MTH 9821 Numerical Methods for Finance I
Lecture 7 - Black Scholes PDE

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1 Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$

1.1 Log-normal model (Geometric Brownian Motion)

$S(t)$ satisfies the PDE

$$dS = (\mu - q)Sdt + \sigma S dX$$

where $X(t)$ is a Weiner process.

(1) $X(0) = 0$;

(2) $X(t_2) - X(t_1) \sim N(0, t_2 - t_1)$;

(3) $X(t_3) - X(t_2)$ and $X(t_2) - X(t_1)$ are independent, $\forall t_1 < t_2 < t_3$;

We can see this PDE as two parts: drift term and stochastic term.
Without the stochastic term, the infinitesimal return of the asset between t and dt

$$\begin{aligned}
&\frac{S(t+dt) - S(t)}{S(t)} = (\mu - q)dt \\
dt \to 0, \quad &\frac{S(t+dt) - S(t)}{dt} = (\mu - q)S \\
&\Rightarrow S' = (\mu - q)S \\
&\Rightarrow \int_0^t \frac{S'}{S}dt = \int_0^t (\mu - q)dt \\
&\Rightarrow \ln(S(t)) - \ln(S(0)) = \ln\left(\frac{S(t)}{S(0)}\right) = (\mu - q)t
\end{aligned}$$

1.2 Portfolio

Let $V(S,t)$ is the value of a dynamically replicable derivative security underlying that pays continuous devidends at rate q.
Set up the following portfolio

- long derivative security
- short Δ units of underlying asset

$$\begin{aligned}
\Pi &= V - \Delta S \\
d\Pi &= dV - \Delta dS - \Delta q S dt
\end{aligned}$$

Using Ito's Lemma,

$$\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}dt \\
\Rightarrow \quad d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - q\Delta S\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS \\
&\text{choose } Delta = \frac{\partial V}{\partial S} \\
&= \left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - qS\frac{\partial V}{\partial S}\right)dt
\end{aligned}$$

Note that with the choice of Δ, Pi is deterministic, i.e., non-random.
For no-arbitrage, Π must grow at the risk-free rate r over the time dt

$$\begin{aligned}
& d\Pi = r\Pi dt = r(V - \Delta S)dt = r\left(V - S\frac{\partial V}{\partial S}\right)dt \\
& \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - qS\frac{\partial V}{\partial S} = rV - rS\frac{\partial V}{\partial S} \\
\Rightarrow \quad & \boxed{\tfrac{\partial V}{\partial t} + \tfrac{\sigma^2 S^2}{2}\tfrac{\partial^2 V}{\partial S^2} + (r-q)S\tfrac{\partial V}{\partial S} - rV = 0} \quad \textbf{Black} - \textbf{Scholes PDE}
\end{aligned}$$

Question: Why Black-Scholes PDE does not hold for American options?
The above process fails when

- $d\Pi = dV - \Delta dS - \Delta q S dt$. With early exercise, Π is then not differentiable.
- $d\Pi = r\Pi dt$. If exercise early, "$\leq$".

1.3 Apply BS PDE

The Black-Scholes PDE:

- is good for European call, put and a lot other securities;
- satisfies different securities with different boundary conditions;

1.3.1 Various Securities

- If $V = S$ with $q = 0$, then $\frac{\partial^2 V}{\partial S^2} = 0$, $\frac{\partial V}{\partial t} = 0$, $\frac{\partial V}{\partial t} = 1$.
  The Black-Scholes PDE still holds.

- If $q \neq 0$, $V = e^{-q(T-t)}S$, thus

$$\begin{aligned}
dS &= (\mu - q)Sdt + \sigma SdX \\
S_1(t) &= e^{-q(T-t)}S \\
\Rightarrow dS_1 &= \frac{\partial S_1}{\partial t}dt + \frac{\partial S_1}{\partial S}dS + \frac{1}{2}\frac{\partial^2 S_1}{\partial S^2}\sigma^2 S^2 dt \\
&= qe^{-q(T-t)}Sdt + e^{-q(T-t)}dS \\
&= qS_1dt + e^{-q(T-t)}\big((\mu - q)Sdt + \sigma SdX\big) \\
&= \mu S_1dt + \sigma S_1dX
\end{aligned}$$

  $S_1(t)$ is a non-dividend-paying asset with the same drift as $S(t)$ and the same value at T as $S(T)$.
  $V = e^{-q(T-t)}S$ is known as the equivalent solution, also satisfies BS PDE.

- If $V = f(S)$, then PDE becomes ODE

$$\begin{aligned}
&\frac{\sigma^2 S^2}{2}f''(S) + (r-q)Sf'(S) - rf(S) = 0 \qquad\qquad f(S) = CS^\alpha \\
\Rightarrow\quad &\frac{\sigma^2}{2}\alpha(\alpha - 1) + \alpha(r - q) - r = 0 \quad \text{solve for } \alpha
\end{aligned}$$

  This is the case of a perpetual American option.

1.3.2 Different Boundary Conditions

The value of European plain vanilla options satisfies the BS PDE with the following boundary conditions:

$$\begin{aligned}
\text{Call:}\quad & V(S,T) = \max(S-K, 0),\ \forall S > 0 \\
& V(0,t) = 0,\ \forall 0 < t < T \\
\text{Put:}\quad & V(S,T) = \max(K-S, 0),\ \forall S > 0 \\
& V(0,t) = Ke^{-r(T-t)},\ \forall 0 < t < T \\
\Rightarrow \text{Goal:}\quad & \text{Find } V(S,0).
\end{aligned}$$

2 Properties of Black-Scholes PDE

(1) BS PDE is a parabolic PDE

$$\frac{\partial V}{\partial t} = C_2(S,t)\frac{\partial^2 V}{\partial S^2} + C_1(S,t)\frac{\partial V}{\partial S} + C_2(S,t)V$$

i.e., no cross partial derivative of S and t.

(2) BS PDE is a linear PDE

$$L_{BS}(c_1V_1 + c_2V_2) = c_1L_{BS}(V_1) + c_2L_{BS}(V_2)$$

(3) BS PDE is a backward parabolic PDE
Boundary conditions are given at time T, solutions must be found at time 0.

(4) BS PDE has nonconstant but homogeneous coefficients.

- the coefficient of $\frac{\partial V}{\partial S}$ is cS;
- the coefficient of $\frac{\partial^2 V}{\partial S^2}$ is cS^2;

Remark:
The simplest parabolic PDE is the heat PDE

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial X^2}$$

We know the solution to heat PDE.

3 Financial Intuition of Black-Scholes PDE

$$\begin{aligned}
\frac{\partial V}{\partial t} &= -(r-q)S\frac{\partial V}{\partial S} + rV - \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} \\
\frac{\partial V}{\partial t} &= -(r-q)\Delta S + rV - \frac{\sigma^2 S^2}{2}\Gamma \\
&= -r\Delta S + q\Delta S + rV - \frac{\sigma^2 S^2}{2}\Gamma \\
\Rightarrow \quad \frac{\partial V}{\partial t} &= r(V-\Delta S) + q\Delta S - \frac{\sigma^2 S^2}{2}\Gamma \\
\Rightarrow \quad dV &= r(V-\Delta S)dt + q\Delta S dt - \frac{\sigma^2 S^2}{2}\Gamma dt
\end{aligned}$$

Note that using forward finite difference approximation,

$$\frac{\partial V}{\partial t} \approx \frac{V(t+dt) - V(t)}{dt} =: \frac{dV}{dt}$$

Consider again the hedging portfolio

$$\Pi = V - \Delta S$$

As discussed earlier, Π is deterministic, grows at the risk-free rate r. We can then replicate V by

- long Δ units of underlying asset
- long cash position with value $V - \Delta S$

From the PDE:

$$dV = r(V - \Delta S)dt + q\Delta S dt - \frac{\sigma^2 S^2}{2}\Gamma dt$$

Interpretation:

dV: change in the value of the derivative security is:

$r(V - \Delta S)dt$: interest at the cash position in the replicating portfolio

$q\Delta S dt$: dividends on the long Δ units underlying asset position in the replicating portfolio

$\frac{\sigma^2 S^2}{2}\Gamma dt$: slippage, loss due to portfolio rebalancing

4 Solution to the Heat PDE

Please refer to Section 8.8 in the Primer.