Introduction : Brownian Motion ; Local Martingales

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Recall notations & concepts : $(\Omega, \mathcal{F}, P)$

① **Stochastic process** : a family of r.v $X = \{X_t ; 0 \le t < \infty\}$

i.e. $X_t(\cdot) : \Omega \longrightarrow \mathbb{R}$; $t \longmapsto X_t(\omega)$ is the **SAMPLE PATH**

② **Filtration of X** : $\mathcal{F}_t^X = \sigma(X_s : 0 \le s \le t)$

⇝ information about the process X that has been revealed up to time t.

⇝ $\mathcal{F}_s^X \subseteq \mathcal{F}_t^X$ for all $s \le t$

⇝ we say that X is **adapted** w.r.t. $\{\mathcal{F}_t\}$ iff
$\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for all $t \ge 0$.

③ **Markov Property** : A stochastic process is called **Markovian** if

$$P\left[X_t \in A \mid \mathcal{F}_s^X\right] = P\left[X_t \in A \mid X_s\right]$$

for all $A \in \mathcal{B}(\mathbb{R})$, $0 < s < t$.

④ **Martingale Property** : A stochastic process with $E|X_t| < \infty$, $(\forall)\ t \ge 0$ is said to be :

$$\left.\begin{array}{ll} \textbf{martingale} \rightsquigarrow \text{ if } & E[X_t \mid \mathcal{F}_s] = X_s \\ \textbf{submartingale} \rightsquigarrow \text{ if } & E[X_t \mid \mathcal{F}_s] \ge X_s \\ \textbf{supermartingale} \rightsquigarrow \text{ if } & E[X_t \mid \mathcal{F}_s] \le X_s \end{array}\right\} \begin{array}{l} \text{a.s.} \\ (\forall)\ s \le t \end{array}$$

⑤ **Stopping time** : A random variable $\tau : \Omega \to [0, \infty]$ is called

⑤ **Stopping time** : A random variable $\tau : \Omega \to [0,\infty]$ is called a **stopping time** of the filtration $\{\mathcal{F}_t\}$ if the event $\{\tau \le t\} \in \mathcal{F}_t$ for $(\forall)$ $0 \le t < \infty$.

$\rightsquigarrow$ the determination of whether τ has occurred by time t can be made by looking at the information $\mathcal{F}_t$, without anticipation of the future

ex : if X has continuous paths, A = closed set of the real line

"the hitting time" $\tau_A = \min\{t \ge 0 : X_t \in A\}$ is a S.T.

⑥ **Local Martingale** : An adapted process $X = \{X_t, \mathcal{F}_t : 0 \le t < \infty\}$ is called a **Local Martingale** if there exists an increasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of S.T with $\lim_{n\to\infty} \tau_n = \infty$ such that the stopped process

$\{X_{t\wedge\tau_n}, \mathcal{F}_t : 0 \le t < \infty\}$ is a martingale $(\forall)$ $n \ge 1$.

! $\begin{cases} \text{• it can be shown that every martingale is a local mart.} \\ \text{but there exist local martingales which are not mart.} \end{cases}$

example of local martingales that are not mart!

• W_t = standard B.M. $\tau = \min\{t : W_t = -1\}$

$\rightsquigarrow$ $W_{t\wedge\tau}$ is a martingale

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$\rightsquigarrow$ a time change leads to a process

$$X_t = \begin{cases} W_{\left(\frac{t}{1-t}\right)\wedge\tau} & \text{for } 0 \le t < 1 \\ -1 & \text{for } 1 \le t < \infty \end{cases}$$

$\rightsquigarrow$ X_t is continuous; its expectation is discontinuous

$$EX_t = \begin{cases} 0 & \text{for } 0 \le t < 1 \\ -1 & \text{for } 1 \le t < \infty \end{cases} \neq EX_{t\wedge\tau_k}$$

hence X is not a martingale; however it can be shown to be a local martingale by picking a "localizing sequence" $\tau_k = \min\{t : X_t = k\}$ if there is such t, otherwise $\tau_k = k$

- for this sequence $\lim_{k\to\infty} \tau_k = \infty$
- $X_{t\wedge\tau_k}$ is martingale for all k.

- another example: Stochastic Integrals are local mart. but not necessarily martingales.

(7) Brownian Motion : the prototypical example of :

(a) process with stationary, independent increments

(b) Markov process

(c) Martingale with continuous paths

(d) Gaussian process (with $\mathrm{Cov}(W_s, W_t) = \min(s,t)$)

- despite its continuity, the sample path $t \longmapsto W_t(\omega)$ is

- despite its continuity, the sample path $t \longmapsto W_t(\omega)$ is NOT DIFFERENTIABLE anywhere on $[0,\infty)$

- fix a partition $0 = t_0^{(n)} < t_1^{(n)} < \dots < t_{2^n}^{(n)} = t$ of $[0,t]$ such that $t_k^{(n)} = k t 2^{-n}$

$$V_p^{(n)}(\omega) \triangleq \sum_{k=1}^{2^n} \left| W_{t_k^{(n)}}(\omega) - W_{t_{k-1}^{(n)}}(\omega) \right|^p \quad , p>0$$

"the Variation of order p" of the sample path $t \mapsto W_t(\omega)$ along the n^{th} partition.

- for $p=1$: $V_1^{(n)}(\omega)$ = length of the polygonal approx to the Brownian motion

- for $p=2$: $V_2^{(n)}(\omega)$ = "quadratic variation"

Th With probability 1 we have :

$$\lim_{n\to\infty} V_p^{(n)} = \begin{cases} \infty & \text{for} \quad 0<p<2 \\ t & \text{for} \quad p=2 \\ 0 & \text{for} \quad p>2 \end{cases}$$

Remark : Arbitrary (local) martingales with continuous sample paths do not behave much differently :

Th For every nonconstant (local) martingale M (cont)

$$\sum_{k=1}^{2^n} \left| M_{t_k^{(n)}} - M_{t_{k-1}^{(n)}} \right|^1 \xrightarrow{P} \infty \quad \text{as} \quad n\to\infty$$

$$\sum_{k=1}^{2^n} \left| M_{t_k^{(n)}} - M_{t_{k-1}^{(n-1)}} \right|^1 \xrightarrow{P} \infty \quad \text{as} \quad n \to \infty$$

$$\sum_{k=1}^{2^n} \left| M_{t_k^{(n)}} - M_{t_{k-1}^{(n)}} \right|^2 \xrightarrow{P} \langle M \rangle_t \quad \text{as} \quad n \to \infty$$

where $\langle M \rangle$ is a process with continuous, nondecreasing paths

Remark: The process $\langle M \rangle$ = the quadratic variation of M is the unique process with cont, nondecreasing paths s.t

$$\boxed{M_t^2 - \langle M \rangle_t = \text{local martingale}}$$

- in particular, if M is a square integrable mart, i.e if $E M_t^2 < \infty$ for all $t \geq 0$ then

$$\boxed{M_t^2 - \langle M \rangle_t = \text{martingale}}$$

Corrolary: Every (local) martingale M, with cont. sample paths and of finite first variation is necessarily constant.

- for any two (local) martingales M and N, with cont. sample paths we have

$$\sum_{k=1}^{2^n} \left(M_{t_k^{(n)}} - M_{t_{k-1}^{(n)}} \right) \left(N_{t_k^{(n)}} - N_{t_{k-1}^{(n)}} \right) \xrightarrow{P} \langle M, N \rangle_t \triangleq \frac{1}{4} \left[\langle M+N \rangle_t - \langle M-N \rangle_t \right]$$

as $n \to \infty$

Remark: The process $\langle M, N \rangle$ is cont. and of bdd variation;

(bded variation = difference of two nondecreasing processes)
it is the unique process with these properties s.t.

$$M_t N_t - \langle M, N \rangle_t = \text{local martingale}$$

$\langle M, N \rangle$ = cross-variation of M and N

- if M, N are independent, then $\langle M, N \rangle \equiv 0$.
- for square integrable martingales, $\langle \cdot, \cdot \rangle$ plays the role of an inner product, and $M_t N_t - \langle M, N \rangle_t$ = martingale

(we say M, N are orthogonal if $\langle M, N \rangle \equiv 0$)

Burkholder - Davis - Gundy Inequalities

Let M be a local martingale with continuous sample paths,
$\langle M \rangle$ = the quadratic variation of M
$M_t^* = \max_{0 \le s \le t} |M_s|$

Then for any $p > 0$ and any stopping time τ we have:

$$k_p \, E \langle M \rangle_\tau^p \le E (M_\tau^*)^{2p} \le K_p E \langle M \rangle_\tau^p$$

where k_p, K_p are universal constants (depend only on p)

- the case $p = 1$: $k_p = 1$, $K_p = 4$

Lemma 1: Let M be a local martingale with $M_0 = 0 \Rightarrow$

$$\boxed{E\langle M\rangle_t \le E(M_t^*)^2 \le 4\,E\langle M\rangle_t \quad \text{for all } t}$$

Proof: since M is local mart $\Rightarrow (\exists)\ \{\tau_n\}$ s.t. $\tau_n \uparrow \infty$ s.t.

$$\left.\begin{array}{l} M^2_{t\wedge\tau_n} - \langle M\rangle_{t\wedge\tau_n} \\ M_{t\wedge\tau_n} \end{array}\right\} \text{ are martingales } \Rightarrow$$

$$E\langle M\rangle_{t\wedge\tau_n} = E\,M^2_{t\wedge\tau_n}\ ;$$

$$E\left(M^*_{t\wedge\tau_n}\right)^2 \le 4\,E\left(M^2_{t\wedge\tau_n}\right) \quad (\text{Doob inequality})$$

$$E\langle M\rangle_{t\wedge\tau_n} \le E\left(M^*_{t\wedge\tau_n}\right)^2 \le 4\,E\left(M^2_{t\wedge\tau_n}\right) = 4\,E\langle M\rangle_{t\wedge\tau_n}$$

- let $n \to \infty$ and apply Monotone convergence theorem ◊.

$$\left(X_n \ge 0,\ \text{non decreasing} \Rightarrow \lim_{n\to\infty} E X_n = E \lim_{n\to\infty} X_n \right)$$

Lemma 2: A local martingale M is a square integrable martingale iff $E M_0^2 < \infty$ and $\langle M\rangle$ is integrable, in which case $M_t^2 - \langle M\rangle_t$ = martingale.

Proof: • no loss of generality to assume $M_0 = 0$

- $\langle M\rangle$ is integrable $\Rightarrow E(M_t^*)^2 < \infty$

$\Rightarrow M^2 - \langle M\rangle$ - local martingale dominated by

$\Rightarrow$ $M_t^2 - \langle M \rangle_t$ = local martingale dominated by

$$(M_T^*)^2 + \langle M \rangle_T \quad \text{for } t \le T$$

$\Rightarrow$ $M_t^2 - \langle M \rangle_t$ = proper martingale

(use dominated convergence theorem)

- $M_t^2 - \langle M \rangle_t$ is local martingale $\Rightarrow (\exists) \{\tau_n\}$ $\tau_n \uparrow \infty$

$$M^2_{t \wedge \tau_n} - \langle M \rangle_{t \wedge \tau_n} = \text{proper martingale}$$

$$E\left[M^2_{t \wedge \tau_n} - \langle M \rangle_{t \wedge \tau_n} \right] = 0 \quad \text{for all } t, \tau_n$$

$$\text{DCT}: \quad 0 = \lim_{n \to \infty} E\left[M^2_{t \wedge \tau_n} - \langle M \rangle_{t \wedge \tau_n} \right] = E\left[\lim_{n \to \infty} \left(\qquad \right) \right]$$

$$= E\left[M_t^2 - \langle M \rangle_t \right]$$

- Conversly, if M is square integrable mart, Doob's ineq gives: $E(M_t^*)^2 \le 4 E(M_t^2) < \infty$

$$\Rightarrow \quad E\langle M \rangle_t = E(M_t^2) < \infty$$