MTH 9821 Numerical Methods for Finance I
Lecture 10 –Finite Difference Valuation For Options II

1 Finite Difference Solution for BS PDE on a Fixed Computational Domain

Recall that heat PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < t < t_{final},\ x_{left} < x < x_{right}$$

with boundary conditions

$$\begin{aligned} u(x,0) &= f(x) \\ u(x_{left},\tau) &= g_{left}(\tau) \\ u(x_{right},\tau) &= g_{right}(\tau) \end{aligned}$$

For BS PDE, by changing of variables,

$$\begin{aligned} &x = \ln\left(\frac{S}{K}\right), \quad \tau = \frac{(T-t)\sigma^2}{2}; \\ &a = \frac{r-q}{\sigma^2} - \frac{1}{2}, \quad b = \left(\frac{r-q}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2q}{\sigma^2} \\ &V(S,t) = exp(-ax - b\tau)u(x,\tau) \end{aligned}$$

1.1 Boundary Conditions

Let

$$\begin{aligned} S_{left} &= S(0)\ \exp\left((r-q-\frac{\sigma^2}{2})T - 3\sigma\sqrt{T}\right) \\ S_{right} &= S(0)\ \exp\left((r-q-\frac{\sigma^2}{2})T + 3\sigma\sqrt{T}\right) \\ x_{left} &= \ln\left(\frac{S_{left}}{K}\right) = \ln\left(\frac{S_0}{K}\right) + (r-q-\frac{\sigma^2}{2})T - 3\sigma\sqrt{T} \\ x_{right} &= \ln\left(\frac{S_{right}}{K}\right) = \ln\left(\frac{S_0}{K}\right) + (r-q-\frac{\sigma^2}{2})T + 3\sigma\sqrt{T} \end{aligned}$$

1.2 Discretization

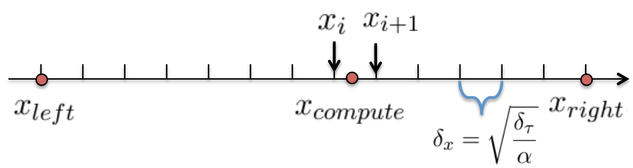
On τ-axis,

$$\tau_{final} = \frac{T\sigma^2}{2}$$

On x-axis

$$x_{compute} = \ln\left(\frac{S_0}{K}\right)$$

will NOT be on the grid, so that the left and right boundaries x_{left} and x_{right} are exact.



- Choose M and α_{temp}, thus

$$\delta_\tau = \frac{\tau_{final}}{M}$$
$$\delta_{x_{temp}} = \sqrt{\frac{\delta_\tau}{\alpha_{temp}}}$$

- Choose number of intervals on $x-$axis N in such a way

$$N = \lfloor \frac{x_{right} - x_{left}}{\delta_{x_{temp}}} \rfloor$$

i.e., if $x_{right} - x_{left}$ is not integer times of $\delta_{x_{temp}}$, we make δ_x a little bigger.
- Thus,

$$\frac{x_{right} - x_{left}}{N}, \qquad \alpha = \frac{\delta_\tau}{(\delta_x)^2}$$

Remark:

$$\delta_x \geq \delta_{x_{temp}} \quad \Rightarrow \quad \alpha \leq \alpha_{temp}$$

we slightly decrease α to ensure convergence of Forward Euler.

1.3 Valuation

Let i such that $x_i < x_{compute} < x_{i+1}$, then

$$S_i = Ke^{x_i}, \quad S_{i+1} = Ke^{x_{i+1}}$$

We use linear interpolation to find approximate value of the option at spot.

Method 1. Use linear interpolation to find $V_{approx}(S_0, 0)$ using V_i and V_{i+1}

$$V_i = \exp(-ax_i - b\tau_{final})U^M(i)$$
$$V_{i+1} = \exp(-ax_{i+1} - b\tau_{final})U^M(i+1)$$

where $U^M(i)$ and $U^M(i+1)$ are the finite difference approximations of $u(x_i, \tau_{final})$ and $u(x_{i+1}, \tau_{final})$ The approximate value of the option at S_0 is

$$V_{approx}(S_0, 0) = \frac{(S_{i+1} - S_0)V_i + (S_0 - S_i)V_{i+1}}{S_{i+1} - S_i}$$

Method 2. Use linear interpolation to find $u(x_{compute}, \tau_{final})$ using $u(x_{i+1}, \tau_{final})$ and $u(x_i, \tau_{final})$

$$u(x_{compute}, \tau_{final}) = \frac{(x_{i+1} - x_{compute})u(x_i, \tau_{final}) + (x_{compute} - x_i)u(x_{i+1}, \tau_{final})}{x_{i+1} - x_i}$$

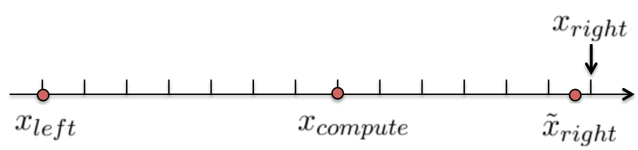
The approximate value of the option at S_0 is

$$V_{approx,2}(S_0, 0) = \exp(-ax_{compute} - b\tau_{final})u(x_{compute}, \tau_{final})$$

Remark: the purpose of fixed domain

Fix computational domain is useful especially for barrier options.

- For single-barrier options, we can still adjust δ_x as last lecture mentioned to ensure that $x_{compute}$ is on the grid.



e.g., for a down-and-out call option, $S_0 = 42,\ K = 40,\ B = 30$. Therefore,

$$x_{left} = \ln\left(\frac{30}{40}\right), \qquad x_{compute} = \ln\left(\frac{42}{40}\right)$$
$$\tilde{x}_{right} = \ln\left(\frac{42}{40}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T + 3\sigma\sqrt{T}$$

From M and α_{temp}, we first adjust α_{temp} such that $x_{compute} - x_{left}$ is integer times of δ_x, then

$$N_{right} = \lceil \frac{\tilde{x}_{right} - x_{compute}}{\delta_x} \rceil$$
$$x_{right} = x_{compute} + N_{right}\delta_x$$

- For double-barrier options, it's impossible to let x_{left}, $x_{compute}$ and x_{right} all on the grid. Here, we need to use fixed computational domain as discussed above.
  e.g., for a down-and-out & up-and-out call option, $S_0 = 42,\ K = 40,\ B_1 = 30,\ B_2 = 50$, with \$2 rebate. Therefore, we have

$$x_{left} = \ln\left(\frac{30}{40}\right), \quad x_{compute} = \ln\left(\frac{42}{40}\right), \quad x_{right} = \ln\left(\frac{50}{40}\right)$$
$$V(30, t) = V(50, t) = 2, \quad \forall 0 < t < T$$

2 Finite Difference Valuation for American Options

$V(S,t)$ = value of an American option

Question: What PDE (if any) does $V(S,t)$ satisfy?

$$
\begin{aligned}
ds &= (r-q)Sdt + \sigma S dX \\
\Pi &= V - \Delta S, \quad \text{where } \Delta = \frac{\partial V}{\partial S} \\
\Rightarrow d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - qS \frac{\partial V}{\partial S} \right) dt
\end{aligned}
\tag{$\star$}
$$

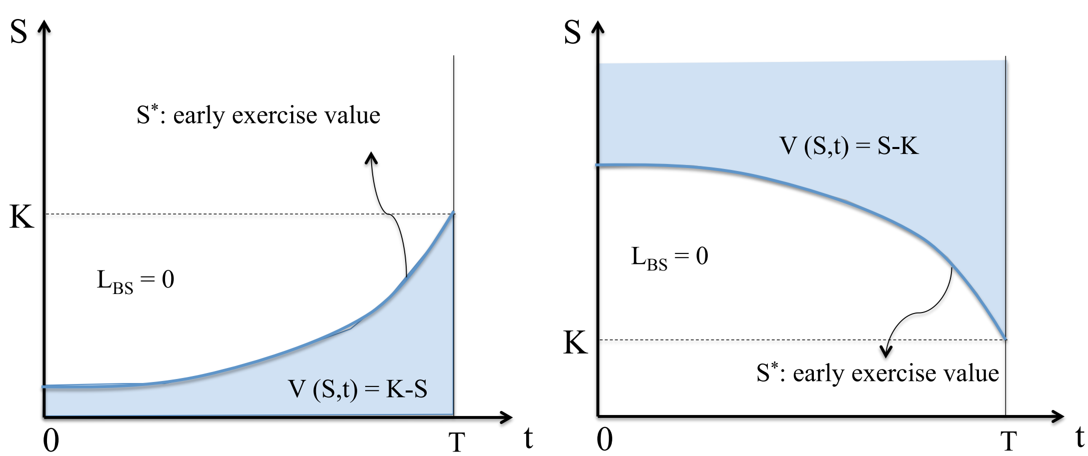
Remark:
($\star$) holds true if the option is not exercised between t and $t + dt$.
($\star$) will underestimate the actual change $d\Pi$ if there's early exercise.

Therefore, the value of Π cannot grow at a rate higher than the risk-free rate, i.e.,

$$d\Pi \leq r\Pi dt$$

Hint:
Let actual change in Π be $d\Pi^\star$, then $d\Pi \leq d\Pi^\star$.
Under risk neutral probability, $d\Pi^\star = r\Pi dt$.
Hence, $d\Pi \leq d\Pi^\star = r\Pi dt$.

Early Exercise Region of American Put and Call Options



Early exercise region of an American put option   Early exercise region of an American call option

As shown above, for American put options, for example, early exercise happens when

$$V(S,t) = K - S, \quad \forall S < S^*(t)$$

If no early exercise, then BS PDE still holds.

We follow the routine as in the FD valuation of European options:
By changing of variables

$$x = \ln\left(\frac{S}{K}\right), \; \tau = \frac{(T-t)\sigma^2}{2}$$
$$V(S,t) = \exp(-ax - b\tau)u(x,\tau)$$

2.1 Boundary conditions

In the American option case,

$$\frac{\partial u}{\partial t} \geq \frac{\partial^2 u}{\partial x^2}$$

For American put options, the value of the option satisfies

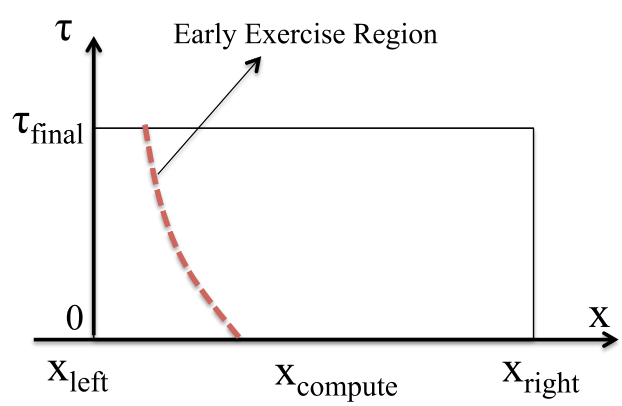
$$V(S,t) \geq \max(K - S, 0) \Rightarrow \; u(x,\tau) \geq Ke^{ax+b\tau} \max(1 - e^x, 0)$$

We denote $g(x,\tau) = Ke^{ax+b\tau} \max(1 - e^x, 0)$.
Similarly, for American call options,

$$u(x,\tau) \geq Ke^{ax+b\tau} \max(e^x - 1, 0)$$

Also, as $S \to 0$, it's optimal to exercise the American put option, and therefore,

$$g_{left}(\tau) = K\exp(ax_{left} + b\tau)(1 - \exp(x_{left})), \quad \forall 0 \leq \tau \leq \tau_{final}$$



2.2 Finite Difference Schemes for American options

2.2.1 Forward Euler

for $m = 0 : M - 1$

  for $n = 1 : N - 1$

$$\underbrace{U_n^{m+1}}_{\text{American Option}} = \max\left(\underbrace{\alpha U_{n+1}^m + (1-2\alpha)U_n^m + \alpha U_{n-1}^m}_{\text{European Option}}, \quad \underbrace{g(x_n, \tau_{m+1})}_{\text{Early Exercise Premium}}\right);$$

  end

end

2.2.2 Implicit Finite Difference Methods (Backward Euler, Crank-Nicolson)

European Options	American Options
Find U^{m+1} s.t.	Find U^{m+1} s.t.
$\mathbf{AU^{m+1} = b^m}$	$\mathbf{AU^{m+1} \geq b^m}$
	(linear complementary formulation)
	$U^{m+1} \geq g^{m+1}$ (intrinsic value)

Question: Can we do this in the following way?

- solve "=" using Cholesky Decomposition

- look pointwise at each node, if U <intrinsic value, change its value to intrinsic value

No! When using Cholesky, if we change the value at any node, the "=" will not hold.

Solution: Use SOR, more specifically, projected SOR.
Change value if $U < IV$, then by iterating, the result will become less and less garbled.

Crank-Nicolson

$$A = \begin{pmatrix} 1+\alpha & -\frac{\alpha}{2} & 0 & 0 & \dots & 0 \\ -\frac{\alpha}{2} & 1+\alpha & -\frac{\alpha}{2} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\frac{\alpha}{2} & 1+\alpha & -\frac{\alpha}{2} \\ 0 & \dots & \dots & 0 & -\frac{\alpha}{2} & 1+\alpha \end{pmatrix}$$

Use SOR for solving $Ax = b$.

Recall that

> **for** $j = 1 : p$
>
> $$x_{n+1}(j) = (1-\omega)x_n(j) - \tfrac{\omega}{A(j,j)}\Big[\sum_{k=1}^{j-1} A(j,k)x_{n+1}(k) + \sum_{k=1}^{j-1} A(j,k)x_n(k) + \tfrac{\omega b(j)}{A(j,j)}\Big];$$
>
> **end**

In this case,

> **for** $j = 1 : N-1$
>
> $$y_{n+1}(j) = (1-\omega)y_n(j) + \tfrac{\alpha\omega}{1(1+\alpha)}\big(y_{n+1}(j-1) + y_n(j+1)\big) + \tfrac{\omega}{1+\alpha}b^m(j);$$
>
> **end**

Projected SOR for Crank-Nicolson for American Options

Projected SOR wiht initial guess $y_0 = U^m$ and consecutive approximation stopping criterion

> **for** $j = 1 : N-1$
>
> $$y_{n+1}(j) = \max\Big[g(x_j, \tau_{m+1}),\ (1-\omega)y_n(j) + \tfrac{\alpha\omega}{1(1+\alpha)}\big(y_{n+1}(j-1) + y_n(j+1)\big) + \tfrac{\omega}{1+\alpha}b^m(j)\Big];$$
>
> **end**