Finite Expiration American Put

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Let us recall the results for the Perpetual American Put

on : $dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t$

Perpetual American Put : $v_*(x) = \max\limits_{\tau \in \mathcal{T}} \tilde{E}\left[e^{-r\tau}(K - S_\tau)\right]$

- Strategy : • if $S_t \leq L$ exercise immediately
  • if $S_t > L$ wait

$$\rightsquigarrow v_L(x) = \begin{cases} K - x & \text{if } 0 \leq x \leq L \\ (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & \text{if } x > L \end{cases}$$

$\rightsquigarrow$ for $L^* = \dfrac{2r}{2r+\sigma^2} K$ the strategy is **OPTIMAL**

Analytic Characterization of v_* $\left(v_{L^*}(x) \right)$

① $v(x) \geq (K-x)^+$

② $rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0$

③ for each $x > 0$: equality holds in either ① or ②
and $v'(L^*) = -1$ (smooth pasting)

Probabilistic Characterization of $V_t = v_{L^*}(S_t)$

① $V_t \geq (K - S_t)^+$

② $e^{-rt} V_t$: $\tilde{\mathbb{P}}$ supermartingale

③ there exists a stopping time τ^* such that

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$$V_0 = \tilde{E}\left[e^{-r\tau^*}(K-S_{\tau^*})^+\right]$$

$\tau^* = \min\{t \geq 0 : S_t \in S\}$

S = stopping region $\quad C$ = continuation region

Now let's move on to finite expiration American Put:

- T = maturity

⇝ the owner of the option can exercise any time up to T

- intrinsic value : $(K-S_t)^+$
- value of the put : $v(t,x) = \max\limits_{t \leq \tau \leq T} E\left[e^{-r(\tau-t)}(K-S_\tau)^+ \,\middle|\, S_t = x\right]$

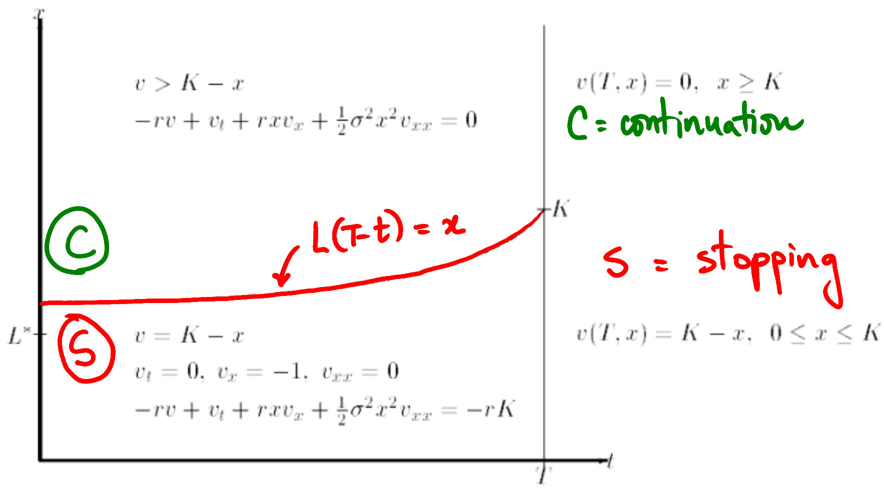
Analytical Characterization of the Put Price $v(t,x)$

① $v(t,x) \geq (K-x)^+ \quad (\forall)\ t \in [0,T]$

② $r v(t,x) - v_t(t,x) - r x v_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0$

③ equality holds in either ① or ②

$v_x(t,x) = -1$ on the curve $L(T-t)$



Probabilistic Characterization of $V_t = v(t, S_t)$

① $V_t \geq (K - S_t)^+$ $\quad (\forall)$ $0 \leq t \leq T$

② $e^{-rt} V_t$: $\tilde{\mathbb{P}}$ supermartingale

③ V_t is the smallest process that satisfies ① & ②