ENGSCI 233 - Iteration

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Introduction

This module focuses on numerical solution methods for Ordinary Differential Equations (ODEs).

As well as algorithmic implementation, we will consider:

- Accuracy
- Stability
- Convergence

1. Iterative Methods for ODEs

First-Order ODE

Consider a first-order linear ODE of the generalised form:

$$\frac{dy}{dt} = f\left(t, y\right)$$

where the derivative of y with respect to t can be represented as a function of both these variables, $f\left(t,y\right)$.

The aim is to iteratively solve this ODE numerically for $y\left(t\right)$, starting from some initial condition $y^{\left(0\right)}=y\left(t^{\left(0\right)}\right)$.

Notation in ENGSCI 233/331

The notation I will try to use consistently throughout this topic:

- t is the independent variable
- y is the dependent variable
- ullet h is the step size along t
- ullet k is the iteration or step number
- k = 0 represents an initial state
- $t^{(k+1)} = t^{(k)} + h$ (condiant h)
- $\bullet \ y^{(k)} = y\left(t^{(k)}\right)$

Euler as a Runge-Kutta Method

Runge-Kutta (RK) methods use a sum of n weighted derivative evaluations to iterate the solution. Expressed in a general form:

$$y^{(k+1)} = y^{(k)} + h \sum_{i=0}^{n-1} \alpha_i f_i$$
 extincte of slope across the expectation

where α_i are weights assigned to each derivative evaluation, f_i .

Expressing the Euler method $(\underline{n-1})$ as an RK method:

$$y^{(k+1)} = y^{(k)} + hf_0$$
 ($\alpha_0 = 1$)
$$f_0 = f\left(t^{(k)}, y^{(k)}\right)$$
 ($k_0 = f_0$)

This derivative is evaluated at the start of the step.

Improved Euler as a Runge-Kutta Method

The Improved Euler method (n = 2) expressed as an RK method:

$$y^{(k+1)} = y^{(k)} + h\left(\frac{f_0}{2} + \frac{f_1}{2}\right)$$

$$f_0 = f\left(t^{(k)}, y^{(k)}\right)$$

$$f_1 = f\left(t^{(k)} + h, y^{(k)} + hf_0\right)$$
(corrector)

Where we evaluate the f_1 derivative is dependent on the f_0 derivative evaluation. This is an explicit method i.e. derivative evaluations depend only on previously evaluated ones.

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Derivative Evaluations in Explicit Runge-Kutta Methods

The derivative evaluations in an explicit Runge-Kutta method depend on previous ones. This can be generalised to:

$$f_i = f\left(t^{(k)} + \beta_i h, y^{(k)} + h \sum_{j=0}^{n-1} \gamma_{ij} f_j\right)$$

where:

- β are nodes i.e. how far along the independent variable we evaluate the derivative.
- γ form the RK matrix i.e. how our derivative evaluation depends on the other derivative evaluations.

Improved Euler (Again)

Expressing the derivative evaluations of the Improved Euler method (n=2) in our most generalised form:

$$y^{(k+1)} = y^{(k)} + h \begin{pmatrix} \frac{1}{2} & f_0 + \frac{1}{2} & f_1 \end{pmatrix}$$

$$f_0 = f \left(t^{(k)} + 0 & h, y^{(k)} + h & f_0 + 0 & f_1 \end{pmatrix}$$

$$f_1 = f \left(t^{(k)} + 0 & h, y^{(k)} + h & f_0 + 0 & f_1 \end{pmatrix}$$

Allowing us to write out the nodes and RK matrix as:

Butcher Tableau

A convenient tool for displaying α , β and γ is known as the Butcher Tableau (developed by a staff member at UoA):

$$\begin{array}{c|c} \beta^T & \gamma \\ \hline & \alpha \end{array}$$

RK methods are commonly presented in this format. It is easy to re-construct the governing equations from the tableau.

Explicit RK Methods

For an explicit RK method, γ must be a lower triangular matrix, as well as zero along the diagonal. This is because each derivative calculated only depends on the *prior* derivatives.

Euler Method

The Euler method (n = 1) governing equation:

$$y^{(k+1)} = y^{(k)} + hf_0$$

 $f_0 = f(t^{(k)}, y^{(k)})$

The Butcher tableau for the Euler method:



Improved Euler Method

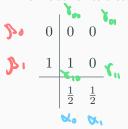
The Improved Euler method (n=2) expressed as an RK method:

$$y^{(k+1)} = y^{(k)} + h\left(\frac{f_0}{2} + \frac{f_1}{2}\right)$$

$$f_0 = f\left(t^{(k)}, y^{(k)}\right)$$

$$f_1 = f\left(t^{(k)} + hy^{(k)} + hf_0\right)$$

The Butcher tableau for the Improved Euler method:



The Classic RK4 Method

The classic RK4 method uses n=4 derivative evaluations and is fourth-order accurate:

$$y^{(k+1)} = y^{(k)} + h\left(\frac{f_0 + 2f_1 + 2f_2 + f_3}{6}\right) \quad \text{as a fine state }$$

where:

$$f_0 = f\left(t^{(k)}, y^{(k)}\right)$$

$$f_1 = f\left(t^{(k)} + \frac{h}{2}, y^{(k)} + \frac{hf_0}{2}\right)$$

$$f_2 = f\left(t^{(k)} + \frac{h}{2}, y^{(k)} + \frac{hf_1}{2}\right)$$

$$f_3 = f\left(t^{(k)} + h, y^{(k)} + hf_2\right)$$

$$\begin{array}{lll}
R &= & \frac{1}{2} \\
R &= & \frac{1}{2} \\
R &= & \frac{1}{2} \\
0 &= & \frac{1}{2$$

The Classic RK4 Method

The Butcher tableau for the classic RK4 method:

