Stochastic Calculus : Ito integral for simple integrands

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10:14 AM

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- W_t = Brownian Motion
- we want to define $I_t(X) = \int_0^t X_s \, dW_s$

$\rightsquigarrow$ regular Calculus does not work for Brownian Motion

[Lebesgue - Stieltjes integral : $\int_0^t f(s)\, dg(s) = \int_0^t f(s) g'(s)\, ds$

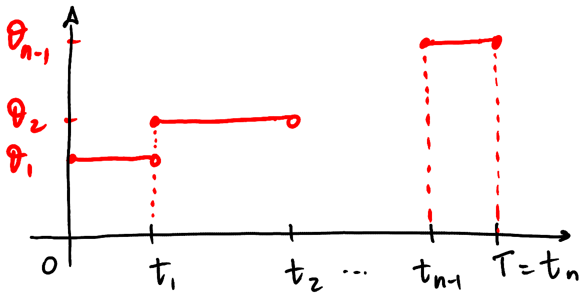
however $t \longmapsto W_t(\omega)$ is not differentiable]

- we shall begin by defining the integral in the case when X = simple process

__Definition__: A process X is called **SIMPLE** if there is a partition $0 = t_0 < t_1 < \dots < t_n = T$ such that

$X_s(\omega) = \theta_j(\omega)$ for $t_j \le s < t_{j+1}$ where

θ_j is bounded, $\mathcal{F}_{t_j}$ measurable.



$\rightsquigarrow$ a path of the simple process X

- We define the Itô Integral as a sum :

$$I_T(X) = \int_0^T X_s\, dW_s = \sum_{j=0}^{n-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right)$$

$\rightsquigarrow$ interpretation : • $t_0, t_1, t_2, \dots, t_n$ = Trading times

• $\theta_0, \theta_1, \dots, \theta_{n-1}$ = position (nr of shares)

$\rightsquigarrow$ the gain from each trading at each time t is given by:

$$I_t(X) = \begin{cases} \theta_0 (W_t - W_{t_0}) & ; 0 \le t \le t_1 \\ \theta_0 (W_{t_1} - W_{t_0}) + \theta_1 (W_t - W_{t_1}) & ; t_1 \le t \le t_2 \\ \theta_0 (W_{t_1} - W_{t_0}) + \theta_1 (W_{t_2} - W_{t_1}) + \theta_2 (W_t - W_{t_2}) & ; t_2 \le t \le t_3 \\ \text{and so on} & \end{cases}$$

so if $t_k \le t \le t_{k+1}$

$$I_t(X) = \sum_{j=0}^{k-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) + \theta_k \left(W_t - W_{t_k}\right)$$

• in particular we can take $t = t_n = T$

$$\Rightarrow \quad I_T(X) = \int_0^T X_s\, dW_s = \sum_{j=0}^{n-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) \quad (*)$$

Properties of the stochastic integral:

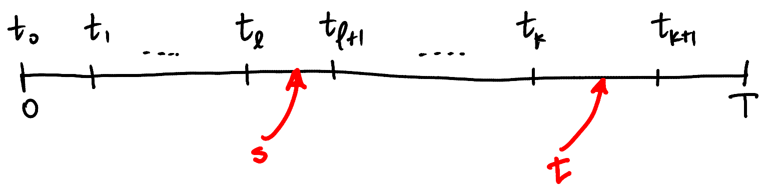
① The Itô integral $(*)$ is a martingale.

Proof: we want to prove that

$$E\left[I_t(X) \mid \mathcal{F}_s \right] = I_s(X) \quad \text{for } s < t$$

$$I_t(X) = \sum_{j=0}^{k-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) + \theta_k \left(W_t - W_{t_k}\right)$$

$$I_s(X) = \sum_{j=0}^{\ell-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) + \theta_\ell \left(W_s - W_{t_\ell}\right)$$



- assume $s \in [t_\ell, t_{\ell+1})$, $t \in [t_k, t_{k+1})$

- first we'll cover the case $t_{\ell+1} < t_k$

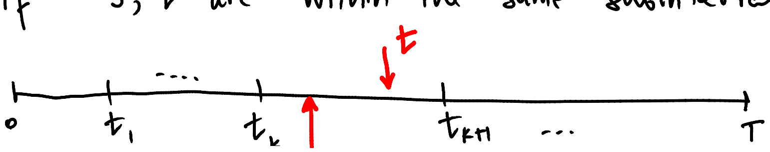
we are going to break down $I_t(X)$ as follows

$$I_t(X) = \sum_{j=0}^{\ell-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) + \theta_\ell \left(W_{t_{\ell+1}} - W_{t_\ell}\right) + \sum_{j=\ell+1}^{k-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) + \theta_{t_k}\left(W_t - W_{t_k}\right)$$

$$\Rightarrow E\left[I_t(X) \,\middle|\, \mathcal{F}_s\right] = \sum_{j=0}^{\ell-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) + E\left[\theta_\ell \left(W_{t_{\ell+1}} - W_{t_\ell}\right) \,\middle|\, \mathcal{F}_s\right] + \underbrace{E\left[\sum_{j=\ell+1}^{k-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) + \theta_k \left(W_t - W_{t_k}\right) \,\middle|\, \mathcal{F}_s\right]}_{=0}$$

$$= I_s(X)$$

- if s, t are within the same subinterval the proof is easier!



$$0 \quad t_1 \quad t_k \; \overset{|}{s} \quad t_{k+1} \quad \cdots \quad T$$

$$E\left(I_t(X) \mid \mathcal{F}_s\right) = \sum_{j=0}^{k-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) + E\left[\theta_k \left(W_t - W_{t_k}\right) \mid \mathcal{F}_s\right]$$

$$= \sum_{j=0}^{k-1} \theta_j \left(W_{t_{j+1}} - W_{t_j}\right) + \theta_k \left(W_s - W_{t_k}\right) = I_s(X)$$

② Itô Isometry : $\boxed{E\, I_t^2(X) = \int_0^t X_s^2 \, ds}$

Proof • for $t_k \le t < t_{k+1}$ we shall use the following notation (to make calculations easier)

$D_j = W_{t_{j+1}} - W_{t_j}$ for $j = \overline{0, k-1}$ and

$D_k = W_t - W_{t_k}$ so the Ito integral becomes

$$\Rightarrow I_t(X) = \sum_{j=0}^{k} \theta_j D_j \Rightarrow I_t^2(X) = \sum_{j=0}^{k} \theta_j^2 D_j^2 + 2 \sum_{0 \le i < j \le k} \theta_i \theta_j D_i D_j$$

$$\Rightarrow E\, I_t^2(X) = \sum_{j=0}^{k} E\, \theta_j^2 D_j^2 + 2 \sum_{0 \le i < j \le k} E\, \theta_i \theta_j D_i D_j$$

• $\left.\begin{array}{l} \theta_j \text{ is } \mathcal{F}_{t_j} \text{ measurable} \\ D_j \text{ is independent of } \mathcal{F}_{t_j} \end{array}\right\} \Rightarrow \theta_j, D_j$ independent

$$\Rightarrow E\, \theta_j^2 D_j^2 = E\, \theta_j^2 \cdot \underbrace{E\, D_j^2}_{t_{j+1} - t_j} = E\, \theta_j^2 \left(t_{j+1} - t_j\right)$$

• $E\, \theta_i \theta_j D_i D_j = E\left[E\left(\theta_i \theta_j D_i D_j \mid \mathcal{F}_{t_j}\right)\right]$

$$\bullet\ \mathbb{E}\,\theta_i\theta_j D_i D_j = \mathbb{E}\left[\mathbb{E}\left(\theta_i\theta_j D_i D_j \mid \mathcal{F}_{t_j}\right)\right]$$

$$= \mathbb{E}\Big[\theta_i\theta_j D_i \underbrace{\mathbb{E}(D_j)}_{=0}\Big] = 0$$

$$\Rightarrow \mathbb{E}\,I_t^2(X) = \mathbb{E}\left[\sum_{j=0}^{k-1}\theta_j^2(t_{j+1}-t_j) + \theta_k^2(t-t_k)\right]$$

$$\boxed{\mathbb{E}\,I_t^2(X) = \mathbb{E}\int_0^t X_s^2\,ds}$$

③ Quadratic Variation : $\langle I(X)\rangle_t = \int_0^t X_s^2\,ds$

- so far we know $I_t(X)$ = martingale
- $\mathbb{E}\left[I_t^2(X) - \int_0^t X_s^2\,ds\right] = 0 \Rightarrow I_t^2(X) - \int_0^t X_s^2\,ds$ = martingale

$$\Rightarrow \langle I(X)\rangle_t = \int_0^t X_s^2\,ds$$

(because of the uniqueness of the Doob-Meyer decomposition)

! Recall that for any square integrable martingale M

$$M_t^2 - \langle M\rangle_t = \text{martingale}$$

($\langle M\rangle_t$ = the unique cont, nondecreasing process that makes the above decomposition true)

④ Cross-Variation $\langle I(X), I(Y)\rangle_t = \int_0^t X_s Y_s\,ds$

(similar explanation)

(5) Differential form of the Itô integral :

- $I_t(X) = \int_0^t X_s \, dW_s \iff dI_t(X) = X_t \, dW_t$

- $\langle I(X) \rangle_t = \int_0^t X_s^2 \, ds \iff d\langle I(X) \rangle_t = X_t^2 \, dt$

furthermore :

$$d\langle I(X) \rangle_t = dI_t(X) \cdot dI_t(X) = X_t^2 \cdot \underbrace{dW_t \cdot dW_t}_{= dt} = X_t^2 \, dt$$

(Recall $d\langle W \rangle_t = dW_t \cdot dW_t = dt$)

- $\langle I(X), I(Y) \rangle_t = \int_0^t X_s Y_s \, ds \iff d\langle I(X), I(Y) \rangle_t = X_t Y_t \, dt$

$$d\langle I(X), I(Y) \rangle = dI(X) \cdot dI(Y) = X_t \cdot Y_t \underbrace{dW_t \cdot dW_t}_{dt} = X_t Y_t \, dt$$