MTH 9821 Numerical Methods for Finance I
Lecture 2

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1 Cholesky Decomposition

Definition:

Given a symmetric positive definite nonsingular matrix A, there is a unique upper triangular matrix U with positive entries on the main diagonal such that

$$A = U^t U$$

Remark:

$\star$ In Cholesky decomposition, we try to solve $\frac{n(n+1)}{2}$ unknowns from $\frac{n(n+1)}{2}$ equations (upper triangular part).

$\star$ In LU decomposition, we try to solve n^2 unknowns from n^2 equations.

Lemma: A nonsingular symmetric matrix A has Cholesky decomposition, then A is spd.

Proof:

It's enough to show that $x^t A x > 0, \ \forall x \neq 0$.

Let U be a nonsingular upper triangular matrix such that $A = U^t U$.

$$x^t A x = x^t U^t U x = (Ux)^t U x = ||Ux||^2 \geq 0 \ \forall x \text{ with } Ux \neq 0$$

Note: $Ux = 0 \iff x = 0$ since U is nonsingular.

Lemma: If A is spd, then there exists an upper triangular nonsingular matrix U such that $A = U^t U$ (and $U(i,i) > 0, \ \forall i = 1:n$.

We can proof by solving $A = U^t U$ for a 4×4 example.

$$\underbrace{\begin{pmatrix} A(1,1) & A(1,2) & A(1,3) & A(1,4) \\ A(2,1) & A(2,2) & A(2,3) & A(2,4) \\ A(3,1) & A(3,2) & A(3,3) & A(3,4) \\ A(4,1) & A(4,2) & A(4,3) & A(4,4) \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} U(1,1) & 0 & 0 & 0 \\ U(1,2) & U(2,2) & 0 & 0 \\ U(1,3) & U(2,3) & U(3,3) & 0 \\ U(1,4) & U(2,4) & U(3,4) & U(4,4) \end{pmatrix}}_{U^t} \underbrace{\begin{pmatrix} U(1,1) & U(1,2) & U(1,3) & U(1,4) \\ 0 & U(2,2) & U(2,3) & U(2,4) \\ 0 & 0 & U(3,3) & U(3,4) \\ 0 & 0 & U(4,3) & U(4,4) \end{pmatrix}}_{U}$$

$U(1,1)^2 = A(1,1) \implies U(1,1) = \sqrt{A(1,1)}$

$\left(\text{Since A is spd, } A(1,1) = e_1^t A e_1 > 0\right)$

$U(1,k) = \frac{A(1,k)}{U(1,1)} \ \forall k = 1:4 \implies U(1,2:4) = \frac{A(1,2:4)}{U(1,1)} = \frac{A(1,2:4)}{\sqrt{A(1,1)}}$

Denote $U(2:4,2:4) = U_1$,we have

$U_1^t U_1 = A(2:4,2:4) - \big(U(1,2:4)\big)^t U(1,2:4)$
$\quad = A(2:4,2:4) - \frac{A(1,2:4)^t}{\sqrt{A(1,1)}}\frac{A(1,2:4)}{\sqrt{A(1,1)}} = \frac{A(1,2:4)^t A(1,2:4)}{A(1,1)}$
$\Big(U_1^t U_1 = A(2:n,2:n) - \frac{A(1,2:n)^t A(1,2:n)}{A(1,1)}\Big)$
In order to show that the matrix U such that $U^tU = A$ exists, it's equal to show that the matrix $U_1^tU_1$ is spd, given A is spd.
Proof is shown in Class Handout Lemma 11.4.

Pseudocode: *Cholesky Decomposition*

Input:

A = nonsingular symmetric positive definite matrix of size n

Output:

U = upper triangular matrix

such that $U^tU = A$

for $i = 1:(n-1)$
  $U(i,i) = \sqrt{A(i,i)};$
  for $k = i+1:n$
    $U(i,k) = \frac{A(i,k)}{U(i,i)};$
  end
  for $j = (i+1):n$
    for $k = \mathbf{j}:n$
      $A(j,k) = A(j,k) - U(i,j)U(i,k);$
    end
  end
end
$U(n,n) = \sqrt{A(n,n)}$

⋆ Operation Count of Cholesky Decomposition: $\frac{n^3}{3} + O(n^2)$
⋆ Operation saving comes from $k = j:n$, only need to update the upper triangular part.

$$
\begin{aligned}
\textit{Operation Count} &= \sum_{i=1}^{n-1}\left((n-i) + \sum_{j=i+1}^{n}\sum_{k=j}^{n} 2\right) \\
&= \sum_{i=1}^{n-1}\left((n-i) + \sum_{j=i+1}^{n} 2(n-j+1)\right) \\
(\textit{Let } p = n-j+1) &= \sum_{i=1}^{n-1}\left((n-i) + \sum_{p=1}^{n-i} p\right) \\
&= \sum_{i=1}^{n-1}\big[(n-1) + (n-i)^2 + (n-i)\big] \\
(\textit{Let } q = n-i) &= \sum_{q=1}^{n-1} q^2 + 2q \\
&= \frac{(n-1)n(2n-1)}{6} + 2\cdot\frac{n(n-1)}{2} \\
&= \frac{2n^3}{6} + O(n^2) = \frac{n^3}{3} + O(n^2)
\end{aligned}
$$

Uniqueness: Cholesky decomposition of a spd matrix is unique.
Proof is similar to the uniqueness proof of LU decomposition.

2 Use of Cholesky Decomposition

Given a matrix A,

$$A \begin{cases} \text{symmetric} & \longrightarrow \begin{cases} \mathit{Success} \\ \mathit{Fail} \longrightarrow \text{LU with row pivoting} \end{cases} \\ \text{asymmetric} & \longrightarrow \text{LU with row pivoting} \end{cases}$$

2.1 Using Cholesky Decompostion for Solving Linear System

Solve $\mathbf{Ax} = \mathbf{b}$

$\mathbf{A}$: $n \times n$ spd matrix given;

$\mathbf{b}$: $n \times 1$ vector given;

Find $\mathbf{x}$: $n \times 1$ vector

Thus,

$$\begin{aligned} Ax &= b \\ \iff \quad U^tUx &= b \\ \iff \quad U^ty = b, \quad Ux &= y \end{aligned}$$

$\mathbf{x} = \textbf{linear\_solve\_LU\_cholesky}\ (\mathbf{A}, \mathbf{b})$

$$\begin{aligned} U &= Cholesky(A) \\ y &= forward\_subst(U^t,\ b) \\ x &= backward\_subst(U,\ y) \\ (x &= backward\_subst(U,\ forward\_subst(U^t,\ b))) \end{aligned}$$

2.2 Solve p Linear Systems

$Ax_i = b_i,\ \forall i = 1:p$

$$\begin{aligned} U &= Cholesky(A); \\ \text{for } i &= 1:p \\ y &= forward\_subst(U^t,\ b_i) \\ x_i &= backward\_subst(U,\ y) \\ \text{end} & \end{aligned}$$

Remark:
⋆ Implement Cholesky before the for loop
⋆ Operation count: $2pn^2 + \frac{n^3}{3} + O(n^2)$

2.3 Least Square

A is an $m \times n$ matrix with $m > n$, it has linearly independent colunns.
Least square solution:

$$\text{Solve} \quad \underbrace{A^tA}_{m\times m\ spd\ matrix} \quad x = A^tb$$

$$\min ||b - Ax||, \quad x \in \mathbb{R}^n$$

$$\mathbf{x} = \textbf{linear\_solve\_cholesky}(\mathbf{A^tA}, \mathbf{A^ty})$$

2.4 Find Normal Random Sample with Correlating

- Find normal random variables $X_1, X_2, \ldots, X_n$ with spd covariance matrix A.

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = U^t \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}$$

  - Linear Transformation Property: $Y = MX \;\Rightarrow\; \Sigma_Y = MM^t\Sigma_X$.

  - n=2 case:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \Rightarrow \Sigma_X = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = \Omega_X$$

- Given samples $Z_1, \ldots, Z_{2N}$ of Z, the pairs $\left(Z_{2i+1}, \rho Z_{2i+1} + \sqrt{1-\rho^2}Z_{2i+2}\right)$, $i = 0 : (N-1)$ are independent and with correlation.

  - Remark: This can be used for Monte Carlo simulation. Brownian motion cannot be used here because random variables generated have no correlation.

3 Cholesky Decomposition for Banded Matrices

A is a banded matrix of band m iff $A(j,k) = 0 \;\forall |j-k| > m$.
Lemma: A is spd of band $m \Rightarrow$ Cholesky factor U of A is also banded with band m. e.g., 8×8 matrix with band 3

$$\begin{pmatrix}
x & x & x & x & 0 & 0 & 0 & 0 \\
x & x & x & x & x & 0 & 0 & 0 \\
x & x & x & x & x & x & 0 & 0 \\
x & x & x & x & x & x & x & 0 \\
0 & x & x & x & x & x & x & x \\
0 & 0 & x & x & x & x & x & x \\
0 & 0 & 0 & x & x & x & x & x \\
0 & 0 & 0 & 0 & x & x & x & x
\end{pmatrix}$$

Remark:
⋆ In the above 8×8 matrix, only shaded entries need to be updated in this step.
⋆ m^2 entries need to be updated

Pseudocode: *Cholesky Decomposition for banded matrices*

Input:

A = nonsingular symmetric positive definite banded matrix of banded m and size n

Output:

U = upper triangular matrix

such that $U^tU = A$

for $i = 1 : (n-1)$
  $U(i,i) = \sqrt{A(i,i)};$
  for $k = i+1 : \min(i+m, n)$
    $U(i,k) = \frac{A(i,k)}{U(i,i)};$
  end
  for $j = (i+1) : \min(i+m, n)$
    for $k = \mathbf{j} : \min(i+m, n)$
      $A(j,k) = A(j,k) - U(i,j)U(i,k);$
    end
  end
end
$U(n,n) = \sqrt{A(n,n)}$

Operation Count: nm^2

4 Efficient Decomposition of a Tridiagonal Matrix

$$A = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} x & x & 0 & 0 & 0 & 0 \\ x & \boxed{x} & x & 0 & 0 & 0 \\ 0 & x & x & x & 0 & 0 \\ 0 & 0 & x & x & x & 0 \\ 0 & 0 & 0 & 0 & x & x \end{pmatrix}$$

Tridiagonal: only one entry needed to be updated, i.e., no difference for Cholesky and LU. But in LU, we have an advantage: 1's on main diagonal of L.

Method 1: $\mathbf{x} = \textbf{linear\_solve\_cholesky\_tridiagonal\_spd}(\mathbf{A}, \mathbf{b})$

$$\begin{aligned} U &= Cholesky(A) \\ y &= forward\_subst(U^t,\ b) \\ x &= backward\_subst(U,\ y) \\ (x &= backward\_subst(U,\ forward\_subst(U^t,\ b))) \end{aligned}$$

Pseudocode: *Cholesky Decomposition for Solving spd Tridiagonal Linear System*

(Cholesky Decomposition)

for $i = 1 : (n-1)$

$U(i,i) \ = \ \sqrt{A(i,i)};$

$U(i,i+1) \ = \ \frac{A(i,i+1)}{U(i,i)};$

$A(i+1,i+1) \ = \ A(i+1,i+1) \ - \ U(i,i+1)^2;$

end

$U(n,n) \ = \ \sqrt{A(n,n)}$

(Forward Substitution)

$y(1) = \frac{b(1)}{U(1,1)};$

for $i = 2 : n$

$y(i) = \frac{b(i) - U(i-1,i)y(i-1)}{U(i,i)}$

end

(Backward Substitution)

$x(n) = \frac{y(n)}{U(n,n)};$

for $i = (n-1) : 1$

$x(i) = \frac{y(i) - U(i,i+1)x(i+1)}{U(i,i)}$

end

Operation Count: $4n + 3n + 3n = 10n + O(1)$

Method 2: $\mathbf{x} = \mathbf{linear\_solve\_LU\_no\_pivoting}(\mathbf{A}, \mathbf{b})$

$$
\begin{aligned}
[L, U] &= lu(A) \\
y &= forward\_subst(L,\ b) \\
x &= backward\_subst(U,\ y) \\
(x &= backward\_subst(U,\ forward\_subst(L,\ b)))
\end{aligned}
$$

Remark:: We can do LU without pivoting for spd tridiagonal matrices, since all eigenvalue is greater than 0, then all leading principal minor is greater than 0.

Pseudocode: *LU Decomposition for Solving spd Tridiagonal Linear System*

(LU Decomposition no Pivoting)

for $i = 1 : (n-1)$

$\quad U(i,i) = A(i,i)$, $U(i,i+1) = A(i,i+1)$;

$\quad L(i+1,i) = \frac{A(i+1,i)}{U(i,i)}$, $L(i,i) = 1$;

$\quad A(i+1,i+1) = A(i+1,i+1) \ - \ L(i+1,i)U(i,i+1)$;

end

$U(n,n) = A(n,n)$, $L(n,n) = 1$;

(Forward Substitution)

$y(1) = \frac{b(1)}{L(1,1)} = b(1)$;

for $i = 2 : n$

$\quad y(i) = \frac{b(i) - L(i,i-1)y(i-1)}{L(i,i)} = b(i) - L(i,i-1)y(i-1)$

end

(Backward Substitution)

$x(n) = \frac{y(n)}{U(n,n)}$;

for $i = (n-1) : 1$

$\quad x(i) = \frac{y(i) - U(i,i+1)x(i+1)}{U(i,i)}$

end

Operation Count: $3n + 2n + 3n = 8n + O(1)$

Method 3: Direct Calculation of Cholesky Factor

For

$$B_N = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

Cholesky factor

$$\begin{aligned} U_N(i,i) &= \sqrt{\frac{i+1}{i}}; \\ U_N(i,i+1) &= -\sqrt{\frac{i}{i+1}} = -\frac{1}{U_N(i,i)}; \end{aligned}$$

⋆ Operation Count for Direct Calculating Cholesky Factor $3n + n = 4n$

⋆ Operation Count for Method 3 $4n + 3n + 3n = 10n + O(1)$, i.e., no saving than implementing Cholesky decomposition