Convergence of random variables

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Consider $X_1, X_2, \ldots$ iid sequence of random variables such that : (coin tossing example)

$$X_n = \begin{cases} 1 & \text{with prob } 1/2 \\ 0 & \text{with prob } 1/2 \end{cases}$$

In the long run we expect the proportion of "1"s to converge to 1/2, that is :

$$\frac{X_1(w) + X_2(w) + \cdots + X_n(w)}{n} \longrightarrow \frac{1}{2} \quad \text{as } n \to \infty$$

where w is the "realization" of the random experiment

$$w = (w_1, w_2, w_3, \ldots)$$

- in particular, for $w = (1, 1, 1, \ldots\ldots)$ the convergence fails

Conclusion : $\frac{X_1 + X_2 + \cdots + X_n}{n} \longrightarrow \frac{1}{2}$

is not true point wise, however we know that the probability of the set of w where convergence fails has measure 0

Definition 1: $X_n \to X$ almost surely as $n \to \infty$

Definition 1: $X_n \to X$ almost surely as $n \to \infty$ if $P\left(\lim_{n\to\infty} X_n = X \right) = 1$

Definition 2: $X_n \to X$ in probability as $n \to \infty$

$$P\left(|X_n - X| \geq \varepsilon \right) \longrightarrow 0 \quad \text{for all } \varepsilon > 0$$

Definition 3: $X_n \xrightarrow{d} X$ in distribution as $n \to \infty$ if for every continuous, bounded $h : \mathbb{R} \to \mathbb{R}$

$$E\, h(X_n) \longrightarrow E\, h(X)$$

(equivalent to: $F_{X_n}(x) \longrightarrow F_X(x)$ for each x where F_X is cont.)

Definition 4: $X_n \xrightarrow{L^p} X$ as $n \to \infty$ if

$$E\, |X_n - X|^p \longrightarrow 0 \quad \text{as } n \to \infty$$

Relationship between different types of convergence

$$\begin{tikzcd}
(1) \arrow[r, Rightarrow, "?\ \checkmark"] & (2) \arrow[r, Rightarrow] & (3) \\
& (4) \quad (p>0) \arrow[u, Rightarrow, "?\ \checkmark"']
\end{tikzcd}$$

Proof $(1 \Rightarrow 2)$ $X_n \longrightarrow X$ a.s is equivalent to

$Y_n = X_n - X \longrightarrow 0$ a.s.

- $P\left(\lim_{n\to\infty} Y_n = 0 \right) = P\left(\bigcap_{\varepsilon>0} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{ |Y_n| < \varepsilon \} \right) = 1$

$\lim_{n\to\infty} Y_n(\omega) = 0 \iff$

$\left[(\forall)\ \varepsilon > 0\ (\exists)\ N \text{ such that for all } n \geq N \quad | Y_n(\omega) | < \varepsilon \right]$

$$P\left(\bigcap_{\varepsilon>0} \underbrace{\bigcup_{N=1}^{\infty} \bigcap_{n\geq N} \{ |Y_n| < \varepsilon \}}_{A_\varepsilon} \right) = 1$$

- since $A_\varepsilon = \bigcup_{N=1}^{\infty} \bigcap_{n\geq N} \{ |Y_n| < \varepsilon \}$ decrease with $\varepsilon \downarrow 0$

we have that $P(A_\varepsilon) = 1$ for all $\varepsilon > 0$

or equivalently $P(A_\varepsilon^c) = 0$ for all $\varepsilon > 0$

$$A_\varepsilon^c = \bigcap_{N=1}^{\infty} \underbrace{\bigcup_{n\geq N} \{ |Y_n| \geq \varepsilon \}}_{B_N:\ \text{decreasing as } N\to\infty}$$

for all N

$$0 = P(A_\varepsilon^c) = P\left(\bigcap_{N=1}^{\infty} B_N\right) = \lim_{N\to\infty} P(B_N)$$

$$\Rightarrow \lim_{N\to\infty} P\left(\bigcup_{n=N}^{\infty}\{|Y_n| \geq \varepsilon\}\right) = 0$$

however

$$0 \leq P(|Y_N| \geq \varepsilon) \leq P\left(\bigcup_{n=N}^{\infty}\{|Y_n| \geq \varepsilon\}\right)$$

so $\quad 0 \leq \lim_{N\to\infty} P(|Y_N| \geq \varepsilon) \leq \lim_{N\to\infty} P\left(\bigcup_{n=N}^{\infty}\{|Y_n| \geq \varepsilon\}\right) = 0$

$\Rightarrow \quad \lim_{N\to\infty} P(|Y_N| \geq \varepsilon) = 0 \quad$ hence $Y_n \xrightarrow{P} 0$

Proof: $\quad (4) \Rightarrow (2)$

$$X_n \xrightarrow{L^p} X \quad \Rightarrow \quad X_n \xrightarrow{P} X$$

- we have to prove that

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

- we have $\quad \lim_{n\to\infty} E|X_n - X|^p = 0$

$$0 \leq P(|X_n - X| \geq \varepsilon) \leq \frac{E|X_n - X|^p}{\varepsilon^p} \to 0 \quad (p > 0)$$

(Markov's inequality)

Lemma: $\quad X_n \xrightarrow{P} X$, then there is a subsequence

Lemma: $X_n \xrightarrow{P} X$, then there is a subsequence $\{n_k\}$ ($n_k \to \infty$, as $k \to \infty$) such that $X_{n_k} \to X$ a.s. as $k \to \infty$

Notation:

$$\limsup_{N\to\infty} A_N = \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} A_n = \lim_{N\to\infty} \left(\bigcup_{n\geq N} A_n \right)$$

"translation":

$\limsup_{N\to\infty} A_N$ = "A_N occurs infinitely often"

(= "A_N i.o.")

Borel Cantelli lemma: $\{A_n\}$ sequence of events

(a) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \text{, i.o.}) = 0$

(b) If $P(A_n \text{ i.o}) = 0$ and if A_n are mutually indep. $\Longrightarrow$ $\sum_{n=1}^{\infty} P(A_n) < \infty$

Remark:

$$\boxed{\begin{array}{l} X_n \to X \text{ a.s.} \iff P\left(|X_n - X| \geq \varepsilon \text{ i.o.}\right) = 0 \\ \text{if } \sum_{n=1}^{\infty} P\left(|X_n - X| \geq \varepsilon\right) < \infty \implies X_n \to X \text{ a.s.} \end{array}}$$

- if $P\left(|X_n - X| \geq \varepsilon\right) \simeq \frac{1}{n}$ then

$\rightsquigarrow$ $X_n \xrightarrow{P} X$ since $\lim_{n\to\infty} P\left(|X_n - X| \geq \varepsilon\right) = \lim_{n\to\infty} \frac{1}{n} = 0$

$\rightsquigarrow$ if also $X_1, X_2, \ldots$ are mutually independent then we do not have convergence a.s. since

$$\sum_{n=1}^{\infty} P\left(|X_n - X| \geq \varepsilon\right) = \infty$$

Th: $X_n \xrightarrow{P} X$ and $|X_n| \leq Y$, $Y \in L^p$

Then $X_n, X \in L^p$ and $X_n \xrightarrow{L^p} X$

Th: $X_n \xrightarrow{P} X \iff \lim_{n\to\infty} E \dfrac{|X_n - X|}{|X_n - X| + 1} = 0$

L^p spaces $\quad p \geq 1 \quad (\Omega, \mathcal{F}, P)$

Def: $X \in L^p$ if $|X|^p$ is integrable.

- in L^p: random var X, Y such that $X = Y$ a.s are one and the same

$^{1/p}$

- L^p is a normed space : $\|X\|_p := \left(E|X|^p\right)^{1/p}$
  - $\|X\|_p \geq 0$
  - $\|X\|_p = 0 \iff X = 0$ (a.s)
  - $\|c \cdot X\|_p = |c| \cdot \|X\|_p \quad (\forall)\ c \in \mathbb{R}$
  - $\|X+Y\|_p \leq \|X\|_p + \|Y\|_p$
  - If $X_n \in L^p$ and $X_n \xrightarrow{L^p} X$ then $X \in L^p$
  - when $p = 2$ L^p has additional properties.
  - $L^q \subseteq L^p$ for $q > p$

Hölder inequality, : $\frac{1}{p} + \frac{1}{q} = 1$

$$\|XY\|_1 \leq \|X\|_p \cdot \|Y\|_q$$

Proof of $\|X+Y\|_p \leq \|X\|_p + \|Y\|_p$ for $p \geq 1$

- for $p = 1$: trivial
- for $p > 1$:

$$|X+Y|^p \leq |X+Y| \cdot |X+Y|^{p-1} \leq \left(|X| + |Y|\right)|X+Y|^{p-1}$$

- for $\frac{1}{p} + \frac{1}{q} = 1$ we must have $q = \frac{p}{p-1}$

Apply Hölder inequality as follows

$$\| X \cdot |X+Y|^{p-1} \|_1 \leq \|X\|_p \cdot \| |X+Y|^{p-1} \|_q$$

$$\| Y \, |X+Y|^{p-1} \|_1 \leq \|Y\|_p \cdot \| |X+Y|^{p-1} \|_q$$

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adding up these inequalities we get

$$\|X+Y\|_p^p \leq \|X\|_p \cdot \| |X+Y|^{p-1} \|_q + \|Y\|_p \, \| |X+Y|^{p-1} \|_q$$

$$= \left(\|X\|_p + \|Y\|_p \right) \cdot \left(E\,|X+Y|^{\overbrace{q(p-1)}^{p}} \right)^{\frac{1}{q}}$$

$$= \left(\|X\|_p + \|Y\|_p \right) \cdot \left(E\,|X+Y|^p \right)^{\frac{p-1}{p}}$$

$$= \left(\|X\|_p + \|Y\|_p \right) \frac{\|X+Y\|_p^p}{\|X+Y\|_p}$$

$$\Rightarrow \|X+Y\|_p \leq \|X\|_p + \|Y\|_p$$

Th L^2 is complete

Def: A metric space M is complete if every Cauchy sequence is convergent and the limit is in M

i.e. if $\|X_n - X_m\| \longrightarrow 0$ as $n, m \to \infty$ independently then there is some $Y \in M$ s.t $\|X_n - Y\| \longrightarrow 0$

(Th) L^2 is Hilbert space.

→ the inner product is defined as

$$\langle X, Y\rangle = E(X \cdot Y), \quad \sqrt{\langle X, X\rangle} = \|X\|_2$$

→ Brief review of the definition of Hilbert spaces

Def : A Hilbert space is an abstract vector space possessing the structure of an inner product

• Hilbert spaces are required to be complete

→ $\langle \cdot, \cdot \rangle$ is linear :

$$\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$$

→ $\langle x, x\rangle \geq 0$ where equality holds for $x = 0$.

→ $\|x\| = \sqrt{\langle x, x\rangle}$

→ $d(x,y) = \|x - y\|$ the distance between x, y

$$\left[\ d(x,z) \leq d(x,y) + d(y,z)\ \right]$$
(triangle inequality)

Cauch-Schwartz ineq : $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

→ if $x \perp y \Rightarrow \|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$

$\rightsquigarrow$ if $x \perp y \Rightarrow \|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$
(Pythagorean theorem)