Binomial Asset Pricing Model

Wednesday, September 07, 2011
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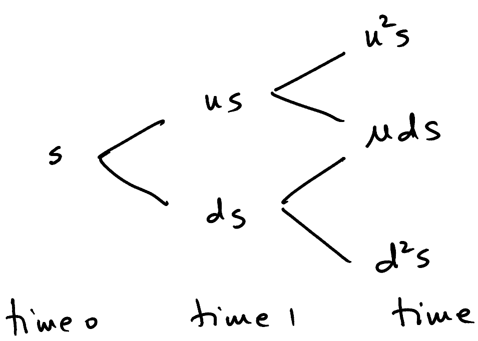
- 1 risky asset (stock)
- 1 riskless asset (money market account)
- 1 period interest rate $r \geq 0$

Parameters : $u, d, r \geq 0$: **Assumption:** $d < 1+r < u$

Number of periods : $n \geq 1$

Initial stock price $S_0 = s$.



At time n there are $n+1$ possible prices : $u^n s, u^{n-1}d s, \ldots, d^n s$

Why do we assume $d < 1+r < u$?

$\rightsquigarrow$ if $d \geq 1+r$: Stock value $\geq$ Money market value
thus one could construct an arbitrage opportunity by borrowing at rate r and investing in the stock, thus creating a profit from 0 initial value.

$\rightsquigarrow$ if $u \leq 1+r$: Stock value $\leq$ Money Market Value
thus one could construct an arbitrage opportunity by selling short the stock and putting the money in the bank.

Problem: Assume that successive movements of the stock price are independent and that the probability to go up in any one time period is $\tilde{p}$. Under what condition on $u, d, r, \tilde{p}$ the discounted stock price is a MARTINGALE ?

Remark: Discounted stock price = present value of the stock price

S_m = stock price at time $m \le n$.

Define

$$X_i = \begin{cases} 1 & \text{if the stock went up at the } i^{th} \text{ period} \\ 0 & \text{otherwise} \end{cases}$$

By assumption: $X_1, X_2, \dots, X_n$ are i.i.d

$P(X_i = 1) = \tilde{p}$ (not given in advance)

$\rightsquigarrow$ $\tilde{p}$ does not describe the stock price in reality, but it is the probability that makes the discounted stock price a martingale.

- discounted stock price: $V_0 = s$

$$V_m := \frac{1}{(1+r)^m} S_m = \frac{1}{(1+r)} s \cdot u^{\sum_{i=1}^{m} X_i} d^{m - \sum_{j=1}^{m} X_j}$$

- we want to determine $\tilde{p}$ so that $\{V_m\}_{0 \le m \le n}$ is a martingale, that is

$$\tilde{E}\left[V_m \mid X_1, X_2, \dots, X_{m-1} \right] = V_{m-1} \quad \text{for all } m = 1, 2, \dots, n$$

(the martingale approach)

(the martingale property)

$$\tilde{E}\left[V_m \mid x_1, \dots, x_{m-1} \right] = \frac{1}{(1+r)^m}\, s\, \tilde{E}\left[u^{\sum_{i=1}^{m} x_i}\, d^{\,m - \sum_{i=1}^{m} x_i} \,\Big|\, x_1, \dots, x_{m-1} \right]$$

$$= \frac{1}{(1+r)^m} \cdot s \cdot u^{\sum_{i=1}^{m-1} x_i} \cdot d^{\,m-1-\sum_{i=1}^{m-1} x_i}\, \tilde{E}\left[u^{x_m}\, d^{\,1-x_m} \right]$$

(since x_m is independent of $x_1, \dots, x_{m-1}$)

$$= V_{m-1} \cdot \frac{1}{1+r} \cdot \left[u \cdot \tilde{p} + d \cdot (1-\tilde{p}) \right]$$

- So in order for the process $\{V_m\}_{0 \le m \le n}$ to be a martingale we need to impose the following condition : $\frac{1}{1+r}\left[u\tilde{p} + d(1-\tilde{p}) \right] = 1$

- solving the above equation for $\tilde{p}$ we get

$$\tilde{p} = \frac{1+r-d}{u-d}$$

$\rightsquigarrow$ this is the "unique" probability under which the discounted stock price becomes a martingale

<u>Remark</u> : Thinking about a martingale as a "fair game" we can say that if $\tilde{p} = \frac{1+r-d}{u-d}$ then the discounted stock price is "fair" (that is on average performs the same as the riskless asset, i.e not better, not worse). Ultimately this implies (it is not obvious) that for the above probability not only the discounted stock price is

a martingale but also the present value of any derivative written on this stock is a martingale

$\rightarrow$ we shall discuss these issues in detail for the Black-Scholes model in the next semester.

- So, to price any derivative security in the setting of the Binomial asset pricing model we compute:

$$\rightsquigarrow \quad E\Big[\text{discounted } \underbrace{\text{payoff}}_{\text{function of } S_m}\Big]$$

<u>example</u>: Call option with maturity n, strike price K

$\rightsquigarrow$ price of the option : $\tilde{E}\left[\frac{1}{(1+r)^n}(S_n - K)^+\right]$

<u>One period models</u>

<u>Def</u>: An opportunity to lock in a risk free profit is called an **ARBITRAGE** opportunity.

(Arbitrage = sure win betting scheme)

<u>Law of One Price</u>: Consider two investments with costs C_1 and C_2. If the (discounted) payoff from the first investment is always equal to that of the second then either $C_1 = C_2$ or there is arbitrage

<u>Generalized Law of One Price</u>: Consider two investments with costs C_1 and C_2. If $C_1 < C_2$ and the (discounted) payoff of the first investment is always greater or equal to that of the other investment then there is arbitrage.

<u>Pricing Based on the "NO ARBITRAGE" condition</u>.

<u>Example</u>: 1-period binomial model

$$S \begin{cases} uS \\ dS \end{cases} \qquad d < 1+r < u$$

We want to price a contingent claim that pays:

$$\begin{cases} a & \text{if } S_1 = uS \\ b & \text{if } S_1 = dS \end{cases}$$

Method: construct a replicating portfolio and use the "Law of One Price" to price the contingent claim

<u>Investment 1</u>	<u>Investment 2</u>
· $C_1 = x$: initial amount	$C_2 = ?$
· bank : $yS - x$	payoff $= \begin{cases} a & \text{if } S_1 = uS \\ b & \text{if } S_1 = dS \end{cases}$
· stock : y shares	

· we want to match investments

· at time 1:

$$\text{if} \quad \begin{aligned} S(1) = uS &\rightsquigarrow \\ S(1) = dS &\rightsquigarrow \end{aligned} \begin{cases} yuS + (yS - x)(1+r) = a \\ ydS + (yS - x)(1+r) = b \end{cases}$$

$S(1) = dS \rightsquigarrow \Big[\; ydS + (yS - x)(1+r) = b$

$\rightsquigarrow \; y = \dfrac{a-b}{S(u-d)}$

$\rightsquigarrow \; (yS - x)(1+r)(d-u) = ad - bu$

$\Rightarrow \; x = yS + \dfrac{ad-bu}{(1+r)(d-u)} = \dfrac{a-b}{u-d} - \dfrac{ad-bu}{(1+r)(u-d)}$

$= \dfrac{(a-b)(1+r) - (ad-bu)}{(1+r)(u-d)}$

$= \dfrac{1}{1+r}\left[a \cdot \underbrace{\dfrac{1+r-d}{u-d}}_{\tilde{p}} + b \underbrace{\dfrac{u-1-r}{u-d}}_{1-\tilde{p}} \right]$

$= \dfrac{1}{1+r} \tilde{E}\left[\text{payoff} \right]$ assuming $\tilde{p} = \dfrac{1+r-d}{u-d}$

- at time 0 : $C_1 = x$
- at time 0 : $C_2 = \tilde{E}\left[\text{discounted payoff}\right]$

$\rightsquigarrow$ according to the NO ARBIRAGE rule we must have $C_1 = C_2 \Rightarrow x = \tilde{E}\left[\text{discounted payoff}\right]$

$= \dfrac{1}{1+r}\left[a \cdot \dfrac{1+r-d}{u-d} + b \dfrac{u-1-r}{u-d} \right]$

and this happends only as $\tilde{p} = \dfrac{1+r-d}{u-d}$

<u>example</u> :

- $\boxed{a=1, b=0} \Rightarrow x = \dfrac{1}{1+r} \dfrac{1+r-d}{u-d}$; $y = \dfrac{1}{S(u-d)}$

• $\boxed{a=0,\ b=1} \Rightarrow x = \frac{1}{1+r}\,\frac{u-1-r}{u-d}\ ;\ \ y = \frac{-1}{S(u-d)}$

Conclusion: we have a portfolio which has the same payoff (value) as the claim. By the law of one price the cost of the claim should be equal to the investment in this portfolio, i.e x.

The expected return from buying this claim with respect to $\tilde{P}$ is equal to zero:

$$\tilde{E}(\text{return}) = \frac{\frac{1}{1+r}\tilde{E}(\text{payoff}) - \text{cost}}{\text{cost}} = 0$$

Thus, the name, risk neutral probabilities

Conclusions for this model:

- every contingent claim can be replicated (or hedged)
- there is a unique set of "risk neutral" probabilities under which the expected return of every claim is 0
- Cost of the claim $= \frac{1}{1+r}\tilde{E}[\text{Payoff}]$

Definition: We say the market is complete if each contingent claim can be hedged.

example: • 1 step binomial model is COMPLETE

General 1-period model.

- suppose we have : m assets

- the prices at time 0 form a vector :

$$S_0 = (S_0^1, \dots, S_0^m)^t = \begin{pmatrix} S_0^1 \\ \vdots \\ S_0^m \end{pmatrix}$$

- at time 1 : the market can be in one of n possible states

D_{ij} = the value of the i^{th} asset if the market is in state j

$D = [D_{ij}]$ $\quad m \times n$ matrix

$\rightsquigarrow$ an alternative notation is : $D = (D^{(1)}, \dots, D^{(n)})$ where $D^{(j)}$ is the j-th column of D

ex : in the BAP :

$$\left.\begin{array}{ll} m = 2 & \text{— money market} \\ & \text{— stock} \\ n = 2 & \begin{array}{l} su \\ sd \end{array} \end{array}\right\} \rightsquigarrow D = \begin{bmatrix} 1+r & 1+r \\ us & ds \end{bmatrix}$$

- Portfolio : $\theta = (\theta_1, \dots, \theta_m)^t$

θ_i = nr of shares of asset i

- Market value of portfolio at time 0 :

$$S_0 \cdot \theta = S_0^1 \cdot \theta_1 + S_0^2 \theta_2 + \dots + S_0^m \theta_m$$

- Market value of the portfolio at time 1 :

$\rightsquigarrow$ if the state of the market is i :

$$D^{(i)} \cdot \theta = \sum^{m} D_{ij} \theta_j$$

$\rightsquigarrow$ more generaly : Market value at time 1 : $D^t \cdot \theta$

since all possible values at time 1 can be represented by a vector :

$$\begin{pmatrix} D^{(1)t} \cdot \theta \\ \vdots \\ D^{(m)t} \cdot \theta \end{pmatrix} = D^t \cdot \theta \in \mathbb{R}^n$$

Notation : let $x = (x_1, \dots, x_d)^t$

- $x \geqslant 0$ if $x_i \geqslant 0$ for all i
- $x > 0$ if $x \geqslant 0$ and $x_i > 0$ for at least 1 i
- $x \gg 0$ if $x_i > 0$ for all i

<u>Definition</u> : Arbitrage is a portfolio $\theta \in \mathbb{R}^m$ such that either $S_0 \cdot \theta \leq 0$ and $D^t \cdot \theta > 0$ or $S_0 \cdot \theta < 0$ and $D^t \cdot \theta \geqslant 0$

<u>Definition</u> : A vector $\psi \gg 0$ such that $S_0 = D\psi$ is called a state price vector

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \rightsquigarrow S_0 = \psi_1 D^{(1)} + \psi_2 D^{(2)} + \dots + \psi_n D^{(n)}$$

- to interpret ψ_i, suppose we can find a portfolio $\theta^{(i)}$ such that

$$\theta^{(i)} \cdot D^{(j)} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

- the value of this portfolio is the indicator function of the event that at time 1 the market is in state i

$$S_0 \cdot \theta^{(i)} = \left(\sum_{j=1}^{n} \psi_j D^{(j)} \right) \cdot \theta_{(i)} = \psi_i$$

therefore, ψ_i is the marginal cost at time 0 of obtaining an additional unit of wealth at time 1 if the market is in state i

<u>Def</u> $\theta^{(i)}$ = Arrow Debreu securities

<u>Example</u> : · in the Binomial model

$$D = \begin{pmatrix} 1+r & 1+r \\ us & ds \end{pmatrix} \qquad H = \begin{pmatrix} \theta^{(1)} & \theta^{(2)} \end{pmatrix}$$

$\Rightarrow$ according to the definition of H we get

$$H^t D = D^t \cdot H = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

· so we want to solve this eq for ④

$\rightsquigarrow$ ④ $= (D^t)^{-1} = \frac{1}{(1+r)(u-d)s} \begin{pmatrix} -ds & us \\ 1+r & -(1+r) \end{pmatrix}$

$$= \begin{pmatrix} \frac{-d}{(1+r)(u-d)} & \frac{u}{(1+r)(u-d)} \\ \frac{1}{s(u-d)} & \frac{-1}{s(u-d)} \end{pmatrix}$$

· compare this with the previous example.

$\rightsquigarrow$ the first column is $\begin{pmatrix} yS-x \\ y \end{pmatrix}$ when $a=1$ and $b=0$

$\rightsquigarrow$ the second column is $\begin{pmatrix} yS-x \\ y \end{pmatrix}$ when $a=0$ $b=1$

· to find ψ (the state price vector) $[\, S_0 = D\psi \,]$

$$\begin{pmatrix} 1 \\ s \end{pmatrix} = \begin{pmatrix} 1+r & 1+r \\ us & ds \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$ $\rightsquigarrow$ solve for ψ_1 and ψ_2

$$\psi_1 = \frac{1+r-d}{(1+r)(u-d)} \qquad \psi_2 = \frac{u-1-r}{(1+r)(u-d)}$$

also notice that $\psi_1 + \psi_2 = \frac{1}{1+r} = \psi_0$

$\rightsquigarrow$ to make ψ into a probability vector one has to normalize it

$$\frac{\psi}{\psi_0} = \left(\frac{\psi_1}{\psi_0}, \frac{\psi_2}{\psi_0} \right) = \left(\frac{1+r-d}{u-d}, \frac{u-1-r}{u-d} \right)$$

(same thing as $\tilde{p}$)

- Set $\Omega = \{1,2,\dots,n\}$ = market states

  $0 < \tilde{\mathbb{P}}(\{i\}) = \dfrac{\psi_i}{\psi_0}$ = risk neutral probabilities

- Suppose we can construct a portfolio $\bar{\theta}$ with payoff 1 regardless of the state of the market at time 1

$$\Rightarrow \quad D^t \cdot \bar{\theta} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \Rightarrow \bar{\theta} = (D^t)^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = (4) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$\bar{\theta} = \sum_{i=1}^{n} \theta^{(i)}$ (will work if D^t has an invese)

$\Rightarrow$ the cost of such portfolio is

$$S_0 \cdot \bar{\theta} = (D\psi)^t \bar{\theta} = \psi^t \cdot D^t \cdot \bar{\theta} = \sum_{i=1}^{n} \psi_i = \psi_0$$

$\rightsquigarrow$ ψ_0 represents the discount on riskless borrowing

(in the Binomial asset pricing example $\psi_0 = \frac{1}{1+r}$)

Suppose that there is a state price vector ψ, thus also risk neutral probabilities. Then

$$\tilde{E}\, S_1^i = \sum_{j=1}^{m} D_{ij} \frac{\psi_j}{\psi_0} = \frac{S_0^i}{\psi_0}, \quad S_0^i = \psi_0 \tilde{E}\, S_1^i$$

$\rightsquigarrow$ any asset price is equal to the discounted expected payoff (with respect to the risk neutral measure)

$\rightsquigarrow$ for any portfolio we have $S_0 \cdot \theta = \psi_0 \tilde{E}(S_1 \cdot \theta)$

$\rightsquigarrow$ for any portfolio we have $S_0 \cdot \theta = \psi_0 \tilde{E}(S_1 \cdot \theta)$

Definition: A contingent claim is attainable at $t=1$ if it can be hedged (i.e. replicated)

Theorem: If there is no arbitrage then the unique price at time 0 (price of an attainable claim) is $\psi_0 \tilde{E}(\text{payoff})$ where the expectation is taken with respect to the risk neutral measure such that

$$S_0^i = \psi_0 \tilde{E} S_1^i \qquad \text{for all } i$$

and ψ_0 = is the discount on riskless borrowing.

Conclusion: To price any attainable contingent claim we need just any one risk neutral measure. The cost then is $\psi_0 \tilde{E}[\text{payoff}]$

Definition: If a market has n possible states at time 1 then any probability measure $\tilde{\mathbb{P}}(\{i\}) = \tilde{p}_i > 0 \quad i = 1, 2, .., n$ such that $S_0^j = \psi_0 \tilde{E} S^j$ for all $j = 1, 2, ..., m$ is called a risk neutral measure or equivalent martingale measure

Proposition: A market of m tradeble assets in a 1 period model with n possible market states at time 1 is COMPLETE iff $m \geq n$ and rank of D is n.

Remark: • $\rightsquigarrow$ the Binomial model is complete since $m = n = 2$

Remark: • $\rightsquigarrow$ the Binomial model is complete since $m = n = 2$ and D has an inverse

• if the market is complete and there is no arbitrage then the risk neutral probabilities are unique.

Conclusion:

1) The market is arbitrage free iff there exist a risk neutral prob.
2) The arbitrage free market is complete iff the risk neutral prob is unique
3) The arbitrage free price of an attainable claim is

$$\psi_0 \tilde{E}(\text{payoff})$$

Reference for single period model:

A. Etheridge: "A course in financial calculus" sections 1.5 and 1.6