PPr701 Project Report

on

Plasma Instabilities in Hall Thrusters

submitted by

Jwalit N. Panchal

Roll no: P0211522

Under the Guidance of

Prof. Bhooshan Paradkar

Department of Physical Science

Centre for Excellence in Basic Sciences(UM-DAE CEBS)

Mumbai- 400 098

November 2024

Nurturing talent...promoting excellence

मौलिक विज्ञान प्रकर्ष केन्द्र

CERTIFICATE

This is to certify that CEBS student **Jwalit Nalin Panchal** has undertaken project work from 5^{th} August, 2024 to 30^{th} November 2024, under the guidance of ~~Prof.~~/**Dr. Bhooshan Paradkar**, School of Physical Science, UM-DAE CEBS.

This submitted project report titled **Plasma Instabilities in Hall Thrusters** is towards the academic requirements of the M.Sc. program's 7^{th} Semester Project Course at UM-DAE CEBS.

Jwalit Panchal

Name and Signature of Student with Date 30/11/2024

Bhooshan Paradkar 30/11/2024

Name and Signature of Advisor with Date

Acknowledgments

The following project would not have been possible without the priceless inputs from and the many interactions with my project guide, Prof. Bhooshan Paradkar. His constant guidance, making sure I reach certain bench-marks, has proved vital for the completion of this work. I thank him from the bottom of my heart for introducing me to such a wonderful and rich research-area.

I would also like to thank my mother, father and brother for supporting me throughout, invariably.

Abstract

The efficiency of hall-thrusters can be greatly advanced by tackling various plasma-instabilities that arise in it. It is found that effective electron-transport in Hall-thruster plasma is more than expected from a pure collisional theory, due to these instabilities. These lead to energy losses that could otherwise be used to ionize more of the propellant gas; thus producing stronger thrust. In the following report, attempt is made at identifying the dominant perturbation wave-modes for two such instabilities, namely Ion Acoustic Instability (IAW) and Modified Two-stream Instability (MTSI). A detailed derivation of a general 3-D dispersion relation (DR) is presented; after this its special cases are shown. MTSI's dominant modes' variation with temperature is presented in detail. This DR is solved numerically, through iterative procedure to identify regions of stability.

Contents

1 Introduction

Definition- A quasi-neutral gas of charged and neutral particles, which exhibits collective behavior is called a Plasma. [1]

1.1 Collective Behavior

Plasma is ionized gas, it contains ions, electrons and neutrals. Do not the ions and electrons recombine to give neutrals? The thermal energy of ions and electrons is much greater than the attractive potential energy between ions and electrons. This thermal energy prevents ions from capturing electrons. This is expressed mathematically as:

$$\frac{e\Phi}{K_B T} \ll 1 \tag{1}$$

1.2 Debye Shielding

To make sense of the above statement, one should understand *quasi-neutrality*. In that vein let us do the following analysis. Plasma consists of ions, electrons and neutrals. At this system's scale of length, it can be regarded as a neutral system. There is a chance that local regions of charge concentration arise within the system spontaneously, or an external electric-potential is supplied (using whatever means possible, like immersing electrode within the plasma bulk); in either case an electric-field will appear. Will this $\tilde{\mathbf{E}}$ be permitted within all of the bulk, or will it be confined to the locality from which it arose?

Due to this $\tilde{\mathbf{E}}$ exterior charges will rush to form a sheath around the region. Denote by m_i mass of ion and by m_e mass of an electron. Generally $\frac{m_e}{m_i} << 1$, so it is reasonable to suppose that the sheath around the region of $\tilde{\mathbf{E}}$, will be formed by either the **deficit or excess** of electrons for positive or negative electric-potential respectively. Consider a 1-D case. Denote by $\mathbf{n_i}$ and $\mathbf{n_e}$ the density of ions and electrons respectively. Denote by $\phi(x)$ the electric-potential. Whenever $\phi \to 0$, $n_e \to n_\infty$ where, n_∞ is density of ions, which is fixed, since they do not move.

At thermal-equilibrium, the distribution function for species with charge q will be:

$$f(u) = Ae^{-\frac{mu^2+q\phi}{KT_q}} \ , \ where \ \mathbf{A} = \sqrt{\frac{\mathbf{m}}{\mathbf{2\pi KT}}} \tag{2}$$

Integrate (2) over all velocities for electrons, i.e., q is now $-e$:

$$n_e = A\int_{-\infty}^{\infty} e^{-\frac{mu^2}{KT_e}} e^{\frac{e\phi}{KT_e}} du \implies n_e = n_\infty e^{\frac{e\phi}{KT_e}} \ , \ since \ A\int_{-\infty}^{\infty} e^{-\frac{mu^2}{KT_e}} = 1 \tag{3}$$

Consider at $x = 0$ an imbalance in charges equal to $-e(n_i - n_e)$, then Poisson's equation is:

$$\triangle\phi = -\frac{e}{\epsilon_0}(n_i - n_e) \tag{4}$$

In 1-D, (4) with (3) and $n_i = n_\infty$ becomes:

$$\frac{d^2\phi}{dx^2} = \frac{e \cdot n_\infty}{\epsilon_0}(e^{\frac{e\phi}{KT_e}} - 1) \tag{5}$$

For $\frac{e\phi}{KT_e} << 1$, $e^{\frac{e\phi}{KT_e}} = 1 + \frac{e\phi}{KT_e} + \frac{1}{2!}\left(\frac{e\phi}{KT_e}\right)^2 + \ldots$

$$\implies \frac{d^2\phi}{dx^2} = \frac{e \cdot n_\infty}{\epsilon_0}(\frac{e\phi}{KT_e}) \tag{6}$$

$$\implies \frac{d^2\phi}{dx^2} = \frac{e^2 \cdot n_\infty}{\epsilon_0 \cdot KT_e} \cdot \phi \tag{7}$$

(7) is like an eigenvalue equation, with the scalar $\frac{\boldsymbol{\epsilon_0 \cdot KT_e}}{\mathbf{e^2 \cdot n_\infty}}$ the square of characteristic length of the system. Denote by λ_D this characteristic length, then:

$$\lambda_D = \sqrt{\frac{\epsilon_0 \cdot KT_e}{e^2 \cdot n_\infty}} \tag{8}$$

So, the $\tilde{\mathbf{E}}$ will die-out within a length of λ_D. If length of the system $L >> \lambda_D$, then most the bulk will not have any electric-field, and $\phi \approx 0$, thus $n_e \approx n_i \approx n_\infty$.

1.3 Plasma Frequency

Consider the following image:

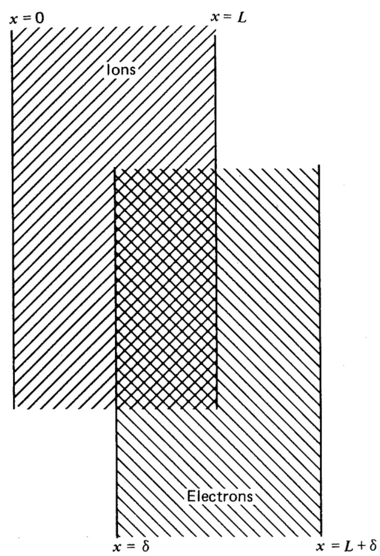


Figure 1: [2] A slab of ions and of electrons.

Think of a fixed slab of ions and a slab of electrons, each of length L. In electron-slab they move together, and can pass through the ion-slab smoothly. For simplicity consider 1-D case, and origin is at the left wall of these slabs. Imagine the electron-slab is perturbed to the right by distance δ. Consider:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{9}$$

$$\frac{dE}{dx} = 4\pi n_0 e \quad ,\texttt{in 1} - \texttt{D} \tag{10}$$

Consider that $L > \delta > \lambda_D$, then it is reasonable to impose the boundary condition $E(0) = E(L+\delta) = 0$. Thus, integrate (10):

$$E(x) = 4\pi n_0 e \cdot x + C \tag{11}$$

$$\texttt{above condition @0} \implies C = 0 \tag{12}$$

$$E(x) = 4\pi n_0 e \cdot x \texttt{ when x } \in [0, \delta] \tag{13}$$

Imposing condition of continuity within $[\delta, L]$, one gets:

$$E(x) = \begin{cases} 4\pi n_0 e \cdot x & ; x \in [0, \delta] \\ 4\pi n_0 e \cdot \delta & ; x \in [\delta, L] \\ 4\pi n_0 e \cdot (\delta + L - x) & ; x \in [L, L+\delta] \end{cases} \tag{14}$$

Because $L > \delta$, suppose that throughout most of the slab electric-field is $4\pi n_0 e \cdot \delta$. The charge per unit area in the slab is: $-en_0L$; thus force per unit area will be: $-4\pi n_0^2 e^2 \cdot \delta \cdot L$. This should be equal to mass per unit area times the acceleration, $\ddot{\delta}$. So,

$$m_e n_0 L \cdot \ddot{\delta} = -4\pi n_0^2 e^2 \cdot \delta \cdot L \tag{15}$$

$$\implies \ddot{\delta} = \frac{-4\pi n_0 e^2}{m_e}\delta \tag{16}$$

Observe (16) is S.H.O. equation, and it looks like an eigenvalue equation, so its characteristic frequency will be:

$$\omega_e = \sqrt{\frac{4\pi n_0 e^2}{m_e}} \tag{17}$$

This is the plasma frequency.

1.4 Ratio of Electrostatic Potential energy to Thermal Energy

Consider this ratio:

$$\frac{e\Phi}{K_B T} \ll 1 \tag{18}$$

The typical dependence of Φ on r, the distance from source, would be $\sim \frac{1}{r}$. The typical length scale over which the potential drops considerably is $r = \lambda_{D,e}$. So, rewriting the above as:

$$\begin{aligned}\frac{e\Phi}{K_B T} &\sim \frac{e^2}{K_B\ T\ \lambda_{D,e}\ \epsilon_0} \\ &\sim \frac{\epsilon_0}{n\lambda_{D,e}^2} \cdot \frac{1}{\lambda_{D,e}\epsilon_0} \qquad \left(\because \lambda_{D,e}^2 = \frac{\epsilon_0 K_B T}{n\ e^2}\right) \\ \frac{e\Phi}{K_B T} &\sim \frac{1}{n\ \lambda_{D,e}^3}\end{aligned} \tag{19}$$

Thus, (18) is equivalent to:

$$\begin{aligned}\frac{1}{n\ \lambda_{D,e}^3} &\ll 1 \\ \implies n\ \lambda_{D,e}^3 &\gg 1\end{aligned} \tag{20}$$

This in words is: for an ionized gas to be considered a plasma, the number of particles in a cube of side-length equal to Debye-length must be mucg larger than one. The thermal energy dominates the electro-static potential energy.

2 Introduction to Hall-thrusters

In the figure one can see characteristic shape of a Hall thruster. The two L shapes along with the middle T shape make a hollow cavity; one needs to rotate the whole diagram about the **axis of hall thruster** to imagine this. Xenon gas is pumped from the passages in the diagram. Electrons are emitted from cathode. An electric-field is applied in the axial-direction and a magnetic-field is applied in the radial-direction. This creates an $\vec{E} \times \vec{B}$ configuration.

Since the Larmor radius for electrons is much smaller than the dimensions of the hall-thruster cavity, electrons exhibit gyration. Ions, which are very massive, have Larmor radius greater than the

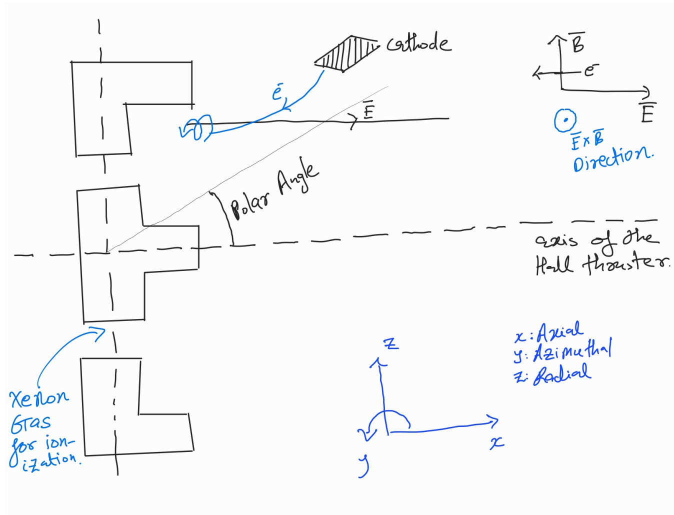


Figure 2: The above image is a schematic diagram of a H.T. The coordinate system that will be used is also drawn.

dimensions of the thuster, and thus do not exhibit considerable gyration. One says **electrons are magnetized** and that **ion are unmagnetized**. So, in the equation of momentum for electrons magnetic-term will appear, whereas for ions it will not. And this will create a lot of difference between the two-species. More about this in 3.2.

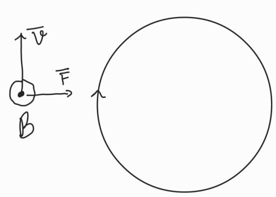


Figure 3: Well-known gyration of charge when moving with constant speed

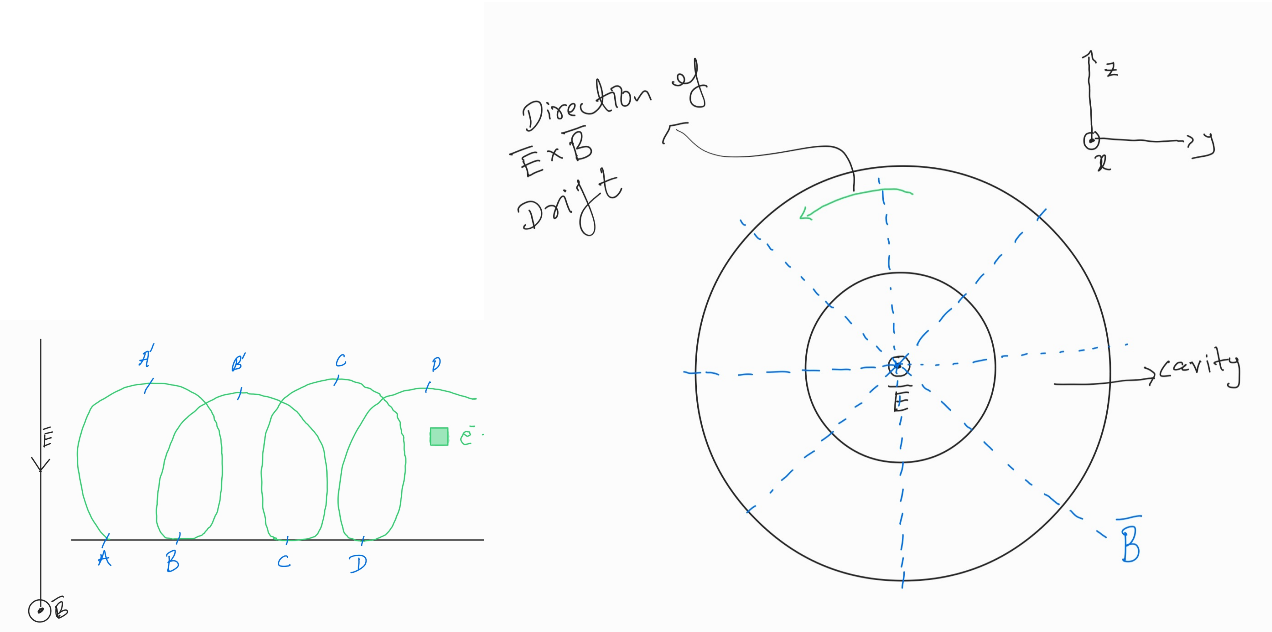
The above figure shows the conventional gyration expected from a charged particle when it traverses a magnetic-field perpendicular to its velocity. The speed is constant here. But in the configuration of a H.T. one observes $\vec{E} \times \vec{B}$ drift. One can understand it like this:

1. initially electron is at rest at point A, refer to figure 4. As electron accelerates, going in the 1st half of its motion, the Larmor radius keeps increasing. In this phase its velocity has some non-zero component along the electric field direction.

2. It reaches a stage where its velocity has no component along the electric-field, this is the point A'. Due to inertia of motion, electron continues to move, and now curves towards point B.

3. As it curves, now some component of its velocity is opposite to the force exerted on it by electric field. So, in this phase it decelerates and its Larmor radius keeps decreasing. The electron fails to reach point A, instead reaches point B. The cycle repeats from B and so on.

Observe every one cycle the electron gains a net displacement in the $\vec{E} \times \vec{B}$ direction. This is depicted in the following figures:
Why is this $\vec{E} \times \vec{B}$ configuration important feature of a hall-thruster? Such an orientation of the fields increases the *residence-time* of electrons in the cavity of the hall-thruster. Residence-time is



(a) Origin of the drift

(b) Cross-sectional view of the hall-thruster

Figure 4: $\vec{E} \times \vec{B}$ drift visualized

the time spent by electrons in the cavity of hall-thruster, during which they collide with income gas (Xenon for example) and ionize it. Electrons are emitted from cathode near the exit-end of the thruster. They are attracted by the axially-aligned $\vec{E}$-field. Xenon gas is pumped from the anode side of the hall-thruster. Refer to fig-2 for clarity. If magnetic-field was not applied to the incoming electrons, they would traverse the length of the thruster cavity and ionize whatever amount of xenon gas they meet. However, once a perpendicular $\vec{B}$ is applied the electrons gyrate in the azimuthal direction; with this gyration, they also drift in the azimuthal direction. These two motions greatly enhance the residence-time of electrons in the thruster cavity. Thus, now they encounter more volume of gas to ionize. So propellant ionization is enhanced.

3 3D Dispersion Relation

The governing equation for this system of plasma are[3]:

$$\textbf{equation of continuity for electrons} \quad \frac{\partial \rho_e}{\partial t} + \vec{\nabla} \cdot (\rho_e \vec{v}_e) = 0 \tag{21}$$

$$\textbf{equation of continuity for ions} \quad \frac{\partial \rho_i}{\partial t} + \vec{\nabla} \cdot (\rho_i \vec{v}_i) = 0 \tag{22}$$

The equations of momentum:

$$\textbf{for electrons} \quad \frac{\partial \vec{v}_e}{\partial t} + \left(\vec{v}_e \cdot \vec{\nabla}\right) \vec{v}_e = -\frac{K_B T_e}{m_e}\, \vec{\nabla} \ln n_e - \frac{e}{m_e}\left(\vec{E} + \vec{v}_e \times \vec{B}\right) - \nu_e \vec{v}_e \tag{23}$$

$$\textbf{for ion} \quad \frac{\partial \vec{v}_i}{\partial t} + \left(\vec{v}_i \cdot \vec{\nabla}\right) \vec{v}_i = -\frac{K_B T_i}{m_i}\, \vec{\nabla} \ln n_i + \frac{e}{m_i}\vec{E} - \nu_i \vec{v}_i \tag{24}$$

3.1 Linearizing Equation of continuity

Consider the equation of continuity for electrons:

$$\frac{\partial \rho_e}{\partial t} + \vec{\nabla} \cdot (\rho_e \vec{v}_e) = 0 \tag{25}$$

$$m_e \frac{\partial n_e}{\partial t} + m_e \vec{\nabla} \cdot (n_e \vec{v}_e) = 0 \qquad \because \rho_e = m_e n_e \tag{26}$$

$$\implies \boxed{\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{v}_e) = 0} \tag{27}$$

Consider the perturbation in these quantities of the form $n_e = n_0 + n'_e$ and $\vec{v}_e = \vec{v}_{e0} + \vec{v}'_e$. Then linearise (27) as:

$$\frac{\partial}{\partial t}(n_0 + n'_e) + \vec{\nabla} \cdot ((n_0 + n'_e)(\vec{v}_{e0} + \vec{v}'_e)) = 0 \tag{28}$$

$$\frac{\partial}{\partial t} n'_e + \cancelto{0}{\vec{\nabla} \cdot (n_0 \vec{v}_{e0})} + \vec{\nabla} \cdot (n_0 \vec{v}'_e) + \vec{\nabla} \cdot (n'_e \vec{v}_{e0}) + \cancelto{0\,:\,\text{2nd order perturbation term}}{\vec{\nabla} \cdot (n'_e \vec{v}'_e)} = 0 \tag{29}$$

Consider the third non-zero term on the LHS of the above (29):

$$\vec{\nabla} \cdot (n'_e \vec{v}_{e0}) = \partial_i (n'_e v_{e0_i}) \tag{30}$$

$$= \cancelto{0}{n'_e \partial_i v_{e0_i}} + v_{e0_i} \partial_i (n'_e) \tag{31}$$

$$\implies \boxed{\vec{\nabla} \cdot (n'_e \vec{v}_{e0}) = \left(\vec{v}_{e0} \cdot \vec{\nabla}\right) n'_e} \tag{32}$$

Consider the second non-zero term on the LHS, by similar arguments:

$$\implies \boxed{\vec{\nabla} \cdot (n_0 \vec{v}'_e) = n_0 \left(\vec{\nabla} \cdot \vec{v}'_e\right)} \tag{33}$$

So, (27) becomes for electrons:

$$\boxed{\frac{\partial n'_e}{\partial t} + n_0 \left(\vec{\nabla} \cdot \vec{v}'_e\right) + \left(\vec{v}_{e0} \cdot \vec{\nabla}\right) n'_e = 0} \tag{34}$$

Similarly, for ions the linearised equation of continuity becomes:

$$\boxed{\frac{\partial n'_i}{\partial t} + n_0 \left(\vec{\nabla} \cdot \vec{v}'_i\right) + \left(\vec{v}_{i0} \cdot \vec{\nabla}\right) n'_i = 0} \tag{35}$$

3.2 Linearizing Equation of Momentum

The equation of momentum is:

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v}\right] = -\vec{\nabla} P + \vec{\nabla} \cdot \underline{\underline{\Pi}} - \rho \vec{\nabla} \psi \tag{36}$$

Here we assume stresses to be zero, so the second term on the RHS is zero. Recall that electrons are magnetized whereas ions are not magnetized. This means that the conservative force term in the RHS will contain magnetic-field term for electron's equation of momentum, but not in the case of ions. Also, note:

$$\vec{\nabla} P = \vec{\nabla}\left(\frac{n_{mol}}{V} RT\right) \tag{37}$$

$$= \vec{\nabla}\left(\frac{N}{V}\frac{R}{N_A}T\right) \qquad \left(\because n_{mol} = \frac{N}{N_A}\right) \tag{38}$$

$$= \vec{\nabla}\left(nK_BT\right) \qquad \left(\because \frac{N}{V} = n \texttt{ and } \frac{R}{N_A} = K_B\right) \tag{39}$$

$$\tag{40}$$

we are considering species that have thermalized within themselves, i.e. T is constant, then:

$$\vec{\nabla} P = K_BT\ \vec{\nabla} n \tag{41}$$

Considering all these, for electrons that are magnetized (36) is:

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right] = -\vec{\nabla}P + \vec{\nabla}\cdot\underline{\underline{\Pi}} - \rho\vec{\nabla}\psi \tag{42}$$

$$m_e\ n_e\left[\frac{\partial \vec{v}_e}{\partial t} + \left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right] = -K_BT_e\ \vec{\nabla}n_e - n_e\ e\ \left(\vec{E} + \vec{v}_e\times\vec{B}\right) - m_e\ n_e\ \nu_e\vec{v}_e \tag{43}$$

$$\left[\frac{\partial \vec{v}_e}{\partial t} + \left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right] = -\frac{K_BT_e}{m_e}\ \frac{\vec{\nabla}n_e}{n_e} - \frac{e}{m_e}\left(\vec{E} + \vec{v}_e\times\vec{B}\right) - \nu_e\vec{v}_e \tag{44}$$

$$\left[\frac{\partial \vec{v}_e}{\partial t} + \left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right] = -\frac{K_BT_e}{m_e}\ \vec{\nabla}\ln n_e - \frac{e}{m_e}\left(\vec{E} + \vec{v}_e\times\vec{B}\right) - \nu_e\vec{v}_e \tag{45}$$

To linearize consider: $\vec{v}_e = \vec{v}_{e0} + \vec{v}_e'$; $n_e = n_{e0} + n_e'$; $\vec{E} = \vec{E}_0 + \vec{E}'$; $\vec{B} = \vec{B}'$, with these (45) becomes:

$$\begin{aligned}\left[\frac{\partial}{\partial t}\left(\vec{v}_{e0} + \vec{v}_e'\right) + \left(\left(\vec{v}_{e0} + \vec{v}_e'\right)\cdot\vec{\nabla}\right)\left(\vec{v}_{e0} + \vec{v}_e'\right)\right] &= -\frac{K_BT_e}{m_e}\ \vec{\nabla}\ln\left(n_{e0} + n_e'\right) \\ &- \frac{e}{m_e}\left(\vec{E}_0 + \vec{E}' + \left(\vec{v}_{e0} + \vec{v}_e'\right)\times\vec{B}\right) - \nu_e\left(\vec{v}_{e0} + \vec{v}_e'\right)\end{aligned} \tag{46}$$

Consider the natural logarithm term on RHS:

$$\vec{\nabla}\ln\left(n_{e0} + n_e'\right) = \vec{\nabla}\ln\left(n_{e0}\left(1 + \frac{n_e'}{n_{e0}}\right)\right) \tag{47}$$

$$= \cancelto{0}{\vec{\nabla}\ln n_{e0}} + \vec{\nabla}\ln\left(1 + \frac{n_e'}{n_{e0}}\right) \tag{48}$$

$$= \vec{\nabla}\left(\frac{n_e'}{n_{e0}} - \frac{1}{2!}\left(\frac{n_e'}{n_{e0}}\right)^2 + \dots\right) \qquad \left(\because \ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \tag{49}$$

$$\approx \vec{\nabla}\left(\frac{n_e'}{n_{e0}}\right) \tag{50}$$

$$= \frac{1}{n_{e0}}\vec{\nabla}n_e' \tag{51}$$

Consider then the following term:

$$\left(\left(\vec{v}_{e0}+\vec{v}_e'\right)\cdot\vec{\nabla}\right)\left(\vec{v}_{e0}+\vec{v}_e'\right)=\left(\vec{v}_{e0}\cdot\vec{\nabla}\right)\vec{v}_{e0}+\left(\vec{v}_{e0}\cdot\vec{\nabla}\right)\vec{v}_e'+\left(\vec{v}_e'\cdot\vec{\nabla}\right)\vec{v}_{e0}+\cancelto{0}{\left(\vec{v}_e'\cdot\vec{\nabla}\right)\vec{v}_e'} \tag{52}$$

So, zeroth order terms in perturbed quantities of (46) become:

$$\frac{\partial}{\partial t}\vec{v_{e0}}+\left(\vec{v}_{e0}\cdot\vec{\nabla}\right)\vec{v_{e0}}=-\frac{e}{m_e}\left(\vec{E}_0+\vec{v}_{e0}\times\vec{B}\right)-\nu_e\ \vec{v}_{e0} \tag{53}$$

The equation involving first order in perturbed quantities is:

$$\boxed{\frac{\partial}{\partial t}\vec{v}_e'+\left(\vec{v}_{e0}\cdot Vec\nabla\right)\vec{v}_e'+\cancelto{0}{\left(\vec{v}_e'\cdot\vec{\nabla}\right)\vec{v}_{e0}}=-\frac{K_B\ T_e}{m_e\ n_{e0}}\ \vec{\nabla}n_e'-\frac{e}{m_e}\left(\vec{E}'+\vec{v}_e'\times\vec{B}\right)-\nu_e\ \vec{v}_e'} \tag{54}$$

Similarly, the linearized equation of momentum for ions in first order of perturbed quantities is:

$$\boxed{\frac{\partial}{\partial t}\vec{v}_i'+\left(\vec{v}_{i0}\cdot\vec{\nabla}\right)\vec{v}_i'=-\frac{K_B\ T_i}{m_i\ n_{i0}}\ \vec{\nabla}n_i'+\frac{e}{m_i}\vec{E}'-\nu_i\ \vec{v}_i'} \tag{55}$$

Also, linearized poisson equation looks as follows:

$$\boxed{\nabla^2\Phi'=-\frac{e}{\epsilon_0}\left(n_i'-n_e'\right)} \tag{56}$$

3.3 Plane-Perturbations

Seek for solutions of the plane-wave form, where quantities with the tilde are constant-amplitudes:

$$n_{e,i}'=\tilde{n_{e,i}}\ e^{\imath\left(\vec{k}\cdot\vec{x}-\omega t\right)} \tag{57}$$

$$\vec{v_{e,i}'}=\tilde{\vec{v_{e,i}}}\ e^{\imath\left(\vec{k}\cdot\vec{x}-\omega t\right)} \tag{58}$$

$$\begin{aligned}\vec{E}'&=-\vec{\nabla}\Phi'\\&=-\vec{\nabla}\left(\tilde{\Phi}\ e^{\imath\left(\vec{k}\cdot\vec{x}-\omega t\right)}\right)\\\vec{E}'&=-\imath\vec{k}\ \tilde{\Phi}\ e^{\imath\left(\vec{k}\cdot\vec{x}-\omega t\right)}\end{aligned} \tag{59}$$

Sub (57) in (34) for electrons, then term-wise, second term becomes:

$$\begin{aligned}\left(\vec{\nabla}n_e'\right)_j&=\frac{\partial n_e'}{\partial x_j} &(60)\\&=\frac{\partial}{\partial x_j}\left(\tilde{n_e}e^{\imath(k_ix_i-\omega t)}\right)\\&=\tilde{n_e}\ \imath k_i\delta_{ij}e^{\imath(k_ix_i-\omega t)}\\&=\imath k_j\tilde{n_e}e^{\imath(k_ix_i-\omega t)}\\\implies\ \vec{v_e}\cdot\vec{\nabla}n_e'&=\imath\ \vec{v_{e0}}\cdot\vec{k}\ \tilde{n_e}e^{\imath\left(\vec{k}\cdot\vec{x}-\omega t\right)} &(61)\end{aligned}$$

Look at 3rd term:

$$\vec{\nabla} \cdot \vec{v_e'} = \frac{\partial}{\partial x_j} v_{ej}' \tag{62}$$

$$= \frac{\partial}{\partial x_j} \left(\tilde{v_{ej}} e^{\imath \left(\vec{k} \cdot \vec{x} - \omega t \right)} \right)$$

$$= \left(\cancelto{0}{\frac{\partial \tilde{v_{ej}}}{\partial x_j}} + \tilde{v_{ej}} \ \imath \ k_i \delta_{ij} \right) e^{\imath \left(\vec{k} \cdot \vec{x} - \omega t \right)}$$

$$\therefore \implies n_0 \vec{\nabla} \cdot \vec{v_e'} = n_0 \imath \tilde{\vec{v_e}} \cdot \vec{k} e^{\imath \left(\vec{k} \cdot \vec{x} - \omega t \right)} \tag{63}$$

The first term that is a time derivative is quite straight forward to evaluate:

$$\frac{\partial n_e'}{\partial t} = -\imath \omega \tilde{n_e} \ e^{\imath \left(\vec{k} \cdot \vec{x} - \omega t \right)} \tag{64}$$

Putting (63), (61) and (64) together in (34), it becomes:

$$-\imath \ \omega \tilde{n_e} \ e^{\imath \left(\vec{k} \cdot \vec{x} - \omega t \right)} + n_0 \imath \tilde{\vec{v_e}} \cdot \vec{k} e^{\imath \left(\vec{k} \cdot \vec{x} - \omega t \right)} + \imath \ \vec{v_{e0}} \cdot \vec{k} \ \tilde{n_e} e^{\imath \left(\vec{k} \cdot \vec{x} - \omega t \right)} = 0 \tag{65}$$

$$\left(-\imath \omega + \imath \vec{v_{e0}} \cdot \vec{k} \right) \tilde{n_e} + \imath \left(\tilde{\vec{v_e}} \cdot \vec{k} \right) n_0 = 0 \tag{66}$$

Define now $\omega - \vec{v_{e0}} \cdot \vec{k} = \bar{w}_e$, then (66) becomes:

$$\boxed{\frac{\tilde{n_e}}{n_0} = \frac{\tilde{\vec{v_e}} \cdot \vec{k}}{\bar{w}_e}} \tag{67}$$

Doing the same substitutions for (35) then one gets:

$$\boxed{\frac{\tilde{n_i}}{n_0} = \frac{\tilde{\vec{v_i}} \cdot \vec{k}}{\bar{w}_i}} \tag{68}$$

Observe in the above number-density ratios, the constant velocity-vector is appears. One can determine these ratios in terms of the perturbation wave-vectors, given velocities can be solved in-terms of the wave-vectors. The next sub-section will exactly do that. It will use the linearized momentum equations to solve for velocity in terms of wave-vectors.

3.4 Solving for Velocity using Momentum Equation

3.4.1 Ion velcoty

The momentum equation of ions has a simpler mathematical-form compared to electrons', as there is no magnetic-field term. After this sub-section, one will see clearly the difference produced by this radially oriented (z direction) magnetic field in the velocity solutions for the two species. Recall the linearized momentum equation for ions, (55):

$$\frac{\partial}{\partial t} \vec{v_i'} + \left(\vec{v}_{i0} \cdot \vec{\nabla} \right) \vec{v_i'} = -\frac{K_B \ T_i}{m_i \ n_{i0}} \vec{\nabla} n_i' + \frac{e}{m_i} \vec{E'} - \nu_i \ \vec{v_i'} \tag{69}$$

One again puts the perturbations in number-density, velocity and electric-field. The only tricky term seems the second-term on the lhs, so lets look at it carefully. Consider its k^{th} component:

$$
\begin{aligned}
\left[\left(\vec{v_{i0}}\cdot\vec{\nabla}\right)\vec{v_i'}\right]_k &= v_{i0_j}\ \frac{\partial}{\partial x_j}v'_{i_k} \qquad (70)\\
&= v_{i0_j}\ \frac{\partial}{\partial x_j}\tilde{v_{i_k}}e^{\imath(k_q x_q-\omega t)}\\
&= v_{i0_j}\ \tilde{v_{i_k}}\ \frac{\partial}{\partial x_j}e^{\imath(k_q x_q-\omega t)}\\
&= v_{i0_j}\ \tilde{v_{i_k}}\ e^{\imath(k_q x_q-\omega t)}\ \imath k_q \cancelto{\delta_{qj}}{\frac{\partial x_q}{\partial x_j}} \quad \text{(collapse } q \text{ running index)}\\
&= v_{i0_j}\ \tilde{v_{i_k}}\ e^{\imath(k_q x_q-\omega t)}\ \imath k_j\\
\left[\left(\vec{v_{i0}}\cdot\vec{\nabla}\right)\vec{v_i'}\right]_k &= \imath\ e^{\imath(k_q x_q-\omega t)}\ \tilde{\vec{v_i}}\vec{k}\cdot\vec{v_{i0}} \qquad (71)
\end{aligned}
$$

The time-derivative term becomes:

$$
\begin{aligned}
\frac{\partial}{\partial t}\vec{v_i'} &= \tilde{\vec{v_i}}\ \frac{\partial}{\partial t}e^{\imath(k_q x_q-\omega t)} \qquad (72)\\
&= -\imath\tilde{\vec{v_i}}\ \omega\ e^{\imath(k_q x_q-\omega t)} \qquad (73)
\end{aligned}
$$

Now consider the 1st term on rhs, consider its k^{th} term:

$$
\begin{aligned}
-\frac{K_B\ T_i}{m_i\ n_{i0}}\ \left[\vec{\nabla}n_i'\right]_k &= -\frac{K_B\ T_i}{m_i\ n_{i0}}\ \tilde{n_i}\frac{\partial}{\partial x_k}e^{\imath(k_q x_q-\omega t)}\\
&= -\frac{K_B\ T_i}{m_i\ n_{i0}}\ \imath k_q\tilde{n_i}\ e^{\imath(k_q x_q-\omega t)}\ \frac{\partial x_q}{\partial x_k}\\
&= -\frac{K_B\ T_i}{m_i\ n_{i0}}\ \imath k_k\tilde{n_i}\ e^{\imath(k_q x_q-\omega t)}\\
&= -\frac{K_B\ T_i}{m_i\ n_{i0}}\ \imath\vec{k}\tilde{n_i}\ e^{\imath(k_q x_q-\omega t)} \qquad (74)
\end{aligned}
$$

Consider the second term in rhs:

$$
\begin{aligned}
\frac{e}{m_i}\vec{E'} &= -\frac{e}{m_i}\tilde{\Phi}\vec{\nabla}e^{\imath(k_q x_q-\omega t)}\\
&= -\frac{e}{m_i}\tilde{\Phi}\ \imath\ \vec{k}e^{\imath(k_q x_q-\omega t)} \qquad (75)
\end{aligned}
$$

The last collisional term on rhs is:

$$
-\nu_i\vec{v_i'} = -\nu_i\tilde{\vec{v_i}}\ e^{\imath(k_q x_q-\omega t)} \qquad (76)
$$

Combining the above results give:

$$\left(-\imath\tilde{\vec{v_i}}\ \omega\ \ +\imath\ \ \tilde{\vec{v_i}}\vec{k}\cdot\vec{v_{i0}}+\nu_i\tilde{\vec{v_i}}\ \right)\ e^{\imath(k_q x_q-\omega t)}=\left(-\frac{K_B\ T_i}{m_i\ n_{i0}}\ \imath\vec{k}\tilde{n}_i\ \ -\frac{e}{m_i}\tilde{\Phi}\ \imath\ \vec{k}\right)e^{\imath(k_q x_q-\omega t)} \tag{77}$$

$$\left(-\imath\tilde{\vec{v_i}}\ \omega\ \ +\imath\ \ \tilde{\vec{v_i}}\vec{k}\cdot\vec{v_{i0}}+\nu_i\tilde{\vec{v_i}}\ \right)=\left(-\frac{K_B\ T_i}{m_i\ n_{i0}}\ \imath\vec{k}\tilde{n}_i\ \ -\frac{e}{m_i}\tilde{\Phi}\ \imath\ \vec{k}\right)$$

$$\left(-\imath\ \omega\ \ +\imath\ \vec{k}\cdot\vec{v_{i0}}+\nu_i\ \right)\tilde{\vec{v_i}}=\left(-\frac{K_B\ T_i}{m_i\ n_{i0}}\ \imath\tilde{n}_i\ \ -\frac{e}{m_i}\tilde{\Phi}\ \imath\ \right)\vec{k}$$

$$\left(\omega-\vec{k}\cdot\vec{v_{i0}}+\imath\nu_i\ \right)\tilde{\vec{v_i}}=\left(\frac{K_B\ T_i}{m_i\ n_{i0}}\tilde{n}_i\ \ +\frac{e}{m_i}\tilde{\Phi}\right)\vec{k} \tag{78}$$

In the (78) identify $\frac{K_B T_i}{m_i}=v_{th,i}^2$ and $\left(\omega-\vec{k}\cdot\vec{v_{i0}}+\imath\nu_i\ \right)=\hat{\omega}_i$, then the equation becomes:

$$\tilde{\vec{v_i}}=\left(v_{th,i}^2\frac{\tilde{n}_i}{n_0}\ \ +\frac{e}{m_i}\tilde{\Phi}\right)\frac{\vec{k}}{\hat{\omega}_i} \tag{79}$$

Observe one very important character of the above equation. The velocity-vector proportional to wave-vector. So they have same direction.

3.4.2 Electron veolcity

Recall (54):

$$\boxed{\frac{\partial}{\partial t}\vec{v_e'}+\left(\vec{v}_{e0}\cdot\vec{\nabla}\right)\vec{v_e'}=-\frac{K_B\ T_e}{m_e\ n_{e0}}\ \vec{\nabla}n_e'-\frac{e}{m_e}\left(\vec{E}'+\vec{v_e'}\times\vec{B}\right)-\nu_e\ \vec{v_e'}} \tag{80}$$

Again, putting perturbations in this equation, one gets:

$$-\imath\omega\tilde{\vec{v_e}}+\imath\vec{k}\cdot\vec{v_{e0}}\ \tilde{\vec{v_e}}=-\frac{e}{m_e}\left(-\imath\vec{k}\ \tilde{\Phi}+\tilde{\vec{v_e}}\times\vec{B}\right)-v_{th,e}^2\frac{\tilde{n}_e}{n_{0e}}\ \imath\vec{k}-\nu_e\tilde{\vec{v_e}} \tag{81}$$

$$\left(-\omega+\vec{k}\cdot\vec{v_{e0}}+\frac{\nu_e}{\imath}\right)\tilde{\vec{v_e}}=\left(\frac{e\vec{k}\tilde{\Phi}}{m_e}-\frac{e\tilde{\vec{v_e}}\times\vec{B}}{\imath m_e}\right)-\frac{\tilde{n}_e}{n_{e0}}v_{th,e}^2\vec{k} \tag{82}$$

$$\hat{\omega}_e\tilde{\vec{v_e}}=\left(\frac{\tilde{n}_e}{n_{e0}}v_{th,e}^2-\frac{e\tilde{\Phi}}{m_e}\right)\vec{k}+\frac{e}{\imath m_e}\tilde{\vec{v_e}}\times\vec{B} \tag{83}$$

Keep in mind that the magnetic-field is: $\vec{B}=(0,0,B_z)$ We consider the three component of (83), starting with x-component:

$$\hat{\omega}_e \tilde{v_{e_x}} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_x + \frac{e}{m_e\ \imath} \left[\tilde{\vec{v}} \times \vec{B} \right]_x \tag{84}$$

$$\hat{\omega}_e \tilde{v_{e_x}} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_x + \frac{e}{m_e\ \imath} \left(\tilde{v_y} B_z - \cancelto{0}{\tilde{v_z} B_y} \right) \tag{85}$$

$$\hat{\omega}_e \tilde{v_{e_x}} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_x + \frac{eB_z}{m_e} \left(\frac{\tilde{v_y}}{\imath} \right) \tag{86}$$

$$\hat{\omega}_e \tilde{v_{e_x}} - \omega_{ce} \left(\frac{\tilde{v_y}}{\imath} \right) = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_x \tag{87}$$

$$\hat{\omega}_e \tilde{v_{e_x}} + \imath\ \omega_{ce} \tilde{v_{e_y}} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_x \tag{88}$$

Consider now the y-component of the (83):

$$\hat{\omega}_e \tilde{v_{e_y}} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_y + \frac{e}{m_e\ \imath} \left[\tilde{\vec{v}} \times \vec{B} \right]_y \tag{89}$$

$$\hat{\omega}_e \tilde{v_{e_y}} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_y + \frac{e}{m_e\ \imath} \left(-\tilde{v_x} B_z + \cancelto{0}{\tilde{v_z} B_x} \right) \tag{90}$$

$$\hat{\omega}_e \tilde{v_{e_y}} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_y - \frac{eB_z}{m_e} \left(\frac{\tilde{v_x}}{\imath} \right) \tag{91}$$

$$\hat{\omega}_e \tilde{v_{e_y}} + \omega_{ce} \left(\frac{\tilde{v_x}}{\imath} \right) = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_y \tag{92}$$

$$-\imath\ \omega_{ce} \tilde{v_{e_x}} + \hat{\omega}_e \tilde{v_{e_y}} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_y \tag{93}$$

Now only the z-component remains, which is quite straight-forward to find, since the magnetic-field is in z-direction, it does not couple z-direction velocity with any other component. We will see later the coupling (mixing) effect that $\vec{B}$ has on x and y- components.

$$\hat{\omega}_e \tilde{v_z} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_z \tag{94}$$

$$\tilde{v_z} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) \frac{k_z}{\hat{\omega}_e} \tag{95}$$

From (88) one can write y-component of the velocity as:

$$\tilde{v_{e_y}} = \left[\left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e} \right) k_x - \hat{\omega}_e \tilde{v_{e_x}} \right] \frac{1}{\imath\ \omega_{ce}} \tag{96}$$

Now, sub. (96) in (93) to get:

$$-\imath\omega_{ce}\tilde{v_{e_x}} + \frac{\hat{\omega}_e}{\imath\omega_{ce}}\left(\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)k_x - \hat{\omega}_e\tilde{v_{e_x}}\right) = \left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)k_y \tag{97}$$

$$\omega_{ce}\tilde{v_{e_x}} + \frac{\hat{\omega}_e}{\omega_{ce}}\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)k_x - \frac{\hat{\omega}_e{}^2}{\omega_{ce}}\tilde{v_{e_x}} = \left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\ \imath k_y \tag{98}$$

$$\omega_{ce}\tilde{v_{e_x}} - \frac{\hat{\omega}_e{}^2}{\omega_{ce}}\tilde{v_{e_x}} = \left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\ \imath k_y - \frac{\hat{\omega}_e}{\omega_{ce}}\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)k_x \tag{99}$$

$$\omega_{ce}\tilde{v_{e_x}} - \frac{\hat{\omega}_e{}^2}{\omega_{ce}}\tilde{v_{e_x}} = \left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\left[\imath k_y - \frac{\hat{\omega}_e}{\omega_{ce}}k_x\right] \tag{100}$$

$$\frac{\omega_{ce}^2\tilde{v_{e_x}} - \hat{\omega}_e{}^2\tilde{v_{e_x}}}{\omega_{ce}} = \left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\left[\frac{\imath k_y\omega_{ce} - \hat{\omega}_e k_x}{\omega_{ce}}\right] \tag{101}$$

$$\tilde{v_{e_x}} = \left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\left[\frac{\imath k_y\omega_{ce} - \hat{\omega}_e k_x}{\omega_{ce}^2 - \hat{\omega}_e{}^2}\right] \tag{102}$$

Sub. (102) in (96) to get:

$$\tilde{v_{e_y}} = \frac{1}{\imath\ \omega_{ce}}\left[\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)k_x - \frac{\hat{\omega}_e}{\omega_{ce}^2 - \hat{\omega}_e{}^2}\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\left[\imath k_y\omega_{ce} - \hat{\omega}_e k_x\right]\right] \tag{103}$$

$$\tilde{v_{e_y}} = \frac{\left(\omega_{ce}^2 - \hat{\omega}_e{}^2\right)^{-1}}{\imath\ \omega_{ce}}\left[\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\left(k_x\omega_{ce}^2 - k_x\hat{\omega}_e{}^2\right) - \left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\left[\imath k_y\omega_{ce}\hat{\omega}_e - \hat{\omega}_e{}^2 k_x\right]\right] \tag{104}$$

$$\tag{105}$$

$$\tilde{v_{e_y}} = \frac{\left(\omega_{ce}^2 - \hat{\omega}_e{}^2\right)^{-1}}{\imath\ \omega_{ce}}$$

$$\left[\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 k_x\omega_{ce}^2 - \frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 k_x\hat{\omega}_e{}^2 - \frac{e\tilde{\Phi}}{m_e}k_x\omega_{ce}^2 + \frac{e\tilde{\Phi}}{m_e}k_x\hat{\omega}_e{}^2 - \frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2\imath k_y\omega_{ce}\hat{\omega}_e + \frac{e\tilde{\Phi}}{m_e}\imath k_y\omega_{ce}\hat{\omega}_e + \frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2\hat{\omega}_e{}^2 k_x - \frac{e\tilde{\Phi}}{m_e}\right.$$

In (106) some terms cancel very neatly and some group out, to yield:

$$\tilde{v_{e_y}} = \frac{\left(\omega_{ce}^2 - \hat{\omega_e}^2\right)^{-1}}{\imath\ \omega_{ce}} \left[k_x \omega_{ce}^2 \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right) - \imath\ \hat{\omega_e}\omega_{ce}\ k_y \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\right] \tag{107}$$

$$\tilde{v_{e_y}} = \frac{\left(\omega_{ce}^2 - \hat{\omega_e}^2\right)^{-1}}{\imath\ \omega_{ce}} \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right) \left[k_x \omega_{ce}^2 - \imath\ \hat{\omega_e}\omega_{ce}\ k_y\right] \tag{108}$$

$$\tilde{v_{e_y}} = \frac{k_x \omega_{ce}^2 - \imath\ \hat{\omega_e}\omega_{ce}\ k_y}{\imath\ \omega_{ce}\left(\omega_{ce}^2 - \hat{\omega_e}^2\right)} \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right) \tag{109}$$

$$\tilde{v_{e_y}} = \frac{\imath\ k_x \omega_{ce} + \hat{\omega_e} k_y}{\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)} \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right) \tag{110}$$

So, now one can write $\tilde{\vec{v_e}}$ in terms of wave vector as:

$$\tilde{\vec{v_e}} = \left(\frac{\tilde{n_e}}{n_{e0}} v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right) \left(\frac{-\imath k_y \omega_{ce} + \hat{\omega_e} k_x}{\hat{\omega_e}^2 - \omega_{ce}^2}, \frac{\imath\ k_x \omega_{ce} + \hat{\omega_e} k_y}{\hat{\omega_e}^2 - \omega_{ce}^2}, \frac{k_z}{\hat{\omega_e}}\right) \tag{111}$$

Observe the differences between (111) and (79). In (111) there is a clear mixing of wave-vector in x and y components, which is not the case for (79). This is because electrons are magnetized, that is to say, magnetic-force is involved in their equation of momentum, unlike ions. The z-directed magnetic field mixes two transverse components. The mixing occurs via electron-cyclotron frequency and it takes up the imaginary-part of the velocity. Also see that when $\omega_{ce} \to 0$ (111) takes exactly the same mathematical form as (79). So, removing magnetic-field will cause electrons' and ions' velocities to behave in similar manner.

Having obtained electron and ion velocities in terms of wave-vectors, it now possible to rewrite the number density ratios (67) and (68) in the same terms. This is the material of next section.

3.5 Number Density Ratios

For ion it is very straight forward. Substitute (79) in (68):

$$\frac{\tilde{n_i}}{n_0} = \frac{\tilde{\vec{v_i}} \cdot \vec{k}}{\bar{w_i}} \tag{112}$$

$$\frac{\tilde{n_i}}{n_0} = \left(v_{th,i}^2 \frac{\tilde{n_i}}{n_0} + \frac{e}{m_i}\tilde{\Phi}\right) \frac{\vec{k} \cdot \vec{k}}{\bar{w_i}\hat{\omega_i}} \tag{113}$$

$$\frac{\tilde{n_i}}{n_0} = \left(v_{th,i}^2 \frac{\tilde{n_i}}{n_0} + \frac{e}{m_i}\tilde{\Phi}\right) \frac{k^2}{\bar{w_i}\hat{\omega_i}} \tag{114}$$

After solving for $\frac{\tilde{n_i}}{n_0}$, one gets:

$$\boxed{\frac{\tilde{n_i}}{n_0} = \left(\frac{k^2}{\bar{\omega_i}\hat{\omega_i} - v_{th,e}^2 k^2}\right) \frac{e\tilde{\Phi}}{m_i}} \tag{115}$$

Number density ratio for electrons is algebraically more cumbersome, it is as follows:

$$\frac{\tilde{n_e}}{n_0} = \frac{1}{\bar{w_e}}\tilde{\vec{v_e}} \cdot \vec{k} \tag{116}$$

$$\frac{\tilde{n_e}}{n_0} = \frac{1}{\bar{w_e}}\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\left(\frac{-\imath k_y\omega_{ce} + \hat{\omega}_e k_x}{\hat{\omega_e}^2 - \omega_{ce}^2}, \frac{\imath\ k_x\omega_{ce} + \hat{\omega}_e k_y}{\hat{\omega_e}^2 - \omega_{ce}^2}, \frac{k_z}{\hat{\omega}_e}\right) \cdot (k_x, k_y, k_z) \tag{117}$$

$$\frac{\tilde{n_e}}{n_0} = \frac{1}{\bar{w_e}}\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\left(\frac{-\imath k_y k_x\omega_{ce} + \hat{\omega}_e k_x^2}{\hat{\omega_e}^2 - \omega_{ce}^2} + \frac{\imath\ k_x k_y\omega_{ce} + \hat{\omega}_e k_y^2}{\hat{\omega_e}^2 - \omega_{ce}^2} + \frac{k_z^2}{\hat{\omega}_e}\right) \tag{118}$$

$$\frac{\tilde{n_e}}{n_0} = \frac{1}{\bar{w_e}}\left(\frac{\tilde{n_e}}{n_{e0}}v_{th,e}^2 - \frac{e\tilde{\Phi}}{m_e}\right)\left(\frac{\hat{\omega}_e}{\hat{\omega_e}^2 - \omega_{ce}^2}\left(k_x^2 + k_y^2\right) + \frac{k_z^2}{\hat{\omega}_e}\right) \tag{119}$$

$$\tag{120}$$

$$\left(\frac{\hat{\omega}_e}{\hat{\omega_e}^2 - \omega_{ce}^2}\left(k_x^2 + k_y^2\right) + \frac{k_z^2}{\hat{\omega}_e}\right)\frac{e\tilde{\Phi}}{m_e\bar{\omega}_e} = \frac{\tilde{n_e}}{n_0}\left[\frac{v_{th,e}^2}{\bar{\omega_e}}\left(\frac{\hat{\omega}_e}{\hat{\omega_e}^2 - \omega_{ce}^2}\left(k_x^2 + k_y^2\right) + \frac{k_z^2}{\hat{\omega}_e}\right) - 1\right] \tag{121}$$

$$\left(\frac{v_{th,e}^2\hat{\omega_e}^2\left(k_x^2 + k_y^2 + k_z^2\right) - \omega_{ce}^2 k_z^2 v_{th,e}^2 - \bar{\omega_e}\hat{\omega_e}^3 + \bar{\omega_e}\hat{\omega}_e\omega_{ce}^2}{\cancel{\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)\bar{\omega_e}\hat{\omega}_e}}\right)\left(\frac{\tilde{n_e}}{n_0}\right) = \left(\frac{e\tilde{\Phi}}{m_e}\right)\frac{\hat{\omega_e}^2\left(k_x^2 + k_y^2 + k_z^2\right) - \omega_{ce}^2 k_z^2 v_{th,e}^2}{\cancel{\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)\bar{\omega_e}\hat{\omega}_e}} \tag{122}$$

$$\frac{\tilde{n_e}}{n_0} = \left(\frac{e\tilde{\Phi}}{m_e}\right)\left(\frac{\hat{\omega_e}^2 k^2 - \omega_{ce}^2 k_z^2 v_{th,e}^2}{v_{th,e}^2\hat{\omega_e}^2 k^2 - \omega_{ce}^2 k_z^2 v_{th,e}^2 - \bar{\omega_e}\hat{\omega_e}^3 + \bar{\omega_e}\hat{\omega}_e\omega_{ce}^2}\right) \tag{123}$$

$$\frac{\tilde{n_e}}{n_0} = \left(-\frac{e\tilde{\Phi}}{m_e}\right)\left(\frac{\hat{\omega_e}^2 k^2 - \omega_{ce}^2 k_z^2 v_{th,e}^2}{-v_{th,e}^2\hat{\omega_e}^2 k^2 + \omega_{ce}^2 k_z^2 v_{th,e}^2 + \bar{\omega_e}\hat{\omega}_e\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)}\right) \tag{124}$$

So, just to summarize, ion-number-density and electron-number-density are:

$$\frac{\tilde{n_i}}{n_0} = \left(\frac{k^2}{\bar{\omega_i}\hat{\omega}_i - v_{th,e}^2 k^2}\right)\frac{e\tilde{\Phi}}{m_i} \tag{125}$$

$$\frac{\tilde{n_e}}{n_0} = \left(-\frac{e\tilde{\Phi}}{m_e}\right)\left(\frac{\hat{\omega_e}^2 k^2 - \omega_{ce}^2 k_z^2 v_{th,e}^2}{-v_{th,e}^2\hat{\omega_e}^2 k^2 + \omega_{ce}^2 k_z^2 v_{th,e}^2 + \bar{\omega_e}\hat{\omega}_e\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)}\right) \tag{126}$$

Observe that in the limit $\omega_{ce} \to 0$, (126) takes the same mathematical for as (125).

3.6 Poisson Equation

Recall (56):

$$\nabla^2\Phi' = -\frac{e}{\epsilon_0}\left(n_i' - n_e'\right) \tag{127}$$

It can be converted to an eigen-value equation of Φ' and then reduced as follows:

$$\nabla^2\Phi' = -k^2\Phi' = -\frac{e\ n_0}{\epsilon_0}\left(\frac{n_i'}{n_0} - \frac{n_e'}{n_0}\right) \tag{128}$$

$$-k^2\tilde{\Phi} = -\frac{e\ n_0}{\epsilon_0}\left(\frac{\tilde{n_i}}{n_0} - \frac{\tilde{n_e}}{n_0}\right) \tag{129}$$

Substitute (126) and (125) in (129) to get:

$$1 = \frac{\left(\frac{e^2 n_0}{\epsilon_0 m_i}\right)}{\bar{\omega_i}\hat{\omega_i} - v_{th,i}^2 k^2} + \frac{\left(\frac{e^2 n_0}{\epsilon_0 m_e}\right)\left(\hat{\omega_e}^2 - \omega_{ce}^2 \frac{k_z^2}{k^2}\right)}{\omega_{ce}^2 k_z^2 v_{th,e}^2 - v_{th,e}^2 \hat{\omega_e}^2 k^2 + \bar{\omega_e}\hat{\omega_e}\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)} \tag{130}$$

Identify $\frac{e^2 n_0}{\epsilon_0 m_i} = \omega_{p,i}^2$ and $\frac{e^2 n_0}{\epsilon_0 m_e} = \omega_{p,e}^2$ then (130) becomes:

$$\boxed{1 = \frac{\omega_{p,i}^2}{\bar{\omega_i}\hat{\omega_i} - v_{th,i}^2 k^2} + \frac{\omega_{p,e}^2\left(\hat{\omega_e}^2 - \omega_{ce}^2 \frac{k_z^2}{k^2}\right)}{\omega_{ce}^2 k_z^2 v_{th,e}^2 - v_{th,e}^2 \hat{\omega_e}^2 k^2 + \bar{\omega_e}\hat{\omega_e}\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)}} \tag{131}$$

4 Ion-Acoustic Wave IAW

In this section a special case of (131) will be considered. Set $k_z = 0$, i.e., the perturbation wave is a $x - y$ plane wave, and does not have any component along $z-$direction. Consider the following approximations:

1. $k_z = 0$,
2. weak collisions among electrons $\nu_e \approx 0$, so that $\bar{\omega_e} \approx \hat{\omega}_e$,
3. neglect ion-collisions, i.e., $\nu_i = 0\ Hz$, so that $\bar{\omega_e} = \hat{\omega}_e$,
4. neglect ion-thermal velocity $v_{th,i} = 0.0\ ms^{-1}$,
5. perturbation wave-number is in the range such that, $\frac{\omega_c}{v_{th,e}} \ll k \ll \frac{\omega_c}{v_{e,0}}$. The latter inequality can be reduced, by identifying $v_{e,0} \cdot k = \hat{\omega}_e$, to: $\omega_{ce} \gg \hat{\omega}_e$.

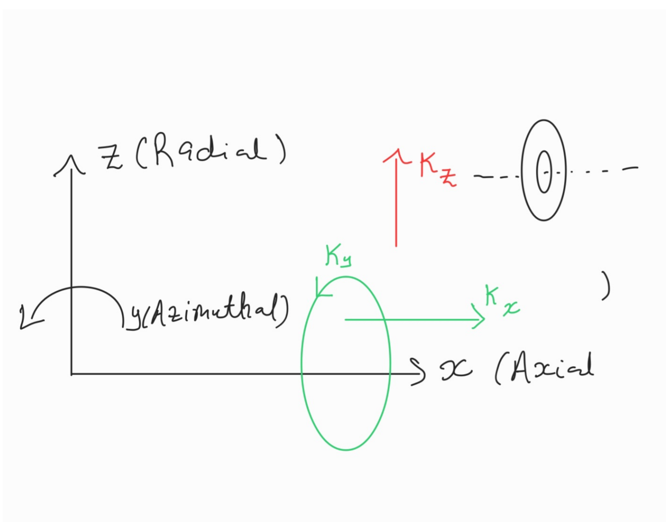


Figure 5: Green directions are ones in which the k-components are non-zero; so, perturbation propagates in these directions. Red direction is one in which k_z is zero, and so perturbation cannot propagate in that direction. The hall thruster is show in the top-right corner.

Consider the (131) and apply the above approximations carefully:

$$1 = \frac{\omega_{p,i}^2}{\bar{\omega_i}\hat{\omega_i} - v_{th,i}^2 k^2} + \frac{\omega_{p,e}^2\left(\hat{\omega_e}^2 - \omega_{ce}^2\frac{k_z^2}{k^2}\right)}{\omega_{ce}^2 k_z^2 v_{th,e}^2 - v_{th,e}^2\hat{\omega_e}^2 k^2 + \bar{\omega_e}\hat{\omega_e}\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)} \tag{132}$$

$$1 = \frac{\omega_{p,i}^2}{\bar{\omega_i}^2} + \frac{\omega_{p,e}^2}{\frac{\bar{\omega_e}\hat{\omega_e}\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)}{\hat{\omega_e}^2} - v_{th,e}^2 k^2} \quad \left(\text{using 3, and taking } \hat{\omega_e}^2 \text{down in second term on rhs}\right) \tag{133}$$

$$1 = \frac{\omega_{p,i}^2}{\bar{\omega_i}^2} + \frac{\omega_{p,e}^2}{v_{th,e}^2 k^2}\left(\frac{1}{\frac{\bar{\omega_e}\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)}{\hat{\omega_e}\ v_{th,e}^2 k^2} - 1}\right) \quad \text{(take out the thermal velocity term)} \tag{134}$$

$$1 = \frac{\omega_{p,i}^2}{\bar{\omega_i}^2} + \frac{\omega_{p,e}^2}{v_{th,e}^2 k^2}\left(\frac{1}{-\frac{\omega_{ce}^2}{v_{th,e}^2 k^2} - 1}\right) \quad \left(\because \omega_{ce} \gg \hat{\omega_e} \text{ and } \nu_e \approx 0 \implies \bar{\omega_e} \approx \hat{\omega_e}\right) \tag{135}$$

$$1 = \frac{\omega_{p,i}^2}{\bar{\omega_i}^2} + \frac{1}{\lambda_{D,e}^2 k^2}\left(\frac{1}{\cancelto{0}{\frac{\left(\frac{\omega_{ce}}{v_{th,e}}\right)^2}{k^2}} - 1}\right) \quad \left(\because \frac{\omega_c}{v_{th,e}} \ll k \text{ and } \frac{v_{th,e}}{\omega_{p,e}} = \lambda_{D,e}\right) \tag{136}$$

$$1 = \frac{\omega_{p,i}^2}{\bar{\omega_i}^2} - \frac{1}{\lambda_{D,e}^2 k^2} \tag{137}$$

Desire is to solve (137) for $\bar{\omega_i}$:

$$\boxed{\hat{\omega_i} = \pm\frac{k\ \lambda_{D,e}\omega_{p,i}}{\sqrt{1 + k^2\lambda_{D,e}^2}}} \tag{138}$$

Recall that $\bar{\omega_i} = \omega - \vec{k}\cdot\vec{v_{i,0}}$, so then:

$$\boxed{\omega = \vec{k}\cdot\vec{v_{i,0}} \pm \frac{k\ \lambda_{D,e}\omega_{p,i}}{\sqrt{1 + k^2\lambda_{D,e}^2}}} \tag{139}$$

It is now possible to plot a graph of ω vs. k. In the large k limit $k^2\lambda_{D,e}^2 \gg 1$ so ignore the one in denominator on rhs of (139) to get:

$$\omega = \vec{k}\cdot\vec{v_{i,0}} \pm \omega_{p,i} \tag{140}$$

Now consider the case of small $k \ll 1$, in that case ignore the $k^2\lambda_{D,e}^2$ to get:

$$\omega = \vec{k}\cdot\vec{v_{i,0}} \pm \lambda_{D,e}\omega_{p,i}\ k \tag{141}$$

One always study the system from the frame of reference of the ions. In that case set $\vec{v_{i,0}} = 0\ ms^{-1}$ This means for $k \ll 1$ ω is linear in k and w saturates to a value $\omega_{p,i}$ for large k values. Such behavior is seen in the following image. The parameters were chosen as $T_e \approx 1\ eV \approx 12\ kK$; number density $n_e = 5\times 10^5\ m^{-3}$:
Consider the small k limit, $\omega = k\cdot\lambda_{D,e}\omega_{p,i}$. Consider this:

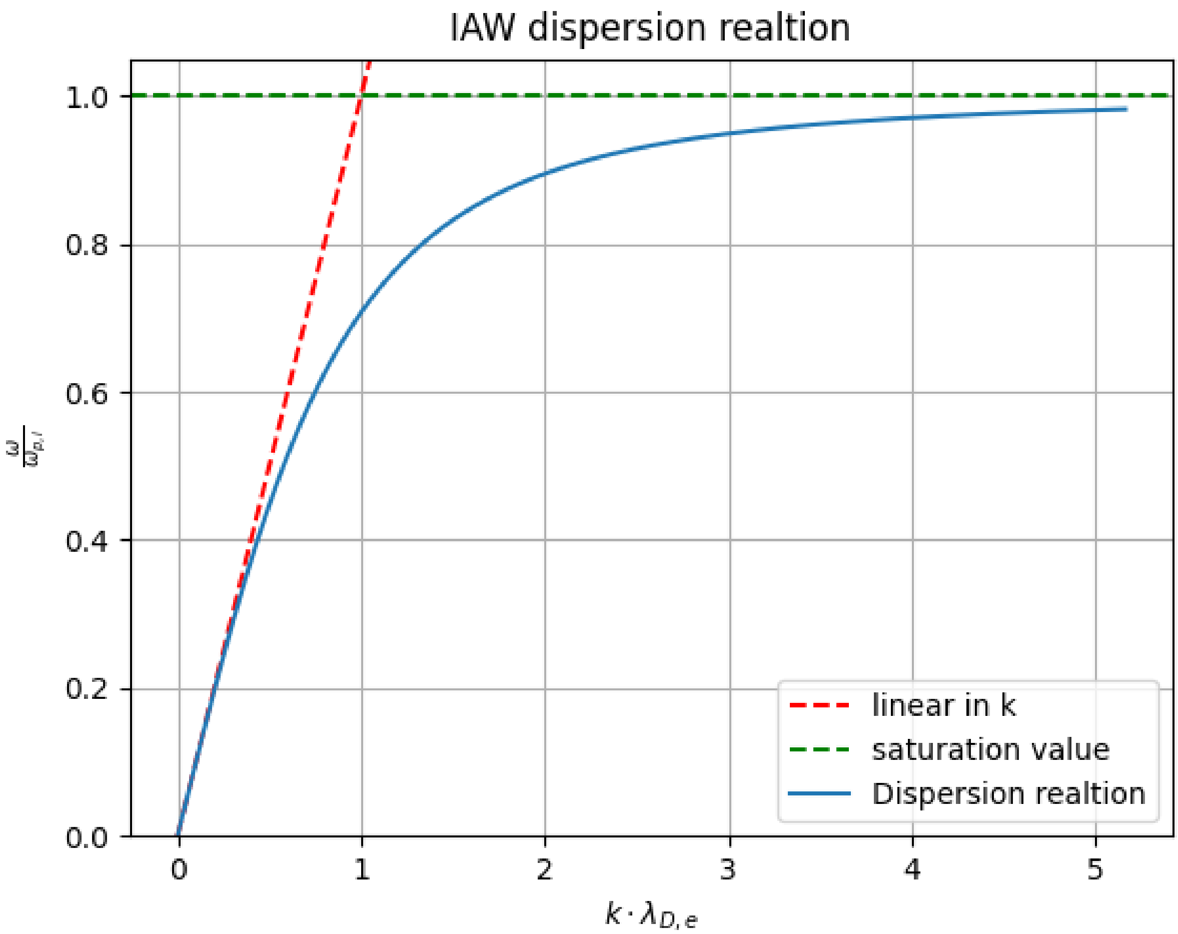


Figure 6: Observe the linear rise for small k values (the red line) and as k becomes large the graph reaches a saturation value (green line).

$$\omega = k \cdot \lambda_{D,e}\omega_{p,i} \tag{142}$$

$$= k \cdot \sqrt{\frac{\epsilon_0 K_B T_e}{n_0 e^2}} \cdot \sqrt{\frac{e^2 n_0}{\epsilon_0 m_i}} \tag{143}$$

$$= k\sqrt{\frac{K_b T_e}{m_i}} \tag{144}$$

The radical in (144) is the Bohm-speed. This is also the sound speed in the plasma. To make this clear, use $p = n_0 K_B T$. Multiply and divide n_0 in the radical:

$$\omega = k\sqrt{\frac{K_b T_e n_0}{m_i n_0}} = k\sqrt{\frac{P}{\rho}} \tag{145}$$

$$\omega = k \cdot c_s \tag{146}$$

This is the sound-speed, for an ideal gas with specific-heat capacity ratio γ, $v = \sqrt{\frac{\gamma \cdot P}{\rho}}$. This is what is acoustic about it, that in small k limit the wave-speed is the sound-speed in plasma. The ion-plasma frequency puts an upper bound on the frequency of the perturbation. This is the ionic part of the name.

5 Modified Two-Stream Instability, MTSI

For this specialization consider initially the following approximations:

1. $k_x = 0$,
2. collisionless electrons and ions, i.e., $\nu_e = \nu_i = 0$, so that $\hat{\omega_{e,i}} = \bar{\omega_{e,i}} = \omega - \vec{k} \cdot \vec{v_{e,i,0}}$,
3. no drifting ions, i.e., $v_{i,0} = 0$,
4. cold-ions, i.e., ion-thermal veoclity is zeor, $\vec{v_{th,e}} = 0$.

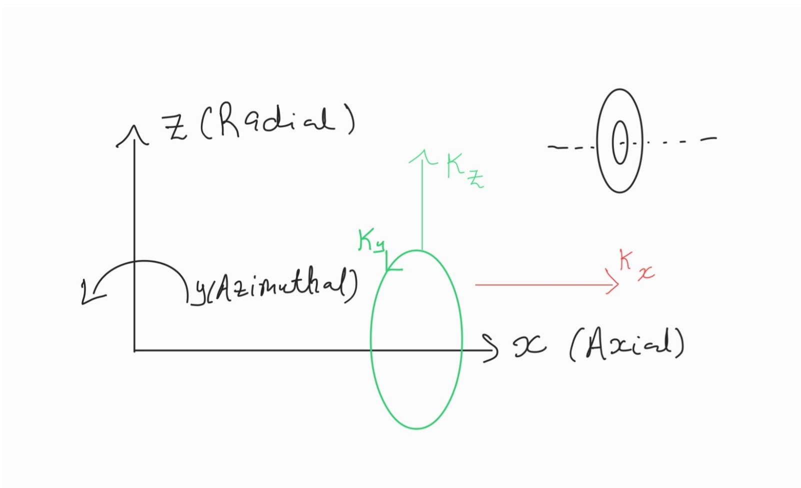


Figure 7: Green directions are ones in which the k-components are non-zero; so, perturbation propagates in these directions. Red direction is one in which k_x is zero, and so perturbation cannot propagate in that direction. The hall thruster is show in the top-right corner.

The points 2 and 3 imply that for ions $\omega = \hat{\omega}_i$. So eq.(131) reduces as follows:

$$1 = \frac{\omega_{p,i}^2}{\bar{\omega}_i \hat{\omega}_i - v_{th,i}^2 k^2} + \frac{\omega_{p,e}^2 \left(\hat{\omega}_e{}^2 - \omega_{ce}^2 \frac{k_z^2}{k^2} \right)}{\omega_{ce}^2 k_z^2 v_{th,e}^2 - v_{th,e}^2 \hat{\omega}_e{}^2 k^2 + \bar{\omega}_e \hat{\omega}_e \left(\hat{\omega}_e{}^2 - \omega_{ce}^2 \right)} \tag{147}$$

$$1 = \frac{\omega_{p,i}^2}{\omega_i^2} + \frac{\omega_{p,e}^2 \left(\hat{\omega}_e{}^2 - \omega_{ce}^2 \frac{k_z^2}{k^2} \right)}{\hat{\omega}_e{}^2 \left(\hat{\omega}_e{}^2 - \omega_{ce}^2 \right) - v_{th,e}^2 k^2 \left(\hat{\omega}_e{}^2 + \omega_{ce}^2 \frac{k_z^2}{k^2} \right)} \tag{148}$$

$$1 = \frac{\omega_{p,i}^2}{\omega_i^2} + \frac{\omega_{p,e}^2}{\hat{\omega}_e{}^2 \frac{\left(\hat{\omega}_e{}^2 - \omega_{ce}^2 \right)}{\left(\hat{\omega}_e{}^2 - \omega_{ce}^2 \frac{k_z^2}{k^2} \right)} - v_{th,e}^2 k^2} \tag{149}$$

Now, one can introduce another approximation: consider elecrons to be cold, i.e., $\vec{v_{th,e}} = 0 \ ms^{-1}$, then (149) reduces to:

$$1 = \frac{\omega_{p,i}^2}{\omega_i^2} + \frac{\omega_{p,e}^2}{\hat{\omega_e}^2 \frac{\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)}{\left(\hat{\omega_e}^2 - \omega_{ce}^2 \frac{k_z^2}{k^2}\right)}} \tag{150}$$

$$1 = \frac{\omega_{p,i}^2}{\omega_i^2} + \frac{\omega_{p,e}^2}{\hat{\omega_e}^2 \frac{\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)k^2}{\left(\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)k_z^2 + k_y^2 \hat{\omega_e}^2\right)}} \tag{151}$$

$$1 = \frac{\omega_{p,i}^2}{\omega_i^2} + \frac{\omega_{p,e}^2 \left(\left(\hat{\omega_e}^2 - \omega_{ce}^2\right)k_z^2 + k_y^2 \hat{\omega_e}^2\right)}{\hat{\omega_e}^2 \left(\hat{\omega_e}^2 - \omega_{ce}^2\right)k^2} \tag{152}$$

$$1 = \frac{\omega_{p,i}^2}{\omega_i^2} + \frac{\omega_{p,e}^2}{\left(\omega - \vec{k} \cdot \vec{v_{e,0}}\right)^2} \cdot \left(\frac{k_z^2}{k^2}\right) + \frac{\omega_{p,e}^2}{\left(\left(\omega - \vec{k} \cdot \vec{v_{e,0}}\right)^2 - \omega_{ce}^2\right)} \cdot \left(\frac{k_y^2}{k^2}\right) \tag{153}$$

One can make the above dimensionless. Use the electron Debye length to make wave-vectors dimensionless, use the ion-plasma frequency to make the frequency (ω_i) dimensionless. Their product $\lambda_{D,e}\omega_{p,i} = u_B$ is called Bohm-speed. Use this to make velocity terms dimensionless.

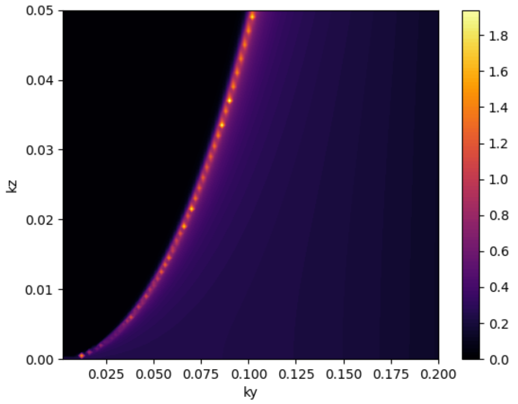
$$0 = 1 - \frac{1}{\left(\frac{\omega_i^2}{\omega_{p,i}^2}\right)} - \left(\frac{\omega_{p,e}^2}{\omega_{p,i}^2}\right) \frac{1}{\frac{\left(\omega - \vec{k} \cdot \vec{v_{e,0}}\right)^2}{\omega_{p,i}^2}} \cdot \left(\frac{k_z^2 \lambda_{D,e}^2}{k^2 \lambda_{D,e}^2}\right) - \left(\frac{\omega_{p,e}^2}{\omega_{p,i}^2}\right) \frac{1}{\left(\frac{\left(\omega - \vec{k} \cdot \vec{v_{e,0}}\right)^2}{\omega_{p,i}^2} - \left(\frac{\omega_{ce}}{\omega_{p,i}}\right)^2\right)} \cdot \left(\frac{k_y^2 \lambda_{D,e}^2}{k^2 \lambda_{D,e}^2}\right) \tag{154}$$

$$0 = 1 - \frac{1}{\tilde{\omega}^2} - \left(\frac{m_i}{m_e}\right) \frac{1}{\left(\frac{\omega}{\omega_{p,i}} - \frac{\vec{k} \cdot \vec{v_{e,0}} \lambda_{D,e}}{u_B}\right)^2} \cdot \left(\frac{\tilde{k_z}^2}{\tilde{k}^2}\right) - \left(\frac{m_i}{m_e}\right) \frac{1}{\left(\left(\frac{\omega}{\omega_{p,i}} - \frac{\vec{k} \cdot \vec{v_{e,0}} \lambda_{D,e}}{u_B}\right)^2 - \left(\frac{\omega_{ce}}{\omega_{p,i}}\right)^2\right)} \cdot \left(\frac{\tilde{k_y}^2}{\tilde{k}^2}\right) \tag{155}$$

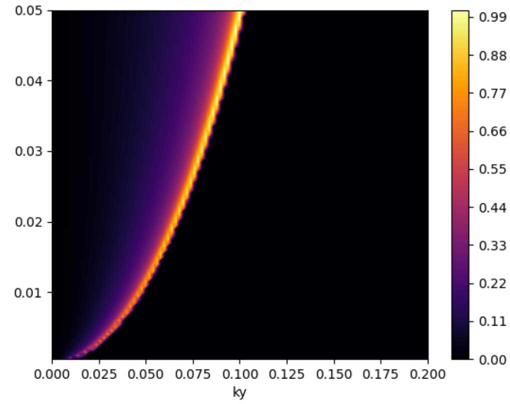
$$0 = 1 - \frac{1}{\tilde{\omega}^2} - \left(\frac{m_i}{m_e}\right) \frac{1}{\left(\tilde{\omega} - \tilde{\vec{k}} \cdot \tilde{\vec{v_0}}\right)^2} \cdot \left(\frac{\tilde{k_z}^2}{\tilde{k}^2}\right) - \left(\frac{m_i}{m_e}\right) \frac{1}{\left(\left(\tilde{\omega} - \tilde{\vec{k}} \cdot \tilde{\vec{v_0}}\right)^2 - \left(\frac{\omega_{ce}}{\omega_{p,i}}\right)^2\right)} \cdot \left(\frac{\tilde{k_y}^2}{\tilde{k}^2}\right) \tag{156}$$

For a given value of k_y and k_z, it is possible to find the roots $\tilde{\omega}^2$ of (156). It is not possible to isolate it on one side and solve for it. Hence, numerical methods need to be employed. We will chose a pair (k_y, k_z) in the domain-space and then using *optimize* module in *scipy*, the roots can be obtained. The roots are complex, with the real part $\tilde{\omega}_r$ representing the frequency of the perturbation and the imaginary part $\tilde{\omega}_i$ representing the growth rate of the perturbation.The DR function for MTSI is a complex-valued function of vector-variables (k_y, k_z). So, one needs to pass a 1-D array of size two as the output that will be given by the *minimize* function in the scipy library. The *minimize* function takes as input the function to be minimized, I gave this as the MTSI DR. Then one can pass the 1-D array of size two as a compounded complex number to the DR, as in say the array-variable is named L, then pass the ω into DR as: $\omega = L[0] + 1j * L[1]$.

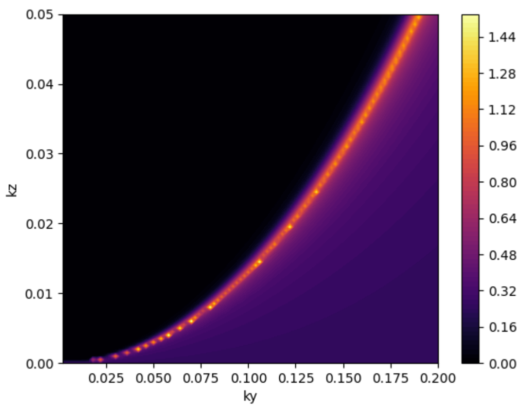
The figures below display the variation of the roots of (156) with respect to temperature. The heat-plots indicate magnitude of frequency and growth-rate of perturbations with wave-vectors in the domain $(k_y, k_z) \in [0.0, 0.20] \times [0.0, 0.05]$. The plots are draw taking the $B = 20\ mT$, $E = 10\ kV\ m^{-1}$ and number-density as $n = 5 \times 10^{16}\ m^{-3}$. The mass of ions is taken as the nuclear-mass of Xenon $m_i = 131.293 \times m_p$. Permittivity was used as vacuum's. The debye-length was found to be $\lambda_{D,e} = 3.44 \times 10^{-5}\ m \approx 34\ \mu m$. The ion-plasma frequency is $\omega_{p,i} = 2.58 \times 10^7\ Hz \approx 26\ MHz$.



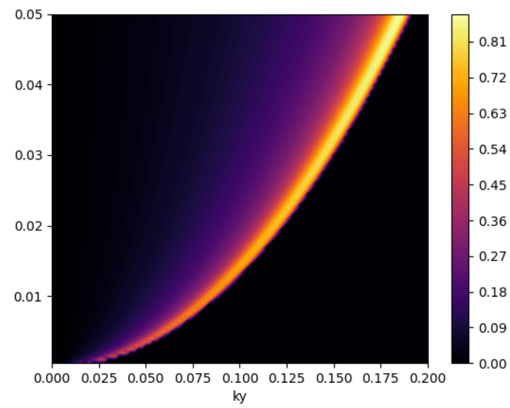
(a) Temp = 1 eV



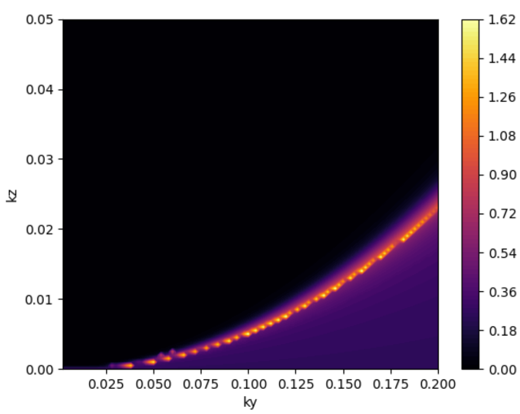
(b) Temp = 1 eV



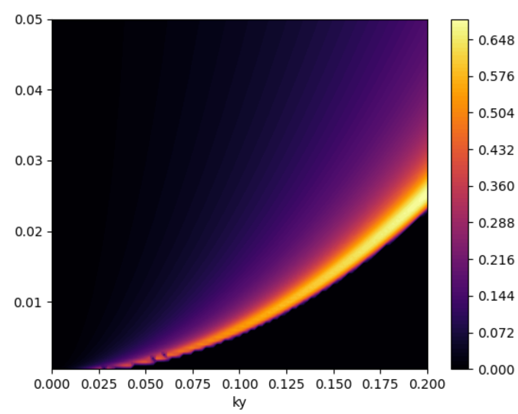
(c) Temp = 10 eV



(d) Temp = 10 eV

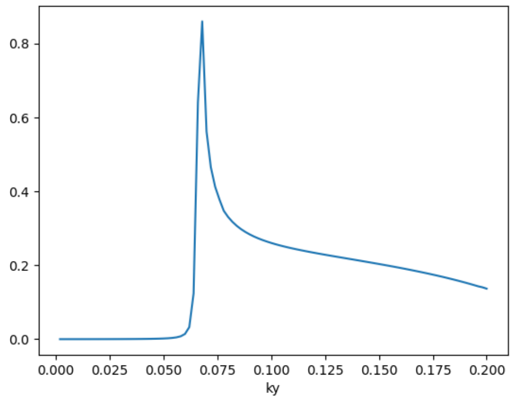


(e) Temp = 50 eV

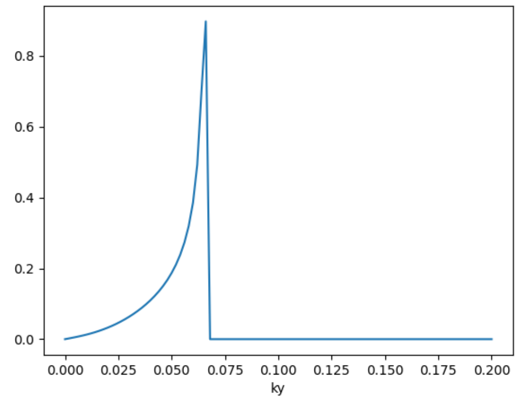


(f) Temp = 50 eV

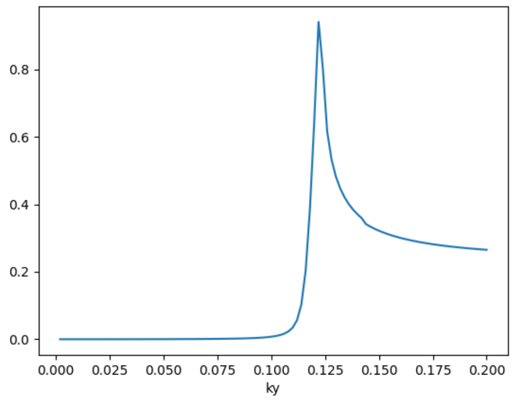
Figure 8: Real-Part Imaginary-part



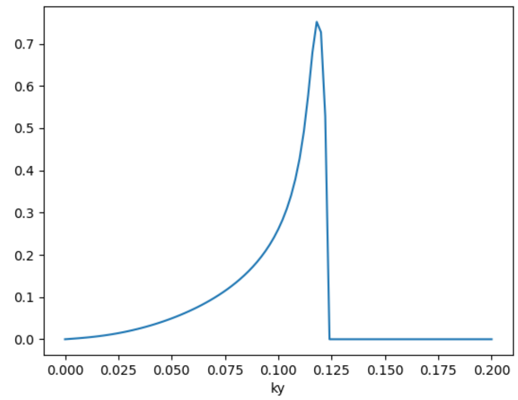
(a) Temp = 1 eV



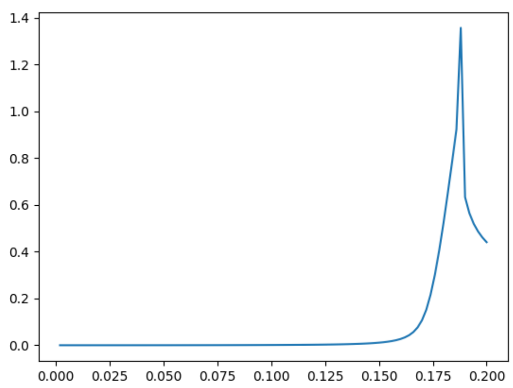
(b) Temp = 1 eV



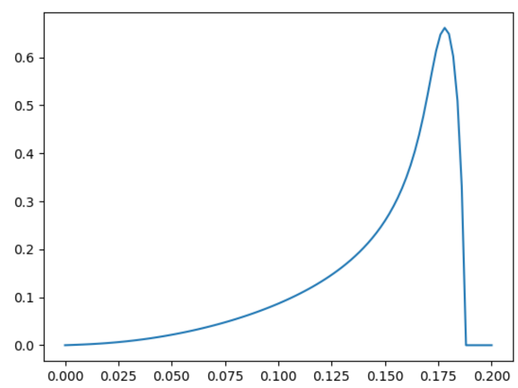
(c) Temp = 10 eV



(d) Temp = 10 eV



(e) Temp = 50 eV



(f) Temp = 50 eV

Figure 9: Real-Part Imaginary-part

In the heat-plots observe as electron-temperature is increased from 1 eV to 10 eV the *orange-band* moves to the right. Consider the images of the real-part of root. See that above the orange-band it is all black. This means that perturbations with wave-vector coordinates (k_y, k_z) in this region have zero frequency. Below this orange-band there are areas of non-zero frequency. This is reflected in the second image-page, where k_z cuts are displayed. Observe in fig.9 (a), (c) and (e) for $k_y < k_0$ the graph is zero. Then near k_0 there is a sharp rise and then again the graph falls.

Now consider the images of the imaginary part. Observe, in these images the upper-region has non-zero value. So those frequencies corresponding to this region are growing modes. But, as noted from the real-part images, the frequencies here are zero. Similarly, looking below the orange-belt, one sees a sharp drop to zero. This is reflected in fig.9 (b), (d) and (f): the graph grows gradually with k_y, reaches a maximum and then drops sharply.

6 Conclusion and Future Scope

In this report a detailed derivation of the fluid dispersion-relation is done. This is eq.(131). Then taking special cases of this dispersion-relation the DR for 1) IAW and 2) MTSI were obtained. For IAW the small k and large k limits were analyzed; in the former case, ω depended on k linearly and in the latter ω reaches a saturation. This saturation happens to be the ion-plasma frequency $\omega_{p,i}$. It is noted in fig-6.

For MTSI the DR (153) could not be solved for ω analytically, unlike IAW. So it needs to be solved numerically. This was done using python, scipy module *optimize*. In the *optimize* module there is the function called *minimize*, it is used to find points in the domain of a function, where the function-value is minimum.

The roots were found to be complex. Then, real and imaginary parts of the roots were plotted for a range of k_y and k_z values. We noted the strong dependence of the roots on electron-temperature. This dependence happens through the *Bohm speed*. In passing, Bohm speed happens to be the same as speed of sound in the plasma.

In the future it is possible to solve the complete 3-D DR using the same minimization procedure that was applied to MTSI. Due to time constraints the **Electron Cyclotron Drift Instability** (ECDI) could not be analyzed similarly. This could be done as a future work. Another promising study could be including anisotropic-pressure in the derivation of the (131). In this report isotropic pressure was considered. Anisotropy in pressure will reflect in the full 3-D Dr from the

References

[1] Francis Chen. *Introduction to Plasma Physics and Controlled Fusion.* Springer. Chap. 1.

[2] D. Nicholson. *Introduction to Plasma Theory.* Chap. 1.

[3] Federico Petronio. “Plasma instabilities in Hall Thrusters : a theoretical and numerical study”. MA thesis. de l’Institut Polytechnique de Paris (EDIPP), 2024.