdot dx\, dy.$$

$$= \int_0^1 \left[\frac{-1}{x+y} + \frac{y}{(x+y)^2}\right]_0^1 \cdot dy.$$

$$= \int_0^1 \left(\frac{-1}{1+y} + \cancel{\frac{1}{y}} + \frac{y}{(1+y)^2} - \cancel{\frac{1}{y}}\right) dy$$

$$= \int_0^1 \frac{-1-y+y}{(1+y)^2} \cdot dy = \int_0^1 \frac{-1}{(1+y)^2}\, dy$$

$$= \left[\frac{1}{1+y}\right]_0^1 = \frac{1}{2} - 1$$

$$= -\frac{1}{2}.$$

Hence L.H.S $\neq$ R.H.S.

Since $\frac{x-y}{(x+y)^3}$ is discontinuous at $(0,0)$, a point on the boundary of the region, change of order of integration doesnot give the same result.

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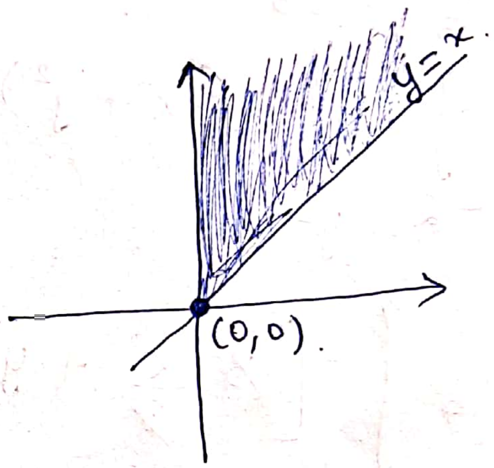
2) Evaluate

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$$ by changing the order of integration.

The region is bounded by $y = x$, $y = \infty$, $x = 0$, $x = \infty$.



$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \cdot dy \cdot dx$$

$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y} \cdot dx \cdot dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot [x]_0^y \cdot dy = \int_0^\infty \frac{e^{-y}}{\cancel{y}} \cdot \cancel{y} \cdot dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$= -\left[e^{-\infty} - e^{-0}\right]$$

$$= -[0 - 1]$$

$$= 1.$$

3) Evaluate $\iint (3x^2 + y^2)\, dA$ over the region bounded by $-2 \leq y \leq 3$, $y^2 - 3 \leq x \leq y + 3$.

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Given $\int_{-2}^{3}\int_{y^2-3}^{y+3}(3x^2+y^2)\,dx\,dy$.

The region is bounded by $x=y^2-3$, $x=y+3$
$y=-2$ and $y=3$.

$$\int_{-2}^{3}\int_{y^2-3}^{y+3}(3x^2+y^2)\,dx\,dy$$

$$=\int_{-2}^{3}\left[\cancel{3}\cdot\frac{x^3}{\cancel{3}}+y^2x\right]_{y^2-3}^{y+3}dy$$

$$=\int_{-2}^{3}(y+3)^3+y^2(y+3)-\left[(y^2-3)^3+y^2(y^2-3)\right]dy$$

$$=\int_{-2}^{3}\left(y^3+9y^2+27y+27+y^3+3y^2-y^6+9y^4-27y^2+27\right.$$
$$\left.-y^4+3y^2\right)dy$$

$$=\int_{-2}^{3}\left(-y^6+8y^4+2y^3-12y^2+27y+54\right)dy$$

$$=\frac{2375}{7}.$$

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4) Change the order of integration in

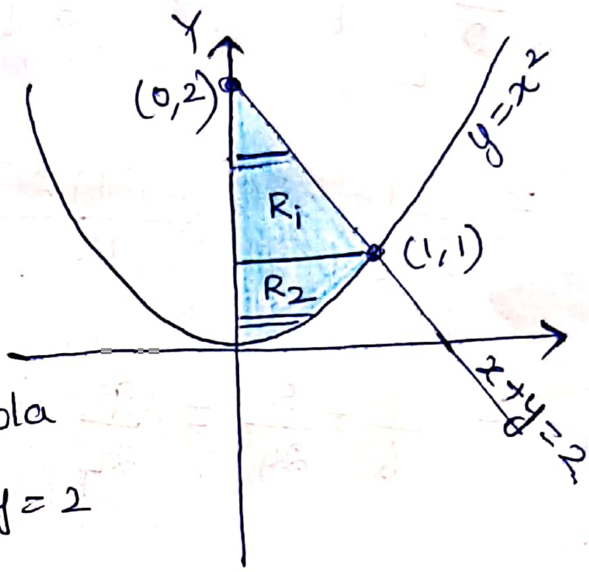
$$\int_0^1 \int_{x^2}^{2-x} xy \cdot dy\, dx$$ and hence evaluate.

The region of integration is bounded by

$y = x^2$, $y = 2-x$, $x = 0$, $x = 1$.

$x + y = 2$.

x	1	0	2
y	1	2	0



The point of intersection of the parabola $y = x^2$ and the line $x + y = 2$ is $(1, 1)$.

$$\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = \int_1^2 \int_{x=0}^{2-y} xy\, dx\, dy + \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy.$$

$$= \int_1^2 y \cdot \left[\frac{x^2}{2}\right]_0^{2-y} dy + \int_0^1 y \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy.$$

$$= \frac{1}{2}\int_1^2 y \cdot (2-y)^2 \cdot dy + \frac{1}{2}\int_0^1 y^2 \cdot dy$$

$$= \frac{1}{2}\int_1^2 y(4 + y^2 - 4y)\, dy + \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1$$

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$$= \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\,dy + \frac{1}{6}$$

$$= \frac{1}{2}\left[4\cdot\frac{y^2}{2} - 4\frac{y^3}{3} + \frac{y^4}{4}\right]_1^2 + \frac{1}{6}.$$

$$= \frac{1}{2}\left[2\cdot[2^2-1^2] - \frac{4}{3}[2^3-1^3] + \frac{1}{4}[2^4-1^4]\right] + \frac{1}{6}$$

$$= \frac{1}{2}\left[\frac{72 - 112 + 45}{12}\right] + \frac{1}{6}$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8}.$$

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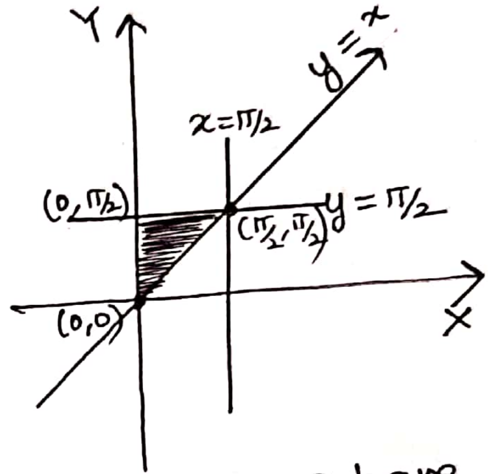
5) change the order of integration in

$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$$ and hence evaluate.

$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$$



The region is bounded by

$y=x,\ y=\pi/2,\ x=0,\ x=\pi/2$.

By changing the order of integration we have

$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx = \int_0^{\pi/2}\int_0^{y} \frac{\sin y}{y}\, dx\, dy.$$

$$= \int_0^{\pi/2} \frac{\sin y}{y}\left[x\right]_0^y . dy.$$

$$= \int_0^{\pi/2} \frac{\sin y}{\cancel{y}} . \cancel{y}\, dy$$

$$= \left[-\cos y\right]_0^{\pi/2}$$

$$= -\left[\cos \pi/2 - \cos 0\right]$$

$$= -\left[0-1\right] = 1$$

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TRIPLE INTEGRALS.

1) Evaluate $\int_2^3 \int_1^2 \int_2^5 xy^2 \, dz \, dy \, dx$.

Solution.

$$\int_2^3 \int_1^2 \int_2^5 xy^2 \, dz \, dy \, dx = \int_2^3 \int_1^2 xy^2 \left[z\right]_2^5 dy \, dx$$

$$= \int_2^3 \int_1^2 3xy^2 \, dy \, dx$$

$$= 3 \int_2^3 \left[x \cdot \frac{y^3}{3}\right]_1^2 \cdot dx$$

$$= \int_2^3 7x \cdot dx = 7\left[\frac{x^2}{2}\right]_2^3$$

$$= \frac{7}{2}(9-4) = \frac{35}{2}.$$

2) Evaluate

$$\int_0^1 \int_0^1 \int_0^1 (x+y+z) \, dz \, dy \, dx.$$

$$\int_0^1 \int_0^1 \int_0^1 (x+y+z) \, dz \, dy \, dx = \int_0^1 \int_0^1 \left[xz + yz + \frac{z^2}{2}\right]_0^1 dy \, dx$$

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$$= \int_0^1 \int_0^1 \left(x+y+\frac{1}{2}\right) dy\, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} + \frac{1}{2}y \right]_0^1 dx$$

$$= \int_0^1 x + \frac{1}{2} + \frac{1}{2}\, dx = \int_0^1 (1+x)\, dx$$

$$= \left[x + \frac{x^2}{2} \right]_0^1 = 1 + \frac{1}{2}$$

$$= \frac{3}{2}.$$

3) Evaluate the following triple integral.

$$\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dy \cdot dx \cdot dz.$$

Solution:

$$\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$$

$$= \int_{-1}^{1} \int_{0}^{z} \left(xy + \frac{y^2}{2} + zy \right)_{x-z}^{x+z} dx\, dz$$

$$= \int_{-1}^{1} \int_{0}^{z} x \cdot [x+z-x+z] + \frac{1}{2}\left[(x+z)^2 - (x-z)^2 \right] + z \cdot [x+z-x+z]\, dx\, dz$$

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$$= \int_{-1}^{1}\int_{0}^{z} 2xz + \frac{1}{2}\left(\cancel{x^2}+\cancel{z^2}+2xz-\cancel{x^2}-\cancel{z^2}+2xz\right)+2z^2 \cdot dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left(2xz + \frac{4xz}{2} + 2z^2\right) dx \cdot dz$$

$$= \int_{-1}^{1} \cancel{2}z \cdot \left[\frac{x^2}{\cancel{2}}\right]_0^z + \cancel{2}x \left[\frac{x^2}{\cancel{2}}\right]_0^z + 2z^2\left[x\right]_0^z \cdot dz$$

$$= \int_{-1}^{1}\left(z^3 + z^3 + 2z^3\right) dz = \int_{-1}^{1} 4z^3 \cdot dz$$

$$= \cancel{4}\left[\frac{z^4}{\cancel{4}}\right]_{-1}^{1}$$

$$= \left[z^4\right]_{-1}^{1} = 1 - 1$$

$$= 0$$

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4) Find the Volume of the Solid bounded by the Surfaces $z=0$, $z=1-x^2-y^2$, $y=0$, $y=1-x$, $x=0$ and $x=1$.

$$\text{Volume} = \int_0^1\int_0^{1-x}\int_0^{1-x^2-y^2} dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \left[z\right]_0^{1-x^2-y^2} dy\,dx$$

$$= \int_0^1\int_0^{1-x} \left(1-x^2-y^2\right) dy\,dx$$

$$= \int_0^1 \left(y - x^2y - \frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \int_0^1 (1-x) - x^2(1-x) - \frac{(1-x)^3}{3}\cdot dx$$

$$= \int_0^1 \frac{3 - \cancel{3x} - 3x^2 + 3x^3 - 1 + \cancel{3x} - 3x^2 + x^3}{3}\cdot dx.$$

$$= \int_0^1 \frac{4x^3 - 6x^2 + 2}{3}\cdot dx = \frac{1}{3}\left[\cancel{4}\cdot\frac{x^4}{\cancel{4}} - \frac{\cancel{6}^{\,2}x^3}{\cancel{3}} + 2\right]_0^1$$

$$= \frac{1}{3}\left[1 - 2 + 2\right] = \frac{1}{3}.$$

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5) The temperature at a point (x,y,z) of a solid E bounded by the planes $x=0$, $y=0$, $z=0$ and the plane $x+y+z=1$ is $\frac{1}{(1+x+y+z)^3}$ degree celsius.

Find the average temparature over the solid.

Solution:

The plane $x+y+z=1$ has intercepts $(1,0,0)$ $(0,1,0)$ and $(0,0,1)$

The triple integral of the Temperature $= \int_0^1 \int_0^{1-x} \int_{z=0}^{1-x-y} \frac{1}{(1+x+y+z)^3} \, dz\, dy\, dx$.

$$= \int_0^1 \int_0^{1-x} \left[\frac{-1}{2(1+x+y+z)^2} \right]_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{-1}{2(1+x+y+1-x-y)^2} + \frac{1}{2(1+x+y)^2} \, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{-1}{8} + \frac{1}{2(1+x+y)^2} \, dy\, dx$$

$$= \int_0^1 \left[\frac{-1}{8} y - \frac{1}{2(1+x+y)} \right]_0^{1-x} dx.$$

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$$= \int_0^1 -\frac{1}{8}(1-x) - \frac{1}{2(1+\cancel{x}+1-\cancel{x})} + \frac{1}{2(1+x)} \cdot dx.$$

$$= \int_0^1 -\frac{1}{8}(1-x) - \frac{1}{4} + \frac{1}{2(1+x)} \cdot dx$$

$$= \int_0^1 \frac{-3}{8} + \frac{x}{8} + \frac{1}{2(1+x)} \cdot dx$$

$$= \left[-\frac{3}{8}x + \frac{1}{8} \cdot \frac{x^2}{2} + \frac{1}{2}\log(1+x) \right]_0^1$$

$$= \left[-\frac{3}{8} + \frac{1}{16} + \frac{1}{2}\log 2 - 0 \right]$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}.$$

$$\text{Volume} = \iiint dv$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx.$$

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$$= \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} . dx$$

$$= \int_0^1 1-x - x(1-x) - \frac{(1-x)^2}{2} . dx$$

$$= \int_0^1 \frac{2-2x-\cancel{2x}+2x^2-1+\cancel{2x}-x^2}{2} . dx$$

$$= \int_0^1 \frac{x^2-2x+1}{2} \, dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} + x\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - 1 + 1\right] = \frac{1}{6}.$$

Hence the average temperature over the solid is

$$= \frac{\frac{1}{2}\log 2 - \frac{5}{16}}{\frac{1}{6}}$$

$$= 6 \cdot \left[\frac{\log 2}{2} - \frac{5}{16}\right]$$

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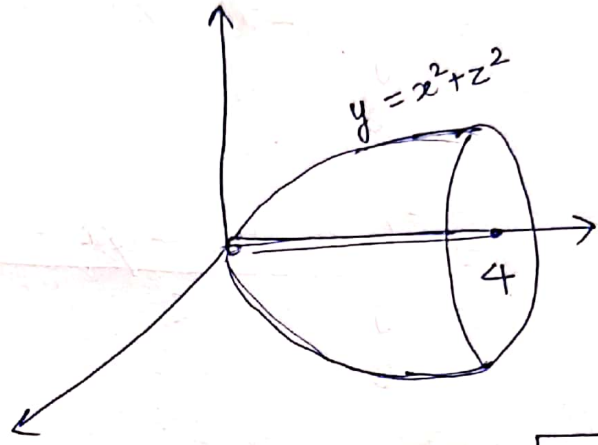
6) Evaluate the triple integral $\iiint_E \sqrt{x^2+z^2}\, dx\, dy\, dz$

where E is the region bounded by the paraboloid

$y = x^2 + z^2$ and the plane $y = 4$.



<u>Solution</u> :

From $y = x^2 + z^2$

we obtain

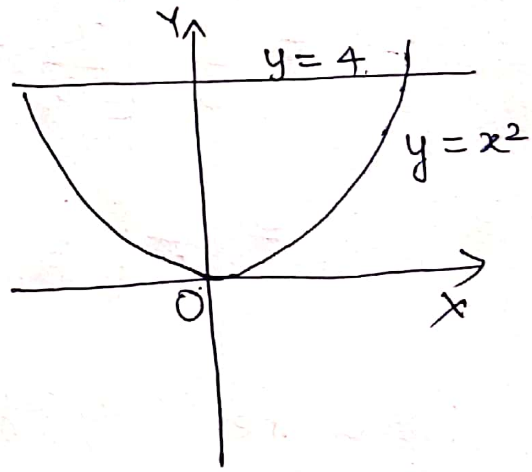
$$z = \pm\sqrt{y - x^2}.$$

So, the lower boundary surface of E is $z = -\sqrt{y-x^2}$

The upper surface is $z = \sqrt{y-x^2}$.

The trace of $y = x^2 + z^2$ in the plane $z=0$ is

the parabola $y = x^2$



$$\iiint_E \sqrt{x^2+z^2}\, .dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy\, dx.$$

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We will change the order of integration for easy computation.

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,.\,dz\,dy\,dx.$$

$$= \int_{-2}^{2}\int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}}\int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\,.\,dy\,dz\,dx.$$

$$= \int_{-2}^{2}\int_{z=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+z^2}\,\Big[\,y\,\Big]_{x^2+z^2}^{4}\,dz\,dx.$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\,dz\,dx.$$

We will use the polar substitution $x = r\cos\theta$, $z = r\sin\theta$, $dz\,dx = r\,dr\,d\theta$ in the xz plane.

$$\therefore \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\,.\,dz\,dx$$

$$= \int_{0}^{2\pi}\int_{r=0}^{2} (4-r^2)\,r\,.\,r\,.\,dr\,d\theta.$$

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$$= \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 \cdot d\theta$$

$$= \frac{64}{15} \int_0^{2\pi} d\theta = \frac{64}{15} \left[\theta\right]_0^{2\pi} = \frac{128\pi}{15}$$

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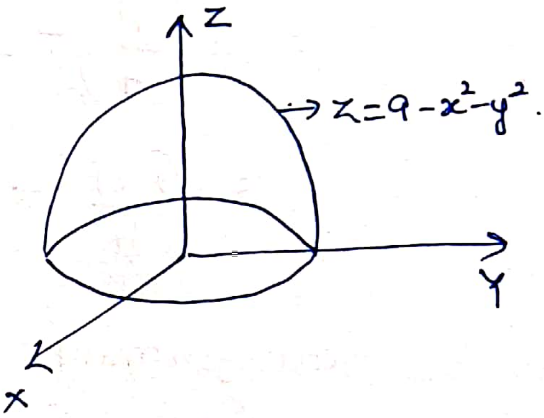
CHANGE OF VARIABLES IN TRIPLE INTEGRALS CYLINDRICAL AND SPHERICAL.

1) Use cylindrical coordinates to evaluate $\iiint_V (x^2+y^2)\,dx\,dy\,dz$ taken over the region V bounded by the paraboloid $z = 9 - x^2 - y^2$ and the plane $z = 0$.



Use cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

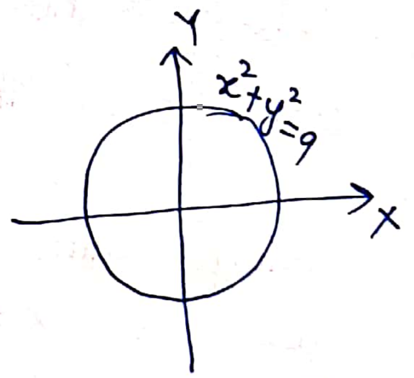
$$dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

Paraboloid becomes $z = 9 - r^2$

z limits : 0 to $9-r^2$

r : 0 to 3

θ : 0 to 2π



Projection on XY-plane.

$$\iiint_V (x^2+y^2)\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r^2 . r\,dz\,dr\,d\theta.$$

$$= \int_0^{2\pi}\int_0^{3} r^3 . \left[z\right]_0^{9-r^2} . dr\,d\theta.$$

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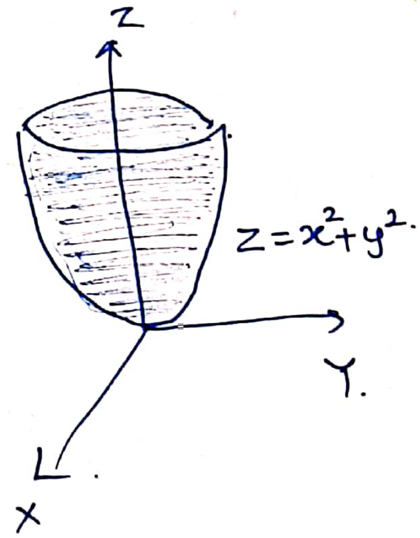
$$= \int_0^{2\pi} \int_0^3 r^3 (9 - r^2) \, dr \, d\theta$$

$$= \int_0^{2\pi} 9 \cdot \left[\frac{r^4}{4}\right]_0^3 - \left[\frac{r^6}{6}\right]_0^3 \cdot d\theta$$

$$= 9 \cdot \left(\frac{3^4}{4}\right) - \frac{3^6}{6} \int_0^{2\pi} d\theta$$

$$= \left(\frac{729}{4} - \frac{729}{6}\right)(2\pi) = \frac{243\pi}{2}.$$

2) By transforming into cylindrical coordinates evaluate the integral $\iiint (x^2+y^2+z^2) \, dx \, dy \, dz$. taken over the region $0 \leq z \leq x^2+y^2 \leq 1$



Using cylindrical coordinates
we have $x = r\cos\theta$

$y = r\sin\theta$

$z = z.$

$dx \, dy \, dz = r \, dz \, dr \, d\theta.$

$$\iiint (x^2+y^2+z^2) \, dx \, dy \, dz.$$

$$= \int_0^{2\pi} \int_0^1 \int_{z=0}^{r^2} (r^2 + z^2) \, r \cdot dz \, dr \, d\theta.$$

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$$= \int_0^{2\pi}\int_0^1\int_0^{r^2} (r^3 + z^2 r)\, dz\, dr\, d\theta$$

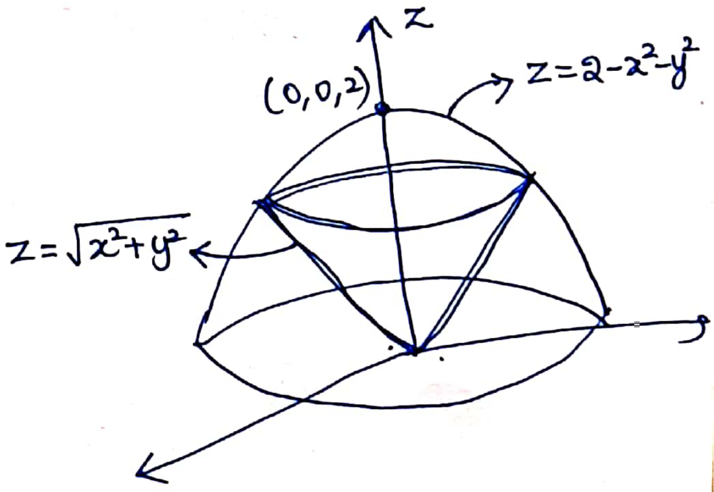
$$= \int_0^{2\pi}\int_0^1 \left[r^3 . z + r . \frac{z^3}{3}\right]_0^{r^2} . dr\, d\theta.$$

$$= \int_0^{2\pi}\int_0^1 \left(r^5 + \frac{r^7}{3}\right) dr\, d\theta.$$

$$= \int_0^{2\pi} \left[\frac{r^6}{6} + \frac{1}{3} . \frac{r^8}{8}\right]_0^1 d\theta = \int_0^{2\pi}\left(\frac{1}{6} + \frac{1}{3} . \frac{1}{8}\right) d\theta$$

$$= \frac{5}{24} \times 2\pi = \frac{5\pi}{12}.$$

3) Calculate the Volume of the solid bounded by the paraboloid $z = 2 - x^2 - y^2$ and the conic Surface $z = \sqrt{x^2+y^2}$.



First we investigate intersection of two Surfaces.

Since $z = 2 - x^2 - y^2$
$= 2 - r^2$

and $z = \sqrt{x^2 + y^2}$
$= r$

we have
$2 - r^2 = r$
$r^2 + r - 2 = 0$
$(r+2)(r-1) = 0.$

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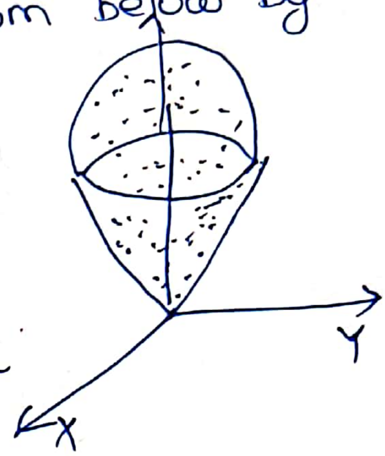
Since $r \geq 0$, we have $r = 1$.

$\therefore$ Therefore both the surfaces intersect at $z = 1$

And the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$.

The region of integration is bounded from above by the paraboloid and from below by the cone.



The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

$$\text{Volume} = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r}^{2-r^2} r \cdot dz \, dr \, d\theta.$$

$$= \int_0^{2\pi} \int_0^1 r \cdot \left[z \right]_r^{2-r^2} \cdot dr \, d\theta.$$

$$= \int_0^{2\pi} \int_0^1 r(2 - r^2 - r) \, dr \, d\theta.$$

$$= \int_0^{2\pi} \int_0^1 (2r - r^3 - r^2) \, dr \, d\theta.$$

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$$= \int_0^{2\pi} \left[2 \cdot \frac{r^2}{2} - \frac{r^4}{4} - \frac{r^3}{3} \right]_0^1 d\theta$$

$$= \left(1 - \frac{1}{4} - \frac{1}{3}\right) \int_0^{2\pi} d\theta = \frac{5}{12}(2\pi) = \frac{5\pi}{6}.$$

4) Evaluate $\iiint_V xyz(x^2+y^2+z^2)^{n/2}\, dx\, dy\, dz$ taken through the positive octant of the sphere $x^2+y^2+z^2 = b^2$ provided $n+5>0$.

Using spherical coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$\iiint_V xyz(x^2+y^2+z^2)^{n/2}\, dx\, dy\, dz$ becomes

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{b} (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)(r\cos\theta)\, r^2 \sin\theta\, (r^2)^{n/2}\, dr\, d\theta\, d\phi.$$

$$= \int_{r=0}^{b} r^{n+5}\, dr \int_0^{\pi/2} \sin^3\theta\cos\theta\, d\theta \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi.$$

$$= \left[\frac{r^{n+6}}{n+6} \right]_0^b \cdot \frac{1}{2}\beta(2,1) \cdot \frac{1}{2}\beta(1,1)$$

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$$= \frac{b^{n+6}}{n+6} \cdot \left(\frac{1}{2} \cdot \frac{\sqrt{2}\cdot\sqrt{1}}{\sqrt{3}}\right)\left(\frac{1}{2}\ \frac{\sqrt{1}\cdot\sqrt{1}}{\sqrt{2}}\right)$$

$$= \frac{b^{n+6}}{8(n+6)} \quad \text{provided } n+5>0.$$

5) Find the Volume of the portion of The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in The first octant using Spherical polar coordinates.

Using Generalized spherical polar coordinates we have

$$x = ar\sin\theta\cos\phi$$

$$y = br\sin\theta\sin\phi$$

$$z = cr\cos\theta$$

$$dx\,dy\,dz = abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\text{Volume} = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= abc\cdot\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\sin\theta\left[\frac{r^3}{3}\right]_0^1 d\theta\,d\phi$$

$$= \frac{abc}{3}\int_{\phi=0}^{\pi/2}\left[-\cos\theta\right]_0^{\pi/2}\cdot d\phi = \frac{abc}{3}\int_0^{\pi/2}(-0+1)\,d\phi$$

$$= \frac{abc}{3}\cdot\frac{\pi}{2} = \frac{\pi abc}{6}$$

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CENTER OF MASS AND MOMENT OF INERTIA

1) Find the total mass of the region in the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$ with density at any point given by xyz.

Given density function $\rho = xyz$.

$$\text{Mass} = \iiint \rho \, dx \, dy \, dz.$$

$$= \int_0^1 \int_0^1 \int_0^1 xyz \, dx \, dy \, dz.$$

$$= \int_0^1 \int_0^1 yz \cdot \left[\frac{x^2}{2}\right]_0^1 dy \, dz$$

$$= \frac{1}{2} \int_0^1 \int_0^1 yz \, dy \, dz$$

$$= \frac{1}{2} \int_0^1 y \, dy \int_0^1 z \, dz$$

$$= \frac{1}{2} \cdot \left[\frac{y^2}{2}\right]_0^1 \left[\frac{z^2}{2}\right]_0^1$$

$$= \frac{1}{8}.$$

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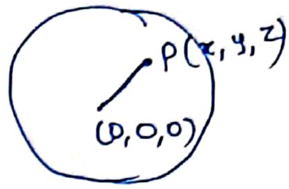
2) Compute the mass of a sphere of radius b if the density varies inversely as the square of the distance from the center.



$$\text{Density} \propto \frac{1}{(\text{Distance from center})^2}$$

$$= \frac{k}{x^2+y^2+z^2}$$

$$\text{Mass} = \iiint \frac{k}{x^2+y^2+z^2}\, dx\, dy\, dz.$$

(Using Put spherical coordinates)

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= k \int_{\phi=0}^{2\pi}\int_{0}^{\pi}\int_{0}^{b} \frac{\cancel{r^2}\sin\theta}{\cancel{r^2}} \cdot dr\, d\theta\, d\phi$$

$$= k \cdot \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{b} dr.$$

$$= k\, [\phi]_0^{2\pi}\, [-\cos\theta]_0^{\pi}\; b$$

$$= k \cdot 2\pi\, (-\cos\pi + \cos 0)\, b$$

$$= 4\pi k b.$$

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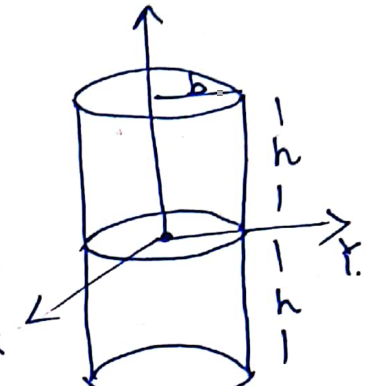
3) Compute the moment of inertia of a right circular cylinder of altitude 2h and radius b relative to the diameter of its median section with density equals to k, a constant.

Moment of inertia of the cylinder relative to the x-axis



$$I_{xx} = \iiint_V \rho\,(y^2+z^2)\,dx\,dy\,dz.$$

$$= \int_0^{2\pi}\int_0^{b}\int_{-h}^{h} k\,(r^2\sin^2\theta + z^2)\,r\cdot dz\,dr\,d\theta.$$

$$= k\int_0^{2\pi}\int_0^{b} r^3\cdot\sin^2\theta\,\big[z\big]_{-h}^{h} + r\cdot\left[\frac{z^3}{3}\right]_{-h}^{h}\cdot dr\,d\theta.$$

$$= k\cdot\int_0^{2\pi}\int_0^{b} r^3\sin^2\theta\,(2h) + \frac{r}{3}\cdot\left(h^3-(-h)^3\right)\,dr\,d\theta.$$

$$= k\int_0^{2\pi}\int_0^{b}\left(2h\sin^2\theta\, r^3 + \frac{2h^3}{3}r\right)dr\,d\theta$$

$$= k\cdot\int_0^{2\pi} 2h\sin^2\theta\cdot\left[\frac{r^4}{4}\right]_0^{b} + \frac{\cancel{2}h^3}{3}\left[\frac{r^2}{\cancel{2}}\right]_0^{b}\cdot d\theta.$$

$$= k\cdot\int_0^{2\pi}\frac{h}{2}\sin^2\theta\cdot b^4 + \frac{h^3}{3}\cdot b^2\,d\theta.$$

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$$= K \cdot \int_0^{2\pi} \frac{h \cdot b^4}{2} \left(\frac{1 - \cos 2\theta}{2} \right) + \frac{h^3}{3} b^2 \cdot d\theta$$

$$= K \cdot \left[\frac{h \cdot b^4}{4} \left(\theta - \frac{\sin 2\theta}{2} \right)_0^{2\pi} + \frac{h^3}{3} b^2 \left[\theta \right]_0^{2\pi} \right]$$

$$= K \left[\frac{h b^4}{4} \times 2\pi + \frac{h^3}{3} b^2 (2\pi) \right]$$

$$= K \cdot \left(\frac{\pi h b^4}{2} + \frac{2}{3} \pi h^3 b^2 \right).$$

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