Class 8

14 July 2021 10:10

Parseval's formula:

1. Parseval's formula for fourier series of $f(x)$ in $(-l, l)$ is,

$$\int_{-l}^{l} (f(x))^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$$

2. Parseval's formula for half range cosine series of $f(x)$ in $(0, l)$ is

$$\int_0^l (f(x))^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

3. Parseval formula for half range sine series of $f(x)$ in $(0, l)$ is

$$\int_0^l (f(x))^2 dx = \frac{l}{2}\sum_{n=1}^{\infty} b_n^2.$$

Proof: F.S expansion of $f(x)$ in $(-l, l)$,

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{l}\right) + \sum b_n \sin\frac{n\pi x}{l} \longrightarrow (1)$$

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$\int_{-l}^{l} f(x)dx = a_0 l.$

(1) $\times f(x)$ and integrating from $-l$ to l,

$$\int_{-l}^{l} (f(x))^2 dx = \frac{a_0}{2}\left(\int_{-l}^{l} f(x)dx\right) + \sum a_n \int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$+ \sum b_n \int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$

$$= \frac{a_0}{2}\cdot a_0 l + \sum_{n=1}^{\infty} a_n \cdot l a_n + \sum_{n=1}^{\infty} b_n \cdot l \cdot b_n.$$

$$= l\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right\}$$

$$\boxed{\int_{-l}^{l} (f(x))^2 dx = l\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}}$$

Half range Fourier cosine series expansion of $f(x)$ in $(0, l)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \longrightarrow (2)$$

$a_0 = \frac{2}{l}\int_0^l f(x)dx$

$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

(2)·x f(x) and integrating from 0 to l.

$$\int_0^l (f(x))^2 dx = \frac{a_0}{2}\cdot\int_0^l f(x)\,dx + \sum_{n=1}^{\infty} a_n \cdot \int_0^l \cos\left(\frac{n\pi x}{l}\right) f(x)\,dx.$$

$$= \frac{a_0}{2}\cdot\frac{a_0 l}{2} + \sum_{n=1}^{\infty} a_n \frac{l a_n}{2} = \frac{l}{2}\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right\}$$

$$\therefore \boxed{\int_0^l (f(x))^2 dx = \frac{l}{2}\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right\}}$$

6. Find the half-range cosine series of $f(x) = (x-1)^2$ in $0 < x < 1$. Hence prove that $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

$$(0, l) \equiv (0, 1) \Rightarrow \boxed{l = 1}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{1}\int_0^1 (x-1)^2\,dx = 2\left\{\frac{(x-1)^3}{3}\right\}_0^1 = \frac{2}{3}\left(0 - (-1)^3\right) = \frac{2}{3}(0-(-1))$$

$$a_0 = \frac{2}{3} \Rightarrow \boxed{\frac{a_0}{2} = \frac{1}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = 2\cdot\int_0^1 \underset{u}{(x-1)^2}\,\underset{v}{\cos n\pi x}\,dx$$

$$= 2\left\{ (x-1)^2\cdot\cancel{\left(\frac{\sin n\pi x}{n\pi}\right)}^{0} - 2(x-1)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + 2\cdot\cancel{\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)}^{0}\right\}_0^1$$

$$= \frac{2\times 2}{n^2\pi^2}\cdot\left((x-1)\cos n\pi x\right)_0^1 = \frac{4}{n^2\pi^2}\left(0 - (-\cos 0)\right) = \boxed{\frac{4(1)}{n^2\pi^2} = a_n}$$

$$\boxed{a_n = \frac{4}{n^2\pi^2}}$$

∴ Half range cosine series of $(x-1)^2$ in $(0, 1)$

$$\boxed{(x-1)^2 = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x}$$

By Parseval's identity for Fourier cosine series,

$$\int_0^l (f(x))^2 dx = \frac{l}{2}\left(\frac{a_0^2}{2} + \sum a_n^2\right)$$

$$\Rightarrow \int_0^1 (x-1)^4\,dx = \frac{1}{2}\left(\frac{\cancel{4}^{2}}{9}\times\frac{1}{\cancel{2}} + \sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}\right)$$

$$\Rightarrow \int_0^1 (x-1)^4\, dx = \frac{1}{2}\left(\frac{\cancel{4}^2}{9}\times\frac{1}{\cancel{2}} + \sum_{n=1}^{\infty} \frac{16}{n^4\pi^4}\right)$$

$$\Rightarrow \left[\frac{(x-1)^5}{5}\right]_0^1 = \frac{1}{2}\left(\frac{2}{9} + \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\right)$$

$$\Rightarrow 0-\left(\frac{-1}{5}\right) = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{5}-\frac{1}{9} = \frac{4}{45} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\cancel{4}}{45}\times\frac{\pi^4}{\cancel{8}\,2} = \frac{\pi^4}{90}$$

$$\therefore \boxed{\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}}$$

7. Prove that in $0 < x < 2$, prove that $x = 1 - \frac{8}{\pi^2}\left\{cos(\frac{\pi x}{2}) + \frac{1}{3^2}cos(\frac{3\pi x}{2}) + \ldots\right\}$ and deduce that $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots = \frac{\pi^4}{96}$.

$(0,2) \equiv (0,l) \Rightarrow l = 2$

$f(x) = x$.

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = \frac{1}{2}(4-0) = 2.$$

$$\boxed{a_0 = 2} \Rightarrow \boxed{\frac{a_0}{2} = 1}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2}\int_0^2 x\cdot\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left\{x\cdot\frac{\cancel{\sin\frac{n\pi x}{2}}^{\,0}}{\left(\frac{n\pi}{2}\right)} - 1\cdot\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right)\right\}_0^2 = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi x}{2}\right)_0^2$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - \cos 0) = \boxed{\frac{4}{n^2\pi^2}((-1)^n - 1) = a_n}$$

$\therefore$ Half range Fourier cosine series expansion of x in $(0,2)$ is,

$$x = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}((-1)^n-1)\cos\left(\frac{n\pi x}{2}\right)$$

$$= 1 + \left\{\frac{-8}{1^2\cdot\pi^2}\cos\frac{\pi x}{2} + 0 - \frac{8}{3^2\pi^2}\cos\frac{3\pi x}{2} + 0 - \frac{8}{5^2\pi^2}\cos\frac{5\pi x}{2} + \cdots\right\}$$

$$x = 1 - \frac{8}{\pi^2}\left\{ \cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) - - - \right\}$$

By Parseval's formula, $\int_0^l (f(x))^2\,dx = \frac{l}{2}\left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right\}$

$$\int_0^2 x^2\,dx = \frac{\cancel{2}}{\cancel{2}}\left\{ \frac{4}{2} + \sum_{n=1}^{\infty} \frac{16}{n^4\pi^4}\left((-1)^n - 1\right)^2 \right\}$$

$$\Rightarrow \left.\frac{x^3}{3}\right|_0^2 = 2 + \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\left((-1)^n - 1\right)^2$$

$$\Rightarrow \frac{1}{3}(8-0) = 2 + \frac{16}{\pi^4}\left\{ (-2)^2 + \frac{1}{2^4}(0) + \frac{1}{3^4}(-2)^2 - - - - \right\}$$

$$\Rightarrow \frac{8}{3} - 2 = \frac{16}{\pi^4}\left\{ 4 + \frac{4}{3^4} + - - - \right\}$$

$$\Rightarrow \frac{\cancel{2}}{3} \times \frac{\pi^4}{\cancel{16}\,8} = 4 + \frac{4}{3^4} + - - - - - -$$

$$= \frac{\pi^4}{24} = 4\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + - - - - \right)$$

$$\therefore\ 1 + \frac{1}{3^4} + \frac{1}{5^4} - - - = \frac{\pi^4}{24\times 4} = \frac{\pi^4}{96}$$

By using sin series for $f(x) = 1$ in $0 < x < \pi$ S.T $\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$.

$f(x) = 1$. F.S.S. in $(0,\pi) = (0,l) \Rightarrow l = \pi$.

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{\pi}\int_0^{\pi} 1.\sin nx\,dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi}$$

$$= -\frac{2}{n\pi}(\cos n\pi - \cos 0) = -\frac{2}{n\pi}\left((-1)^n - 1\right) = \boxed{\frac{2}{n\pi}\left(1-(-1)^n\right) = b_n}$$

$\therefore$ F.S.S. expansion of $f(x) = 1$ in $(0,\pi)$ is

$$1 = \sum \frac{2}{n\pi}\left(1-(-1)^n\right)\sin nx.$$

Parseval's identity for F.S.S in $(0,l)$ is,

$$\int_0^l (f(x))^2\,dx = \frac{l}{2}\sum b_n^2$$

$$\Rightarrow \int_0^\pi 1\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)^2$$

$$\Rightarrow (x)_0^\pi = \cancel{\frac{\pi}{2}}\,\frac{\cancel{4}\,2}{\pi^{\cancel{2}}}\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)^2}{n^2}$$

$$= \pi = \frac{2}{\pi}\left\{\frac{4}{1^2} + 0 + \frac{4}{3^2} + 0 + \frac{4}{5^2} + \cdots\right\}$$

$$= \frac{\pi^2}{2} = 4\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right\}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$$

Class 9

14 July 2021 12:28

In some practical situations a function $y=f(x)$ is specified through a table of corresponding values of x and y. In such problems a_0, a_n, b_n cannot be evaluated by the usual method of integration. They can be evaluated by following properties of definite integrals.

The average value of a function $f(x)$ over an interval $(a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

where $b-a$ is length of the interval.

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{2}{2}\cdot\frac{1}{l}\int_c^{c+2l} f(x)\,dx = 2 \times \text{average value of } f(x)$$

length of the interval $(c, c+2l) = 2l$.

$$= 2\,\frac{\Sigma y}{N}$$ where N is number of values of y.

(average value of $f(x)$ in $(c, c+2l) = \frac{1}{2l}\int_c^{c+2l} f(x)\,dx$)

$$\boxed{a_0 = \frac{2}{N}\Sigma y}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$\frac{\pi x}{l} = \theta$.

$$= 2 \times \text{average value of } f(x)\cos\left(\frac{n\pi x}{l}\right) = 2\cdot\frac{\Sigma y\cos\left(\frac{n\pi x}{l}\right)}{N}$$

$$\therefore \boxed{a_n = \frac{2}{N}\Sigma y\cos\left(\frac{n\pi x}{l}\right)}$$

Similarly $\boxed{b_n = \frac{2}{N}\Sigma y\sin\left(\frac{n\pi x}{l}\right)}$

If we take $\theta = \frac{\pi x}{l}$ then, $\boxed{a_n = \frac{2}{N}\Sigma y\cos n\theta}$ and $\boxed{b_n = \frac{2}{N}\Sigma y\sin n\theta}$.

The Fourier series expansion of $f(x)$ over $(c, c+2l)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right\} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)$$

where $\frac{a_0}{2}$ is called constant term.

$a_1\cos\left(\frac{\pi x}{l}\right) + b_1\sin\left(\frac{\pi x}{l}\right)$ is called first harmonic.

$a_2\cos\left(\frac{2\pi x}{l}\right) + b_2\sin\left(\frac{2\pi x}{l}\right)$ is called second harmonic.

<u>Note</u>: If the values of y at $x=c$ and $x=c+2l$ are given we must leave one of them $\because f(x) = f(x+2l)$

$$y_c = y_{c+2l}.$$

We must have one of them $f(x) = f(x+2l)$

$$y_c = y_{c+2l}.$$

Working procedure

1) Write the period of f(x) from the given range of values.

2) If period is 2π, depending on the harmonics required prepare relevant table of Σy, $\Sigma y\cos x$, $\Sigma y\cos 2x$ ----, $\Sigma y\sin x$, $\Sigma y\sin 2x$... and compute the harmonics.

3. If the period is not 2π equate it with $2l$ and obtain the value of l. Find Σy, $\Sigma y\cos\theta$, $\Sigma y\cos 2\theta$ ----, $\Sigma y\sin\theta$, $\Sigma y\sin 2\theta$ ... and compute the harmonics where $\theta = \frac{\pi x}{l}$

Note: 1) Fourier series upto first harmonic is $f(x) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$.

" " " second " " $f(x) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta$.

2) Amplitude of n^{th} harmonic $= \sqrt{a_n^2 + b_n^2}$

1. Find the direct current part and amplitude of the first harmonic from the following table consisting of the variations of periodic current:

t sec	0	T/6	T/3	T/2	2T/3	5T/6	T
A(amps)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

length of the interval $= T = 2l \Rightarrow \boxed{l = \frac{T}{2}}$

Let $\theta = \frac{\pi t}{l} = \frac{\pi t}{\left(\frac{T}{2}\right)} = \frac{2\pi t}{T} = \frac{2\pi}{T} \times t$.

	t	θ	A	A cosθ	A sinθ
1	0	0	1.98	1.98	0
2	$\frac{T}{6}$	$\frac{\pi}{3} = 60°$	1.30	0.65	1.1258
3	$\frac{T}{3}$	$\frac{2\pi}{3} = 120°$	1.05	−0.525	0.9093
4	$\frac{T}{2}$	$\pi = 180°$	1.30	−1.30	0
5	$\frac{2T}{3}$	$\frac{4\pi}{3} = 240°$	−0.88	0.44	0.7621
6	$\frac{5T}{6}$	$\frac{5\pi}{3} = 300°$	−0.25	−0.125	0.2165
	~~T~~	~~$2\pi = 360°$~~	~~1.98~~	~~1.98~~	~~0~~
			4.5	1.12	3.0134
			$= \Sigma A$	$= \Sigma A\cos\theta$	$= \Sigma A\sin\theta$

$\boxed{N = 6}$

→ Don't consider for calculation

Consider any one $y_{at\ 0}$ or $y_{at\ T}$.

$$a_0 = \frac{2}{N} \Sigma A = \frac{2}{6}(4.5) = 1.5$$

$$a_n = \frac{2}{N} \Sigma A\cos\theta = \frac{2}{6}(1.12) = 0.3733.$$

$$b_n = \frac{2}{N} \Sigma A\sin\theta = \frac{2}{6}(8.0134) = 1.0046.$$

$\therefore$ Direct current part = constant term = $\frac{a_0}{2} = \frac{1.5}{2} = 0.75$ amp.

amplitude of I harmonic = $\sqrt{a_1^2+b_1^2} = \sqrt{(0.3733)^2 + (1.0046)^2}$

$$= \underline{\underline{1.0717}}.$$

2. The following table gives displacement 'u' (in min) of a sliding piece from a fixed reference point for every 30 degrees of rotation of the crank.

θ	0	30	60	90	120	150	180	210	240	270	300	330
u	298	356	373	337	254	155	80	51	60	93	147	221
	1	2	3	4	5	6	7	8	9	10	11	12

$\boxed{N = 12}$

Expand u as a Fourier series upto first harmonic.

θ	u	$u\cos\theta$	$u\sin\theta$
0	298	298	0
30°	356	308.3050	178
60°	373	186.5	323.0275
90°	337	0	337
120°	254	−127	219.9705
150°	155	−134.2339	77.5
180°	80	−80	0
210	51	−44.1673	−25.5
240	60	−30	−51.9615
270	93	0	−93
300	147	73.5	−127.3057
330	221	191.3916	−110.5
	$\Sigma u = 2425$	$\Sigma u\cos\theta = 642.2954$	$\Sigma u\sin\theta = 727.2308$

$$a_0 = \frac{2}{N} \Sigma u = \frac{2}{12} \cdot (2425) = \frac{2425}{6} = 404.1667.$$

$$a_1 = \frac{2}{N} \Sigma u\cos\theta = \frac{2}{12}(642.2954) = 107.0492.$$

$$b_1 = \frac{2}{N} \Sigma u\sin\theta = \frac{2}{12} \cdot (727.2308) = 121.2051$$

$$b_1 = \frac{2}{N}\Sigma\, u\sin\theta = \frac{2}{12}\cdot(727.2308) = 121.2051$$

$\therefore$ F-S expansion of $f(x)$ upto I harmonic is,

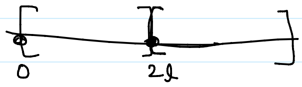
$$f(x) = \frac{a_0}{2} + a_1\cos\theta + a_2\sin\theta$$

$$\boxed{f(x) = 202.0834 + 107.0492\cos\theta + 121.2051\sin\theta}$$

3. Find the first two harmonics for the function $f(\theta)$ given by the following table.

θ	0	60	120	180	240	300	360
$f(\theta)$	0.8	0.6	0.4	0.7	0.9	1.1	0.8

1 2 3 4 5 6



θ	$f(\theta)$	$f(\theta)\cos\theta$	$f(\theta)\cos2\theta$	$f(\theta)\sin\theta$	$f(\theta)\sin2\theta$
0	0.8	0.8	0.8	0	0
60°	0.6	0.3	−0.3	0.5196	0.5196
120°	0.4	−0.2	−0.2	0.3464	−0.3464
180	0.7	−0.7	0.7	0	0
240	0.9	−0.45	−0.45	−0.7794	0.7794
300	1.1	0.55	−0.55	−0.9526	−0.9526
	$\Sigma f(\theta) = 4.5$	$\Sigma f(\theta)\cos\theta = 0.3$	$\Sigma f(\theta)\cos2\theta = 0$	$\Sigma f(\theta)\sin\theta = -0.8660$	$\Sigma f(\theta)\sin2\theta = 0$.

$$a_0 = \frac{2}{N}\Sigma f(\theta) = \frac{2}{6}(4.5) = \frac{1}{3}\times 4.5 = \underline{\underline{1.5}}$$

$$a_1 = \frac{2}{N}\Sigma f(\theta)\cos\theta = \frac{2}{6}\times 0.3 = 0.1.$$

$$a_2 = \frac{2}{N}\Sigma f(\theta)\cos2\theta = 0$$

$$b_1 = \frac{2}{N}\Sigma f(\theta)\sin\theta = \frac{1}{3}(-0.8660) = -0.2887.$$

$$b_2 = \frac{2}{N}\Sigma f(\theta)\sin2\theta = 0.$$

$\therefore$ First harmonics $\longrightarrow$ $a_1\cos\theta + b_1\sin\theta = 0.1\cos\theta - 0.2887\sin\theta$.

II harmonics $\longrightarrow$ $a_2\cos2\theta + b_2\sin2\theta = 0$.

Class 10

15 July 2021 10:58

Complex form of Fourier series:

F.S expansion of f(x) where f(x) is a periodic function of period $2l$ is,

$f(x) =$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}}$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{ a_n\left(\frac{e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}}}{2}\right) + b_n\left(\frac{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}}{2i}\right)\right\}$$

$$= C_0 + \sum_{n=1}^{\infty}\left\{ e^{\frac{in\pi x}{l}}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) + e^{-\frac{in\pi x}{l}}\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)\right\}$$

$$f(x) = C_0 + \sum_{n=1}^{\infty} e^{\frac{in\pi x}{l}}\,\frac{1}{2}(a_n - ib_n) + e^{-\frac{in\pi x}{l}}\,\frac{1}{2}(a_n + ib_n)\Big\}$$

$$\boxed{f(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{\frac{in\pi x}{l}} + C_{-n} e^{-\frac{in\pi x}{l}}} \longrightarrow (1)$$

$$C_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left\{\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l} - i\,\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\right\}$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)$$

$$\boxed{C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}}$$ $n = 1, 2, 3 \cdots$

$$C_{-n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)$$

$$\boxed{C_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{in\pi x}{l}}}$$ $n = 1, 2, 3 \cdots$

$$\boxed{C_0 = \frac{a_0}{2} = \frac{1}{2}\cdot\frac{1}{l}\int_{-l}^{l} f(x)\,dx} = \frac{1}{2l}\int_{-l}^{l} f(x)\cdot e^{\frac{i\pi x n}{l}\times(0)}$$ $n = 0$

$$\boxed{c_0 = \frac{a_0}{2} = \frac{1}{2}\cdot\frac{1}{l}\int_{-l}^{l} f(x)\,dx} = \frac{1}{2l}\int_{-l}^{l} f(x)\cdot e^{\frac{i\pi x n}{l}\times(0)} \qquad n=0$$

$$\boxed{C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{-in\pi x}{l}}\,dx} \qquad n = 0, \pm 1, \pm 2 \dots$$

Sub in (1),

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{\frac{in\pi x}{l}}}$$

1. Find the complex Fourier series of $f(x) = \cos ax, -\pi < x < \pi$ where 'a' is a non-integer.

WKT complex F.S of $f(x)$ is $f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{\frac{in\pi x}{l}} \longrightarrow (1)$

where $C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{-in\pi x}{l}}\,dx$ $\qquad l = \pi$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax \cdot e^{-inx}\,dx \qquad \int e^{ax}\cos bx\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\cos ax\,dx \qquad = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$a = -in \quad b = a.$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2+a^2}\left(-in\cos ax + a\sin ax\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\frac{1}{(-n^2+a^2)}\left\{e^{-in\pi}\left(-in\cos a\pi + a\sin a\pi\right) - e^{+in\pi}\left(-in\cos a\pi - a\sin a\pi\right)\right\}$$

$$= \frac{1}{2\pi}\frac{1}{a^2-n^2}\left\{-in\cos a\pi\left(e^{-in\pi} - e^{in\pi}\right) + a\sin a\pi\left(e^{-in\pi} + e^{in\pi}\right)\right\}$$

$$= \frac{1}{\pi(a^2-n^2)}\left\{+in\cos a\pi\left(\frac{e^{in\pi} - e^{-in\pi}}{2i}\right)\times i + a\sin a\pi\left(\frac{e^{in\pi} + e^{-in\pi}}{2}\right)\right\}$$

$$= \frac{1}{\pi(a^2-n^2)}\left\{-n\cos a\pi\cdot \cancel{\sin n\pi}^{\,0} + a\sin a\pi\cos n\pi\right\}$$

$$\boxed{c_n = \frac{(-1)^n \cdot a \sin a\pi}{\pi (a^2-n^2)}}$$

by ①. Complex F.S of $\cos ax$ is,

$$\cos ax = \sum_{n=-\infty}^{\infty} \frac{(-1)^n a \sin a\pi}{(a^2-n^2)\pi} e^{\frac{in\pi x}{\pi}}$$

$$\therefore \boxed{\cos ax = \frac{a \sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(a^2-n^2)} e^{inx}}$$

2 Find the complex form of the Fourier series of the $f(x) = e^{-ax}$, in $-1 \le x \le 1$. $l = 1$

Comp. form of F.S of $f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \longrightarrow ①$$

where $c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx.$

$$= \frac{1}{2}\int_{-1}^{1} e^{-ax} . e^{-in\pi x} dx = \frac{1}{2}\int_{-1}^{1} e^{-(a+in\pi)x} dx$$

$$= \frac{1}{2}\left[\frac{e^{-(a+in\pi)x}}{-(a+in\pi)}\right]_{-1}^{1} = \frac{-1}{2} \cdot \frac{1}{a+in\pi} \left\{ e^{-(a+in\pi)} - e^{+(a+in\pi)} \right\}$$

$$= \frac{-1}{a+in\pi} \left\{ \frac{e^{-(a+in\pi)} - e^{(a+in\pi)}}{2} \right\}$$

$e^{in\pi} = \cos n\pi + i\cancel{\sin n\pi}^{0}$
$= (-1)^n.$

$$= \frac{-1}{(a+in\pi)} \left[\frac{e^{-a} e^{-in\pi} - e^{a} e^{in\pi}}{2} \right]$$

$e^{-in\pi} = \cos n\pi - i\cancel{\sin n\pi}^{0}$
$= (-1)^n.$

$$= \frac{-1}{in\pi + a} \left[\frac{e^{-a}(-1)^n - e^{a}(-1)^n}{2} \right]$$

$$= \frac{-(a-in\pi)}{a^2-(in\pi)^2} (-1)^n \left(\frac{e^{-a} - e^{a}}{2} \right)$$

$a-(in\pi)$ $\qquad$ — 2

$$= \frac{+(a-in\pi)}{a^2+n^2\pi^2}(-1)^n\cdot\left(\frac{e^a-e^{-a}}{2}\right) = \boxed{\frac{(a-in\pi)(-1)^n}{n^2\pi^2+a^2}\sinh a = C_n}$$

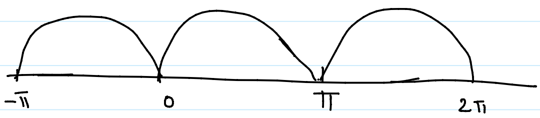
$\therefore$ by (1),

$$e^{-ax} = \sum_{n=-\infty}^{\infty}\frac{(a-in\pi)}{n^2\pi^2+a^2}(-1)^n\cdot\sinh a\cdot e^{in\pi x}$$

i.e. $$\boxed{e^{-ax} = \sinh a\sum_{n=-\infty}^{\infty}\frac{(a-in\pi)}{n^2\pi^2+a^2}(-1)^n e^{in\pi x}}$$

3. Find the complex form of the Fourier series of the periodic function $f(x) = sinx, 0 < x < \pi, f(x+\pi) = f(x)$.

$2l = \pi \Rightarrow l = \pi/2$.



Complex F.S of $f(x)$ is,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{+\frac{in\pi x}{l}}$$

where $$C_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-\frac{in\pi x}{l}} = \frac{1}{\pi}\int_0^{\pi}\sin x\cdot e^{-\frac{in\pi x}{(\pi/2)}}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin x\cdot e^{-2inx}\,dx.$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{-2inx}\sin x\,dx$$

$a = -2in \qquad b = 1.$

$$= \frac{1}{\pi}\left\{\frac{e^{-2inx}}{(-2in)^2+1}(-2in\sin x - 1.\cos x)\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{e^{-2in\pi}}{-4n^2+1}(-\cancel{2in\sin\pi}^{0} - \cos\pi) - \frac{1}{(-4n^2+1)}(-\cancel{2in\sin 0}^{0} - \cos 0)\right\}$$

$$= \frac{1}{\pi}\frac{1}{(1-4n^2)}\left\{e^{-2in\pi}(1) + 1\right\}$$

$\underbrace{\cos 2n\pi}_{=1} + i\,\cancel{\sin 2n\pi}^{0}$

$$= \frac{1}{\pi}\left\{\frac{1}{1-4n^2)}\right\} 2. \quad = \boxed{\frac{2}{\pi(1-4n^2)} = C_n}$$

$\therefore$ Complex F.S is,

$$\sin x = \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)} e^{\frac{in\cancel{\pi}x}{(\cancel{\pi}/2)}}$$

$$\boxed{\sin x = \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{1}{1-4n^2} \cdot e^{2inx}}$$