Three definitions of the Poisson Process

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① Definition : $\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity λ

if (i) $N(0)=0$

(ii) for every $m\geq 2$, $0\leq t_0 < t_1 < \dots < t_m$ the increments

$N(t_1)-N(t_0), N(t_2)-N(t_1), \dots\dots, N(t_m)-N(t_{m-1})$

are independent

(iii) for each $s,t\geq 0$ the random variable

$N(t+s)-N(t)$ has a Poisson distribution with parameter

λs, i.e

$$P\Big(N(t+s)-N(t)=k\Big)=\frac{(\lambda s)^k}{k!}\cdot e^{-\lambda s} \qquad k=0,1,2,\dots$$

- very similar definition with the definition of the Brownian Motion $\Big($ for BM : $\underbrace{W(t+s)-W(t)\sim N(0,s)}_{\text{comes from CLT}}$ $\Big)$

- where does the Poisson distribution come from?

$\leadsto$ fix a large n, consider a random walk $W_k^{(n)}=\sum_{i=1}^{m} X_i^{(n)}$

where $X_1^{(n)}, X_2^{(n)}, \dots$ i.i.d Bernoulli r.v with parameter

$p_n\in(0,1)$.

$\leadsto$ let $N^{(n)}(t)=W^{(n)}_{[tn]}$ $\Rightarrow$ $N^{(n)}(t+s)-N^{(n)}(t)=W^{(n)}_{[(t+s)n]}-W^{(n)}_{[tn]}$

is the number of "successes" in independent sequence of trials that occured after $[tn]$ and before $[(t+s)n]$-th trial

$\Rightarrow N^{(n)}(t+s) - N^{(n)}(t) \sim \text{Bin}\left([(t+s)n] - [tn], p_n\right)$

- assume that $p_n \cdot n \longrightarrow \lambda$ as $n \to \infty$

$p_n \simeq \frac{\lambda}{n}$

$\Rightarrow N^{(n)}(t+s) - N^{(n)}(t) \sim \text{Poisson}(\lambda s)$

(this is a consequence of the following th.)

<u>Theorem</u> : [Poisson approximation to binomial distribution]

Let $X_1^{(n)}, X_2^{(n)}, \ldots$ iid Bernoulli r.v with parameter p_n and $p_n \cdot n \longrightarrow \lambda$ as $n \to \infty$. Then for $s > 0$

$$P\left(\sum_{i=1}^{[sn]} X_i^{(n)} = k\right) \longrightarrow \frac{(\lambda s)^k}{k!} e^{-\lambda s} \qquad k = 0, 1, 2, \ldots$$

(2) <u>Definition</u> : The process $\{N(t)\}_{t \geq 0}$ with values in $\mathbb{N} \cup \{0\}$ is a Poisson process with intensity λ if :

(i) $N(0) = 0$

(ii) $\{N(t)\}_{t \geq 0}$ has stationary ; independent increments

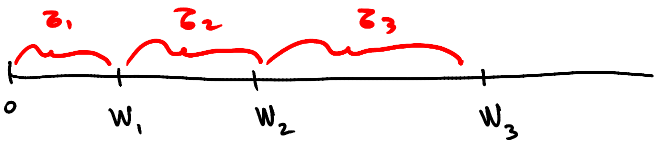
(iii) $P(N(h) = 1) = \lambda h + o(h)$ as $h \downarrow 0$

(iv) $P(N(h) = 2) = o(h)$ as $h \downarrow 0$

[Recall : a function is said to be $o(h)$ if $\lim_{h \to 0} \frac{f(h)}{h} = 0$]

(3) <u>Construction of the Poisson Process</u>

- let $z_1, z_2, \ldots$ be iid exponential random variables with parameter $\lambda > 0$



- $W_n = \sum_{i=1}^{n} z_i$ with $W_0 = 0$

- think about counting events (such as customer arrivals, defaults, etc)

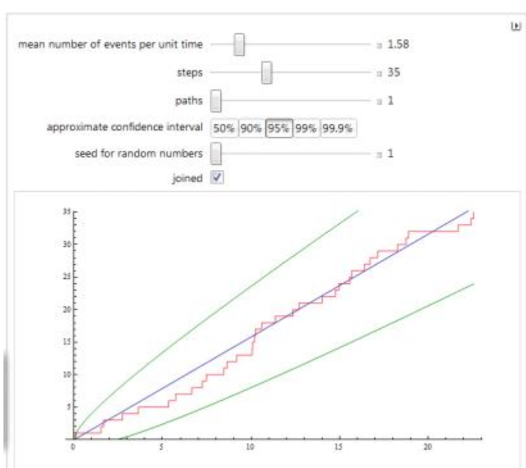
  z_i = waiting time between the arrival $(i-1)$ and i

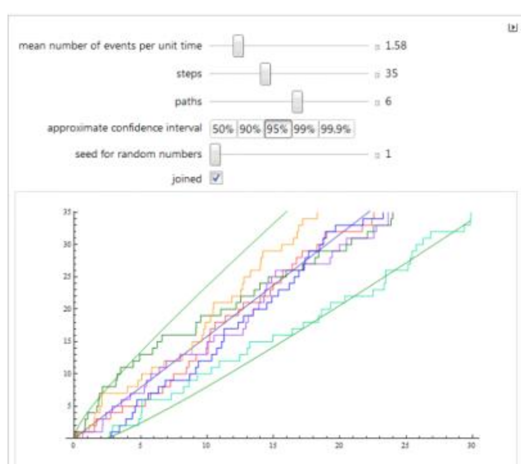
- define

$$N(t) = \begin{cases} 0 & \text{if } \quad 0 \le t < W_1 \\ 1 & \text{if } \quad W_1 \le t < W_2 \\ \vdots & \\ n & \text{if } \quad W_n \le t < W_{n+1} \\ \vdots & \end{cases}$$

$\rightarrow$ $N(t)$ is a Poisson process with intensity λ

[Simulating the Poisson Process (Wolfram demosntration)]





This Demonstration shows simulated paths of the Poisson process. You can see how the cumulative number of events increases as time lapses. You can adjust the mean number of events per unit time. The Demonstration also shows the mean of the process (the blue line) and approximate confidence intervals (the green curves). The confidence intervals are based on the normal approximation to the Poisson distribution. The Poisson process is a special case of a continuous-time Markov chain.

Remark (by Alex !!) $N(t)$ NOT A MARTINGALE !!

Is $M(t) = N(t) - \lambda t$ a martingale ?

$$E\big(M(t+s) \mid \text{path up to time } t \big) = M(t)$$

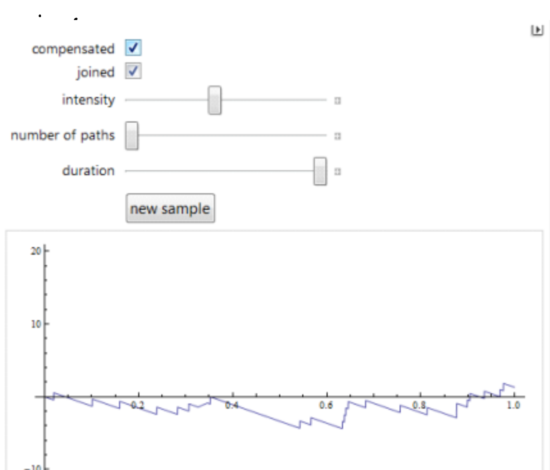
$$E\big(N(t+s) - \lambda(t+s) \mid \text{path up to time } t \big) =$$

$$= E\big(N(t+s) - N(t) + N(t) \mid \text{path up to time } t \big) - \lambda(t+s)$$

$$= \underbrace{E\big(N(t+s) - N(t) \mid \text{path up to time } t \big)}_{\lambda s} + N(t) - \lambda(t+s)$$

$$= N(t) - \lambda t = M(t)$$

[The Poisson process (compensated) (Wolfram demonstration)]



$M(t) = N(t) - \lambda t$ compensated process

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Remark: $N(t)$ is right continuous : $\lim_{s \downarrow t} N(s) = N(t)$

$s \downarrow t$

Lemma 1: The time of the n-th jump, $W_n \sim$ Gamma distribution with parameter λn

$$g_n(s) = \mathbb{1}_{\{s>0\}} \frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s}$$

Lemma 2: $$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-(\lambda t)} \qquad k = 0, 1, 2, \dots$$

Proof: $$P(N(t) = 0) = P(W_1 > t) = e^{-\lambda t}$$

- let $k \geq 1$

$$P(N(t) = k) = P(N(t) \geq k) - P(N(t) \geq k+1)$$

$$P(N(t) \geq k) = P(W_k \leq t) = \int_0^t \frac{(\lambda s)^{k-1}}{(k-1)!} \lambda e^{-\lambda s} \, ds$$

$$P(N(t) \geq k+1) = P(W_{k+1} \leq t) = \int_0^t \frac{(\lambda s)^k}{k!} \lambda e^{-\lambda s} \, ds$$

$$= -\int_0^t \frac{(\lambda s)^k}{k!} \, d\left(e^{-\lambda s}\right) = -\frac{(\lambda s)^k}{k!} e^{-\lambda s} \Big|_0^t + \int_0^t \frac{(\lambda s)^{k-1}}{(k-1)!} \lambda e^{-\lambda s} \, ds$$

$$= P(N(t) \geq k) - \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

- from these calculations and the construction we conclude that our process satisfies (i), (iii), (iv) of the Definition 2

$\rightsquigarrow$ how about (ii) ?

(consequence of the memoryless property of the exponential distribution)

Lemma 3 : Let $\tau \sim \exp(\lambda) \Rightarrow$

$$P(\tau > t+s \mid \tau > t) = P(\tau > s)$$

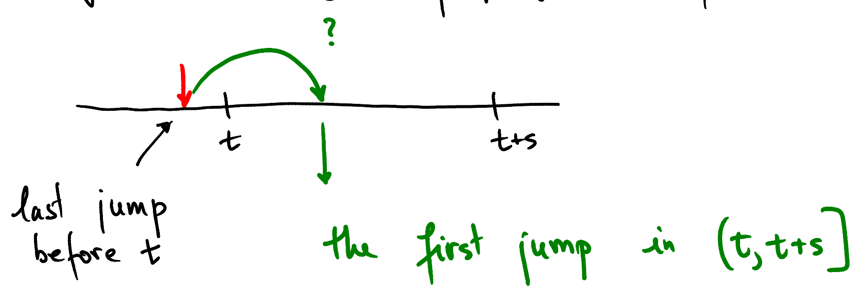
Explanation on how lemma 3 $\Rightarrow$ (ii)

· consider $N(t+s) - N(t) = \#$ of jumps in $(t, t+s]$

$\rightsquigarrow$ look at the history of the process up to time t

· the waiting times are independent, so the only relevant info from the past seems to be the time of the last jump. in $(0,t]$

· by the memoryless property this info is also irrelevant



$\rightsquigarrow$ as if we restart the clock at time t.

$N(t+s) - N(t)$ does not depend on the past and has the same distribution as $N(s) - N(0)$

(hence : independence & stationarity)

Remarks : Poisson process is the simplest example of "PURE JUMP" processes.

· it is often used when counting random events of the same nature (arrivals, # of customers, defaults, etc) nature (arrivals, # of customers, defaults, etc)

- it allows a number of modifications
  - variable rate $\lambda(t)$ intead of λ
  - compound Poisson Process

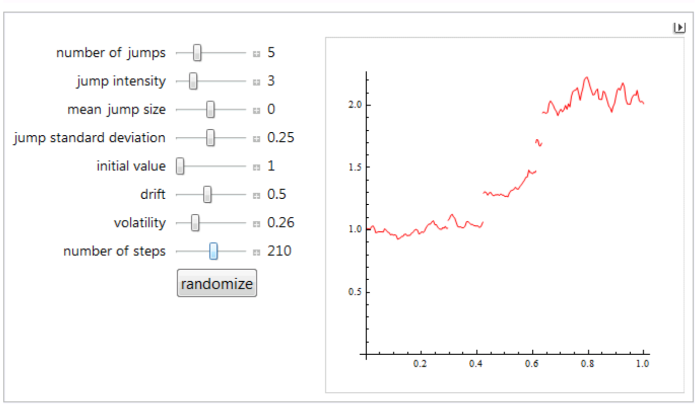
  ....

- Poisson process can be combined with GBM and give a reasonable model for the dynamics of stock prices.

$\rightsquigarrow$ see
- Sheldon Ross: "An Elementary Intoduction to Financial Math" (section 8.4)
- Shreve II: Ch. 11 (will discuss it in the Spring)

Merton's Jump Diffusion Model (Wolfram demosntration)

Merton's Jump Diffusion Model



This Demonstration displays one path of Merton's "jump diffusion" stochastic process. This process extends the notion of the standard Black–Scholes model by allowing discrete jumps in addition to a Brownian process motion as the source of randomness. The jumps occur at random times. The interarrival times of the jumps follow an exponential distribution, while the size of the jumps has a normal distribution. Setting the mean size of jumps and the standard deviation to zero (the default) yields a path of a Black–Scholes process (exponential Wiener process).

Further Properties of a Poisson Process.

① $\{N_1(t)\}_{t\geq 0}$, $\{N_2(t)\}_{t\geq 0}$ are independent, Poisson processes with intensities λ_1 and λ_2

Cf. $N(t) = N_1(t) + N_2(t) \Rightarrow \{N(t)\}$ is Poisson process

Set $N(t) = N_1(t) + N_2(t) \Rightarrow \{N(t)\}_{t\geq 0}$ is Poisson process with the intensity $\lambda_1 + \lambda_2$

② If $\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity λ.
Suppose that each jump is classified to be type I with probability p and type II with probability $1-p$

$N_1(t)$ = # of jumps of type I up to time t
$N_2(t)$ = # of jumps of type II up to time t

$\Rightarrow \{N_1(t)\}_{t\geq 0}$ is Poisson process with intensity λp

$\{N_2(t)\}_{t\geq 0}$ is Poisson process with intensity $\lambda(1-p)$

Moreover N_1, N_2 are independent !!!

③ Conditional Distribution of arrival times

Given $N(t)=n$, the arrival times $W_1, W_2, \dots, W_n$ have the same distribution as the order statistics corresponding to n, iid uniform r.v on $(0,t)$

Remark: To sample arrival times simply generate n iid r.v uniform on $(0,1)$: $U_1, U_2, \dots, U_n$, then multiply by t and finally order them :

$W_1 = t\cdot U_{(1)}$, $W_2 = t\cdot U_{(2)}$, ..., $W_n = t\cdot U_{(n)}$

where $U_{(1)} \leq U_{(2)} \leq \dots \leq U_{(n)}$

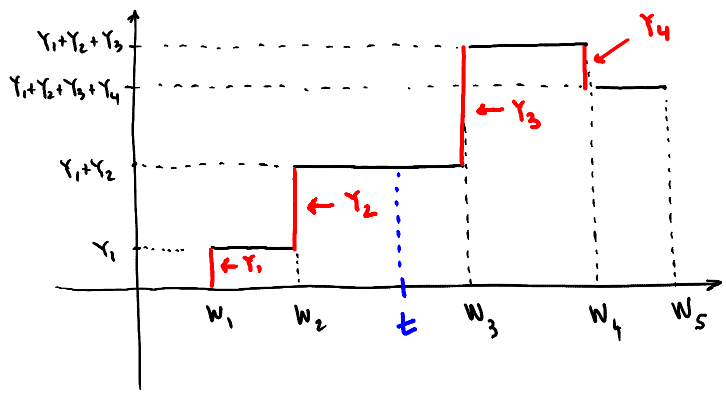
(are the ordered $U_1, U_2, \dots, U_n$)

Compound Poisson Process :

- $\{N(t)\}_{t\geq 0}$ Poisson process with intensity λ
- $Y_1, Y_2, \dots$, iid random variables independent of N

The Compound Poisson Process is defined as : $\{Q(t)\}_{t\geq 0}$

- $Q(t) = 0$ if $N(t) = 0$
- $Q(t) = \sum_{i=1}^{N(t)} Y_i$ if $N(t) \geq 1$



- say $W_2 \leq t < W_3 \Rightarrow N(t) = 2 \Rightarrow Q(t) = \sum_{i=1}^{2} Y_i$

$\rightsquigarrow$ this process has stationary and independent increments
but the distribution of $\underbrace{Q(t+s) - Q(t)}_{\text{depends on the distr. of } Y_1}$ is not Poisson !!!

- $\mathbb{E}\, Q(t) = \mathbb{E}\Big[\, \mathbb{E}\big[\, Q(t) \mid N(t) \big] \Big] = \mathbb{E}\big[(\mathbb{E}Y_1) \cdot N(t) \big] = \lambda t \;\mathbb{E}(Y_1)$
- $\operatorname{Var} Q(t) = \mathbb{E}\Big[\operatorname{Var}\big(Q(t) \mid N(t) \big) \Big] + \operatorname{Var}\Big[\mathbb{E}\big(Q(t) \mid N(t) \big) \Big]$

$$= \mathbb{E}\left[N(t) \cdot Var(Y_1) \right] + Var\left(N(t) \cdot \mathbb{E}(Y_1) \right)$$

$$= Var(Y_1) \cdot \mathbb{E}(N(t)) + \left(\mathbb{E}(Y_1)\right)^2 \cdot Var(N(t))$$

$$= Var(Y_1)\ \lambda t + \left(\mathbb{E}(Y_1)\right)^2 \cdot \lambda t$$

$$= \left(\mathbb{E}(Y_1^2) - (\mathbb{E}(Y_1))^2\right) \lambda t + \left(\mathbb{E}(Y_1)\right)^2 \lambda t$$

$$= \mathbb{E}(Y_1^2) \cdot \lambda t$$

- $\mathbb{E}\, e^{uQ(t)} = \mathbb{E}\left[\mathbb{E}\left[e^{uQ(t)} \middle| N(t) \right] \right] =$

$$= \mathbb{E}\left[\mathbb{E}\left[e^{u \cdot \sum_{i=1}^{N(t)} Y_i} \middle| N(t) \right] \right]$$

$$= \mathbb{E}\left[\left[\varphi_{Y_1}(u) \right]^{N(t)} \right]$$

$$= \sum_{n=0}^{\infty} \varphi_{Y_1}(u)^n \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\left(\varphi_{Y_1}(u) \cdot \lambda t\right)^n}{n!}$$

$$= e^{-\lambda t} \cdot e^{\varphi_{Y_1}(u) \cdot \lambda t} \qquad \varphi_Y(u) = \mathbb{E}\, e^{uY}$$

A pricing model with jumps

- $Y_1, Y_2, \ldots$ iid independent of $N(t)$
- $Q(t)$: compound Poisson process
- r = interest rate (cont)
- $S(t)$ = stock price $\qquad S^*(t) = GBM(\mu, \sigma)$

$$\boxed{S(t) = S^*(t) \cdot e^{Q(t)}}$$

$$\boxed{S(t) = S^*(t)\cdot e^{Q(t)}}$$

. to price any security we need the risk neutral measure !!

$M(t) = e^{-rt}\, S(t)$ discounted stock price.

$$\tilde{E}\left[M(t+s) \mid \text{path up to time } t \right] \overset{?}{=} M(t)$$

$$\tilde{E}\left[e^{-r(t+s)}\, S(t+s) \mid \dots \right] =$$

$$= \tilde{E}\left[e^{-r(t+s)}\, S^*(t+s)\cdot e^{Q(t+s)} \mid \dots \right] =$$

$$= \tilde{E}\left[e^{-r(t+s)}\, \frac{S^*(t+s)}{S^*(t)} \cdot S^*(t) \cdot e^{Q(t+s)-Q(t)} \cdot e^{Q(t)} \mid \dots \right]$$

$$= e^{-r(t+s)}\, S^*(t)\cdot e^{Q(t)}\, \tilde{E}\left[\frac{S^*(t+s)}{S^*(t)} \cdot e^{Q(t+s)-Q(t)} \right]$$

$$= e^{-r(t+s)}\, \underbrace{S^*(t)\, e^{Q(t)}}_{S(t)}\, \underbrace{\tilde{E}\left[\frac{S^*(t+s)}{S^*(t)} \right]} \cdot \underbrace{\tilde{E}\left[e^{Q(t+s)-Q(t)} \right]}_{E\, e^{Q(s)}}$$

$$\frac{S^*(t+s)}{S^*(t)} = e^{\mu^* s + \sigma^* \sqrt{s}\, Z}$$

$$= e^{-rs} \cdot M(t) \cdot e^{\mu^* s + \frac{\sigma^{*2}}{2} s} \cdot e^{\lambda s \left(\varphi_{Y_1}(1) - 1 \right)}$$

we need : $-rs + \mu^* s + \frac{\sigma^{*2}}{2} s + \lambda s \left(\varphi_Y(1) - 1 \right) = 0$

we need : $-rs + \mu^* s + \frac{\sigma^{*2}}{2} s + \lambda s \left(\varphi_{Y_1}(1) - 1 \right) \overset{!}{=} 0$

$$\begin{cases} \sigma^* = \sigma \\ \mu^* = r - \frac{\sigma^2}{2} + \lambda - \lambda \underbrace{E e^{Y_1}}_{\varphi_{Y_1}(1)} \end{cases}$$

- security payoff $f(S(t))$
- price : $e^{-rt} \tilde{E} f(S(t))$

  where $S(t) = S(0)\, e^{\mu^* t + \sigma\sqrt{t}\, Z + Q(t)}$