Fundamental Theorems of Asset Pricing : multidimensional model

Wednesday, February 22, 2012
2:31 PM

Let $W_t = (W_1(t), W_2(t), \dots, W_d(t))^T$ - d-dim Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{P})$

Girsanov's Th (multidimensional)

Let $\theta_t = (\theta_1(t), \theta_2(t), \dots, \theta_d(t))^T$ - d-dim adapted process

Define
$$Z_t = \exp\left\{ -\int_0^t \theta_u \, dW_u - \frac{1}{2}\int_0^t \|\theta_u\|^2 du \right\}$$

$$\widetilde{W}_t = W_t + \int_0^t \theta_u \, du$$

Assume $\mathbb{E}\int_0^T \|\theta_u\|^2 \cdot Z_u^2 \, du < \infty$. Then $EZ_T = 1$ and

$\widetilde{W}_t$ is Brownian motion under $\widetilde{\mathbb{P}}$, where

$d\widetilde{\mathbb{P}}(\omega) = Z_T(\omega)\, d\mathbb{P}(\omega)$.

Remark :
$$\int_0^t \theta_u \, dW_u = \sum_{j=1}^{d} \int_0^t \theta_j(u)\, dW_j(u)$$

$$\|\theta_u\| = \left(\sum_{j=1}^{d} \theta_j^2(u) \right)^{1/2}$$

Multidimensional Market Model

Assume there are m stocks, each with :

$$dS_i(t) = \alpha_i(t) S_i(t)\, dt + S_i(t) \cdot \sum_{j=1}^{d} \sigma_{ij}(t)\, dW_j(t) \qquad i = 1, \dots, m.$$

$$dS_i(t) = \alpha_i(t) S_i(t)\, dt + S_i(t) \cdot \sum_{j=1}^{d} \sigma_{ij}(t)\, dW_j(t) \qquad i = 1, \dots, m.$$

$$(\alpha_1(t), \alpha_2(t), \dots, \alpha_m(t))^T = \text{mean rate of return vector} \in \mathbb{R}^m$$

$$[\sigma_{ij}(t)] = \text{volatility matrix} \in \mathbb{R}^{m \times d}$$

Notation :

- $\sigma_i(t) = \sqrt{\sum_{j=1}^{d} \sigma_{ij}^2(t)} \neq 0$

(martingale)

- $B_i(t) = \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}\, dW_j(u) \qquad i = 1, \dots, m$

- $d\langle B_i \rangle_t = \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)}\, dt = dt$

$\Rightarrow$ $B_i(t)$ is Brownian Motion

$\leadsto$ $dS_i(t) = \alpha_i(t) S_i(t) + S_i(t) \sigma_i(t)\, dB_i(t)$

where $B_1, B_2, \dots, B_m$ are correlated.

Existence of a risk-neutral measure

Def : $\tilde{\mathbb{P}}$ is called **RISK NEUTRAL PROBABILITY** iff

(i) $\tilde{\mathbb{P}} \sim \mathbb{P}$

(ii) under $\tilde{\mathbb{P}}$, the discounted stock price is martingale, for all i.

($\tilde{S}_i(t) = D(t) \cdot S_i(t)$: for $i = 1, \dots, m$)

Remark : if the discounted stock price is $\tilde{\mathbb{P}}$-martingale so is the discounted wealth process.

⇝ easy to see that the discounted stock price follows the eq:

$$d\tilde{S}_i(t) = \tilde{S}_i(t)\left(\alpha_i(t) - R(t)\right)dt + \tilde{S}_i(t)\sum_{j=1}^{d} \sigma_{ij}(t)\, dW_j(t)$$

⇝ in the multidimensional case the market price of risk $\theta_i(t)$ need not always exist !!!

Market Price of Risk Equations:

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\,\theta_j(t) \qquad i = \overline{1,\dots,m}$$

⇝ if one cannot solve this system of equations there is arbitrage lurking in the market.
(the model is bad and it should not be used for pricing)

- let's rewrite the discounted stock equations in matrix form.

$$d(D_t S_t) = D_t S_t \left[(\alpha_t - \mathbb{1}_m R_t)\, dt + \sigma_t\, dW_t \right]$$

- we want to compare the above eq with the following:

$$d(D_t S_t) = D_t S_t\, \sigma_t \left[\underbrace{\theta_t\, dt + dW_t}_{d\tilde{W}_t} \right]$$

where $\theta_t \in \mathbb{R}^d$, $\alpha_t \in \mathbb{R}^m$, $1_m = (1, 1, \dots, 1)^T \in \mathbb{R}^m$
$W_t \in \mathbb{R}^d$, $\sigma_t \in \mathbb{R}^{m \times d}$

- the market price of risk equation becomes:

$$(*) \quad \boxed{\alpha_t - 1_m R_t = \sigma_t \theta_t}$$

where θ_t is unknown.

(wealth)

Definition: An **ARBITRAGE** is a portfolio value process X_t satisfying $X(0) = 0$ and

$$P(X_T \geq 0) = 1, \quad P(X_T > 0) > 0.$$

- Now let's discuss the equation $(*)$ and how it relates to the concept of arbitrage!

Case 1: **Unique solution**

- for every t and $\mathbb{P}$-almost all ω $(*)$ has a unique solution
- using θ_t we can construct $\tilde{\mathbb{P}}$ (uniquely)
- no arbitrage

$\rightsquigarrow$ the market is **COMPLETE**

(every contingent claim can be hedged)

Case 2: **No Solution**

- (*) has no solution, therefore there is no risk neutral measure
- there must be some way to form an arbitrage

$\rightsquigarrow$ BAD MODEL !!!

Case 3: Infinitely many solutions

- different θ_t lead to different $\widetilde{\mathbb{P}}$ which lead to different prices for the contingent claims.
- any contingent claim that has multiple prices cannot be hedged
- the market admits no arbitrage

$\rightsquigarrow$ the market is INCOMPLETE !

First Fundamental Theorem of Asset Pricing [Case 1,3]

If a market model has at least one risk-neutral measure, then it does not admit arbitrage.

Second Fundamental Theorem of Asset Pricing [Case 1]

Consider a market model that has a risk-neutral measure. The model is complete iff the risk-neutral measure is unique.

Proof of the first theorem

- $\widetilde{\mathbb{P}}$ - risk neutral $\Rightarrow$ $\widetilde{X}_t = D_t X_t$ is martingale under $\widetilde{\mathbb{P}}$ for any portfolio value process
- assume $X_0 = 0$ $\Rightarrow$ $\widetilde{\mathbb{E}}[D_T X_T] = 0$

- assume $X_0 = 0 \Rightarrow \tilde{\mathbb{E}}[D_T X_T] = 0$
- suppose X_T is such that : $\tilde{\mathbb{P}}[X_T \geq 0] = 1$ and $\tilde{\mathbb{P}}[X_T > 0] > 0$

$$\tilde{\mathbb{E}}[D_T X_T] = \tilde{\mathbb{E}}\left[D_T X_T \mathbb{1}(X_T > 0)\right] + \underbrace{\tilde{\mathbb{E}}\left[D_T X_T \mathbb{1}(X_T = 0)\right]}_{=0} = 0$$

- since $\tilde{\mathbb{P}}(X_T > 0) > 0 \Rightarrow \tilde{\mathbb{E}}\left(D_T X_T \mathbb{1}(X_T > 0)\right) > 0$
  which is a contradiction

$\Rightarrow$ therefore if $\tilde{\mathbb{P}}$ exists then there is no arbitrage.

Proof of the Second theorem

"$\Rightarrow$"

- first assume that the market is complete we wish to prove the risk neutral measure is unique.
- suppose there are 2 such risk neutral measures $\tilde{\mathbb{P}}_1, \tilde{\mathbb{P}}_2$
- let $A \in \mathcal{F}_T$, consider the following contingent claim :

$$V_T = \mathbb{1}_A \frac{1}{D_T}$$

- because the market is complete $\rightsquigarrow$ V_T can be hedged (there is a replicating portfolio such that $X_T = V_T$)
- since both $\tilde{\mathbb{P}}_1, \tilde{\mathbb{P}}_2$ are risk-neutral $\Rightarrow$

  $D_t X_t$ is martingale under both $\tilde{\mathbb{P}}_1$ and $\tilde{\mathbb{P}}_2$

$$\tilde{\mathbb{P}}_1(A) = \tilde{\mathbb{E}}_1[D_T V_T] = \tilde{\mathbb{E}}_1[D_T X_T] = X_0 = \tilde{\mathbb{E}}_2[D_T X_T]$$

$$= \tilde{\mathbb{E}}_2[D_T V_T] = \tilde{\mathbb{P}}_2(A)$$

- since A is an arbitrary event in $\mathcal{F}_T \Rightarrow \tilde{\mathbb{P}}_1 = \tilde{\mathbb{P}}_2$

- since A is an arbitrary event in $\mathcal{F}_T \Rightarrow \tilde{\mathbb{P}}_1 = \tilde{\mathbb{P}}_2$

"$\Leftarrow$" • suppose there is only one risk neutral measure $\tilde{\mathbb{P}}$

we want to prove the market is complete.

- $\tilde{\mathbb{P}}$ unique : means the market price of risk eq have an unique solution : $\theta_t = \sigma_t^{-1}(\alpha_t - R_t \mathbb{1}_m)$

- in order to show that every contingent claim can be hedged we must be able to find $(\Delta_1(t), \dots, \Delta_m(t))^T = \Delta_t$ the hedging portfolio.

- from the Martingale Representation Th :

$$D_t V_t = V_0 + \sum_{j=1}^{d} \int_0^t \tilde{\Gamma}_j(u)\, d\tilde{W}_j(u)$$

$$D_t X_t = X_0 + \sum_{i=1}^{m} \int_0^t \Delta_i(u)\, d(D_u S_i(u))$$

$$= X_0 + \sum_{i=1}^{m} \sum_{j=1}^{d} \int_0^t \Delta_i(u) D_u S_i(u) \sigma_{ij}(u)\, dW_j(u)$$

$$= X_0 + \sum_{j=1}^{d} \int_0^t \sum_{i=1}^{m} \Delta_i(u) D_u S_i(u) \sigma_{ij}(u)\, dW_j(u)$$

$$\Rightarrow \tilde{\Gamma}_j(t) = \sum_{i=1}^{m} \Delta_i(t) D_t S_i(t) \sigma_{ij}(t) \qquad j = 1, 2, \dots, d$$

- in matrix form this system of equations becomes

$$\sigma^T y = c$$

where $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ & & & \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{md} \end{pmatrix}$ d

$$y = \begin{pmatrix} \Delta_1 S_1 \\ \Delta_2 S_2 \\ \vdots \\ \Delta_m S_m \end{pmatrix} \qquad c = \frac{1}{D_t} \begin{pmatrix} \tilde{\Gamma}_1 \\ \tilde{\Gamma}_2 \\ \vdots \\ \tilde{\Gamma}_d \end{pmatrix}$$

- we know that σ^{-1} exists since the system $\alpha_t - 1_m R_t = \sigma_t \theta_t$ has a unique solution.

$\Rightarrow$ y can be solved uniquely so we can identify a hedging portfolio $(\Delta_1, \Delta_2, \dots, \Delta_m)^T$.

- example: $m = 2$, $d = 1$, suppose the market price of risk eq are:

$$\begin{cases} \alpha_1 - r = \sigma_1 \theta \\ \alpha_2 - r = \sigma_2 \theta \end{cases}$$

$\rightsquigarrow$ these eq have solution if $\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2} = \theta$

$\rightsquigarrow$ if the above eq does not hold then one can arbitrage one stock against the other!

- say $\mu = \frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} > 0$

$$\Delta_1(t) = \frac{1}{S_1(t)\sigma_1} \qquad \Delta_2(t) = \frac{-1}{S_2(t)\sigma_2}$$

- initial capital required: $\frac{1}{\sigma_1} - \frac{1}{\sigma_2}$

$$dX_t = \Delta_1(t)\,dS_1(t) + \Delta_2(t)\,dS_2(t) + r\left(X_t - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t)\right)dt$$

$$= \frac{\alpha_1(t)-r}{\sigma_1}dt + dW_t - \frac{\alpha_2(t)-r}{\sigma_2}dt - dW_t + rX_t\,dt$$

$$= \mu\,dt + rX_t\,dt$$

$\Rightarrow$ $d\tilde{X}_t = d(D_tX_t) = \mu D_t\,dt > 0$

$\Rightarrow$ this is an arbitrage portfolio.