PDE connections ; Feynman - Kac theorem

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- we have seen that under "suitable" conditions on $\beta(t,x)$, $\sigma(t,x)$ the SDE :

$$dX_t = \beta(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$$

has a unique solution X_t !

Theorem : Let $\{X_t\}$ be a solution of the above SDE with the initial condition $X_0 = x_0$. Then $\{X_t\}$ is a Markov process, i.e for any Borel measurable function $h(x)$ there exists a function $g(t,x)$ such that

$$\mathbb{E}\left[h(X_T) \mid \mathcal{F}_t \right] = g(t, X_t)$$

$$g(t,x) = \mathbb{E}\left[h(X_T) \mid X_t = x \right]$$

- example : Risk - neutral pricing formula for a contingent claim V_T that depends on the final value of the stock price : $V_T = f(S_T)$

$$V_t = \tilde{\mathbb{E}}\left[e^{-\int_t^T r_s\,ds} V_T \mid \mathcal{F}_t \right]$$

- since X_t the replicating portfolio is build so that it matches V_t then V_t is Markovian as well so there exist a function $v(t,x)$ such that

$$V_t = v(t, S_t)$$

- Feynman-Kac Theorem relates the SDE with PDE
- when the PDE is solved it produces the function $g(t,x)$
- Black-Scholes-Merton PDE is a special case of the relationship between SDE & PDE !!!

Feynman-Kac Theorem : Suppose the following SDE

$$dX_t = \beta(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

has a unique solution.

Let $h(\cdot)$ be Borel measurable, $T>0$ fixed. Define

$$g(t,x) = \mathbb{E}\left[h(X_T) \mid X_t = x \right]$$

Then $g(t,x)$ satisfies :

$$\boxed{g_t(t,x) + \beta(t,x)\, g_x(t,x) + \frac{1}{2}\sigma^2(t,x)\, g_{xx}(t,x) = 0}$$

with the terminal condition $g(T,x) = h(x)$.

Proof :

- consider the process defined as :

$$g(t, X_t) = \mathbb{E}\left[h(X_T) \mid \mathcal{F}_t \right] = \mathbb{E}\left[h(X_T) \mid X_t \right]$$

$$\mathbb{E}\left[g(t, X_t) \mid \mathcal{F}_s \right] = \mathbb{E}\left[\mathbb{E}\left[h(X_T) \mid \mathcal{F}_t \right] \mid \mathcal{F}_s \right] =$$

$$= \mathbb{E}\left[h(X_T) \mid \mathcal{F}_s \right] = g(s, X_s)$$

- $\boxed{g(t, X_t) = \text{martingale}}$
- use Ito formula for $g(t, X_t)$

$$dg(t,X_t) = g_t(t,X_t)\,dt + g_x(t,X_t)\,dX_t + \frac{1}{2} g_{xx}(t,X_t)\,dX_t \cdot dX_t$$

$$= g_t(t,X_t)\,dt + g_x(t,X_t)\,\beta(t,X_t)\,dt + g_x(t,X_t)\,\sigma(t,X_t)\,dW_t + \frac{1}{2} g_{xx}(t,X_t)\,\sigma^2(t,X_t)\,dt$$

$$= g_x(t,X_t)\,\sigma(t,X_t)\,dW_t + \underbrace{\left[g_t(t,X_t) + g_x(t,X_t)\beta(t,X_t) + \frac{1}{2} g_{xx}(t,X_t)\sigma^2(t,X_t)\right]}_{\equiv 0} dt$$

- to ensure that $g(t, X_t)$ is martingale then

$$g_t + \beta g_x + \frac{1}{2} g_{xx}\,\sigma^2 = 0 \qquad \forall\ (t,x)$$

<u>Theorem</u>: (Discounted Feynman-Kac)

Consider the SDE:

$$dX_t = \beta(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$$

let $h(\cdot)$ be a Borel measurable function, r be a

Fix $T>0$, and let $t\in[0,T]$ be given. Define the function:

$$f(t,x) = \mathbb{E}\left[e^{-r(T-t)} h(X_T) \mid X_t = x\right]$$

Then $f(t,x)$ satisfies the PDE:

$$f_t(t,x) + \beta(t,x) f_x(t,x) + \frac{1}{2}\sigma^2(t,x) f_{xx}(t,x) = r f(t,x)$$

with the terminal condition $f(T,x) = h(x)$ for all x.

Proof sketch: first identify the martingale

Proof: strategy: $\left\{\begin{array}{l} \bullet \text{ identify the martingale} \\ \bullet \text{ write the Itô formula for the above mart.} \\ \bullet \text{ set the } dt \text{ term equal to } 0. \end{array}\right.$

Question: Is $f(t,X_t)$ a martingale?

$$\mathbb{E}\left[f(t,X_t)\,|\,\mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[e^{-r(T-t)}h(X_T)\,|\,\mathcal{F}_t\right]\,|\,\tilde{\mathcal{F}}_s\right] =$$

$$= e^{-r(T-t)}\,\mathbb{E}\left[h(X_T)\,|\,\mathcal{F}_s\right] = e^{-r(T-t)}\cdot e^{r(T-s)} f(s,X_s)$$

$$= e^{rt}\cdot e^{-rs} f(s,X_s)$$

$\Rightarrow$ $f(t,X_t)$ is not a martingale, however

$e^{-rt} f(t,X_t)$ is one, since

$$\mathbb{E}\left[e^{-rt}f(t,X_t)\,|\,\mathcal{F}_s\right] = \underbrace{e^{-rs}f(s,X_s)}_{=\tilde{f}(s,X_s)}$$

- so we have identified a martingale: $\tilde{f}(t,X_t)$, now let's write the Itô formula for it.

$$d\tilde{f}(t,X_t) = d\left(e^{-rt}\cdot f(t,X_t)\right)$$

$$= e^{-rt}\,df(t,X_t) + f(t,X_t)\,d(e^{-rt}) + \underbrace{df(t,X_t)\cdot d(e^{-rt})}_{=0}$$

$$= e^{-rt}\left(f_t(t,X_t)\,dt + f_x(t,X_t)\,dX_t + \tfrac{1}{2} f_{xx}(t,X_t)\,dX_t\cdot dX_t\right) +$$

$$+ f(t,X_t)\cdot\left(-r e^{-rt}\,dt\right)$$

$$= e^{-rt} f_t(t,X_t)\,dt + e^{-rt}\beta(t,X_t) f_x(t,X_t)\,dt + e^{-rt}\sigma(t,X_t) f_{x}(t,X_t)\,dW_t$$

$$+ \tfrac{1}{2}\sigma^2(t,X_t) f_{xx}(t,X_t)\,dt - r e^{-rt} f(t,X_t)\,dt$$

$$= e^{-rt}\left[f_t(t,X_t) + \beta(t,X_t) f_x(t,X_t) + \frac{1}{2}\sigma^2(t,X_t) f_{xx}(t,X_t) - r f(t,X_t) \right] dt$$

$$+ \; e^{-rt} \sigma(t,X_t) \, f_x(t,X_t) \, dW_t$$

- since $\tilde{f}(t,X_t)$ is a martingale $\rightsquigarrow$ dt term should be 0

$$\Rightarrow \quad f_t(t,X_t) + \beta(t,X_t) f_x(t,X_t) + \frac{1}{2}\sigma^2(t,X_t) f_{xx}(t,X_t) - r f(t,X_t) = 0$$

for every path of X_t so if we replace X_t with a dummy variable x, we get:

$$\boxed{f_t(t,x) + \beta(t,x) f_x(t,x) + \frac{1}{2}\sigma^2(t,x) f_{xx}(t,x) = r f(t,x)}$$

for all $x, t \in [0,T]$.

- for the final condition recall the definition of $f(t,x)$

$$f(T,x) = \mathbb{E}\left[e^{-r(T-T)} h(X_T) \,\middle|\, \mathcal{F}_T \right] = h(X_T)$$

Multidimensional Feynman-Kac Theorem

Let $W_t = (W_1(t), W_2(t))$ - 2-dim Brownian Motion

- consider two SDEs:

$$\begin{cases} dX_1(t) = \beta_1(t, X_1(t), X_2(t))\, dt + \sigma_{11}(t, X_1(t), X_2(t))\, dW_1(t) + \sigma_{12}(t, X_1(t), X_2(t)) \cdot dW_2(t) \\ dX_2(t) = \beta_2(t, X_1(t), X_2(t))\, dt + \sigma_{21}(t, X_1(t), X_2(t))\, dW_1(t) + \sigma_{22}(t, X_1(t), X_2(t))\, dW_2(t) \end{cases}$$

Define:

$$g(t, x_1, x_2) = \mathbb{E}\left[h(X_1(T), X_2(T)) \,\middle|\, X_1(t) = x_1,\; X_2(t) = x_2 \right]$$

$$f(t, x_1, x_2) = \mathbb{E}\left[e^{-r(T-t)} h(X_1(T), X_2(T)) \,\middle|\, X_1(t) = x_1,\ X_2(t) = x_2 \right]$$

Then

$$g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} + \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) g_{x_1 x_1} + (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) g_{x_1 x_2} + \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2) g_{x_2 x_2} = 0$$

$$f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} + \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) f_{x_1 x_1} + (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) f_{x_1 x_2} + \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2) f_{x_2 x_2} = r f$$

with the terminal conditions :

$$g(T, x_1, x_2) = f(T, x_1, x_2) = h(X_1(T), X_2(T))$$