MTH 9821 Numerical Methods for Finance I
Lecture 1

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1 Linear System Solutions

Solve $\mathbf{Ax} = \mathbf{b}$

$\mathbf{A}$: $n \times n$ matrix given;

$\mathbf{b}$: $n \times 1$ vector given;

Find $\mathbf{x}$: $n \times 1$ vector

Routines for solving $Ax = b$

$$x = linear\_solve\_LU\_no\_pivoting\ (A, b)$$
$$x = linear\_solve\_LU\_row\_pivoting\ (A, b)$$
$$x = linear\_solve\_Cholesky\ (A, b)$$

NLA Methods for Solving $\mathbf{Ax} = \mathbf{b}$

- Direct Method:
  - LU/LU with row pivoting;
  - Cholesky;
- Iterative Method:
  - Jacobi;
  - Gauss-Siedel;
  - SOR;

Remark:
⋆ Direct methods are fast for 2D problems, while iterative methods are useful for 3D problems.
⋆ Financial application problems normally can be solved using direct methods.

2 Linear Solve with LU No Pivoting

$\mathbf{x} = $ **linear_solve_LU_no_pivoting** $(\mathbf{A}, \mathbf{b})$

$$\begin{aligned} [L,\ U] &= lu(A) \\ y &= forward\_subst(L,\ b) \\ x &= backward\_subst(U,\ y) \\ (x &= backward\_subst(U,\ forward\_subst(L,\ b))) \end{aligned}$$

LU decomposition of A:

Find L lower triangular matrix with $1's$ on main diagonal

U upper triangular matrix

such that $A = LU$

Thus,

$$\begin{aligned} & Ax = b \\ \iff & LUx = b \\ \iff & Ly = b, \quad Ux = y \end{aligned}$$

2.1 Forward and Backward Substitution

Forward Substitution

L lower triangular nonsingular matrix

L is nonsingular $\iff L(i,\ i) \neq 0, \quad \forall i = 1:n$

$(0 \neq det(L) = \prod_{i=1}^{n} L(i,\ i)\ )$

Solve $Lx = b$

1st row: $L(1,1)x(1) = b(1) \implies x(1) = \frac{b(1)}{L(1,\ 1)} \quad \big(L(1,\ 1) \neq 0\big)$

jth row: $\sum_{k=1}^{j} L(j,\ k)x(k) = b(j) \quad \big(L(j,\ k) = 0,\ \forall k > j,\ \text{since } L \text{ is lower triangular}\big)$

$$x(j) = \frac{b(j) - \sum_{k=1}^{j-1} L(j,\ k)x(k)}{L(j,\ j)} \quad \big(L(j,\ j) \neq 0\big)$$

Backward Substitution

U upper triangular nonsingular matrix

Solve $Ux = b$

nth row: $U(n,n)x(n) = b(n) \implies x(n) = \frac{b(n)}{L(n,\ n)} \quad \big(U(n,\ n) \neq 0\big)$

jth row: $\sum_{k=j}^{n} U(j,\ k)x(k) = b(j) \quad \big(U(j,\ k) = 0,\ \forall k < j,\ \text{since } U \text{ is upper triangular}\big)$

$$x(j) = \frac{b(j) - \sum_{k=j+1}^{n} U(j,\ k)x(k)}{U(j,\ j)} \quad \big(U(j,\ j) \neq 0\big)$$

Remark:
⋆ Operation Count for forward and backward substitution: $n^2 + O(n)$

2.2 LU Decomposition

The LU decomposition of an $n \times n$ nonsingular matrix A requires finding

L lower triangular with $L(i,\ i) = 1,\ \forall i = 1:n$

U upper triangular

such that $A = LU$

Remark:
⋆ $L(i,\ i) = 1,\ \forall i = 1:n$ is to make sure that the LU decomposition is unique.

- **Existence:**

  A has LU decomposition iff all the leading principal minors of A are nonzero.

  Leading principal minors: $det\big(A(1:i,\ 1:i)\big), \quad i = 1:n$

- **Uniqueness:**
  If it exists, the LU decomposition of a matrix is unique.

Proof for Uniqueness of LU Decomposition

$$
\begin{aligned}
& A \;=\; L_1U_1 \;=\; L_2U_2 \\
& L_2^{-1}L_1\boxed{U_1U_1^{-1}} \;=\; \boxed{L_2^{-1}L_2}\,U_2U_1^{-1} \\
& \underbrace{L_2^{-1}L_1}_{\text{lower trangular}} \;=\; \underbrace{U_2U_1^{-1}}_{\text{upper triangular}} \;=\; \underbrace{D}_{diagonal} \\
\Rightarrow\;\; & L_2D \;=\; L_2L_2^{-1}L_1 \;=\; L_1 \\
& (L_2D)\,(i,\ i) = D(i,\ i)L_2(i,i) \;=\; L_1(i,\ i) \\
& \text{Since } L_1(i,\ i) = 1,\ L_2(i,\ i) = 1 \\
\Rightarrow\;\; & D(i,\ i) = 1 \\
\Rightarrow\;\; & D = I \\
\Rightarrow\;\; & L_1 \;=\; L_2, \quad U_1 \;=\; U_2
\end{aligned}
$$

LU Decomposition

$$
\underbrace{\begin{pmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{pmatrix}}_{A}
=
\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ x & x & 1 & 0 & 0 \\ x & x & x & 1 & 0 \\ x & x & x & x & 1 \end{pmatrix}}_{L}
\underbrace{\begin{pmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \end{pmatrix}}_{U}
$$

Step 1: Calculate 1st column of L (L_1) and 1st row of U (U_1);

$$
\begin{aligned}
& U(1,\ k) \;=\; A(1,\ k), \quad \forall k = 1:n \\
& L(j,\ 1)U(1,\ 1) \;=\; A(j,\ 1), \quad \forall j = 2:n \\
\Rightarrow & \; L(j,\ 1) \;=\; \frac{A(j,\ 1)}{U(1,\ 1)}, \quad \forall j = 2:n \\
& \big(U(1,\ 1) \;=\; 0 \text{ iff } A(1,\ 1) = 0 \quad \Rightarrow \quad \text{LU no pivoting won't work}\big)
\end{aligned}
$$

Step 2: Repeat on $(n-1)\times(n-1)$ matrix
By block multiplication,

$$
\begin{aligned}
& \big(L(2:n,\ 2:n) = L_1, \quad U(2:n,\ 2:n) = U_1\big) \\
& A(2:n,\ 2:n) \;=\; L(2:n,\ 1)U(1,\ 2:n) + L_1U_1 \\
\Rightarrow\;\; & L_1U_1 \;=\; A(2:n,\ 2:n) \;-\; L(2:n,\ 1)U(1,\ 2:n)
\end{aligned}
$$

Pseudocode: *LU without pivoting*

Input:

A = nonsingular matrix of size n with LU decomposition

Output:

L = lower triangular matrix with entries 1 on main diagonal

U = upper triangular matrix

such that $LU = A$

```
for i = 1 : (n − 1)
    for j = i : n
        U(i, j) = A(i, j);
        L(j, i) = A(j, i) / U(i, i)
    end
    for j = (i + 1) : n
        for k = (i + 1) : n
            A(j, k) = A(j, k) − L(j, i)U(i, k);
        end
    end
end
L(n, n) = 1;
U(n, n) = A(n, n)
```

Operation Count of LU no Pivoting: $\frac{2}{3}n^3 + O(n^2)$

LU Decomposition of Banded Matrices

- **Banded Matrix**

  Def: A is a matrix of band m iff

$$A(j,\ k) \ = \ 0, \quad \forall 1 \leq j, k \leq n \text{ with } |j-k| > m$$

  e.g. diagonal matrix: band 0;

  tridiagonal matrix: band 1;

- **LU decomposition of banded matrix with band** m e.g., 8×8 matrix with band 3

$$\begin{pmatrix}
x & x & x & x & 0 & 0 & 0 & 0 \\
x & x & x & x & x & 0 & 0 & 0 \\
x & x & x & x & x & x & 0 & 0 \\
x & x & x & x & x & x & x & 0 \\
0 & x & x & x & x & x & x & x \\
0 & 0 & x & x & x & x & x & x \\
0 & 0 & 0 & x & x & x & x & x \\
0 & 0 & 0 & 0 & x & x & x & x
\end{pmatrix}$$

Step 1:

$$U(1,\ k) \ = \ A(1,\ k), \quad \forall k = 1 : (m+1)$$

$$\big(A(1,\ m+2) = 0, \quad A(1,\ m+1) \neq 0\big)$$

$$L(j,\ 1) \ = \ \frac{A(j,\ 1)}{U(1,\ 1)}, \quad \forall j = 2 : (n+1)$$

Step 2:

$$A(j,\ k) \ = \ A(j,\ k) \ - \ L(j,\ 1)U(1,\ k)$$

$$\forall j = 2 : (2+m-1)$$

$$\forall k = 2 : (2+m-1)$$

Remark:

⋆ In the above 8×8 matrix, only shaded entries need to be updated in this step.

⋆ m^2 entries need to be updated

Pseudocode: *LU Decomposition of Banded Matrices*

Input:

A = nonsingular matrix of size n of band m

Output:

L = lower triangular matrix with entries 1 on main diagonal

U = upper triangular matrix

such that $LU = A$

```
for i = 1 : (n − 1)
    for j = i : min(i + m, n)
        U(i, j) = A(i, j);
        L(j, i) = A(j, i) / U(i, i)
    end
    for j = (i + 1) : min(i + m, n)
        for k = (i + 1) : min(i + m, n)
            A(j, k) = A(j, k) − L(j, i)U(i, k);
        end
    end
end
L(n, n) = 1;
U(n, n) = A(n, n)
```

Operation Count: $(n-1)(m+2m^2) = \mathbf{2m^2n + O(mn)}$

2.3 Tridiagonal Matrices

Note:

- If tridiagonal matrix A is strictly diagonally dominated, then A has LU decomposition.

> **Why?**
> All leading principal minors are nonzero since $A(1:i,\ 1:i)$ is strictly diagonally dominated and therefore nonsingular, $\forall i = 1:n$.

- Only 1 entry need to be updated while doing LU decomposition.

Pseudocode for solving linear system of tridiagonal matrices

Step 1: LU Decomposition for Tridiagonal Matrix

Pseudocode: *LU Decomposition of Tridiagonal Matrices*
Input:
A = nonsingular tridiagonal matrix of size n
Output:
L = lower triangular matrix with entries 1 on main diagonal
U = upper triangular matrix
such that $LU = A$

for $i = 1:(n-1)$
  $U(i,\ i) = A(i,\ i),\quad U(i,\ i+1) = A(i,\ i+1);$
  $L(i,\ i) = 1,\quad L(i+1,\ i) = A(i+1,\ i)U(i,\ i)\ A(i+1,\ i+1) = A(i+1,\ i+1) - L(i+1,\ i)U(i,\ i+1);$
end
$L(n,\ n) = 1;$
$U(n,\ n) = A(n,\ n)$

$\star$ Operation Count: $3n + O(1)$

Step 2: Forward Substitution and Backward Substitution for Tridiagonal Matrix

Pseudocode: *Forward Substitution*
Input:
$L =$ lower triangular matrix with entries 1 on main diagonal
$b =$ column vector of size n
Output:
$y =$ column vector of size n
such that $Ly = b$

$y(1) = \frac{b(1)}{L(1,\ 1)} = b(1);$
for $j = 2 : n$
 $y(j) = b(j) - \frac{L(j,\ j-1)y(j-1)}{L(j,\ j)} = b(j) - L(j,\ j-1)y(j-1);$
end

⋆ Operation Count: $2n + O(1)$

Pseudocode: *Backward Substitution*
Input:
$U =$ upper triangular matrix
$y =$ column vector of size n
Output:
$x =$ column vector of size n
such that $Ux = y$

$x(n) = \frac{y(n)}{U(n,\ n)};$
for $j = 1 : (n-1)$
 $x(j) = y(j) - \frac{U(j,\ j+1)x(j+1)}{U(j,\ j)};$
end

⋆ Operation Count: $3n + O(1)$

⋆ Operation Count of linear solve for a tridiagonal matrix: $8n + O(1)$

2.4 Efficient Use of LU Decomposition without Row Pivoting

- **Solve** p **linear systems,** $Ax_i \ = \ b_i, \quad \forall i = 1:p$

$$
\begin{aligned}
[L,\ U] &= lu(A); \\
\text{for } i &= 1:p \\
y &= forward\_subst(L,\ b_i) \\
x_i &= backward\_subst(U,\ y) \\
\text{end} &
\end{aligned}
$$

- **Find** A^{-1} **where** $A^{-1} = col(a_k)_{k=1:n},\ AA^{-1} = I$

$$
\begin{aligned}
&\text{It's equivalent to solving } Aa_k = e_k, \quad \forall k = 1:n \\
[L,\ U] &= lu(A); \\
\text{for } i &= 1:p \\
y &= forward\_subst(L,\ e_i) \\
x_i &= backward\_subst(U,\ y) \\
\text{end} &
\end{aligned}
$$

Remark:
⋆ Implement LU before the for loop

3 LU Decomposition with Row Pivoting

Find P: permutation matrix;
L: lower triangular matrix with $1's$ on main diagonal;
U: upper triangular matrix;
Such that PA = LU

Existence
Any nonsingular matrix has an LU decomposition with row pivoting.
Not Unique

x = linear_solve_LU_row_pivoting $(\mathbf{A}, \mathbf{b})$

$$
\begin{aligned}
[P,\ L,\ U] &= lu\_row\_pivoting(A) \\
y &= forward\_subst(L,\ Pb) \\
x &= backward\_subst(U,\ y) \\
(x &= backward\_subst(U,\ forward\_subst(L,\ Pb)))
\end{aligned}
$$

$$
\begin{aligned}
Ax &= b \\
\Rightarrow \quad PAx &= Pb \\
\Rightarrow \quad LUx &= Pb \\
\Rightarrow \quad Ly &= Pb \\
Ux &= y
\end{aligned}
$$

Permutation Matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} r_2 \\ r_4 \\ r_1 \\ r_3 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 & 4 & 1 & 3 \end{pmatrix}$$

Example:
LU decomposition with row pivoting of matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix}$$

Rule:
Largest entry in the first column of the updated matrix A must be the first largest in absolute value of all the entries of that first column.

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix} \xrightarrow{P(3\ 2\ 1\ 4)} \begin{pmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{pmatrix}$$

$$\boxed{P_{index} = (3\ 2\ 1\ 4)}$$

$$U(1,i) = A(1,i),\ \forall i = 1:4; \quad L(j,1) = A(j,1)/U(1,1),\ \forall j = 2:4$$

$$U = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4/8 & 1 & 0 & 0 \\ 2/8 & x & 1 & 0 \\ 6/8 & x & x & 1 \end{pmatrix}$$

$$L_1U_1 = \begin{pmatrix} 3 & 3 & 1 \\ 7 & 9 & 5 \\ 7 & 9 & 8 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} \begin{pmatrix} 7 & 9 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \end{pmatrix} \xrightarrow{P(3\ 4\ 1\ 2)} \begin{pmatrix} \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \end{pmatrix}$$

$$\boxed{P_{index} = (3\ 4\ 1\ 2)}$$

Only switch the solved part of L

$$L \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/4 & x & 1 & 0 \\ 1/2 & x & x & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/4 & -\frac{3}{7} & 1 & 0 \\ 1/2 & -\frac{2}{7} & x & 1 \end{pmatrix}$$

$$L_2U_2 = \begin{pmatrix} -\frac{5}{4} & -\frac{5}{4} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} - \begin{pmatrix} -\frac{3}{7} \\ -\frac{2}{7} \end{pmatrix} \begin{pmatrix} \frac{9}{4} & \frac{17}{4} \end{pmatrix} = \begin{pmatrix} -\frac{2}{7} & \frac{4}{7} \\ -\frac{6}{7} & -\frac{2}{7} \end{pmatrix} \xrightarrow{P(3\ 4\ 2\ 1)} \begin{pmatrix} -\frac{6}{7} & -\frac{2}{7} \\ -\frac{2}{7} & \frac{4}{7} \end{pmatrix}$$

$$\boxed{P_{index} = (3\ 4\ 2\ 1)}$$

Only switch the solved part of L

$$L \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/2 & -\frac{2}{7} & 1 & 0 \\ 1/4 & -\frac{3}{7} & x & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ 0 & 0 & -\frac{6}{7} & -\frac{2}{7} \\ 0 & 0 & 0 & x \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/2 & -\frac{2}{7} & 1 & 0 \\ 1/4 & -\frac{3}{7} & \frac{1}{3} & 1 \end{pmatrix}$$

$$U(4,4) = \frac{4}{7} - \frac{1}{3}\left(-\frac{2}{7}\right) = \frac{2}{3}$$

Therefore,

$$U = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ 0 & 0 & -\frac{6}{7} & -\frac{2}{7} \\ 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/2 & -\frac{2}{7} & 1 & 0 \\ 1/4 & -\frac{3}{7} & \frac{1}{3} & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$