American Options

Wednesday, March 21, 2012
9:54 AM

Sponsored by the National Grid Foundation nationalgrid The power of action.

- American Options : have early exercise feature
  - ⇝ at least as valuable as the European version
  - ⇝ sometimes the difference is negligible
    (ex : American Call = European Call)
  - ⇝ for put options the early exercise premium can be substantial

- intrinsic value of A.O. = payoff associated with the immediate exercise

- in contrast to European Options the discounted price process is a supermartingale under the risk neutral measure
  - ⇝ if the holder fails to exercise at the optimal exercise date the discounted value process has a tendency to fall.
  - ⇝ during any period of time in which it is not optimal to exercise the discounted price behaves like a martingale.

like a martingale.

Stopping Times: $\tau : \Omega \longrightarrow \mathbb{R}$

$$\{\tau = t\} = \{\omega \in \Omega : \tau(\omega) = t\}$$

Def: τ = stopping time $\iff$ $\{\tau \leq t\} \in \mathcal{F}_t$ for all t

examples:

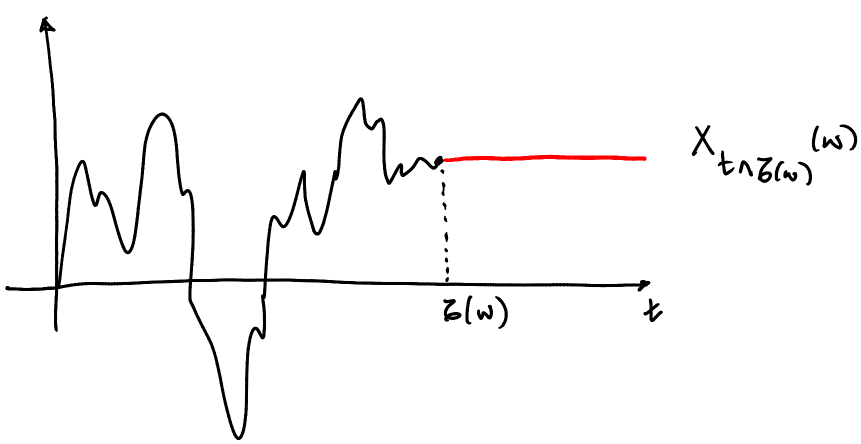
1) first passage time for a continuous process:

$$\tau_m \triangleq \min\{t \geq 0 : X_t = m\}$$ is a stopping time

2) last passage time for a continuous process

$$\rho_m \triangleq \max\{t \geq 0 : X_t = m\}$$ is NOT a stopping time

Stopped Process: $X_{t \wedge \tau} = \begin{cases} X_t & \text{if } t \leq \tau \\ X_\tau & \text{if } t > \tau \end{cases}$



Optional Sampling Theorem

A martingale stopped at a stopping time is still a martingale

A martingale stopped at a stopping time is still a martingale
A supermartingale (or submartingale) stopped at a stopping time is still a supermartingale (or submartingale)

Perpetual American Put Option

- optimal exercise policy is not obvious
- this is not a traded option however the ideas used in analyzing it are useful for more realistic situations.
- assume : $dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t$

  S_t = stock price

  $\tilde{W}_t$ = Brownian Motion under the risk-neutral measure $\tilde{\mathbb{P}}$

- the perpetual American Put pays $(K-S_t)^+$ if the option is exercised at time t.

Definition : $v_*(x) = \max_{\tau \in \mathcal{T}} \tilde{\mathbb{E}}\left[e^{-r\tau}(K-S_\tau)\right]$

= the price of the Perpetual American Put $(x = S_0)$

$\mathcal{T}$ = the family of all stopping times

Remarks :

- if $\tau = \infty$ we interpret $e^{-r\tau}(K-S_\tau) = 0$

  ⇝ if the option is never exercised the payoff is 0

$\rightsquigarrow$ if the option is never exercised the payoff is 0

- $v_*(x)$ = initial capital required for an agent to hedge a short position of the perpetual American Put regardless of the exercise strategy τ used by the owner

- as far as the perpetual option is concerned every date is like every other date , x

Strategy : Exercise the put as soon as S_t reaches the optimal level L^* !

Questions :
- What is the optimal level L^* ?
- How do we know it is optimal ?
- What is the value of the put ?

- to answer all these questions let's investigate first an arbitrary exercise rule L !

Recall some properties of the Brownian Motion with drift

$X_t = \mu t + \widetilde{W}_t$ (Brownian Motion with drift)

$\tau_m = \min\{t \geq 0 \;:\; X_t = m\}$, then

$$\boxed{\widetilde{\mathbb{E}}\left(e^{-r\tau_m}\right) = e^{-m\left(-\mu + \sqrt{\mu^2+2r}\right)}} \quad \text{for all } \lambda > 0.$$

- if $r=0$: $\tilde{\mathbb{P}}(\tau_m<\infty) = e^{m\mu - m|\mu|}$
- if $\mu>0 \Rightarrow \tilde{\mathbb{P}}(\tau_m<\infty) = 1$
- if $\mu<0 \Rightarrow \tilde{\mathbb{P}}(\tau_m<\infty) = e^{-2m|\mu|} < 1$

- suppose the owner sets a level $L<K$ and resolves to exercise the first time S_t reaches level L !

- if $S_0 \le L \rightsquigarrow$ exercise immediately :

  payoff : $K-S_0 \rightsquigarrow v_L(S_0) = K-S_0$

- if $S_0 > L \rightsquigarrow$ wait until $\tau_L = \min\{t\ge 0 : S_t = L\}$

  at time τ_L the payoff is $K - S_{\tau_L} = K-L$

$$\rightsquigarrow \boxed{v_L(S_0) = (K-L)\,\tilde{\mathbb{E}}\left(e^{-r\tau_L}\right)}$$

Lemma $v_L(x) = \begin{cases} K-x & ; \text{if } \quad 0\le x\le L \\ (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & ; \text{if } \quad x>L \end{cases}$

Proof : $S_t = x\exp\left\{\sigma\tilde{W}_t + (r-\tfrac{1}{2}\sigma^2)t\right\}$ $\qquad S_0 = x$

$$\rightsquigarrow \log\frac{S_t}{x} = \sigma\tilde{W}_t + (r-\tfrac{1}{2}\sigma^2)t$$

$$\rightsquigarrow \underbrace{-\frac{1}{\sigma}\log\frac{S_t}{x}}_{=X_t} = -\tilde{W}_t + \underbrace{\frac{-1}{\sigma}(r-\tfrac{1}{2}\sigma^2)}_{=\mu}t$$

$$\underbrace{\qquad\qquad}_{= X_t} \qquad\qquad\qquad = \mu$$

$$\tau_L = \min\{t \geq 0 : S_t = L\} =$$

$$= \min\{t \geq 0 : X_t = \underbrace{-\frac{1}{\sigma} \log \frac{L}{x}}_{m > 0 \;\rightsquigarrow\; \text{since } x > L}\}$$

· we know already that $\mathbb{E} e^{-r\tau_m} = e^{-m(-\mu + \sqrt{\mu^2 + 2r})}$

$$\mu^2 + 2r = \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 + 2r$$

$$= \frac{1}{\sigma^2}\left(r^2 - r\sigma^2 + \frac{1}{4}\sigma^4\right) + 2r$$

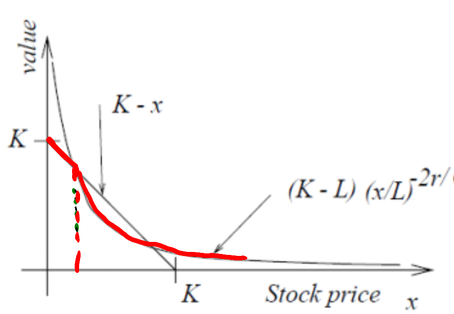
$$= \frac{r^2}{\sigma^2} + r + \frac{1}{4}\sigma^2$$

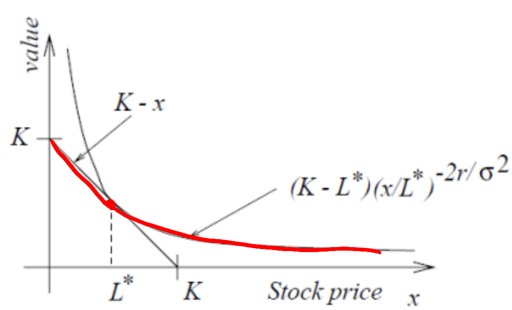
$$= \frac{1}{\sigma^2}\left(r + \frac{1}{2}\sigma^2\right)^2$$

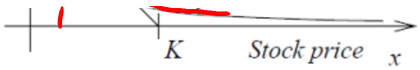
$$\Rightarrow \quad -\mu + \sqrt{\mu^2 + 2r} = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{\sigma}\left(r + \frac{1}{2}\sigma^2\right) = \frac{2r}{\sigma}$$

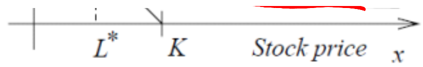
$$\Rightarrow \quad \mathbb{E} e^{-r\tau_L} = \exp\left\{-\frac{1}{\sigma}\left(\log \frac{x}{L}\right) \cdot \frac{2r}{\sigma}\right\} = \left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}$$

Optimal Exercise Rule









$$v_L(x) = \begin{cases} K - x & \text{if} \quad 0 \le x \le L \\ (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & \text{if} \quad x > L \end{cases}$$

- $g(L) \triangleq (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}$ for x fixed

$$g'(L) = -L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2}(K-L)\, L^{\frac{2r}{\sigma^2}-1}$$

$$= -\frac{2r+\sigma^2}{\sigma^2} L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2} K L^{\frac{2r}{\sigma^2}-1}$$

- solve $g'(L) = 0 \Rightarrow \boxed{L_* = \frac{2r}{2r+\sigma^2} K}$

$\rightsquigarrow$ L_* is the optimal choice

Analytical Characterization of the Perpetual Put Price

$$v_{L^*}(x) = \begin{cases} K - x & 0 \le x \le L^* \\ (K-L^*)\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}} & x > L^* \end{cases}$$

$$v'_{L^*}(x) = \begin{cases} -1 & 0 \le x < L^* \\ -(K-L^*)\frac{2r}{\sigma^2 x}\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}} & x > L^* \end{cases}$$

$$v'_{L^*}(L^*+) = -\frac{2r}{\sigma^2} \cdot \frac{1}{L^*}(K - L^*) =$$

$2r \quad 2r+\sigma^2 \quad 1 \quad (K \quad \sigma^2 \quad K) \quad = \quad 1$

$$= - \frac{2r}{\sigma^2} \cdot \frac{\quad}{\sigma^2} \cdot \frac{1}{K} \left(K - \frac{\quad}{2r+\sigma^2} K \right) = -1$$

$\rightsquigarrow$ smooth pasting

$$v''_{L^*}(x) = \begin{cases} 0 & 0 \le x < L^* \\ (K-L^*) \dfrac{2r(2r+\sigma^2)}{\sigma^4 x^2} \left(\dfrac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}} & x > L^* \end{cases}$$

$$v''(L^*-) = 0 \qquad v''(L^*+) = (K-L^*) \frac{2r(2r+\sigma^2)}{\sigma^4 L^*} > 0$$

$\rightsquigarrow$ for $x > L^*$ we verify

$$r\, v_{L^*}(x) - r x\, v'_{L^*}(x) - \frac{1}{2} \sigma^2 x^2 v''_{L^*}(x) =$$

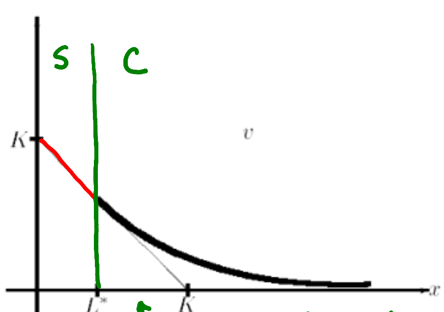
$$= (K-L^*) \left(r + \frac{2r^2}{\sigma^2} - \frac{2r^2 + r\sigma^2}{\sigma^2} \right) \left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}} = 0$$

$\rightsquigarrow$ for $x < L^*$

$$r\, v_{L^*}(x) - r x\, v'_{L^*}(x) - \frac{1}{2} \sigma^2 x^2 v''_{L^*}(x) =$$

$$= r(K-x) + rx = rK$$

$\Rightarrow$ $v_{L^*}(x)$ satisfies the so called linear complementarity conditions



continuation region

stopping region

stopping region

(1) $v(x) \geq (K-x)^+ \qquad x \geq 0$

(2) $r v(x) - r x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0 \qquad (\forall)\ x > 0.$

(3) $\rightsquigarrow$ for each $x > 0$ equality holds in either (1) or (2).

$S = \{ x \geq 0 : v_{L*}(x) = (K-x)^+ \}$ Stopping Set

$C = \{ x \geq 0 : v_{L*}(x) > (K-x)^+ \}$ Continuation Set

Probabilistic Characterization of the Perpetual Put Price

- $e^{-rt} v_{L*}(S_t)$ is a supermartingale under $\tilde{\mathbb{P}}$.
- $e^{-r(t \wedge \tau_{L*})} v_{L*}(S_{t \wedge \tau_{L*}})$ is martingale under $\tilde{\mathbb{P}}$.

Proof:

$$d\left(e^{-rt} v_{L*}(S_t)\right) = e^{-rt} d v_{L*}(S_t) + v_{L*}(S_t)\, d(e^{-rt})$$

$$= e^{-rt} v'_{L*}(S_t)\, dS_t + \frac{1}{2} e^{-rt} v''_{L*}(S_t)\, dS_t \cdot dS_t - r e^{-rt} v_{L*}(S_t)\, dt$$

$$= e^{-rt} \Big[\underbrace{-r v_{L*}(S_t) + r S_t v'_{L*}(S_t) + \frac{1}{2}\sigma^2 S_t^2 v''_{L*}(S_t)}_{\leq 0} \Big] dt \; +$$

$$+ e^{-rt} \sigma S_t v'_{L*}(S_t)\, d\tilde{W}_t$$

$\rightsquigarrow \quad e^{-rt} v_{L^*}(S_t)$ is $\tilde{\mathbb{P}}$-supermartingale

- the dt term is either 0 or $-rk$ depending on whether $S_t > L^*$ or $S_t < L^*$

- $$d\left(e^{-rt} v_{L^*}(S_t)\right) = -e^{-rt} rK \mathbb{1}\{S_t < L^*\}\, dt + e^{-rt} \sigma S_t\, v'_{L^*}(S_t)\, d\tilde{W}_t$$

- $$d\left(e^{-r(t\wedge \tau_{L^*})} v_{L^*}(S_{t\wedge\tau_{L^*}})\right) = e^{-r(t\wedge\tau_{L^*})} \sigma S_{t\wedge\tau_{L^*}}\, v'(S_{t\wedge\tau_{L^*}})\, d\tilde{W}_t$$

<u>Corollary</u>: $\displaystyle v_{L^*}(x) = \max_{\tau\in\mathcal{T}} \tilde{\mathbb{E}}\left[e^{-r\tau}(K-S_\tau)\right]$

<u>Proof</u>: • because $e^{-rt} v_{L^*}(S_t)$ is $\tilde{\mathbb{P}}$-supermartingale $\xRightarrow{\text{Optional Sampling Th}}$

$$v_{L^*}(S_0) \geq \tilde{\mathbb{E}}\left[e^{-r(t\wedge\tau)} v_{L^*}(S_{t\wedge\tau})\right] \quad \text{for all } \tau\in\mathcal{T}$$

• let $t\to\infty$, using Dominated Convergence Th

$$v_{L^*}(S_0) \geq \tilde{\mathbb{E}}\left[e^{-r\tau} v_{L^*}(S_\tau)\right] \geq \tilde{\mathbb{E}}\left[e^{-r\tau}(K-S_\tau)\right]$$

$$\Rightarrow \quad v_{L^*}(S_0) \geq \max_{\tau\in\mathcal{T}} \tilde{\mathbb{E}}\left[e^{-r\tau}(K-S_\tau)\right]$$

• on the other hand if we use $\tau = \tau_{L^*}$ we have

$$v_{L^*}(S_0) = (K-L^*)\, \mathbb{E}\, e^{-r\tau_{L^*}} = \tilde{\mathbb{E}}\left[e^{-r\tau_{L^*}}(K-S_{\tau_{L^*}})\right]$$

$\Rightarrow$

$$\Rightarrow \quad v_{L^*}(x) = \max_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-r\tau}(K-S_\tau)\right] \quad ; \quad S_0 = x.$$

Hedging the Perpetual Put

- let S_0 be given
- sell the put at time 0 for $v(S_0)$
- the value of the portfolio X_t :

$$dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t)\,dt - C_t\,dt$$

or in the discounted form

$$d(e^{-rt}X_t) = e^{-rt}\Delta_t \sigma S_t \, d\widetilde{W}_t - e^{-rt} C_t \, dt$$

- the discounted value of the put satisfies :

$$d\left(e^{-rt} v_{L^*}(S_t)\right) = e^{-rt} v_{L^*}'(S_t)\sigma S_t \, d\widetilde{W}_t - e^{-rt} rk \, \mathbb{1}_{\{S_t < L^*\}} \, dt$$

- therefore : $C_t = rk \, \mathbb{1}_{\{S_t < L^*\}}$

$$\Delta_t = v_{L^*}'(S_t)$$

- <u>Remark</u> : if $S_t < L^*$ then $v_{L^*}(S_t) = k - S_t$

$$\leadsto \Delta_t = v_{L^*}'(S_t) = -1$$

- so to hedge the put when $S_t < L^*$ short one share of stock and hold k in the money market.

- as long as the owner does not exercise you can consume the interest from the money market position

$$C_t = rK\,1_{\{S_t < L^*\}}$$

- the three linear complementarity conditions have counterparts that can be stated probabilistically for $V_t = v(S_t)$

(1) $V_t \geq (K-S_t)^+$ for all $t \geq 0$

(2) $e^{-rt} V_t$: $\tilde{\mathbb{P}}$- supermartingale

(3) there exists a stopping time τ_* such that

$$V_0 = \tilde{\mathbb{E}}\left[e^{-r\tau_*} (K - S_{\tau_*})^+ \right]$$

Remark : condition (3) can be replaced by the following observation :

- The process $V_t = v(S_t)$ is the smallest such that conditions (1) & (2) are satisfied

- this means that if there is any other process Y_t such that $Y_t \geq (K-S_t)^+$ & $e^{-rt} Y_t$ is $\tilde{\mathbb{P}}$ supermart then we have also $Y_t \geq V_t$.

Proof : - let Y_t be such a process satisfying (1) & (2)

Case 1 : if $S_0 \leq L^*$

$\Rightarrow$ $Y_0 \geq (K-S_0)^+ = v(S_0) = V_0$

Case 2 : if $S_0 > L^*$

$$\Rightarrow \quad Y_0 \geq \tilde{\mathbb{E}}\left[e^{-r(t\wedge \tau_{L^*})} Y_{t\wedge \tau_{L^*}} \right] \qquad \text{let } t \to \infty$$

$$\Rightarrow \quad Y_0 \geq \mathbb{E}\left[e^{-r\tau_{L^*}} \underbrace{Y_{\tau_{L^*}}}_{\geq (K - S_{\tau_{L^*}})^+ = (K - L^*)} \right] \geq (K - L^*)\tilde{\mathbb{E}}\, e^{-r\tau_{L^*}} = v(S_0) = V_0$$