Lecture 10 : Jump-Diffusion Processes

Note Title 4/17/2012

- "diffusion" ⇝ Brownian motion component (integral)
- we shall consider processes with finitely many jumps in each finite time interval.
- fundamental jump process is the Poisson process

Recall :
- all jumps of a Poisson process are of size 1
- the Compound Poisson process is like a Poisson process except that the jumps are of random size.

Recall facts about the Poisson Process (PP)

- to construct a PP we begin with a sequence $\tau_1, \tau_2, \ldots$ of independent r.v $\sim \exp(\lambda)$ (the "interarrival times")

$$S_n = \sum_{k=1}^{n} \tau_k$$: the time of the n^{th} jump.

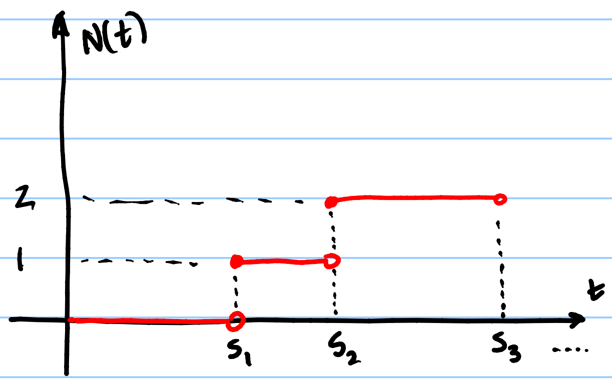
• The PP : $N(t)$ counts the number of jumps that occur at or before time t.

$$N(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < S_1 \\ 1 & \text{if} \quad S_1 \le t < S_2 \\ \cdots & \end{cases}$$

• right-continuous process $\left(N(t) = \lim_{s \downarrow t} N(s) \right)$

• $\mathcal{F}_t = \sigma(N(s) \; ; \; s \le t)$

• we say $N(t)$ has intensity λ



The Distribution of PP's increments

- $S_n = \sum_{k=1}^{n} z_k$
  $z_1, z_2 \dots \sim exp(\lambda)$, indep $\Bigg\} \Rightarrow S_n \sim \Gamma(n,\lambda)$

- more precisely: the density of S_n is:

$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s} \quad \text{for } s \geq 0$$

(notice that for $n=1 \rightsquigarrow g_1(s) = \lambda e^{-\lambda s} \rightsquigarrow exp(\lambda)$)

- $N(t) \sim Poisson(\lambda t)$

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t} \qquad k = 0,1,2,\dots$$

Theorem Let $0 = t_0 < t_1 < t_2 < \dots < t_n$ be given.
The increments $N(t_1) - N(t_0), N(t_2) - N(t_1), \dots, N(t_n) - N(t_{n-1})$ are stationary and independent and

$$\boxed{P\Big(N(t_{j+1}) - N(t_j) = k\Big) = \frac{\lambda^k (t_{j+1} - t_j)^k}{k!} e^{-\lambda(t_{j+1} - t_j)}}$$

$$W_t - W_s \sim N(0, t-s) \sim W_{t-s}$$

$\rightsquigarrow$ in general for any $s<t$:

$$N(t) - N(s) \sim N(t-s) \sim \text{Poisson}(\lambda(t-s))$$

Martingale Property

- $N(t)$ is not a martingale (easy)
- $M(t) = N(t) - \lambda t$: the compensated PP is martingale

Compound Poisson Process (CPP)

- $N(t)$: $PP(\lambda)$ (Poisson process with intensity λ)
- $Y_1, Y_2, \dots$ iid r.v. with $p = EY_i$
  ($Y_1, Y_2, \dots$ independent of each other and $N(t)$)

$\rightsquigarrow$ the CPP is : $Q(t) = \sum_{i=1}^{N(t)} Y_i \qquad t \geq 0$

$\rightsquigarrow$ the jumps of CPP occur at the same time with the jumps of PP

$\rightsquigarrow$ the jump size of PP are always 1, however the jumps of CPP are of size $Y_1, Y_2, \dots, Y_n, \dots$

- like PP, the increments of CPP are independent and stationary

$$Q(t) - Q(s) = \sum_{i=N(s)+1}^{N(t)} Y_i \sim \sum_{i=1}^{N(t-s)} Y_i = Q(t-s)$$

(because $N(t) - N(s) \sim N(t-s)$)

- $$\mathbb{E}\, Q(t) = \mathbb{E}\left[\sum_{i=1}^{N(t)} Y_i\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N(t)} Y_i \,\middle|\, N(t)\right]\right] = \mathbb{E}\left[\beta \cdot N(t)\right] = \beta \lambda t$$

- The compensated CPP is a martingale

$\rightsquigarrow$ $Q(t) - \beta\lambda t$

(easy to check)

Jump Processes and Their Integrals

$(\Omega, \mathcal{F}, P)$ $\{\mathcal{F}_t\}_{t\geq 0}$: filtration

- we say that W is a Brownian Motion relative to $\{\mathcal{F}_t\}$

if $\begin{cases} \cdot\ W_t \text{ is } \mathcal{F}_t \text{ measurable} \\ \cdot\ W_t - W_s \text{ is independent of } \mathcal{F}_s \end{cases}$

- we say that N is a PP relative to $\{\mathcal{F}_t\}$ if

$\begin{cases} \cdot\ N(t) \text{ is } \mathcal{F}_t \text{ measurable} \\ \cdot\ N(t) - N(s) \text{ is independent of } \mathcal{F}_s \end{cases}$

(same goes for CPP)

Jump Process

- we wish to define : $\int_0^t \Phi_s \, dX_s$

where X can have jumps.

- assume all processes are adapted to $\{\mathcal{F}_t\}$

$$X(t) = X(0) + I(t) + R(t) + J(t)$$

- $I(t) = \int_0^t \Gamma(s)\, dW(s)$: stochastic integral
- $R(t) = \int_0^t \Theta(s)\, ds$: Riemann integral
- $X^c(t) = X(0) + I(t) + R(t)$: the continuous part of X

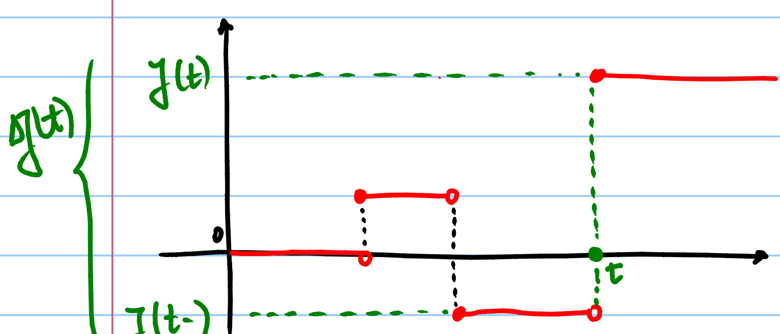
  $\rightsquigarrow \langle X^c \rangle_t = \int_0^t \Gamma^2(s)\, ds$: quadratic var. of X^c

  $$\left(dX_t^c \cdot dX_t^c = \Gamma^2(t)\, dW_t \cdot dW_t = \Gamma^2(t)\, dt \right)$$

- $J(t)$: adapted, right-continuous pure jump process

  $\rightsquigarrow J(0) = 0$; $J(t) = \lim_{s \downarrow t} J(s)$

  $\rightsquigarrow$ the left continuous vesion of such a process will be denoted $J(t-)$



$\Delta J(t)$ = size of the jump

$= J(t) - J(t-)$

- assume J has finitely many jumps on each $(0,T]$.

Def: A process $X(t) = \overbrace{X(0) + I(t) + R(t)}^{X^c(t)} + J(t)$ is called a jump process.

- the jump process X is adapted, right continuous
- the left continuous version of X is

$$X(t-) = X(0) + I(t) + R(t) + J(t-)$$

- the jump size of X at time t is:

$$\Delta X(t) = X(t) - X(t-) = J(t) - J(t-) = \Delta J(t)$$

Def: Let X be a jump process, ϕ adapted process

$$\rightsquigarrow \int_0^t \phi(s)\, dX(s) = \int_0^t \phi(s)\Gamma(s)\, dW_s + \int_0^t \phi(s)\Theta(s)\, ds + \sum_{s \le t} \phi(s)\Delta J(s)$$

or

$$\rightsquigarrow \phi(t)\, dX(t) = \underbrace{\phi(t)\Gamma(t)\, dW(t) + \phi(t)\Theta(t)\, dt}_{\phi(t)\, dX^c(t)} + \phi(t)\, dJ(t)$$

$$\int_0^t \phi_s \, dX_s$$

- *example*: $X(t) = N(t) - \lambda t$ (compensated PP)

for this process $I(t) = 0$; $R(t) = -\lambda t$, $J(t) = N(t)$

- let $\phi(t) = \Delta N(t) = \begin{cases} 1 & \text{if } N \text{ jumps at } t \\ 0 & \text{otherwise} \end{cases} = N(t) - N(t-)$

$R(t) = -\lambda t$

$dR(t) = -\lambda \, dt$

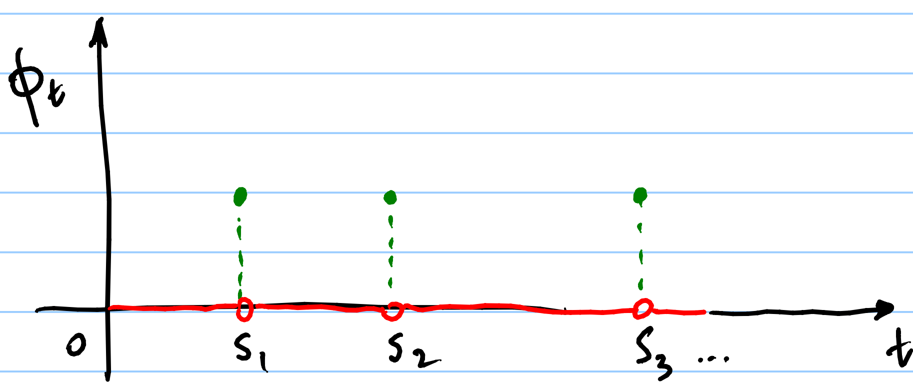
- $\int_0^t \phi(s) \, dX^c(s) = \int_0^t \phi(s) \, dR(s) = -\lambda \int_0^t \phi(s) \, ds = 0$
- $\int_0^t \phi(s) \, dJ(s) = \int_0^t \phi(s) \, dN(s) = \sum_{0 < s \le t} (\Delta N(s))^2 = N(t) = \sum_{s \le t} \phi(s) \cdot \Delta N(s)$

$$\int_0^t \phi(s) \, dX(s) = -\lambda \int_0^t \phi(s) \, ds + \int_0^t \phi(s) \, dN(s) = N(t)$$

Theorem: Assume that the jump process $X(t)$ is a martingale, $\phi(t)$ is left-continuous and adapted with

$$\mathbb{E} \int_0^t \Gamma^2(s) \, \phi^2(s) \, ds < \infty \quad (\forall) \; t \ge 0$$

Then the stochastic integral $\int_0^t \phi(s) \, dX(s)$ is also a martingale.



Quadratic Variation

- recall the definition $\langle X \rangle_t = \lim_{\|\Pi\| \to 0} Q_\Pi(X)$

where $Q_\Pi(X) = \sum_{j=0}^{n-1} \left(X(t_{j+1}) - X(t_j) \right)^2$

$\Pi = \{t_0, t_1, \dots, t_n\} \qquad 0 = t_0 < t_1 < \dots < t_n = T$

- cross-variation $\langle X_1, X_2 \rangle(t) = \lim_{\|\Pi\| \to 0} C_\Pi(X_1, X_2)$

where $C_\Pi(X_1, X_2) = \sum_{j=0}^{n-1} \left(X_1(t_{j+1}) - X_1(t_j) \right)\left(X_2(t_{j+1}) - X_2(t_j) \right)$

Theorem: Let $X(t) = X(0) + I(t) + R(t) + J(t)$ jump process

$$\Rightarrow \langle X \rangle(t) = \langle X^c \rangle(t) + \langle J \rangle(t)$$
$$= \int_0^t \Gamma^2(s)\, ds + \sum_{s \le t} (\Delta J(s))^2$$

- similarly for the cross-variation process we have:

$$\langle X_1, X_2 \rangle(t) = \langle X_1^c, X_2^c \rangle(t) + \langle J_1, J_2 \rangle(t)$$
$$= \int_0^t \Gamma_1(s) \Gamma_2(s)\, ds + \sum_{s \le t} \Delta J_1(s)\, \Delta J_2(s)$$

· in differential notation:

$$dX_1(t) = dX_1^c(t) + dJ_1(t)\ ; \quad dX_2(t) = dX_2^c(t) + dJ_2(t)$$

$$\rightsquigarrow\ dX_1(t)\cdot dX_2(t) = dX_1^c(t)\cdot dX_2^c(t) + \underbrace{dX_1^c(t)\,dJ_2(t)}_{=0} +$$

$$+ \underbrace{dX_2^c(t)\,dJ_1(t)}_{=0} + dJ_1(t)\,dJ_2(t)$$

Remark: if $M(t) = N(t) - \lambda t$ compensated PP

W : Brownian Motion

$$\rightsquigarrow\ \langle M, W\rangle_t = -\lambda\,\underbrace{dt\cdot dW_t}_{=0} + \underbrace{dN(t)\cdot 0}_{=0} = 0$$

$\rightsquigarrow$ we shall see later that $\langle M, W\rangle = 0$ implies that N and W are in fact independent.

Stochastic Calculus for Jump Processes

Ito formula: $f \in C^2$

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(t))\, dX^c_t + \frac{1}{2}\int_0^t f''(X(t))\, d\langle X^c \rangle_t$$
$$+ \sum_{s \le t}' \left[f(X(s)) - f(X(s-)) \right]$$

example: The Geometric Poisson Process

$$S(t) = S(0) \exp\{ \overbrace{N(t) \log(\sigma+1) - \lambda\sigma t}^{X(t) \rightsquigarrow \text{jump process}} \} =$$
$$= S(0)\, e^{-\lambda\sigma t} (\sigma+1)^{N(t)}$$

$\rightsquigarrow$ $S(t) = S(0)\, f(X(t))$ where $f(x) = e^x$

Itô formula gives:

$$S(t) = S(0) - \lambda\sigma \int_0^t S(u)\, du + \sum_{s \le t}' \left[S(u) - S(u-) \right]$$

$$\left(\begin{array}{ll} \text{recall } X^c(t) = -\lambda\sigma t & \rightsquigarrow dX^c(t) = -\lambda\sigma dt \\ & \rightsquigarrow d\langle X^c \rangle(t) = 0 \end{array} \right)$$

- if there is a jump at time u $\rightsquigarrow$

$$S(u) = (\sigma+1)\, S(u-)$$

$$\Rightarrow \quad S(u) - S(u-) = \sigma S(u-)$$

- if there is no jump at time u: $S(u) - S(u-) = 0$

$$\Rightarrow \sum_{u \le t}' \left[S(u) - S(u-) \right] = \sum_{0 < u \le t}' \sigma S(u-)\, \Delta N(u) = \int_0^t \sigma S(u-)\, dN(u)$$

(recall $\Delta N(u) = \begin{cases} 1 & \text{if there is a jump at } u \\ 0 & \text{otherwise} \end{cases}$)

$$\Rightarrow S(t) = S(0) - \lambda\sigma \int_0^t S(u)\, du + \sigma \int_0^t S(u-)\, dN(u)$$

or equivalently

$$S(t) = S(0) - \lambda\sigma \int_0^t S(u-)\, du + \sigma \int_0^t S(u-)\, dN(u)$$

$$= S(0) + \sigma \int_0^t S(u-)\, dM(u)$$

where $M(t) = N(t) - \lambda t$ (Compensated PP)

$dM(t) = dN(t) - \lambda\, dt$

Corollary: Let W be a Brownian motion and N a PP with intensity λ, both defined on the same prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ and relative to the same filtration. Then W and N are independent.

Proof: define $Y(t) = \exp\{\underbrace{u_1 W(t) + u_2 N(t) - \frac{1}{2}u_1^2 t - \lambda(e^{u_2}-1)t}_{X(t) \text{ jump proc.}}\}$

- write the Itô formula for $f(X(t)) = e^{X(t)}$
- show that $Y(t)$ is martingale

$\rightsquigarrow \mathbb{E}\, e^{u_1 W(t) + u_2 N(t)} = \underbrace{e^{\frac{1}{2}u_1^2 t}} \cdot \underbrace{e^{\lambda t(e^{u_2}-1)}}$

$\rightsquigarrow$ moment generating funct of $N(0,t)$ and Poisson (λt)

Ito formula for multiple Jump Processes

$$f(t, X_1(t), X_2(t)) = f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))\,ds$$

$$+ \int_0^t f_{x_1}(s, X_1(s), X_2(s))\,dX_1^c(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s))\,dX_2^c(s)$$

$$+ \frac{1}{2}\int_0^t f_{x_1 x_1}(s, X_1(s), X_2(s))\,d\langle X_1^c\rangle(s) +$$

$$+ \int_0^t f_{x_1 x_2}(s, X_1(s), X_2(s))\,d\langle X_1^c, X_2^c\rangle(s) +$$

$$+ \frac{1}{2}\int_0^t f_{x_2 x_2}(s, X_1(s), X_2(s))\,d\langle X_2^c\rangle(s)$$

$$+ \sum_{0<s\le t}{}' \left[f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-)) \right]$$

Corrollary : (Product Rule)

$$X_1(t)X_2(t) = X_1(0)X_2(0) + \int_0^t X_2(s)\,dX_1^c(s) + \int_0^t X_1(s)\,dX_2^c(s)$$

$$+ \langle X_1^c, X_2^c\rangle(t) + \sum_{0<s\le t}{}' \left[X_1(s)X_2(s) - X_1(s-)X_2(s-) \right]$$

$$= X_1(0)X_2(0) + \int_0^t X_2(s-)\,dX_1(t) + \int_0^t X_1(s-)\,dX_2(s) + \langle X_1, X_2\rangle(t)$$

Change of measure (Girsanov's Th)
general case.

Recall the Girsanov's Th (case with no jumps)

$(\Omega, \mathcal{F}, P)$ $W =$ B.M , X adapted to $\mathcal{F}^W$

$Z_t^X = \mathcal{E}(X)_t = \exp\{ X_t - \frac{1}{2}\langle X \rangle_t \}$

- is a solution of the following SDE

$$\boxed{dZ_t = Z_t \, dX_t \quad \text{with } Z_0 = 1}$$

(Doléans-Dade exponential of X w.r.to W)

- Z_t^X is local martingale

- a prob. measure $Q \sim P$ exists $dQ = Z^X dP$ such that for any M, P local martingale

  $\tilde{M}_t = M_t - \langle M, X \rangle_t$ is a

  continuous Q-local martingale

• in particular, if

$$Z(t) = \exp\left\{ -\int_0^t \Gamma(s)\, dW(s) - \frac{1}{2}\int_0^t \Gamma^2(s)\, ds \right\}$$

or $\quad dZ(t) = -\Gamma(t) Z(t)\, dW(t) = Z(t)\, dX^c(t)$

where $\quad X^c(t) = -\int_0^t \Gamma(s)\, dW(s) \qquad \langle X^c \rangle(t) = \int_0^t \Gamma^2(s)\, ds$

$\rightsquigarrow \quad Z(t) = \exp\left\{ X^c(t) - \frac{1}{2} \langle X^c \rangle(t) \right\}$

• in the stochastic calculus with jumps the above equation becomes

$$(*) \qquad dZ^X(t) = Z^X(t-)\, dX(t)$$

where $\quad X(t) = X^c(t) + J(t)$

• the solution to $(*)$ is the same as R-N in the continuous case except that whenever there is a jump in X there will be a jump in Z^X

$$\Delta Z^X(t) = Z^X(t-)\, \Delta X(t)$$

$$\rightsquigarrow Z^X(t) = Z^X(t-) + \Delta Z^X(t)$$
$$= Z^X(t-)\left(1 + \Delta X(t)\right)$$

- so if X is a jump process,

$Z^X(t) = \mathcal{E}(X)(t)$ becomes

$$= \exp\left\{X^c(t) - \tfrac{1}{2}\langle X^c\rangle(t)\right\} \cdot \prod_{s \le t}\left(1 + \Delta X(s)\right)$$

- this process is the solution to the SDE :

$$\boxed{Z^X(t) = 1 + \int_0^t Z^X(s-)\, dX(s)}$$

Change of measure for a PP process

- $N(t) = PP$ on $(\Omega, \mathcal{F}, P)$

  $\mathbb{E}\, N(t) = \lambda t \qquad (\lambda = \text{intensity})$

- assume the stock price is modelled by a GPP

  $$S(t) = S(0)\, \exp\{N(t) \log(\sigma+1) - \lambda t\}$$

  we have seen already that

  $$S(t) = S(0) - \lambda\sigma \int_0^t S(u-)\, du + \sigma \int_0^t S(u-)\, dN(u)$$

  $$= S(0) + \sigma \int_0^t S(u-)\, dM(u)$$

where $M(t) = N(t) - \lambda t$: compensated PP (martingale)

$\rightsquigarrow$ $S(t)$ is martingale as well.

· assume now the equation of the stock price is

$$S(t) = S(0) \exp\{\alpha t + N(t) \log(\sigma+1) - \lambda\sigma t\}$$

$\rightsquigarrow$ $dS(t) = \alpha S(t)\,dt + \sigma S(t-)\,dM(t)$

$\rightsquigarrow$ $d\left(e^{-rt} S(t)\right) = \left((\alpha - r) S(t)\,dt + \sigma S(t-)\,dM(t)\right) e^{-rt}$

$$= \sigma \left[\frac{\alpha - r}{\sigma} S(t)\,dt + S(t-)\,dM(t) \right] e^{-rt}$$

$$= \sigma S(t-) \underbrace{\left[\frac{\alpha - r}{\sigma}\,dt + dM(t) \right]}_{d\tilde{M}(t)} e^{-rt}$$

· we would like $\tilde{M}$ to be a compensated PP under $\tilde{P}$ with intensity $\tilde{\lambda}$

$$\tilde{M}(t) = M(t) + \frac{\alpha - r}{\sigma} t$$

$$= N(t) - \lambda t + \frac{\alpha - r}{\sigma} t$$

$$= N(t) - \tilde{\lambda} t$$

where $\tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}$: new intensity

Remark : we need $\tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma} > 0$ otherwise

there is arbitrage !

- we want the change of measure that transforms $N(t)$ from a $PP(\lambda)$ to a $PP(\tilde{\lambda})$

- the density that makes this change possible is

$$Z(t) = e^{(\lambda-\tilde{\lambda})t} \cdot \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)} \qquad \left(\tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}\right)$$

- easy to check that :

$$dZ(t) = \frac{\tilde{\lambda}-\lambda}{\lambda} Z(t-)\, dM(t)$$

$\rightsquigarrow$ $Z(t)$ is martingale under $\mathbb{P}$ and $EZ(t)=1$

Theorem : Under prob. measure $\tilde{\mathbb{P}}$ $\quad 0 \le t \le T$
$(d\tilde{\mathbb{P}} = Z_T \, d\mathbb{P})$, the process $N(t)$ is a PP with intensity $\tilde{\lambda}$

Pricing a European Call in a Jump Model.

- asset driven by a Poisson process

$$S(t) = S(0) \exp\left\{\alpha t + N(t)\log(\sigma+1) - \lambda\sigma t\right\}$$

$$= S(0)\, e^{(\alpha-\lambda\sigma)t} (\sigma+1)^{N(t)}$$

$\rightsquigarrow$ $dS(t) = \alpha S(t)\, dt + \sigma S(t-)\, dM(t)$

where $M(t) = N(t) - \lambda t$: compensated PP

$V(T) = (S(T) - K)^+$: payoff at time T

- assume $\lambda > \frac{\alpha - r}{\sigma}$ (to rule out arbitrage)

$\rightsquigarrow \tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}$

$\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P}$ for all $A \in \mathcal{F}$

where $Z_T = e^{(\lambda - \tilde{\lambda})T} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(T)}$

(Z_t is a solution of : $dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t-)\, dM(t)$)

$\tilde{\mathbb{P}}$: risk neutral measure

- under $\tilde{\mathbb{P}}$: $\tilde{M}(t) = N(t) - \tilde{\lambda} t$ is compensated PP (martingale)

$dS(t) = rS(t)\, dt + \sigma S(t-)\, d\tilde{M}(t)$

$d\left(e^{-rt} S(t)\right) = \sigma e^{-rt} S(t-)\, d\tilde{M}(t)$

$\rightsquigarrow$ discounted stock price is a martingale under $\tilde{\mathbb{P}}$

$\rightsquigarrow S(t) = S(0)\, e^{(r - \tilde{\lambda}\sigma)t} (\sigma + 1)^{N(t)}$

- let $V(t)$: the price of the European Call with payoff V_T

$\rightsquigarrow$ $e^{-rt}V(t)$ is a martingale under $\widetilde{\mathbb{P}}$

$$e^{-rt}V(t) = \widetilde{\mathbb{E}}\left[e^{-rT}V(T) \,\middle|\, \mathcal{F}_t\right]$$
$$= \widetilde{\mathbb{E}}\left[e^{-rT}(S(T)-K)^+ \,\middle|\, \mathcal{F}_t\right]$$

where $S(T) = S(t)\, e^{-(r-\tilde{\lambda}\sigma)(T-t)} (\sigma+1)^{N(T)-N(t)}$

$$\Rightarrow V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(t)\, e^{-(r-\tilde{\lambda}\sigma)(T-t)} (\sigma+1)^{N(T)-N(t)} - K\right)^+ \,\middle|\, \mathcal{F}_t\right]$$

$= c(t, S(t))$ $\uparrow$ Markov property.

where

$$c(t, x) = \sum_{j=0}^{\infty} e^{-r(T-t)}\left(x e^{-(r-\tilde{\lambda}\sigma)(T-t)} (\sigma+1)^j - K\right)^+ \cdot \frac{\tilde{\lambda}^j (T-t)^j}{j!} e^{-\tilde{\lambda}(T-t)}$$

(since $N(T)-N(t) \sim N(T-t) \sim \text{Poisson}(\tilde{\lambda}(T-t))$ under $\widetilde{\mathbb{P}}$)

- so $e^{-rt}c(t, S(t))$: $\widetilde{\mathbb{P}}$ martingale

$\xrightarrow{\text{Ito}}$ we need to compute $d(e^{-rt}c(t,S(t)))$ and set the dt term equal to 0.

$$dS(t) = \underbrace{(r - \tilde{\lambda}\sigma) S(t)\, dt}_{dS^c(t) \;\rightsquigarrow\; \text{the cont. part.}} + \sigma S(t-)\, dN(t)$$

- on the other hand if the stock jumps at t

$$\Delta S(t) = S(t) - S(t-) = \sigma S(t-) \rightsquigarrow S(t) = (\sigma + 1) S(t-)$$

$$\Rightarrow e^{-rt} c(t, S(t)) = c(0, S(0)) + \int_0^t e^{-ru} \Big[-rc(u, S(u))\, du + c_t(u, S(u))\, du + $$

$$+ c_x(u, S(u))\, dS^c(u) \Big] + \sum_{u \le t} e^{-ru} \Big[c(u, S(u)) - c(u, S(u-)) \Big]$$

(note that c_{xx} term is missing because $d\langle S^c \rangle(t) \equiv 0$)

$$\Rightarrow e^{-rt} c(t, S(t)) = c(0, S(0)) + \int_0^t e^{-ru} \Big(-rc(u, S(u)) + c_t(u, S(u)) + $$

$$+ (r - \tilde{\lambda}\sigma) S(u)\, c_x(u, S(u)) \Big)\, du + $$

$$\underbrace{\int_0^t e^{-ru} \Big[c(u, (\sigma+1) S(u-)) - c(u, S(u-)) \Big]\, d\big[N(t)}_{\text{not a martingale yet !!!}} - \tilde{\lambda} u + \tilde{\lambda} u \big]$$

- to make it a martingale you need to compensate it with $-\tilde{\lambda} t$

$$\int [\ \ldots\ ]\, d\big[\underbrace{N(u) - \tilde{\lambda} u}_{\tilde{M}_u}\big] + \int [\ \ \ ]\, d(\tilde{\lambda} u)$$

$\Rightarrow$ we need to set the dt term $= 0$.

$$-rc(t,S(t)) + c_t(t,S(t)) + (r-\tilde{\lambda}\sigma)S(t)\,c_x(t,S(t)) + \tilde{\lambda}\Big(c(t,(\sigma+1)S(t)) - c(t,S(t))\Big) = 0$$

$\rightsquigarrow$ replace $S(t)$ with the dummy var x:

$$-rc(t,x) + c_t(t,x) + (r-\tilde{\lambda}\sigma)\,x\,c_x(t,x) + \tilde{\lambda}\Big(c(t,(\sigma+1)x) - c(t,x)\Big) = 0$$

"differential - difference" eq.

- to complete the pricing of this option we must construct a hedging portfolio:
  - at time 0 invest $X(0) = c(0,S(0))$
  - at time t: invest in stock & money market so that: $X(t) = c(t,S(t))$

or equivalently: $e^{-rt}X(t) = e^{-rt}c(t,S(t))$

$\rightsquigarrow$ match the differentials:

$$d\left(e^{-rt}c(t,S(t))\right) = e^{-rt}\left[c(t,(\sigma+1)S(t-)) - c(t,S(t-))\right] d\tilde{M}(t)$$

$$d\left(X(t)\right) = \Gamma(t-)\,dS(t) + r\left[X(t) - \Gamma(t)S(t)\right]dt$$

$$\Rightarrow d\left(e^{-rt}X(t)\right) = e^{-rt}\left[-rX(t)\,dt + dX(t)\right]$$

$$= e^{-rt}\left[\Gamma(t-)\,dS(t) - r\Gamma(t)S(t)\,dt\right]$$

$$= e^{-rt}\sigma\Gamma(t-)S(t-)\,d\tilde{M}(t)$$

$$\Rightarrow \Gamma(t-) = \frac{c(t,(\sigma+1)S(t-)) - c(t,S(t-))}{\sigma S(t-)}$$

- this hedging position we should hold at all times whether they are jump times or not.

i.e define $\Gamma(t) = \dfrac{c(t,(\sigma+1)S(t)) - c(t,S(t))}{\sigma S(t)}$

$$\Rightarrow e^{-rt}X(t) = X(0) + \int_0^t e^{-ru}\left[c(u,(\sigma+1)S(u-)) - c(u,S(u-))\right] d\tilde{M}(u)$$

Change of measure for the CPP

$Y_i = \{y_1, y_2, \ldots, y_M\}$ $\qquad p_m = P(Y_i = y_m) > 0$

$$Q(t) = \sum_{i=1}^{N(t)} Y_i = \sum_{m=1}^{M} y_m \cdot N_m(t)$$

where $N_1, N_2, \ldots, N_M$ are indep. PP.

with intensities $\lambda p_1, \lambda p_2, \ldots, \lambda p_M$

$$Z_1(t) = e^{(\lambda_1 - \tilde{\lambda}_1)t} \left(\frac{\tilde{\lambda}_1}{\lambda_1}\right)^{N_1(t)} \qquad \lambda_1 = \lambda \cdot p_1$$

$$Z(t) = \prod_{m=1}^{M} Z_m(t)$$