Stochastic calculus : Itô integral for general integrands; Itô formula

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- we no longer assume X - simple process
- we are interested in defining $I_t(X) = \int_0^t X_s \, dW_s$

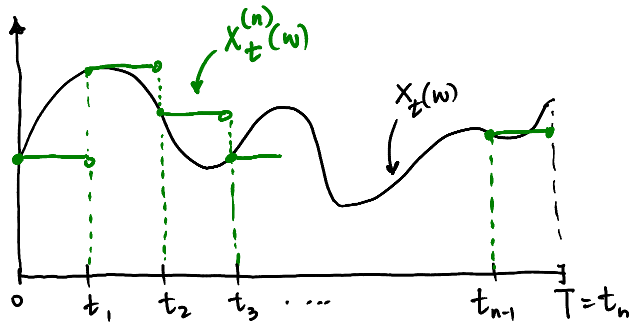
(or alternatively, $\int_0^t X_s \, dM_s$ where

M_t = (local) martingale with continuous paths)

- the idea is to approximate X by a sequence of simple processes!
- assume X is adapted with respect to $\{\mathcal{F}_t\}$
- the approximation of X by simple processes work s.t

$$\lim_{n\to\infty} E\int_0^T |X_s^{(n)} - X_s|^2 \, ds = 0$$

- we partition the interval $[0,T]$ in n subintervals



- for each $X_t^{(n)}$ the Itô integral has been defined for $0 \le t \le T$

Definition : Itô integral for general integrands.

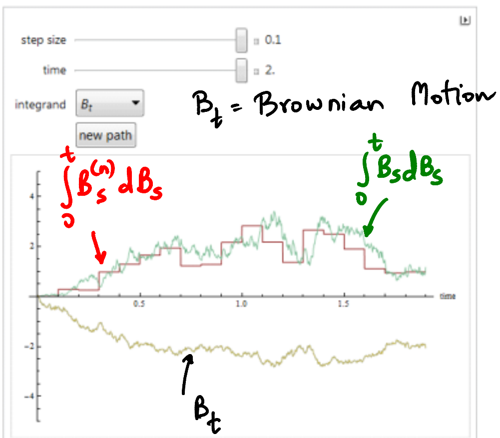
$$\int_0^t X_s \, dW_s = \lim_{n\to\infty} \int_0^t X_s^{(n)} \, dW_s$$

(the convergence is in $L^2(dt \times dP)$)

The only assumption necessary for this definition is that : $\int_0^t X_s^2 \, ds < \infty$ a.s. $(\forall)\, t$

[Ito integral (Wolfram demonstration project)]

The Itô Integral and Itô's Lemma



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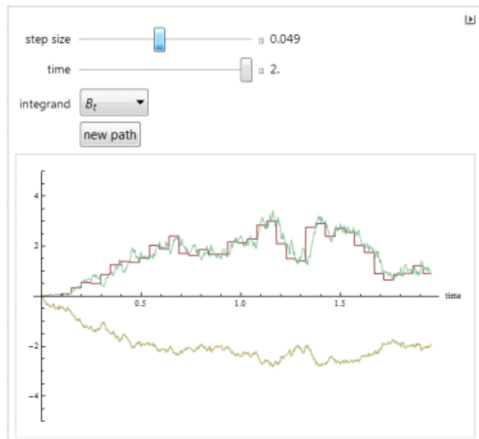
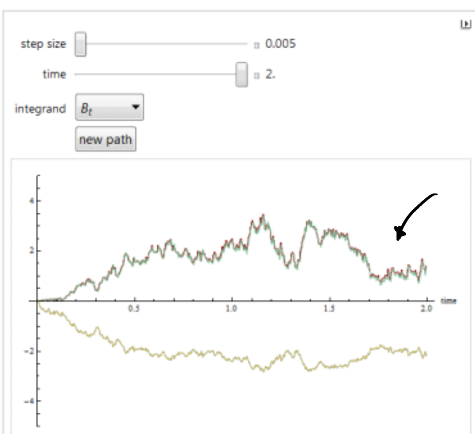


The Itô Integral and Itô's Lemma

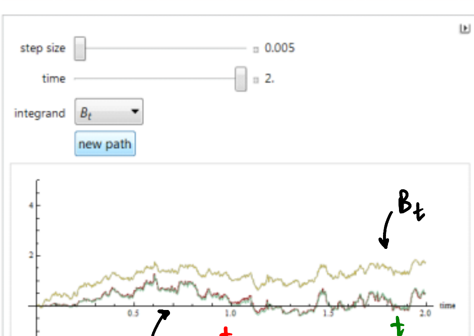


$\int_0^t B_s^{(n)} ds \simeq \int_0^t B_s dB_s$

for all $t \in [0,T]$

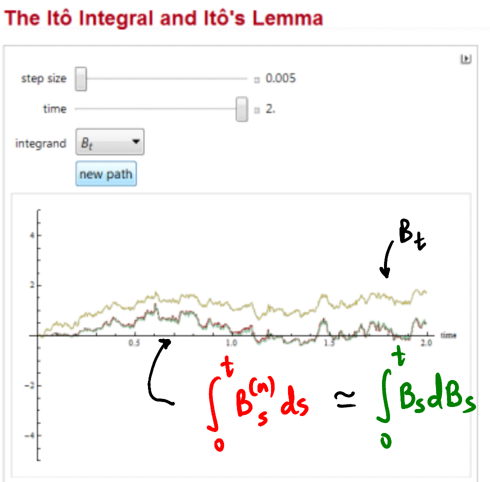
- same will happen if we look at other paths !

The Itô Integral and Itô's Lemma



This are the same processes as above (different paths : ω)

$\Big(\int_0^t B_s^{(n)} dB_s \sim \int_0^t B_s dB_s$



This are the same processes as above (different paths : ω)

Properties of Itô integral . $\quad I_t(X) = \int_0^t X_s dW_s \;;\quad \int_0^t X_s^2 ds < \infty$

(1) "Continuity" : the paths $t \mapsto I_t(X)$ are continuous

(2) "Adaptivity" : for each t, I_t is $\mathcal{F}_t$ measurable

(3) "Linearity" : for any $a, b \in \mathbb{R}$ $\quad I_t(aX + bY) = a I_t(X) + b I_t(Y)$

(4) "Local martingale" : $I_t(X) := $ local martingale ,

if moreover $E \int_0^t X_s^2 ds < \infty \Rightarrow I_t(X) = $ martingale

(5) "Itô isometry" : $E\, I_t^2(X) = E \int_0^t X_s^2 ds$

(6) "Quadratic Variation" : $\langle I(X) \rangle_t = \int_0^t X_s^2 ds$

$\int_0^t X_s\, dW_s \quad : \quad \int_0^t X_s^2 ds < \infty$ a.s

$I_t^M(x) = \int_0^t X_s\, dM_s \quad : \quad \int_0^t X_s^2\, d\langle M \rangle_s < \infty$

$$d\langle I^M \rangle_t = dI_t^M \cdot dI_t^M = X_t^2\, dM_t \cdot dM_t = X_t^2\, d\langle M \rangle_t$$

- this is a classic example of an Itô integral :

$$\int_0^t W_s\, dW_s = ?$$

- according to the definition :

$$\int_0^t W_s \, dW_s = \lim_{n\to\infty} \int_0^t W_s^{(n)} \, dW_s$$

where

$$W_t^{(n)} = \begin{cases} W_{t_0} & \text{if } t_0 \le t < t_1 \\ W_{t_1} & \text{if } t_1 \le t < t_2 \\ \quad \cdots & \\ W_{t_{n-1}} & \text{if } t_{n-1} \le t \le t_n = T \end{cases}$$

$$\Rightarrow \int_0^T W_s \, dW_s = \lim_{n\to\infty} \sum_{j=0}^{n-1} W_{t_j} \left(W_{t_{j+1}} - W_{t_j} \right)$$

- so we shall be interested in understanding this sum

$$I_n = \sum_{j=0}^{n-1} W_{t_j} \left(W_{t_{j+1}} - W_{t_j} \right) = \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_{j+1}} + W_{t_j} \right) \left(W_{t_{j+1}} - W_{t_j} \right)$$

$$= \sum_{j=0}^{n-1} W_{t_{j+1}} \left(W_{t_{j+1}} - W_{t_j} \right) - \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j} \right)^2$$

$$= \sum_{j=0}^{n-1} W_{t_{j+1}}^2 - \sum_{j=0}^{n-1} W_{t_{j+1}} W_{t_j} - \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j} \right)^2$$

$$= W_T^2 + \sum_{j=0}^{n-1} W_{t_j}^2 - \sum_{j=0}^{n-1} W_{t_{j+1}} W_{t_j} - \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j} \right)^2$$

$$= W_T^2 - \underbrace{\sum_{j=0}^{n-1} W_{t_j} \left(W_{t_{j+1}} - W_{t_j} \right)}_{= I_n} - \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j} \right)^2$$

$$\Rightarrow I_n = W_T^2 - I_n - \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j} \right)^2$$

$$\Rightarrow 2 I_n = W_T^2 - \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j} \right)^2$$

$$\sum_{j=0} \quad t_{j+1} \quad t_j$$

$$\Rightarrow I_n = \frac{1}{2} W_T^2 - \frac{1}{2} \underbrace{\sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j} \right)^2}_{\xrightarrow[n\to\infty]{} \langle W \rangle_T = T}$$

$$\lim_{n\to\infty} I_n = \frac{1}{2} W_T^2 - \frac{1}{2} T$$

$$\rightsquigarrow \boxed{\int_0^T W_t \, dW_t = \frac{1}{2} W_T^2 - \frac{1}{2} T} \quad \text{a.s.}$$

- in general the above Itô integral becomes:

$$\boxed{I_t(W) = \int_0^t W_s \, dW_s = \frac{1}{2} W_t^2 - \frac{1}{2} t}$$

- let's contrast this integral with regular calculus.

$$\int_0^t g(s) \, d\, g(s) = \int_0^t g(s) \, g'(s) \, ds = \frac{1}{2} g^2(s) \Big|_0^t$$

- so if $g(0) = 0$ we get:

$$\int_0^t g(s) \, dg(s) = \frac{1}{2} g^2(s)$$

- the extra term $-\frac{1}{2} t$ comes from the quadratic var. of Brownian motion, since BM is not differentiable we cannot use the regular calculus.

Itô - Doeblin Formula : (the Chain Rule of Stoch. Calculus)

① • suppose $f \in \mathcal{C}^2$

$$\boxed{\begin{aligned} d f(W_t) &= f'(W_t) \, dW_t + \frac{1}{2} f''(W_t) \, dt \\ \Longleftrightarrow f(W_t) - f(W_0) &= \int_0^t f'(W_s) \, dW_s + \frac{1}{2} \int_0^t f''(W_s) \, ds \end{aligned}}$$

Itô integral   Lebesgue integral

More generally, if we replace W with M (local) martingale with continuous sample paths, Itô formula becomes

$$\boxed{f(M_t) - f(M_0) = \int_0^t f'(M_s)\, dM_s + \frac{1}{2}\int_0^t f''(M_s)\, d\langle M\rangle_s}$$

example: • $f(x) = x^2 \rightsquigarrow f'(x) = 2x \rightsquigarrow f''(x) = 2$

$$W_t^2 - W_0^2 = \int_0^t 2W_s\, dW_s + \frac{1}{2}\int_0^t 2\, ds$$

$$\Rightarrow W_t^2 = 2\int_0^t W_s\, dW_s + t \Rightarrow \boxed{\int_0^t W_s\, dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t}$$

Idea of the proof.

- consider the partition: $0 = t_0 < t_1 \dots < t_n = T$ and perform a Taylor expansion

$$f(W_T) - f(W_0) = \sum_{j=0}^{n-1}\left(f(W_{t_{j+1}}) - f(W_{t_j})\right)$$

$$\simeq \underbrace{\sum_{j=0}^{n-1} f'(W_{t_j})(W_{t_{j+1}} - W_{t_j})}_{\downarrow\ \int_0^T f'(W_t)\, dW_t} + \frac{1}{2}\underbrace{\sum_{j=0}^{n-1} f''(W_{t_j})(W_{t_{j+1}} - W_{t_j})^2}_{\downarrow\ \int_0^T f''(W_t)\, dt}$$

- here is an important application of Itô formula:

Th Let M be a (local) martingale with continuous paths such that $\langle M\rangle_t = t$. Then M is a Brownian Motion.

Proof: • we have to show that:

- M has independent increments

- M has independent increments
- $M_t - M_s \sim N(0, t-s)$ for $s<t$

$\leadsto$ both these claims will follow as soon as it is shown that $E\left[e^{i\theta(M_t - M_s)} \,\middle|\, \mathcal{F}_s \right] = e^{-\frac{1}{2}\theta^2(t-s)}$ $(\forall)\theta$

- take $f(x) = e^{i\theta x} \to f'(x) = i\theta e^{i\theta x} \longrightarrow f''(x) = -\theta^2 e^{i\theta x}$

$\leadsto$ according to the Itô formula:

$$f(M_t) - f(M_s) = \int_s^t f'(M_u)\, dM_u + \frac{1}{2}\int_s^t f''(M_u)\, d\langle M\rangle_u$$

$$\Rightarrow e^{i\theta M_t} = e^{i\theta M_s} + \int_s^t i\theta e^{i\theta M_u}\, dM_u - \frac{1}{2}\int_s^t \theta^2 e^{i\theta M_u}\, du$$

$$\Rightarrow E\left[e^{i\theta(M_t - M_s)} \,\middle|\, \mathcal{F}_s \right] = 1 + i\theta e^{-i\theta M_s}\, \overbrace{E\left[\int_s^t e^{i\theta M_u}\, dM_u \,\middle|\, \mathcal{F}_s \right]}^{* \ =0}$$

$$- \frac{1}{2}\theta^2 e^{-i\theta M_s} \cdot E\left[\int_s^t e^{i\theta M_u}\, du \,\middle|\, \mathcal{F}_s \right]$$

$(*)$

$I_t = \int_0^t e^{i\theta M_u}\, dM_u$ is a martingale

$\Rightarrow E[I_t \mid \mathcal{F}_s] = I_s$

but also $I_t = I_s + \int_s^t e^{i\theta M_u}\, dM_u$ $\Rightarrow$

$\Rightarrow E\left[\int_s^t e^{i\theta M_u}\, dM_u \,\middle|\, \mathcal{F}_s \right] = 0$ a.s.

$$\Rightarrow \underbrace{E\left[e^{i\theta(M_t - M_s)} \,\middle|\, \mathcal{F}_s \right]}_{\triangleq\ g(t)} = 1 - \frac{1}{2}\theta^2 \int_s^t \underbrace{E\left[e^{i\theta(M_u - M_s)} \,\middle|\, \mathcal{F}_s \right]}_{g(u)} du$$

- $g(t) = 1 - \frac{1}{2}\theta^2 \int^t g(u)\, du$ $\left(g'(t) = -\frac{1}{2}\theta^2 g(t) \right)$

$$\cdot \quad g(t) = 1 - \frac{1}{2}\theta^2 \int_s^t g(u)\,du \qquad \left(g'(t) = -\frac{1}{2}\theta^2 g(t) \right)$$

this equation has an unique solution

$$g(t) = e^{-\frac{1}{2}\theta^2 (t-s)}$$

$$\Rightarrow \quad E\left[e^{i\theta(M_t - M_s)} \,\middle|\, \mathcal{F}_s \right] = e^{-\frac{1}{2}\theta^2 (t-s)}$$

$\Rightarrow$ • $M_t - M_s$ is independent of $\mathcal{F}_s$

• $M_t - M_s \sim N(0, t-s)$

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② Ito formula for $f(t,x)$

• assume the partial derivatives exist and are cont.

$$(f_t, f_x, f_{xx}) \qquad f_{xt} \qquad f_{tt}$$

$$\boxed{f(t, W_t) - f(0, W_0) = \int_0^t f_t(s, W_s)\,ds + \int_0^t f_x(s, W_s)\,dW_s + \frac{1}{2}\int_0^t f_{xx}(s, W_s)\,dt}$$

· proof is based on the Taylor expansion, once again.

Sketch of the proof :

. on the partition $0 = t_0 < t_1 < t_2 \dots < t_n = T$

$$f(t_{j+1}, W_{t_{j+1}}) - f(t_j, W_{t_j}) = f_t(t_j, W_{t_j})(t_{j+1} - t_j) + f_x(t_j, W_{t_j})(W_{t_{j+1}} - W_{t_j})$$

$$+ \frac{1}{2} f_{xx}(t_j, W_{t_j})(W_{t_{j+1}} - W_{t_j})^2 + f_{tx}(t_j, W_j)(t_{j+1} - t_j)(W_{t_{j+1}} - W_{t_j}) +$$

$$+\frac{1}{2} f_{tt}(t_j, W_{t_j})(t_{j+1}-t_j)^2 \quad : \quad \text{for } j=\overline{0,n-1}$$

$$\Rightarrow f(T, W_T) - f(0, W_0) = \sum_{j=0}^{n-1} f_t(t_j, W_{t_j})(t_{j+1}-t_j) +$$

$$+ \sum_{j=0}^{n-1} f_x(t_j, W_{t_j})(W_{t_{j+1}} - W_{t_j}) + \frac{1}{2}\sum_{j=0}^{n-1} f_{xx}(t_j, W_{t_j})(W_{t_{j+1}} - W_{t_j})^2$$

$$+ \underbrace{\sum_{j=0}^{n-1} f_{tx}(t_j, W_{t_j})(t_{j+1}-t_j)(W_{t_{j+1}} - W_{t_j})}_{\to 0} + \underbrace{\frac{1}{2}\sum_{j=0}^{n-1} f_{tt}(t_j, W_{t_j})(t_{j+1}-t_j)^2}_{\to 0}$$

since $dt \cdot dW_t = 0$ since $dt \cdot dt = 0$

- So as $n \to \infty$

$$f(T, W_T) - f(0,0) = \int_0^T f_t(s, W_s)\, ds + \int_0^T f_x(s, W_s)\, dW_s + \frac{1}{2}\int_0^T f_{xx}(s, W_s)\, ds$$

- in differential form :

$$\boxed{df(t, W_t) = f_t(t, W_t)\, dt + f_x(t, W_t)\, dW_t + \frac{1}{2} f_{xx}(t, W_t)\, dt}$$

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3 Itô formula for semimartingales :

Let X_t be a semimartingale $\qquad V_t = A_t - B_t$

(i.e $X_t = X_0 + M_t + V_t$, where

M_t = cont., local martingale

V_t = cont, bounded variation process)

Then for every function $f \in C^2(\mathbb{R})$ we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dM_s + \int_0^t f'(X_s)\, dV_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle M\rangle_s$$

Proof: (in differential form) $\quad dX_t = dM_t + dV_t$

$$\boxed{df(X_t) = f'(X_t)\,dX_t + \tfrac{1}{2} f''(X_t)\,d\langle X\rangle_t}$$

$$= f'(X_t)(dM_t + dV_t) + \tfrac{1}{2} f''(X_t)(dM_t + dV_t)\cdot(dM_t + dV_t)$$

$$\rightsquigarrow \begin{cases} dM_t \cdot dM_t = d\langle M\rangle_t \\ dM_t \cdot dV_t = 0 & \text{(always cross-variation of a mart with a process of bdded var} = 0) \qquad (dW_t \cdot dt = 0) \\ dV_t \cdot dV_t = 0 \end{cases}$$

- $d\langle X\rangle_t = (dM_t + dV_t)\cdot(dM_t + dV_t)$

$$= dM_t \cdot dM_t + 2\underbrace{dM_t \cdot dV_t}_{=0} + \underbrace{dV_t \cdot dV_t}_{=0}$$

$$\Rightarrow \boxed{df(X_t) = f'(X_t)\,dM_t + f'(X_t)\,dV_t + \tfrac{1}{2} f''(X_t)\,d\langle M\rangle_t}$$

$$df(t, X_t) = f_t(t, X_t)\,dt + f_x(t, X_t)\,dX_t + \tfrac{1}{2} f_{xx}(t, X_t)\,d\langle X\rangle_t$$

Remark: The textbook uses the following terminology:

- suppose X_t satisfies the following eq:

$$X_t = X_0 + \underbrace{\int_0^t \Delta_u \, dW_u}_{\text{local martingale}} + \underbrace{\int_0^t \theta_u \, du}_{\text{bounded variation process}}$$

$\rightsquigarrow$ X_t = Itô process

- as we can see an Itô process is a SEMIMARTINGALE

(we shall see later that any semimartingale is als...

(we shall see later that any semimartingale is also an Ito process

Ito process

- in differential form : $\boxed{dX_t = \Delta_t dW_t + \theta_t dt}$

- one can generalize a stochastic integral w.r.t. X_t as follows:

$$\boxed{\int_0^t \Gamma_u dX_u = \int_0^t \Gamma_u \Delta_u dW_u + \int_0^t \Gamma_u \theta_u du}$$

- also $\langle X \rangle_t = dX_t \cdot dX_t = \int_0^t \Delta_u^2 du$

<u>example</u> : Generalized Brownian Motion :

- let $X_t = \int_0^t \sigma_s dW_s + \int_0^t (\alpha_s - \frac{1}{2}\sigma_s^2) ds$

- then $dX_t = \sigma_t dW_t + (\alpha_t - \frac{1}{2}\sigma_s^2) dt$

$$d\langle X \rangle_t = dX_t \cdot dX_t = \sigma_t^2 dt$$

$\rightsquigarrow$ consider the following process : $S_t = S_0 e^{X_t}$

$f(x) = S_0 e^x \rightsquigarrow f'(x) = f''(x) = f(x)$

$$\begin{aligned} dS_t &= d f(X_t) \\ &= f'(X_t) dX_t + \frac{1}{2} f''(X_t) dX_t \cdot dX_t \\ &= S_t (\sigma_t dW_t + (\alpha_t - \frac{1}{2}\sigma_t^2) dt) + \frac{1}{2} S_t \sigma_t^2 dt \\ &= S_t \sigma_t dW_t + S_t \alpha_t dt \end{aligned}$$

$$\Rightarrow \boxed{dS_t = S_t \sigma_t dW_t + S_t \alpha_t dt}$$

$\rightarrow$ let's consider now the discounted version of S. i.e

$\longrightarrow$ let's consider now the discounted version of S_t i.e

$$e^{-rt} S_t \quad : \quad f(t,x) = e^{-rt} x$$

$$f_t = -r f \qquad f_x = e^{-rt} \qquad f_{xx} = 0$$

$$d(e^{-rt} S_t) = -r e^{-rt} S_t \, dt + e^{-rt} \cdot dS_t$$

$$= -r e^{-rt} S_t \, dt + e^{-rt} S_t \sigma_t \, dW_t + e^{-rt} S_t \alpha_t \, dt$$

$$\Rightarrow d(\underbrace{e^{-rt} S_t}) = e^{-rt} S_t \sigma_t \, dW_t + e^{-rt} S_t (\alpha_t - r) \, dt$$

$\tilde{S}_t$ = discounted stock price.

$$\boxed{d(\tilde{S}_t) = \tilde{S}_t \sigma_t \, dW_t + \tilde{S}_t (\alpha_t - r) \, dt}$$

example : · let M be a local martingale with cont. paths

· consider the process $Z_t = \exp\Big[\underbrace{M_t - \tfrac{1}{2} \langle M \rangle_t}_{\tilde{M}_t}\Big]$

$\tilde{M}_t = M_t - \frac{1}{2} \langle M \rangle_t$ = semi-martingale

$f(x) = e^x = f'(x) = f''(x) \qquad Z_t = e^{\tilde{M}_t}$

$$dZ_t = d f(\tilde{M}_t)$$

$$= f'(\tilde{M}_t) \, d\tilde{M}_t + \tfrac{1}{2} f''(\tilde{M}_t) \, d\langle \tilde{M} \rangle_t$$

$$= Z_t \left(dM_t - \tfrac{1}{2} d\langle M \rangle_t \right) + \tfrac{1}{2} Z_t \cdot d\langle M \rangle_t$$

$$= Z_t \, dM_t$$

$\Rightarrow dZ_t = Z_t \, dM_t$
· since $Z_0 = 1$ $\Bigg\} \Rightarrow \boxed{Z_t = 1 + \int_0^t Z_s \, dM_s}$

· how do we know that Z_t is a martingale ?

- how do we know that Z_t is a martingale?

Karatzas & Shreve

Brownian Motion