American Call Option

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- Stock price : $dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t$
  (no dividends)

- in general for an American Contingent Claim with expiration:
  T = maturity
  $h(S_t)$: intrinsic value (payoff)
  Value of the contingent claim :
  $$v(t,x) = \max_{t \le \tau \le T} E\left[e^{-r\tau} h(S_\tau) \mid S_t = x\right]$$

Analytic Characterization of $v(t,x)$

(1) $v(t,x) \geq h(x)$ $\quad (\forall)\ x \geq 0$

(2) $rv - v_t - rx\,v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} \geq 0$

(3) equality holds in either 1 or 2

Probabilistic Characterization: $V_t = v(t,S_t)$

(1) $V_t \geq h(S_t)$

(2) $e^{-rt} V_t$: $\tilde{P}$ supermartingale

(3) V_t is the smallest process that satisfies 1 & 2

$\rightsquigarrow$ optimal exercise time
$$\tau^* = \min\left\{t \geq 0 : v(t,S_t) = h(S_t)\right\} \wedge T$$

- in particular if $h(x) = (x-K)^+$ $\rightsquigarrow$ American Call

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Lemma: If $h(x)$ is nonnegative, convex function, $h(0)=0$, then

$e^{-rt}h(S_t)$ is a submartingale (under $\tilde{\mathbb{P}}$)

Proof: $h(\cdot)$ convex $\Rightarrow$ $(\forall)$ x_1, x_2 $\alpha \in (0,1)$ we have

$$h(\alpha x_1 + (1-\alpha)x_2) \le \alpha h(x_1) + (1-\alpha) h(x_2)$$

- in particular pick $x_1 = 0 \rightsquigarrow h(x_1) = 0$

$$\Rightarrow h(\lambda x) \le \lambda h(x) \quad (\forall)\ \lambda \in (0,1),\ x>0$$

- we can apply this to the discounded stock price

$$\Rightarrow h\left(e^{-r(t-u)}S_t\right) \le e^{-r(t-u)}h(S_t)$$

$$\Rightarrow \tilde{\mathbb{E}}\left[e^{-r(t-u)}h(S_t) \,|\, \mathcal{F}_u\right] \geq \tilde{\mathbb{E}}\left[h\left(e^{-r(t-u)}S_t\right) \,|\, \mathcal{F}_u\right]$$

$$\underset{\substack{\uparrow \\ \text{Jensen}}}{\geq} \tilde{h}\left(\tilde{\mathbb{E}}\left[e^{-r(t-u)}S_t \,|\, \mathcal{F}_u\right]\right) = h(S_u)$$

$$\Rightarrow \tilde{\mathbb{E}}\left[e^{-rt}h(S_t) \,|\, \mathcal{F}_u\right] \geq e^{-ru}h(S_u)$$

$\Rightarrow e^{-rt}h(S_t)$ is submartingale.

Theorem: The price of an American Call on an asset not paying dividends is the same with the price of the European Call on the same asset with the same expiration.

Proof: take $h(x) = (x-K)^+ \Rightarrow e^{-rt}h(S_t)$: $\tilde{\mathbb{P}}$ submartingale

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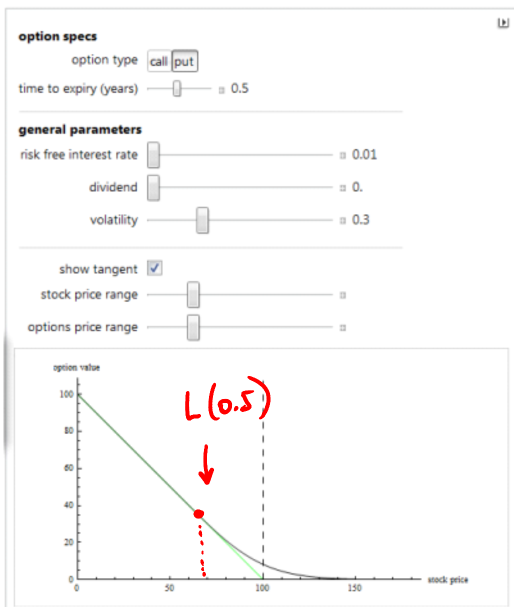
$$V_t = \max_{t \le \tau \le T} \mathbb{E}\left[e^{-r(\tau - t)} (S_\tau - K)^+ \,\middle|\, \mathcal{F}_t \right] \le \mathbb{E}\left[e^{-r(T-t)} (S_T - K)^+ \,\middle|\, \mathcal{F}_t \right] \le V_t$$

$$\Rightarrow \quad V_t = \mathbb{E}\left[e^{-r(T-t)} (S_T - K)^+ \,\middle|\, \mathcal{F}_t \right]$$

$\uparrow$ value of the European Call.

[Early Exercise of American Derivative Securities (Wolfram Demonstrations Project)]

Early Exercise of American Options



This Demonstration shows the optimal value for the exercise of an American option (call or put) in the Black–Scholes model. Unlike the European option, the American option allows early exercise. One can show that for all put options there is a price of the underlying stock such that when the stock is at (or below) this price, the option should be exercised. For call options on a stock that pays a nonzero continuous dividend, there is a stock price such that the option should be exercised when the stock price is at or above this optimal price. It is never optimal to exercise call options that pay no dividend.

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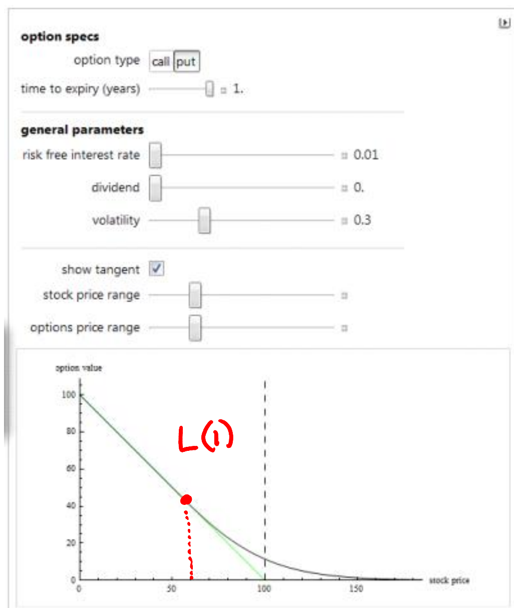

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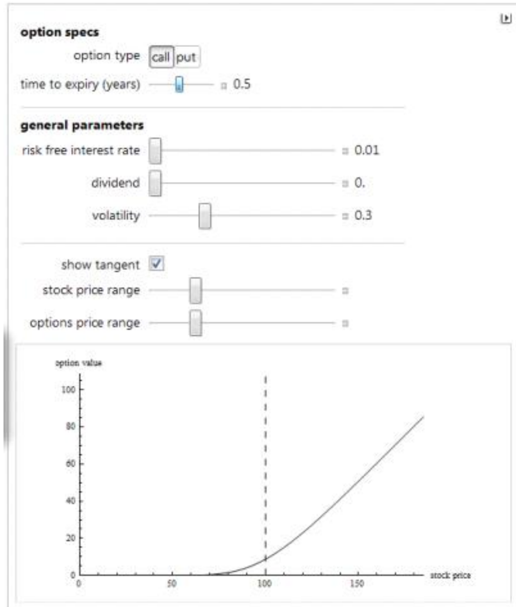

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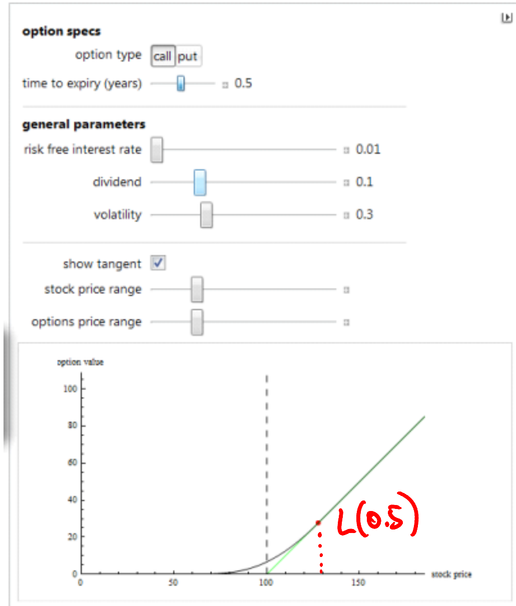

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