Moment Generating Functions; Characteristic Functions

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- Independence
- Basic Properties of Characteristic Functions
- Examples
- Uniqueness Theorem. Independence & Characteristic Func
- Gaussian random variables : def, density, char. & moment generating func.

Independence

Def : $(\Omega, \mathcal{F}, P)$ prob. space
and $\mathcal{G}_1, \mathcal{G}_2$ are sub σ-algebras of $\mathcal{F}$
We say $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent iff

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

for all $A_1 \in \mathcal{G}_1$ and $A_2 \in \mathcal{G}_2$

Def : Let X_1, X_2 be random variables on $(\Omega, \mathcal{F}, P)$
We say that X_1, X_2 are independent iff
$\sigma(X_1)$ and $\sigma(X_2)$ are independent sub-σ-algebras.

$$P(X_1 \in B_1, X_2 \in B_2) = P(X_1 \in B_1) \cdot P(X_2 \in B_2)$$

$$(X_1 \in B) \in \sigma(X_1) = \{ X_1^{-1}(B) \mid B \in \mathcal{B} \}$$

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Moment Generating Functions

X : random variable on $(\Omega, \mathcal{F}, P)$

$$M_X(t) = E\, e^{tX} = \int_\Omega e^{tX}\, dP = \int_{\mathbb{R}} e^{tx}\, dP_X(x)$$

$$X : \Omega \longrightarrow \mathbb{R}$$

$$(\Omega, \mathcal{F}, P) \qquad (\mathbb{R}, \mathcal{B}, P_X)$$

$$\left(\quad P_X(B) = P(X \in B) \quad \right)$$

ex : $X \sim$ Bernoulli (p) $= \begin{cases} 1 & \text{with prob } p \\ 0 & 1-p \end{cases}$

$$\delta_y(B) = \begin{cases} 1 & \text{if } y \in B \\ 0 & \text{otherwise} \end{cases}$$

$$P_X = p \cdot \delta_1 + (1-p)\delta_0$$

$$M_X(t) = \int_{\mathbb{R}} e^{tx}\, dP_X(x) = \int_{\mathbb{R}} e^{tx} \left(p\, d\delta_1(x) + (1-p)\, d\delta_0(x) \right)$$

$$= p \cdot \int_{\mathbb{R}} e^{tx}\, d\delta_1(x) + (1-p) \int_{\mathbb{R}} e^{tx}\, d\delta_0(x)$$

$$= p \cdot e^{t \cdot 1} + (1-p) e^{t \cdot 0} = p e^{t} + (1-p)$$

ex : $X \sim \text{Bin}(2, p)$

$$P_X = (1-p)^2 \cdot \delta_0 + 2p(1-p)\delta_1 + p^2 \delta_2$$

$$B = (-1, 1] \qquad P_X(B) = (1-p)^2 + 2p(1-p)$$

ex : $X \sim \exp(\lambda)$ $\qquad f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

$$M_X(t) = \int_{\mathbb{R}} e^{tx}\, dP_X(x) = \int_{[0,\infty)} e^{tx} \lambda e^{-\lambda x}\, dx = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx$$

$$dP_X(x) = dF_X(x) = f_X(x)\, dx \qquad = P_X((-\infty, x])$$

$$f = F' \qquad F(x) = P(\underbrace{X \leq x})$$

$$M_X(t) = \frac{\lambda}{t-\lambda} e^{-(\lambda - t)x} \Big|_0^\infty = \frac{\lambda}{\lambda - t}$$

works for $t < \lambda$

· Recall that if X, Y are indep. iff

$$E e^{uX + vY} = E e^{uX} \cdot E e^{vY} \quad \text{for all } u, v$$

$$\left(M_{(X,Y)}(u,v) = M_X(u) \cdot M_Y(v) \right)$$

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- $$E X^k = \left. \frac{d^k M(t)}{dt^k} \right|_{t=0}$$

- if $M_X(t) = M_Y(t) \Rightarrow P_X = P_Y$

<u>example</u> : X : $f_X(x) = \frac{1}{2x^2} \mathbb{1}_{\{|x| \geq 1\}}(x)$

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{2x^2} \mathbb{1}_{\{|x| \geq 1\}}(x)\, dx$$

$$= \underbrace{\int_{-\infty}^{-1} e^{tx} \cdot \frac{1}{2x^2}\, dx}_{\substack{\text{converges for } t \geq 0 \\ \text{diverges for } t < 0}} + \underbrace{\int_{1}^{\infty} e^{tx} \frac{1}{2x^2}\, dx}_{\substack{\text{converges for } t \leq 0 \\ \text{diverges for } t > 0}} = \begin{cases} \infty & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

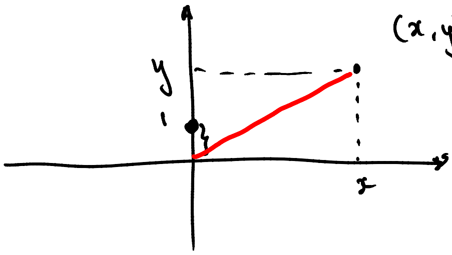
In general, if the tails of X have only power decay $P(|X| \geq x) \sim \frac{c}{|x|^\alpha}$ $\alpha > 0$

then the MGF does not exist!

We shall use the Characteristic function instead.

$M_X(t) : \mathbb{R} \longrightarrow \mathbb{R}$ $\qquad$ $M_X(t) = E\, e^{tX}$

$\varphi_X(t) : \mathbb{R} \longrightarrow \mathbb{C}$ $\qquad$ $\varphi_X(t) = E\, e^{itX}$



$(x,y) \longleftrightarrow z = x + iy$

$i = \sqrt{-1}$

$$|z| = \sqrt{x^2 + y^2}$$

$$e^{ix} = 1 + (ix) - \frac{x^2}{2} + \dots + \frac{(ix)^n}{n!} + \dots$$

$$\boxed{\text{Lemma : } \quad e^{ix} = \cos x + i \sin x}$$

$$\left| e^{ix} \right| = \sqrt{\cos^2 x + \sin^2 x} = 1$$

Def : $\varphi_X(t) = E\, e^{i\langle t, X\rangle}$ $\quad$ if $X = (X_1, X_2, \dots, X_n)$, $t = (t_1, t_2, \dots, t_n)$

Theorem : P_X distr. on $\mathbb{R}^n$ then $\varphi_X(t)$ is continuous, bounded on $\mathbb{R}^n$ with $\varphi_X(0) = 1$

Proof

$$|\varphi_X(t)| = \left| \int_{\mathbb{R}^n} e^{i\langle t, x\rangle} \, dP_X(x) \right| \le \int_{\mathbb{R}^n} \underbrace{\left| e^{i\langle t, x\rangle} \right|}_{=1} dP_X(x) = 1$$

$\varphi_X(0) = 1$

- $t_k \longrightarrow t$ as $k \to \infty$
- the function $x \longmapsto e^{i\langle t,x\rangle}$ is bded

$$\varphi_X(t_k) = \int_{\mathbb{R}^n} e^{i\langle t_k,x\rangle}\, dP_X(x) \longrightarrow \int_{\mathbb{R}^n} e^{i\langle t,x\rangle}\, dP_X(x)$$

$$\lim_{k\to\infty} \varphi_X(t_k) = \varphi_X(t)$$

Lebesgue Dominated Convergence Th.

example : $Z \sim N(0,1)$ $\quad \varphi_Z(t) = ?$

$$\varphi_Z(t) = E\, e^{itZ} = \int_{-\infty}^{+\infty} e^{itz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= \int_{-\infty}^{+\infty} \cos(tz) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz + \underbrace{i \int_{-\infty}^{+\infty} \underbrace{\sin(tz) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}_{\text{odd}}\, dz}_{=0}$$

$$\varphi_X'(t) = -\int_{-\infty}^{+\infty} z \sin(tz) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= \int^{+\infty} \sin(tz) \frac{1}{\sqrt{2\pi}}\, d\left(e^{-\frac{z^2}{2}}\right) =$$

$$= \int_{-\infty} \sin(tz) \frac{1}{\sqrt{2\pi}} \, d\left(e^{-\frac{z^2}{2}}\right) =$$

$$= -t \int_{-\infty}^{+\infty} \cos(tz) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

$$\Rightarrow \begin{cases} \varphi_Z'(t) = -t\,\varphi_Z(t) \\ \varphi_Z(0) = 1 \end{cases} \Rightarrow \boxed{\varphi_Z(t) = e^{-\frac{t^2}{2}}}$$

ex: $Y = AX + b$ $\qquad$ A $m \times n$ matrix

$b \in \mathbb{R}^m \quad X \in \mathbb{R}^n$

$$\varphi_Y(t) = E\, e^{i\langle t, Y\rangle} = E\, e^{i\langle t, AX+b\rangle}$$

$$= e^{i\langle t, b\rangle} E\, e^{i\langle t, AX\rangle} = e^{i\langle t, b\rangle} \cdot E\, e^{i\langle A^T t, X\rangle}$$

$$\Rightarrow \varphi_Y(t) = e^{i\langle t, b\rangle} \cdot \varphi_X(A^T t)$$

$n = m = 1$

$$\varphi_Y(t) = e^{itb} \varphi_X(at)$$

· If $X = \sigma Z + \mu$; $\varphi_Z(t) = e^{-\frac{t^2}{2}}$

$$\Rightarrow \varphi_X(t) = e^{it\mu} \cdot \varphi_Z(\sigma t) = e^{it\mu} \cdot e^{-\frac{\sigma^2 t^2}{2}}$$

$$\rightsquigarrow \varphi_X(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$$

Theorem : $X_1, X_2, \dots, X_n$ independent iff

$$\varphi_{(X_1,\dots,X_n)}(t_1,\dots,t_n) = \prod_{i=1}^{n} \varphi_{X_i}(t_i)$$

Uniqueness : $\varphi_X(t) = \varphi_Y(t) \Rightarrow P_X = P_Y$

Def Gaussian random variables

$X = (X_1, X_2, \dots, X_n)$ is said to be Gaussian if

for all $a \in \mathbb{R}^n \Rightarrow \langle a, X \rangle = \sum_{i=1}^{n} a_i X_i$ has a one dim. normal distr.

X is Gaussian iff

$$\varphi_X(t) = e^{i\langle t,\mu\rangle - \frac{1}{2}\langle t, Ct\rangle}$$

$\mu \in \mathbb{R}^n$ $\quad C \in \mathbb{R}^n \times \mathbb{R}^n$ (symmetric)

$\rightsquigarrow$ covariance matrix

Recall $\left(\varphi_X(t) = e^{it\mu - \frac{t^2\sigma^2}{2}} \quad \text{one dim. normal var} \right)$

Theorem : $X =$ Gaussian vector

$X_1, X_2, \ldots, X_n$ are independent iff they are uncorrelated!

Proof : Clue $\rightsquigarrow$ $C = \begin{pmatrix} \sigma_1^2 & 0 & 0 & \cdots \\ 0 & \sigma_2^2 & 0 & \cdots \\ \vdots & & & \\ 0 & & & \sigma_n^2 \end{pmatrix}$

$$\langle t, Ct \rangle = \sum_{i=1}^{n} t_i^2 \sigma_i^2$$

$$e^{-\frac{1}{2}\langle t, Ct \rangle} = \prod_{i=1}^{n} e^{-\frac{1}{2} t_i^2 \sigma_i^2}$$

Theorem : $X =$ Gaussian $\quad \mu =$ mean vector

Then there exis indep r.v. $Y_1, Y_2, \ldots, Y_n$

$Y_i \sim N(0, \sigma_i^2)$ and A an orthogonal matrix

such that $\boxed{X = \mu + AY}$

Proof : $C =$ symmetric

$\Rightarrow$ there is A : $\quad C = ADA^T$

$\Rightarrow$ there is A : $\quad C = A D A^T$

$$Y = A^T (X - \mu)$$

- easy to check that D is the covariance matrix of Y