Exotic Options & Feynman-Kac formula

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- for path dependent options 1-dim F-K does not work since the value process is not markovian anymore.

examples: Asian Option, Lookback (floating strike) Option.
Barrier Option (up-and-out)

(1) Asian Option $V_T = \left(\frac{1}{T}\int_0^T S_t\,dt - K\right)^+$ where

- $dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t$ ($\tilde{W}_t$ = BM under $\tilde{\mathbb{P}}$)

- K = strike price, T = expiration time

Remark: V_T depends on the path of S_t up to time T so it is not a Markov process so we can no longer say $V_t = v(t, S_t)$ because we need to consider the whole path of S up to the current time.

- define: $Y_t = \int_0^t S_u\,du \rightsquigarrow dY_t = S_t\,dt$

- the pair (S_t, Y_t) is given by a pair of SDE:

$$\begin{cases} dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t \\ dY_t = S_t\,dt \end{cases}$$

- Y_t alone is not Markov because it depends on S_t, however

- Y_t alone is not Markov because it depends on S_t, however the pair (S_t, Y_t) is a 2-dim Markov process.

- $V_t = \tilde{\mathbb{E}}\left[e^{-r(T-t)} V_T \,\middle|\, \mathcal{F}_t \right] = \tilde{\mathbb{E}}\left[e^{-r(T-t)} \left(\frac{1}{T} Y_T - K\right)^+ \,\middle|\, \mathcal{F}_t \right]$

$$\underset{\substack{\uparrow \\ \text{2-dim Markov property}}}{=} v(t, S_t, Y_t)$$

- terminal condition becomes : $v(T, x, y) = \left(\frac{y}{T} - K\right)^+$

- $e^{-rt} v(t, S_t, Y_t)$: martingale (easy)

- $d\left(e^{-rt} v(t, S_t, Y_t)\right) = e^{-rt} dv(t, S_t, Y_t) + v(t, S_t, Y_t)\, d(e^{-rt})$

$$= e^{-rt}\Big(v_t(t, S_t, Y_t)\,dt + v_x(t, S_t, Y_t)\, dS_t + v_y(t, S_t, Y_t)\, dY_t +$$

$$+ \frac{1}{2} v_{xx}(t, S_t, Y_t)\, dS_t \cdot dS_t + v_{xy}(t, S_t, Y_t) \underbrace{dS_t \cdot dY_t}_{=0} + \frac{1}{2} v_{yy} \underbrace{dY_t \cdot dY_t}_{=0} \Big)$$

$$- r v(t, S_t, Y_t) e^{-rt} dt$$

$$= e^{-rt}\Big(-r v(t, S_t, Y_t) + v_t(t, S_t, Y_t) + r S_t v_x(t, S_t, Y_t) + S_t v_y(t, S_t, Y_t) +$$

$$+ \frac{1}{2} \sigma^2 S_t^2 v_{xx}(t, S_t, Y_t) \Big)\, dt + e^{-rt} \sigma S_t v_x(t, S_t, Y_t)\, d\tilde{W}_t$$

- set the dt term equal to 0!

$$\Rightarrow v_t(t, S_t, Y_t) + r S_t v_x(t, S_t, Y_t) + S_t v_y(t, S_t, Y_t) + \frac{1}{2}\sigma^2 S_t^2 v_{xx}(t, S_t, Y_t) = r v(t, S_t, Y_t)$$

- replace $S_t = x$, $Y_t = y$

the PDE becomes

$$v_t(t, x, y) + r x\, v_x(t, x, y) + x\, v_y(t, x, y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x, y) = r v(t, x, y)$$

The PDE becomes

$$v_t(t,x,y) + rx\, v_x(t,x,y) + x\, v_y(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = r\, v(t,x,y)$$

with the boundary conditions

$$\begin{cases} v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T} - K\right)^+ \\ \lim\limits_{y \downarrow -\infty} v(t,x,y) = 0 \qquad 0 \le t \le T, \quad x \ge 0 \\ v(T,x,y) = \left(\frac{y}{T} - K\right)^+ \end{cases}$$

Remark:

$$d\left(e^{-rt} v(t,S_t,Y_t)\right) = e^{-rt}\sigma S_t\, v_x(t,S_t,Y_t)\, d\tilde{W}_t$$

- recall the equation of wealth invested in the stock + money market: (replicating portfolio)

$$d\left(e^{-rt} X_t\right) = e^{-rt}\sigma S_t \Delta_t\, d\tilde{W}_t$$

time 0 for $v(0,S_0,Y_0)$

we use this as the initial capital for a hedging portfolio

$$X_0 = v(0,S_0,Y_0)$$

- at time t, we use the portfolio process:

$$\Delta_t = v_x(t,S_t,Y_t)$$

then we will have perfect replication:

$$d\left(e^{-rt} X_t\right) = d\left(e^{-rt} v(t,S_t,Y_t)\right)$$

$$\rightsquigarrow \quad X_T = V_T = v(T,S_T,Y_T) = \left(\frac{1}{T} Y_T - K\right)^+$$

② Lookback Options (floating strike)

$$V_T = \left(\max_{0 \le t \le T} S_t\right) - S_T$$

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$$Y_t = \max_{0 \le u \le t} S_u \quad \Rightarrow \quad V_T = Y_T - S_T$$

$$V_t = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} (Y_T - S_T) \,\middle|\, \mathcal{F}_t \right] = v(t, S_t, Y_t)$$

- we do not have an explicit SDE for the process Y_t however it can be proven that (S_t, Y_t) is a 2-dim Markov process.

- moreover : $dY_t \cdot dY_t = 0$, $dY_t \cdot dS_t = 0$

(Y_t is not a Lebesgue integral , as a matter of fact Y_t increases over time on a set of zero Lebesgue meas)

$$d\left(e^{-rt} v(t, S_t, Y_t)\right) = e^{-rt}\, dv(t, S_t, Y_t) + v(t, S_t, Y_t)\, d(e^{-rt})$$

$$= e^{-rt}\left(v_t\, dt + v_x\, dS_t + v_y\, dY_t + \tfrac{1}{2} v_{xx}\, dS_t \cdot dS_t \right) - r e^{-rt} v\, dt$$

$$= e^{-rt}\left(\overbrace{-rv + v_t + v_x r S_t + \tfrac{1}{2} v_{xx} \sigma^2 S_t^2}^{=0} \right) dt +$$

$$+ e^{-rt} v_x \sigma S_t\, d\widetilde{W}_t + \underbrace{e^{-rt} v_y\, dY_t}_{=0}$$

- Remark : we do not have $dY_t = \theta_t\, dt$ so we must consider this term separately !

- $e^{-rt} v(t, S_t, Y_t)$ = martingale $\Rightarrow$

$$\begin{cases} -r v(t, S_t, Y_t) + r S_t v_x(t, S_t, Y_t)) + \tfrac{1}{2} \sigma^2 S_t^2 v_{xx}(t, S_t, Y_t) = r v(t, S_t, Y_t) \\ v_y(t, S_t, Y_t)\, dY_t = 0 \end{cases}$$

$\Rightarrow$ the PDE becomes :

$$v_t(t,x,y) + rx\,v(t,x,y) + \frac{1}{2}\sigma^2 x^2 v(t,x,y) = r\,v(t,x,y)$$

where $\begin{cases} 0 \le t \le T \\ 0 \le x \le y \end{cases}$

- boundary conditions : $\begin{cases} v(t,0,y) = e^{-r(T-t)}\,y \\ v_y(t,y,y) = 0 \\ v(T,x,y) = y - x \; ; \; 0 \le x \le y \end{cases}$

Barrier Options : (Up-and-Out Call)

- stock price follows GBM (r,σ) under $\widetilde{\mathbb{P}}$ (risk neutral measure)

$$dS_t = rS_t + \sigma S_t\, d\widetilde{W}_t \quad : \quad S_t = S_0 \exp\left\{\sigma \widetilde{W}_t + \left(r - \tfrac{1}{2}\sigma^2\right)t\right\}$$

- T = expiration time
- K = strike price ; B = barrier $(K < B)$
- payoff : $\begin{cases} (S_T - K)^+ & \text{if } S_T \le B \\ 0 & \text{otherwise} \end{cases}$

Notation : $\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$

$$\widehat{W}_t = \widetilde{W}_t + \alpha t \quad \text{(Brownian Motion with drift under } \widetilde{\mathbb{P}}\text{)}$$

$$\rightsquigarrow S_t = S_0\, e^{\sigma \widehat{W}_t}$$

- define $\widehat{M}_T = \max_{0 \le t \le T} \widehat{W}_t \Rightarrow \max_{0 \le t \le T} S_t = S_0\, e^{\sigma \widehat{M}_T}$

- $V_T = (S_T - K)^+ \mathbb{1}_{\{\max_{0 \le t \le T} S_t \le B\}}$

$$= (S_T - K)\, \mathbb{1}_{\{S_0 e^{\sigma \widehat{W}_T} \ge K,\; S_0 e^{\sigma \widehat{M}_T} \le B\}}$$

$$= (S_T - K)\, \mathbb{1}\{S_0 e^{\sigma \widehat{W}_T} \geq K\ ,\ S_0 e^{\sigma \widehat{M}_T} \leq B\}$$

$$= (S_T - K)\, \mathbb{1}\{\widehat{W}_T \geq k\ ,\ \widehat{M}_T \leq b\}$$

where $k = \frac{1}{\sigma} \log \frac{K}{S_0}$ $\quad b = \frac{1}{\sigma} \log \frac{B}{S_0}$

- we would like to use Markov property to describe the price of this option :

$$V_t = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} V_T \,|\, \mathcal{F}_t \right]$$

- however, V_T depends on the whole path $(\widehat{M}_T)$ so we cannot use Markov property right away.

- to fix this problem define the following stopping time :

$$\boxed{\rho = \min\{ t \geq 0 \ :\ S_t = B\}}$$: Time of the Knock-out !!!

$\rightsquigarrow$ clearly $\begin{cases} S_t < B & \text{for all } t < \rho \\ S_\rho = B \end{cases}$

$$\rightsquigarrow \quad e^{-r(t \wedge \rho)} V_{t\wedge\rho} = \begin{cases} e^{-rt} V_t & \text{if } 0 \leq t \leq \rho \\ e^{-r\rho} V_\rho & \text{if } \rho < t \leq T \end{cases}$$

is a $\widetilde{\mathbb{P}}$-martingale (since $e^{-rt} V_t$ is $\widetilde{\mathbb{P}}$ martingale)

<u>Theorem</u> : $V_t = v(t, S_t)$ for all $0 \leq t \leq \rho$

<u>in particular</u> $e^{-rt} v(t, S_t)$ is a $\widetilde{\mathbb{P}}$ martingale up to time ρ.

$\rightsquigarrow$ for $t \leq \rho$

- $d\left(e^{-rt} v(t, S_t)\right) = e^{-rt}\left[-rv + v_t + rS_t v_x + \frac{1}{2}\sigma^2 S_t^2 v_{xx}\right] dt +$

$$+ \; e^{-rt} \sigma S_t v_x \, d\tilde{W}_t$$

- set the dt term $=0$, replace S_t with a dummy x

$$v_t(t,x) + rx\, v_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = r\,v(t,x)$$

with the following terminal condition

$$v(t,0) = 0 \qquad 0 \le t \le T$$

$$v(t,B) = 0 \qquad 0 \le t < T$$

$$v(T,x) = (x-K)^+ , \qquad 0 \le x \le B$$