Term structure models

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Sponsored by the National Grid Foundation nationalgrid The power of action.

- we shall consider $\{W_t\}$ a Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t : 0 \le t \le T^*\}$ a filtration generated by W.

- given: $\{r_t\}$: adapted interest rate process $: 0 \le t \le T^*$

$$\rightsquigarrow \boxed{\beta_t = \exp\left\{ \int_0^t r_s \, ds \right\}}$$ the accumulation factor

- in a term structure model we take the zero-coupon bond of various maturities the be the primitive assets.
  (assume these bonds are default-free and pay \$1 at maturity)

Theorem: Fundamental Theorem of Asset Pricing

A term structure model is free of arbitrage if and only if there is a probability measure $\tilde{\mathbb{P}}$ on Ω (a risk neutral measure) $(\tilde{\mathbb{P}} \sim \mathbb{P})$ such that for each $T \in (0, T^*]$, the process

$$\frac{B(t,T)}{\beta_t} \text{ is } \tilde{\mathbb{P}} \text{ martingale} ; \quad 0 \le t \le T. \qquad \frac{1}{\beta_t} = D_t$$

where $B(t,T)$ = price process of a bond with maturity T (zero-coupon, \$1 face value)

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martingale representation Th.

$\rightsquigarrow$ martingale property:

$$\frac{B(t,T)}{\beta_t} = \tilde{\mathbb{E}}\left[\frac{1}{\beta_T} \,\Big|\, \mathcal{F}_t \right] = B(0,T) + \int_0^T \gamma_t \, d\tilde{W}_t$$

$$\frac{B(t,T)}{\beta_t} = \tilde{\mathbb{E}}\left[\frac{1}{\beta_T} \,\Big|\, \mathcal{F}_t\right] = B(0,T) + \int_0^t \gamma_s \, d\tilde{W}_s$$

(Arbitrage free bond pricing formula)

⇝ Itô formula :

$$d\left(\frac{B(t,T)}{\beta_t}\right) = B(t,T)\, d\left(\frac{1}{\beta_t}\right) + \frac{1}{\beta_t}\, dB(t,T) + \underbrace{dB(t,T)\cdot d\frac{1}{\beta_t}}_{=0}$$

$$\beta_t = \exp\left\{\int_0^t r_s\, ds\right\} \leadsto \frac{1}{\beta_t} = \exp\left\{-\int_0^t r_s\, ds\right\}$$

$$\Rightarrow d\left(\frac{1}{\beta_t}\right) = -r_t \cdot \frac{1}{\beta_t}\, dt$$

$$\Rightarrow d\left(\frac{B(t,T)}{\beta_t}\right) = -r_t \frac{B(t,T)}{\beta_t}\, dt + \frac{1}{\beta_t}\, dB(t,T)$$

we also have

$$d\left(\frac{B(t,T)}{\beta_t}\right) = \gamma_t \, d\tilde{W}_t$$

$\Rightarrow$

$$dB(t,T) = r_t B(t,T)\, dt + \beta_t \gamma_t \, d\tilde{W}_t$$

⇝ $B(t,T)$ shall always be thought as :

$$dB(t,T) = \mu(t,T) B(t,T)\, dt + \rho(t,T) B(t,T)\, dW_t$$

for some functions $\mu(t,T)$, $\rho(t,T)$

$$\leadsto d\left(\frac{B(t,T)}{\beta_t}\right) = B(t,T)\, d\left(\frac{1}{\beta_t}\right) + \frac{1}{\beta_t}\, dB(t,T)$$

$$= \left[\mu(t,T) - r_t\right] \frac{B(t,T)}{\beta_t}\, dt + \rho(t,T) \frac{B(t,T)}{\beta_t}\, dW_t$$

Remark : $\frac{B(t,T)}{\beta_t}$ is $\mathbb{P}$ martingale iff $\mu(t,T) = r_t$

$$\rightsquigarrow \quad d\left(\frac{B(t,T)}{\beta_t}\right) = \rho(t,T)\, \frac{B(t,T)}{\beta_t}\, d\tilde{W}_t$$

$$\rightsquigarrow \quad d\left(B(t,T)\right) = r_t\, B(t,T)\, dt + \rho(t,T)\, B(t,T)\, d\tilde{W}_t$$

Terminology :

Def 1: [Term-Structure Mode] : any mathematical model which determines (at least theoretically) the stochastic process $B(t,T)$ $\quad 0 \le t \le T$ for all $T \in (0, T^*]$.

Def 2 : [Yield to maturity] : $0 \le t \le T \le T^*$ $\quad Y(t,T)$ - $\mathcal{F}_t$ measurable such that : $B(t,T) \exp\{(T-t)\, Y(t,T)\} = 1$

or
$$\boxed{Y(t,T) = -\frac{1}{T-t} \log B(t,T)}$$

Forward Rate Agreement

- $0 \le t \le T < T+\varepsilon \le T^*$
- plan at time t : borrow \$1 at time T with repayment at $T+\varepsilon$ at an interest rate agreed upon at time t.
- this plan involves :

$\rightsquigarrow$ at time t :
- buy one zero-coupon bond with maturity T
- short $\frac{B(t,T)}{B(t,T+\varepsilon)}$ zero-coupon bonds with maturity $T+\varepsilon$

value at t : $B(t,T) - \frac{B(t,T)}{B(t,T+\varepsilon)} \cdot B(t,T+\varepsilon) = 0$

$\rightsquigarrow$ at time T : • receive \$1

$\rightsquigarrow$ at time $T+\varepsilon$: pay $\dfrac{B(t,T)}{B(t,T+\varepsilon)}$

• the effective interest rate on the \$1 you receive at time T is

$$R(t,T,T+\varepsilon) = -\frac{\log B(t,T+\varepsilon) - \log B(t,T)}{\varepsilon}$$

or

$$\frac{B(t,T)}{B(t,T+\varepsilon)} = \exp\left\{ \varepsilon\, R(t,T,T+\varepsilon) \right\}$$

The forward rate : $\boxed{f(t,T) = \lim_{\varepsilon \downarrow 0} R(t,T,T+\varepsilon) = -\frac{\partial}{\partial T} \log B(t,T)}$

Remark ① $f(t,T) = -\frac{\partial}{\partial T} \log B(t,T)$ for all $T \in (0,T^*]$

$$\rightsquigarrow \int_t^T f(t,u)\,du = -\int_t^T \frac{\partial}{\partial u} \log B(t,u)\,du = -\log B(t,u)\Big|_t^T$$

$$\rightsquigarrow \int_t^T f(t,u)\,du = -\log B(t,T)$$

$$\rightsquigarrow \boxed{B(t,T) = \exp\left\{ -\int_t^T f(t,u)\,du \right\}}$$

② $B(t,T) = \tilde{\mathbb{E}}\left[\exp\left\{ -\int_t^T r_u\,du \right\} \Big| \mathcal{F}_t \right]$

$$\rightsquigarrow \frac{\partial}{\partial T} B(t,T) = \tilde{\mathbb{E}}\left[-r_T \exp\left\{ -\int_t^T r_u\,du \right\} \Big| \mathcal{F}_t \right]$$

in particular for $T=t$ we have :

$$\frac{\partial}{\partial T} B(t,T)\Big|_{T=t} = -r_t$$

$\rightsquigarrow$ also from the previous remark we get easily that

$$\frac{\partial}{\partial T} B(t,T)\Big|_{T=t} = -f(t,t)$$

$$\Rightarrow \boxed{r_t = f(t,t)}$$

Computing Arbitrage Free Bond Prices :

(Heath - Jarrow - Morton method)

$f(t,T)$: forward rate curve

. suppose $f(t,T)$ is given by :

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW_u$$

or in differential form :

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t$$

$$B(t,T) = \exp\left\{-\int_t^T f(t,u)\,du\right\}$$

2 time variables

$$\rightsquigarrow d\left(-\int_t^T f(t,u)\,du\right) = f(t,t)\,dt - \int_t^T df(t,u)\,du$$

$$= r_t\,dt - \int_t^T \left[\alpha(t,u)\,dt + \sigma(t,u)\,dW_t\right] du$$

$$= r_t\,dt - \left(\int_t^T \alpha(t,u)\,du\right) dt - \left(\int_t^T \sigma(t,u)\,du\right) dW_t$$

$$= r_t\,dt - \underbrace{\left(\int_t^T \alpha(t,u)\,du\right)}_{=\alpha^*(t,T)} dt - \underbrace{\left(\int_t^T \sigma(t,u)\,du\right)}_{=\sigma^*(t,T)} dW_t$$

$$I_t = -\int_t^T f(t,u)\,du \qquad g(x) = e^x = g'(x) = g''(x)$$

Ito formula

$$\rightsquigarrow \quad dg(I_t) = g(I_t)\,dI_t + \frac{1}{2} g(I_t)\cdot dI_t \cdot dI_t$$

$$\Rightarrow \quad dB(t,T) = B(t,T)\left[r_t\,dt - \alpha^*(t,T)\,dt - \sigma^*(t,T)\,dW_t\right] + \frac{1}{2} B(t,T)\left[\sigma^*(t,T)\right]^2 dt$$

$$\Rightarrow \quad \boxed{dB(t,T) = B(t,T)\left[r_t - \alpha^*(t,T) + \frac{1}{2}\left(\sigma^*(t,T)\right)^2\right] dt - B(t,T)\,\sigma^*(t,T)\,dW_t}$$

No Arbitrage Condition

- based on the above equation $\mathbb{P}$ is risk neutral measure iff

$$\boxed{\alpha^*(t,T) = \frac{1}{2}\left(\sigma^*(t,T)\right)^2 \quad (\forall)\ 0 \le t \le T \le T^*}$$

that is :

$$(*) \qquad \boxed{\int_t^T \alpha(t,u)\,du = \frac{1}{2}\left(\int_t^T \sigma(t,u)\,du\right)^2 \quad (\forall)\ 0 \le t \le T \le T^*}$$

- differentiating with respect to T we obtain :

$$\alpha(t,T) = \sigma(t,T) \cdot \int_t^T \sigma(t,u)\,du \qquad (\forall)\ 0 \le t \le T \le T^*$$

- if $(*)$ does not hold $\rightsquigarrow$ $\mathbb{P}$ is not risk neutral therefore a change of measure is needed.

- let's rewrite the equation of the bond price in the discounted version.

$$D_t = \frac{1}{\beta_t} = \exp\left\{-\int_0^t r_u \, du\right\}$$

$$dB(t,T) = B(t,T)\left[r_t \, dt - \alpha^*(t,T) + \frac{1}{2}\left(\sigma^*(t,T)\right)^2\right] dt - \sigma^*(t,T) B(t,T) \, dW_t$$

$$\rightsquigarrow \quad d\,D_t B(t,T) = D_t B(t,T) \Big[\underbrace{-\alpha^*(t,T) + \frac{1}{2}\left(\sigma^*(t,T)\right)^2}_{\text{suppose} \uparrow \text{ is } \neq 0} \Big] dt - D_t B(t,T) \, \sigma^*(t,T) \, dW_t$$

$$d\tilde{W}_t = \theta_t \, dt + dW_t$$

we want to rewrite $\quad d\left(D_t B(t,T)\right) = -D_t B(t,T) \, \sigma^*(t,T) \, d\tilde{W}_t$

$\rightsquigarrow$ we must solve the eq: $\quad$ for $(\forall)$ $0 \le t \le T \le T^*$

$$(**) \qquad -\alpha^*(t,T) + \frac{1}{2}\left(\sigma^*(t,T)\right)^2 = -\sigma^*(t,T) \, \theta_t$$

(market price of risk equation)

Remarks: • there are infinite many eq since for every T you get one equation

- there is only one process θ_t : market price of risk
- there are as many θ_t as we have sources of uncertainty

To solve the market price of risk equations recall that

$$\frac{\partial}{\partial T} \alpha^*(t,T) = \alpha(t,T) \qquad \frac{\partial}{\partial T} \sigma^*(t,T) = \sigma(t,T)$$

- differentiating in $(**)$ with respect to T we get

$$-\alpha(t,T) + \sigma^*(t,T)\,\sigma(t,T) = -\sigma(t,T)\cdot\theta_t$$

or equivalently:

$$\alpha(t,T) = \sigma(t,T)\left[\sigma^*(t,T) + \theta_t\right]$$

Theorem: [Heath-Jarrow-Morton no-arbitrage condition]

For each $T \in (0,T^*]$. Let $\alpha(u,T)$, $\sigma(u,T)$, $0 \le u \le T$ be adapted processes, and assume $\sigma(u,T) > 0$ $\forall\, u,T$.

Let $f(0,T)$, be a deterministic function and define

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW_u \qquad (\forall)\ 0 \le t \le T \le T^*$$

Then $f(t,T)$ is a family of forward rate processes for a term structure model without arbitrage if and only if there is an adapted process θ_t, $0 \le t \le T^*$ satisfying the market price of risk equations

$$\boxed{\alpha(t,T) = \sigma(t,T)\left[\sigma^*(t,T) + \theta_t\right] \qquad (\forall)\quad 0 \le t \le T \le T^*}$$

Remark: if θ_t exists then:

$$\theta_t = -\frac{-\alpha^*(t,T) + \frac{1}{2}\left(\sigma^*(t,T)\right)^2}{\sigma^*(t,T)} = \frac{\alpha^*(t,T)}{\sigma^*(t,T)} - \frac{1}{2}\sigma^*(t,T)$$

does not depend on T

HJM under Risk-Neutral Measure

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t$$

$$= \sigma(t,T)\,\sigma^*(t,T)\,dt + \sigma(t,T)\underbrace{\left[\theta_t\,dt + dW_t\right]}_{d\widetilde{W}_t}$$

$$= \sigma(t,T)\,\sigma^*(t,T)\,dt + \sigma(t,T)\,\underbrace{[\theta_t\,dt + dW_t]}_{d\tilde{W}_t}$$

$$\rightsquigarrow \boxed{df(t,T) = \sigma(t,T)\,\sigma^*(t,T)\,dt + \sigma(t,T)\,d\tilde{W}_t}$$

- $$d\left(D_t B(t,T)\right) = -\sigma^*(t,T)\,D_t B(t,T)\,\underbrace{[\theta_t\,dt + dW_t]}_{d\tilde{W}_t}$$

$$\rightsquigarrow \boxed{d\left(D_t B(t,T)\right) = -\sigma^*(t,T)\,D_t B(t,T)\,d\tilde{W}_t}$$

$$\rightsquigarrow \boxed{dB(t,T) = r_t B(t,T)\,dt - \sigma^*(t,T)\,B(t,T)\,d\tilde{W}_t}$$

where $r_t = f(t,t)$ $\quad D_t = \exp\left\{-\int_0^t r_u\,du\right\}$

$$\Rightarrow \boxed{B(t,T) = B(0,T)\exp\left\{\int_0^t r_u\,du - \int_0^t \sigma^*(u,T)\,d\tilde{W}_u - \frac{1}{2}\int_0^t \left(\sigma^*(u,T)\right)^2 du\right\}}$$

Relation to Affine Yield Models

- every term-structure model driven by Brownian Motion is an HJM model $\rightsquigarrow$ there are forward rates

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW_u$$

- risk neutral measure given by $\theta_t = \dfrac{\alpha^*(t,T)}{\sigma^*(t,T)} - \dfrac{1}{2}\sigma^*(t,T)$

$\rightsquigarrow$ apply this model to Hull-White and Cox-Ingersol-Ross models

- for both of these models

$$dr_t = \beta(t,r_t)\,dt + \gamma(t,r_t)\,d\tilde{W}_t$$

$$\boxed{dr_t = \beta(t,r_t)\,dt + \gamma(t,r_t)\,d\tilde{W}_t}$$

($\tilde{W}_t$: Brownian Motion under risk neutral measure $\tilde{\mathbb{P}}$)

Hull-White model : $\beta(t,r) = a_t - b_t r$; $\gamma(t,r) = \sigma_t$

CIR model : $\beta(t,r) = a - br$; $\gamma(t,r) = \sigma\sqrt{r}$

($a, b, \sigma > 0$ constants)

$\leadsto$ bond prices are of the form :

$$B(t,T) = e^{-r_t C(t,T) - A(t,T)}$$

H-W model :
$$\begin{cases} C(t,T) = \displaystyle\int_t^T e^{-\int_t^s b_u\,du}\,ds \\ A(t,T) = \displaystyle\int_t^T \left(a_s C(s,T) - \tfrac{1}{2}\sigma_s^2 C^2(s,T)\right) ds \end{cases}$$

- in particular if a_t, b_t, σ_t are constants (a, b, σ) we obtain the Vasicek model for which : $\beta(t,r) = a - br$, $\gamma(t,r) = \sigma$

V model :
$$\begin{cases} C(t,T) = \frac{1}{b}\left(1 - e^{-b(T-t)}\right) \\ A'(t,T) = -a\,C(t,T) + \frac{1}{2}\sigma^2 C^2(t,T) \end{cases}$$

- let's check the HJM no arbitrage condition for V model

Recall : (1) $df(t,T) = \sigma(t,T)\sigma^*(t,T)\,dt + \sigma(t,T)\,d\tilde{W}_t$
(under risk neutral measure)

- also $f(t,T) = -\frac{\partial}{\partial T}\log B(t,T) = r_t \frac{\partial}{\partial T} C(t,T) + \frac{\partial}{\partial T} A(t,T)$

- let's use Ito formula for the forward rate :

(2) $df(t,T) = d\left(r_t \cdot \frac{\partial}{\partial T} C(t,T) + \frac{\partial}{\partial T} A(t,T)\right)$ $= \sigma(t,T) \cdot \sigma^*(t,T)\, dt + \sigma(t,T)\, d\tilde{W}_t$

$$= \frac{\partial}{\partial T} C(t,T)\, dr_t + r_t \frac{\partial}{\partial T} C'(t,T)\, dt + \frac{\partial}{\partial T} A'(t,T)\, dt$$

$$= \left[\frac{\partial}{\partial T} C(t,T)\, \beta(t,r_t) + r_t \frac{\partial}{\partial T} C'(t,T) + \frac{\partial}{\partial T} A'(t,T)\right] dt$$

$$+ \frac{\partial}{\partial T} C(t,T)\, \gamma(t,r_t)\, d\tilde{W}_t$$

- we compare (1) with (2) by matching the corresponding terms

$$\xrightarrow{d\tilde{W}_t} \quad \sigma(t,T) = \frac{\partial}{\partial T} C(t,T)\, \gamma(t,r_t)$$

$$\xrightarrow{dt} \quad \sigma(t,T)\, \sigma^*(t,T) = \frac{\partial}{\partial T} C(t,T)\, \beta(t,r_t) + r_t \frac{\partial}{\partial T} C'(t,T) + \frac{\partial}{\partial T} A'(t,T)$$

- the no arbitrage condition becomes :

$$\frac{\partial}{\partial T} C(t,T)\, \beta(t,r_t) + r_t \frac{\partial}{\partial T} C'(t,T) + \frac{\partial}{\partial T} A'(t,T) = \frac{\partial}{\partial T} C(t,T)\, \gamma(t,r_t) \cdot \int_t^T \frac{\partial}{\partial v} C(t,v) \cdot \gamma(t,r_t)\, dv$$

$$= \frac{\partial}{\partial T} C(t,T)\, \gamma^2(t,r_t) \cdot \left[C(t,T) - \underbrace{C(t,t)}_{=0}\right]$$

$$= \left(\frac{\partial}{\partial T} C(t,T)\right) \cdot \gamma^2(t,r_t) \cdot C(t,T)$$

NO ARBITRAGE condition

$$\rightsquigarrow \boxed{\frac{\partial}{\partial T} C(t,T)\, \beta(t,r_t) + r_t \frac{\partial}{\partial T} C'(t,T) + \frac{\partial}{\partial T} A'(t,T) = \left(\frac{\partial}{\partial T} C(t,T)\right) \cdot \gamma^2(t,r_t) \cdot C(t,T)}$$

- in particular for the Vasicek model

$$B(t,T) = \exp\{-r\, C(t,T) - A(t,T)\}$$

$$\begin{cases} C(t,T) = \frac{1}{b}\left(1 - e^{-b(T-t)}\right) \end{cases}$$

$$\begin{cases} C(t,T) = \frac{1}{b}\left(1 - e^{-b(T-t)}\right) \\ A'(t,T) = -a\, C(t,T) + \frac{1}{2}\sigma^2 C^2(t,T) \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial T} C(t,T) = e^{-b(T-t)} \\ \frac{\partial}{\partial T} A'(t,T) = \left(\frac{\sigma^2}{b} - a\right) e^{-b(T-t)} - \frac{\sigma^2}{a}\, e^{-2b(T-t)} \end{cases}$$

$$\rightsquigarrow \quad \sigma(t,T) = \sigma\, e^{-b(T-t)}$$

$$\sigma^*(t,T) = \int_t^T \sigma(t,u)\, du = \sigma \int_t^T e^{-(u-t)}\, du = \frac{\sigma}{b}\left(1 - e^{-b(T-t)}\right)$$

$$\rightsquigarrow \quad \frac{\partial}{\partial T} C(t,T)\, \beta(t, r_t) + r_t \frac{\partial}{\partial T} C'(t,T) + \frac{\partial}{\partial T} A'(t,T) =$$

$$= e^{-b(T-t)}\left(a - b r_t\right) + r_t \cdot b\, e^{-b(T-t)} + \left(\frac{\sigma^2}{b} - a\right) e^{-b(T-t)} - \frac{\sigma^2}{b} e^{-2b(T-t)}$$

$$= \frac{\sigma^2}{b}\left(e^{-b(T-t)} - e^{-2b(T-t)}\right) = \sigma(t,T)\, \sigma^*(t,T)$$

as expected.