Fundamental Theorems

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2:02 PM

- today we shall expand on the theme
  "Brownian Motion is the fundamental martingale with continuous sample paths"

① **Theorem**: Let M be a continuous local martingale with $\langle M\rangle_\infty = \infty$. There exists a Brownian Motion W, s.t.

$$\boxed{M_t = W_{\langle M\rangle_t}}$$

Proof (Sketch)

- suppose $\langle M\rangle$ is strictly increasing $\rightsquigarrow$ $\langle M\rangle$ has an inverse
  (as a function of t: $t \longmapsto \langle M\rangle_t$)

$\rightsquigarrow$ let's call T: the inverse function of $\langle M\rangle$
(also continuous, increasing)

- it is not too hard to see that: $W_s = M_{T(s)}$
  is a martingale (with respect to the filtration $\mathcal{G}_s = \mathcal{F}_{T(s)}$)
  with continuous sample paths

- $\langle W\rangle_s = \langle M\rangle_{T(s)} = s$ $\rightsquigarrow$ W_s is a Brownian Motion

- furthermore: $W_{\langle M\rangle} = M_{T(\langle M\rangle)} = M_t$

- furthermore : $W_{\langle M\rangle_t} = M_{T(\langle M\rangle_t)} = M_t$

② **Theorem** : M = local martingale with cont. sample paths and quadratic variation of the form $\langle M\rangle_t = \int_0^t X_s^2\,ds$ for some adapted process X. Then M admits the representation :

$$M_t = M_0 + \int_0^t X_s\,dW_s$$

for some Brownian Motion W.

Proof : (homework)

- hint : if X takes values in $\mathbb{R}\setminus\{0\}$ one can use

$$W_t \overset{\Delta}{=} \int_0^t \frac{1}{X_s}\,dM_s$$

③ **Theorem** : Let W be a Brownian Motion, $\mathcal{F}_t^W = \sigma(W_s : 0\le s\le t)$ (the Brownian Motion filtration). For every local martingale M with respect to the filtration $\{\mathcal{F}_t^W\}$ admits a representation of the form

$$\boxed{M_t = M_0 + \int_0^t X_s\,dW_s} \qquad t\ge 0$$

for a measurable process X which is adapted to $\{\mathcal{F}_t^W\}$ and satisfies $\int_0^T X_s^2\,ds < \infty$ a.s.

In particular, every such M has continuous sample paths.

Remark : If M happends to be a square integrable martingale

Remark: If M happends to be a square integrable martingale then X can be chosen to satisfy $\mathbb{E}\int_0^T X_s^2\, ds < \infty$.