Stochastic Differential Equations (SDE)

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SDE : $dX_t = \beta(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$

- $\beta(t,x)$, $\sigma(t,x)$ are the "drift", "diffusion"
- $X_0 = x_0$: initial condition
- solutions of such SDE's are called Itô diffusion.

Problem : find X_T such that :

$$\begin{cases} X_0 = x_0 \\ X_T = X_0 + \int_0^T \beta(t, X_t)\, dt + \int_0^T \sigma(t, X_t)\, dW_t \end{cases}$$

- for suitable drift $\beta(t,x)$, and diffusion $\sigma(t,x)$ coefficients we would like to answer questions concerning the existence, the uniqueness, and various other properties of the solution

example : • Generalized Geometric Brownian Motion :

SDE : $dS_t = \underbrace{\alpha_t S_t}_{\beta(t,S_t)} dt + \underbrace{\sigma_t S_t}_{\sigma(t,S_t)} dW_t$

drift diffussion

$$\rightsquigarrow \quad S_t = S_0 \exp\left\{ \int_0^t \sigma_s\, dW_s + \int_0^t \alpha_s - \frac{1}{2}\sigma_s^2\, ds \right\}$$

is the solution of the above SDE

(can be verified using Itô formula)

- for the generalized GBM : $\begin{cases} \beta(t,x) = \alpha_t \cdot x \\ \sigma(t,x) = \sigma_t \cdot x \end{cases}$ linear functions of x

<u>Linear Systems</u> : $\begin{cases} \beta(t,x) = A_t x + a_t \\ \sigma(t,x) = B_t x + b_t \end{cases}$

$$\boxed{dX_t = (A_t X_t + a_t)\,dt + (B_t X_t + b_t)\,dW_t}$$

⇝ this eq can also be solved explicitly

⇝ solution is a Gaussian process if $X_0 \sim$ normal distr. and X_0 is independent of W

⇝ recall the "Discounting" technique + Itô lemma

$$D_t = \exp\left\{-\int_0^t A_s\,ds\right\} : \quad dD_t = -A_t D_t\,dt$$

$$d(D_t X_t) = D_t\,dX_t + X_t\,dD_t + \underbrace{dX_t \cdot dD_t}_{=0}$$

$$= D_t(A_t X_t + a_t)\,dt + D_t(B_t X_t + b_t)\,dW_t - A_t D_t X_t\,dt$$

$$= D_t a_t\,dt + D_t(B_t X_t + b_t)\,dW_t$$

(1) • in particular if $B_t \equiv 0$ we can write the solution as:

$$D_t X_t = X_0 + \int_0^t D_s a_s\,ds + \int_0^t D_s b_s\,dW_s$$

$$\Rightarrow X_t = \frac{X_0}{D_t} + \frac{1}{D_t}\int_0^t D_u a_u\,du + \frac{1}{D_t}\int_0^t D_u b_u\,dW_u$$

- assumming $X_0 \sim N(m, \sigma^2)$ independent of W

$$\begin{cases} \bullet\ EX_t = \frac{1}{D_t}\left[m + \int_0^t D_u a_u\, du\right] \\ \bullet\ Cov(X_s, X_t) = E\left[(X_t - EX_t)\cdot(X_s - EX_s)\right] = \end{cases}$$

$$= E\left[\frac{1}{D_t}\left(X_0 - m + \int_0^t D_u b_u\, dW_u\right)\cdot\frac{1}{D_s}\left(X_0 - m + \int_0^s D_u b_u\, dW_s\right)\right]$$

$$= \frac{1}{D_t D_s} Var(X_0) + \frac{1}{D_t D_s}\left[\underbrace{E\left(\int_0^{s\wedge t} D_u b_u\, dW_u\right)^2}_{= \int_0^{s\wedge t} D_u^2 b_u^2\, du} + \underbrace{E\left(\int_{s\wedge t}^{s\vee t} D_u b_u\, dW_u\right)}_{=0}\right]$$

$$= \frac{1}{D_t D_s}\left[\sigma^2 + \int_0^{s\wedge t} D_u^2 b_u^2\, du\right]$$

(2) • for $A_t \equiv a_t \equiv b_t = 0 \rightsquigarrow dX_t = B_t X_t\, dW_t$

solution of this eq is given by:

$$X_t = X_0 \exp\left\{\int_0^t B_s\, dW_s - \frac{1}{2}\int_0^t B_s^2\, ds\right\}$$

(3) • for $A_t \equiv -\alpha < 0$, $b_t = b > 0$, $a_t = B_t = 0$

(Langevin equation)

$$dX_t = -\alpha X_t\, dt + b\, dW_t$$

this equation leads to the so called

Ornstein - Uhlenbeck process (Brownian Motion with proportional restoring force)

$$\rightsquigarrow \quad d\left(e^{\alpha t} X_t\right) = b\, e^{\alpha t}\, dW_t$$

③ Hull-White interest rate model :

$$dR_t = (a_t - b_t R_t)\,dt + \sigma_t\, d\widetilde{W}_t$$

($\widetilde{W}_t$: Brownian motion under the risk neutral measure $\widetilde{\mathbb{P}}$)

a_t, b_t, σ_t are deterministic positive functions

$$\begin{cases} \beta(t,r) = a_t - b_t \cdot r \\ \sigma(t,r) = \sigma_t \end{cases}$$

- initial condition $R_0 = r$

- to solve this eq, we can use the "discounting" once again!

$$D_t = e^{+\int_0^t b_s\,ds} \quad : \quad dD_t = b_t D_t\, dt$$

$$d(D_t R_t) = D_t\, dR_t + R_t\, dD_t + \underbrace{dR_t \cdot dt}_{=0}$$

$$= D_t(a_t - b_t R_t)\,dt + D_t \sigma_t\, d\widetilde{W}_t + D_t b_t R_t\, dt$$

$$= D_t a_t\, dt + D_t \sigma_t\, d\widetilde{W}_t$$

$$\Rightarrow D_t R_t = R_0 + \int_0^t D_u a_u\, du + \int_0^t D_u \sigma_u\, dW_u$$

④ Cox-Ingersoll-Ross interest rate model

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$$dR_t = (a - bR_t)\,dt + \sigma\sqrt{R_t}\,d\tilde{W}_t \quad ; \quad R_0 = r$$

a, b, σ : positive constants

- although there is no formula for R_t, there is a unique solution nonetheless !!

$\rightsquigarrow$ R_t can be approximated by Monte Carlo simulation

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What does "suitable" conditions on $\beta(t,x)$, $\sigma(t,x)$ mean ?

Theorem : Suppose $\beta(t,x)$, $\sigma(t,x)$ satisfy the "Lipschitz" and "linear growth" conditions.

(1) • $|\beta(t,x) - \beta(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le k\,|x-y| \quad (\forall)\, x,y$

(2) • $|\beta(t,x)| + |\sigma(t,x)| \le K\,(1 + |x|) \quad (\forall)\, x$

for some $k, K > 0$. Then there exists a unique process X that satisfies :

$$X_t = X_0 + \int_0^t \beta(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s$$

- X_t has continuous paths, is adapted to the filtration $\{\mathcal{F}_t\}$ of the Brownian Motion W

- Condition (2) ensures the solution exists (i.e it does not "explode" : $|X_t(\omega)| \to \infty$ in a finite time)

- Condition (1) guarantees uniqueness of the solution

Example :

- $dX_t = X_t^2\, dt$, $X_0 = 1$

  ⤳ corresponds to a drift : $\beta(x) = x^2$
  (which does not satisfy condition (2))

  ⤳ the solution : $X_t = \frac{1}{1-t}$: $0 \le t < 1$

  $\lim_{t \to 1} X_t = \infty$: X_t explodes in finite time !!

- $dX_t = 3 X_t^{2/3}\, dt$; $X_0 = 0$

  - this equation has more than 1 solution :

  ⤳ $X_t = \begin{cases} 0 & \text{for } t \le a \\ (t-a)^3 & \text{for } t > a \end{cases}$ where $a > 0$.

  - in this case : $b(x) = 3x^{2/3}$ does not satisfy the Lipschitz condition ((1)) at $x = 0$.