# MTH 9821 Numerical Methods for Finance I

# Lecture 9 –Finite Difference Valuation For European Options

## 1 Finite Difference Valuation of European Options

Recall Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \quad \forall S > 0,\ \forall 0 < t < T$$

with boundary conditions

$$\begin{aligned} &\text{(Call Option)} \quad V(S,T) = \max(S-K, 0), \quad \forall S > 0 \\ &\text{(Put Option)} \quad V(S,T) = \max(K-S, 0), \quad \forall S > 0 \end{aligned}$$

### 1.1 Black Sholes PDE to Heat PDE

Recall that by changing of variables, we can reduce it to the heat PDE:

$$\begin{aligned} &x = \ln\left(\frac{S}{K}\right), \quad \tau = \frac{(T-t)\sigma^2}{2}; \\ &a = \frac{r-q}{\sigma^2} - \frac{1}{2}, \quad b = \left(\frac{r-q}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2q}{\sigma^2} \\ &V(S,t) = exp(-ax - b\tau)u(x,\tau) \end{aligned}$$

where $u(x,\tau)$ satisfies the heat PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < t < t_{final},\ x_{left} < x < x_{right}$$

with boundary conditions

$$\begin{aligned} u(x,0) &= f(x) \\ u(x_{left},\tau) &= g_{left}(\tau) \\ u(x_{right},\tau) &= g_{right}(\tau) \end{aligned}$$

**Summing up, we need to find the following in order to use the heat PDE engine to solve the BS PDE.**

- $x_{left}$, $x_{right}$
- $g_{left}(\tau)$, $g_{right}(\tau)$
- $f(x)$

**Question: Why reuse the heat PDE engine?**

- clean, reliable;
  Some PDE solver have unstable solutions when using different discretization.
- portable;

## 1.2 Computational Domain

In BS PDE, $S > 0$. By changing of variable, $x \in (-\infty, \infty)$. We need to change the domain of $S$ into bounded region as in heat PDE, i.e., $x_{left} < x < x_{right}$.

**Intuition:**

Recall that $S$ follows lognormal distribution, then by setting $S$ within 3 standard deviations from mean, we covered 99.73% of the entire $S$, i.e.,

$$\begin{aligned} S(T) &= S(0) \ \exp\left((r - q - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\right) \\ P(-3 \leq Z \leq 3) &= 0.9973 \end{aligned}$$

Assume that

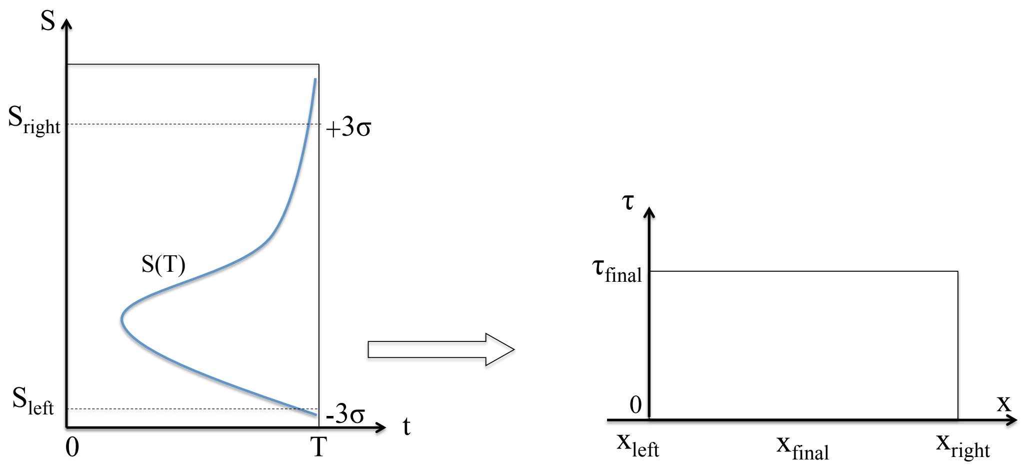
$$S(0) \ \exp\left((r - q - \frac{\sigma^2}{2})T - 3\sigma\sqrt{T}\right) \leq S \leq S(0) \ \exp\left((r - q - \frac{\sigma^2}{2})T + 3\sigma\sqrt{T}\right)$$

we get the *temporary* left and right end points as follows:

$$\begin{aligned} \tilde{x}_{left} &= \ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T - 3\sigma\sqrt{T} \\ \tilde{x}_{right} &= \ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T + 3\sigma\sqrt{T} \end{aligned}$$

and

$$0 < \tau < \tau_{final}, \quad \text{where } \tau_{final} = \frac{T\sigma^2}{2}$$



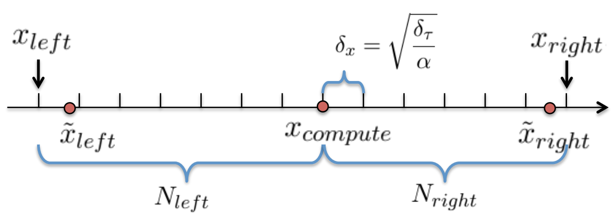

Also, we want $S_0$ on the grid, define $x_{compute}$ as

$$x_{compute} = \ln\left(\frac{S_0}{K}\right)$$

We start with $\alpha$ and $M$, and try to get the domain on $x$-axis.

$$\text{For} \quad \alpha = \frac{\delta_\tau}{(\delta_x)^2} \;\Rightarrow\; \delta_x = \sqrt{\frac{\delta_\tau}{\alpha}}$$
$$\text{Given} \quad M,\; \delta_\tau = \frac{\tau_{final}}{M}$$

Start from $x_{compute}$, we go left/right on $x$-axis by $\delta_x$, until we hit or overhit $\tilde{x}_{left}$ and $\tilde{x}_{right}$.




$$\begin{aligned}
N_{right} &= ceil\left[\frac{\tilde{x}_{right} - x_{compute}}{\delta_x}\right]; \\
N_{left} &= ceil\left[\frac{x_{compute} - \tilde{x}_{left}}{\delta_x}\right]; \\
\text{where} \quad ceil(x) &= \text{smallest integer that is greater than or equal to } x \\
N &= N_{right} + N_{left}; \\
x_{right} &= x_{compute} + N_{right}\delta_x; \\
x_{left} &= x_{compute} - N_{left}\delta_x;
\end{aligned}$$

### 1.3 Boundary Conditions

Recall Put-Call Parity

$$C - P = Se^{-q(T-t)} - Ke^{-r(T-t)}$$

Boundary conditions:

| | Put | Call |
|---|---|---|
| $S \to 0$ | $V(S,t) \approx Ke^{-r(T-t)} - Se^{-q(T-t)}$ | $V(S,t) \approx 0$ |
| $S \to \infty$ | $V(S,t) \approx 0$ | $V(S,t) \approx Se^{-q(T-t)} - Ke^{-r(T-t)}$ |

**For put options:**
Try to find $u(x,0) = f(x),\ u(x_{left},\tau) = g_{left}(\tau),\ u(x_{right},\tau) = g_{right}(\tau)$.

$$\begin{aligned} V(S,T) &= \max(K-S,0) \\ V(S,\tau) &= \exp(-ax-b\tau)u(x,\tau) \end{aligned}$$

- $t = T,\ \tau = 0$, try to find $u(x,0) = f(x)$

$$\begin{aligned} V(S,0) &= \exp(-ax)u(x,0) \\ \Rightarrow \max(K-S,0) &= \exp(-ax)u(x,0) \\ \Rightarrow \max(K-Ke^x,0) &= e^{-ax}u(x,0) \qquad \left(x = \ln\left(\frac{S}{K}\right) \Rightarrow S = Ke^x\right) \\ \Rightarrow f(x) = u(x,0) &= Ke^{ax}\max(1-e^x,0). \end{aligned}$$

- $S \to 0$, try to find $u(x_{left},\tau) = g_{left}(\tau)$

$$\begin{aligned} V(S,t) &\approx Ke^{-r(T-t)} - Se^{-q(T-t)} \\ \Rightarrow \exp(-ax-b\tau)u(x,\tau) &\approx Ke^{-r(T-t)} - Se^{-q(T-t)} \\ \Rightarrow \exp(-ax_{left}-b\tau)u(x_{left},\tau) &\approx Ke^{-r(T-t)} - Se^{-q(T-t)} \\ \Rightarrow g_{left}(\tau) = u(x_{left},\tau) &= K\exp(ax_{left}+b\tau)\left(\exp\left(-\frac{2r\tau}{\sigma^2}\right) - \exp\left(x_{left} - \frac{2q\tau}{\sigma^2}\right)\right) \end{aligned}$$

- $S \to \infty$, try to find $u(x_{right},\tau) = g_{right}(\tau)$

$$V(S,t) \approx 0 \quad \Rightarrow g_{right}(\tau) = u(x_{right},\tau) = 0$$

**For call options:**

$$\begin{aligned} f(x) &= Ke^{ax}\max(e^x-1,0) \\ g_{left}(\tau) &= 0 \\ g_{right}(\tau) &= K\exp(ax_{right}+b\tau)\left(\exp\left(x_{right} - \frac{2q\tau}{\sigma^2}\right) - \exp\left(-\frac{2r\tau}{\sigma^2}\right)\right) \end{aligned}$$

## 1.4 Convergence and RMS Error

### 1.4.1 Pointwise Convergence

By discretization, we have nodes

$$x_k \;=\; x_{left} + k\delta_x, \quad k = 0 : N$$

Since $S_k = Ke^{x_k}$, we have

$$\begin{aligned}V_{approx}(S_k, 0) &= \exp(-ax - b\tau_{final})U^M(k) \\ V_{exact}(S_k, 0) &= V_{BS}(S_k)\end{aligned}$$

where $U^M(k) = u(x_k, \tau_{final})$.
In particular, $x_{N_{left}} = x_{compute} = \ln\left(\frac{S_0}{K}\right)$,

$$\begin{aligned}V_{approx}(S_0, 0) &= \exp(-ax - b\tau_{final})U^M(N_{left}) \\ V_{exact}(S_0, 0) &= V_{BS}(S_0)\end{aligned}$$

The pointwise relative error of the finite difference solution is

$$error\_pointwise \;=\; \Big|V_{exact}(S_0, 0) - V_{approx}(S_0, 0)\Big|.$$

### 1.4.2 Root-Mean-Square (RMS) Error

$$error\_RMS \;=\; \sqrt{\frac{1}{N_{RMS}} \sum_{0 \le k \le N,\, \frac{V_{exact}(S_k,0)}{S_0} > 0.00001} \frac{\big|V_{approx}(S_k, 0) - V_{exact}(S_k, 0)\big|^2}{\big|V_{exact}(S_k, 0)\big|^2}}$$

where $N_{RMS}$ is the number of nodes $k$ such that $V_{exact}(S_k, 0) > 0.00001 \cdot S_0$.

**Remark:**
If the error doesnot decrease as the discretization level increases, there must be something wrong.

## 2 Finite Difference Approximation for the Greeks

$$
\begin{aligned}
x_{compute} = x_{N_{left}} &\longrightarrow S_0 = Ke^{x_{N_{left}}} \\
x_{compute} - \delta_x = x_{(N_{left}-1)} &\longrightarrow S_{-1} = S_0 e^{-\delta_x} = Ke^{x_{N_{left}}-\delta_x} \\
x_{compute} + \delta_x = x_{(N_{left}+1)} &\longrightarrow S_1 = S_0 e^{\delta_x} = Ke^{x_{N_{left}}+\delta_x}
\end{aligned}
$$

Let $V_{-1},\ V_0,\ V_1$ be the approximate values of the option at the nodes $S_{-1},\ S_0$, and $S_1$, respectively, corresponding to the finite difference solution, i.e.,

$$
\begin{aligned}
V_{-1} &= \exp\big(-ax_{(N_{left}-1)} - b\tau_{final}\big)\ U^M\big(N_{left}-1\big) \\
V_0 &= \exp\big(-ax_{N_{left}} - b\tau_{final}\big)\ U^M\big(N_{left}\big) \\
V_1 &= \exp\big(-ax_{(N_{left}+1)} - b\tau_{final}\big)\ U^M\big(N_{left}+1\big)
\end{aligned}
$$

### 2.1 Delta

Three approximations for the Delta of the option, using forward, backward, and central approximations:

$$
\begin{aligned}
\Delta_{forward} &= \frac{V_1 - V_0}{S_1 - S_0} \\
\Delta_{backward} &= \frac{V_0 - V_{-1}}{S_0 - S_{-1}} \\
\Delta_{central} &= \frac{V_1 - V_{-1}}{S_1 - S_{-1}}
\end{aligned}
$$

Note that central approximation is NOT of second order convergence, since $S_0$ is not the midpoint of $S_1$ and $S_{-1}$.

### 2.2 Gamma

Central difference approximation of the Gamma of the option:

$$
\Gamma_{central} = \frac{\frac{V_1 - V_0}{S_1 - S_0} - \frac{V_0 - V_{-1}}{S_0 - S_{-1}}}{\frac{S_1 + S_0}{2} - \frac{S_{-1} + S_0}{2}}
$$

### 2.3 Theta

Recall that $t = T - \frac{2\tau}{\sigma^2}$, $\tau_{final} = \frac{T\sigma^2}{2}$, thus $T = \frac{2\tau_{final}}{\sigma^2}$, and then

$$
t = \frac{2(\tau_{final} - \tau)}{\sigma^2}
$$

The next to last time step on the $\tau-$axis, i.e., $\tau_{final} - \delta_\tau$, corresponding to time on the $t-$axis

$$
\begin{aligned}
\delta_t &= \frac{2(\tau_{final} - (\tau_{final} - \delta_\tau))}{\sigma^2} = \frac{2\delta_\tau}{\sigma^2} \\
V_{0,\delta_t} &= \exp\big(-ax_{N_{left}} - b(\tau_{final} - \delta_\tau)\big)\ U^{M-1}\big(N_{left}\big) \\
\Rightarrow \Theta_{forward} &= \frac{V_{0,\delta_t} - V_0}{\delta_t}
\end{aligned}
$$

where $U^{M-1}\big(N_{left}\big)$ is the finite difference approximation of $u\big(x_{N_{left}}, \tau_{final} - \delta_\tau\big)$

## 3 Moreover

- If $S_0$ is not on the grid, then
  - We can interpolate to get $S_0$
  - Given $M$ and $\alpha$, $x_{right} - x_{left}$ may not be of the integer times of $\delta_x$. To solve this problem, we can decrease $\alpha$ for a little bit. (Always decrease $\alpha$!)
  - Greeks:
    - Delta: we need to interpolate and get $S_0$
    - Gamma: two extra grids are needed
- Barrier Options:
  - One of the boundary conditions should be exact "=".
- Implied volatility

$$\text{Solve} \quad V_{market} - V_{FD}(\sigma) = 0$$

Step 1. Decide what $M, N$ should be

Step 2. Secant Method

$$\sigma_0 = 0.05, \quad \sigma_1 = 0.50$$
$$\sigma_{m+2} = \sigma_{m+1} - \frac{V_{FD}(\sigma_{m+1}) - V_{market}}{\frac{V_{FD}(\sigma_{m+1}) - V_{FD}(\sigma_m)}{\sigma_{m+1} - \sigma_m}}$$