Markov Property

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Definition: $\{X(t)\}_{t\geq 0}$ is a Markov process if for every non-negative Borel measurable function f there is another Borel measurable function g s.t.

$$E\big(f(X(t))\,|\,\mathcal{F}_s\big) = g(X(s)) \quad \text{for all } s\leq t$$

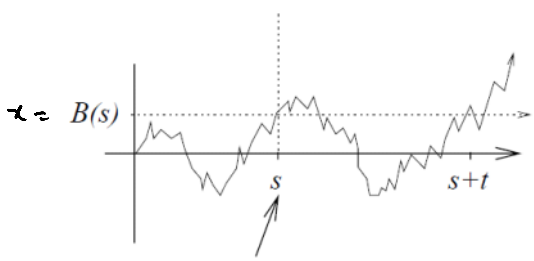
- The main point is that given the whole past of the process up to time s, its future behaviour depends only on the value of the process at time s.

Th Let $\{B(t)\}_{t\geq 0}$ be a Brownian Motion and $\{\mathcal{F}_t\}_{t\geq 0}$ be its filtration $\Rightarrow$ $\{B(t)\}_{t\geq 0}$ is a Markov process.

Proof: $$E\big(f(B(t))\,|\,\mathcal{F}_s\big) = E\Big(f\big(\underbrace{B(t)-B(s)}_{\text{indep. of } \mathcal{F}_s} + \underbrace{B(s)}_{\mathcal{F}_s \text{ meas.}}\big)\,\Big|\,\mathcal{F}_s\Big)$$

$$= \int f(y+B(s))\cdot\frac{1}{\sqrt{2\pi(t-s)}}\cdot e^{-\frac{y^2}{2(t-s)}}\,dy$$

- define $g(x) = \dfrac{1}{\sqrt{2\pi(t-s)}}\displaystyle\int e^{-\frac{y^2}{2(t-s)}}\cdot f(x+y)\,dy$



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• in other words : $g(x) = \mathbb{E}\left[f(\underbrace{B(s+t) - B(s)}_{\sim N(0,t)} + x) \right]$

• but $B(t) \sim N(0,t)$ so we can write $g(x)$ as.

$$g(x) = \mathbb{E}\left[f(B(t) + x) \right] = \mathbb{E}^x\left[f(B(t)) \right]$$

where $\mathbb{E}^x$ corresponds to the probability measure such that

$$\mathbb{P}^x\big(B(0) = x \big) = 1$$

[$\Rightarrow$ the distribution of $B(t)$ under $\mathbb{P}^x$ is the same as the distribution of $x + B(t)$ under $\mathbb{P}$]

Strong Markov Property:

(Th) Let $\{B(t)\}$ be a Brownian Motion and $\{\mathcal{F}_t\}_{t\geq 0}$ its filtration.
Let τ be a stopping time $\big($i.e $\{\tau \leq t\} \in \mathcal{F}_t$ $(\forall)$ $t \geq 0\big)$
Then for every Borel meas f, there exist a Borel meas g such that

$$\mathbb{E}\left[f(B(\tau+t)) \,\middle|\, \mathcal{F}_\tau \right] = g(B(\tau)) = \mathbb{E}^{B(\tau)}\left[f(B(t)) \right]$$

Remark : $\tau + t$ is a stopping time w.r.to $\{\mathcal{F}_t\}_{t\geq 0}$

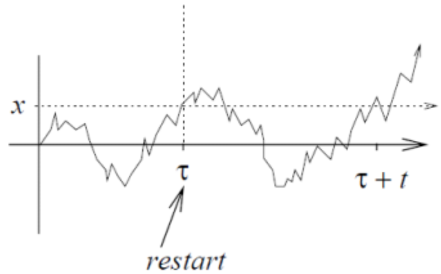
$\{\tau + t \leq u\} = \{\tau \leq u - t\} \in \mathcal{F}_{u-t} \subseteq \mathcal{F}_u$ for all $u \geq t$

• for $u < t \Rightarrow \{\tau + t \leq u\} = \phi \in \mathcal{F}_u$ for all $u < t$

ex: $\tau = \min\{t \geq 0 : B(t) = x\} \rightsquigarrow B(\tau) = x$ a.s.

SMP

$$\Rightarrow E\left[f(B(\tau+t)) \,\middle|\, \mathcal{F}_\tau\right] = g(\underbrace{B(\tau)}_{x}) = E^x\left[f(B(t))\right]$$



Transition density: $p(t,x,y)$ = prob that B.M changes from state x to y in time t.

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$$

$$\rightsquigarrow g(x) = E^x f(B(t)) = \int_{-\infty}^{+\infty} f(y)\cdot p(t,x,y)\, dy$$

MP $\quad E\left[f(B(t+s)) \,\middle|\, \mathcal{F}_s\right] = \int_{-\infty}^{+\infty} f(y)\, p(t, B(s), y)\, dy = g(B(s))$

SMP $\quad E\left[f(B(t+\tau)) \,\middle|\, \mathcal{F}_\tau\right] = \int_{-\infty}^{+\infty} f(y)\, p(t, B(\tau), y)\, dy = g(B(\tau))$

$$\mathcal{F}_\tau := \{ A \cap (\tau \leq t) \text{ for all } A \in \mathcal{F}_t \}.$$

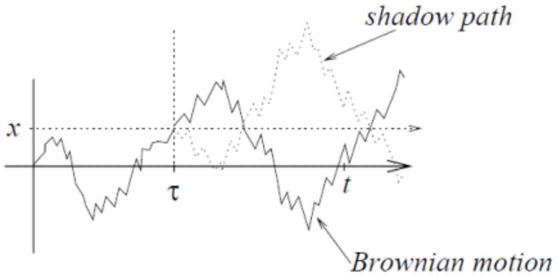
- An important consequence of the SMP is the Reflection Principle

Lemma: $\{B(t)\}_{t\geq 0}$ Brownian Motion. Then $(\forall)\ x > 0$

$$P\left(\max_{0\le s\le t} B(s) > x\right) = 2\,P\big(B(t) > x\big)$$

Proof:

$B(\tau) =$



$\tau = \min\{t \ge 0 : B(t) = x\}$

$\rightsquigarrow$ stopping time

$\Downarrow$ SMP

$B(t) - B(\tau)$ is BM

$\rightsquigarrow$ independent of $\mathcal{F}_\tau$

$$P\left(\max_{s\le t} B(s) > x\right) = P(\tau < t) =$$

$$= \underbrace{P(\tau < t,\ B(t) > x)}_{\downarrow} + P(\tau < t,\ B(t) \le x)$$

$$= P(B(t) > x) + \underbrace{P\Big(\underbrace{B(t) \le B(\tau) = x}_{\text{in } \mathcal{F}_\tau} \,\Big|\, \underbrace{\tau < t}_{\text{in } \mathcal{F}_\tau}\Big)}_{=\frac{1}{2}} \cdot P(\tau < t)$$

$$= P(B(t) > x) + \frac{1}{2} P(\tau < t)$$

$$P(\tau < t) = P(B(t) > x) + \frac{1}{2} P(\tau < t)$$

$$\Downarrow$$

$$\Rightarrow \boxed{P(\tau < t) = 2\,P(B(t) > x)} = \frac{2}{\sqrt{2\pi t}} \int_x^\infty e^{-\frac{y^2}{2t}}\,dy$$

Th Reflection Principle

$\{B(t)\}_{t\ge 0}$ Brownian Motion, τ = stopping time w.r.t. $\{\mathcal{F}_t\}_{t\ge 0}$

$$\Rightarrow \tilde{B}(t) = \begin{cases} B(t) & ; \quad t \le \tau \end{cases}$$

$$\Rightarrow \quad \tilde{B}(t) = \begin{cases} B(t) & ; \quad t \le \tau \\ 2B(\tau) - B(t) & ; \quad t > \tau \end{cases}$$ (the reflected path)

is also a Brownian Motion !!!

<u>First Passage Time</u>:

- $\tau := \min\{t \geq 0 : B(t) = x\}$
- for a fixed $\theta > 0$: $\boxed{Y(t) := \exp\{\theta B(t) - \frac{1}{2}\theta^2 t\}}$

Check : $\{Y(t)\}_{t\geq 0}$ is martingale w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$

$$E\left[Y(t) \mid \mathcal{F}_s\right] = E\left[\exp\{\theta B(t)\} \cdot \exp\{-\tfrac{1}{2}\theta^2 t\} \mid \mathcal{F}_s\right] =$$

$$= E\left[\exp\{\theta \cdot \underbrace{(B(t) - B(s))}_{\text{indep of } \mathcal{F}_s} + \theta B(s)\} \mid \mathcal{F}_s\right] \cdot \exp\{-\tfrac{1}{2}\theta^2 t\}$$

$$= \exp\{\theta B(s)\} \cdot E\left[\exp\{\theta \underbrace{(B(t) - B(s))}_{\sim N(0, t-s)}\}\right] \cdot \exp\{-\tfrac{1}{2}\theta^2 t\}$$

$$= \exp\{\theta B(s)\} \cdot \exp\{\tfrac{1}{2}\theta^2 (t-s)\} \exp\{-\tfrac{1}{2}\theta^2 t\}$$

$$= \exp\{\theta \cdot B(s) - \tfrac{1}{2}\theta^2 s\} = Y(s)$$

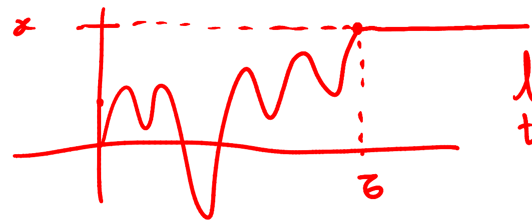
- so $\{Y(t)\}_{t\geq 0}$ is martingale $\xrightarrow{\text{O.S.P}}$ $\{Y(t\wedge\tau)\}_{t\geq 0}$ is martingale

$$\Rightarrow \quad E\left[\exp\{\theta B(t\wedge\tau) - \tfrac{1}{2}\theta^2 (t\wedge\tau)\}\right] = 1$$

- $$\lim_{t\to\infty} \exp\{-\tfrac{1}{2}\theta^2 (t\wedge\tau)\} = \begin{cases} e^{-\frac{1}{2}\theta^2 \tau} & \text{if } \tau < \infty \\ 0 & \text{if } \tau = \infty \end{cases}$$

- $0 \le Y(t\wedge\tau) \le e^{\theta x}$ so we can use DCT to get

$$\boxed{E\left[\exp\left\{\theta x - \tfrac{1}{2}\theta^2\tau\right\}\cdot 1_{\{\tau<\infty\}}\right] = 1}$$



$$\lim_{t\to\infty} B(t\wedge\tau) = x = B(\tau)$$

- as $\theta \downarrow 0$ we get $E\,1_{\{\tau<\infty\}} = 1 \Rightarrow P(\tau<\infty) = 1$

$$E\left[e^{\theta x - \frac{1}{2}\theta^2\tau}\right] = 1$$

$$E\left[e^{-\frac{1}{2}\theta^2\tau}\right] = e^{-\theta x}$$

- for $\alpha = \frac{1}{2}\theta^2 \Rightarrow E\left[e^{-\alpha\tau}\right] = e^{-x\sqrt{2\alpha}}$ $\quad \alpha > 0$

(moment generating function)

$\Rightarrow -E\left[\tau e^{-\alpha\tau}\right] = -\frac{x}{\sqrt{2\alpha}}\, e^{-x\sqrt{2\alpha}}$: for $\alpha \downarrow 0$ we obtain

$$E[\tau] = \infty$$

- we can obtain the same results using the distr. of τ:

$$P(\tau \le t) = \frac{2}{\sqrt{2\pi t}}\int_x^\infty e^{-\frac{y^2}{2t}}\,dy = \frac{2}{\sqrt{2\pi}}\int_{\frac{x}{\sqrt{t}}}^\infty e^{-\frac{z^2}{2}}\,dz$$

$\sqrt{t}$

(for a change of variable $z = \frac{y}{\sqrt{t}}$; $dz = \frac{dy}{\sqrt{t}}$)

$$\Rightarrow P(\zeta < \infty) = \lim_{t\to\infty} P(\zeta \le t) = \frac{2}{\sqrt{2\pi}} \lim_{t\to\infty} \int_{\frac{x}{\sqrt{t}}}^{\infty} e^{-\frac{z^2}{2}} dz = 1$$

- also $f_\zeta(t) = \frac{\partial}{\partial t} P(\zeta \le t) = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}}$

(the density of ζ)

$$\Rightarrow E e^{-\alpha\zeta} = \int_0^\infty e^{-\alpha t} f_\zeta(t)\, dt = \int_0^\infty e^{-\alpha t} \cdot \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} dt$$

$B^2(t) - t$ = martingale

(Th) $\left.\begin{array}{l} M(t) = \text{martingale}, \ (\text{cont}) \\ M^2(t) - t = \text{martingale} \end{array}\right\} \Rightarrow \{M(t)\} = \text{is a B.M.}$