ENGSCI 233 - Iteration and Stability
Lecture 2

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Accuracy

Numerical Accuracy of RK Methods

Generally, RK methods that include more derivative evaluations, n, will have better numerical accuracy. Methods covered so far:

RK Method	n	Order	Truncation error
Euler	1	first	$\mathcal{O}\left(h^2\right)$
Improved Euler	2	second	$\mathcal{O}\left(h^3\right)$
Classic RK4	4	fourth	$\mathcal{O}\left(h^5\right)$

Note that the number of derivative evaluations, n, is not always equal to the order of accuracy.

Accuracy vs Cost

There are two main ways of improving numerical accuracy:

- Use a higher order method.
- Use a smaller step size.

However, both will come at a higher computational cost:

- Higher order methods require more derivative evaluations.
- Smaller step sizes require extra iterations.

Truncation Error of Euler Method

Consider the Taylor series expansion of $y\left(t+h\right)$:

$$y\left(t+h\right)=y\left(t\right)+h\frac{dy\left(t\right)}{dt}+\frac{h^2}{2!}\frac{d^2y\left(t\right)}{dt^2}+\frac{h^3}{3!}\frac{d^3y\left(t\right)}{dt^3}+\dots$$

Truncating this series after the first derivative:

$$y\left(t+h\right)=y\left(t\right)+h\frac{dy\left(t\right)}{dt}+\mathcal{O}\left(h^2\right)$$

This is the Euler method, with a truncation error $\mathcal{O}\left(h^2\right)$. This is the error per step in absence of floating point error.

ENGSCI 331 will cover the derivation of higher-order methods.

Euler Method: Local vs Global Error

We found previously the local truncation error of the Euler method:

$$\epsilon_l \propto h^2$$

The global truncation error is that at a fixed t. The number of steps required to reach a given t scales inversely with the step size e.g. halving the step size requires double the number of steps.

Therefore, we can express the global truncation error as:

$$\epsilon_g \propto h^2 \cdot \frac{1}{h}$$

$$\epsilon_g \propto h$$

In general, we will consider only the local error.

Stability

Introduction

Explicit solution schemes can encounter a problem with *stability* for certain ODEs.

An unstable solution is one which does not adequately represent the general behaviour of the exact solution.

For example, an ODE with an exponential decay solution may have a numerical solution that grows with each step, rather than decaying. This would represent an unstable solution.

Worked Example

Consider a first-order linear ODE of the form:

$$\frac{dy}{dt} = ay$$

The general solution to this ODE is:

$$y = Ce^{at}$$

If $a < 0$ then this solution will decay to zero.

Euler Method

The first step of the Euler method is:

$$y^{(1)} = y^{(0)} + h\,a\,y^{(0)}$$
$$y^{(1)} = (1 + h\,a)\,y^{(0)}$$

The next step of the Euler method is:

$$y^{(2)} = (1 + h\,a)\,y^{(1)}$$
$$y^{(2)} = (1 + h\,a)^2\,y^{(0)}$$

Th Euler update step can therefore be generalised to:

$$y^{(k+1)} = (1 + ha)^k\,y^{(0)}$$

Euler Method: Stability Criteria

Let's consider the behaviour of $y^{(k+1)}$ for $h > 0$ and $a < 0$:

Stability	Condition	$\|1 + ha\|$	$\|y^{(k+1)}\|$
Stable	$-2 < h\,a < 0$	< 1	$< \|y^{(k)}\|$
Neutral	$h\,a = -2$	$= 1$	$= \|y^{(k)}\|$
Unstable	$h\,a < -2$	> 1	$> \|y^{(k)}\|$

In order to achieve a stable solution (i.e. the stability criteria), we need to set the step size such that $h < -\frac{2}{a}$.

Example: Solution and Stability Criteria

Consider the first-order linear ODE:

$$\frac{dy}{dt} = -10y$$

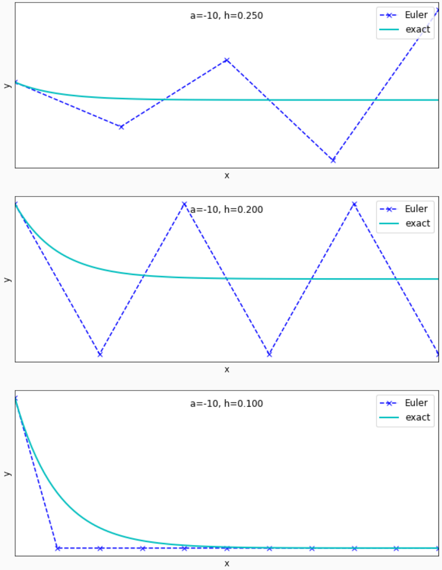
The general solution to this ODE is:

$$y = Ce^{-10t}$$

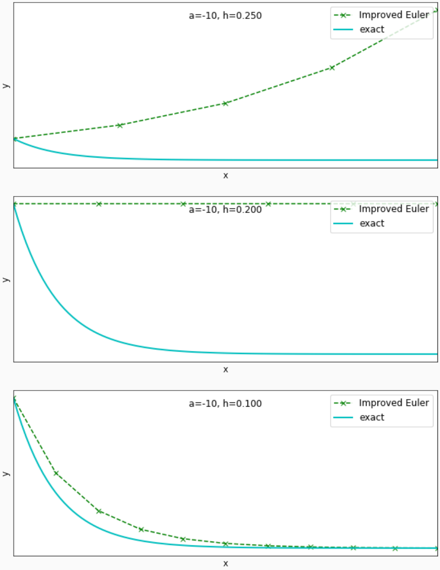
Based on the previously derived stability criteria for this form of ODE, a stable Euler (or Improved Euler method it turns out) solution requires:

$$h < \frac{1}{5}$$

Example: Euler Solution



Example: Improved Euler Solution



Convergence

Convergence Analysis

As noted previously, we often use numerical methods when the exact solution to a problem is not known.

How do we know if a numerical solution method is sufficiently close to the exact solution?

We can solve for varying step sizes, and look for convergent behaviour. Also known as a *convergence analysis*.

Convergence Analysis: Example

Consider the initial value problem:

$$\frac{dy}{dt} = -10y \quad , \; y(0) = 1$$

Solving up to $y(0.2)$ for a variety of step sizes, h, and plotting $y(0.2)$ against $\frac{1}{h}$: