Delta hedging (BAP); Geometric Brownian Motion; Change of measure

Wednesday, September 14, 2011
9:04 AM

Sponsored by the National Grid Foundation nationalgrid
The power of action.

① BAP: Δ-hedging for European claims

- discounted value of the European claim is martingale
- self financing condition
- discrete stochastic integral

② GBM: definition; limit of binomial models

③ Change of measure:

- in discrete setting ⇝ BAP
- in continuous setting ⇝ shift of a normal distr.

① · at time n: $S(n) = S(0)\, u^{i} d^{n-i}$
(where $i = 0, 1, 2, \dots n$)

- contingent claim pays $C_{i,n}$ at time n
- the cost of replicating portfolio = the cost of contingent claim

$$C_{0,0} := \frac{1}{1+r}\left[C_{1,1}\, \frac{1+r-d}{u-d} + C_{0,1}\, \frac{u-1-r}{u-d} \right]$$

$$= \frac{1}{1+r}\left[C_{1,1}\, \tilde{p} + C_{0,1}\,(1-\tilde{p}) \right]$$

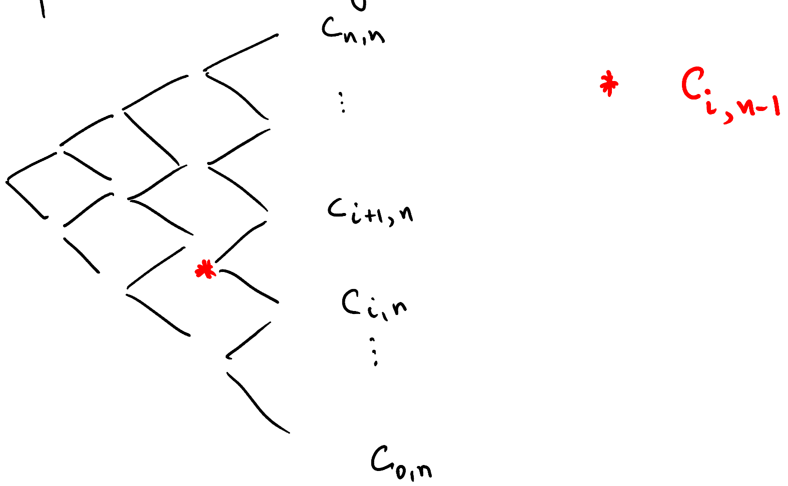
for $\tilde{p} = \dfrac{1+r-d}{u-d}$: the risk neutral prob.

· the portfolio consists of:

$$\begin{cases} \Delta_{0,0} = \dfrac{C_{1,1} - C_{0,1}}{S(u-d)} & \text{shares of stock} \\[2ex] C_{0,0} - \Delta_{0,0} S & \text{shares of MMA} \end{cases}$$

$$\left(C_{0,0} - \Delta_{0,0} S = - \frac{(C_{1,1}\, d - C_{0,1}\, u)}{(1+r)(u-d)} \right)$$

· we shall extend this argument by adjusting the replicating portfolio as you travel along the tree (that is, at every moment in time the replicating portfolio has to change to maintain the martingale prop.)



· working backwards along the tree :

$$\begin{cases} C_{i,n-1} = \frac{1}{1+r}\left[C_{i+1,n}\,\tilde{p} + C_{i,n}\,(1-\tilde{p}) \right] \\ \Delta_{i,n-1} = \dfrac{C_{i+1,n} - C_{i,n}}{S(n-1)\,(u-d)} \quad \text{nr of shares at } n-1 \end{cases}$$

$$\begin{cases} C_{i,n-2} = \frac{1}{1+r}\left[C_{i+1,n-1}\,\tilde{p} + C_{i,n-1}\,(1-\tilde{p}) \right] \\ \quad = \frac{1}{(1+r)^2}\left[C_{i+2,n}\,\tilde{p}^2 + 2\,C_{i+1,n}\,\tilde{p}(1-\tilde{p}) + C_{i,n}(1-\tilde{p})^2 \right] \\ \Delta_{i,n-2} = \dfrac{C_{i+1,n-1} - C_{i,n-1}}{S(n-2)\,(u-d)} \end{cases}$$

and so on ....

$$C_{0,0} = \frac{1}{(1+r)^n} \sum_{k=0}^{n} C_{k,n} \binom{n}{k} \tilde{p}^k (1-\tilde{p})^{n-k}$$

$\rightsquigarrow$ in general, if we call V_m the value of the replicating portfolio at time m then we have established that $\left\{ \frac{V_m}{(1+r)^m} \right\}_{0 \le m \le n}$ is a martingale with respect to the risk neutral probability.

- More formally, we have a sample space Ω which consists of all paths of length n on the binomial tree

$$w = (w_1, w_2, \dots, w_n) \in \Omega$$

$$w_i = \begin{cases} 1 & \text{if stock goes up} \\ 0 & \text{if stock goes down} \end{cases}$$

$$\tilde{P}(w) = \tilde{p}^{\sum_{i=1}^{n} w_i} (1-\tilde{p})^{n - \sum_{i=1}^{n} w_i}$$

$$V_m(w) = C_{i,m} \quad \text{if} \quad \sum_{j=1}^{m} w_j = i$$

$$\Rightarrow \quad \frac{V_m}{(1+r)^m} = \tilde{E}\left[\frac{V_{m+1}}{(1+r)^{m+1}} \,\middle|\, S(0), S(1), \dots, S(m) \right]$$

Conclusions:

(i) By the Law of One Price the time m value of a European claim with payoff vector $(C_{i,n})_{0 \le i \le n}$ is equal to V_m the cost of replicating portfolio; $0 \le m \le n$

(ii) the discounted value of such claim is a martingale under the risk neutral measure.

Self-financing condition :

$$V_m = \Delta_m S(m) + \left(V_m - \Delta_m S(m) \right)$$

- Δ_m = nr of shares of stock in the replicating portfolio held between time m and $m+1$

- at time $m+1$ the value of this portfolio

⇝ before rebalancing : $\Delta_m S(m+1) + (1+r)\left(V_m - \Delta_m S(m)\right)$

⇝ after rebalancing : $\Delta_{m+1} S(m+1) + \left(V_{m+1} - \Delta_{m+1} S(m+1)\right)$

- the self-financing condition is :

  Value before rebalancing = Value after rebalancing

$$\Delta_m S(m+1) + (1+r)\left(V_m - \Delta_m S(m)\right) = \Delta_{m+1} S(m+1) + \left(V_{m+1} - \Delta_{m+1} S(m+1)\right)$$

$$\Rightarrow \Delta_m \frac{S(m+1)}{(1+r)^{m+1}} + \frac{V_m}{(1+r)^m} - \Delta_m \frac{S(m)}{(1+r)^m} = \frac{V_{m+1}}{(1+r)^{m+1}}$$

that is : for all $0 \le m \le n-1$

$$\frac{V_{m+1}}{(1+r)^{m+1}} - \frac{V_m}{(1+r)^m} = \Delta_m \left[\frac{S(m+1)}{(1+r)^{m+1}} - \frac{S(m)}{(1+r)^m} \right]$$

- adding up these equalities

$$\Rightarrow \frac{V_n}{(1+r)^n} = V_0 + \sum_{m=0}^{n-1} \Delta_m \left[\frac{S(m+1)}{(1+r)^{m+1}} - \frac{S(m)}{(1+r)^m} \right]$$

⇝ discounded value of the claim is represented as stochastic integral with respect to the discounted stock price (discrete version of S.I)

Definition: Suppose $X = \{X_n\}_{n\geq 0}$ is a martingale with respect to $\mathbb{P}$ and the history of X. Let for each $n \geq 0$ a random variable Δ_n be a function of $X_1, \dots, X_n$.

$$Y_n := Y_0 + \sum_{m=0}^{n-1} \Delta_m (X_{m+1} - X_m) \longrightarrow Y_t = Y_0 + \int_0^t \Delta_s \, dX_s$$

where Y_0 is a constant.

We say that $\{Y_n\}_{n\geq 0}$ is a discrete stochastic integral of $\{\Delta_n\}_{n\geq 0}$ with respect to $\{X_n\}_{n\geq 0}$

Lemma: $\{Y_n\}_{n\geq 0}$ is a martingale with respect to the history of $\{X_n\}_{n\geq 0}$

Proof: $E\left[Y_{n+1} \mid X_0, \dots, X_n\right] = Y_n$ for all n.

$$E\left[Y_0 + \sum_{m=0}^{n} \Delta_m (X_{m+1} - X_m) \;\middle|\; X_0, \dots, X_n \right] =$$

$$= E\left[Y_n + \Delta_n (X_{n+1} - X_n) \;\middle|\; X_0, \dots, X_n \right]$$

$$= Y_n + \Delta_n E\left[X_{n+1} - X_n \mid X_n \right] = Y_n$$

(2) GBM.

Definition: $\{S(t)\}_{t\geq 0}$ continuous process is a Geometric Brownian Motion (μ, σ) if

(i) $\dfrac{S(t+y)}{S(t)}$ independent of all history up to time t
$(y>0)$

(ii) $\ln\left(\dfrac{S(t+y)}{S(t)}\right) \sim N(\mu y, \sigma^2 y)$

$Z \sim N(0,1)$

$$\ln\left(\frac{S(t+y)}{S(t)}\right) = \mu y + Z \cdot \sigma \cdot \sqrt{y}$$

$$\rightsquigarrow S(t+y) = S(t) \cdot \exp\{\mu y + Z \cdot \sigma\sqrt{y}\}$$

· in particular : $\boxed{S(t) = S(0)\exp\{\mu t + Z \cdot \sigma\sqrt{t}\}}$

ex : $P(S(t+2) > S(t+1) > S(t)) = ?$

$$= P\left(\frac{S(t+2)}{S(t+1)} > 1 > \frac{S(t)}{S(t+1)}\right)$$

$$= P\left(\frac{S(t+2)}{S(t+1)} > 1 \; ; \; \frac{S(t+1)}{S(t)} > 1\right)$$

independent events

$$= P\left(\frac{S(t+2)}{S(t+1)} > 1\right) \cdot P\left(\frac{S(t+1)}{S(t)} > 1\right)$$

$$\ln\left(\frac{S(t+2)}{S(t+1)}\right), \ln\left(\frac{S(t+1)}{S(t)}\right) \sim N(\mu, \sigma^2)$$

$$\rightsquigarrow E\,S(t) = \exp\left\{\mu t + \frac{\sigma^2 t}{2}\right\}$$

$$\rightsquigarrow \mathrm{Var}\, S(t) = \exp\{2\mu t + \sigma^2 t\} \cdot \left(e^{\sigma^2 t} - 1\right)$$

$\rightsquigarrow \operatorname{Var} S(t) = \exp\{2\mu t + \sigma^2 t\} \cdot \left(e^{\sigma^2 t} - 1\right)$

<u>GBM (μ, σ) as limit of BAP</u>

$h =$ small time increment

$$u = e^{\sigma\sqrt{h}} \; ; \quad d = e^{-\sigma\sqrt{h}} \; ; \quad p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\cdot\sqrt{h}\right)$$

- for $i = 0, 1, 2 \dots$ assume the process $\{S(ih)\}_{i \geq 0}$ is defined such that $S(0) > 0$ (constant) and

$$S((i+1)h) = \begin{cases} u\, S(ih) & \text{with probability } p \\ d\, S(ih) & \text{with probability } 1-p \end{cases}$$

Moreover, assume that the succesive movements are independent. Notice that the independence assumption can be rephrased as follows :

$\dfrac{S((i+k)h)}{S(ih)}$ is independent of $S(h), S(2h), \dots, S(ih)$ (for all , $k > 0$)

Define : $X_i = \begin{cases} 1 & \text{if } S((i+1)h) = u\, S(ih) \\ 0 & \text{if } S((i+1)h) = d\, S(ih) \end{cases}$

Then $\{X_i\}_{1 \leq i \leq n}$ are independent and

- $S(nh) = S(0)\, u^{\sum_{i=1}^{n} X_i}\, d^{\,n - \sum_{i=1}^{n} X_i}$

- if $t = nh \Rightarrow n = \dfrac{t}{h}$ so we can rewrith $S(nh)$ as

t/h

- if $t = nh \Rightarrow n = \frac{t}{h}$ so we can rewrite $S(nh)$ as

$$S(t) = S(0) \left(\frac{u}{d}\right)^{\sum_{i=1}^{t/h} X_i} \cdot d^{t/h}$$

$$\leadsto \log\left(\frac{S(t)}{S(0)}\right) = \frac{t}{h} \log d + \log\frac{u}{d} \left(\sum_{i=1}^{t/h} X_i\right)$$

(replace u, d by $e^{\pm\sigma\sqrt{h}}$)

$$= \frac{t}{h} \cdot \left(-\sigma\sqrt{h}\right) + 2\sigma\sqrt{h} \cdot \left(\sum_{i=1}^{t/h} X_i\right)$$

$$= -\frac{t\sigma}{\sqrt{h}} + 2\sigma\sqrt{h} \cdot \underbrace{\sum_{i=1}^{t/h} X_i}_{\sim \text{ Normally distributed (CLT)}} \cdot$$

- let's calculate $E \log \frac{S(t)}{S(0)}$, $\operatorname{Var} \log \frac{S(t)}{S(0)}$ to justify the $GBM(\mu, \sigma)$ approximation of BAP as $h \to 0$

$$E \log \frac{S(t)}{S(0)} = -\frac{t\sigma}{\sqrt{h}} + 2\sigma\sqrt{h} \quad \frac{t}{h} \cdot p$$

$$= -\frac{t\sigma}{\sqrt{h}} + 2\sigma\sqrt{h} \cdot \frac{t}{h} \cdot \frac{1}{2}\left(1 + \frac{\mu}{\sigma} \cdot \sqrt{h}\right)$$

$$= \mu t$$

$$\operatorname{Var} \log \frac{S(t)}{S(0)} = 4\sigma^2 h \cdot \frac{t}{h} \cdot p \cdot (1-p) \xrightarrow[\text{as } h \to 0]{} \sigma^2 t$$

- Therefore, as $h \to 0$: $\log \frac{S(t)}{S(0)} \sim N(\mu t, \sigma^2 t)$

which leads immediately to the conclusion that which leads immediately to the conclusion that $S(t)$ follows a $GBM(\mu, \sigma)$.

<u>Risk neutral measure for the price process that follows $GBM(\mu,\sigma)$</u>

- let $S(t)$ follow $GBM(\mu,\sigma)$.

$\rightsquigarrow$ we know that in the BAP model the risk neutral probability is given by $\tilde{p} = \frac{1+r^*-d}{u-d}$ where

r^* is the one period rate

$\rightsquigarrow$ in our approximation : $u = e^{\sigma\sqrt{h}}$; $d = e^{-\sigma\sqrt{h}}$

and the one period rate becomes $r^* = e^{\frac{rt}{n}} - 1$

$\rightsquigarrow$ let us continue to approximate u, d, r^* as follows

$$\begin{cases} u = e^{\sigma\sqrt{h}} \cong 1 + \sigma\sqrt{h} + \frac{1}{2}\sigma^2 h \\ d = e^{-\sigma\sqrt{h}} \cong 1 - \sigma\sqrt{h} + \frac{1}{2}\sigma^2 h \\ 1+r^* \simeq 1 + \frac{rt}{n} \end{cases}$$

use these approximations in the formula $\tilde{p}$

$$\tilde{p} \cong \frac{1 + \frac{rt}{n} - \left(1 - \sigma\sqrt{h} + \frac{1}{2}\sigma^2 h\right)}{2\sigma\sqrt{h}}$$

$$= \frac{1}{2}\left(1 + \frac{r - \frac{\sigma^2}{2}}{\sigma}\sqrt{\frac{t}{n}}\right)$$

also recall that while approximating BAP with GBM (μ, σ) we assumed also that

$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\frac{t}{n}}\right)$$

⇝ this implies that if we start with the discrete risk neutral process, then as $n \to \infty$ we shall obtain a GBM $\left(r - \frac{\sigma^2}{2}, \sigma\right)$

⇝ it is not surprising that if $S(t)$ follows GBM (μ, σ) then all claims are priced using $\underline{GBM\left(r - \frac{\sigma^2}{2}, \sigma\right)}$ the risk neutral GBM

Example: Price an European Call Option with
K = strike price T = maturity

⇝ payoff $= (S(T) - K)^+$

$$\text{price at time } 0 = \tilde{E}\left[e^{-rT}(S(T) - K)^+\right]$$

- model under risk neutral measure : $GBM\left(r - \frac{\sigma^2}{2}, \sigma\right)$

$$S(T) = S(0)\exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + Z \cdot \sigma\sqrt{T}\right\}$$

- $$\tilde{E}\left[e^{-rT}(S(T) - K)^+\right] = \int^{+\infty} e^{-rT}\left(\underbrace{S(0)\exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + x\,\sigma\sqrt{T}\right\}}_{} - K\right)^+ \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

ex : $$E\, f(Z) = \int_{-\infty}^{+\infty} f(x)\cdot \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\, dx$$

$Z \sim N(0,1)$

③ Change of measure in the dicrete setting.

Ω : finite sample space.

- we can define 2 different probability measures on it

1) $(\forall)$ $w \in \Omega$ set its probability to be p_w, $p_w \geq 0$; $\sum_{w\in\Omega} p_w = 1$

2) $(\forall)$ $w \in \Omega$ set its probability to be $\tilde{p}_w$; $\tilde{p}_w \geq 0$; $\sum_{w\in\Omega} \tilde{p}_w = 1$

- if $A \subseteq \Omega$ (is an event), then

$$\mathbb{P}(A) = \sum_{w\in A} p_w \quad \text{(for measure 1)} \quad \text{and}$$

$$\tilde{\mathbb{P}}(A) = \sum_{w\in A} \tilde{p}_w \quad \text{(for measure 2)}$$

- also, if $X : \Omega \to \mathbb{R}$

$$E X = \sum_{w\in\Omega} X(w)\, p_w \quad ; \quad \tilde{E} X = \sum_{w\in\Omega} X(w)\, \tilde{p}_w$$

(for measure 1) (for measure 2)

- thus, naturally, the same event will have different measures and the same variable will have different distributions

- notice that if $p_w > 0$ for all w we can always write

· notice that if $p_\omega > 0$ for all ω we can always write

$$\tilde{\mathbb{P}}(A) = \sum_{\omega \in A}' \tilde{p}_\omega \cdot \frac{p_\omega}{p_\omega} = \sum_{\omega \in A}' \frac{\tilde{p}_\omega}{p_\omega} \cdot p_\omega$$

and

$$\tilde{E}\, X = \sum_{\omega \in \Omega}' X(\omega)\, \tilde{p}_\omega = \sum_{\omega \in \Omega}' X(\omega) \frac{\tilde{p}_\omega}{p_\omega} \cdot p_\omega = E\left[X \cdot Z\right]$$

where $Z(\omega) = \frac{\tilde{p}_\omega}{p_\omega}$ for all $\omega \in \Omega$

$Z : \Omega \longrightarrow \mathbb{R}^+$ is called

RADON - NIKODYM derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$

· we will also use the following notations for the Radon-Nikodym derivative :

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \quad \text{or} \quad d\tilde{\mathbb{P}} = Z \cdot d\mathbb{P}$$

Example : Consider two n-period BAP models with the same u, d but different probabilities, call them p and $\tilde{p}$

Ω = set of all paths of length n $\qquad \omega = (\omega_1, \omega_2, \dots, \omega_n)$

1) $\mathbb{P}(\omega) = p^{\sum_{i=1}^{n} \omega_i} (1-p)^{n - \sum_{i=1}^{n} \omega_i} > 0$

2) $\tilde{\mathbb{P}}(\omega) = \tilde{p}^{\sum_{i=1}^{n} \omega_i} (1-\tilde{p})^{n - \sum_{i=1}^{n} \omega_i} > 0$

$$Z(\omega) = \frac{d\tilde{\mathbb{P}}(\omega)}{d\mathbb{P}(\omega)} = \left(\frac{\tilde{p}}{p}\right)^{\sum_{i=1}^{n} \omega_i} \left(\frac{1-\tilde{p}}{1-p}\right)^{n - \sum_{i=1}^{n} \omega_i}$$

· the random variable $\sum_{i=1}^{n}{}' \omega_i$ has binomial distribution with parameters (n, p) in case 1) and $(n, \tilde{p})$ in case 2)

Definition: The probability measure $\tilde{\mathbb{P}}$ is said to be **absolutely continuous** with respect to measure $\mathbb{P}$ if

$$\mathbb{P}(A) = 0 \implies \tilde{\mathbb{P}}(A) = 0$$

Notation: $\tilde{\mathbb{P}} \ll \mathbb{P}$ $\left(\text{or } \tilde{\mathbb{P}} \propto \mathbb{P}\right)$

Definition: If $\tilde{\mathbb{P}} \ll \mathbb{P}$ and $\mathbb{P} \ll \tilde{\mathbb{P}}$ we say that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are **equivalent**

Notation: $\tilde{\mathbb{P}} \sim \mathbb{P}$

- in our simple setting $\tilde{\mathbb{P}} \ll \mathbb{P}$ if $p_\omega = 0$ implies $\tilde{p}_\omega = 0$.
- $\tilde{\mathbb{P}} \sim \mathbb{P}$ if they completely agree on "impossible" and "sure" events.

Example: Let $\Omega = \{1,2,3,4,5\}$

$\mathbb{P}$: $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{3}$, $p_3 = \frac{1}{4}$, $p_4 = \frac{1}{6}$, $p_5 = 0$

$\tilde{\mathbb{P}}$: $\tilde{p}_1 = \frac{1}{3}$, $\tilde{p}_2 = \frac{1}{3}$, $\tilde{p}_3 = \frac{1}{3}$, $\tilde{p}_4 = \tilde{p}_5 = 0$

Questions: Is $\mathbb{P} \ll \tilde{\mathbb{P}}$? Is $\tilde{\mathbb{P}} \ll \mathbb{P}$?

- $p_4 \neq 0$, $\tilde{p}_4 = 0 \implies \mathbb{P}$ is not absolutely cont. w.r.to $\tilde{\mathbb{P}}$

$\implies \mathbb{P}$ and $\tilde{\mathbb{P}}$ are not equivalent

- on the other hand $\tilde{\mathbb{P}} \ll \mathbb{P}$ since whenever $\mathbb{P}(\omega) = 0$ $\tilde{\mathbb{P}}(\omega)$ is also 0.
- since $\tilde{\mathbb{P}} \ll \mathbb{P}$ then $\dfrac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ makes sense

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(1) = \frac{\tilde{p}_1}{p_1} = \frac{4}{3} \quad ; \quad \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(2) = 1 \quad ; \quad \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(3) = \frac{4}{3}$$

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(4) = \frac{\tilde{p}_4}{p_4} = 0$$

. notice that $\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}(4)$ is undefined

Theorem : Radon - Nikodym

let $\tilde{\mathbb{P}} \ll \mathbb{P}$ then $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ exists and for every event A

$$\tilde{\mathbb{P}}(A) = \mathbb{E}\left[1_A \cdot \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right]$$

Continuous example (Shreve $\underline{\text{II}}$ p. 37-38)

- $X \sim N(0,1)$ on $(\Omega, \mathcal{F}, \mathbb{P})$
  ($\mathcal{F}$ = set of all events)
- θ = constant $\rightsquigarrow$ define $Y = X + \theta \sim N(\theta, 1)$

on the same probability space.

Goal : Find a probability $\tilde{\mathbb{P}}$ on the same sample space so that under $\tilde{\mathbb{P}}$ the random variable Y has standard normal distribution.

Question : What should $Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ be ?

we want $\tilde{\mathbb{P}}(Y \le y) = \phi(y)$!!!

$$\tilde{\mathbb{P}}(Y \le y) = \tilde{\mathbb{E}}\left[1_{\{w : Y \le y\}} \right] \overset{R-N}{=} \mathbb{E}\left[1_{\{w : X+\theta \le y\}} \cdot Z \right]$$

we want $\tilde{\mathbb{P}}(Y \le y) = \phi(y)$!!!

$$\tilde{\mathbb{P}}(Y \le y) = \tilde{E}\left[1_{\{w: Y \le y\}}\right] \overset{R\text{-}N}{=} E\left[1_{\{w: X+\theta \le y\}} \cdot Z\right]$$

$$= E\left[1_{\{w: X \le y-\theta\}} \cdot Z\right]$$

- if Z depends on X we can writh the expectation as

$$= \int_{-\infty}^{y-\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \underbrace{f(x)}_{} \, dx$$

$\hookrightarrow$ represents $Z = f(X)$

- and we want the result to be equal to $\phi(y)$

$\leadsto$ also recall that we must have $\int_{-\infty}^{+\infty} f(x)\, dx = 1$ (as a consequence of it being a density).

$$\phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz = \int_{-\infty}^{y-\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\theta)^2}{2}} \, dx$$

- change the variable $z = x+\theta \Rightarrow dz = dx$

- to complete the example solve for:

$$\tilde{\mathbb{P}}\left[Y \le y\right] = \phi(y)$$

$$\Rightarrow \int_{-\infty}^{y-\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} f(x)\, dx = \int_{-\infty}^{y-\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\theta)^2}{2}} \, dx$$

$\leadsto$ for a good guess as to $f(x) = ?$ solve the following

$$\frac{1}{\sqrt{\quad}} e^{-\frac{x^2}{2}} f(x) = \frac{1}{\sqrt{\quad}} e^{-\frac{(x+\theta)^2}{2}}$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\theta)^2}{2}}$$

$$\Rightarrow \boxed{f(x) = e^{-x\theta - \frac{\theta^2}{2}}}$$

$\leadsto$ the R-N derivative should be : $\boxed{Z = f(X) = e^{-X\theta - \frac{\theta^2}{2}}}$

Check that it satisfies the requirement : $EZ = 1$

$$EZ = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot e^{-x\theta - \frac{\theta^2}{2}} \, d\theta = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\theta)^2}{2}} dx = 1$$

$$\Rightarrow \tilde{\mathbb{P}}(\Omega) = \tilde{E}\, 1_\Omega = E\left[1_\Omega \cdot Z\right] = E Z = 1$$

Therefore $\tilde{\mathbb{P}}$ is an actual probability measure on $(\Omega, \mathcal{F})$ and under $\tilde{\mathbb{P}}$, Y has a standard normal distribution!

- another intuitive approach is to look for $Z = \frac{d\tilde{P}}{dP}$ simply by taking the ratio of the corresponding densities under $\tilde{\mathbb{P}}$ and $\mathbb{P}$

$\leadsto$ under $\tilde{\mathbb{P}}$: $Y \sim N(0,1) \leadsto \tilde{f}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

under $\mathbb{P}$: $Y \sim N(\theta, 1) \leadsto f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2}}$

$$\Rightarrow \frac{\tilde{f}(y)}{f(y)} = e^{-\frac{y^2}{2} + \frac{(y-\theta)^2}{2}} = e^{-y\theta + \frac{\theta^2}{2}} =$$

. but recall that $y = x + \theta$

$$\Rightarrow \quad Z = \frac{\tilde{f}(X+\theta)}{f(X+\theta)} = e^{-X\theta - \frac{\theta^2}{2}}$$

(same result as before but much more intuitive)

· the reason we can do this will be more clear as we shall be able to represent $\begin{cases} d\mathbb{P} = f \cdot dm \\ d\tilde{\mathbb{P}} = \tilde{f} \cdot dm \end{cases}$

(where dm = Lebesgue measure)

$\leadsto$ we shall discuss this soon !!!