Bond Hedging

Summary of methods available

In the following, we describe the hedging methods implemented in R. A zero yield curve is fitted each day on the coupon bonds such as the zero yield curve series are available.

Notations

We use the following notations:

* t: date on which we want to estimate the hedge ratios
* $T \rightarrow r(t,T) = r_T(t)$ is the zero yield curve T estimated by curve fitting at date t.
* $B = (b_i)_{1 \leq i \leq N}$ is the portfolio of N bonds to be hedged, with maturities T_i and prices $P^b(t,T_i)$
* $C = (c_1, c_2, c_3)$ is the hedge CTD's bonds (also included in the portfolio B), with maturities τ_i and prices $P^c(t,\tau_i)$
* $h = (h_{i,1}, h_{i,2}, h_{i,3})_{1 \leq i \leq N}$ the hedge ratios to be estimated

1 PCA hedging on zero yields

In this part, we describe how we hedge the portfolio B by applying PCA to zero yields.

1.1 Stripping the coupon bonds portfolio

First, we strip each coupon bond to a portfolio of zero-coupon bonds. For example, we consider the coupon bond b with maturity T and p remaining cash flows at $(T_1, \ldots T_p = T)$. Its price is given by:

$$P^b(t,T) = \sum_{j=1}^{p} c\ P^z(t,T_j) + 100 \times P^z(t,T_p) \tag{1}$$

where c is the coupon amount and $P^z(t,T_j)$ is the price of the zero-coupon bond Z_j. The replicating portfolio of zeros is then:

$$b \equiv [c\ .\ Z_1, \ldots, c\ .\ Z_{p-1}, (100+c)\ .\ Z_p]$$

For the whole portfolio B, we consider the zero-coupon equivalent for each bond $b \in B$ and aggregate the cash flows such as the B-equivalent portfolio contains M zeros at maturities $(T_1, \ldots, T_M)$.

1.2 Factor Model

Description

Let $r_i = r(t,T_i)$, $i = 1, \ldots, M$ be the zero coupon rate at each cash flow maturity and $dr_i = r(t,T_i) - r(t-1, T_i - 1)$ the corresponding change in yield. We assume that K common factors drive the dynamic of the term structure:

$$\forall i \in [1,M], \quad dr_i = \sum_{\text{[redacted]}}^{K} v_{i,j} F_j + \epsilon_i \tag{2}$$

where ϵ_i is a white noise. For J observations, we can write the equation in matrix form:

$$dr = F\ v' + \epsilon \tag{3}$$

where $dr = [dr_1, \ldots, dr_M]$ is $J \times M$, F the factors are $J \times K$ and v the factor loadings are $M \times K$.

Principal Component Analysis (PCA) Estimation

One way to identify the factors is to use PCA. In fact, PCA extracts from the data orthogonal factors that explain most of the variance: the first principal component has the largest possible variance, ...
To estimate the principal components, we center the data dr and do its Singular Value Decomposition (SVD):

$$dr = u\ S\ v' \tag{4}$$

The K-PCA factors F of size $T \times K$ are:

$$F = dr\ v_{:,1:k} \tag{5}$$

i.e the first factor is $F_1 = \sum_{i=1}^{M} dr_i\ v_{i,1}$ and v gives the factor loadings.

1.3 Factor Neutrality

Portfolio Immunization

We would like to hedge each bond using the 3 CTD's. For example for a coupon bond B with price $P^b(t,T)$, we would like to find the optimal $h = (h_1, h_2, h_3)$ such as the change in price:

$$dP^b = dP_1^c\ h_1 + dP_2^c\ h_2 + dP_3^c\ h_3 \tag{6}$$

In our factor model, K orthogonal factors drive the term structure dynamic. Therefore, any change in bond price dP can be approximated in first order by:

$$dP = \frac{\partial P}{\partial F_1} F_1 + \ldots + \frac{\partial P}{\partial F_K} F_K \tag{7}$$

Let us assume that the bond sensitivity to the factors are known, then Eq. 6 becomes:

$$F1 \left(-\frac{\partial P^b}{\partial F_1} + h_1 \frac{\partial P_1^c}{\partial F_1} + h_2 \frac{\partial P_2^c}{\partial F_1} + h_3 \frac{\partial P_3^c}{\partial F_1} \right) + F_2\ () + \ldots + F_K\ () = 0 \tag{8}$$

More precisely, Eq. 8 means that we would like to immunize the portfolio $(-b,\ h_1 c_1,\ h_2 c_2,\ h_3 c_3)$ against the factors.

Solving for hedge Ratios

The factors being orthogonal, it yields:

$$\begin{cases} \left[\frac{\partial P_1^c}{\partial F_1}, \frac{\partial P_2^c}{\partial F_1}, \frac{\partial P_3^c}{\partial F_1}\right] \cdot h' & = \frac{\partial P^b}{\partial F_1} \\ \vdots & = \vdots \\ \left[\frac{\partial P_1^c}{\partial F_K}, \frac{\partial P_2^c}{\partial F_K}, \frac{\partial P_3^c}{\partial F_K}\right] \cdot h' & = \frac{\partial P^b}{\partial F_K} \end{cases} \tag{9}$$

in matrix form:

$$\underset{K \times 3}{\left(\frac{\partial P_j^c}{\partial F_i}\right)_{i,j}} \cdot \underset{3 \times 1}{h'} = \underset{K \times 1}{\left(\frac{\partial P^b}{\partial F_i}\right)_i} \tag{10}$$

Therefore, the hedge ratio h is uniquely defined if the number of factors $K = 3$ the number of hedge instruments.

Generalization to several bonds

We generalize to the case where we hedge jointly for all the bonds in the portfolio $B = (b_1, \ldots, b_N)$. For N bonds Eq. 10 becomes:

$$\begin{array}{ccccc} \left(\frac{\partial P_j^c}{\partial F_i}\right)_{i,j} & . & h' & = & \left(\frac{\partial P_j^b}{\partial F_i}\right)_{i,j} \\ \\ K \times 3 & . & 3 \times N & & K \times N \end{array} \tag{11}$$

When $K = 3$, each bond b_i has an unique hedge $h_{,i}$ that solves Eq. 10 and the portfolio hedge ratios h $(3 \times N)$ are uniquely determined.
Moreover, we allow for the case where $K > 3$ and the system is over-determined. We find h by least squares on the whole portfolio. We use a weighted least squares with $w_k = S_{k,k}$ the PCA singular values which represent the % of variance explained by each factor. In fact, it guarantees that we neutralize more the first factors (natural PCA weighting).

1.4 Factor Duration

In order to solve for the hedge ratios, we need to estimate the bond sensitivity to the factors: $\frac{\partial P_j}{\partial F_i}$.

Zero Coupon Bond

More precisely, for a zero coupon bond:

$$\frac{\partial P^z(t,T_i)}{\partial F_1} = \frac{\partial P^z(t,T_i)}{\partial r_i} \times \frac{\partial r_i}{\partial F_1} \tag{12}$$

From Eq. 2:

$$\frac{\partial r_i}{\partial F_1} = v_{i,1}$$

Moreover, for a zero-coupon bond:

$$\frac{\partial P^z(t,T_i)}{\partial r_i} = -(T_i - t)P^z(t,T_i)$$

Therefore, it yields:

$$\frac{\partial P^z(t,T_i)}{\partial F_1} = -(T_i - t)\ v_{i,1}\ P^z(t,T_i) \tag{13}$$

Coupon Bond

For a coupon bond of price $P^b(t,T)$, maturity T, payments at $T_1, \ldots, T_p = T$ the same method yields:

$$\frac{\partial P^b(t,T)}{\partial F_1} = -c\sum_{j=1}^{p}(T_j - t)\ v_{j,1}\ P^z(t,T_j) - 100\ (T_p - t)\ v_{p,1}\ P^z(t,T_p) \tag{14}$$

1.5 Conclusion

To conclude, we estimate in closed form the coupon bond sensitivity to the factors. Consequently, we manage to solve for the hedge ratios that immunize the portfolio against changes in the factors by solving Eq. 11.

2 Hedging using yields to maturity

In this part, we describe another approach that consists in appliying PCA to bond yields to maturity.

Yield to maturity time series

The first step is to generate yield to maturity time series for the coupon bonds in the portfolio B. For a given bond b at time t, we compute the yield $y^b(s,T)$ at time $s < t$ by discounting the bond's future cash flows as of time t on the yield curve fitted at time s.

For example, if at time t, the bond b has p remaining cash flows at $(T_1, \ldots T_p = T)$. Its synthetic price at time s is given by:

$$P^b(s,T) = \sum_{j=1}^{p} c\ P^z(s,T_j) + 100 \times P^z(s,T_p) \tag{15}$$

and the corresponding yield to maturity is $y^b(s,T)$.

2.1 Regression based hedging (Least Squares)

It uses linear regression on the yield to maturity time series to compute the hedge ratios. For a bond b with yield to maturity y^b. We look for the hedge ratios h that explain the bond change in yield:

$$dy^b = dy^c * h \tag{16}$$

Then we back out the hedge ratios in price space:

$$h_j = h_j \frac{\partial P_j^c/\partial y}{\partial P^b/\partial y}, \qquad j = 1,2,3 \tag{17}$$

2.2 PCA hedging

Description

We assume that yields to maturity follow a factor model. For the portfolio $B = (b_1, \ldots, b_N)$ with yield to maturity $y = (y_1, \ldots, y_N)$ we assume:

$$dy = F\ v' + \epsilon \tag{18}$$

PCA estimation

Here again, the factor are estimated using PCA on the centered yield changes dy (factors that explain better the variance).

$$dy = u\ S\ v' \tag{19}$$

The K-PCA factors F of size $T \times K$ are:

$$F = dy\ v_{:,1:k} \tag{20}$$

Factor Neutrality

In order to find the optimal hedge ratios that solves Eq. 11, we need to estimate the bond sensitivity to the factors. For a coupon bond of price $P^b(t,T)$ and maturity T_i, the price sensitivity to the first factor is:

$$\begin{aligned}
\frac{\partial P^b(t,T_i)}{\partial F_1} &= \frac{\partial P^b(t,T_i)}{\partial y_i} \times \frac{\partial y_i}{\partial F_1} \\
&= \frac{\partial P^b(t,T_i)}{\partial y_i} \times v_{i,1}
\end{aligned} \tag{21}$$

where $\frac{\partial P^b(t,T_i)}{\partial y_i}$ is the coupon bond dpdy which is known. Consequently, we can solve Eq. 11 in the same way.

3 Refinements

We consider the following refinements in our methodology.

Decaying the data

We allow for an exponentially decayed data. In fact, we would like to weight more the more recent information. We choose the decay δ such as the halflife

$$\text{halflife} = -\frac{\ln 2}{\ln(\text{decay})} \tag{22}$$

Then, we apply the same hedging methodology to the decayed data. More precisely, we estimate the PCA factors by doing the Eigenvalue Decomposition (EVD) of the decayed yield covariance.

PCA on correlation

We also consider doing PCA on the yield correlation. In fact, PCA captures the direction of maximum variance. Therefore, it is impacted by each yield idiosyncratic volatility. By considering the correlation, we estimate factors that are not biased by the yield volatilities.

TDG Implementation

TDG uses the PCA on yield to maturity approach. However, it deals differently with the volatilities. First, it rescales the factor loadings such as $diag(v\ v') = 1$ (rescale by the volatility per asset in the factors). However, this rescaling distorts the eigenvectors v_i which are not orthogonal anymore.

TDG includes again the volatilities when solving for the hedge ratios:

$$\left(\frac{\partial P_j^c}{\partial F_i}\,\sigma_j^c\right)\,.\,h' = \left(\frac{\partial P_j^b}{\partial F_i}\,\sigma_j^b\right) \tag{23}$$

where σ_j^b is the volatility of the yield $y^b(t,T_j)$ (bond b_j).

4 Benchmarks

Finally, we describe the methods currently used for hedging that we use to benchmark the methods already described.

Interpolation based hedging

We consider a bond b of maturity T. The hedge ratios $h = (h_1, h_2, h_3)$ for b are given by:

$$h_j = w \frac{\partial P_j^c / \partial y}{\partial P^b / \partial y}, \qquad j = 1, 2, 3 \tag{24}$$

where w is given by time-to-maturity linear interpolation:

$$\begin{cases} w = (1,0,0) & \text{if } T \leq \tau_1 \\ w = (0,0,1) & \text{if } T \geq \tau_3 \\ w = (0, 1-\alpha, \alpha) & \text{if } \tau_2 \leq T \leq \tau_3 \\ \vdots \end{cases} \tag{25}$$

where τ's are the CTD maturities, and

$$\alpha = \frac{T - \tau_2}{\tau_3 - \tau_2}$$

Similarly, w is given by linear interpolation whenever a bond is between 2 CTD's.

PCA on CMT yields

Another approach used by the UK desk is to apply PCA on par yield time series at the bond's maturities computed from the zero curves. The hedge ratios are then produced by neutralizing the 3 first factors in the case of 3 hedge instruments.

1. Why hedge with futures?
2. Delta Markets
3. Delta Grab.