Brownian Bridge

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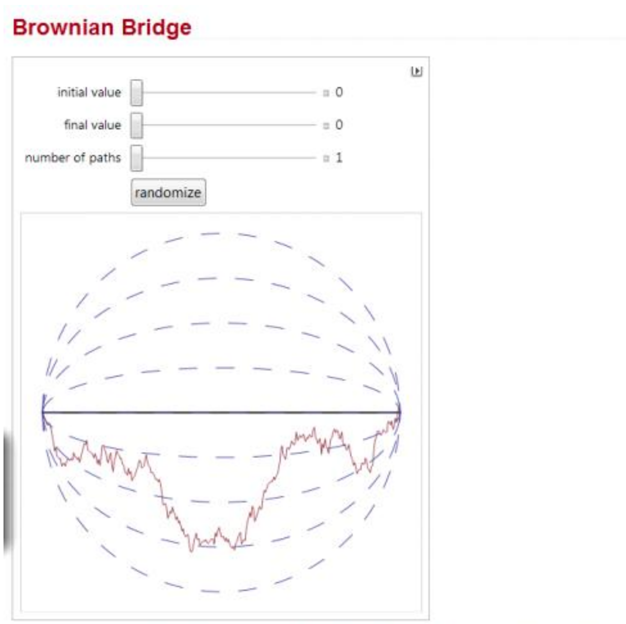
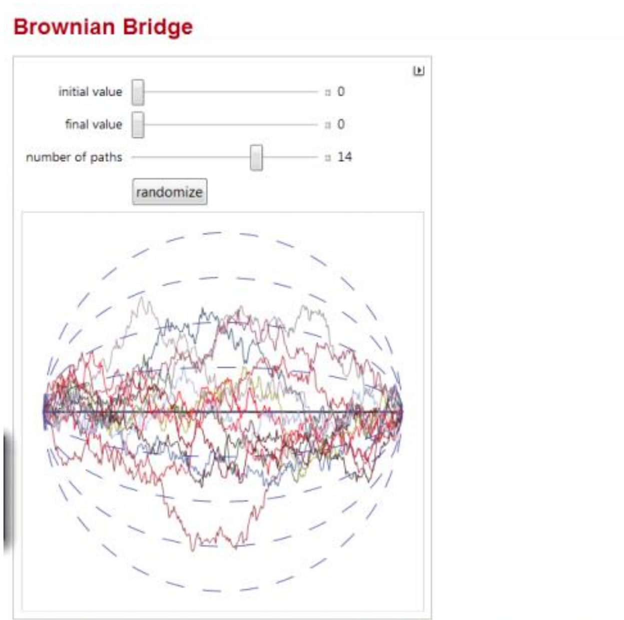
[Brownian Bridge (Wolfram Demonstrations Project)]



A Brownian bridge is a continuous stochastic process with a probability distribution that is the conditional distribution of a Wiener process given prescribed values at the beginning and end of the process. This Demonstration displays a specified number of paths of a Brownian bridge process connecting two values, chosen by the user, at the beginning and end. It also shows (as dashed lines) "small" positive and negative integer multiples of the standard deviation of the process.



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Remarks :

- Brownian motion is "Tied down" at the origin ($W_0 = 0$)
- Brownian Bridge is "Tied down" both at the origin and at the end of a fixed time interval $[0,T]$

  $\rightsquigarrow$ $B_0 = 0$; $B_T = 0$

(we can be flexible about T and about the "tied down" values)

- Brownian Bridge is used in Monte Carlo simulation.

Definition : Let W be a standard Brownian Motion. Fix $T > 0$

we define the Brownian Bridge from 0 to 0 on $[0,T]$ as

$$\boxed{X_t = W_t - \frac{t}{T} W_T}$$

Remark : • because we need W_T in the definition of X_t
$\rightsquigarrow$ X_t is not adapted to $\mathcal{F}_t$ (filtration of B.M.)

• X_t = Gaussian process
(that is : $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ are jointly normally distr.
for all $0 < t_1 < t_2 < \dots < t_n < T$)

$$E X_t = E\left[W_t - \frac{t}{T} W_T \right] = 0$$

$$E X_s X_t = E\left[W_s - \frac{s}{T} W_T \right]\left[W_t - \frac{t}{T} W_T \right]$$

$$= E\left[W_s W_t - \frac{t}{T} W_s W_T - \frac{s}{T} W_t W_T + \frac{st}{T^2} W_T^2 \right]$$

$$= s \wedge t - 2 \frac{st}{T} + \frac{st}{T}$$

$$= s \wedge t - \frac{st}{T}$$

$\Rightarrow$ X_t is Gaussian with mean 0 and covariance function given by $\boxed{c(s,t) = s \wedge t - \frac{st}{T}}$

Definition Let W_t be a standard Brownian Motion. Fix $T > 0$, $a \in \mathbb{R}$, and $b \in \mathbb{R}$. We define the **Brownian Bridge from a to b on $[0,T]$** as

$$\boxed{X_t^{a \to b} = a + \frac{(b-a)t}{T} + X_t}$$

where X_t is the Brownian Bridge from 0 to 0 on $[0,T]$

$$m^{a \to b}(t) = E X_t^{a \to b} = a + \frac{(b-a)t}{T}$$

$$c^{a \to b}(s,t) = \operatorname{Cov}\left(X_s^{a \to b}, X_t^{a \to b} \right) = s \wedge t - \frac{st}{T}$$

Brownian Bridge as a Scaled Stochastic Integral

Remark: we cannot write the Brownian Bridge as a stochastic integral because:

$$E X_t^2 = c(t,t) = t - \frac{t^2}{T} = \frac{t(T-t)}{T}$$

$$\left(E X_t^2 \text{ is } \begin{cases} \cdot \text{ increasing for } & 0 \le t \le \frac{T}{2} \\ \cdot \text{ decreasing for } & \frac{T}{2} \le t \le T \end{cases} \right)$$

· if X_t where a stochastic integral say $\int_0^t \Delta_t \, dW_t$ then $E X_t^2 = E \int_0^t \Delta_t^2 \, dt$ increasing

· define $\boxed{I_t = \int_0^t \frac{1}{T-u} \, dW_u \quad ; \quad Y_t = (T-t) I_t}$

· easy to see that I_t is a Gaussian process with mean 0 and covariance matrix:

$$c^I(s,t) = \int_0^{s \wedge t} \frac{1}{(T-u)^2} \, du = \underline{\frac{1}{T - s\wedge t} - \frac{1}{T}}$$

Recall: $c^I(s,t) = E\left[I_s \cdot I_t \right] = E\left[I_{s\wedge t}^2 + \left(\int_{s\wedge t}^{s \vee t} \frac{1}{T-u} dW_u \right) I_{s \wedge t} \right] = E\left(I_{s\wedge t}^2 \right)$

· therefore the process Y_t will also be Gaussian with mean 0 and covariance function (assume $0 \le s \le t < T$)

$$c^Y(s,t) = E\left[(T-s)(T-t) \cdot I_s \cdot I_t \right] =$$

$$= (T-s)(T-t) \cdot \left(\frac{1}{T-s} - \frac{1}{T} \right) = (T-s)(T-t) \frac{s}{T(T-s)}$$

$$= \frac{(T-t)s}{T}$$

$$\Rightarrow \quad c^Y(s,t) = s - \frac{st}{T}$$

- if we assume $t < s$ the roles of s, t are reversed in the above eq

$$\Rightarrow \quad \boxed{c^Y(s,t) = s \wedge t - \frac{st}{T} = c^X(s,t)}$$

(we have obtained the same covariance formula for the Brownian Bridge)

Formally : $Y_t = \begin{cases} (T-t)\displaystyle\int_0^t \frac{1}{T-u}\, dW_u & \text{if } 0 \le t < T \\ 0 & \text{if } t = T \end{cases}$

has the same distribution with the Brownian Bridge from 0 to 0 on $[0,T]$

- in differential form : (product rule)

$$dY_t = \left(\int_0^t \frac{1}{T-u}\, dW_u\right) \cdot d(T-t) + (T-t) \cdot \frac{1}{T-t}\, dW_t + \underbrace{\frac{1}{T-t}\, dW_t \cdot d(T-t)}_{=0}$$

$$= -\left(\int_0^t \frac{1}{T-u}\, dW_u\right) \cdot dt + dW_t$$

$$= -\frac{Y_t}{T-t}\, dt + dW_t$$

$$\boxed{dY_t = -\frac{Y_t}{T-t}\, dt + dW_t}$$