Martingales ; Stopping Times

Wednesday, November 23, 2011
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Let $(\Omega, \mathcal{F}, P)$ be a probability space.

<u>Def 1</u>: A non-decreasing sequence of σ-algebras
$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}$ is called a FILTRATION

<u>Def 2</u>: A sequence of random variables $\{M_n\}_{n \geq 0}$ on $(\Omega, \mathcal{F}, P)$ is said to be a MARTINGALE with respect to the filtration $\{\mathcal{F}_n\}_{n \geq 0}$ if

(i) $E|M_n| < \infty$ for all $n \geq 0$

(ii) M_n is $\mathcal{F}_n$ measurable

(iii) $E[M_{n+1} \mid \mathcal{F}_n] = M_n$ for all $n \geq 0$, a.s.

("$\leq$" gives a SUPERMARTINGALE
"$\geq$" gives a SUBMARTINGALE)

<u>Example</u>: $\rightsquigarrow$ the random walk $W_n = \sum_{i=1}^{n} X_i$, $W_0 = 0$

$X_1, X_2, \dots$ iid on $(\Omega, \mathcal{F}, P)$ $EX = \mu$

- $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n) \rightsquigarrow \{\mathcal{F}_n\}_{n \geq 0}$ is a filtration
  (also called the NATURAL FILTRATION)

- if $\mu = 0 \Rightarrow \{W_n\}$ is martingale w.r.to $\{\mathcal{F}_n\}$
- if $\mu > 0 \Rightarrow \{W_n\}$ is submartingale w.r.to $\{\mathcal{F}_n\}$
- if $\mu < 0 \Rightarrow \{W_n\}$ is supermartingale w.r.to $\{\mathcal{F}_n\}$

1. if $\mu < 0 \Rightarrow \{W_n\}$ is supermartingale w.r.to $\{\mathcal{F}_n\}$

- let $X \in L^1(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_n\}_{n\geq 0}$ a filtration

$\rightsquigarrow$ define $M_n := E[X \mid \mathcal{F}_n]$ for all n

Claim: $\{M_n\}_{n\geq 0}$ is martingale w.r.to $\{\mathcal{F}_n\}$

(i) $E|M_n| = E(|E(X|\mathcal{F}_n)|) \underset{\substack{\uparrow \\ \text{conditional Jensen's ineq.}}}{\leq} E(E[|X| \mid \mathcal{F}_n]) = E|X| < \infty$

(ii) M_n is $\mathcal{F}_n$ meas by the definition of conditional exp.

(iii) $E[M_{n+1} | \mathcal{F}_n] = E[E[X|\mathcal{F}_{n+1}] | \mathcal{F}_n] \underset{\substack{\uparrow \\ \text{Tower property}}}{=} E[X|\mathcal{F}_n] = M_n$

Definition: A martingale $\{M_n\}_{n\geq 0}$ is said to be **CLOSED** by a random variable X if $E|X| < \infty$ and $M_n = E[X|\mathcal{F}_n]$ a.s.

Question: How do we know is a martingale is closed by a random variable?

- the answer is given by the **UNIFORM INTEGRABILITY** of the martingale $\{M_n\}$

Def: $\{M_n\}$ is **UNIFORMLY INTEGRABLE** **(U.I)** if:

$$\lim_{N\to\infty} \sup_n E\left[|M_n| \mathbb{1}_{\{|M_n| > B\}}\right] = 0$$

Martingale Convergence Theorem

(a) Let $\{M_n\}_{n\geq 0}$ be a UI martingale. Then

$$\lim_{n\to\infty} M_n = M_\infty \quad \text{exists. a.s.}$$

where $M_\infty \in L^1$, and $M_n \xrightarrow{L^1} M_\infty$. Moreover

$M_n = E[M_\infty | \mathcal{F}_n]$ a.s for all n

(that is the martingale is closed by M_∞)

(b) Conversely, let $X \in L^1$, $M_n = E[Y | \mathcal{F}_n]$ a.s for all n

then $\{M_n\}$ is U.I. martingale

Remark on how do we know if a family of r.v. is U.I.

(1) if $\sup_n E|X_n|^p < \infty$ for some $p > 1 \Rightarrow \{X_n\}$ is UI

(2) if $\{X_n\}$ are dominated a.s. by $Y \in L^1 \Rightarrow \{X_n\}$ is U.I

Martingale inequalities: (Doob)

- let's call $M_n^* = \sup_{j\leq n} |M_j|$ where $\{M_n\}$ martingale

$$\rightsquigarrow \quad P(M_n^* \geq \alpha) \leq \frac{E|M_n|}{\alpha}$$

- also for any $p > 1$ we have

$$E\{(M_n^*)^p\} \leq \left(\frac{p}{p-1}\right)^p E|M_n|^p$$

or otherwise said in the notation of L^p norms

$$\| M_n^* \|_p \le \frac{p}{p-1} \| M_n \|_p$$

- let $\varphi(\cdot)$ be a convex function, $\{M_n\}_{n\ge 0}$ martingale w.r.t. $\{\mathcal{F}_n\}$. Then $\{\varphi(M_n)\}_{n\ge 0}$ is a submartingale w.r.t. $\{\mathcal{F}_n\}$ provided that $\varphi(M_n) \in L^1(\Omega, \mathcal{F}, P)$

<u>Proof</u> . to prove the submartingale property we shall use Jensen inequality for conditional expectations

$$E\left[\varphi(M_{n+1}) | \mathcal{F}_n \right] \ge \varphi\left(E(M_{n+1} | \mathcal{F}_n) \right) = \varphi(M_n) \quad \text{a.s.}$$

$$\varphi(M_n) = \varphi\left(E(M_{n+1} | \mathcal{F}_n) \right) \le E\left[\varphi(M_{n+1}) | \mathcal{F}_n \right]$$

M_n : martingale : $E M_n = E M_{n-1} = \dots = E M_0$

submartingale : $E M_n \ge E M_{n-1} \ge \dots \ge E M_0$

supermartingale : $E M_n \le E M_{n-1} \le \dots \le E M_0$

<u>Doob Decomposition Theorem</u> :

$\{X_n\}_{n\ge 1}$ is a submartingale on $(\Omega, \mathcal{F}, P)$ then X_n can be written as : $X_n = M_n + A_n$ with the following prop.

(i) $\{M_n\}$ is martingale

(ii) $A_{n+1} \ge A_n$ a.s. for all $n \ge 1$. , $A_0 = 0$

$\{A_n\}$ predictable $\left(A_n \text{ is } \mathcal{F}_{n-1} \text{ measurable } (\forall) n \right)$

(iii) the decomposition is unique

Proof: define the sequence A_n as follows:

$\cdot \; A_n - A_{n-1} = E\left[X_n - X_{n-1} \mid \mathcal{F}_{n-1} \right] \geq 0$ a.s.

(because of the submartingale prop.)

$\rightsquigarrow A_n = A_{n-1} + E\left[X_n - X_{n-1} \mid \mathcal{F}_{n-1} \right]$

- non decreasing. $(A_0 = 0)$.
- $\mathcal{F}_{n-1}$ measurable $(A_n = \text{predictable})$

$\rightsquigarrow M_n = X_n - A_n$

$\cdot$ do we have $E(M_n \mid \mathcal{F}_{n-1}) = M_{n-1}$?

$$A_n = A_{n-1} + E(X_n - X_{n-1} \mid \mathcal{F}_{n-1})$$

$$E(M_n \mid \mathcal{F}_{n-1}) = E(X_n - A_n \mid \mathcal{F}_{n-1}) =$$

$$= E(X_n \mid \mathcal{F}_{n-1}) - A_{n-1} - E(X_n - X_{n-1} \mid \mathcal{F}_{n-1}) =$$

$$= X_{n-1} - A_{n-1} = M_{n-1}$$

- uniqueness follows from the predictability of $\{A_n\}$

$\rightsquigarrow$ suppose there is another decomposition

$X_n = N_n + B_n$ s.t $\{N_n\}$ is martingale and

$X_n = N_n + B_n$ s.t $\{N_n\}$ is martingale and $\{B_n\}$ is predictable, non-decreasing $B_0 = 0$

$$\Rightarrow M_n = E\left[M_{n+1} \middle| \mathcal{F}_n\right] = E\left[X_{n+1} - A_{n+1} \middle| \mathcal{F}_n\right] =$$

$$= E\left[N_{n+1} + B_{n+1} - A_{n+1} \middle| \mathcal{F}_n\right] =$$

$$= N_n + B_{n+1} - A_{n+1}$$

$$\Rightarrow \underbrace{M_n - N_n}_{\text{martingale}} = B_{n+1} - A_{n+1} \quad \text{a.s.} \quad \text{for all } n$$

martingale $\Rightarrow \{B_n - A_n\}$ is martingale as well

$$\Rightarrow E\left[B_{n+1} - A_{n+1} \middle| \mathcal{F}_n\right] = B_n - A_n$$

but $B_{n+1} - A_{n+1}$ is also $\mathcal{F}_n$ measurable

so $E\left[B_{n+1} - A_{n+1} \middle| \mathcal{F}_n\right] = B_{n+1} - A_{n+1}$

$$\Rightarrow B_{n+1} - A_{n+1} = B_n - A_n = \dots = B_0 - A_0 = 0 \quad \text{a.s.}$$

$\Rightarrow B_n = A_n$ as for all n so the decomposition is unique

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Another way in which martingales can arise:

Let $(\Omega, \mathcal{F}, P)$ be a prob. space, $\{\mathcal{F}_n\}$ = filtration

- Q is another probability on $(\Omega, \mathcal{F})$
- $Q \ll P$ on each $\mathcal{F}_n$ $\Rightarrow$ $Z_n = \left.\dfrac{dQ}{dP}\right|_{\mathcal{F}_n}$

$\leadsto \{Z_n\}$ is a non-negative martingale with respect to $\{\mathcal{F}_n\}$

$\rightsquigarrow \{Z_n\}$ is a non-negative martingale with respect to $\{\mathcal{F}_n\}$

$(EZ_n = 1)$

- if $Q \ll P$ on $\mathcal{F}$ $\Rightarrow$ $Z = \frac{dQ}{dP}$ closes $\{Z_n\}$

that is: $Z_n = E[Z \mid \mathcal{F}_n]$ a.s.

- Another example of a martingale:

Def: Let $\{M_n\}$ be a martingale, $\{H_n\}$ predictable w.r.t. $\{\mathcal{F}_n\}$

The sequence:

$$Y_n = Y_0 + \sum_{k=1}^{n} H_k (M_k - M_{k-1}) \qquad n \geq 0 \qquad (Y_0 = M_0)$$

is called **STOCHASTIC INTEGRAL** of H w.r.t. M.

Th: If $\{H_n\}$ is bounded $\Rightarrow$ $\{Y_n\}$ is a martingale

Proof: easy!

Stopping times:

Def: A random time $\tau: \Omega \to \mathbb{N} \cup \{\infty\}$ is called a **STOPPING TIME** with respect to a filtration $\{\mathcal{F}_n\}$ if for every $n \geq 0$: $\{\tau \leq n\} \in \mathcal{F}_n$

- in other words: τ is **NON-ANTICIPATIVE**

Lemma 1: τ is a stopping time iff $\{\tau = n\} \in \mathcal{F}_n$ for all n

Lemma 1: τ is a stopping time iff $\{\tau = n\} \in \mathcal{F}_n$ for all n

- in discrete time the above lemma is often taken as the definition of the stopping time
- in continuous time it not true.

Lemma 2: If τ_1, τ_2 are stopping times then $\tau_1 \vee \tau_2$, $\tau_1 \wedge \tau_2$, $\tau_1 + \tau_2$ are also stopping times

Optional Sampling Theorem:

$\{M_n\}$ martingale on $(\Omega, \mathcal{F}, P)$ w.r.t. $\{\mathcal{F}_n\}_n$

τ is a bounded stopping time w.r.t. $\{\mathcal{F}_n\}$

($\tau \leq K$ a.s. for a fixed constant K)

Then: $\mathbb{E} M_\tau = \mathbb{E} M_0$

Proof:
$$M_\tau = \sum_{n=0}^{\infty}{}' \; 1_{\{\tau = n\}} = \sum_{n=0}^{\infty} M_n 1_{\{\tau = n\}}$$

$$\mathbb{E}[M_\tau] = \mathbb{E}\left[\sum_{n=0}^{\infty}{}' M_n 1_{\{\tau=n\}}\right] = \sum_{n=1}^{K}{}' \mathbb{E}\left[M_n 1_{\{\tau=n\}}\right]$$

(because M_n martingale: $\mathbb{E}[M_K \mid \mathcal{F}_n] = M_n$)

$$= \sum_{n=1}^{K}{}' \mathbb{E}\left[\mathbb{E}[M_K \mid \mathcal{F}_n] \cdot 1_{\{\tau=n\}}\right]$$

$$= \sum_{n=1}^{k}{}' E\left[E\left[M_k 1_{\{\tau = n\}} \,\middle|\, \tilde{\mathcal{F}}_n \right]\right]$$

$$= \sum_{n=1}^{K}{}' E\left[M_k 1_{\{\tau = n\}} \right]$$

$$= E\left[\sum_{n=1}^{k}{}' M_k 1_{\{\tau = n\}} \right] = E\left[M_k \cdot \sum_{n=1}^{K}{}' 1_{\{\tau = n\}} \right]$$

- $\tau \leq K$ a.s. $\Rightarrow$ $P(\tau \leq K) = 1$

$$= E[M_k] = E[M_0]$$

- this theorem seems limited, since it applies only to bdd stopping times.
- however, every stopping time τ can be truncated such as:

$\tau_n = \tau \wedge n$ stopping time.

$\tau_n \leq n \rightsquigarrow \tau_n$ is bounded.

- so even if τ is not bdd one can use the OST for the truncated version of τ : τ_n

$\rightsquigarrow EM_{\tau_n} = EM_0$

- also notice that $\lim_{n \to \infty} \tau_n = \tau$ a.s.

Lemma: let (M_n) be a martingale (sub or super)

Lemma: Let $(M_n)_{n\geq 0}$ be a martingale (sub or super) with respect to $\{\mathcal{F}_n\}_{n\geq 0}$, and τ be a stopping time. Then $\{M_{\tau\wedge n}\}_{n\geq 0}$ is a martingale w.r.t. $\{\mathcal{F}_n\}_{n\geq 0}$

- Moreover, if $\{M_n\}$ is U.I martingale $\Rightarrow$ $EM_\tau = EM_0$

  since $M_{\tau\wedge n} \longrightarrow M_\tau$ as $n\to\infty$ a.s.

" $M_{\tau\wedge n}$ is called STOPPED MARTINGALE "

$$M_{\tau\wedge n} = \begin{cases} M_n & \text{if } \tau > n \\ M_\tau & \text{if } \tau \leq n \end{cases}$$