Multivariate Stochastic Calculus

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Def: d-dimensional Brownian Motion is a process $W_t = (W_t^1, W_t^2, \dots, W_t^d)'$ with the following prop:

(i) each W_t^i is a Brownian Motion

(ii) if $i \neq j \Rightarrow W_t^i$ and W_t^j are independent

- associated with a d-dim Brownian Motion we have a filtration $\{\mathcal{F}_t\}$ such that

(iii) (Information accumulates) : $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$

(iv) (Adaptivity) : W_t is $\mathcal{F}_t$ measurable for all t

(v) (Independence of future increments) : for all $t \leq u$
$W_u - W_t$ is independent of $\mathcal{F}_t$

(i) $\rightsquigarrow$ $dW_t^i \cdot dW_t^i = dt$

(ii) $\rightsquigarrow$ $dW_t^i \cdot dW_t^j = 0$ for $i \neq j$
(we need to justify this claim)

Proof: $\underbrace{0 = t_0 < t_1 < \dots < t_n = T}_{\Pi}$ partition of $[0,T]$

- the sample cross-variation:

$$C_\Pi = \sum_{k=0}^{n-1} \left[W_{t_{k+1}}^i - W_{t_k}^i \right] \cdot \left[W_{t_{k+1}}^j - W_{t_k}^j \right]$$

$$\cdot \ E\, C_\Pi = \sum_{k=0}^{n-1} E\left[W_{t_{k+1}}^i - W_{t_k}^i \right] \cdot E\left[W_{t_{k+1}}^j - W_{t_k}^j \right] = 0$$

$$\cdot \ \text{Var}\, C_\Pi = E\, C_\Pi^2 = E \sum_{k=0}^{n-1} \left(W_{t_{k+1}}^i - W_{t_k}^i\right)^2 \left(W_{t_{k+1}}^j - W_{t_{k+1}}^j\right)^2$$

$$= \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \leq \|\Pi\| \cdot T \xrightarrow[\text{as } \|\Pi\| \to 0]{} 0$$

$$\left.\begin{array}{l} E\,C_\Pi = 0 \\ \mathrm{Var}\,C_\Pi \longrightarrow 0 \end{array}\right\} \Rightarrow C_\Pi \longrightarrow 0 \quad \text{as } \|\Pi\| \to 0$$

Itô-Doeblin Formula for Multiple Processes

X_t, Y_t : Itô processes $\qquad d=2 \qquad W_t = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \in \mathbb{R}^2$

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t \theta_u^1\,du + \int_0^t \sigma_u^{11}\,dW_u^1 + \int_0^t \sigma_u^{12}\,dW_u^2 \\ Y_t = Y_0 + \displaystyle\int_0^t \theta_u^2\,du + \int_0^t \sigma_u^{21}\,dW_u^1 + \int_0^t \sigma_u^{22}\,dW_u^2 \end{cases}$$

- in vector form :

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \int_0^t \theta_u\,du + \int_0^t \sigma_u\,dW_u$$

$$\theta_t = \begin{pmatrix} \theta_t^1 \\ \theta_t^2 \end{pmatrix} \in \mathbb{R}^2 \qquad \sigma_t = \begin{pmatrix} \sigma_t^{11} & \sigma_t^{12} \\ \sigma_t^{21} & \sigma_t^{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

$$d\langle X\rangle_t = dX_t \cdot dX_t = (\sigma_t^{11})^2\,dt + (\sigma_t^{12})^2\,dt$$

$$d\langle Y\rangle_t = dY_t \cdot dY_t = (\sigma_t^{21})^2\,dt + (\sigma_t^{22})^2\,dt$$

$$d\langle X, Y\rangle_t = dX_t \cdot dY_t = \left(\sigma_t^{11}\cdot\sigma_t^{21} + \sigma_t^{12}\cdot\sigma_t^{22}\right)dt$$

Two-dimensional Itô formula : $\quad f(t,x,y)$ is $\mathcal{C}^2$:

$f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ are defined & continuous

$$\boxed{\begin{aligned} df(t,X_t,Y_t) &= f_t(t,X_t,Y_t)\,dt + f_x(t,X_t,Y_t)\,dX_t + f_y(t,X_t,Y_t)\,dY_t \\ &\quad + \tfrac{1}{2} f_{xx}(t,X_t,Y_t)\,d\langle X\rangle_t + f_{xy}(t,X_t,Y_t)\,d\langle X,Y\rangle_t + \tfrac{1}{2} f_{yy}(t,X_t,Y_t)\,d\langle Y\rangle_t \end{aligned}}$$

example : $f(x,y) = x \cdot y \Rightarrow f_x = y$, $f_y = x$

$f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = 1$

$$\boxed{d(X_t \cdot Y_t) = Y_t \, dX_t + X_t \, dY_t + d\langle X, Y \rangle_t}$$

stochastic version of the product rule

example : Correlated Stock Prices

- $\dfrac{dS^1_t}{S^1_t} = \alpha_1 \, dt + \sigma_1 \, dW^1_t \qquad \Longleftrightarrow \quad dS^1_t = \alpha_1 S^1_t \, dt + \sigma_1 S^1_t \, dW^1_t$

- $\dfrac{dS^2_t}{S^2_t} = \alpha_2 \, dt + \sigma_2 \left[\underbrace{\rho \, dW^1_t + \sqrt{1-\rho^2} \, dW^2_t}_{= \, dW^3_t} \right]$

where W^1 and W^2 are independent B.M

$\sigma_1 \geq 0$, $\sigma_2 \geq 0$ $\qquad -1 \leq \rho \leq 1$

- to analyze the second stock price, define :

$$\boxed{W^3_t = \rho \, W^1_t + \sqrt{1-\rho^2} \, W^2_t} \quad \text{martingale}$$

- is W^3 also a Brownian Motion also ?

$\rightsquigarrow$ we need to check if $d\langle W^3 \rangle_t = dt$

$$d\langle W^3 \rangle_t = dW^3_t \cdot dW^3_t = \rho^2 \, dt + (1-\rho^2) \, dt = dt$$

$\Rightarrow$ W^3_t is also a Brownian Motion.

So we can represent S^2 as :

$$\frac{dS^2_t}{S^2_t} = \alpha_2 \, dt + \sigma_2 \, dW^3_t \qquad (\text{GBM})$$

- $dW^1_t W^3_t = \underbrace{W^1_t \, dW^3_t}_{\text{martingale}} + \underbrace{W^3_t \, dW^1_t}_{\text{martingale.}} + \underbrace{dW^1_t \cdot dW^3_t}_{\rho dt}$

t. $\qquad$ t

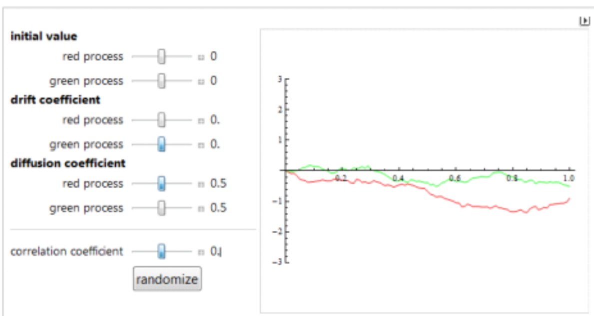
$$W^1_t W^3_t = \int_0^t W^1_s \, dW^3_s + \int_0^t W^3_s \, dW^1_s + \rho t$$

$$\Rightarrow E\, W^1_t W^3_t = \rho t \qquad \operatorname{corr}(W^1_t, W^3_t) = \frac{E(W^1_t \cdot W^3_t)}{t} = \rho$$

ρ = correlation between W^1 and W^3

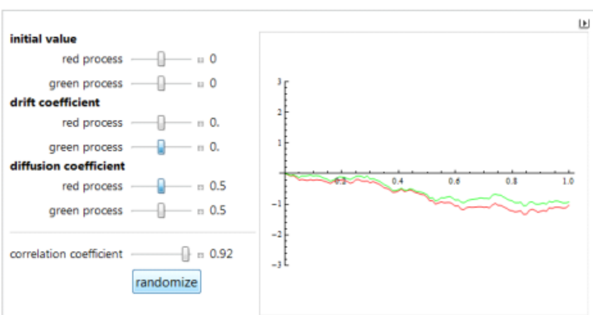
[Correlated Wiener Processes (Wolfram Demonstrations Project)]()

Correlated Wiener Processes



This Demonstration displays the paths of two correlated Wiener processes.

Correlated Wiener Processes



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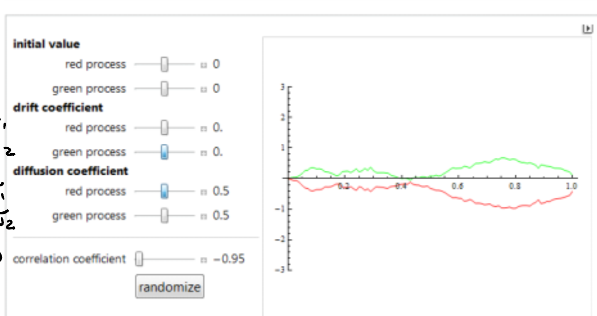
Correlated Wiener Processes

α_1 α_2 σ_1 σ_2 ρ



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