Itô Formula : Examples

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Itô formula (Recall from last time)

X_t = semi martingale ; $f(t,x) \in \mathcal{C}^2$

$$\boxed{df(t,X_t) = f_t(t,X_t)\,dt + f_x(t,X_t)\,dX_t + \frac{1}{2} f_{xx}(t,X_t)\,d\langle X\rangle_t}$$

- in particular if X_t = Itô process

that is : $X_t = X_0 + \int_0^t \Delta_u\, dW_u + \int_0^t \theta_u\, du$

Δ_t, θ_t some adapted processes

- $\int_0^t \Delta_u^2\, du < \infty$

$$\boxed{\begin{aligned} dX_t &= \Delta_t\, dW_t + \theta_t\, dt \\ d\langle X\rangle_t &= dX_t \cdot dX_t = \Delta_t^2\, dt \end{aligned}}$$

- $df(t,X_t) = f_t(t,X_t)\,dt + f_x(t,X_t)\left(\Delta_t dW_t + \theta_t\, dt\right) + \frac{1}{2} f_{xx}(t,X_t)\Delta_t^2\, dt$

$= f_x(t,X_t)\Delta_t\, dW_t + \left(f_t(t,X_t) + f_x(t,X_t)\theta_t + \frac{1}{2} f_{xx}(t,X_t)\Delta_t^2\right) dt$

Examples :

- last time we have seen the **Generalized Geometric BM**

$\rightsquigarrow$ the process that satisfies :

$S_t = S_0 \exp\left\{ \int_0^t \sigma_s\, dW_s + \int_0^t \left(\alpha_s - \frac{1}{2}\sigma_s^2\right) ds \right\}$

$dS_t = S_t \sigma_t\, dW_t + S_t \alpha_t\, dt$

$\rightsquigarrow$ if $\alpha_t \equiv 0 \Rightarrow dS_t = S_t \sigma_t\, dW_t \rightsquigarrow S_t$ = martingale

$\rightsquigarrow$ if α_t, σ_t are constant $\Rightarrow S_t$ = Geometric BM

$dS_t = \sigma S_t\, dW_t + \alpha S_t\, dt$

- let's assume : S_t is the stock price at time t

- Δ_t nr of shares held at time t (adapted to the filtration of B.M.)
- $X_t - \Delta_t S_t$: invested in the money market (constant interest rate r)
- the wealth process X_t satisfies :

$$\boxed{dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t)\,dt}$$

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in the discrete time setting the above eq. becomes :

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

$$\rightsquigarrow \underbrace{X_{n+1} - X_n}_{dX_n} = \Delta_n \underbrace{(S_{n+1} - S_n)}_{dS_n} + r(X_n - \Delta_n S_n)$$

- we can use Itô formula to describe :

$\rightsquigarrow$ the discounted stock price : $e^{-rt} S_t = \tilde{S}_t$

$\rightsquigarrow$ the discounted wealth : $e^{-rt} X_t = \tilde{X}_t$

- for both processes the function we work with is

$$f(t,x) = e^{-rt}x : \quad f_t = -re^{-rt}x, \quad f_x = e^{-rt}, \quad \boxed{f_{xx} = 0}$$

- $$df(t,S_t) = f_t(t,S_t)\,dt + f_x(t,S_t)\,dS_t + \tfrac{1}{2} f_{xx}(t,S_t)\,d\langle S\rangle_t$$

$$= -re^{-rt} S_t \,dt + e^{-rt}(S_t \sigma\, dW_t + S_t \alpha\, dt)$$

$$\Rightarrow d(e^{-rt} S_t) = e^{-rt} S_t \sigma \, dW_t + e^{-rt} S_t (\alpha - r)\,dt$$

$$\rightsquigarrow \boxed{d(\tilde{S}_t) = \tilde{S}_t \sigma\, dW_t + \tilde{S}_t(\alpha - r)\,dt}$$

- $$df(t,X_t) = f_t(t,X_t)\,dt + f_x(t,X_t)\,dX_t + \tfrac{1}{2} f_{xx}(t,X_t)\,d\langle X\rangle_t$$

$$= -re^{-rt} X_t\,dt + e^{-rt}(\Delta_t\, dS_t + r(X_t - \Delta_t S_t)\,dt)$$

$$= e^{-rt}\Delta_t\, dS_t - re^{-rt}\Delta_t S_t\,dt$$

$$= e^{-rt}\Delta_t\left(S_t\sigma\, dW_t + S_t\alpha\, dt\right) - re^{-rt}\Delta_t S_t\, dt$$

$$= e^{-rt}\Delta_t S_t\sigma\, dW_t + e^{-rt}\Delta_t S_t\,(\alpha - r)\, dt$$

$$= \Delta_t\, d(\tilde{S}_t)$$

$$\Rightarrow \boxed{d(\tilde{X}_t) = \Delta_t\, d\tilde{S}_t} \qquad\qquad I_t = \int \theta_s\, dM_s$$

- $d\tilde{S}_t = \tilde{S}_t\sigma\, dW_t + (\alpha - r)\tilde{S}_t\, dt$

$\rightsquigarrow$ if $\boxed{\alpha = r}$ $\Rightarrow$ $d\tilde{S}_t = \tilde{S}_t\sigma\, dW_t$: martingale

$$\tilde{S}_t = \tilde{S}_0 \cdot \int_0^t \tilde{S}_u\, \sigma\, d\, W_u \, .$$

__Remark__ : if $\tilde{S}_t$ is martingale $\Rightarrow$ $\tilde{X}_t$ is martingale

__Black - Scholes - Merton eq__ :

- we want to find the price of an European Option
  $(S_T - K)^+$: payoff at maturity

$c(t,x)$ = value of the call option price at time t if the stock price at time t is x $\quad (S_t = x)$

- we shall use Ito formula for $c(t,x)$
  with c_t, c_x, c_{xx} the partial derivatives of c

$$\Rightarrow d\, c(t,S_t) = c_t(t,S_t)\, dt + c_x(t,S_t)\, dS_t + \frac{1}{2} c_{xx}(t,S_t)\, d\langle S\rangle_t$$

- recall in our model $S_t := \text{GBM}(\alpha,\sigma)$

$$\left(dS_t = S_t\sigma dW_t + S_t\alpha\, dt \;\; ; \;\; d\langle S\rangle_t = S_t^2\sigma^2 dt \right)$$

So :

$$d\, c(t,S_t) = c_t(t,S_t)\, dt + c_x(t,S_t)\left(S_t\sigma\, dW_t + S_t\alpha\, dt\right) +$$
$$+ \frac{1}{2} c_{xx}(t,S_t)\, S_t^2\sigma^2\, dt$$

$$= c_x(t,S_t)\, S_t\, \sigma\, dW_t + \left(c_t(t,S_t) + c_x(t,S_t)\, S_t\, \alpha + \frac{1}{2} c_{xx}(t,S_t)\, S_t^2 \sigma^2 \right) dt$$

- On the other hand $c(t,S_t)$ must match the wealth eq in order to create a perfect hedge :

$$dX_t = \Delta_t S_t \sigma\, dW_t + \Delta_t S_t \alpha\, dt + \left(rX_t - r\Delta_t S_t \right) dt$$

Delta - Hedging Rule : $\boxed{\Delta_t = c_x(t,S_t)}$

Black - Scholes PDE :

$$c_t(t,S_t) + c_x(t,S_t)\, \alpha S_t + \frac{1}{2} c_{xx}(t,S_t)\, \sigma^2 S_t^2 = rX_t + \Delta_t S_t (\alpha - r)$$

- recall we have set : $X_t = c(t,S_t)$; $\Delta_t = c_x(t,S_t)$

$\rightsquigarrow$
$$c_t(t,S_t) + c_x(t,S_t)\, \alpha S_t + \frac{1}{2} c_{xx}(t,S_t)\, \sigma^2 S_t^2 = r\, c(t,S_t) + c_x(t,S_t)\, (\alpha - r)\, S_t$$

$\Rightarrow$
$$\boxed{c_t(t,S_t) + c_x(t,S_t)\, r S_t + \frac{1}{2} c_{xx}(t,S_t)\, \sigma^2 S_t^2 = r\, c(t,S_t)}$$

- in particular if we replace S_t with x in the above eq we obtain :

$$c_t(t,x) + c_x(t,x)\, rx + \frac{1}{2} c_{xx}(t,x)\, \sigma^2 x^2 = r\, c(t,x)$$

$$c(T,x) = (x-K)^+ \qquad \text{(Terminal condition)}$$

a PDE that contains no random element !!

$\rightsquigarrow$ B-S-M PDE : backwards parabolic PDE

one needs boundary conditions at $x=0$ and $x=\infty$

- $c_t(t,0) = r\, c(t,0) \Rightarrow c(t,0) = e^{rt} c(0,0)$
- easy to see that $c(t,0) = 0$ All $t \in [0,T]$ (since $C(T,0) = 0$)

- easy to see that $c(t,0)=0$ $\forall$ $t\in[0,T]$ (since $C(T,0)=0$)
- as $x\to\infty$:

$$\lim_{x\to\infty}\left[c(t,x)-\left(x-e^{-r(T-t)}K\right)\right]=0$$

(that is $c(t,x)$ grows at the same rate as x)

- the solution of B-S PDE

$$c(t,x)=x\,N\left(d_+(T-t,x)\right)-Ke^{-r(T-t)}N\left(d_-(T-t,x)\right)$$

where $d_\pm(\tau,x)=\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K}+\left(r\pm\frac{\sigma^2}{2}\right)\tau\right]$

The Greeks: the derivatives of $c(t,x)$

Δ_t $\quad c_x(t,x)=N\left(d_+(T-t,x)\right)$

Θ_t $\quad c_t(t,x)=-rKe^{-r(T-t)}N\left(d_-(T-t,x)\right)-\frac{\sigma x}{2\sqrt{T-t}}N'\left(d_+(T-t,x)\right)$

Γ_t $\quad c_{xx}(t,x)=N'\left(d_+(T-t,x)\right)\cdot\frac{\partial}{\partial x}d_+(T-t,x)$