Girsanov's Theorem

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① Previously seen : change of measure

$(\Omega, \mathcal{F}, \mathbb{P})$: probability space

Z : nonnegative random variable, such that $\mathbb{E}Z = 1$

- based on Z we create a new prob. measure $\tilde{\mathbb{P}}$ s.t.

$$\tilde{\mathbb{P}}(A) = \mathbb{E}(\mathbb{1}_A Z) = \int_A Z(\omega)\, d\mathbb{P}(\omega)$$

- if X : random variable : $\boxed{\tilde{\mathbb{E}}(X) = \mathbb{E}(ZX)}$

  (expectation under $\tilde{\mathbb{P}}$)

- if moreover $\mathbb{P}(Z>0)=1$: $\mathbb{P} \sim \tilde{\mathbb{P}}$ and $\boxed{\mathbb{E}(X) = \tilde{\mathbb{E}}\left(\frac{1}{Z}X\right)}$

Z : Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ w.r.to $\mathbb{P}$

Notation : $Z(\omega) = \dfrac{d\tilde{\mathbb{P}}(\omega)}{d\mathbb{P}(\omega)}$ or $d\tilde{\mathbb{P}}(\omega) = Z(\omega)\, d\mathbb{P}(\omega)$

$$\frac{1}{Z(\omega)} d\tilde{\mathbb{P}}(\omega) = d\mathbb{P}(\omega)$$

- in particular if $X \sim N(0,1)$ under $\mathbb{P}$ and

$$Z = \exp\left\{-\theta X - \tfrac{1}{2}\theta^2\right\} \quad \text{for some } \theta \in \mathbb{R} \qquad (\mathbb{E}Z = 1)$$

$\Rightarrow Y = X + \theta \sim N(0,1)$ under $\tilde{\mathbb{P}}$

(that is : $\tilde{\mathbb{E}}Y = 0$; while $\mathbb{E}Y = \mathbb{E}X + \theta = \theta$)

② Now, we want to perform a similar change of measure in order to change the mean, but this time for a whole process rather than a single variable.

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$(\Omega, \mathcal{F}, \mathbb{P})$ = prob. space ; $\{\mathcal{F}_t\}$ = filtration

$0 \le t \le T$: (T = fixed final time)

- suppose $Z > 0$ a.s. such that $\mathbb{E}Z = 1$
- define $Z_t = \mathbb{E}[Z \mid \mathcal{F}_t]$ $0 \le t \le T$. (martingale) : $Z_t(\omega) = \left.\dfrac{d\tilde{\mathbb{P}}(\omega)}{d\mathbb{P}(\omega)}\right|_{\mathcal{F}_t}$

(Radon - Nikodym derivative process)

Bayes' formula : if Y is $\mathcal{F}_t$ - measurable ; random variable

$$\Rightarrow \boxed{\tilde{\mathbb{E}}(Y \mid \mathcal{F}_s) = \frac{1}{Z_s}\mathbb{E}\left[Z_t Y \mid \mathcal{F}_s\right]} \qquad \underline{s < t}$$

- in particular, if $s = 0$: $\boxed{\tilde{\mathbb{E}}(Y) = \mathbb{E}(Z_t Y)}$

Girsanov's theorem (one dimension) on $\boxed{0 \le t \le T}$

Let W_t be a Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}_t\}$ be a filtration of this B.M. Let θ_t be an adapted process.

Define
$$\begin{cases} Z_t = \exp\left\{-\int_0^t \theta_u \, dW_u - \frac{1}{2}\int_0^t \theta_u^2 \, du\right\} \\ \tilde{W}_t = W_t + \int_0^t \theta_u \, du . \end{cases}$$

$Z = \exp\left\{-\theta X - \frac{1}{2}\theta^2\right\}$

$Y = X + \theta$

Assume $\mathbb{E}\int_0^T \theta_u^2 Z_u^2 \, du < \infty$.

Then $\mathbb{E}Z_T = 1$ and $\tilde{W}_t$ is Brownian Motion under $\tilde{\mathbb{P}}$, where $\tilde{\mathbb{P}}$ is defined as : $d\tilde{\mathbb{P}}(\omega) = Z_T(\omega)\, d\mathbb{P}$.

Proof: $\rightsquigarrow$ lets investigate the properties of $\tilde{W}_t$

- $\tilde{W}_0 = 0$ | (assume for the moment

Proof: $\rightsquigarrow$ lets investigate the properties of $\tilde{W}_t$

- $\tilde{W}_0 = 0$
- $\tilde{W}_t$ has continuous paths
- $\langle \tilde{W} \rangle_t = \langle W \rangle_t = t \quad (\forall)\ t$

$\boxed{\text{(assume for the moment that } \mathbb{E} Z_T = 1 \text{ so that the change of measure is well defined)}}$

$$\left(d\tilde{W}_t \cdot d\tilde{W}_t = (dW_t + \theta_t dt) \cdot (dW_t + \theta_t dt) = \underbrace{dW_t \cdot dW_t}_{=dt} + 2 \underbrace{dt \cdot dW_t}_{=0} + \underbrace{dt \cdot dt}_{=0} \right)$$

- $\tilde{W}_t$ is martingale under $\tilde{\mathbb{P}}$

- to check the martingale property under $\tilde{\mathbb{P}}$ notice that

$$\tilde{\mathbb{E}}(\tilde{W}_t \mid \mathcal{F}_s) = \frac{1}{Z_s} \mathbb{E}(Z_t \tilde{W}_t \mid \mathcal{F}_s) \qquad \text{(Bayes formula)} \qquad = \tilde{W}_s$$

$$\mathbb{E}(Z_t \tilde{W}_t \mid \mathcal{F}_s) = Z_s \tilde{W}_s$$

$\rightsquigarrow$ $\tilde{W}_t$: martingale under $\tilde{\mathbb{P}} \iff Z_t \tilde{W}_t$: martingale under $\mathbb{P}$

$\rightsquigarrow$ using Ito lemma we can check

- $d(Z_t \tilde{W}_t) = Z_t d\tilde{W}_t + \tilde{W}_t dZ_t + d\tilde{W}_t \cdot dZ_t \qquad$ (product rule)

- $Z_t = \exp\Big\{ \underbrace{- \int_0^t \theta_u dW_u - \frac{1}{2} \int_0^t \theta_u^2 du}_{X_t} \Big\} \quad : \quad f(x) = e^x \quad : \quad Z_t = f(X_t)$

$$\rightsquigarrow dZ_t = d f(X_t) = e^{X_t} dX_t + \frac{1}{2} e^{X_t} d\langle X \rangle_t$$
$$= Z_t \left(-\theta_t dW_t - \frac{1}{2} \theta_t^2 dt \right) + \frac{1}{2} Z_t \theta_t^2 dt$$
$$= -Z_t \theta_t dW_t$$

$$\rightsquigarrow d\tilde{W}_t \cdot dZ_t = (dW_t + \theta_t dt) \cdot (-Z_t \theta_t dW_t) = -Z_t \theta_t dt$$

$$\Rightarrow d(Z_t \cdot \tilde{W}_t) = Z_t \cdot d\tilde{W}_t + \tilde{W}_t \cdot dZ_t + d\tilde{W}_t \cdot dZ_t$$

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$$= Z_t\left(dW_t + \theta_t dt\right) + \tilde{W}_t\left(-Z_t\theta_t dW_t\right) - Z_t\theta_t dt$$

$$= Z_t dW_t + Z_t\theta_t dt - \tilde{W}_t Z_t \theta_t dW_t - Z_t\theta_t dt$$

$$= Z_t\left[1 - \tilde{W}_t\theta_t\right] dW_t \quad \rightsquigarrow \text{martingale under } \mathbb{P}$$

$\Rightarrow$ $Z_t \cdot \tilde{W}_t$ is martingale under $\mathbb{P}$

Conclusion: $\tilde{W}_t$ is continuous martingale under $\tilde{\mathbb{P}}$ with $d\langle \tilde{W} \rangle_t = dt$, therefore $\tilde{W}_t$ is Brownian Motion under $\tilde{\mathbb{P}}$ on the time interval $[0,T]$

- the assumption $E\int_0^T \theta_u^2 Z_u^2\, du < \infty$ guarantees that the solution of the following SDE is a martingale

$$\begin{cases} dZ_t = -Z_t\theta_t\, dW_t \quad \text{or otherwise} \qquad Z_0 = 1 \\ Z_t = 1 - \int_0^t Z_u\theta_u\, dW_u \end{cases}$$

$\rightsquigarrow$ this stochastic integral is well defined for $\int_0^T Z_u^2\theta_u^2\, du < \infty$ and as such is a LOCAL MARTINGALE

$\rightsquigarrow$ if moreover $E\int_0^T Z_u^2\theta_u^2\, du < \infty \Rightarrow Z_t =$ MARTINGALE
(square integrable)

$$\Downarrow$$

$$\boxed{E Z_T = 1}$$

Remark: The fact that Z is a martingale is essential for the Girsanov's theorem to work.

- Unless Z_t is a martingale the change of measure is not possible.

- Recall that for a change of measure you need the Radon-Nikodým derivative Z_T to satisfy $\underline{\underline{\mathbb{E} Z_T = 1}}$

Theorem: Let M be a continuous local martingale and define $Z_t = \exp\{ M_t - \frac{1}{2} \langle M \rangle_t \}$.

If $\mathbb{E}\left[\exp\left\{ \frac{1}{2} \langle M \rangle_t \right\} \right] < \infty$ for $0 \le t < \infty$,
then $\mathbb{E} Z_t = 1$; $0 \le t < \infty$.

Corollary [Novikov condition]
If θ_t is an adapted process, satisfying

$$\mathbb{E}\left[\exp\left\{ \frac{1}{2} \int_0^T \theta_u^2 \, du \right\} \right] < \infty$$

Then $Z_t = \exp\left\{ -\int_0^t \theta_u \, dW_u - \frac{1}{2} \int_0^t \theta_u^2 \, du \right\}$ is martingale!

- Novikov's condition is a sufficient condition for the Girsanov's theorem, easier to check than

$$\mathbb{E}\left(\int_0^T \theta_u^2 Z_u^2 \, du \right) < \infty$$