Introduction and Course Overview

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Goal: learn the language of probability theory necessary to understand and work with basic financial models

Basic Processes $\rightsquigarrow$ Random Walks (discrete time)
$\rightsquigarrow$ Poisson Process (continuous time)

Motivation for Random Walks

- Binomial asset pricing model = exponentiated RW
- scaled RW $\rightsquigarrow$ Brownian Motion
- Brownian Motion is the driving process for more general diffusions (i.e. Geometric Brownian Motion)

Motivation for Poisson Process

- widely used in insurance and credit risk modelling
- it can be used to model interest rates
- adding a version of a Poisson process to a diffusion allows for modelling price processes with jumps

Plan:

- basic properties of these processes (RW, PP)
- use them as main examples when talking about such notions as Markov Property, Filtrations, Stopping times, Martingales
- most of the class will be devoted to basic notions

and concepts of real analysis necessary for understanding

$\underbrace{\text{Ito calculus, Girsanov th, Feynman - Kac formula}}_{\text{second semester}}$

Random Walks.

$X_1, X_2, \dots$ indep, identically distr. r.v.

Define : $W_0 = 0$

$$W_n = \sum_{j=1}^{n} X_j$$

$W = \{W_n\}_{n \geq 0}$ is called a random walk.

Properties :

(a) Process $W = \{W_n\}$ has independent increments.

Namely, for each $m \in \mathbb{N}$ and

$0 = t_0 < t_1 < t_2 \dots < t_m$

$W_{t_1} = W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, $\dots$, $W_{t_m} - W_{t_{m-1}}$

random variables , independent

example : $t_1 = 2$ $t_2 = 5$ $t_3 = 7$

$W_{t_1} = X_1 + X_2$

$W_{t_2} = X_1 + X_2 + X_3 + X_4 + X_5$

$W_{t_3} = X_1 + X_2 \dots + X_7$

$W_{t_1} = X_1 + X_2$

$W_{t_2} - W_{t_1} = X_3 + X_4 + X_5$

$W_{t_3} - W_{t_2} = X_6 + X_7$

(b) Process $W = \{W_n\}_{n\in\mathbb{N}}$ has stationary increments

$$\underbrace{W_{t+s} - W_t}_{\sum_{i=t+1}^{t+s} X_i} \overset{d}{=} \underbrace{W_s}_{\sum_{i=1}^{s} X_i} \qquad \text{for all } t, s \geq 0$$

<u>Strong Law of Large Numbers</u>

$X_1, X_2, \ldots$ iid $\quad E|X_1| < \infty \quad , \quad \mu = EX_1$

$W_n = \sum_{i=1}^{n} X_i$

Then with prob. 1 $\quad \dfrac{W_n}{n} \xrightarrow{a.s.} \mu$

<u>Central Limit Theorem</u>

$X_1, X_2, \ldots$ iid $\quad E X_1^2 < \infty \quad \mu = EX_1, \ \sigma^2 = \operatorname{Var} X_1$

$W_n = \sum_{i=1}^{n} X_i \quad$, for each $x \in \mathbb{R}$

$$P\left(\frac{W_n - n\mu}{\sigma\sqrt{n}} \leq x \right) \xrightarrow{\text{as } n\to\infty} \underset{\text{standard normal c.d.f.}}{\phi(x)}$$

$W_n \sim N(n\mu, n\sigma^2)$ as n is large enough

<u>Remark</u> : standard def of Brownian Motion

$B = (B_t)_{t\geq 0}$ continuous paths

(i) $B_0 = 0$

(ii) independent increments

(iii) for $t, s \geq 0$

$$B_{t+s} - B_t \sim N(0, s) \sim B_s$$

<u>Calculations</u> : Assume $E|X_1| < \infty$ and the following path is given

$$W_0 = 0, \; W_1 = w_1, \; W_2 = w_2, \; \ldots, \; W_n = w_n$$

- we need to calculate:

(i) $P(W_{n+1} \leq x \mid W_0 = 0, W_1 = w_1, \ldots, W_n = w_n) = ?$

(ii) $E(W_{n+1} \mid W_0 = 0, W_1 = w_1, \ldots, W_n = w_n) = ?$

(i) $P(W_{n+1} \leq x \mid W_0 = 0, W_1 = w_1, \ldots, \boxed{W_n = w_n}) =$

$= P(W_n + X_{n+1} \leq x \mid W_0 = 0, W_1 = w_1, \ldots, W_n = w_n)$

$= P(w_n + X_{n+1} \leq x \mid W_0 = 0, W_1 = w_1, \ldots, W_n = w_n)$

$= P(X_{n+1} \leq x - w_n \mid W_0 = 0, W_1 = w_1, \ldots, W_n = w_n)$

$= P(X_{n+1} \leq x - w_n) = F_{X_{n+1}}(x - w_n) = F_X(x - w_n)$

Markov property. (loss of memory)

(ii) Martingale property

(ii) Martingale property

$$E\left(W_{n+1} \mid W_0 = w_0, W_1 = w_1, \dots, W_n = w_n\right)$$

$$= E\left(W_{n+1} \mid W_n = w_n\right) = E\left(W_n + X_{n+1} \mid W_n = w_n\right)$$

Markov prop.

$$= w_n + \underbrace{E X_{n+1}}_{= \mu}$$

more generally

$$E\left(W_{n+1} \mid W_n\right) = W_n + \mu$$

- if $\mu = 0$: $E\left(W_{n+1} \mid W_n\right) = W_n$
  martingale property (fair game)
- if $\mu > 0$: $E\left(W_{n+1} \mid W_n\right) > W_n$
  sub martingale
- if $\mu < 0$: $E\left(W_{n+1} \mid W_n\right) < W_n$
  super martingale

- $\mu = 0 \rightsquigarrow$ RW is martingale.

$$\underbrace{E\left[E\left(W_{n+1} \mid W_n\right)\right]}_{E(W_{n+1})} = E\left[W_n\right]$$

$$E\left(W_{n+1}\right) = E\left(W_n\right)$$

Simple Random Walk : $X_1, X_2, \dots$ iid.

$$p = P(X_1 = 1) \qquad p + q = 1$$
$$q = P(X_1 = -1)$$

$$\mu = EX_1 = p \cdot 1 + q(-1) = p + p - 1$$
$$= 2p - 1$$

(Obs : for $p = \frac{1}{2}$ $\rightsquigarrow$ RW is martingale)

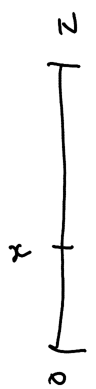
① <u>Gambler's ruin problem</u>.

W_n = wealth at time n $= \sum_{i=1}^{n} X_i$

x = initial wealth

N = target (Goal) !

What is the probability that he reaches his goal ?



Method : conditioning on the first step.

P_i = prob that starting with i dollars the gambler reaches his goal.

$P_0 = 0$, $P_N = 1$

- start with i dollars , after 1 game

$$i \begin{cases} i+1 & \text{with prob } p \\ i-1 & \text{with prob } q \end{cases}$$

$$P_i = p \cdot P_{i+1} + q \, P_{i-1} \qquad i = 1, 2, \ldots, N-1$$

$$P_{i+1} - P_i = \frac{q}{p}\left(P_i - P_{i-1}\right)$$

$$P_2 - P_1 = \frac{q}{p}\left(P_1 - P_0\right) = \frac{q}{p}\cdot P_1$$

$$P_3 - P_2 = \frac{q}{p}\left(P_2 - P_1\right) = \left(\frac{q}{p}\right)^2 \cdot P_1$$

$$\vdots$$

$$P_N - P_{N-1} = \quad \cdots \quad = \left(\frac{q}{p}\right)^{N-1} \cdot P_1$$

$$P_N - P_1 = P_1\left[\frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{N-1}\right]$$

$$P_i - P_1 = P_1\left[\frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{i-1}\right]$$

$$\Rightarrow \quad P_i = \begin{cases} \dfrac{i}{N} & \text{if } p = q \\[2ex] \dfrac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^N} & \text{if } p \neq q \end{cases}$$

<u>First passage times</u> (examples of stopping times)

Let $m \in \mathbb{Z}$ and

$$\tau_m = \min\{n \geq 0 : W_n = m\}$$

$\rightsquigarrow$ the first time the R.W reaches m

$\rightsquigarrow$ we set $\tau_m = \infty$ if the R.W never reaches m

In the gambler's ruin problem the number of games played, τ is random and can be expressed as

$\tau_0 \wedge \tau_N := \min\{\tau_0, \tau_N\}$

(we think of one game as one unit of time)

- informally speaking, a random variable τ taking values $0, 1, 2, \dots, \infty$ is a **stopping time** iff for all $n = 0, 1, 2, \dots$ the event $\{\tau \le n\}$ depends only on the RW path up to time n; i.e we can say whether $\tau \le n$ or not simply by looking at $W_0, W_1, \dots, W_n$

- examples of random times that are **NOT** stopping times

  $\rightsquigarrow$ $\rho_0 = \max\{n \geq 0 : W_n = 0\}$

  i.e the last time $W_n = 0$

  (how can you tell when the last time will occur if you can't see in the future of the random walk.)

Reflection Principle for the Simple Symmetric R.W and the distribution of τ_1

Questions :

(i) • $P(\tau_1 \le 2j-1) = ?$ (for $j = 1, 2, \dots$)

(since one can reach 1 only in an odd nr of steps we don't need to consider the even values)

(ii) • $P(\tau_1 < \infty) = ?$

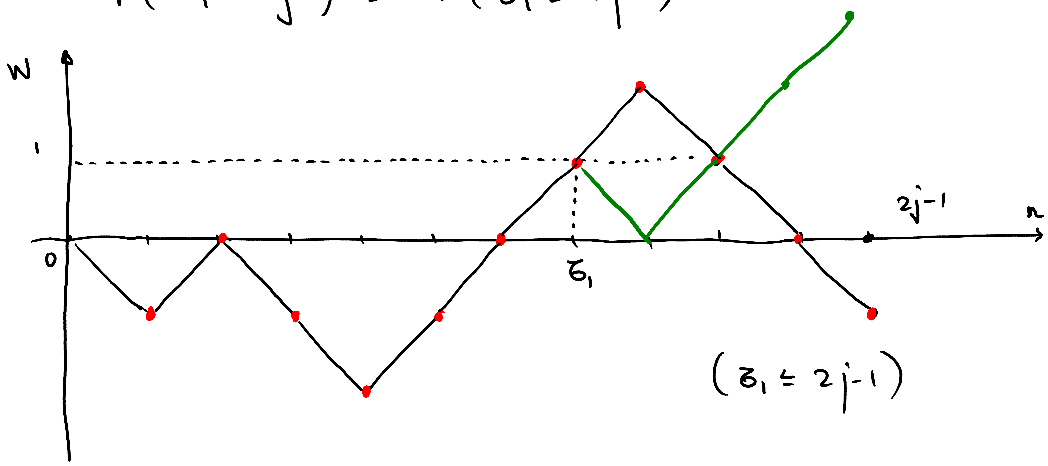
(ii) $\cdot \quad P(\tau_1 < \infty) = ?$

(iii) $\cdot \quad E\,\tau_1 = ?$

(i) we start with $P(\tau_1 \le 2j-1)$ for $j = 1, 2, \dots$

· again we worry about odd numbers only since

$$P(\tau_1 \le 2j) = P(\tau_1 \le 2j-1)$$



Consider a path that reaches 1 before time $2j-1$

From time τ_1 we construct a "reflected path"

(symmetric with the original one with respect to the line $w=1$) ($\rightsquigarrow$ the green path)

$$P(\tau_1 \le 2j-1) = \frac{\text{\# of paths that hit 1 by time } 2j-1}{2^{2j-1}}$$

$$\{\tau_1 \le 2j-1\} = \{\tau_1 \le 2j-1 \,;\, W_{2j-1} > 1\} \cup$$

$$\cup \{\tau_1 \le 2j-1 \,;\, W_{2j-1} < 1\} \cup \{\tau_1 \le 2j-1 \,,\, W_{2j-1} = 1\}$$

The reflection principle states that each path in $\{\tau_1 \le 2j-1 \,;\, W_{2j-1} > 1\}$ corresponds to exactly one path in $\{\tau_1 \le 2j-1 \,;\, W_{2j-1} < 1\}$, namely the reflected path.

Thus

$$P(\tau_1 \le 2j-1) = P(\tau_1 \le 2j-1,\ W_{2j-1} = 1) + 2P(\tau_1 \le 2j-1;\ W_{2j-1} > 1)$$

but $\{W_{2j-1} > 1\} \subset \{\tau_1 \le 2j-1\}$

and $\{W_{2j-1} = 1\} \subset \{\tau_1 \le 2j-1\}$

So

$$P(\tau_1 \le 2j-1) = 2P(W_{2j-1} > 1) + P(W_{2j-1} = 1)$$

$$= P(W_{2j-1} > 1) + P(W_{2j-1} < -1) + P(W_{2j-1} = 1)$$

$$= 1 - P(W_{2j-1} = -1)$$

but $P(W_{2j-1} = -1) = \dfrac{\binom{2j-1}{j}}{2^{2j-1}}$

also $P(\tau_1 = 2j-1) = P(\tau_1 \le 2j-1) - P(\tau_1 \le 2j-3)$

$$= \frac{\binom{2j-3}{j-1}}{2^{2j-3}} - \frac{\binom{2j-1}{j}}{2^{2j-1}} = \frac{1}{2^{2j-1}} \frac{(2j-3)!}{(j-1)!\,j!} \times \left[4j(j-1) - (2j-1)(2j-2)\right]$$

$$= \frac{(2j-3)!}{2^{2j-1}(j-1)!\,j!}\left(\cancel{4j^2} - \cancel{4j} - \cancel{4j^2} + 2j + \cancel{4j} - 2\right)$$

$$= \frac{(2j-2)!}{j!\,(j-1)!} \cdot \frac{1}{2^{2j-1}}$$

(ii) Since we allow τ_1, in principle to take value ∞ we need to worry about $P(\tau_1 < \infty)$

- we already know that

$$P(z_1 \le 2j-1) = 1 - \frac{(2j-1)!}{j!\,(j-1)!} \frac{1}{2^{2j-1}}$$

$$\{z_1 < \infty\} = \bigcup_{j=1}^{\infty} \underbrace{\{z_1 \le 2j-1\}}_{= A_j}$$

- notice that $A_j \subset A_{j+1}$ for $j \ge 1$
- in such cases we say $A_j \uparrow$

  (increasing sequence of sets)

- look at $A_1, A_2 \setminus A_1, A_3 \setminus A_2, \dots$

  these sets are disjoint and their union is still $\{z_1 < \infty\}$ hence

$$P(z_1 < \infty) = \sum_{j=1}^{\infty} P(A_{j+1} \setminus A_j) + P(A_1)$$

$$= \lim_{n \to \infty} \Big(\underbrace{\sum_{j=1}^{n} P(A_{j+1} \setminus A_j) + P(A_1)}_{P(A_n)} \Big)$$

$$= \lim_{n \to \infty} P(A_n)$$

$$= \lim_{n \to \infty} \left[1 - \frac{(2n-1)!}{n!\,(n-1)!} \frac{1}{2^{2n-1}} \right]$$

[Stirling formula](#)

Using the Stirling formula (see the above link) we see that

$$\frac{(2n-1)!}{n!\,(n-1)!} \cdot \frac{1}{2^{2n-1}} \simeq \frac{\left(\frac{2n-1}{e}\right)^{2n-1} \sqrt{2\pi(2n-1)}}{\left(\frac{n}{e}\right)^{n} \left(\frac{n-1}{e}\right)^{n-1} \sqrt{2\pi n}\,\sqrt{2\pi(n-1)}} \frac{1}{2^{2n-1}}$$

$$= \frac{2^{2n-1}\left(n-\frac{1}{2}\right)^{2n-1}}{2^{2n-1}\, n^n\, (n-1)^{n-1}} \cdot \frac{\sqrt{2n-1}}{\sqrt{2\pi}\,\sqrt{n}\,\sqrt{n-1}}$$

$$= \left(\frac{n-\frac{1}{2}}{n}\right)^n \left(\frac{n-\frac{1}{2}}{n-1}\right)^{n-1} \frac{\sqrt{2n-1}}{\sqrt{2\pi}\,\sqrt{n}\,\sqrt{n-1}}$$

$$= \left(1-\frac{1}{2n}\right)^n \left(1+\frac{1}{2(n-1)}\right)^{n-1} \frac{\sqrt{2n-1}}{\sqrt{2\pi}\,\sqrt{n}\,\sqrt{n-1}}$$

$$\simeq \frac{1}{\sqrt{e}}\,\sqrt{e}\,\frac{1}{\sqrt{\pi}\,\sqrt{n}} = \frac{1}{\sqrt{\pi n}} \longrightarrow 0 \quad \text{as } n \to \infty$$

therefore $P(\tau_1 < \infty) = 1$

Remark: We have also proved that if $A = \bigcup_{j=1}^{\infty} A_j$ and $A_j \uparrow$ then $P(A) = \lim_{j\to\infty} P(A_j)$

(iii) Finally we shall try to find $E\tau_1$

$$E\tau_1 = \sum_{j=1}^{\infty} (2j-1)\, P(\tau_1 = 2j-1)$$

· much easier however is to approach this calculati[on] using the following property of expected value

Lemma: let $X \in \{0,1,2,\dots\}$ Then

$$EX = \sum_{n=0}^{\infty} P(X > n)$$

Proof: $EX = \sum_{n=1}^{\infty} n\, P(X=n)$

$= P(X=1) + 2P(X=2) + 3P(X=3) + \cdots$

$$\begin{aligned} = P(X=1) &+ P(X=2) + P(X=3) + \cdots \\ &+ P(X=2) + P(X=3) + \cdots \\ &\qquad\quad + P(X=3) + \cdots \end{aligned}$$

$= P(X>0) + P(X>1) + P(X>2) + \cdots$

$= \sum_{n=0}^{\infty} P(X>n)$

So $\quad E z_1 = \sum_{n=0}^{\infty} P(z_1 > n)$

$P(z_1 > 2j) = P(z_1 > 2j-1) = 1 - P(z_1 \le 2j-1)$

$\Rightarrow E z_1 = P(z_1 > 0) + \sum_{j=1}^{\infty} P(z_1 > 2j) + \sum_{j=1}^{\infty} P(z_1 > 2j-1)$

$= 1 + 2\sum_{j=1}^{\infty} P(z_1 > 2j-1)$

$= 1 + 2\sum_{j=1}^{\infty} \underbrace{\frac{(2j-1)!}{j!\,(j-1)!}\,\frac{1}{2^{2j-1}}}_{\simeq \frac{1}{\sqrt{\pi j}}}$

. we use Stirling approximation again, hence.

$E z_1 = 1 + 2\sum_{j=1}^{\infty} \frac{1}{\sqrt{\pi j}} = \infty$

<u>Conclusion</u>:

$P(z_1 < \infty) = 1$ however $E z_1 = \infty$