Feynman - Kac (applications)

Wednesday, March 07, 2012
10:13 AM

① Options on GBM : $dS_t = \alpha S_t\, dt + \sigma S_t\, dW_t$

- under the risk neutral measure :

$$dS_t = r S_t\, dt + \sigma S_t\, d\tilde{W}_t$$

- $V_t = \tilde{E}\left[e^{-r(T-t)} h(S_T) \,\middle|\, \mathcal{F}_t \right] = v(t, \overset{x}{\overset{\downarrow}{S_t}}) = \tilde{E}\left[e^{-r(T-t)} h(S_T) \,\middle|\, S_t \right]$

- $v(t,x)$ must satisfy :

B-S-M : $\boxed{v_t + r x\, v_x + \frac{1}{2} \sigma^2 x^2 v_{xx} = r v}$ $\qquad v(T,x) = h(x)$

Remark : we can use B-S-M eq as long as the derivative security has a payoff that depends on the final stock price

$V_T = h(S_T)$ and the stock follows GBM (r, σ) under $\tilde{\mathbb{P}}$

- if σ in this example is replaced by $\sigma(t,x)$ then S_t is no longer GBM so the B-S-M eq no longer applies, however one can still solve the eq :

$$v_t(t,x) + r x\, v_x(t,x) + \frac{1}{2} \sigma^2(t,x)\, x^2\, v_{xx}(t,x) = r v(t,x)$$

(numerically !)

- in practice one assumes σ = constant
- the value σ that makes the theoretical price given by BSM agrees with the market price is called IMPLIED VOLATILITY

agree with the market price is called IMPLIED VOLATILITY

- implied volatility is different for options with different strikes
  ⇝ implied volatility is a convex function of K
  (volatility smile)

example : if $dS_t = rS_t\,dt + \sigma S_t^{\delta}\,d\tilde{W}_t$

where $\delta \in (0,1)$: Constant elasticity of variance
⇝ chosen such that the model gives a good fit to option prices across different strikes.

Interest Rate Models :

- suppose the interest rate R_t is described by :

$$dR_t = \beta(t,R_t)\,dt + \gamma(t,R_t)\,d\tilde{W}_t$$

$\tilde{W}_t$ = BM under the risk-neutral measure $\tilde{\mathbb{P}}$

- such models for R_t are called "short-rate" models.
- when R_t is determined by 1 SDE the model is said to have one factor.

(one factor models tend to not capture complicated yield curve behaviour)

- $D_t = \exp\left\{-\int_0^t R_s\,ds\right\}$: the discount process
- $\frac{1}{D_t} = \exp\left\{\int_0^t R_s\,ds\right\}$: the money market price process

$D_t = \exp\left[-\int_0^t R_s\,ds\right]$

$\rightsquigarrow \quad dD_t = -R_t D_t\,dt \quad ; \quad d\left(\frac{1}{D_t}\right) = \frac{R_t}{D_t}\,dt$

(smooth processes)

- $B(t,T) = \frac{1}{D_t}\tilde{\mathbb{E}}\left[D_T \mid \mathcal{F}_t\right]$: the zero-coupon bond price (face value $=1$)

$$= \tilde{\mathbb{E}}\left[e^{-\int_t^T R_s\,ds} \mid \mathcal{F}_t\right]$$

- $Y(t,T) = -\frac{1}{T-t}\log B(t,T)$ the yield curve

(or equivalently : $B(t,T) = e^{-(T-t)\,Y(t,T)}$)

$Y(t,T)$ = constant rate of continuously compounding interest between times t and T that is consistent with the bond price $B(t,T)$

(long rate)

- since R_t is given by a SDE $\rightsquigarrow$ R_t = Markov process

$$\Rightarrow B(t,T) = f(t,R_t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R_s\,ds} \mid \mathcal{F}_t\right] = \tilde{\mathbb{E}}\left[e^{-\int_t^T R_s\,ds} \mid R_t\right]$$

for some function $f(t,r)$ that satisfies :

$$f_t(t,r) + \beta(t,r)\,f_r(t,r) + \tfrac{1}{2}\sigma^2(t,r)\,f_{rr}(t,r) = r\,f(t,r)$$

where $f(T,r) = B(T,T) = 1$ for all r.

- $D_t B(t,T) = D_t f(t,R_t) = \tilde{E}\left[D_T \mid \mathcal{F}_t\right]$: $\tilde{P}$ - martingale

$$d(D_t f(t,R_t)) = f(t,R_t)\, dD_t + D_t\, df(t,R_t)$$

$$= D_t \left[-R_t f(t,R_t)\,dt + f_t(t,R_t)\,dt + f_r(t,R_t)\,dR_t + \tfrac{1}{2} f_{rr}(t,R_t)\, dR_t \cdot dR_t \right]$$

$$= D_t \Big[-R_t f(t,R_t)\,dt + f_t(t,R_t)\,dt + f_r(t,R_t)\beta(t,R_t)\,dt + f_r(t,R_t)\sigma(t,R_t)\,d\tilde{W}_t$$
$$+ \tfrac{1}{2} f_{rr}(t,R_t)\, \sigma^2(t,R_t)\,dt \Big]$$

$$= D_t \left(-R_t f(t,R_t) + f_t(t,R_t) + \beta(t,R_t) f_r(t,R_t) + \tfrac{1}{2}\sigma^2(t,R_t) f_{rr}(t,R_t) \right) dt$$
$$+ D_t\, \sigma(t,R_t)\, f_x(t,R_t)\, d\tilde{W}_t$$

- since $D_t f(t,R_t)$ is martingale $\Rightarrow$ dt term $= 0$

$$\Rightarrow \quad f_t(t,R_t) + \beta(t,R_t) f_r(t,R_t) + \tfrac{1}{2}\sigma^2(t,R_t) f_{rr}(t,R_t) = R_t f(t,R_t)$$

for all paths of R_t

- so replacing R_t with a dummy variable r we get the following eq :

$$f_t(t,r) + \beta(t,r) f_r(t,r) + \tfrac{1}{2}\sigma^2(t,r) f_{rr}(t,r) = r f(t,r)$$

with the terminal condition $f(T,r) = 1$.

Remark : the above SDE is different than the regular F-K where we have fixed interest rate r, and the dummy var is x.

① Hull-White interest rate model:

$$dR_t = \underbrace{(a_t - b_t R_t)} dt + \sigma_t \, d\tilde{W}_t$$

(a_t, b_t, σ_t deterministic, positive functions)

- the PDE becomes:

$$f_t(t,r) + \underbrace{(a_t - b_t r)}_{\text{drift.}} f_r(t,r) + \frac{1}{2}\sigma_t^2 f_{rr}(t,r) = r f(t,r)$$

- solution of this PDE has the form:

$$f(t,r) = \exp\{-r\,C(t,T) - A(t,T)\}$$

for some nonrandom functions $C(t,T)$, $A(t,T)$ to be determined.

$$\Rightarrow Y(t,T) = -\frac{1}{T-t}\log f(t,R_t) = \frac{1}{T-t}\left(R_t\,C(t,T) + A(t,T)\right)$$

(affine function of R_t)

- to identify $C(t,T)$ and $A(t,T)$ we calculate:

$$\begin{cases} f_t(t,r) = \left[-r\,C'(t,T) - A'(t,T)\right]\cdot f(t,r) \\ f_r(t,r) = -C(t,T) f(t,r) \\ f_{rr}(t,r) = C^2(t,T) f(t,r) \end{cases}$$

$\rightsquigarrow$ substitute these formulas in the above PDE

$\leadsto$ substitute these formulas in the above PDE

$$\Rightarrow \left[\underbrace{\left(-C'(t,T) + b_t C(t,T) - 1\right)}_{*} r - \underbrace{A'(t,T) - a_t C(t,T) + \tfrac{1}{2}\sigma_t^2 C^2(t,T)}_{**} \right] f(t,r) = 0$$

- this eq must hold for all r so the term that multiplies r should be $= 0$.

$$\begin{cases} * : & -C'(t,T) + b_t C(t,T) - 1 = 0 \\ ** : & -A'(t,T) - a_t C(t,T) + \tfrac{1}{2}\sigma_t^2 C^2(t,T) = 0 \end{cases}$$

$$C(T,T) = A(T,T) = 0$$

$$\Rightarrow \quad C(t,T) = \int_t^T e^{-\int_t^s b_u \, du} \, ds$$

$$A(t,T) = \int_t^T a_s C(s,T) - \tfrac{1}{2}\sigma_s^2 C^2(s,T) \, ds$$

Conclusion : $\boxed{B(t,T) = e^{-R_t C(t,T) - A(t,T)}}$

② Cox - Ingersoll - Ross interest rate model

$$dR_t = (a - bR_t)\,dt + \sigma\sqrt{R_t}\, d\tilde{W}_t$$

(a, b, σ are positive constants)

- the PDE for bond prices becomes :

$$f_t(t,r) + (a - br) f_r(t,r) + \tfrac{1}{2}\sigma^2 r f_{rr}(t,r) = r f(t,r)$$

- just like before assume $f(t,r) = \exp\{-rC(t,T) - A(t,T)\}$ and verify that it satisfies the above PDE for a specific choice of $C(t,T)$ and $A(t,T)$.

- so the PDE becomes :

$$\Big[\big(\underbrace{-C'(t,T) + b\,C(t,T) + \tfrac{1}{2}\sigma^2 C^2(t,T) - 1}_{*}\big) r - \underbrace{A'(t,T) - a\,C(t,T)}_{**} \Big] f(t,r) = 0$$

$$\begin{cases} * \quad -C'(t,T) + b\,C(t,T) + \tfrac{1}{2}\sigma^2 C^2(t,T) - 1 = 0 \\ ** \quad -A'(t,T) - a\,C(t,T) = 0 \end{cases}$$

$C(T,T) = A(T,T) = 0$

$$C(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma \cosh(\gamma(T-t)) + \frac{1}{2} b \sinh(\gamma(T-t))}$$

$$A(t,T) = -\frac{2a}{\sigma^2} \log\left[\frac{\gamma e^{\frac{1}{2}b(T-t)}}{\gamma \cosh(\gamma(T-t)) + \frac{1}{2} b \sinh(\gamma(T-t))} \right]$$

where $\gamma = \frac{1}{2}\sqrt{b^2 + 2\sigma^2}$; $\sinh u = \dfrac{e^u - e^{-u}}{2}$; $\cosh u = \dfrac{e^u + e^{-u}}{2}$