Risk neutral pricing

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- We shall assume in this section that we have a final time T, up to which we study the stock price.

Recall: we have used the Generalized Geometric B.M to model a stock price

$$\begin{cases} dS_t = \sigma_t S_t \, dW_t + \alpha_t S_t \, dt \\ \text{or, otherwise} \\ S_t = S_0 \exp\left\{ \int_0^t \sigma_u \, dW_u + \int_0^t \left(\alpha_u - \frac{1}{2}\sigma_u^2\right) du \right\} \end{cases}$$

- suppose we have an adapted interest rate process R_t
- we define the discount process D_t such that

$$D_t = \exp\left\{ -\int_0^t R_s \, ds \right\}$$

$$dD_t = -R_t D_t \, dt \qquad \text{(easy to check with Ito lemma)}$$

Remarks: D_t : random process

$\langle D \rangle_t = 0$ (the discount process is "smooth")

Discounted Stock Price : $\tilde{S}_t = D_t \cdot S_t$

$$\begin{aligned} d\tilde{S}_t &= d(D_t \cdot S_t) \\ &= D_t \, dS_t + S_t \, dD_t + dS_t \cdot dD_t \\ &= D_t \left(S_t \sigma_t \, dW_t + S_t \alpha_t \, dt \right) + S_t \left(-R_t D_t \, dt \right) + \underbrace{\left(\sigma_t S_t \, dW_t + \alpha_t S_t \, dt \right) \cdot \left(-D_t R_t \, dt \right)}_{=0} \\ &= \underbrace{D_t S_t}_{\tilde{S}_t} \sigma_t \, dW_t + \underbrace{D_t S_t}_{\tilde{S}_t} (\alpha_t - R_t) \, dt \end{aligned}$$

$$\Rightarrow \boxed{d\tilde{S}_t = \tilde{S}_t \sigma_t \, dW_t + \tilde{S}_t (\alpha_t - R_t) \, dt} \qquad \text{Generalized GBM}$$

- consider the following adapted process :

$$\theta_t := \frac{\alpha_t - R_t}{\sigma_t} \quad : \text{Market Price of Risk}$$

(θ_t is well defined if $\sigma_t(\omega) \neq 0$)

$$\Rightarrow d\tilde{S}_t = \tilde{S}_t \sigma_t \, dW_t + \tilde{S}_t \theta_t \sigma_t \, dt$$

$$= \tilde{S}_t \sigma_t \underbrace{\left(dW_t + \theta_t \, dt \right)}_{d\tilde{W}_t}$$

- according to Girsanov's theorem $\tilde{W}_t$ is BM under $\tilde{\mathbb{P}}$

where $Z_t = \exp\left\{ -\int_0^t \theta_u \, dW_u - \frac{1}{2}\int_0^t \theta_u^2 \, du \right\}$

$$d\tilde{\mathbb{P}}(\omega) = Z_T(\omega) \, d\mathbb{P}(\omega)$$

- so under $\tilde{\mathbb{P}}$ the discounted stock price $\tilde{S}_t$ is a martingale.

$$\boxed{d\tilde{S}_t = \tilde{S}_t \sigma_t \, d\tilde{W}_t}$$

(assuming that $E\int_0^T \tilde{S}_t^2 \sigma_t^2 \, dt < \infty$)

Remark : $\tilde{\mathbb{P}}$ is called RISK-NEUTRAL PROBABILITY since the discounted stock price is a martingale under $\tilde{\mathbb{P}}$.

- we have also studied the wealth of an investor in this stock, and we have seen that it satisfies the following eq.

$$dX_t = \underbrace{\Delta_t \, dS_t} + \underbrace{R_t \left[X_t - \Delta_t S_t \right] dt}$$

Capital gains from stock — Interest earnings

$$= \Delta_t (S_t \sigma_t dW_t + S_t \alpha_t dt) + R_t X_t dt - \Delta_t S_t R_t dt$$

$$= \Delta_t S_t \sigma_t dW_t + \Delta_t S_t (\alpha_t - R_t) dt + R_t X_t dt$$

- the discounted weath process follows by the product rule:

$$d(\tilde{X}_t) = d(D_t X_t) = D_t dX_t + X_t dD_t + \underbrace{dX_t \cdot dD_t}_{=0}$$

$$= D_t \left[\Delta_t S_t \sigma_t dW_t + \Delta_t S_t (\alpha_t - R_t) dt + R_t X_t dt \right] - X_t D_t R_t dt$$

$$= \Delta_t \tilde{S}_t \sigma_t dW_t + \Delta_t \tilde{S}_t (\alpha_t - R_t) dt$$

$$= \Delta_t d\tilde{S}_t$$

- recall that under $\tilde{\mathbb{P}}$: $\tilde{S}$ is martingale
  $\Rightarrow$ under $\tilde{\mathbb{P}}$: $\tilde{X}$ is martingale

$\tilde{S}_t$ martingale under $\tilde{\mathbb{P}}$: $\boxed{d\tilde{S}_t = \tilde{S}_t \sigma_t d\tilde{W}_t}$

$\Rightarrow \boxed{d\tilde{X}_t = \Delta_t d\tilde{S}_t = \Delta_t \tilde{S}_t \sigma_t d\tilde{W}_t} \Big\} \Rightarrow$ martingale

stochastic integral

(if the conditions of Girsanov's cond. are satisfied)

<u>Risk-neutral valuation</u> $V_t :=$ value process of the contingent claim.

- example :
  - $\rightsquigarrow$ call option : $V_T = (S_T - K)^+$
  - $\rightsquigarrow$ lookback : $V_T = \max S_t$

$\rightsquigarrow$ lookback : $V_T = \max_{t \leq T} S_t$

- $\tilde{X}_t$: discounted wealth (martingale under $\tilde{\mathbb{P}}$)

  $\Rightarrow \quad X_0 = \tilde{\mathbb{E}}(\tilde{X}_T)$

- if X_t is replicating the contingent claim then

  $X_T = V_T$ value of the option (a.s.)

  (would be path dependent)

- to identify a hedging portfolio we need to find Δ_t s.t.

  $X_T = V_T$ a.s.

$$D_t X_t = \tilde{\mathbb{E}}\left[D_T X_T \middle| \mathcal{F}_t \right] = \tilde{\mathbb{E}}\left[D_T V_T \middle| \mathcal{F}_t \right] \qquad X_t = V_t$$

$\rightsquigarrow$ martingale under $\tilde{\mathbb{P}}$

$$\Rightarrow \quad D_t V_t = \tilde{\mathbb{E}}\left[D_T V_T \middle| \mathcal{F}_t \right]$$

$$\boxed{V_t = \tilde{\mathbb{E}}\left[e^{-\int_t^T R_u du} V_T \middle| \mathcal{F}_t \right]}$$

Risk neutral pricing formula

- So, the risk neutral pricing formula is based on the fact that the discounted value process is a martingale under $\tilde{\mathbb{P}}$

  ($\tilde{S}_t$, $\tilde{X}_t$, $\tilde{V}_t = D_t V_t$: martingales under $\tilde{\mathbb{P}}$)

$\rightsquigarrow$ according the the "Martingale Representation Th"

there exists a process $\tilde{\Gamma}_t$; for $0 \leq t \leq T$ such that

$$\tilde{V}_t = V_0 + \int_0^t \tilde{\Gamma}_u \, d\tilde{W}_u \qquad \text{for } 0 \leq t \leq T. \qquad d\tilde{V}_t = \tilde{\Gamma}_t \, d\tilde{W}_t$$

- now let's compare the "hedging portfolio" value : X_t with the contingent claim value V_t

$$\left.\begin{aligned} d\tilde{X}_t &= \Delta_t \sigma_t \tilde{S}_t \, d\tilde{W}_t \\ d\tilde{V}_t &= \tilde{\Gamma}_t \, d\tilde{W}_t \end{aligned}\right\} \rightsquigarrow \Delta_t \sigma_t \tilde{S}_t = \tilde{\Gamma}_t$$

in order to have $X_t = V_t$ $(\forall) t$

$$\Rightarrow \quad \Delta_t = \frac{\tilde{\Gamma}_t}{\sigma_t \tilde{S}_t} = \frac{\tilde{\Gamma}_t}{\sigma_t D_t S_t} \qquad 0 \le t \le T$$

Key Assumptions that make the hedge possible

- $\sigma_t \neq 0$ (if $\sigma_t = 0$ the randomness of B.M. does not enter the stock, so the stock is no longer an effective hedging instrument)
- $\{\mathcal{F}_t\}$ is generated by the underlying B.M. (the Brownian Motion is the only source of randomness in the contingent claim)

$\rightsquigarrow$ such a model is said to be **COMPLETE**

Remark: the martingale representation theorem justifies the Risk-neutral pricing formula but does not provide a practical method for finding a hedging portfolio.