· Discrete Probability Distr. & Basic Concepts

$(\Omega, \mathcal{F}, P)$: prob. space.

Ω : set of all elementary outcomes of a random experiment.

ex : experiment : "toss a coin"

$\Omega = \{\text{head}, \text{tail}\}$

· for now, Ω either finite or countable.

$\mathcal{F}$ = set of all events (σ-algebra)

A = event $\qquad A \subseteq \Omega$

· for now $\mathcal{F}$ = all subsets of Ω

$|\Omega| = 4 \rightsquigarrow |\mathcal{F}| = 2^4$

$n \longrightarrow 2^n$

ex 1) $\Omega = \{H, T\}$

$\emptyset$ = impossible event $\qquad \Omega$ = sure event

$\{H\}$, $\{T\}$

2) $\Omega = \{1, 2, 3, 4, 5, 6\}$

random experiment.

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$|\mathcal{F}| = 2^6$

$\emptyset, \Omega$

$A = \{ \text{even nr} \} = \{2, 4, 6\}$

$A^c = \Omega \setminus A = \{1, 3, 5\}$

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<u>Def</u> $\mathcal{F}$ = σ-algebra

1) $\phi, \Omega \in \mathcal{F}$

2) if $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$

3) if $A_1, A_2, \ldots \in \mathcal{F} \Longrightarrow \bigcup_{n \geq 1} A_n \in \mathcal{F}$

operations with events :

$A^c = \Omega \setminus A$ = opposite event

A, B : $A \cup B$ = A or B

$A \cap B$ = A and B

A, B disjoint if $A \cap B = \phi$

. $P : \mathcal{F} \longrightarrow [0,1]$ probability <u>measure</u>

(1) $P(\Omega) = 1$; $P(\phi) = 0$

(2) for every sequence of (pairwise) disjoint events

$A_1, A_2, \ldots$

$$P\left(\bigcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} P(A_n)$$

$A_1 = \{w_1\}$ $A_2 = \{w_2\}$ ....

$\Omega = \{w_1, w_2, w_3, \ldots \}$

$$\Omega = \{ \omega_1, \omega_2, \omega_3, \ldots \}$$

- if A and B are not disjoint

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- if A, B, C are not pairwise disjoint

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Problem 1 : $(\Omega, \mathcal{F}, P)$ $n = 3$

$$\Omega = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \ldots$$

$$P(\text{exactly one } H) = P(\{HTT, THT, TTH\}) = \frac{3}{8}$$

$$n = 10 \rightsquigarrow 10 \cdot \frac{1}{2^{10}}$$

$$P(\text{at most one } H) = P(\text{no } H) + P(\text{exactly one } H)$$

$$P(\text{at least one } H \text{ and at least one tail}) = 1 - \frac{2}{8}$$

Random Variables $(\Omega, \mathcal{F}, P)$

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X real valued function on Ω

ex : $\Omega = \{HHH, HHT, \dots, TTT\}$

$X =$ nr of H

$X(HHT) = 2$

$$\omega \longmapsto X(\omega) \in \mathbb{R}$$

$$X : \Omega \longrightarrow \mathbb{R}$$

- distribution of X : $P(X = x) = ?$

$X =$ nr of H : possible outcomes : $\{0, 1, 2, 3\}$

$X : \Omega \longmapsto \{0, 1, 2, 3\}$

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$P(X=0) = P(TTT) = \frac{1}{8}$

Cumulative Distribution function :

$$F(x) = P(X \le x)$$

Expected Value of X (measure of location)

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$$EX = \sum_{n \geq 1}^{\infty} x_n \cdot P(X = x_n)$$

if the possible outcomes are $\{x_1, x_2, x_3, \dots\}$

Variance of X (measure of spread)

$$\text{Var}\, X = EX^2 - (EX)^2 = E(X - EX)^2$$

- X, Y r.v ; a, b real n.r

$$E[aX + bY] = a\,EX + b\,EY$$

$$\text{Var}[aX + bY] = a^2\, \text{Var}\, X + b^2\, \text{Var}\, Y + 2ab\, \text{Cov}(X, Y)$$

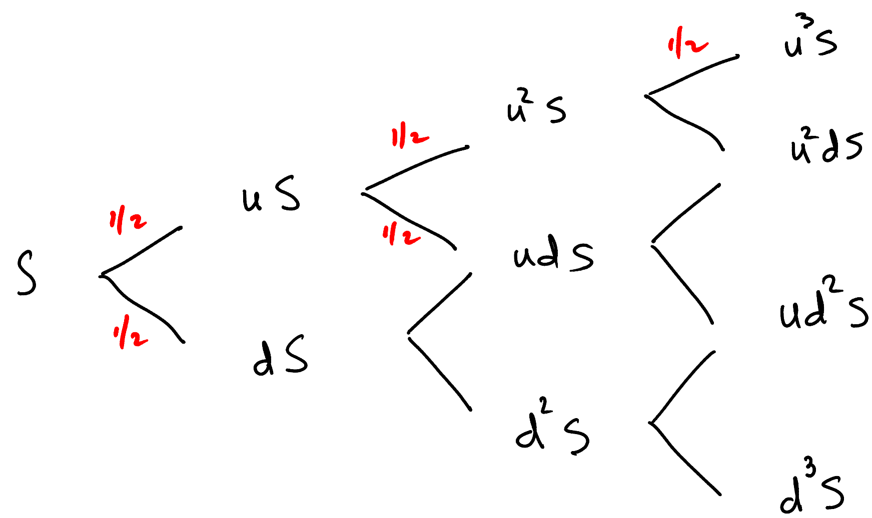
- If X, Y are independent $\rightsquigarrow$ $\text{Cov}(X, Y) = 0$

<u>example</u> : Binomial Asset Pricing Model. $n = 3$

S_0 : initial price

$S \uparrow$: $S_n = u \cdot S_{n-1}$

$S \downarrow$: $S_n = d \cdot S_{n-1}$



Distribution of S_3

S_3	u^3S	u^2dS	ud^2S	d^3S
prob.	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$P\left(S_n = u^k d^{n-k} \cdot S\right) = \binom{n}{k} p^k q^{n-k}$$

$p =$ Prob $S\uparrow$ $\qquad$ $q =$ Prob $S\downarrow$

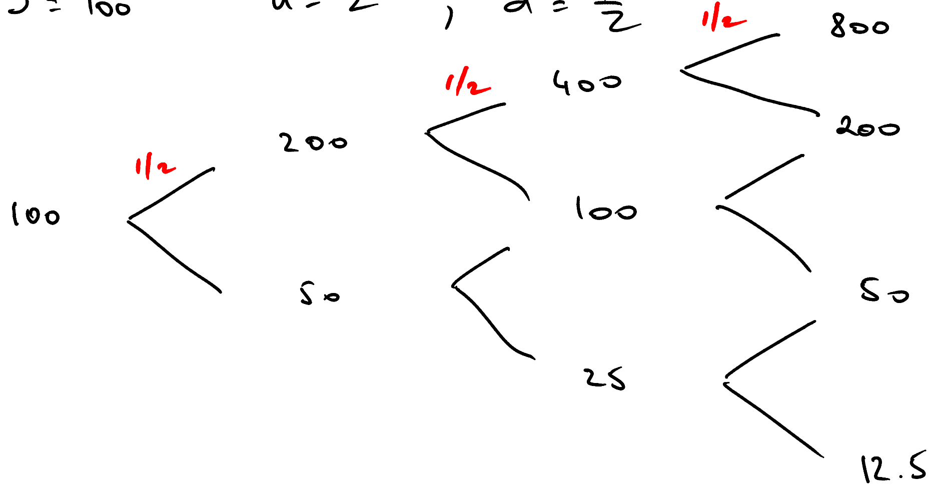
payoff of a call option : $(S_3 - K)^+$

$$E\,(S_3 - K)^+$$

$$X \rightsquigarrow EX = \sum_n{}' x_n P(X = x_n)$$

$$E f(X) = \sum_n{}' f(x_n)\, P(X = x_n)$$

$S = 100 \qquad u = 2 \,, \quad d = \frac{1}{2}$



S_3	$(S_3 - 100)^+$
800	700
200	100
50	0
12.5	0

· $n = 3 \,, \quad K = 100$

$$E(S_3 - 100)^+ = 700 \cdot \frac{1}{8} + 100 \cdot \frac{3}{8} = \frac{1000}{8}$$

$$X \sim Bin(n, p)$$

$$EX = n \cdot p$$

$$Var = n \cdot p(1 - p)$$