Discrete Distributions, Conditional Prob., Joint Distributions

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Ω ⇝ based on the coin toss experiment

1) Bernoulli trials :

$$X = \begin{cases} 1 & \text{if } H \quad (\text{success}) \\ 0 & \text{if } T \quad (\text{failure}) \end{cases}$$

· enough to mention $p = P(X=1)$

2) Binomial Distribution . Bin (n,p)

$X_1, \dots, X_n$ independent Bernoulli trials with p

$X = \sum_{i=1}^{n} X_i \sim \text{Bin}(n,p) \rightsquigarrow$ range $\{0,1,2,\dots,n\}$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

⇝ nr of successes in a sequence of n - indep Bernoulli trials

3) Geometric Distribution

⇝ the number of tails before the appearance of the first head. $X : \Omega \longrightarrow \dots$

⇝ range = $\{0,1,2,3,\dots\}$

$$P(X=k) = (1-p)^k p$$

4) Poisson distribution : fix $\lambda > 0$

→ range : $\{0,1,2,\dots\}$

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

λ = rate of "successes"

$\lambda = n \cdot p = E\,\text{Bin}(n,p)$

$\downarrow$ E Poisson (λ)

$X \sim \text{Bin}(n,p)$ n large.

$\sim N(np, np(1-p))$

$$\frac{X - np}{\quad} \sim N(0,1)$$

$\sqrt{np(1-p)}$

$$\bar{X} = \frac{X}{n} \sim N\left(p, \frac{p(1-p)}{n}\right)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) \cdot P(B)$$

iff A and B are independent.

- in general : $P(A \cap B) = P(A) \cdot P(B \mid A)$
$= P(B) \cdot P(A \mid B)$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0$$

Problem 2 from the quiz.

A : we see at most one H

B : at least one H and at least one T

- toss the coin n times $(n \geq 3)$
- Are A and B independent?

$A = \{\text{no H}, \text{exactly one H}\} \rightsquigarrow P(A) = \frac{n+1}{2^n}$

$B = \Omega \setminus \{\text{no H}, \text{no T}\} = 1 - \frac{2}{2^n}$

$A \cap B = \{\text{exactly one H}\} \rightsquigarrow P(A \cap B) = \frac{n}{2^n}$

$\rightsquigarrow$ do not verify $P(A \cap B) = P(A) \cdot P(B)$

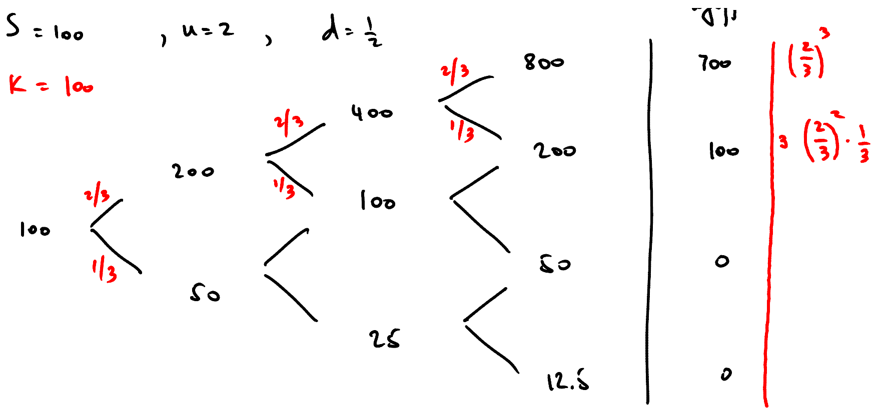
$n = 3 \rightsquigarrow P(A) = \frac{4}{8} = \frac{1}{2} \quad P(B) = \frac{6}{8} = \frac{3}{4}$

$P(A \cap B) = \frac{3}{8}$

A, B are indep for $n = 3$!!!

$S = 100 \quad , u = 2 \quad , \quad d = \frac{1}{2}$

$K = 100$



$$P(\text{call option is in the money}) = \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2$$

$$P(\text{call option in the money} \mid S_1 = 50)$$

$$= \frac{P(S_1 = 50,\ S_2 = 100,\ S_3 = 200)}{P(S_1 = 50)} = \frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}}{\frac{1}{3}}$$

$$P(\text{call in the money} \mid S_2 = 100) = \frac{2}{3}$$

$$P(S_1 = 50 \mid \text{option in the money}) = ?$$

$$= \frac{P(S_1 = 50,\ \text{option in the money})}{P(\text{option in the money})} = \frac{\frac{1}{3}\cdot\left(\frac{2}{3}\right)^2}{\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2}$$

$$= \frac{\frac{1}{3}}{\frac{2}{3} + 1} = \frac{1}{5}$$

Bayes Formula.

$\{E_i\}_{i \in I}$ partition of Ω , $A \in \mathcal{F}$

$$P(A) = \sum_{i \in I}{}' \underbrace{P(A \mid E_i) \cdot P(E_i)}_{P(A \cap E_i)} = P\left(\bigcup_{i \in I} (A \cap E_i)\right)$$

$$\bigcup_{i \in I} (A \cap E_i) = A \cap \left(\bigcup_{i \in I} E_i\right) = A \cap \Omega = A$$

$$P(A) = \sum P(A \mid E_i) P(E_i)$$

$i \in I$

$$P(E_n \mid A) = \frac{P(E_n \cap A)}{P(A)}$$

$$= \frac{P(E_n) \cdot P(A \mid E_n)}{\sum_{i \in I} P(E_i) P(A \mid E_i)}$$

Def : A, B are indep iff $P(A \mid B) = P(A)$

- A, B mutually exclusive : $A \cap B = \phi$
  $P(A \cap B) = 0$
  if $P(A) = 0$ $\qquad P(A \cap B) = P(B) \cdot \underbrace{P(A \mid B)}_{=0}$

- if $P(A) > 0$, $P(B) > 0 \Rightarrow P(A) \cdot P(B) > 0$

Independence of r.v.

$$X : \{ -1, 0, 1 \} \quad \begin{matrix} -1 \to \frac{1}{3} & 0 \to \frac{1}{3} & 1 \to \frac{1}{3} \end{matrix} \qquad ; \qquad Y = X^2 : \{ 0, 1 \} \quad \begin{matrix} 0 \to \frac{1}{3} & 1 \to \frac{2}{3} \end{matrix}$$

$$\mathrm{Cov}(X, Y) = E(X - EX)(Y - EY) = \underbrace{EXY}_{=0} - \underbrace{EX}_{=0} \cdot EY$$

$EX = 0$

$E(X \cdot Y) = E(X^3) = 0$

Def A collection of discrete var $\{X_\alpha\}_{\alpha \in A}$ is said to be mutually independent if for any subset of $J \subseteq A$

$$P\left(\bigcap_{i \in J} \{ X_i = x_i \} \right) = \prod_{i \in J} P(X_i = x_i)$$

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$$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i)$$

joint distribution marginal distr.

- if $X_1, X_2, \dots, X_n$ are indep

$$E(X_1 X_2 \dots X_n) = \prod_{i=1}^{n} E X_i$$

ex :

$Y \backslash X$	0	1
0	q^2	pq
1	pq	p^2

are X, Y indep ? ✓

$p + q = 1$

$$P(X=0) = q^2 + pq = q(q+p) = q$$
$$P(X=1) = p$$
$$P(Y=0) = q^2 + pq = q$$

Problem 4) from the quiz.

$X =$ score of the first die

$Y =$ score of the second.

distribution of $X \mid X+Y = 8$

$$P(X=1 \mid X+Y=8) = 0$$
$$P(X=2 \mid X+Y=8) = \frac{P(X=2,\ X+Y=8)}{P(X+Y=8)}$$
$$= \frac{P(X=2,\ Y=6)}{P(X+Y=8)} = \frac{P(X=2)\cdot P(Y=6)}{P(X+Y=8)} = \frac{\frac{1}{6}\cdot\frac{1}{6}}{\frac{5}{36}} = \frac{1}{5}$$

Conditional distribution & conditional expectations.

$$P_{X|Y}(x|y) = P(X=x \mid Y=y) = \frac{P_{(X,Y)}(x,y)}{P_Y(y)}$$

$$E(X \mid Y=y) = \sum_x{}' x\, P_{X|Y}(x|y) \rightsquigarrow \text{function of } y.$$

$E(X \mid Y=y) = \sum_x{}' \, x \, P_{X|Y}(x|y) \rightsquigarrow$ function of y.

- $E(X \mid Y) \rightsquigarrow$ function of Y (RV) (ω)

$$E\left[E(X \mid Y)\right] \overset{?}{=} \sum_y E[X \mid Y=y] \cdot P_Y(y)$$

$$= \sum_y{}' \sum_x{}' \, x \cdot P_{X|Y}(x|y) \cdot P_Y(y)$$

$$= \sum_y{}' \sum_x{}' \, x \, P_{(X,Y)}(x,y)$$

$$= \sum_x{}' \sum_y{}' \, x \, P_{(X,Y)}(x,y)$$

$$= \sum_x{}' \, x \cdot \underbrace{\left(\sum_y P_{(X,Y)}(x,y)\right)}_{P_X(x)} = \sum_x{}' \, x \cdot P_X(x)$$

$$= EX$$

ex: nr of eggs $\sim$ Poisson (λ)

- survival prob is p
- Find the expected number of survived eggs

$S =$ nr of survived eggs

$N =$ initial nr.

$S \leq N \qquad S \sim \text{Bin}(N, p)$

$E(S \mid N = 100) = 100 \cdot p$

$E(S \mid N) = N \cdot p$

$ES = E(E(S|N)) = E(N \cdot p) = \lambda p$

Continuous Distributions

range

range

$X \sim \text{Bin}(n,p) \rightsquigarrow \{0,1,2,\dots,n\}$

Y cont $\rightsquigarrow$ interval I

$f(y)$ density over the interval I

- $f(y) \geq 0$ for all $y \in I$
- $\int_I f(y)\,dy = 1$

<u>ex</u> : Uniform distr. over $I = [a,b]$

$$f(y) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq y \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$EY = \frac{a+b}{2}$$

- exponential distr : $(0,\infty)$

$$f(y) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- normal distr : $\mathbb{R}$

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ - \frac{(x-\mu)^2}{2\sigma^2} \right\}$$

$$EY = \mu \qquad \text{Var } Y = \sigma^2$$

$X \sim N(\mu_1, \sigma_1^2)$ $\qquad$ $Y \sim N(\mu_2, \sigma_2^2)$

- <u>indep.</u>

$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$\text{Var}(X+Y) = \text{Var } X + \text{Var } Y$

Moment Generating Functions.

$$M_X(t) = E\, e^{Xt}$$

$$M_X'(t) = E\, X e^{Xt} \quad \rightsquigarrow \quad EX = M_X'(0)$$

$$M_X''(t) = E\, X^2 e^{Xt} \quad \rightsquigarrow \quad EX^2 = M_X''(0)$$