Fixed Income Risk Management

2. Girsanov, Numeraires, and All That

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1 Arbitrage asset pricing in a nutshell

This is a technical intermezzo in preparation for next few themes: valuation of options on interest rates, CMS based instruments, and term structure modeling. We start by reviewing briefly some basic concepts of arbitrage pricing theory, just enough to cover our upcoming needs. For a full account of this theory, I encourage you to take the course in continuous time finance offered in this program. In particular, I will be skipping over a lot of technicalities while discussing the probabilistic concepts underlying this framework, and, again, I recommend further study for a more in depth understanding of these concepts. Next, we will discuss the technique of *change of numeraire*, which will play a key role in the following lectures.

1.1 Self-financing portfolios and arbitrage

We consider a model of a *frictionless* financial market which consists of N assets $I_1, \ldots, I_N$. By frictionless we mean that each of the assets is *liquid*, i.e. at each time any bid or ask order can be immediately executed, and there are no *transaction costs*, i.e. each bid and ask order for the security I_i at time t is executed at the same unique price level. Very much like the formulation of Newtonian gravity in vacuum, this is obviously a gross oversimplification of reality, and much work has been done to relax these assumptions. From the conceptual point of view, however, it leads to a profound and workable framework of asset pricing theory.

We model the price processes of these assets by $S_1(t), \ldots, S_N(t)$, i.e. $S_i(t)$ denotes the price of asset I_i at time t. We emphasize that these processes represent market observable asset prices, and not merely some convenient state variables. We assume that each price process is a diffusion process. In other words, there is an underlying probability space $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathsf{P})$ generated by a multidimensional Wiener process $W_1(t), \ldots, W_d(t)$, and for each $j = 1, \ldots, N$,

$$dS_j(t) = \Delta_j(t, S(t))dt + \sum_{1\leq k\leq d} C_{jk}(t, S(t))dW_k(t), \tag{1}$$

with suitable drift and diffusion coefficients Δ_j and C_{jk}, respectively.

In order to develop an intuition for the concepts explained below we recall the basic example from the world of equity derivatives.

Example 1.1 (Black-Scholes model) *In this classic model of equity derivatives, $S_1(t) = B(t)$ is the riskless money market account, and $S_2(t) = S(t)$ is a risky stock, with the dynamics given by*

$$\begin{aligned} dB(t) &= rB(t)dt, \\ dS(t) &= \mu S(t)dt + \sigma S(t)dW(t). \end{aligned} \tag{2}$$

The rate of return r on the money market account is called the *riskless rate*, while μ is the rate of return on the risky asset. As we already know, there is really no such thing as riskless rate (with the closest proxy being the overnight OIS rate), but its presence in the Black-Scholes model helps one understand the general framework of risk neutral valuation. The modeling framework for interest rate derivatives, which is the subject of these lectures, does not require invoking the risk free rate explicitly.

A *portfolio* is specified by the weights $w_1(t), \ldots, w_N(t)$, of the assets at time t. We assume, of course, that the weights are non-negative, and they add up to one. The value process of the portfolio is given by

$$V(t) = \sum_{1\leq i\leq N} w_i(t)S_i(t). \tag{3}$$

A portfolio is *self-financing*, if

$$dV(t) = \sum_{1\leq i\leq N} w_i(t)dS_i(t), \tag{4}$$

or, equivalently,

$$V(t) = V(0) + \int_0^t \sum_{1 \le i \le N} w_i(s) dS_i(s). \tag{5}$$

In other words, the price process of a self-financing portfolio does not allow for infusion or withdrawal of capital. It is entirely driven by price processes of the constituent instruments and their weights.

A fundamental assumption of arbitrage pricing theory is that financial markets (or at least, their models) are free of arbitrage opportunities[1]. An *arbitrage opportunity* arises if one can construct a self-financing portfolio such that:

(a) The initial value of the portfolio is zero, $V(0) = 0$.

(b) With probability one, the portfolio has a non-negative value at maturity, $\mathsf{P}(V(T) \ge 0) = 1$.

(c) With a positive probability, the value of the portfolio at maturity is positive, $\mathsf{P}(V(T) > 0) > 0$.

We say the model is *arbitrage free* if it does not allow arbitrage opportunities. Requiring arbitrage freeness has important consequences for price dynamics.

1.2 Complete markets

A random variable X on the probability space $(\Omega, (\mathscr{F}_t)_{t \ge 0}, \mathsf{P})$ is square integrable if $\mathsf{E}[X^2] < \infty$. In other words, a square integrable random variable has a well defined variance. A financial market is called *complete* if each such random variable can be obtained as the terminal value of a self-financing trading strategy, i.e. if there exists a self-financing portfolio such that $X = V(T)$.

This somewhat technical sounding condition has a natural interpretation. If we think about X as the price of a (possibly very complicated) financial claim, market completeness means that it can be replicated by way of a self-financing trading strategy. All the models in this course assume complete markets.

[1]This assumption is, mercifully, violated frequently enough so that much of the financial industry can sustain itself exploiting the market's lack of respect for arbitrage freeness.

1.3 The fundamental theorems

A key concept in modern asset pricing theory is that of a *numeraire*. A numeraire is any tradeable asset with price process $\mathcal{N}(t)$ such that $\mathcal{N}(t) > 0$, for all times t. The *relative price* process of asset I_i is defined by

$$S_i^{\mathcal{N}}(t) = \frac{S_i(t)}{\mathcal{N}(t)} \,. \tag{6}$$

In other words, the relative price of an asset is its price expressed in the units of the numeraire.

A probability measure Q is called an *equivalent martingale measure* for the above market, with numeraire $\mathcal{N}(t)$, if it has the following properties:

(a) Q is equivalent to P, i.e.

$$d\mathsf{P}(\omega) = D_{\mathsf{PQ}}(\omega)\, d\mathsf{Q}(\omega) \,,$$

and

$$d\mathsf{Q}(\omega) = D_{\mathsf{QP}}(\omega)\, d\mathsf{P}(\omega) \,,$$

with some $D_{\mathsf{PQ}}(\omega) > 0$ and $D_{\mathsf{QP}}(\omega) > 0$.

(b) The relative price processes $S_i^{\mathcal{N}}(t)$ are martingales under Q,

$$S_i^{\mathcal{N}}(s) = \mathsf{E}^{\mathsf{Q}}\left[S_i^{\mathcal{N}}(t) \middle| \mathscr{F}_s \right] . \tag{7}$$

We now formulate (without proof) two important theorems.

Theorem 1.2 (First Fundamental Theorem of arbitrage free pricing) *A market is arbitrage free if and only if there exists an equivalent martingale measure* Q.

This theorem is formulated in a somewhat cavalier way, as we have suppressed some important technical assumptions. What the theorem says is that arbitrage freeness means the existence of a numeraire $\mathcal{N}(t)$, and an equivalent measure such that the relative price process is a martingale under this measure. In other words, in an arbitrage free market, we can express the prices of all assets in the units of a single asset so that the prices are martingales.

Consider, for example, the Black-Scholes model, Example 1.1. First, we choose the money market account as the numeraire,

$$\begin{aligned} \mathcal{N}(t) &= B(t) \\ &= e^{rt}. \end{aligned}$$

With this choice of numeraire,

$$dS^B(t) = (\mu - r)S^B(t)dt + \sigma S^B(t)dW(t),$$

where $S^B(t)$ denotes the relative price process,

$$\begin{aligned} S^B(t) &= \frac{S(t)}{B(t)} \\ &= e^{-rt}S(t). \end{aligned}$$

Next, we use Girsanov's theorem to change the probability measure so that the relative price process is driftless,

$$dS^B(t) = \sigma S^B(t)dW(t).$$

Explicitly, this amounts to the following change of measure

$$\frac{d\mathsf{Q}}{d\mathsf{P}}(t) = \exp\Big(-\lambda W(t) - \frac{1}{2}\lambda^2\Big),$$

where

$$\lambda = \frac{\mu - r}{\sigma}$$

is known as the *market price of risk*.

Theorem 1.3 (Second Fundamental Theorem of arbitrage free pricing) *An arbitrage free market is complete if and only if the equivalent martingale measure* Q *is unique.*

In other words, in a complete, arbitrage free market, to a given numeraire $\mathcal{N}(t)$ corresponds one and only one martingale measure Q.

1.4 Change of numeraire

An important consequence of Theorem 1.2 is the arbitrage pricing law:

$$\frac{V(s)}{\mathcal{N}(s)} = \mathsf{E}^{\mathsf{Q}}\Big[\frac{V(T)}{\mathcal{N}(T)}\,\Big|\mathscr{F}_s\Big]. \tag{8}$$

One is, of course, free to use a different numeraire $\mathcal{N}(t) \to \mathcal{N}'(t)$. Girsanov's theorem (see the Appendix for a summary) implies that there exists a martingale measure Q' such that

$$\frac{V(s)}{\mathcal{N}'(s)} = \mathsf{E}^{\mathsf{Q}'}\Big[\frac{V(T)}{\mathcal{N}'(T)}\,\Big|\mathscr{F}_s\Big], \tag{9}$$

and thus the Radon-Nikodym derivative is given by the ratio of the numeraires:

$$\begin{aligned}\frac{d\mathsf{Q}'}{d\mathsf{Q}}\Big|_t &= \frac{\frac{\mathcal{N}(0)}{\mathcal{N}(t)}}{\frac{\mathcal{N}'(0)}{\mathcal{N}'(t)}}\\ &= \frac{\mathcal{N}(0)}{\mathcal{N}(t)}\frac{\mathcal{N}'(t)}{\mathcal{N}'(0)}\ .\end{aligned} \tag{10}$$

The choice of numeraire and the corresponding martingale measure is very much a matter of convenience, and is motivated by the problem at hand. We will see in the following lectures how this important technique works in practice. In the meantime, let us review some of the most important numeraires encountered in interest rates modeling.

2 Examples of numeraires

2.1 Spot numeraire

The *spot numeraire* (or *rolling numeraire*) is simply a $\$1$ deposited in a bank and accruing the (riskless) instantaneous rate. Its price process $\mathcal{N}(t)$ is given by

$$\mathcal{N}(t) = \exp\left(\int_0^t r(s)ds\right). \tag{11}$$

Here,

$$r(t) = f(t,t), \tag{12}$$

where $f(t,s)$ is the instantaneous forward rate introduced in Lecture 1. The special case of a constant riskless rate $r(t) = r$ plays a key role in the Black-Scholes model, and the rolling numeraire is the riskless bond $B(t)$ mentioned before.

2.2 Forward numeraire

The T*-forward numeraire* is simply the zero coupon bond for maturity T. Its price at $t < T$ is given by

$$\mathcal{N}_T(t) = P\left(t,t,T\right). \tag{13}$$

We will see in Lecture 3 that the T-forward numeraire arises naturally in pricing instruments based of forwards maturing at T. Forward rates for maturity at T are martingales under the measure associated with this numeraire.

2.3 Annuity numeraire

Consider a forward starting swap which settles in T_0 and matures T years from now, respectively. The *annuity numeraire* associated with this swap is defined as the process of the annuity (or level) function of the swap. Recall from Lecture 1 that the latter pays $\$1$ (per annum) on each coupon day of the swap, accrued according to the swap's day count day conventions. Its present value for the valuation date T_{val} as observed at time $t \leq T_{\text{val}}$ is given by:

$$A(t, T_{\text{val}}, T_0, T) = \sum_{j=1}^{n_c} \alpha_j P(t, T_{\text{val}}, T_j^c), \tag{14}$$

where the summation runs over the coupon dates of the swap. We define the annuity numeraire (associated with the given swap) as

$$\mathcal{N}_{T_0,T}(t) = A(t, t, T_0, T). \tag{15}$$

The annuity numeraire arises as the natural numeraire when valuing swaptions. In Lecture 3 we will see that the swap rate $S(t, T_0, T)$ is a martingale under the measure associated with the annuity numeraire.

3 Change of numeraire technique

Choice of a numeraire is a matter of convenience and is dictated by the valuation problem at hand. Asset valuation leads frequently to complicated stochastic processes, and one way of making the problem easier is to eliminate the drift term from the stochastic differential equation defining the process. The change of numeraire technique allows us to achieve precisely this: modify the probability law (the measure) of the process so that, under this new measure, the process is driftless, i.e. it is a martingale.

Consider a financial asset whose dynamics is given in terms of the state variable $X(t)$. Under the measure P this dynamics reads:

$$dX(t) = \Delta^{\mathsf{P}}(t)dt + C(t)dW^{\mathsf{P}}(t). \tag{16}$$

Our goal is to relate this dynamics to the dynamics of the same asset under an equivalent measure Q:

$$dX(t) = \Delta^{\mathsf{Q}}(t)dt + C(t)dW^{\mathsf{Q}}(t). \tag{17}$$

Remember that the diffusion coefficients in these equations are unaffected by the change of measure! We assume that P is associated with the numeraire $\mathcal{N}(t)$ whose dynamics is given by:

$$d\mathcal{N}(t) = A_{\mathcal{N}}(t)dt + B_{\mathcal{N}}(t)dW^{\mathsf{P}}(t), \tag{18}$$

while Q is associated with the numeraire $\mathcal{M}(t)$ whose dynamics is given by:

$$d\mathcal{M}(t) = A_{\mathcal{M}}(t)dt + B_{\mathcal{M}}(t)dW^{\mathsf{P}}(t). \tag{19}$$

According to Girsanov's theorem (see Appendix A), the Radon-Nikodym derivative

$$D(t) = \frac{d\mathsf{Q}}{d\mathsf{P}}\Big|_t \tag{20}$$

is a martingale under the measure P, which satisfies the stochastic differential equation:

$$dD(t) = \theta(t)D(t)dW^{\mathsf{P}}(t), \tag{21}$$

with

$$\theta(t) = \frac{\Delta^{\mathsf{Q}}(t) - \Delta^{\mathsf{P}}(t)}{C(t)} \ . \tag{22}$$

Explicitly, the likelihood process $D(t)$ is given by the Doléans-Dade exponential of the martingale $\int_0^t \theta(s)dW^{\mathsf{P}}(s)$:

$$\begin{aligned} D(t) &= \mathcal{E}\Big(\int_0^{\cdot} \theta(s)dW^{\mathsf{P}}(s)\Big)(t) \\ &= \exp\Big(\int_0^t \theta(s)dW^{\mathsf{P}}(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds\Big). \end{aligned} \tag{23}$$

On the other hand, from the fundamental theorem of asset pricing we infer that

$$D(t) = \frac{\mathcal{N}(0)}{\mathcal{M}(0)}\,\frac{\mathcal{M}(t)}{\mathcal{N}(t)} \ . \tag{24}$$

Since $D(t)$ is a martingale under P, we conclude that the process $\mathcal{M}(t)/\mathcal{N}(t)$ is driftless under P. As a consequence,

$$d\left(\frac{\mathcal{M}(t)}{\mathcal{N}(t)}\right) = \frac{\mathcal{M}(t)}{\mathcal{N}(t)}\,\left(\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)} - \frac{B_{\mathcal{N}}(t)}{\mathcal{N}(t)}\right) dW^{\mathsf{P}}(t).$$

Comparing this with (20) we infer that

$$\theta(t)\,\frac{\mathcal{M}(t)}{\mathcal{N}(t)}=\frac{\mathcal{M}(t)}{\mathcal{N}(t)}\,\left(\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)}-\frac{B_{\mathcal{N}}(t)}{\mathcal{N}(t)}\right). \tag{25}$$

This leads to the following drift transformation law:

$$\begin{aligned}\Delta^{\mathsf{Q}}(t)-\Delta^{\mathsf{P}}(t)&=C(t)\Big(\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)}-\frac{B_{\mathcal{N}}(t)}{\mathcal{N}(t)}\Big)\\&=\frac{d}{dt}\int_0^t dX(s)d\Big(\log\frac{\mathcal{M}(s)}{\mathcal{N}(s)}\Big).\end{aligned} \tag{26}$$

The formula above expresses the change in the drift in the dynamics of the state variable, which accompanies a change of numeraire, in terms of the processes themselves.

We can rewrite (26) in a more intrinsic form. Note that the integral in the equation above defines the quadratic covariation between X and $\log(\mathcal{M}/\mathcal{N})$. Consequently, the change of numeraire formula can be stated in the elegant, easy to remember form:

$$\Delta^{\mathsf{Q}}(t)=\Delta^{\mathsf{P}}(t)+\frac{d}{dt}\,\Big[X,\log\frac{\mathcal{M}}{\mathcal{N}}\Big](t). \tag{27}$$

A Girsanov's theorem

What is now known as Girsanov's theorem is a culmination of efforts by a number of researchers studying the effect of "change of variables" in the measure P on the properties of martingales under that measure. Girsanov's theorem plays a key conceptual role in arbitrage free pricing theory, a fact that will be explained in the next chapter.

We consider a Brownian motion $W(t)$, and the associated probability space $(\Omega,\mathscr{F},\mathsf{P})$, where Ω is the sample space, $\mathscr{F}=(\mathscr{F}_t)_{t\geq 0}$, is the filtered information set, and P is the probability measure. By E (or E^P, when we want to be precise) we denote the expected value with respect to the measure P.

We say that a measure Q on Ω is *absolutely continuous* with respect to P if there exists a positive function D (called the *Radon-Nikodym derivative*) such that

$$\mathsf{Q}\left(A\right)=\int_A D\left(\omega\right)d\mathsf{P}\left(\omega\right), \tag{28}$$

for $A \subset \Omega$, or

$$\frac{d\mathsf{Q}}{d\mathsf{P}}(\omega) = D(\omega). \tag{29}$$

In other words, the "volume element" $d\mathsf{Q}$ is always proportional to the "volume element" $d\mathsf{P}$, with the proportionality factor being a positive function throughout the probability space. In the context of a Brownian motion, we also require that the Radon-Nikodym derivative respect the filtration by time, i.e. the identity above holds if we condition on the information up to time t:

$$\frac{d\mathsf{Q}}{d\mathsf{P}}(\omega)\Big|_t = D(\omega, t). \tag{30}$$

Two probability measures Q and P are called *equivalent*, if Q is absolutely continuous with respect to P and P is absolutely continuous with respect to Q.

Consider now a diffusion process:

$$dX(t) = \Delta(X(t), t)\, dt + C(X(t), t)\, dW(t). \tag{31}$$

A natural question arises: can we transform a diffusion process into a diffusion process with a different drift,

$$dX(t) = \widetilde{\Delta}(X(t), t)\, dt + C(X(t), t)\, d\widetilde{W}(t). \tag{32}$$

by a change to an equivalent probability measure Q? In particular, can we make the new process a martingale? Recall that if the process $X(t)$ is a *martingale*, the diffusion above is driftless, i.e. $\widetilde{\Delta}(X(t), t) = 0$. Recall that a process $X(t)$ is a martingale if $\mathsf{E}^{\mathsf{Q}}[|X(t)|] < \infty$, for all t, and

$$X(s) = \mathsf{E}^{\mathsf{Q}}[X(t)|\mathscr{F}_s], \tag{33}$$

where $\mathsf{E}^{\mathsf{Q}}[\,\cdot\,|\mathscr{F}_s]$ denotes the conditional expected value. In other words, given all information up to time s, the expected value of future values of a martingale is $X(s)$. An affirmative answer to this question is provided by Girsanov's theorem.

One might heuristically proceed like this. Write

$$\begin{aligned} dX(t) &= \widetilde{\Delta}(t)dt + C(t)\left(\frac{\Delta(t) - \widetilde{\Delta}(t)}{C(t)}\, dt + dW(t)\right) \\ &= \widetilde{\Delta}(t)dt + C(t)d\widetilde{W}(t), \end{aligned} \tag{34}$$

where

$$\begin{aligned}\widetilde{W}(t) &= W(t) + \int_0^t \frac{\Delta(s) - \widetilde{\Delta}(s)}{C(s)}\, ds \\ &\equiv W(t) - \int_0^t \theta(s)\, ds.\end{aligned} \tag{35}$$

This looks like a new Brownian motion! Girsanov's theorem asserts that, under some technical assumptions on the drift and diffusion coefficients, $\widetilde{W}(t)$ is indeed a Brownian motion provided that the probability measure is modified appropriately.

More precisely, define the stochastic process:

$$D(t) = \exp\left(\int_0^t \theta(s)\, dW(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds\right). \tag{36}$$

Note that we have changed our notation: as always when dealing with stochastic processes, we have suppressed the argument ω in D, and made the dependence on t explicit. We now define the equivalent measure Q with

$$\left.\frac{d\mathsf{Q}}{d\mathsf{P}}\right|_t = D(t). \tag{37}$$

Theorem A.1 (Girsanov's theorem) *Assume that the following technical condition (*Novikov's condition*) holds:*

$$\mathsf{E}^{\mathsf{P}}\left[\exp\left(\frac{1}{2}\int_0^t \theta(s)^2 ds\right)\right] < \infty. \tag{38}$$

Then

(a) The process $D(t)$ is a martingale under P. Furthermore, it satisfies the following stochastic differential equation:

$$dD(t) = \theta(t)D(t)dW(t). \tag{39}$$

(b) $\widetilde{W}(t)$ is a Wiener process under Q.

We have stated Girsanov's theorem for a one-dimensional Brownian motion. This assumption is not essential and, using a bit of linear algebra, one can easily formulate a version of Girsanov's theorem for an arbitrary multidimensional Brownian motion.

References

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