UE20MA151- ENGINEERING MATHEMATICS

UNIT 4- INVERSE LAPLACE TRANSFORMS

CLASS WORK PROBLEMS

1	Definition and Inverse Laplace transform of standard functions.
2	General properties of Inverse Laplace transforms: Linearity, first shift, Change of scale.
3-4	Inverse Laplace transform of derivatives, Integrals, Multiplication by s, Division by s, Second shifting theorem.
5-6	Problems based on above discussed properties.
7	Inverse Laplace transforms using partial fractions.
8	Convolution theorem: Statement, applications, and problems.
9-10	Applications of Laplace transforms to solve Differential equations.

Find the following inverse Laplace transform.

1. $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$
2. $\mathcal{L}^{-1}\left\{\frac{1}{2s-5}\right\}$
3. $\mathcal{L}^{-1}\left\{\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}\right\}$
4. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+20}\right\}$
5. $\mathcal{L}^{-1}\left\{\frac{16+3s}{s^2-8s+20}\right\}$
6. $\mathcal{L}^{-1}\left\{\frac{3s-1}{(s-2)^2}\right\}$

Find the inverse Laplace transform of the following function using partial fractions.

7. $\frac{3s+7}{s^2-2s-3}$
8. $\frac{s}{(s+1)^2(s^2+1)}$
9. $\frac{s}{s^4+s^2+1}$
10. Obtain f(t) when $F(s) = \frac{s}{s^4+4a^4}$.

Calculate the inverse transform of the following functions using suitable properties.

11. $log\left(1+\frac{1}{s^2}\right)$

12. $\frac{s}{(s^2+a^2)^2}$

13. $\frac{1}{s^3\ (s^2+1)}$

14. $G(s)\ =\ \frac{1-e^{-sT}}{s(1+e^{-sT})}$

Apply convolution theorem to evaluate the following.

15. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$

16. Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$

17. Find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$

18. Find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Solve the following differential equations using Laplace transform.

19. Solve $\frac{d^2x}{dt^2}\ -3\ \frac{dx}{dt}\ +2x\ =\ 4e^{2t}$, given that x(0)=-3 and x'(0)=5.
    Ans: $4e^{2t}-7e^{t}+4te^{2t}$

20. The current $i(t)$ in an electrical circuit is given by the differential equation.

$$\frac{d^2i}{dt^2}\ +\ 2\ \frac{di}{dt}\ =\ \begin{cases} 0 & 0<t<10 \\ 1 & 10<t<20 \\ 0 & t>20 \end{cases} \text{ and } i(0)\ =\ 0, i'(0)\ =\ 0.$$

21. Solve the equation for the response $i(t)$, given that.
    $\frac{di}{dt}\ +2i+5\int_0^t i\ dt\ =\ u(t)$ and i(0) = 0.

22. Consider a series RLC circuit where R=20W, L=0.05H and C=$10^{-4}F$ and is driven by an alternating emf given by $E\ =\ 100t$. Given that both the circuit current *I* and the capacitor charge *q* are zero at time *t=0*, find an expression for *i(t)* in the region t>0.
    [Ri +L$\frac{di}{dt}\ +\ \frac{1}{c}\int idt\ =\ E$]

23. The deflection of a beam of length L, clamped, horizontally at both ends and loaded at $x = \frac{L}{4}$ by a weight W is given by $EI \frac{d^4y}{dx^4} = W\,\delta\left(x - \frac{L}{4}\right)$. Find the deflection curve, given that $y = \frac{dy}{dx} = 0$ when $x = 0 \; and \; x = L.$

24. A mechanical system with two degrees of freedom satisfies the equations.

$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4,\; 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0.$ Use Laplace transforms to determine x and y at any instant, given that x, y and the first order derivative of them with respect to ‘t’ vanish at t=0.

INVERSE LAPLACE TRANSFORMS

1) Find $L^{-1}\left\{\frac{1}{s^5}\right\}$.

$$L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$$

2) Find $L^{-1}\left\{\frac{1}{2s-5}\right\}$

$$L^{-1}\left\{\frac{1}{2s-5}\right\} = L^{-1}\left\{\frac{1}{2(s-5/2)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-5/2}\right\}$$
$$= \frac{1}{2}e^{5t/2}.$$

3) Find $L^{-1}\left\{\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}\right\}$

$$L^{-1}\left\{\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}\right\}$$
$$= 3L^{-1}\left\{\frac{s}{s^2+4}\right\} - 8L^{-1}\left\{\frac{1}{s^2+4}\right\} - 4L^{-1}\left\{\frac{s}{s^2-16}\right\} + 24L^{-1}\left\{\frac{1}{s^2-16}\right\}$$
$$= 3\cdot\cos 2t - 4\sin 2t - 4\cosh 4t + 6\sinh 4t$$

4) Find $L^{-1}\left\{\frac{1}{s^2-4s+20}\right\}$

Consider.
$s^2-4s+4+16$.
$(s-2)^2+16$

$$L^{-1}\left\{\frac{1}{s^2-4s+20}\right\} = L^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$
$$= e^{2t}L^{-1}\left\{\frac{1}{s^2+16}\right\}. = \frac{e^{+2t}\sin 4t}{4}$$

5) $L^{-1}\left\{\frac{16+3s}{s^2-8s+20}\right\}$

$s^2-8s+20$

$s^2-8s+16+4$

$(s-4)^2+4$.

$L^{-1}\left\{\frac{16+3s}{s^2-8s+20}\right\} = L^{-1}\left\{\frac{16+3s}{(s-4)^2+4}\right\}$.

$= L^{-1}\left\{\frac{28+3(s-4)}{(s-4)^2+4}\right\} = e^{4t}\left[L^{-1}\left\{\frac{28+3s}{s^2+4}\right\}\right]$

$\Rightarrow e^{4t}.\left\{14\sin 2t + 3\cos 2t\right\}$

6) $L^{-1}\left\{\frac{3s-1}{(s-2)^2}\right\}$

$L^{-1}\left\{\frac{3s-1}{(s-2)^2}\right\} = L^{-1}\left\{\frac{3(s-2)+5}{(s-2)^2}\right\}$.

$= e^{2t}L^{-1}\left\{\frac{3s+5}{s^2}\right\}$

$= e^{2t}\left[L^{-1}\left\{\frac{3}{s}\right\} + L^{-1}\left\{\frac{5}{s^2}\right\}\right]$

$= 3e^{2t} + 5te^{2t}$

7) $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

Consider

$$\frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}.$$

$$3s+7 = A(s+1) + B(s-3)$$

Put $s = -1$,

$$4 = -4B \Rightarrow \boxed{B = -1}$$

Put $s = 3$,

$$16 = 4A \Rightarrow \boxed{A = 4}$$

$$L^{-1}\left\{\frac{3s+7}{(s-3)(s+1)}\right\} = L^{-1}\left\{\frac{4}{s-3} + \frac{-1}{s+1}\right\}.$$

$$= 4 \cdot e^{3t} - e^{-t}.$$

8) $$L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}$$

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 \quad \text{——— (1)}$$

(1) can be written as

$$s = (A+C)s^3 + (A+B+2C+D)s^2 + (A+2D+C)s + (A+B+D).$$

Equating the corresponding coefficients of

s^3 : $A+C = 0$. ——— (2)

s^2 : $A+B+2C+D = 0$ ——— (3)

coeff of s : $A + 2D + C = 1$ — (4)

Constants $A + B + D = 0$. — (5)

From (2) & (4)

$$2D = 1 \Rightarrow \boxed{D = \tfrac{1}{2}}$$

(3) can be written as

$$A + B + 2C + D = 0$$

$$\Rightarrow 2C = 0 \quad (\because \text{ from } (5) \; A+B+D=0)$$

$$\boxed{C = 0}$$

$\therefore$ (2) becomes $A + C = 0$

$$\boxed{A = 0}$$

Sub all these values in (5)

$$A + B + D = 0$$

$$\Rightarrow 0 + B + \tfrac{1}{2} = 0$$

$$\boxed{B = -\tfrac{1}{2}}$$

$$\therefore L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\} = L^{-1}\left\{\frac{-1/2}{(s+1)^2} + \frac{1/2}{s^2+1}\right\}$$

$$= -\tfrac{1}{2} L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \tfrac{1}{2} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= -\frac{1}{2} \cdot e^{-t} \cdot L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2} \cdot \sin t$$

$$= \frac{-1}{2} e^{-t} t + \frac{1}{2} \sin t$$

$$= \frac{\sin t - t \cdot e^{-t}}{2}$$

9) $L^{-1}\left\{\frac{s}{s^4 + s^2 + 1}\right\}$

$$s^4 + s^2 + 1 = (s^2+1)^2 - s^2$$
$$= (s^2+1-s)(s^2+1+s)$$

Also,

$$2s = (s^2+1+s) - (s^2+1-s)$$

$$\therefore \frac{s}{(s^2+1-s)(s^2+1+s)} = \frac{1}{2}\left[\frac{(s^2+1+s)-(s^2+1-s)}{(s^2+1-s)(s^2+1+s)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right]$$

$$L^{-1}\left\{\frac{s}{(s^2+1-s)(s^2+1+s)}\right\} = \frac{1}{2} \cdot L^{-1}\left\{\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right\}.$$

$$= \frac{1}{2} \cdot \left[L^{-1}\left\{\frac{1}{(s-\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\} - L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= \frac{1}{2}\left\{ e^{t/2} \cdot L^{-1}\left\{ \frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right\} - e^{-t/2} L^{-1}\left\{ \frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right\} \right\}$$

$$= \frac{1}{2}\left\{ e^{t/2} \cdot \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t - e^{-t/2} \cdot \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \right\}.$$

$$= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \cdot \left\{ \frac{e^{t/2} - e^{-t/2}}{2} \right\}.$$

$$= \underline{\underline{\frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \sinh \frac{t}{2}}}$$

10) obtain $f(t)$ when $F(s) = \dfrac{s}{s^4+4a^4}$.

Consider

$$s^4 + 4a^4 = (s^2 + 2a^2)^2 - 4a^2s^2$$

$$= (s^2 + 2a^2 + 2as)(s^2 + 2a^2 - 2as).$$

Also,

$$4as = (s^2 + 2a^2 + 2as) - (s^2 + 2a^2 - 2as).$$

$$\therefore \frac{s}{s^4+4a^4} = \frac{1}{4a}\left\{ \frac{(s^2+2a^2+2as) - (s^2+2a^2-2as)}{(s^2+2a^2+2as)(s^2+2a^2-2as)} \right\}$$

$$= \frac{1}{4a}\left\{ \frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as} \right\}$$

$$L^{-1}\left\{ \frac{s}{s^4+4a^4} \right\} = \frac{1}{4a}\left[L^{-1}\left\{ \frac{1}{s^2+2a^2-2as} \right\} - L^{-1}\left\{ \frac{1}{s^2+2a^2+2as} \right\} \right]$$

$$= \frac{1}{4a}\left\{ L^{-1}\left[\frac{1}{(s-a)^2+a^2} \right] - L^{-1}\left[\frac{1}{(s+a)^2+a^2} \right] \right\}.$$

$$= \frac{1}{4a}\left[e^{at}\, L^{-1}\left\{ \frac{1}{s^2+a^2} \right\} - e^{-at}.\, L^{-1}\left\{ \frac{1}{s^2+a^2} \right\} \right]$$

$$= \frac{1}{4a}\left[e^{at}.\frac{\sin at}{a} - e^{-at}.\frac{\sin at}{a} \right]$$

$$= \frac{1}{4a}.\frac{\sin at}{a}\left[e^{at} - e^{-at} \right]$$

$$= \frac{\sin at}{2a^2}.\left[\frac{e^{at} - e^{-at}}{2} \right] = \frac{\sin at . \sinh at}{2a^2}$$

11) Find $L^{-1}\left\{ \log\left(1+\frac{1}{s^2}\right) \right\}$.

Let $F(s) = \log\left(1+\frac{1}{s^2}\right) = \log\left(\frac{s^2+1}{s^2}\right)$.

$$F(s) = \log(s^2+1) - \log s^2$$

$$F'(s) = \frac{2s}{s^2+1} - \frac{2\cancel{s}}{s^{\cancel{2}}}$$

$$-F'(s) = \frac{2}{s} - \frac{2s}{s^2+1}$$

Taking inverse laplace on both sides

$$L^{-1}\{-F'(s)\} = L^{-1}\left\{\frac{2}{s}\right\} - 2L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$t \cdot f(t) = 2 - 2\cos t$$

$$f(t) = \frac{2-2\cos t}{t} = \frac{2(1-\cos t)}{t}$$

12) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$.

We know that $\frac{d}{ds}\left\{\frac{1}{s^2+a^2}\right\} = \frac{-2s}{(s^2+a^2)^2}$.

$$\Rightarrow \frac{s}{(s^2+a^2)^2} = \frac{-1}{2}\frac{d}{ds}\left\{\frac{1}{s^2+a^2}\right\}$$

Take inverse laplace on both sides.

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{-1}{2}L^{-1}\left\{\frac{d}{ds}\left\{\frac{1}{s^2+a^2}\right\}\right\}$$

$$= \frac{-1}{2}\cdot\left\{-t \cdot L^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right\}$$

$$= \frac{1}{2}t\frac{\sin at}{a} = \frac{t\sin at}{2a}$$

13) $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$.

Let $F(s) = \frac{1}{s^2+1}$

$\therefore L^{-1}\{F(s)\} = f(t) = \sin t$.

$$L^{-1}\left\{\frac{F(s)}{s^3}\right\} = \int_0^t \int_0^t \int_0^t L^{-1}\{F(s)\} \cdot dt \cdot dt\, dt$$

$$= \int_0^t \int_0^t \int_0^t \sin t\, dt \cdot dt\, dt$$

$$= \int_0^t \int_0^t \left[-\cos t\right]_0^t \cdot dt\, dt$$

$$= \int_0^t \int_0^t (-\cos t + 1)\, dt\, dt$$

$$= \int_0^t \left[-\sin t + t\right]_0^t dt$$

$$= \int_0^t (-\sin t + t)\, dt$$

$$= \left[+\cos t + \frac{t^2}{2}\right]_0^t$$

$$= \cos t + \frac{t^2}{2} - 1.$$

14) Find inverse laplace transform of

$$G(s) = \frac{1-e^{-sT}}{s(1+e^{-sT})}$$

We know that $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$$\therefore (1+e^{-sT})^{-1} = 1 - e^{-sT} + e^{-2sT} - e^{-3sT} + \dots$$

$$\therefore \frac{1-e^{-sT}}{s(1+e^{-sT})} = \frac{1}{s}(1-e^{-sT})(1+e^{-sT})^{-1}$$

$$= \frac{1}{s}(1-e^{-sT})(1 - e^{-sT} + e^{-2sT} - e^{-3sT} + \dots).$$

$$= \frac{1}{s}\big(1 - e^{-sT} + e^{-2sT} - e^{-3sT} + \dots$$
$$- e^{-sT} + e^{-2sT} - e^{-3sT} + e^{-4sT} - \dots\big)$$

$$= \frac{1}{s}(1 - 2e^{-sT} + 2e^{-2sT} - 2e^{-3sT} + \dots).$$

$$\therefore L^{-1}\left\{\frac{1-e^{-sT}}{s(1+e^{-sT})}\right\}$$

$$= L^{-1}\left\{\frac{1}{s}(1 - 2e^{-sT} + 2e^{-2sT} - 2e^{-3sT} + \dots)\right\}$$

$$= 1 - 2\cdot u(t-T) + 2u(t-2T) - 2u(t-3T) + \dots$$

Apply Convolution Theorem to evaluate The following :

15) $L^{-1}\left\{ \frac{1}{s^2(s^2+a^2)} \right\}$.

Let $F(s) = \frac{1}{s^2}$ and $G(s) = \frac{1}{s^2+a^2}$

$f(t) = L^{-1}\{F(s)\} = t$ and $g(t) = L^{-1}\{G(s)\}$

$= \frac{1}{a}\sin at$.

By convolution Theorem we have.

$L^{-1}\{F(s)\cdot G(s)\} = f(t) * g(t)$

$$= \int_0^t g(u)\cdot f(t-u)\cdot du.$$

$$= \int_0^t \frac{1}{a}\sin au \cdot (t-u)\cdot du.$$

$$= \frac{1}{a}\left[\int_0^t t\cdot \sin au \cdot du - \int_0^t u\cdot \sin au \cdot du\right].$$

$$= \frac{1}{a}\left[t\left[-\frac{\cos au}{a}\right]_0^t - \left[u\left(-\frac{\cos au}{a}\right) - \left(\frac{-1}{a}\right)\cdot\frac{\sin au}{a}\right]_0^t\right].$$

$$= \frac{1}{a}\cdot\left[t\cdot\left(-\frac{\cos at}{a}+\frac{1}{a}\right)-\left[-\frac{t\cdot\cos at}{a}+\frac{1}{a^2}\sin at\right]\right]$$

$$= \frac{1}{a}\cdot\left\{\frac{-t\cos \cancel{at}}{\cancel{a}}+\frac{t}{a}+\cancel{\frac{t\cos at}{a}}-\frac{1}{a^2}\sin at\right\}.$$

$$= \frac{1}{a^3}\cdot\left[at-\sin at\right]$$

16) Find $L^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$

Let $F(s)=\frac{1}{s-a}$ and $G(s)=\frac{1}{s-b}$.

$\therefore f(t) = L^{-1}\{F(s)\}$

$= L^{-1}\left\{\frac{1}{s-a}\right\}$

$= e^{at}$.

$g(t) = L^{-1}\{G(s)\}$

$= L^{-1}\left\{\frac{1}{s-b}\right\}$

$= e^{bt}$.

$L^{-1}\{F(s)\cdot G(s)\} = f(t) * g(t)$

$= \int_0^t f(u)\cdot g(t-u)\,du.$

$$= \int_0^t e^{au} \cdot e^{b(t-u)} \, du$$

$$= \int_0^t e^{au-bu} \cdot e^{bt} \, du$$

$$= e^{bt} \cdot \int_0^t e^{(a-b)u} \, du$$

$$= e^{bt} \cdot \left[\frac{e^{(a-b)u}}{a-b} \right]_0^t$$

$$= \frac{e^{bt}}{(a-b)} \left[e^{(a-b)t} - 1 \right]$$

$$= \frac{e^{at} - e^{bt}}{a-b}.$$

17) Find $L^{-1} \left\{ \frac{s^2}{(s^2+4)^2} \right\}$.

Let $F(s) = \frac{s}{s^2+4}$ $\quad G(s) = \frac{s}{s^2+4}$

$\Rightarrow f(t) = \cos 2t$ $\quad g(t) = \cos 2t$.

$$L^{-1}\{F(s)\cdot G(s)\} = f(t) * g(t)$$

$$= \int_0^t f(u)\cdot g(t-u)\cdot du$$

$$= \int_0^t \cos 2u \cdot \cos 2(t-u)\cdot du.$$

$$\cos A \cdot \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right].$$

$$= \frac{1}{2}\int_0^t \cos(\cancel{2u} + 2t - \cancel{2u}) + \cos(2u - 2t + 2u)\, du$$

$$= \frac{1}{2}\int_0^t \cos 2t + \cos(4u-2t)\cdot du$$

$$= \frac{1}{2}\cdot\left[\cos 2t\cdot[t]_0^t + \left(\frac{\sin(4u-2t)}{4}\right)_0^t\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}(\sin 2t + \sin 2t)\right]$$

$$= \frac{1}{2}\left(t\cos 2t + \frac{2}{4}\sin 2t\right).$$

$$= \frac{t\cos 2t}{2} + \frac{\sin 2t}{4}.$$

18) Find $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$.

Let $F(s) = \dfrac{s}{s^2+a^2}$ $\qquad G(s) = \dfrac{s}{s^2+b^2}$.

$f(t) = \cos at$ $\qquad g(t) = \cos bt$.

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t f(u)\cdot g(t-u)\,du.$$

$$= \int_0^t \cos au \cdot \cos b(t-u)\,du.$$

$$\cos A \cdot \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right].$$

$$= \frac{1}{2}\int_0^t \cos(au+bt-bu) + \cos(au-bt+bu)\,du$$

$$= \frac{1}{2}\left[\frac{\sin(a-b)u+bt}{a-b} + \frac{\sin(a+b)u-bt}{a+b}\right]_0^t$$

$$\Rightarrow \frac{1}{2}\left[\frac{\sin at}{a-b} - \frac{\sin bt}{a-b} + \frac{\sin at}{a+b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b}+\frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b}-\frac{1}{a-b}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{2a}{a^2-b^2}\right) + \sin bt\left(\frac{-2b}{a^2-b^2}\right)\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2} \quad (a \neq b).$$

Extra problem

1) Find $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$.

We know That

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\sin at}{2a} \quad \text{(Refer } 12^{th} \text{ prob)}$$

Also We know That

$$L^{-1}\{s.F(s)\} = f'(t) \quad \text{provided } f(0)=0.$$

Let $F(s) = \frac{s}{(s^2+a^2)^2} \Rightarrow f(t) = \frac{t\sin at}{2a}$

$$L^{-1}\left\{s.\frac{s}{(s^2+a^2)^2}\right\} = \frac{d}{dt}\left\{\frac{t\sin at}{2a}\right\} \quad \text{Also check. } f(0)=0.$$

$$= \frac{1}{2a}\{t.a\cos at + \sin at\}$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{\sin at + at\cos at}{2a}$$

Solve the following differential equations using Laplace transform.

19) solve $\frac{d^2x}{dt^2} - 3 \cdot \frac{dx}{dt} + 2x = 4e^{2t}$. given that $x(0) = -3$ and $x'(0) = 5$.

Given

$$\frac{d^2x}{dt^2} - 3 \cdot \frac{dx}{dt} + 2x = 4e^{2t}$$

Take Laplace on both sides.

$$L\{x''(t)\} - 3L\{x'(t)\} + 2L\{x\} = L\{4e^{2t}\}.$$

$$s^2 \cdot L\{x(t)\} - s x(0) - x'(0) - 3 \cdot (s L\{x(t)\} - x(0))$$
$$+ 2L\{x(t)\} = \frac{4}{s-2}.$$

$$s^2 \cdot L\{x(t)\} + 3s - 5 - 3sL\{x(t)\} - 9$$
$$+ 2L\{x(t)\} = \frac{4}{s-2}.$$

$$(s^2 - 3s + 2) L\{x(t)\} = \frac{4}{s-2} - 3s + 14$$

$$(s^2 - 3s + 2) L\{x(t)\} = \frac{4 - (3s)(s-2) + 14(s-2)}{s-2}$$

$$L\{x(t)\} = \frac{4 - 3s^2 + 6s + 14s - 28}{(s-2)(s^2-3s+2)}$$

$$L\{x(t)\} = \frac{-3s^2 + 20s - 24}{(s-2)(s^2-3s+2)}$$

$$x(t) = L^{-1}\left\{\frac{-3s^2+20s-24}{(s-2)(s-2)(s-1)}\right\}$$

$$\frac{-3s^2+20s-24}{(s-2)^2(s-1)} = \frac{A}{s-1} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$$

$$-3s^2+20s-24 = A(s-2)^2 + B(s-2)(s-1) + C(s-1)$$

Put $s = 1$, $\quad -3+20-24 = A \Rightarrow \boxed{A = -7}$

Put $s = 2$, $\quad -12+40-24 = C \Rightarrow \boxed{C = 4}$

Put $s = 0$, $\quad -24 = 4A + 2B - C$

$$\boxed{B = 4}$$

$$x(t) = L^{-1}\left\{\frac{-7}{s-1} + \frac{4}{(s-2)} + \frac{4}{(s-2)^2}\right\}$$

$$x(t) = -7e^{t} + 4e^{2t} + 4e^{2t}t.$$

20). The current $I(t)$ in an electrical circuit is given by the differential equation.

$$\frac{d^2 i}{dt^2} + 2 \cdot \frac{di}{dt} = \begin{cases} 0 & 0 < t < 10 \\ 1 & 10 < t < 20 \\ 0 & t > 20 \end{cases} \quad \text{and}$$

$i(0) = 0$, $i'(0) = 0$.

$$\frac{d^2 i}{dt^2} + 2 \frac{di}{dt} = u(t-10) - u(t-20).$$

Taking Laplace on both sides.

$$L\{i''(t)\} + 2 L\{i'(t)\} = L\{u(t-10) - u(t-20)\}$$

$$s^2 \cdot L\{i(t)\} - s\, i(0) - i'(0) + 2 \cdot (s \cdot L\{i(t)\} - i(0))$$

$$= \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}.$$

$$(s^2 + 2s) L\{i(t)\} = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$$L\{i(t)\} = \frac{1}{(s^2+2s)} \left(\frac{e^{-10s} - e^{-20s}}{s} \right).$$

$$i(t) = L^{-1}\left\{ \frac{e^{-10s} - e^{-20s}}{s^2(s+2)} \right\}. \quad \text{——— (1)}$$

We need to find the inverse Laplace of this expression. First, we concentrate on the $\frac{1}{s^2(s+2)}$ part and ignore the $(e^{-10s} - e^{-20s})$ part for now.

Now, we find the partial fractions:

$$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}.$$

$$1 = A(s)(s+2) + B(s+2) + C(s^2).$$

Put $s = 0$, $1 = 2B \Rightarrow B = 1/2$.

$s = -2$, $1 = 4C \Rightarrow C = 1/4$

$s = 1$, $1 = 3A + 3B + C \Rightarrow A = -1/4$.

So, $$\frac{1}{s^2(s+2)} = \frac{-1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)}$$

$$L^{-1}\left\{\frac{1}{s^2(s+2)}\right\} = L^{-1}\left\{\frac{-1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)}\right\}.$$

$$= \frac{-1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t}$$

Now,

$$i(t) = L^{-1}\left\{\frac{e^{-10s} - e^{-20s}}{s^2(s+2)}\right\} \quad \text{from } (1)$$

becomes

$$i(t) = L^{-1}\left\{\frac{e^{-10s}}{s^2(s+2)}\right\} - L^{-1}\left\{\frac{e^{-20s}}{s^2(s+2)}\right\}$$

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By Second shifting Theorem we have

$$L^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a).$$

$$\therefore\ i(t) = \left[\frac{-1}{4} + \frac{1}{2}(t-10) + \frac{1}{4}e^{-2(t-10)}\right] \cdot u(t-10)$$

$$- \left[\frac{-1}{4} + \frac{1}{2}(t-20) + \frac{1}{4}e^{-2(t-20)}\right] u(t-20).$$

$$= \frac{1}{4}\left(2t - 21 + e^{-2(t-10)}\right) u(t-10)$$

$$+ \frac{1}{4}\left(41 - 2t - e^{-2(t-20)}\right) u(t-20).$$

21) Solve the equation for the response $i(t)$ given that

$$\frac{di}{dt} + 2i + 5\int_0^t i\, dt = u(t) \text{ and } i(0) = 0.$$

Take Laplace on both sides

$$L\{i'(t)\} + 2L\{i(t)\} + 5L\left\{\int_0^t i\,dt\right\} = L\{u(t)\}.$$

$$s \cdot L\{i(t)\} - i(0) + 2L\{i(t)\} + \frac{5 \cdot L\{i(t)\}}{s} = \frac{1}{s}.$$

Put $i(0) = 0$,

Multiply Throughout by s,

$$s^2 L\{i(t)\} + 2s L\{i(t)\} + 5 L\{i(t)\} = 1$$

$$(s^2+2s+5)\, L\{i(t)\} = 1$$

$$L\{i(t)\} = \frac{1}{s^2+2s+5}$$

$$i(t) = L^{-1}\left\{\frac{1}{s^2+2s+5}\right\}.$$

$$= L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= e^{-t} \cdot \frac{1}{2} \sin 2t .$$

22) Consider a series RLC circuit where $R = 20W$
$L = 0.05H$ and $C = 10^{-4} F$ and is driven by
an alternating emf given by $E = 100t$.
Given that both the circuit current I and the capacitor charge q are zero at time $t=0$.
find an expression for $i(t)$ in the region
$t > 0$. $\left[Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt = E\right]$

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Given.

$$Ri + L\frac{di}{dt} + \frac{1}{c}\int_0^t i\,dt = E$$

$$20i + 0.05\frac{di}{dt} + \frac{1}{10^{-4}}\int_0^t i\,dt = 100t.$$

$\times$ by 20,

$$400\,i(t) + i'(t) + 200000\int_0^t i(t)\,dt = 2000t.$$

Take Laplace on both sides.

$$400\,L\{i(t)\} + L\{i'(t)\} + 200000\,L\int_0^t i(t)\,dt = 2000\,L\{t\}.$$

$$400\,L\{i(t)\} + sL\{i(t)\} - i(0) + 200000\frac{L\{i(t)\}}{s} = \frac{2000}{s^2}.$$

Using the fact $i(0)=0$ and multiplying by s throughout

$$400\,s\,L\{i(t)\} + s^2 L\{i(t)\} + 200000\,L\{i(t)\} = \frac{2000}{s}.$$

$$(s^2+400s+200000)\,L\{i(t)\} = \frac{2000}{s}$$

$$L\{i(t)\} = \frac{2000}{s(s^2+400s+200000)}$$

$$i(t) = L^{-1}\left\{\frac{2000}{s(s^2+400s+200000)}\right\}$$

Consider $F(s) = \dfrac{1}{s^2+400s+200000}$.

$$= \frac{1}{s^2+400s+40000+160000}.$$

$$F(s) = \frac{1}{(s+200)^2+400^2}.$$

$$\Rightarrow f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+200)^2+400^2}\right\}.$$

$$= e^{-200t}\,L^{-1}\left\{\frac{1}{s^2+400^2}\right\}.$$

$$= e^{-200t}\cdot\frac{1}{400}\sin 400t.$$

$$i(t) = L^{-1}\left\{ \frac{2000}{s(s^2+400s+200000)} \right\}$$

$$= 2000 \int_0^t e^{-200t} \frac{1}{400} \sin 400t \, dt$$

$$= \frac{2000}{400} \int_0^t e^{-200t} \sin 400t \, dt$$

$$= 5 \left[\frac{e^{-200t}}{(-200)^2+(400)^2} \left[-200 \sin 400t - 400 \cos 400t \right] \right]_0^t$$

$$= 5 \cdot \left[\frac{e^{-200t}}{200000} (-200 \sin 400t - 400 \cos 400t) - \frac{1}{200000} (-400 \cos 400(0)) \right]$$

$$= \frac{5}{200000} \left[e^{-200t} (-200 \sin 400t - 400 \cos 400t) + 400 \right]$$

$$\boxed{i(t) = \frac{1}{100} - \frac{e^{-200t}}{100} \cos 400t - \frac{e^{-200t}}{200} \sin 400t}$$

23) The deflection of a beam of length L, clamped, horizontally at both ends and loaded at $x = \frac{l}{4}$ by a weight W is given by $E \cdot I \frac{d^4y}{dx^4} = W\,\delta(x - \frac{l}{4})$. Find the deflection curve, given that $y = \frac{dy}{dx} = 0$ when $x = 0$ and $x = l$.

The differential equation for deflection is

$$\frac{d^4y}{dx^4} = \frac{W}{E \cdot I}\,\delta(x - \frac{l}{4}).$$

Applying Laplace transform on both sides.

$$L \cdot \{y^{iv}(x)\} = \frac{W}{EI} L\{\delta(x - \frac{l}{4})\}.$$

$$s^4 \cdot L\{y(x)\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{W}{EI} e^{-ls/4}.$$

Given

$y(0) = 0 \quad y'(0) = 0.$

$y(l) = 0 \quad y'(l) = 0.$

So, let us assume That $y''(0) = c_1$ and $y'''(0) = c_2$.

$$s^4 \cdot L\{y(x)\} - s \cdot c_1 - c_2 = \frac{W}{EI} e^{-\frac{ls}{4}}$$

$$s^4 \cdot L\{y(x)\} = \frac{W}{EI} e^{-ls/4} + sc_1 + c_2$$

$$L\{y(x)\} = \frac{W}{EI} \frac{e^{-ls/4}}{s^4} + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

$$y(x) = L^{-1}\left\{\frac{W}{EI} \frac{e^{-ls/4}}{s^4}\right\} + c_1 L^{-1}\left\{\frac{1}{s^3}\right\} + c_2 L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$y(x) = \frac{W}{EI} L^{-1}\left\{\frac{e^{-ls/4}}{s^4}\right\} + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3!}$$

$$y(x) = \frac{W}{EI} \frac{(x - l/4)^3}{3!} u\left(x - \frac{l}{4}\right) + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{6}$$

$$y(x) = \begin{cases} \frac{W}{EI} \frac{(x - l/4)^3}{6} (0) + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{6} & \text{for } 0 < x < l/4 \\ \frac{W}{EI} \frac{(x - l/4)^3}{6} + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{6} & \text{for } x > l/4 \\ & \frac{l}{4} < x \leq l. \end{cases}$$

When $x = l$

$$y(l) = \frac{W}{EI}\frac{(l - l/4)^3}{6} + \frac{c_1 l^2}{2} + \frac{c_2 l^3}{6}$$

$$y(l) = \frac{W}{EI}\left(\frac{27 l^3}{64 \times 6}\right) + \frac{c_1}{2} l^2 + \frac{c_2}{6} l^3.$$

For $x = l$

$$y(x) = \frac{W}{EI}\frac{(x - l/4)^3}{6} + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{6}.$$

$$y'(x) = \frac{W}{EI}\frac{3(x - l/4)^2}{6} + c_1 \cdot \cancel{\frac{2x}{2}} + c_2 \frac{3x^2}{6}.$$

Put $x = l$

$$y'(l) = \frac{W}{EI}\frac{\cancel{3}(l - l/4)^2}{\cancel{6}_2} + c_1 l + \frac{c_2}{2} l^2$$

$$y'(l) = \frac{W}{EI}\left(\frac{9 l^2}{16 \times 2}\right) + c_1 l + \frac{c_2}{2} l^2$$

Given that $y(l)$ and $y'(l)$ is zero.

$$\therefore \quad 0 = \frac{W}{EI}\left(\frac{27 l^3}{64 \times 6}\right) + \frac{c_1}{2} l^2 + \frac{c_2}{6} l^3 \quad \text{—— (1)}$$

$$0 = \frac{W}{EI}\left(\frac{9 l^2}{16 \times 2}\right) + c_1 l + \frac{c_2}{2} l^2. \quad \text{—— (2)}$$

$$① \Rightarrow \frac{C_1}{2}l^2 + \frac{C_2}{6}l^3 = \frac{-9}{128}\frac{W}{EI}l^3$$

$$② \times \frac{l}{2} \Rightarrow \frac{C_1}{2}l^2 + \frac{C_2}{4}l^3 = -\frac{W}{EI}\left(\frac{9}{64}\right)l^3.$$

$$C_2 l^3\left(\frac{1}{6} - \frac{1}{4}\right) = \frac{W}{EI}l^3\left(\frac{-9}{128} + \frac{9}{64}\right).$$

$$C_2 \cancel{l^3}\left(\frac{-1}{12}\right) = \frac{W}{EI}\cancel{l^3}\left(\frac{9}{128}\right)$$

$$C_2 = \frac{W}{EI}\left(\frac{9}{128} \times -12\right)$$

$$C_2. = \left(\frac{-27}{32}.\right)\frac{W}{EI}$$

$$① \Rightarrow \frac{C_1}{2}l^2 + \frac{C_2}{6}l^3 = -\frac{W}{EI}\left(\frac{9}{128}\right)l^3.$$

$$② \times \frac{l}{3} \Rightarrow \frac{C_1 l^2}{3} + \frac{C_2}{6}l^3 = -\frac{W}{EI}\left(\frac{3}{32}\right)l^3.$$

$$\frac{C_1}{2}l^2 - \frac{C_1}{3}l^2 = \frac{W}{EI}l^3\left(\frac{-9}{128} + \frac{3}{32}\right).$$

$$C_1 l^2\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{W}{EI} l^3\left(\frac{-9}{128}+\frac{3}{32}\right)$$

$$C_1 l^2\left(\frac{1}{6}\right) = \frac{W}{EI} l^3\left(\frac{3}{128}\right).$$

$$C_1 = \frac{W}{EI} l\left(\frac{3\times 6}{128}\right).$$

$$C_1 = \frac{9}{64}\frac{W}{EI} l.$$

Thus the deflection of the beam is

$$y(x) = \begin{cases} C_1\frac{x^2}{2} + C_2\frac{x^3}{6} & \text{for } 0 < x < l/4 \\ \frac{W}{EI}\frac{(x-l/4)^3}{6} + C_1\frac{x^2}{2} + C_2\frac{x^3}{6} & \text{for } x > l/4 \text{ or } l/4 < x < l. \end{cases}$$

where C_1 and C_2 are given by

$$C_1 = \frac{9}{64}\frac{W}{EI} l.$$

$$C_2 = \left(\frac{-27}{32}\right)\frac{W}{EI}.$$

24) A mechanical system with two degrees of freedom satisfies the equations

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4, \quad 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$$

Use Laplace transforms to determine x and y at any instant, given that x, y and the first order derivative of them w.r.t. to 't' vanish at $t=0$.

Consider

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4.$$

Taking Laplace on both sides

$$2 \cdot L\{x''(t)\} + 3\, L\{y'(t)\} = L\{4\}.$$

$$2.\left[s^2 L\{x(t)\} - s\,x(0) - x'(0)\right] + 3\left[s\,L\{y(t)\} - y(0)\right] = \frac{4}{s}$$

Applying the given condition, i.e.,
$x(0) = x'(0) = y(0) = 0$ we have.

$$2s^2 L\{x(t)\} + 3s\,L\{y(t)\} = \frac{4}{s} \quad \text{—— (1)}.$$

Consider

$$2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0.$$

Taking Laplace on both sides.

$$2 \cdot L\{y''(t)\} - 3\, L\{x'(t)\} = 0.$$

$$2\left[s^2 \cdot L\{y(t)\} - s\, y(0) - y'(0)\right]$$
$$-3\left[s \cdot L\{x(t)\} - x(0)\right] = 0.$$

Applying the given condition $y(0) = y'(0) = x(0) = 0$

we have

$$2s^2 L\{y(t)\} - 3s\, L\{x(t)\} = 0 \quad \text{—} (2)$$

We have to solve (1) & (2)

(1) $\times$ 2s

$$\Rightarrow \; 4s^3 L\{x(t)\} + \cancel{6s^2 L\{y(t)\}} = 8$$

(2) $\times$ 3 $\Rightarrow \; \cancel{6s^2 L\{y(t)\}} - 9s\, L\{x(t)\} = 0.$

By subtracting $\Rightarrow (4s^3 + 9s)\, L\{x(t)\} = 8$

$$L\{x(t)\} = \frac{8}{4s^3 + 9s}.$$

$$x(t) = L^{-1}\left\{\frac{8}{4s^3 + 9s}\right\}.$$

$$= \frac{8}{4} L^{-1}\left\{\frac{1}{s\left(s^2 + \frac{9}{4}\right)}\right\}.$$

$$= 2 \cdot \int_0^t \frac{2}{3} \sin\left(\frac{3}{2}\right)t \cdot dt.$$

$$= \frac{4}{3} \int_0^t \sin\left(\tfrac{3}{2}\right)t \, dt$$

$$= \frac{4}{3} \cdot \frac{2}{3} \left[-\cos\left(\tfrac{3}{2}\right)t \right]_0^t$$

$$\boxed{x(t) = \frac{8}{9}\left[1 - \cos\left(\tfrac{3}{2}\right)t\right]}$$

① $\times 3 \Rightarrow 6s^2 L\{x(t)\} + 9s L\{y(t)\} = \frac{12}{s}$.

② $\times 2s \Rightarrow 4s^3 L\{y(t)\} - 6s^2 L\{x(t)\} = 0$.

$$(4s^3 + 9s) \, L\{y(t)\} = \frac{12}{s}$$

$$L\{y(t)\} = \frac{12}{s(4s^3 + 9s}.$$

$$y(t) = L^{-1}\left\{ \frac{12}{s^2(4s^2+9)} \right\}.$$

$$y(t) = \frac{12}{4} L^{-1}\left\{ \frac{1}{s^2(s^2 + 9/4)} \right\}$$

$$= \frac{\cancel{12}^{3}}{\cancel{4}} \cdot \int_0^t \int_0^t \frac{2}{3} \cdot \sin \tfrac{3}{2}(t) . \, dt .$$

$$= 2 \cdot \frac{2}{3} \int_0^t \left[-\cos \tfrac{3}{2} t \right]_0^t \, dt$$

$$= \frac{1}{3} \int_0^t \left(-\cos \tfrac{3}{2}t + 1\right) dt$$

$$= \frac{1}{3} \left[- \frac{\sin \frac{3}{2}t}{\frac{3}{2}} + t \right]_0^t$$

$$= \frac{4}{3}t - \frac{4}{3} \cdot \frac{2}{3} \sin \tfrac{3}{2}t$$

$$\boxed{y(t) = \frac{4}{3}t - \frac{8}{9} \sin \tfrac{3}{2}t}$$