Brownian Motion

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Definition 1: Let $(\Omega, \mathcal{F}, P)$ be a prob. space and for each $\omega \in \Omega$, there is a continuous function $B(t,\omega)$ for $t \geq 0$ such that $B(0,\omega) = 0$. Then $B = \{B(t,\omega)\}_{t \geq 0}$ is a **Brownian Motion** if

(1) for all $0 = t_0 < t_1 < \dots < t_m$ and $m \geq 2$

$B(t_1) - B(t_0),\ B(t_2) - B(t_1), \dots, B(t_m) - B(t_{m-1})$

are independent

(2) each $B(t_i) - B(t_{i-1}) \sim N(0, t_i - t_{i-1})$

- an equivalent definition of B.M. is given in terms of Gaussian processes

Definition 2: $(\Omega, \mathcal{F}, P)$ = prob. space.

For each $\omega \in \Omega$, $B(t,\omega)$ is a continuous such that $B(0,\omega) = 0$.

Then $B = \{B(t,\omega)\}_{t \geq 0}$ is a Brownian Motion if

(1) B is a Gaussian process (i.e. all its finite dimensional distributions are gaussian)

(2) $EB(t) = 0$; $E\,B(t)B(s) = s \wedge t$ $\quad s,t \geq 0$.

Lemma 1: (Joint distribution of BM) (use def. 1)

For all $0 \leq t_1 < t_2 < \dots < t_m$, $m \geq 2$

$(B(t_1), B(t_2), \dots, B(t_m))$ has a Gaussian distribution with mean vector 0 and covariance matrix given by:

$$Cov\left(B(t_i), B(t_j)\right) = t_i \wedge t_j \qquad i,j = 1,2,\dots, m.$$

Proof:

$$\underbrace{\begin{pmatrix} B(t_1) \\ B(t_2) \\ \vdots \\ B(t_m) \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} B(t_1) \\ B(t_2) - B(t_1) \\ \vdots \\ B(t_m) - B(t_{m-1}) \end{pmatrix}}_{X}$$

- vector X is Gaussian $\overset{def.}{\iff}$ $(\forall)\ a \in \mathbb{R}^m$ $\langle a, X\rangle$ is normally distributed.

- to show that Y is Gaussian consider an arbitrary $b \in \mathbb{R}^m$

$$\Rightarrow \langle b, Y\rangle = b^T \cdot Y = b^T \cdot A \cdot X = (A^T b)^T \cdot X = \langle A^T b, X\rangle$$

$\leadsto$ since X is Gaussian $\Rightarrow$ $\langle \underbrace{A^T b}_{a}, X\rangle$ is normally distr.

$\leadsto$ for $i \le j$

$$E\left(B(t_i) \cdot B(t_j)\right) = E\left(\underbrace{B(t_i)\left(B(t_j) - B(t_i)\right)}_{\text{independent}} + B^2(t_i)\right)$$

$$= EB(t_i) \cdot \underbrace{E\left(B(t_j) - B(t_i)\right)}_{=0} + E\left(B^2(t_i)\right) = t_i$$

- similarly, if $i > j \Rightarrow E\left(B(t_i) \cdot B(t_j)\right) = E B^2(t_j) = t_j$

$$\Rightarrow E\, B(t_i) \cdot B(t_j) = t_i \wedge t_j$$

- this lemma shows that the Definition 1 $\Rightarrow$ Definition 2

<u>Remmark</u>: If $Y = AX$ then $C_Y = A C_X A^T$

where C_X = covariance matrix of X

C_Y = covariance matrix of Y

<u>Lemma 2</u>: If BM is given by Definition 2. Then $(\forall) m \geq 2$ for all $0 = t_0 < t_1 < \dots < t_m$ the increments

$B(t_1) - B(t_0)$, $B(t_2) - B(t_1)$, ..., $B(t_m) - B(t_{m-1})$ are independend

and each $B(t_i) - B(t_{i-1}) \sim N(0, t_i - t_{i-1})$

<u>Proof</u>: $Y = AX$, $\det A = 1 \Rightarrow X = A^{-1} Y$

- we use the same reasoning, Y is gaussian $\Rightarrow$ X is gaussia
- to check for independence of coordinates of a gaussian vector it is enough to check that they are uncorrelated

$$0 \leq i < j \leq l < k \leq m$$

$$E\Big(\big(B(t_j) - B(t_i)\big)\big(B(t_k) - B(t_l)\big) \Big) = t_j \wedge t_k - t_j \wedge t_l - t_i \wedge t_k + t_i \wedge t_l$$

$$= t_j - t_j - t_i + t_i = 0$$

- the distribution of $B(t_i) - B(t_{i-1})$ is gaussian with mean 0 and variance given by:

$$\mathrm{Var}\big(B(t_i) - B(t_{i-1})\big) = \mathrm{Cov}\big(B(t_i) - B(t_{i-1}), B(t_i) - B(t_{i-1})\big) =$$

$$= \mathrm{Cov}\big(B(t_i), B(t_i)\big) - 2\,\mathrm{Cov}\big(B(t_i), B(t_{i-1})\big) + \mathrm{Cov}\big(B(t_{i-1}), B(t_{i-1})\big)$$

$$= t_i - 2t_{i-1} + t_{i-1} = t_i - t_{i-1}$$

So the definitions (1) & (2) are equivalent!!

The Characteristic Function of $\underbrace{(B(t_1), \dots, B(t_m))}_{= Y^T}$

- $0 \le t_1 < t_2 < \dots < t_m$, $m \ge 1$
- set $\bar{u} = (u_1, u_2, \dots, u_m)^T$

CF : $\varphi_Y(\bar{u}) = E\, e^{i\langle \bar{u}, Y\rangle} = e^{-\frac{1}{2}\langle \bar{u}, C_Y \bar{u}\rangle}$ the characteristic function

where $\boxed{C_Y = (c_{ij})_{i,j=\overline{1,m}} \qquad c_{ij} = t_i \wedge t_j}$

MGF : $M_Y(\bar{u}) = E\, e^{\langle \bar{u}, Y\rangle} = e^{\langle \bar{u}, C_Y \bar{u}\rangle}$ moment generating function

Joint probability density function : of $(B(t_1), \dots, B(t_m))^T = Y$

$Y = AX$

$$f(y) = \frac{1}{(\sqrt{2\pi})^m \sqrt{\det C}}\, e^{-\frac{1}{2}\langle y, C_Y^{-1} y\rangle}$$

- recall $C_Y = A C_X A^T$ where $C_X = \begin{pmatrix} t_1 & 0 & 0 & \cdots & 0 \\ 0 & t_2 - t_1 & 0 & \cdots & 0 \\ & & & & \\ 0 & 0 & 0 & & t_m - t_{m-1} \end{pmatrix}$

and $A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$; which has an inverse

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 1 & 0 & \cdots & 0 \\ & & & & & \\ -1 & -1 & -1 & -1 & \cdots & 1 \end{pmatrix}$$

$$\Rightarrow \det C_Y = \det A \cdot \det C_X \cdot \det A^T = t_1 (t_2 - t_1) \cdots (t_m - t_{m-1})$$

- $y = Ax \Rightarrow \langle y, C_Y^{-1} y \rangle = \langle Ax, (A^T)^{-1} C_X^{-1} A^{-1} \cdot Ax \rangle =$

$$= \langle A^{-1} A x, C_X^{-1} x \rangle = \langle x, C_X^{-1} x \rangle$$

- since $x_i = y_i - y_{i-1}$ we get

$$f(y_1, y_2, \dots, y_m) = \frac{1}{(\sqrt{2\pi})^m} \cdot \frac{1}{\sqrt{\prod_{i=1}^m (t_i - t_{i-1})}} \cdot e^{-\frac{1}{2} \sum_{i=1}^m \frac{x_i^2}{t_i - t_{i-1}}}$$

$$= \prod_{i=1}^m \frac{1}{\sqrt{2\pi} \sqrt{t_i - t_{i-1}}} e^{-\frac{1}{2} \frac{(y_i - y_{i-1})^2}{t_i - t_{i-1}}}$$

$$= \prod_{i=1}^m p(t_i - t_{i-1}, y_{i-1}, y_i) \quad \text{where}$$

$$\boxed{p(t, x, y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}}$$

Brownian Motion as a limit of a symmetric R.W.

$(\Omega, \mathcal{F}, P)$: prob. space

$X_1, X_2, \dots$ iid r.v $EX = 0$, $\operatorname{Var} X = 1$

$W_0 = 0$; $W_n = \sum_{k=1}^n X_k$ for $n \geq 1$

- define a piecewise linear path :

$$W^{(n)}(t) = \frac{W_{[nt]}}{\sqrt{n}} + \frac{(tn - [tn])}{\sqrt{n}} \cdot \left(W_{[nt]+1} - W_{[nt]} \right)$$

- this defines : $\omega \longmapsto W^{(n)}(t, \omega)$ from Ω into the space of continuous functions on $[0,\infty)$

- $\mathbb{P}_n$ the prob. measure induced by this mapping on $C([0,\infty))$

$$\mathbb{P}_n(A) = \mathbb{P}(\omega : W^{(n)}(t,\omega) \in A) \quad (\forall) A \text{ s.t. } \{W^n(t,\omega) \in A\} \in \mathcal{F}$$

<u>Donsker's Invariance Principle</u>

For every continuous, bdd function g on $C([0,\infty))$

$$\int g \, d\mathbb{P}_n \longrightarrow \int g \, d\mu \qquad \text{as } n \to \infty$$

where μ is a prob. measure on $C([0,\infty))$ such that under μ the process $\{B(t)\}_{t \geq 0}$ is a Brownian Motion.

- in short, $W^{(n)}(\cdot) \Longrightarrow B(\cdot)$

weak convergence of stochastic processes

- what is the difference between $W^{(n)}(t) \longrightarrow B(t)$ and the above statement?

$\rightsquigarrow$ Donsker's th implies not only convergence of 1-dim distributions but also all n-dim joint distributions and much more.

<u>Filtration for Brownian Motion</u>

<u>Definition</u>: Let $(\Omega, \mathcal{F}, P)$ be a prob. space on which is defined a Brownian Motion $\{W(t)\}$ $t \geq 0$. A FILTRATION for the BM is a collection of σ-algebras $\{\mathcal{F}_t\}_{t \geq 0}$ such that:

(1) [Information accumulates] : $(\forall)$ $s < t$: $\mathcal{F}_s \subset \mathcal{F}_t$

(2) [Adaptivity] : $(\forall)$ $t \geq 0$: $W(t)$ is $\mathcal{F}_t$ measurable

(2) [Adaptivity] : $(\forall)\ t \geq 0$: $W(t)$ is $\mathcal{F}_t$ measurable

(3) [Independence of future increments] : $(\forall)\ 0 \leq t < u$
$W(u) - W(t)$ is independent of $\mathcal{F}_t$

Remark: If $\{\mathcal{F}_t\}_{t\geq 0}$ contains only information obtained by observing the B.M then the filtration is called the natural filtration of B.M. : $\sigma(\{W_s : s \leq t\})$

Martingale Property for Brownian Motion

- for every $s<t$: $E[W(t) \mid \mathcal{F}_s] = W(s)$

$$\rightsquigarrow \quad E[W(t) \mid \mathcal{F}_s] = E[W(t) - W(s) + W(s) \mid \mathcal{F}_s] =$$

$$= \underbrace{E[W(t) - W(s)]}_{=0} + W(s) = W(s)$$

Quadratic Variation : $M_k = \sum_{i=1}^{k} X_i \Rightarrow Var\, M_k = k$

- in discrete time :

$$[M, M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2$$

- if M is the symmetric r.w $\Rightarrow [M,M]_k = k$

(it looks like in this case $[M,M]_k$ matches $Var(M_k)$ however they are very different concepts).

$\rightsquigarrow$ $Var(M_k)$ is computed by averaging over all paths
$\rightsquigarrow$ $[M,M]_k$ is computed along a single path

- for the scaled r.w $\Rightarrow [W^{(n)}, W^{(n)}]_t = t$

- in continuous time : for $W(t)$ = B.M.

$$[W, W](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left[W(t_{j+1}) - W(t_j) \right]^2 = T$$

where $\Pi = \{t_0, t_1, t_2, \dots, t_n = T\}$ is a partition of the interval $[0, T]$ $(0 = t_0 < t_1 < \dots < t_n = T)$

$$\|\Pi\| = \max_{i=\overline{1,n}} (t_i - t_{i-1})$$

Proof : $Q_\Pi = \sum_{j=0}^{n-1} \left[W(t_{j+1}) - W(t_j) \right]^2$ for a given Π

- we want to show that as $\|\Pi\| \to 0$ we have :

$$Q_\Pi \longrightarrow T \qquad \text{a.s.}$$

- show first that $\begin{cases} \cdot \ \mathbb{E} Q_\Pi = T \\ \cdot \ \operatorname{Var} Q_\Pi \to 0 \end{cases}$

$$\rightsquigarrow E Q_\Pi = \sum_{j=0}^{n-1} E\big(W(t_{j+1}) - W(t_j)\big)^2 = \sum_{j=0}^{n-1} \operatorname{Var}\big(W(t_{j+1}) - W(t_j)\big)$$

$$= \sum_{j=0}^{n-1} \big(t_{j+1} - t_j\big) = T \qquad \text{for all } \Pi$$

$$\rightsquigarrow \operatorname{Var} Q_\Pi = \sum_{j=0}^{n-1} \operatorname{Var}\Big[\big(W(t_{j+1}) - W(t_j)\big)^2\Big]$$

$$= \sum_{j=0}^{n-1} \Big[E\Big[\big(W(t_{j+1}) - W(t_j)\big)^4\Big] - \big(t_{j+1} - t_j\big)^2\Big]$$

$$= \sum_{j=0}^{n-1} \Big[3\big(t_{j+1} - t_j\big)^2 - \big(t_{j+1} - t_j\big)^2\Big] = 2\sum_{j=0}^{n-1} \big(t_{j+1} - t_j\big)^2 \longrightarrow 0$$

- the arguement is heuristic, and technically only proves convergence in probability.

- next we shall discuss the same concepts in the context of square integrable martingales (which BM is also)

<u>Continuous Square Integrable martingales</u>

<u>Def</u>: Let $\{X_t\}_{t\geq 0}$ be a continuous martingale. We say that $\{X_t\}_{t\geq 0}$ is square-integrable if $E X_t^2 < \infty$ for every $t>0$.

- for any $\{X_t\}_{t\geq 0}$ cont. martingale $\Rightarrow \{X_t^2\}_{t\geq 0}$ is submartingale

$\rightsquigarrow$ in particular, Doob-Meyer decomposition applies to $\{X_t^2\}$

$\Rightarrow X_t^2 = M_t + A_t$, where

$\{M_t\}$ continuous martingale

$\{M_t\}$ = continuous martingale

$\{A_t\}$ = predictable increasing process $\quad (A_0 = 0)$

Definition (*) : X_t = square integrable, cont martingale we define the Quadratic Variation of X to be the process :

$$\langle X \rangle_t \triangleq A_t \quad \left(= [X, X]_t \right)$$

where A_t is the increasing process from the D-M decomposition of X_t^2

. in other words : $X_t^2 - \langle X \rangle_t$ = martingale

ex : X_t = Brownian motion.

$$E\left[X_t^2 \mid \mathcal{F}_s \right] = E\left[(X_t - X_s + X_s)^2 \mid \mathcal{F}_s \right]$$

$$= E\left[(X_t - X_s)^2 + 2 X_s (X_t - X_s) + X_s^2 \mid \mathcal{F}_s \right]$$

$$= \underbrace{E(X_t - X_s)^2}_{t-s} + 2 X_s \underbrace{E(X_t - X_s)}_{=0} + X_s^2$$

$$= t - s + X_s^2$$

$$E\left[X_t^2 \mid \mathcal{F}_s \right] = X_s^2 - s + t$$

$$E\left[X_t^2 - t \mid \mathcal{F}_s \right] = X_s^2 - s$$

$$X_t^2 - t = \text{martingale}$$

$X_t^2 - t$ = martingale

Def: For two continuous martingales X, Y (square int.) we define the CROSS VARIATION process:

$$[X,Y]_t \stackrel{\Delta}{=} \frac{1}{4}\left[\langle X+Y \rangle_t - \langle X-Y \rangle_t \right]$$

- observe that $XY - \langle X, Y \rangle$ is martingale
- two cont. square int. martingales are ORTHOGONAL iff $[X,Y]_t = 0$ a.s.
- the use of quadratic variation in the definition (*) may appear to be unfounded

Recall: that for a partition $\Pi = \{t_0, t_1, \dots, t_n\}$ of $[0,T]$

$$V_T^{(p)}(\Pi) = \sum_{i=0}^{n-1} \left| X_{t_{j+1}} - X_{t_j} \right|^p$$

p^{th} variation of X over the partition Π

- for $p = 2$, as $\|\Pi\| = \max\limits_{i=\overline{0,n-1}} (t_{j+1} - t_j) \longrightarrow 0$

  $V^{(2)}(\Pi)$ converges in some sense to the quadratic var of X

Th Let X = continuous, square integrable martingale

$$\Rightarrow \lim_{\|\Pi\| \to 0} V_t^{(2)}(\Pi) = \langle X \rangle_t \quad \text{in probability}$$

- if X_t = continuous process s.t. $\lim_{\|\Pi\| \to 0} V_t^{(p)}(\Pi) = L_t$ in prob,

  where L_t is a random variable on $[0, \infty)$.

$$\Rightarrow \lim_{\|\Pi\| \to 0} V_t^{(q)}(\Pi) = \begin{cases} \infty & \text{if } q < p \\ 0 & \text{if } q > p \end{cases}$$

- let us return to the Brownian Motion

- for $W(t)$ = B.M. $\Rightarrow$ $\langle W \rangle_T = T$

- for $f(t) = t$ $\Rightarrow$ $\langle f \rangle = 0$

- $$[W, t]_t = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j} \right) \cdot \left(t_{j+1} - t_j \right) \longrightarrow 0$$

- Quadratic var in differential form.

$$dW(t) \cdot dW(t) = dt$$

$$dt \cdot dt = 0$$

$$dW(t) \cdot dt = 0$$