INTEREST RATES AND FX MODELS

3. Caps, Floors, and Swaptions

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1 Introduction

The levels of interest rates fluctuate. Because of the existence of term structure of rates (i.e. the fact that the level of a rate depends on the term of the underlying loan), the dynamics of rates is highly complex. While a good analogy to the price dynamics of an equity is a particle moving in a medium exerting random shocks to it, a natural way of thinking about the evolution of rates is that of a string moving in a random environment where the shocks can hit any location along its length. Additional complications arise from the presence of various spreads between rates (we have encountered some of them in Lecture 1) which reflect credit worthiness of the borrowing entity, liquidity of the instrument, or counterparty risk. In the remainder of these lectures we describe some of the mathematical methodologies to to capture and quantify this dynamics so that it can be used to

(a) put a value on the optionality embedded in various financial instruments,

(b) help risk manage portfolios of fixed income securities, and

(c) help identify mispricings and trading opportunities in the fixed income markets.

We have already taken the first step in this direction, namely learned how to construct the current snapshot of the rates market. This current snapshot serves as the starting point for the stochastic process describing the curve dynamics. The next step is to construct the *volatility cube*, which is used to model the uncertainties in the future evolution of the rates. The volatility cube is built out of implied volatilities of a number of liquidly trading options.

2 Options on LIBOR based instruments

2.1 Caps and floors

Caps and floors are baskets of European calls (called *caplets*) and puts (called *floorlets*) on LIBOR forward rates. They trade over the counter.

Let us consider for example, a 10 year spot starting cap struck at 2.50$\%$. It consists of 39 caplets each of which expires on the 3 month anniversary of today's date. It pays $\max$ (current LIBOR fixing $- 2.50\%, 0) \times$act/360 day count fraction. The payment is made at the end of the 3 month period covered by the LIBOR contract and follows the modified business day convention. Notice that the very first period is excluded from the cap: this is because the current LIBOR fixing is already known and no optionality is left in that period.

In addition to spot starting caps and floors, *forward starting* instruments trade. For example, a 1 year $\times$ 5 years (in the market lingo: "1 by 5") cap struck at 2.50$\%$ consists of 16 caplets struck at 2.50$\%$ the first of which matures one year from today. The final maturity of the contract is 5 years, meaning that the last caplets matures 4 years and 9 months from today (with appropriate business dates adjustments). Unlike in the case of spot starting caps, the first period is included into the structure, as the first LIBOR fixing is of course unknown. Note that the total maturity of the $m \times n$ cap is n years.

The definitions of floors are similar with the understanding that a floorlet pays $\max$ (strike $-$ current LIBOR fixing$\%, 0) \times$ act/360 day count fraction at the end of the corresponding period.

2.2 Eurodollar options

Eurodollar options are standardized contracts traded at the Merc. These are short dated American style calls and puts on Eurodollar futures. At each time options on the eight front (Whites and Reds) quarterly Eurodollar futures contracts and on two front serial futures are listed. Their expirations coincide with the maturity dates of the underlying Eurodollar contracts. The exchange sets the strikes for the options spaced every 25 basis points (or 12.5 bp for the front contracts). The options are cash settled.

Strike	Calls	Puts
98.875	0.5325	0.0525
99.000	0.4175	0.0625
99.125	0.3075	0.0775
99.250	0.2025	0.0975
99.375	0.1125	0.1325
99.500	0.0450	0.1900
99.625	0.0100	0.2800
99.750	0.0025	0.3975
99.875	0.0025	0.5200

Table 1: ED options: March 2012 expirations. Price of the underlying 99.355

In addition to the quarterly and serial contracts, a number of *midcurve* options are listed on the Merc. These are American style calls and puts with expirations between three months and one year on longer dated Eurodollar futures. Their expirations do not coincide with the maturity on the underlying futures contracts, which mature one, two, or four years later.

Strike	Calls	Puts
98.875	0.5275	0.0925
99.000	0.4200	0.1100
99.125	0.3150	0.1300
99.250	0.2175	0.1575
99.375	0.1275	0.1925
99.500	0.0650	0.2250
99.625	0.0250	0.3400
99.750	0.0075	0.4475
99.875	0.0025	0.5650

Table 2: ED options: June 2012 expirations. Price of the underlying 99.31

2.3 Swaptions

European *swaptions* are European calls and puts (in the market lingo they are called *receivers* and *payers*, respectively) on interest rate swaps. A holder of a payer swaption has the right, upon exercise, to pay fixed coupon on a swap of contractually defined terms. Likewise, a holder of a receiver swaption has the right to receive fixed on a swap. Swaptions are traded over the counter.

For example, a 2.50% 1Y $\to$ 5Y ("1 into 5") receiver swaption gives the holder the right to receive 2.50% on a 5 year swap starting in 1 year. More precisely, the option holder has the right to exercise the option on the 1 year anniversary of today (with the usual business day convention adjustments) in which case they enter into a receiver swap starting two business days thereafter. Similarly, a 3.50% 5Y $\to$ 10Y ("5 into 10") payer swaption gives the holder the right to pay 3.50% on a 10 year swap starting in 5 years. Note that the total maturity of the $m \to n$ swaption is $m + n$ years.

Table 3 contains the December 13, 2011 snapshot of the at the money swaption market. The rows in the matrix represent the swaption expiration and the columns represent the tenor of the underlying swap. Each entry in the table represents the swaption premium expressed as a percentage of the notional on the underlying swap.

	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y	20Y	30Y
1M	0.06%	0.11%	0.18%	0.27%	0.37%	0.67%	1.10%	1.70%	2.17%	2.94%
3M	0.10%	0.20%	0.31%	0.48%	0.68%	1.18%	1.91%	2.90%	3.69%	5.02%
6M	0.14%	0.30%	0.47%	0.74%	1.04%	1.73%	2.71%	4.06%	5.17%	6.97%
1Y	0.21%	0.45%	0.75%	1.16%	1.60%	2.51%	3.82%	5.56%	7.05%	9.45%
2Y	0.40%	0.85%	1.37%	1.94%	2.55%	3.66%	5.26%	7.38%	9.23%	12.20%
3Y	0.62%	1.26%	1.91%	2.58%	3.25%	4.50%	6.26%	8.61%	10.64%	13.77%
4Y	0.78%	1.54%	2.28%	3.02%	3.75%	5.11%	7.00%	9.52%	11.66%	15.11%
5Y	0.88%	1.74%	2.56%	3.35%	4.13%	5.58%	7.57%	10.21%	12.49%	16.15%
7Y	0.97%	1.90%	2.78%	3.63%	4.44%	5.97%	8.09%	10.81%	13.16%	16.86%
10Y	1.01%	1.96%	2.86%	3.71%	4.53%	6.08%	8.22%	10.86%	13.12%	16.71%

Table 3: ATM swaption prices

Since a swap can be viewed as a particular basket of underlying LIBOR forwards, a swaption is an option on a basket of forwards. This observation leads to the popular relative value trade of, say, a $2 \to 3$ swaption straddle versus a 2×5 cap / floor straddle. Such a trade my reflect the trader's view on the correlations between the LIBOR forwards or a misalignment of swaption and cap / floor volatilities.

3 Valuation of LIBOR options

3.1 Black's model

The market standard for quoting prices on caps / floors and swaptions is in terms of *Black's model*. This is a version of the Black-Scholes model adapted to handle forward underlying assets. We will now briefly discuss this model and in the following section we will describe some popular extensions of Black's model.

We assume that a forward rate $F(t)$, such as a LIBOR forward or a forward swap rate, follows a driftless lognormal process reminiscent of the basic Black-Scholes model,

$$dF(t) = \sigma F(t) dW(t). \tag{1}$$

Here $W(t)$ is a Wiener process, and σ is the *lognormal volatility*. It is understood here, that we have chosen a numeraire $\mathcal{N}$ with the property that, in the units of that numeraire, $F(t)$ is a tradable asset. The process $F(t)$ is thus a martingale, and we let Q denote the probability distribution.

The solution to this stochastic differential equation reads:

$$F(t) = F_0 \exp\left(\sigma W(t) - \frac{1}{2}\,\sigma^2 t\right). \tag{2}$$

We consider a European call struck at K and expiring in T years. Assuming that the numeraire has been chosen so that $\mathcal{N}(T) = 1$, we can write its today's value as

$$\begin{aligned} P^{\text{call}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\, \mathsf{E}^{\mathsf{Q}}\left[\max\left(F(T) - K, 0\right)\right] \\ &= \mathcal{N}(0)\, \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} \max\left(F_0 e^{\sigma W - \frac{1}{2}\,\sigma^2 T} - K, 0\right) e^{-\frac{W^2}{2T}}\, dW, \end{aligned} \tag{3}$$

where E^{Q} denotes expected value with respect to Q. The last integral can easily be carried out, and we find that

$$\begin{aligned} P^{\text{call}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\left[F_0 N(d_+) - K N(d_-)\right] \\ &\equiv \mathcal{N}(0)\, B_{\text{call}}(T, K, F_0, \sigma). \end{aligned} \tag{4}$$

Here,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \tag{5}$$

is the cumulative normal distribution, and

$$d_\pm = \frac{\log\dfrac{F_0}{K} \pm \dfrac{1}{2}\,\sigma^2 T}{\sigma\sqrt{T}}\ . \tag{6}$$

Similarly, the price of a European put is given by:

$$\begin{aligned} P^{\mathrm{put}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\left[-F_0 N(-d_+) + K N(-d_-)\right] \\ &\equiv \mathcal{N}(0)\, B_{\mathrm{put}}(T, K, F_0, \sigma)\,. \end{aligned} \tag{7}$$

3.2 Valuation of caps and floors

A cap is a basket of options on LIBOR forward rates. Consider the OIS forward rate $F(S, T)$ covering the accrual period $[S, T]$. Its time $t \le S$ value $F(t, S, T)$ can be expressed in terms of discount factors:

$$\begin{aligned} F(t, S, T) &= \frac{1}{\delta}\left(\frac{P(t, t, S)}{P(t, t, T)} - 1\right) \\ &= \frac{1}{\delta}\ \frac{P(t, t, S) - P(t, t, T)}{P(t, t, T)}\ . \end{aligned} \tag{8}$$

The interpretation of this identity is that $F(t, S, T)$ is a tradable asset if we use the zero coupon bond maturing in T years as numeraire. Indeed, the trade is as follows:

(a) Buy $1/\delta$ face value of the zero coupon bond for maturity S.

(b) Sell $1/\delta$ face value of the zero coupon bond for maturity T.

The value of this position in the units of $P(t, t, T)$ is $F(t, S, T)$. An OIS forward rate can thus be modeled as a martingale! We call the corresponding martingale measure the T*-forward measure* and denote it by Q_T.

Consider now a LIBOR forward $L(S, T)$ spanning the same accrual period. Throughout these lectures we will make the assumption that the LIBOR / OIS spread is deterministic (rather than stochastic). This assumption is, clearly, a gross oversimplification of reality but it has some merits. There are no liquidly trading options on this spread, and thus calibrating a model with a stochastic spread is problematic. On the theoretical side, the picture is more transparent with a deterministic spread. Namely, we have from formula (13) of Lecture 1,

$$\begin{aligned} L(t,S,T) &= F(t,S,T) + B(t,S,T) \\ &= \frac{1}{\delta}\,\frac{P(t,t,S) - P(t,t,T) + \delta B(t,S,T)P(t,t,T)}{P(t,t,T)}\,. \end{aligned} \tag{9}$$

This shows that the LIBOR forward is a martingale under the T-forward measure Q_T.

Choosing, for now, the process to be (1), we conclude that the price of a call on $L(S,T)$ (or caplet) is given by

$$P^{\text{caplet}}(T,K,L_0,\sigma) = \delta P_0(0,T)B^{\text{call}}(S,K,L_0,\sigma), \tag{10}$$

where L_0 denotes here today's value of the forward, namely $L(0,S,T) = L_0(S,T)$.

Since a cap is a basket of caplets, its value is the sum of the values of the constituent caplets:

$$P^{\text{cap}} = \sum_{j=1}^{n} \delta_j B^{\text{call}}(T_{j-1},K,L_j,\sigma_j)P_0(0,T_j), \tag{11}$$

where δ_j is the day count fraction applying to the accrual period starting at T_{j-1} and ending at T_j, and L_j is the LIBOR forward rate for that period. Notice that, in the formula above, the date T_{j-1} has to be adjusted to accurately reflect the expiration date of the option (2 business days before the start of the accrual period). Similarly, the value of a floor is

$$P^{\text{floor}} = \sum_{j=1}^{n} \delta_j B^{\text{put}}(T_{j-1},K,L_j,\sigma_j)P_0(0,T_j). \tag{12}$$

What is the at the money (ATM) cap? Characteristic of an ATM option is that the call and put struck ATM have the same value. We shall first derive a put / call parity relation for caps and floors. Let E^{Q_j} denote expected value with respect to the T_j-forward measure Q_{T_j}. Then,

$$\begin{aligned} &P^{\text{floor}} - P^{\text{cap}} \\ &\quad= \sum_{j=1}^{n} \delta_j \left(\mathsf{E}^{\mathsf{Q}_j}\left[\max\left(K - L_j, 0\right)\right] - \mathsf{E}^{\mathsf{Q}_j}\left[\max\left(L_j - K, 0\right)\right]\right) P_0(0,T_j) \\ &\quad= \sum_{j=1}^{n} \delta_j \mathsf{E}^{\mathsf{Q}_j}\left[K - L_j\right] P_0(0,T_j). \end{aligned}$$

Now, the expected value $\mathsf{E}^{\mathsf{Q}_j}[L_j]$ is the current value of the LIBOR forward $L_0(T_{j-1}, T_j)$. Hence we have arrived at the following put / call parity relation:

$$P^{\text{floor}} - P^{\text{cap}} = K \sum_{j=1}^{n} \delta_j P_0(0, T_j) - \sum_{j=1}^{n} \delta_j L_0(T_{j-1}, T_j) P_0(0, T_j), \tag{13}$$

which is the present value of the swap receiving K on the quarterly, act/360 basis. This is an important relation. It implies that:

(a) It is natural to think about a floor as a call option, and a cap as a put option. The underlying asset is the forward starting swap on which both legs pay quarterly and interest accrues on the act/360 basis. The coupon dates on the swap coincide with the payment dates on the cap / floor.

(a) The ATM rate is the break-even rate on this swap. This rate is close to but not identical to the break-even rate on the standard semi-annual swap.

3.3 Valuation of swaptions

Consider a swap that settles at T_0 and matures at T. Let $S(t, T_0, T)$ denote the corresponding (break-even) forward swap rate observed at time $t < T_0$ (in particular, $S_0(T_0, T) = S(0, T_0, T)$). We know from Lecture 1 that the forward swap rate is given by

$$S(t, T_0, T) = \frac{\sum_{1 \le j \le n_{\text{f}}} \delta_j L_j P(t, T_{\text{val}}, T_j^{\text{f}})}{A(t, T_{\text{val}}, T_0, T)}, \tag{14}$$

where $T_{\text{val}} \le T_0$ is the valuation date of the swap (its choice has no impact on the value of the rate). Here, B_j is the LIBOR / OIS spread, and $A(t, T_{\text{val}}, T_0, T)$ is the forward level function:

$$A(t, T_{\text{val}}, T_0, T) = \sum_{1 \le j \le n_{\text{c}}} \alpha_j P(t, T_{\text{val}}, T_j^{\text{c}}). \tag{15}$$

Using formula (4) of Lecture 1, we write this as

$$\begin{aligned} S(t, T_0, T) &= \frac{\sum_{1 \le j \le n_{\text{f}}} \delta_j L_j P(t, t, T_j^{\text{f}})}{A(t, t, T_0, T)} \\ &= \frac{P(t, t, T_0) - P(t, t, T) + \sum_{1 \le j \le n_{\text{f}}} \delta_j B_j P(t, t, T_j^{\text{f}})}{A(t, t, T_0, T)}. \end{aligned} \tag{16}$$

The forward level function $A(t, t, T_0, T)$ is the time t present value of an *annuity* paying $\$1$ on the dates $T_1^{\mathrm{c}}, \ldots, T_{n_{\mathrm{c}}}^{\mathrm{c}}$, as observed at t.

As in the case of a simple LIBOR forward, the interpretation of (16) is that $S(t, T_0, T)$ is a tradable asset if we use the annuity as numeraire. Recall that we are assuming that all the LIBOR / OIS spreads are deterministic. Indeed, consider the following trade:

(a) Buy $\$1$ face value of the zero coupon bond for maturity T_0.

(b) Sell $\$1$ face value of the zero coupon bond for maturity T.

(c) Buy a stream of $\delta_j B_j$ face value zero coupon bonds for maturity T_j^{f}, $j = 1, \ldots, n_{\mathrm{f}}$.

A forward swap rate can thus be modeled as a martingale! We call the martingale measure associated with the annuity numeraire the *swap measure*.

Choosing, again, the lognormal process (1), we conclude that today's value of a receiver swaption is thus given by

$$P^{\mathrm{rec}} = A_0(T_0, T) B^{\mathrm{put}}(T_0, K, S_0, \sigma), \tag{17}$$

and the value of a payer swaption is

$$P^{\mathrm{pay}} = A_0(T_0, T) B^{\mathrm{call}}(T_0, K, S_0, \sigma). \tag{18}$$

Here $A_0(T_0, T) = A(0, 0, T_0, T)$, i.e.

$$A_0(T_0, T) = \sum_{1 \leq j \leq n_{\mathrm{c}}} \alpha_j P_0(0, T_j^{\mathrm{c}}) \tag{19}$$

(all discounting is done to today), and S_0 is today's value of the forward swap rate $S_0(T_0, T)$.

The put / call parity relation for swaptions is easy to establish, namely

$$P^{\mathrm{rec}} - P^{\mathrm{pay}} = \text{PV of the swap paying } K \text{ on the semi-annual, 30/360 basis.}$$

Therefore,

(a) It is natural to think about a receiver as a call option, and a payer as a put option.

(a) The ATM rate is the break-even rate on the underlying forward starting swap.

4 Beyond Black's model

The basic premise of Black's model, that σ is independent of T, K, and F_0, is not supported by the interest rates markets. In fact, option implied volatilities exhibit:

(a) *Term structure*: At the money volatility depends on the option expiration.

(b) *Smile* (or *skew*): For a given expiration, there is a pronounced dependence of implied volatilities on the option strike.

These phenomena became pronounced in the mid nineties or so and, in order to accurately value and risk manage options portfolios, refinements to Black's model are necessary. Modeling term structure of volatility is hard, and not much progress has been made. We will focus on modeling volatility smile.

An improvement over Black's model is a class of models called *local volatility models*. The idea is that even though the exact nature of volatility (it could be stochastic) is unknown, one can, in principle, use the market prices of options in order to recover the risk neutral probability distribution of the underlying asset. This, in turn, will allow us to find an effective ("local") specification of the underlying process so that the implied volatilities match the market implied volatilities. Such specifications can be justified by developing suitable underlying economic models but, for our purposes, these underlying micro models will be irrelevant, and we will regard a local model as a fit to the observed market.

Local volatility models are usually specified in the form

$$dF(t) = C(t, F(t))dW(t), \tag{20}$$

where $C(t, F)$ is a certain effective instantaneous volatility. In general, $C(t, F(t))$ is not given in a parametric form. It is, however, a matter of convenience to work with a parametric specification that fits the market data best. Popular local volatility models which admit analytic solutions include:

(a) The normal model.

(b) The shifted lognormal model.

(c) The CEV model.

We now briefly discuss the basic features of these models.

4.1 Normal model

The dynamics for the forward rate $F(t)$ in the normal model reads

$$dF(t) = \sigma dW(t), \tag{21}$$

under the suitable choice of numeraire. The parameter σ is appropriately called the *normal volatility*. This is easy to solve:

$$F(t) = F_0 + \sigma W(t). \tag{22}$$

This solution exhibits one of the main drawbacks of the normal model: with non-zero probability, $F(t)$ may become negative in finite time. Under typical circumstances, this is, however, a relatively unlikely event.

Prices of European calls and puts are now given by:

$$\begin{aligned} P^{\text{call}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\, B_{\text{n}}^{\text{call}}(T, K, F_0, \sigma), \\ P^{\text{put}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\, B_{\text{n}}^{\text{put}}(T, K, F_0, \sigma). \end{aligned} \tag{23}$$

The functions $B_{\text{n}}^{\text{call}}(T, K, F_0, \sigma)$ and $B_{\text{n}}^{\text{put}}(T, K, F_0, \sigma)$ are given by:

$$\begin{aligned} B_{\text{n}}^{\text{call}}(T, K, F_0, \sigma) &= \sigma\sqrt{T}\Big(d_+ N(d_+) + N'(d_+)\Big), \\ B_{\text{n}}^{\text{put}}(T, K, F_0, \sigma) &= \sigma\sqrt{T}\Big(d_- N(d_-) + N'(d_-)\Big), \end{aligned} \tag{24}$$

where

$$d_\pm = \pm\,\frac{F_0 - K}{\sigma\sqrt{T}}\ . \tag{25}$$

In order to see it, we write (22) as $F(t) = F_0 + \sigma\sqrt{t}X$, with $X \sim N(0,1)$. In the case of a call option,

$$\begin{aligned} \mathsf{E}[(F(T) - K)^+] &= \mathsf{E}[(F_0 + \sigma\sqrt{T}X - K)^+] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma\sqrt{T}X + F_0 - K)^+\, e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-(F_0-K)/\sigma\sqrt{T}}^{\infty} (F_0 - K + \sigma\sqrt{T}x)\, e^{-x^2/2} dx \\ &= \sigma\sqrt{T}\,\frac{1}{\sqrt{2\pi}} \left(d_+ \int_{d_-}^{\infty} e^{-x^2/2} dx + \int_{d_-}^{\infty} x e^{-x^2/2} dx\right) \\ &= \sigma\sqrt{T}\,\frac{1}{\sqrt{2\pi}} \left(d_+ \int_{-\infty}^{d_+} e^{-x^2/2} dx + e^{-d_+^2/2} dx\right) \\ &= \sigma\sqrt{T}\Big(d_+ N(d_+) + N'(d_+)\Big). \end{aligned}$$

The calculation in the case of a put proceeds along the same lines.

The normal model is (in addition to the lognormal model) an important benchmark in terms of which implied volatilities are quoted. In fact, many traders are in the habit of thinking in terms of normal implied volatilities, as the normal model often seems to capture the rates dynamics better than the lognormal (Black's) model.

4.2 Shifted lognormal model

The *shifted lognormal model* (also known as the *displaced diffusion model*) is a diffusion process whose volatility structure is a linear interpolation between the normal and lognormal volatilities. Its dynamics reads:

$$dF(t) = (\sigma_1 F(t) + \sigma_0)\, dW(t).$$

The volatility structure of the shifted lognormal model is given by the values of the parameters σ_1 and σ_0.

Prices of calls and puts are given by the following valuation formulas:

$$\begin{aligned} P^{\text{call}}(T, K, F_0, \sigma_0, \sigma_1) &= \mathcal{N}(0)\, B^{\text{call}}_{\text{sln}}(T, K, F_0, \sigma_0, \sigma_1), \\ P^{\text{call}}(T, K, F_0, \sigma_0, \sigma_1) &= \mathcal{N}(0)\, B^{\text{put}}_{\text{sln}}(T, K, F_0, \sigma_0, \sigma_1). \end{aligned} \tag{26}$$

The functions $B^{\text{call}}_{\text{sln}}(T, K, F_0, \sigma_0, \sigma_1)$ and $B^{\text{put}}_{\text{sln}}(T, K, F_0, \sigma_0, \sigma_1)$ are generalizations of the corresponding functions for the lognormal and normal models:

$$B^{\text{call}}_{\text{sln}}(T, K, F_0, \sigma_0, \sigma_1) = \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) N(d_+) - \left(K + \frac{\sigma_0}{\sigma_1}\right) N(d_-), \tag{27}$$

where

$$d_\pm = \frac{\log \dfrac{\sigma_1 F_0 + \sigma_0}{\sigma_1 K + \sigma_0} \pm \dfrac{1}{2}\, \sigma_1^2 T}{\sigma_1 \sqrt{T}}\ , \tag{28}$$

and

$$B^{\text{put}}_{\text{sln}}(T, K, F_0, \sigma_0, \sigma_1) = -\left(F_0 + \frac{\sigma_0}{\sigma_1}\right) N(-d_+) + \left(K + \frac{\sigma_0}{\sigma_1}\right) N(-d_-). \tag{29}$$

The shifted lognormal model is used by some market practitioners as a convenient compromise between the normal and lognormal models. It captures some aspects of the volatility smile.

4.3 The CEV model

Another model in-between the normal and lognormal models is the *CEV model*[1], whose volatility structure is a power interpolation between the normal and lognormal volatilities. Its dynamics is explicitly given by

$$dF(t) = \sigma F(t)^{\beta} dW(t),$$

where $\beta < 1$. Note that the exponent β is allowed to be negative. In order for the dynamics to be well defined, we have to prevent $F(t)$ from becoming negative (otherwise $F(t)^{\beta}$ would turn imaginary!). To this end, we specify a boundary condition at $F = 0$. It can be

(a) *Dirichlet* (absorbing): $F|_0 = 0$. Solution exists for all values of β, or

(b) *Neumann* (reflecting): $F'|_0 = 0$. Solution exists for $\frac{1}{2} \leq \beta < 1$.

Unlike the models discussed above, where the option valuation formulas can be obtained by purely probabilistic methods, the CEV model requires solving a terminal value problem for a partial differential equation, namely the backward Kolmogorov equation:

$$\begin{aligned} &\frac{\partial}{\partial t} B(t, f) + \frac{1}{2} \sigma^2 f^{2\beta} \frac{\partial^2}{\partial f^2} B(t, f) = 0, \\ &B(T, f) = \begin{cases} (f - K)^+, & \text{for a call}, \\ (K - f)^+, & \text{for a put}, \end{cases} \end{aligned} \tag{30}$$

This equation has to be supplemented by a boundary condition, Dirichlet of Neumann, at zero f.

Pricing formulas for the CEV model can be obtained in a closed (albeit somewhat complicated) form. For example, in the Dirichlet case the prices of calls and puts are:

$$\begin{aligned} P^{\text{call}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\, B^{\text{call}}_{\text{CEV}}(T, K, F_0, \sigma), \\ P^{\text{put}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\, B^{\text{put}}_{\text{CEV}}(T, K, F_0, \sigma). \end{aligned} \tag{31}$$

The functions $B^{\text{call}}_{\text{CEV}}(T, K, F_0, \sigma)$ and $B^{\text{put}}_{\text{CEV}}(T, K, F_0, \sigma)$ are the time $t = 0$ solutions to the terminal value problem (30), and can be expressed in terms of the

[1]CEV stands for "constant elasticity of variance".

cumulative function of the non-central χ^2 distribution:

$$\chi^2\left(x;r,\lambda\right)=\int_0^x p\left(y;r,\lambda\right)dy, \tag{32}$$

whose density is given by a Bessel function [2]:

$$p\left(x;r,\lambda\right)=\frac{1}{2}\left(\frac{x}{\lambda}\right)^{(r-2)/4}\exp\left(-\frac{x+\lambda}{2}\right)I_{(r-2)/2}\left(\sqrt{\lambda x}\right). \tag{33}$$

We also need the quantity:

$$\nu=\frac{1}{2\left(1-\beta\right)}\ ,\quad \text{i.e. } \nu\geq\frac{1}{2}\ . \tag{34}$$

A tedious computation shows then that the valuation formulas for calls and puts under the CEV model with the Dirichlet boundary condition read:

$$\begin{aligned}B^{\text{call}}_{\text{CEV}}(T,K,F_0,\sigma)=F_0\Big(1-\chi^2\Big(\frac{4\nu^2K^{1/\nu}}{\sigma^2T};\ 2\nu+2,\frac{4\nu^2F_0^{1/\nu}}{\sigma^2T}\Big)\Big)\\ -K\chi^2\Big(\frac{4\nu^2F_0^{1/\nu}}{\sigma^2T};\ 2\nu,\frac{4\nu^2K^{1/\nu}}{\sigma^2T}\Big)\ ,\end{aligned} \tag{35}$$

and

$$\begin{aligned}B^{\text{put}}_{\text{CEV}}(T,K,F_0,\sigma)=F_0\chi^2\Big(\frac{4\nu^2K^{1/\nu}}{\sigma^2T};\ 2\nu+2,\frac{4\nu^2F_0^{1/\nu}}{\sigma^2T}\Big)\\ -K\Big(1-\chi^2\Big(\frac{4\nu^2F_0^{1/\nu}}{\sigma^2T};\ 2\nu,\frac{4\nu^2K^{1/\nu}}{\sigma^2T}\Big)\Big)\ ,\end{aligned} \tag{36}$$

respectively.

From these formulas one can deduce that the terminal probability density $g(T,F)$ is given by

$$g(T,F)=\frac{4\nu F_0F^{1/\nu-2}}{\sigma^2T}\ p\left(\frac{4\nu^2F^{1/\nu}}{\sigma^2T};\ 2\nu+2,\ \frac{4\nu^2F_0^{1/\nu}}{\sigma^2T}\right). \tag{37}$$

This is the "transition portion" of the process only. Indeed, the total mass of the density $g(T,F)$ is less than one, meaning that there is a nonzero probability of absorption at zero. Using the series expansion [2]:

$$I_\nu\left(z\right)=\sum_{k\geq0}\ \frac{1}{k!\,\Gamma\left(\nu+k+1\right)}\left(\frac{z}{2}\right)^{2k+\nu}\ , \tag{38}$$

we readily find that

$$\int_0^\infty g(T,F)\,dF = 1 - \frac{1}{\Gamma(\nu)}\,\Gamma\Big(\nu,\ \frac{2\nu^2 F_0^{1/\nu}}{\sigma^2 T}\Big), \tag{39}$$

where

$$\Gamma\left(\nu,\ x\right) = \int_x^\infty t^{\nu-1}e^{-t}dt \tag{40}$$

is the complementary incomplete gamma function [2].

The quantity

$$\frac{1}{\Gamma(\nu)}\,\Gamma\Big(\nu,\ \frac{2\nu^2 F_0^{1/\nu}}{\sigma^2 T}\Big) \tag{41}$$

is the probability of absorption at zero. For example, in the square root process case, i.e. $\nu = 1$, that probability equals $\exp(-\frac{2F_0}{\sigma^2 T})$. The total terminal probability is thus the sum of $g(T,F)$ and the Dirac delta function $\delta(F)$ multiplied by the absorption probability (41).

Similar valuation formulas hold for the Neumann boundary condition but we will not reproduce them here.

References

[1] Gatheral, J.: *The Volatility Surface: A Practitioner's Guide*, Wiley (2006).

[2] Whittaker, E. T., and Watson, G. N.: *A Course of Modern Analysis*, Cambridge University Press (1996).