# MTH 9821 Numerical Methods for Finance I
# Lecture 4

October 3, 2013

## 1 Binomial Tree

$$\begin{tikzcd} & uS \\ S \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & dS \end{tikzcd}$$

We know that

$$\lim_{\mathbf{n}\to\infty} \textbf{Binomial} = \textbf{lognormal}$$
$$s.t. \quad r, \sigma \text{ are satisfied with parameters } u, d, p$$

Here we have 2 constraints with 3 parameters, we can use

1. $ud = 1$
2. $p = 0.5$

## 2 Arbitrage-free one period binomial model derivative security value

Consider time interval $[t, \delta t]$

$$\begin{tikzcd} & S_{up} \\ S_0 \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & S_{down} \end{tikzcd} \qquad \begin{tikzcd} & V_{up} \\ V_0 \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & V_{down} \end{tikzcd}$$

Find $V_0$ in terms of $V_{up},\ V_{down},\ u,\ d,\ S_0,\ \delta t,\ r,\ q$.

**Method 1: Hedging**

Set up a portfolio as:

- Long $V$
- short $e^{-q\delta t}\Delta$ units of underlying asset (with $\Delta$ to be chosen later on).

The payoff of the portfolio will be:

$$
\begin{tikzcd}
 & V_{up} - \Delta S_{up} \\
V_0 - e^{-q\delta t}\Delta S_0 \arrow[ur] \arrow[dr] & \\
 & V_{down} - \Delta S_{down}
\end{tikzcd}
$$

For the hedge, we need to choose $\Delta$ such that

$$
\begin{aligned}
& V_{up} - \Delta S_{up} = V_{down} - \Delta S_{down} \\
\Rightarrow \quad & \Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{V_u - V_d}{S_0(u-d)} \\
& \text{payoff } = V_d - \frac{V_u - V_d}{S_0(u-d)} dS_0 = \frac{uV_d - dV_u}{u-d}
\end{aligned}
$$

By law of one price,

$$
\begin{aligned}
& V_0 - e^{-q\delta t}\Delta S_0 = e^{-r\delta t}\frac{uV_d - dV_u}{u-d} \\
\Rightarrow \ & V_0 = \frac{V_u - V_d}{u-d}e^{-q\delta t} + e^{-r\delta t}\frac{uV_d - dV_u}{u-d} \\
& \quad = e^{-r\delta t}\left[V_u \frac{e^{(r-q)\delta t} - d}{u-d} + V_d \frac{u - e^{(r-q)\delta t}}{u-d}\right] \\
& \quad = e^{-r\delta t}\left[V_u p_{RN,up} + V_d p_{RN,down}\right] \\
& \quad = e^{-r\delta t} E_{RN}\big(V(\delta t)\big)
\end{aligned}
$$

We denote $p_{RN,up}$ and $p_{RN,down}$ as risk neutral probability. It's easy to see that

$$p_{RN,up} + p_{RN,down} = 1$$

We also need

$$
\begin{aligned}
& p_{RN,up} > 0, \quad p_{RN,down} > 0 \\
\Rightarrow \ & d < e^{(r-q)\delta t} < u
\end{aligned}
$$

This is the no-arbitrage condition for the one period model.

**Method 2: Replicating**

Set up a portfolio as:

- Long cash $B$
- Long $e^{-q\delta t}\Delta$ units of underlying asset (with $\Delta$ to be chosen later on).

The payoff of the portfolio will be:

$$
\begin{tikzcd}
 & Be^{r\delta t} + \Delta S_{up} \\
B + \Delta e^{-q\delta t} S_0 \arrow[ur] \arrow[dr] & \\
 & Be^{r\delta t} + \Delta S_{down}
\end{tikzcd}
$$

By replicating, we have

$$
\begin{aligned}
Be^{r\delta t} + \Delta S_{up} &= V_{up} \\
Be^{r\delta t} + \Delta S_{down} &= V_{down}
\end{aligned}
$$

Thus,

$$V_0 = B + \Delta e^{-q\delta t} S_0$$

Note that $V_0$ will be the same as in Method 1.

In this model, we are able to value the derivative security by modeling the process of underlying asset.

# 3 Calibration of Binomial Tree model to a Lognormal Model

Tree model from time 0 to time $T$ with $N$ equal time periods of size $\delta t = \frac{T}{N}$.
Find $u_N,\ d_N,\ p_N$ such that the discrete distribution $S_N(T)$ converges to a lognormal distribution with drift $\mu$ and volatility $\sigma$.

$$\ln\left(\frac{S_N(T)}{S(0)}\right) \longrightarrow (\mu - q - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z, \quad N \to \infty$$

$$S_N(T) = S(0)X_1 \cdot X_2 \cdots X_N$$

$$\text{where } X_i = \begin{cases} u_N, \text{with probability } &= p_N \\ d_N, \text{with probability } &= 1 - p_N \end{cases}$$

$$\ln\left(\frac{S_N(T)}{S(0)}\right) = \sum_{i=1}^{N} \ln((X_i) = \sum_{i=1}^{N} Y_i$$

$$\text{From CLT,} \quad \frac{\frac{1}{N}\sum_{i=1}^{N} Y_i - \mu_Y}{\frac{\sigma_Y}{\sqrt{N}}} \longrightarrow Z$$

$$\Rightarrow \frac{\sum Y_i - N\mu_Y}{\sigma_Y\sqrt{N}} \longrightarrow Z$$

$$\Rightarrow \sum Y_i \approx N\mu_Y + \sigma_Y\sqrt{N}$$

$$\Rightarrow \ln\left(\frac{S_N(T)}{S(0)}\right) \longrightarrow (\mu - q - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z$$

Choose $u_N,\ d_N,\ p_N$ such that $\mu_Y$ and $\sigma_Y$ satisfy

$$N\mu_Y = (\mu - q - \frac{\sigma^2}{2})T \quad \Rightarrow \quad \mu_Y = (\mu - q - \frac{\sigma^2}{2})\delta t$$

$$N\sigma_Y = \sigma\sqrt{\delta} \quad \Rightarrow \quad \sigma_Y = \delta t$$

$$\Rightarrow \quad E(Y) = p_N \ln(u_N) + (1 - p_N)\ln(d_N) = \mu_Y = (\mu - q - \frac{\sigma^2}{2})\delta t$$

$$E(Y^2) = p_N \ln^2(u_N) + (1 - p_N)\ln^2(d_N) = \sigma_Y^2 + (\mu_Y)^2 = (\mu - q - \frac{\sigma^2}{2})^2\delta t^2 + \sigma^2\delta t$$

Since we have two constraints but three unknowns, we need to use proper parametritation method.

**CRR parametritation**

$$ud = 1 \ \Rightarrow \ u \approx e^{\sigma\sqrt{\delta t}},\ d \approx e^{-\sigma\sqrt{\delta t}}$$

by neglecting terms smaller than $\delta t$.

**Remark:**
⋆ Using Taylor Expansion: $u = 1 + \sigma\sqrt{\delta t} + O(\delta t)$.
Other parametrition methods may improve the accuracy, but the $1 + \sigma\sqrt{\delta t}$ will always exist.

**Pseudocode:**

Notation:

$$\begin{tikzcd}[row sep=small] & S_{j+1,i} \\ S_{j,i} \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & S_{j+1,i+1} \end{tikzcd}$$

$S_{j,i}$: price of the underlying asset at time period $j$ if the price goes down $i$ times.

**Pseudocode:** *Binomial Tree Valuation of Non-path-dependent Securities*

**Input**:

$S_0,\ \sigma,\ q,\ r$;

$V_T(S) =$ payoff at $T$ of the derivative security,

e.g. $V_T(S) = (S - K)^+$ for European call option;

$N$: number of time steps;

**Output**:

$V =$ approximation of value of the derivative security.

$\delta t = \frac{T}{N}$;

$u = e^{\sigma\sqrt{\delta t}}, \quad d = \frac{1}{u}$;

$p_{RN,u} = \frac{e^{(r-q)\delta t} - d}{u-d}, \quad p_{RN,d} = 1 - p_{RN,u}$;

**for** $i = 0 : N$

  $V(i) = V_T(Si) = V_T(S_0 u^{N-i} d^i)$

**end**

**for** $j = N - 1 : 0$ *(Going backward in time)*

  **for** $i = 0 : j$

    $V(i) = e^{-r\delta t}\big(p_{RN,u} V(i) + p_{RN,d} V(i+1)\big)$;

    (Note: We keep overwrite $V(i)$ at each time period)

  **end**

**end**

$V = V(0)$;

**How to improve?**

We calculate $\tilde{p}_{RN,u} = e^{-r\delta t} p_{RN,u}$ and $\tilde{p}_{RN,d} = e^{-r\delta t} p_{RN,d}$ before the for loop, and then

$V(i) = \tilde{p}_{RN,u} V(i) + \tilde{p}_{RN,d} V(i+1)$

**How to converge?**

Start with $N = 20$, double number of steps until error hit tolerance level.

**Question:**

**Why we use binomial tree to calculate the approximation of the value of European plain vanilla options since we've found closed form solutions in Black Scholes model?**

- Check with $V_{BS}$ to make sure that the binomial tree model we developed works fine.
- Variance Reduction

  There will be approximation error between a binomial tree model value and the exact value in Black Scholes framework. We can't find closed form solutions for American options in BS, but we can adjust the value in binomial tree model accordingly to reduce the approximation error.

$$\begin{aligned} \text{Error} &= V_{Euro,binom} - V_{Euro,BS} \\ V*_{Ame} &= V_{Ame,binom} - \text{Error} \end{aligned}$$

- It's convenient to calculate greeks using binomial tree model.

# 4 Estimate Greeks using Binomial Trees: $\Delta$, $\Gamma$, $\Theta$

In a $N$ time step binomial tree model, we can use any 2-period branch to calculate the $\Delta$, $\Gamma$, $\Theta$. As in the above pseudocode, instead of saving only $V(0)$, we can save the values in the first two time periods and then calculate the greeks.

$$\begin{tikzcd}
 & & V_{2,0} \\
 & V_{1,0} \arrow[ur] \arrow[dr] & \\
V_{0,0} \arrow[ur] \arrow[dr] & & V_{2,1} \\
 & V_{1,1} \arrow[ur] \arrow[dr] & \\
 & & V_{2,2}
\end{tikzcd}$$

**Delta $\Delta$:**

$$\Delta \approx \frac{V_{1,0} - V_{1,1}}{S_{1,0} - S_{1,1}}$$

**Gamma $\Gamma$:**

$$\Gamma \approx \frac{\frac{V_{2,0}-V_{2,1}}{S_{2,0}-S_{2,1}} - \frac{V_{2,1}-V_{2,2}}{S_{2,1}-S_{2,2}}}{\frac{S_{2,0}+S_{2,1}}{2} - \frac{S_{2,1}+S_{2,2}}{2}} = \frac{\frac{V_{2,0}-V_{2,1}}{S_{2,0}-S_{2,1}} - \frac{V_{2,1}-V_{2,2}}{S_{2,1}-S_{2,2}}}{\frac{S_{2,0}-S_{2,2}}{2}}$$

**Note:**
Here we use $\frac{S_{2,0}-S_{2,2}}{2}$ instead of $S_{1,0}-S_{1,1}$ to get better accuracy.

$$\begin{aligned}
\frac{S_{2,0}-S_{2,2}}{2} &= S_0(u-d)\frac{u+d}{2} \\
S_{1,0}-S_{1,1} &= S_0(u-d) \\
\frac{u+d}{2} &= \frac{e^{\sigma\sqrt{\delta t}} + e^{-\sigma\sqrt{\delta t}}}{2} \\
&\approx \frac{1}{2}\left(1+\sigma\sqrt{\delta t} + \frac{\sigma^2\delta t}{2} + O(\delta t^{3/2}) + 1 - \sigma\sqrt{\delta t} + \frac{\sigma^2 \delta t}{2} + O(\delta t^{3/2})\right) \\
&= 1 + \frac{\sigma^2\delta t}{2} + O(\delta t^{3/2}) \to 1, \text{ as } \delta t \to 0
\end{aligned}$$

$i.e.,\ \dfrac{S_{2,0}-S_{2,2}}{2} \to S_{1,0}-S_{1,1}$

**Theta $\Theta$:**
We need to fix $S$ to get the sensitivity to $t$, we know that

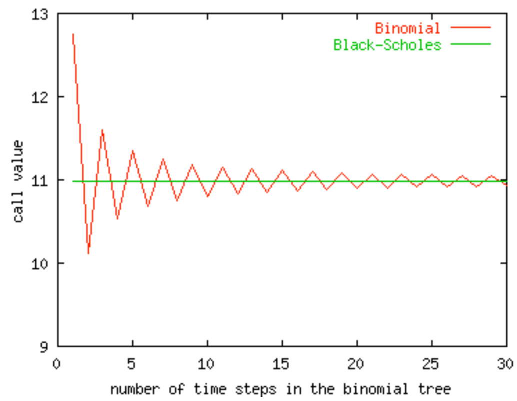
$$S_{2,1} = S_{0,0}ud = S_{0,0}$$

Thus,

$$\Theta \approx \frac{V(2\delta t, S_0) - V(0, S_0)}{2\delta t} \approx \frac{V_{2,1}-V_{0,0}}{2\delta t}$$

# 5 Extensions

## 5.1 Average Binomial Method




As seen in the graph, the approximate value overshoots and undershoots for adjacent $N$'s. A simple improvement of binomial method would be taking average of adjacent approximations, i.e.,

$$V_{average,bin}(N) = \frac{V_{bin}(N) + V_{bin}(N-1)}{2}$$
$$\Delta_{average,bin}(N) = \frac{\Delta_{bin}(N) + \Delta_{bin}(N-1)}{2}$$

## 5.2 Binomial Black Scholes

For binomial model with $N$ steps, instead of using the value of the payoff at $T$, use the BS value of the option on the $N-1$-th time step.

**for** $i = 0 : N$
  $V(i) = V_{BS}(S_0 u^{N-1-i} d^i, \delta t)$
**end**
**for** $j = N-1 : 0$ *(Going backward in time)*
  **for** $i = 0 : j$
    $V(i) = e^{-r\delta t}\big(p_{RN,u}V(i) + p_{RN,d}V(i+1)\big);$
    (Note: We keep overwrite $V(i)$ at each time period)
  **end**
**end**
$V_{BBS} = V(0);$

**Remark**
⋆ BBS model only works for plain vanilla European options (exact BS solutions exist)
⋆ The problem of jumps at maturity will be releaved in BBS model.

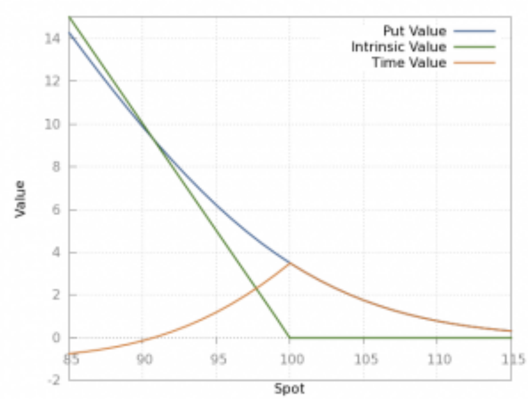
### 5.3 Binomial Black Sholes with Richardson Extrapolation

For even $N$,

$$V_{BBSR}(N) = 2V_{BBS}(N) - V_{BBS}\left(\frac{N}{2}\right)$$
$$\Delta_{BBSR}(N) = 2\Delta_{BBS}(N) - \Delta_{BBS}\left(\frac{N}{2}\right)$$

$$\text{If } V_{BBS}(N) = V_{exact} + \frac{C_1}{N} + \frac{C_2}{N^2} + O\left(\frac{1}{N^3}\right)$$
$$V_{BBS}(N/2) = V_{exact} + \frac{2C_1}{N} + \frac{4C_2}{N^2} + O\left(\frac{1}{N^3}\right)$$
$$V_{BBSR}(N) = 2V_{BBS}(N) - V_{BBS}(N/2) = V_{exact} - \frac{2C_2}{N^2} + O\left(\frac{1}{N^3}\right)$$

# 6 Binomial Tree Model for American Options




For American options, at each time period $j$ we need to compare

- Exercise at period $j$, get intrinsic value;
- Not exercise at period $j$, get discounted value of 2 nodes in $j+1$.

**Pseudocode:** *Binomial Tree Valuation of American Options*

$\delta t = \frac{T}{N}$;
$u = e^{\sigma\sqrt{\delta t}}, \quad d = \frac{1}{u}$;
$p_{RN,u} = \frac{e^{(r-q)\delta t} - d}{u-d}, \quad p_{RN,d} = 1 - p_{RN,u}$;
**for** $i = 0 : N$
    $V(i) = \max\left[K - S_0 u^{N-i} d^i, 0\right]$;
**end**
**for** $j = N-1 : 0$ *(Going backward in time)*
    **for** $i = 0 : j$
        $V(i) = \max\left[e^{-r\delta t}\big(p_{RN,u}V(i) + p_{RN,d}V(i+1)\big), K - S_0 u^{j-i} d^i\right]$;
    **end**
**end**
$V = V(0)$;