SUMMER PROJECT REPORT

INTRODUCTION TO HRG AND ITS APPLICATIONS

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Contents

Abstract

In this report Hadron Resonance Gas (HRG) is being studied. The statistical thermalization model equations are used to predict number densities, particle-yields out of the fireball. Using data from earlier papers and plots as guideposts, this report will do a comparative study of the same. Ideal Hadron Resonance gas will be discussed along with its short-coming. Pressure and number-density under Boltzmann approximation will be discussed.

1 Introduction

This is a report on Hadron Resonance Gas shortened as HRG. It is a form of matter that forms after the thermalization of Quark-Gluon Plasma (QGP). This QGP existed some time after the big-bang, exists today in the core of neutron stars and is formed when heavy-nuclei are collided in colliders like L.H.C. and R.H.I.C.

Bag-model is the one that can be used for estimating the magnitude of thermodynamic quantities like pressure and temperature, that exist within nuclear matter and hadrons. A simple depiction of this model is as follows:

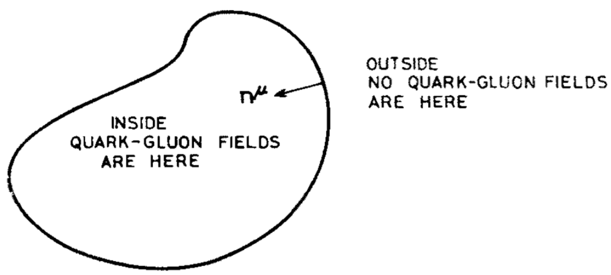


Figure 1: The M.I.T Bag model. Courtesy, image taken from [1]

So in the finite volume within the boundary of the Hadron particle, there are quark-gluon fields. At equilibrium conditions in nuclear-matter, the bag-pressure crushing quark-gluons inwards is equal to the quark-gluon pressure pushing the bag out-wards. Bag-pressure is the pressure exerted by the boundary of the hadron-bag inwards on the quark-gluon particles. In what follows, a simple method will be used instructively to deduce the quark-gluon gas pressure that develops in the bag. And the bag-pressure should be equal to this estimate. For the formation of QGP, the quark-gluon pressure must exceed the bag-pressure. This imbalance in the pressures can happen in two ways:

1. quark-gluon pressure increases due to high temperatures ($\sim 144\ MeV$),
2. quark-gluon pressure increases due to high baryon density. ($\sim 0.72\ fm^{-3}$)

2 Pressure of Mass-less relativistic Quark-Gluon Gas

[1] The set-up one considers now is, a system of quarks and gluons in a spatial volume V and temperature T. Also, this is a relativistic mass-less gas.

2.1 Quarks and Anti-quarks

Consider the phase-space volume occupied by the gas, when the momentum is p and in the interval dp, then volume in phase-space is:

$$V_{phase} = 4\pi p^2 dpV \tag{1}$$

Volume occupied by a state is:

$$V_{state} = (2\pi\hbar)^3 \tag{2}$$

[1]for next two section Ref. [2]

Say the chemical potential of this quark gas is μ_q at temperature T and degeneracy is g_q and since quarks are Fermions, then the infinitesimal number of quarks present in this phase space shell is:

$$dN_q = g_q \cdot \frac{4\pi p^2 dpV}{(2\pi)^3} \cdot \left(\frac{1}{1+e^{\frac{p-\mu_0}{T}}} \right) \qquad where\ \hbar = 1 \tag{3}$$

Note: Presence of Anti-quarks $\equiv$ absence of Quarks in the *negative states*. $\therefore$ if total number of particles in negative states is:

$$N_{tot}^{-} = \frac{g_q V}{(2\pi)^3} \int_{-\infty}^{0} 4\pi p^2 dp \tag{4}$$

Number of quarks in the negative states with chemical potential μ_q is:

$$N_q^{-} = \frac{g_q V}{(2\pi)^3} \int_{-\infty}^{0} 4\pi p^2 dp \cdot \left(\frac{1}{1+e^{\frac{p-\mu_0}{T}}} \right) \tag{5}$$

$\therefore$ number of anti-quarks $N_{\bar{q}}^{-} = N_{tot}^{-} - N_q^{-} = \frac{g_q V}{(2\pi)^3} \cdot \int_{-\infty}^{0} 4\pi p^2 \left(1 - \frac{1}{1+e^{\frac{p-\mu_q}{T}}} \right) dp$. Thus the number density will be:

$$n_{\bar{q}} = \frac{g_q}{(2\pi)^3} \cdot \int_{-\infty}^{0} 4\pi p^2 \left(1 - \frac{1}{1+e^{\frac{p-\mu_q}{T}}} \right) dp \tag{6}$$

$$\implies n_{\bar{q}} = \frac{g_q}{(2\pi)^3} \cdot \int_{-\infty}^{0} 4\pi p^2 \left(\frac{e^{\frac{p-\mu_q}{T}}}{1+e^{\frac{p-\mu_q}{T}}} \right) dp$$

$$\implies n_{\bar{q}} = \frac{g_q}{(2\pi)^3} \cdot \int_{-\infty}^{0} 4\pi p^2 \left(\frac{1}{1+e^{\frac{-p+\mu_q}{T}}} \right) dp$$

Make the substitution, $-p = p_0 \implies dp = -dp_0$, when $p \to -\infty$ then, $p_0 \to \infty$.

$$\implies n_{\bar{q}} = \frac{g_q}{(2\pi)^3} \cdot \int_{0}^{\infty} 4\pi p_0^2 \left(\frac{1}{1+e^{\frac{p_0+\mu_q}{T}}} \right) dp_0$$

Consider $\mu_q = 0$ then,

$$n_{\bar{q}} = \frac{g_q}{(2\pi)^3} \cdot \int_{0}^{\infty} 4\pi \left(\frac{p_0^2}{1+e^{\frac{p_0}{T}}} \right) dp_0 \tag{7}$$

The energy for a relativistic mass-less gas is $E = pc$, and in natural units, $E = p$. The infinitesimal energy dE due to quarks in a volume V at number density $n_{\bar{q}}$ and with each having energy p is:

$$dE = pV dn_q = pcV dn_q$$

$$dE = pV dn_q \quad \text{ since } c = 1 \tag{8}$$

Then the total energy would be,

$$E_{tot} = \int_{0}^{\infty} dE = \frac{g_q V}{(2\pi)^3} \int_{0}^{\infty} 4\pi p_0^3 dp_0 \left(\frac{1}{1+e^{\frac{p_0}{T}}} \right) \tag{9}$$

Let $\frac{p_0}{T} = z$, then, $dp_0 = T dz$

$$E_{tot} = \frac{g_q V T^4}{2\pi^2} \int_{0}^{\infty} \frac{z^3}{1+e^z} dz$$

Now, denote the integral in the above with I, then it can be reduced as:

$$thenI = \int_{0}^{\infty} \frac{z^3}{1+e^z} dz = \int_{0}^{\infty} \frac{z^3 e^{-z}}{1+e^{-z}} dz \tag{10}$$

Notice, the G.P. sum: $1 - e^{-z} + e^{2z} - e^{-3z} + ... = \sum_0^{\infty} (-1)^n e^{-nz} = \frac{1}{1-(-e^{-z})}$

$$\therefore I = \int_0^{\infty} z^3 e^{-z} \left(\sum_0^{\infty} (-1)^n e^{-nz} \right) dz$$

Interchange sum and integral:

$$\sum_0^{\infty} (-1)^n \int_0^{\infty} z^3 e^{-(n+1)z} dz$$

Substitute now $z = \frac{t}{n+1} \implies dz = \frac{dz}{n+1}$, then

$$I = \sum_0^{\infty} \frac{(-1)^n}{(n+1)^4} \int_0^{\infty} t^3 e^{-t} dt$$

$$\implies I = \Gamma(4) \sum_0^{\infty} \frac{(-1)^n}{(n+1)^4}$$

The infinite sum is evaluated as:

$$S = \sum_0^{\infty} \frac{(-1)^n}{(n+1)^4}$$

Letting $n + 1 \to m$ then,

$$S = \sum_1^{\infty} \frac{(-1)^{m-1}}{m^4} = \sum_{1,3,5,...} \frac{1}{m^4} - \sum_{2,4,6,...} \frac{1}{m^4}$$

$$S = \sum_1^{\infty} \frac{1}{m^4} - 2 \sum_{2,4,...} \frac{1}{m^4}$$

Letting in the second term, $m \to 2m$ then,

$$S = \sum_1^{\infty} \frac{1}{m^4} - 2 \sum_1^{\infty} \frac{1}{(2m)^4}$$

$$S = \zeta(4)(1 - 2^{-3})$$

So, the integral becomes:

$$I = \Gamma(4)\zeta(4)\frac{7}{8} \tag{11}$$

Then, the full expression is:

$$E_q = \frac{7}{8} g_q V T^4 \frac{\pi^2}{30}$$

The expression for pressure $P = \frac{E}{3V}$:

$$P_q = \frac{7}{8} g_q T^4 \frac{\pi^2}{90} \tag{12}$$

2.2 Gluon Gas

Gluons are Bosons, and thus the distribution function will be Bose-Einstein Distribution. Following mathematics similar to the above, one finds:

$$P_g = g_g \left(\frac{\pi^2}{90} \right) T^4 \tag{13}$$

Thus, total pressure is then $P_{total} = P_g + p_q$:

$$P_{total} = \left(g_g + \frac{7}{8} g_q \right) \frac{\pi^2}{90} T^4 \tag{14}$$

3 Ideal HRG

3.1 Energy density

In preceding sections the gas was supposed to be massless, i.e., rest-mass $m_0 = 0$. For Ideal Hadron Resonance Gas, Id-HRG $m_0 \neq 0$. Denote momentum by k, then the energy of the relativistic gas is: $E = \sqrt{k^2 + m_0^2}$, letting c = 1. Now, the number density $\left(\frac{dNi}{V}\right)$ is given by:

$$\frac{dN_i}{V} = dn_i = g_i \frac{4\pi k^2 dk}{(2\pi)^3} \left(\exp\left(\frac{\sqrt{k^2 + m_0^2} - \mu_i}{T} \right) + \eta i \right)^{-1} \tag{15}$$

$$d\epsilon_i = E dn_i = \frac{g_i k^2 dk}{(2\pi^2) \left(\exp\left(\frac{\sqrt{k^2+m_0^2} - \mu_i}{T} \right) + \eta_i \right)} \tag{16}$$

$$\epsilon_i = \frac{g_i}{2\pi^2} \int_0^\infty \frac{\sqrt{k^2 + m_0}\ k^2 dk}{\left(\exp\left(\frac{\sqrt{k^2+m_0^2} - \mu_i}{T} \right) + \eta_i \right)} \tag{17}$$

3.2 Pressure Density

Relativistic pressure is given by:

$$p_i = \frac{1}{V} \frac{k^2}{3E} \quad \text{here, } c = 1$$

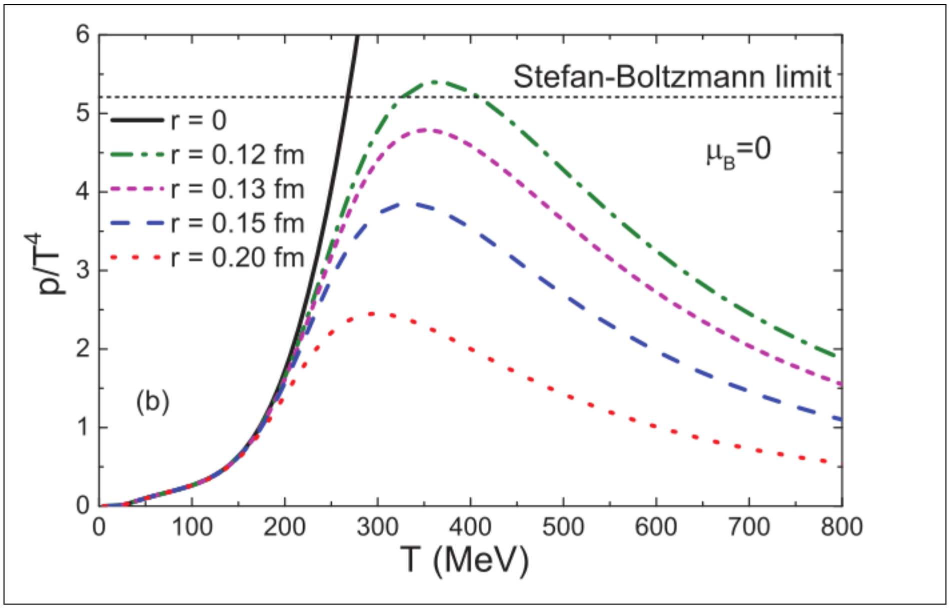
$$dp_i^{id} = p_i dN_i$$

$$p_i^{id} = \frac{g_i}{6\pi^2} \int_0^\infty \frac{k^4}{\sqrt{k^2 + m_0^2}} \frac{dk}{\left(\exp\left(\frac{\sqrt{k^2+m_0^2} - \mu_i}{T} \right) + \eta_i \right)} \tag{18}$$

4 Short coming of Id-HRG Model

The Id-HRG model treats hadron gas as constituted by point-like masses, i.e., the hard-core radii of the hadrons involved are $r_i = 0 \ \forall \ i$. However, this causes one problem. At high temperatures, the id-HRG pressure predicted by this model exceeds the **Stefan-Boltzmann limit** (see eg. Ref [3]). This is apparent from the following image (courtesy Ref. [3]):

One observes that without including the hardcore radii of the particles the pressure blows up. This is non-physical. Thus, to model the real gas system it is imperative to include the hardcore-radii of the particles. As can be seen above, with $r < 0.13\ fm$ the behaviour of the gas goes lower than the Stefan-Boltzmann limit. Thus, to get a physical pressure relation, one must allow for non-zero $r < 0.13\ fm$.



5 Pressure under Boltzmann Approximation

For a relativistic ideal-Hadron gas, pressure was expressed as:

$$p_i^{id} = \frac{g_i}{6\pi^2}\int_0^\infty \frac{k^4}{\sqrt{k^2+m_0^2}} \frac{dk}{\left(\exp\left(\frac{\sqrt{k^2+m_0^2}-\mu_i}{T}\right)+\eta_i\right)} \tag{19}$$

The integral on the R.H.S may or may not have a closed analytical form. To simplify our analysis initially, consider the Boltzmann-approximation $\eta_i = 0$. Then the expression is simplified. Further approximations will be added for analysis.

5.1 Case-1:

Consider the simplest case when rest-mass of each i^{th} particle is 0, i.e., $m_0 = 0$. Also, the i^{th} species chemical potential is zero, i.e., $\mu_i = 0$. Then, the integral becomes:

$$p_i^{id} = \frac{g_i}{6\pi^2}\int_0^\infty k^3 \frac{dk}{(\exp(\frac{k}{T}))} \tag{20}$$

Let $z = \frac{k}{T}$ then, $dk = Tdz$

$$\implies I = \int_0^\infty T^3 z^3 \frac{Tdz}{e^z}$$

$$\implies p_i^{id} = \frac{g_i T^4}{6\pi^2}\int_0^\infty z^3 e^{-z} dz$$

$$\implies p_i^{id} = \frac{g_i T^4}{6\pi^2}\Gamma(4)$$

This can be visualised in the following figure. It was made using MatPlotLib, using Gaussian-Lengendre Quadrature Formula for integration [2]:

[2]the number of points taken are 300, and $a = 0$ and $b = 1e6$

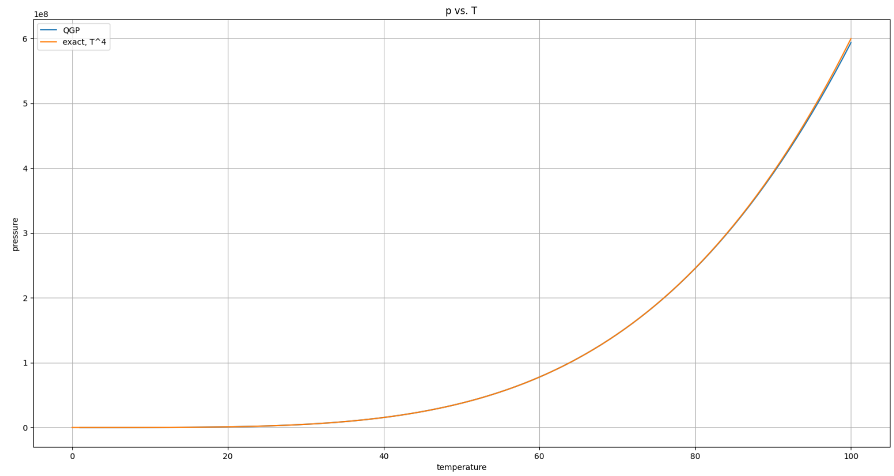


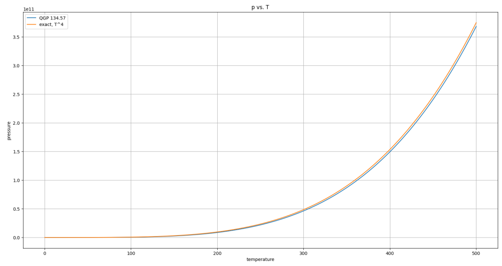
Figure 2: This image shows $6 \cdot T^4$ in orange, which is the expected integral solution analytically; the blue line is the numerically integrated solution.

Particle	mass($\frac{MeV}{c^2}$)
Pion	134.57
Kaon	497.611
Proton	938.272

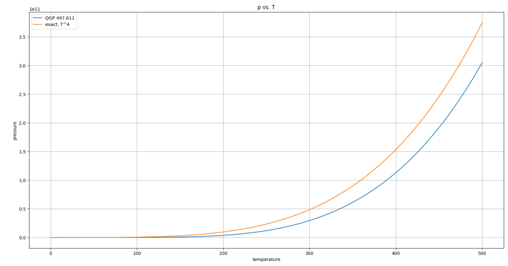
5.2 Case-2:

Unlike in the previous section, in this one, mass will be non-zero and finite, i.e., $m_0 \neq 0$. Due to non-zero mass, (19) does not reduce to the friendly form of (20); so, one does not obtain a nice analytical closed form of the non-zero-mass equation. Then, numerical integration is used. The particles considered here are:

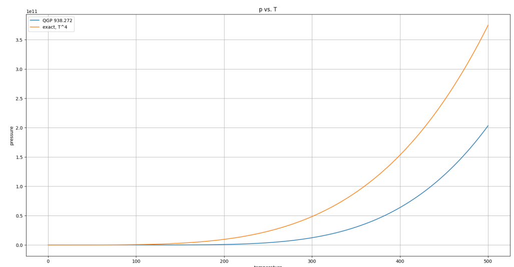
Below figure displays the plots of pressure vs. temperature for different particle-masses. Interesting observation can be made from this:



(a)



(b)



(c)

Figure 3: (a) the P vs. T graph for a Pion, (b) P vs. T for a Kaon and (c) P vs. T for a Proton. The Orange line depicts the function $P = 6 \cdot T^4$. Observe the variation in pressure at the same temperature for particles with different masses.

One can see clearly that **as particle-mass increases, pressure decreases at the same temperature**. This is more apparent in the following simultaneous plot.

One question arises: Why is this behaviour observed? The reason for such behaviour can be understood physically by recalling the **Maxwell-Boltzmann Speed Distribution function**.

5.2.1 Using Bessel Function of Second Kind

[3] Consider (19) with all the physical constants:

$$p_i^{id} = \frac{g_i}{6\pi^2\hbar^3}\int_0^\infty \frac{c^2k^4}{\sqrt{k^2c^2+m_0^2c^4}}\frac{dk}{\left(\exp\left(\frac{\sqrt{k^2c^2+m_0^2c^4}-\mu_i}{K_BT}\right)+\eta_i\right)} \tag{21}$$

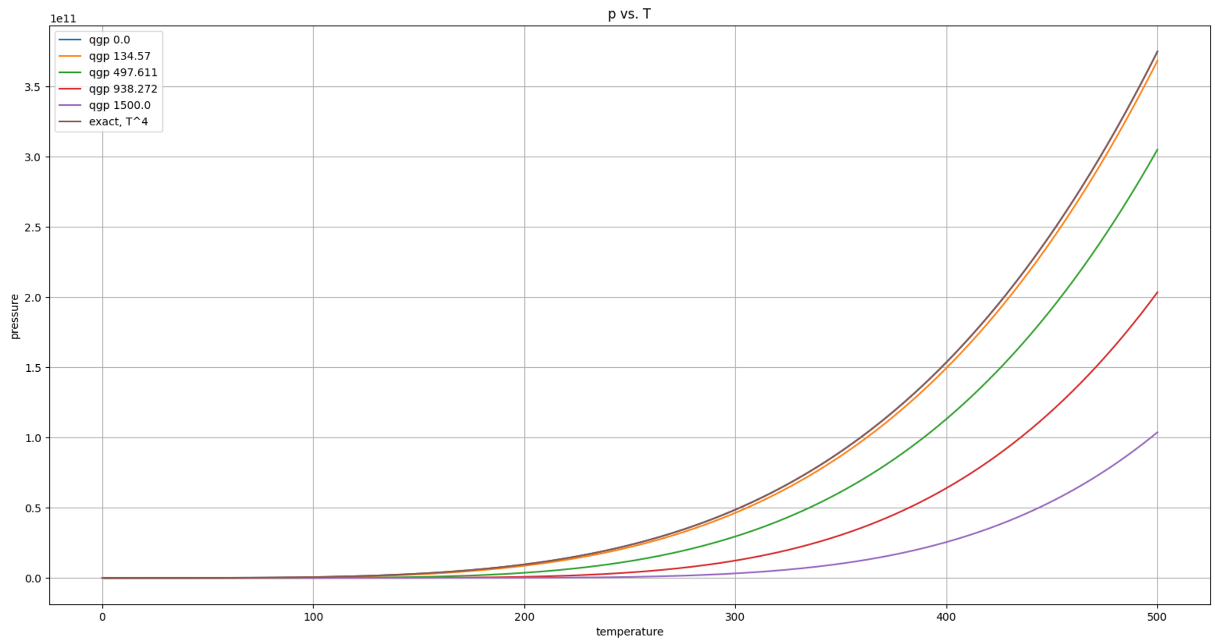
Abstract $m_0^2c^4$ then, and set $\eta_i = 0$:

$$p_i^{id} = \frac{g_i}{6\pi^2\hbar^3}e^{\frac{\mu_i}{K_BT}}\int_0^\infty \frac{c^2k^4}{m_0c^2\sqrt{\left(\frac{k}{m_0c}\right)^2+1}}\frac{dk}{\left(\exp\left(\frac{m_0c^2\sqrt{\left(\frac{k}{m_0c}\right)^2+1}}{K_BT}\right)\right)} \tag{22}$$

Now, let $\frac{k}{m_0c} = \tan\theta \implies dk = m_0c\sec^2\theta d\theta$ and $0 \equiv 0$ and $\infty \equiv \frac{\pi}{2}$, then:

$$p_i^{id} = \frac{g_i}{6\pi^2\hbar^3}e^{\frac{\mu_i}{K_BT}}\int_0^{\frac{\pi}{2}} \frac{m_0^4c^4\tan^4\theta}{m_0\sqrt{\tan^2\theta+1}}\frac{m_0c\sec^2\theta d\theta}{\left(\exp\left(\frac{m_0c^2\sec\theta}{K_BT}\right)\right)} \tag{23}$$

[3] Refer to [4]



see

Figure 4: This figure is simultaneous plot for P vs. T for various masses listen in the top-left corner. One finds as mass increases, at same temperature, corresponding pressure decreases.

$$\implies p_i^{id} = \frac{g_i m_0^4 c^5}{6\pi^2\hbar^3} e^{\frac{\mu_i}{K_B T}} \int_0^{\frac{\pi}{2}} \frac{\tan^3\theta}{1} \frac{\tan\theta \sec\theta d\theta}{\left(\exp\left(\frac{m_0 c^2 \sec\theta}{K_B T}\right)\right)}$$

Substitute $\sec\theta = z \implies dz = \sec\theta \cdot \tan\theta d\theta$ and $\tan\theta = (\sec^2\theta - 1)^{\frac{1}{2}}$, and $0 \equiv 1$ and $\frac{\pi}{2} \equiv \infty$ thus:

$$p_i^{id} = \frac{g_i m_0^4 c^5}{6\pi^2\hbar^3} e^{\frac{\mu_i}{K_B T}} \int_1^{\infty} \frac{(z^2-1)^{3/2} dz}{\left(\exp\left(\frac{m_0 c^2}{K_B T} z\right)\right)} \tag{24}$$

Make another substitution, $\frac{m_0 c^2}{K_B T} z = \tau$, $z = \frac{K_B T}{m_0 c^2}\tau$ and so, $dz = \frac{K_B T}{m_0 c^2} d\tau$, this yields:

$$p_i^{id} = \frac{g_i m_0^4 c^5}{6\pi^2\hbar^3} e^{\frac{\mu_i}{K_B T}} \int_{\frac{m_0 c^2}{K_B T}}^{\infty} \frac{((\frac{K_B T}{m_0 c^2}\tau)^2 - 1)^{3/2} \frac{K_B T}{m_0 c^2} d\tau}{\exp(\tau)} \tag{25}$$

$$\implies p_i^{id} = \frac{g_i m_0^4 c^5}{6\pi^2\hbar^3} e^{\frac{\mu_i}{K_B T}} \left(\frac{K_B T}{m_0 c^2}\right)^4 \int_{\frac{m_0 c^2}{K_B T}}^{\infty} \frac{(\tau^2 - (\frac{m_0 c^2}{K_B T})^2)^{3/2} d\tau}{\exp(\tau)}$$

This is the same mathematical form as:

$$K_n(a) = \frac{2^n n!}{(2n)!} \frac{1}{a^n} \int_a^{\infty} (\tau^2 - a^2)^{(n-\frac{1}{2})} e^{-\tau} d\tau \tag{26}$$

With $a = \frac{m_0 c^2}{K_B T}$, $n = 2$:

$$\implies \int_{\frac{m_0 c^2}{K_B T}}^{\infty} \frac{(\tau^2 - (\frac{m_0 c^2}{K_B T})^2)^{3/2} d\tau}{\exp(\tau)} = K_2\left(\frac{m_0 c^2}{K_B T}\right) \frac{4!}{4 \cdot 2!} \left(\frac{m_0 c^2}{K_B T}\right)^2$$

Thus, (24) becomes:

$$p_i^{id} = \frac{g_i m_0^4 c^5}{6\pi^2\hbar^3} e^{\frac{\mu_i}{K_B T}} \left(\frac{K_B T}{m_0 c^2}\right)^4 K_2\left(\frac{m_0 c^2}{K_B T}\right) \frac{4!}{4 \cdot 2!} \left(\frac{m_0 c^2}{K_B T}\right)^2 \tag{27}$$

$$\implies p_i^{id} = \frac{g_i m_0^2 c (K_B T)^2}{2\pi^2\hbar^3} \exp\left(\frac{\mu_i}{K_B T}\right) K_2\left(\frac{m_0 c^2}{K_B T}\right) \tag{28}$$

Thus, using Bessel-function of second kind, it is possible to write the expression in a concise fashion.

6 Number Density Under Boltzmann Approximation

In the two cases considered above, the chemical potential of the i^{th} species was taken to be zero. Now, consider the case when $\mu_i \neq 0$, and is some finite value. The expression for number-density, with all the physical constants considered, is:

$$dn_i = g_i \frac{4\pi k^2 dk}{(2\pi)^3} \exp\left(\frac{\sqrt{k^2c^2 + m_0^2c^4} - \mu_i}{K_B T}\right)^{-1} \tag{29}$$

$$\implies n_i = \frac{g_i e^{\frac{\mu_i}{K_B T}}}{2\pi^2\hbar^3} \int_0^\infty k^2 \exp\left(-\frac{\sqrt{k^2c^2 + m_0^2c^4}}{K_B T}\right) dk \tag{30}$$

Extract $m_0^2c^4$ from the exponential, then:

$$\implies n_i = \frac{g_i e^{\frac{\mu_i}{K_B T}}}{2\pi^2\hbar^3} \int_0^\infty k^2 \exp\left(-\frac{m_0c^2\sqrt{\left(\frac{k}{m_0c}\right)^2 + 1}}{K_B T}\right) dk \tag{31}$$

Make transformations similar to the previous section, $\frac{k}{m_0c} = \tan\theta \implies dk = m_0 c \sec^2\theta d\theta$ and $0 \equiv 0$ and $\infty \equiv \frac{\pi}{2}$, then

$$\implies n_i = \frac{g_i e^{\frac{\mu_i}{K_B T}}}{2\pi^2\hbar^3} \int_0^{\frac{\pi}{2}} \frac{m_0^2c^2\tan^2\theta \cdot m_0c \cdot \sec^2\theta d\theta}{\exp\left\{\frac{m_0c^2}{K_BT}\right\}\sec\theta} \tag{32}$$

$$\implies n_i = \frac{g_i m_0^3 c^3 e^{\frac{\mu_i}{K_B T}}}{2\pi^2\hbar^3} \int_0^{\frac{\pi}{2}} \frac{\tan\theta\sec\theta \cdot \tan\theta\sec\theta d\theta}{\exp\left\{\frac{m_0c^2}{K_BT}\right\}\sec\theta} \tag{33}$$

Make the substitution $\tau = \frac{m_0c^2}{K_BT}\sec\theta \implies d\tau = \frac{m_0c^2}{K_BT}\sec\theta\tan\theta d\theta$, then:

$$\implies n_i = \frac{g_i m_0^3 c^3 e^{\frac{\mu_i}{K_B T}}}{2\pi^2\hbar^3} \int_{\frac{m_0c^2}{K_BT}}^\infty \frac{\left(\left(\frac{K_BT}{m_0c^2}\right)^2\tau^2 - 1\right)^{\frac{1}{2}} \frac{K_BT}{m_0c^2} \cdot \tau \cdot \frac{K_BT}{m_0c^2} \cdot d\tau}{e^\tau} \tag{34}$$

$$\implies n_i = \frac{g_i m_0^3 c^3 e^{\frac{\mu_i}{K_B T}}}{2\pi^2\hbar^3} \left(\frac{K_BT}{m_0c^2}\right)^3 \int_{\frac{m_0c^2}{K_BT}}^\infty \frac{\left(\tau^2 - \left(\frac{m_0c^2}{K_BT}\right)^2\right)^{\frac{1}{2}} \cdot \tau \cdot d\tau}{e^\tau} \tag{35}$$

Denote by $a = \frac{m_0c^2}{K_BT}$, then:

$$\implies n_i = \frac{g_i m_0^3 c^3 e^{\frac{\mu_i}{K_B T}}}{2\pi^2\hbar^3} \left(\frac{K_BT}{m_0c^2}\right)^3 \int_a^\infty \frac{(\tau^2 - a^2)^{\frac{1}{2}} \cdot \tau \cdot d\tau}{e^\tau} \tag{36}$$

The integral on the right-side is exactly of the following integral form for Bessel-funtion of the second kind:

$$K_n(a) = \frac{2^{n-1}(n-1)!}{(2n-2)!} \cdot \frac{1}{a^n} \cdot \int_a^\infty \frac{\left(\tau^2 - a^2\right)^{n-\frac{3}{2}} \cdot \tau \cdot d\tau}{e^\tau} \tag{37}$$

Here, $n = 2$, so:

$$\int_a^\infty \frac{\left(\tau^2 - a^2\right)^{\frac{1}{2}} \cdot \tau \cdot d\tau}{e^\tau} = K_2(a) \frac{2!}{2 \cdot 1!} \left(\frac{m_0 c^2}{K_B T}\right)^2 \tag{38}$$

Substituting (38) in (36), one obtains:

$$n_i = \frac{g_i m_0^3 c^3 e^{\frac{\mu_i}{K_B T}}}{2\pi^2 \hbar^3} \left(\frac{K_B T}{m_0 c^2}\right)^3 K_2(a) \cdot \frac{2!}{2 \cdot 1!} \left(\frac{m_0 c^2}{K_B T}\right)^2 \tag{39}$$

This, upon cancellations and reorganization yields:

$$n_i = \frac{g_i \cdot m_0^2 c \cdot e^{\frac{\mu_i}{K_B T}}}{2\pi^2 \hbar^3} (K_B T) \cdot K_2 \left(\frac{m_0 c^2}{K_B T}\right) \tag{40}$$

The equation has a neater form if one employs natural units: $\hbar = c = k_B = 1$:

$$n_i = \frac{g_i \cdot m_0^2 T \cdot e^{\frac{\mu_i}{T}}}{2\pi^2} \cdot K_2 \left(\frac{m_0}{T}\right) \tag{41}$$

So, in natural units mass m_0, T and μ_i are expressed in MeV. So, from (41), n_i in natural units is in MeV^3. But that seems strange! Number density as $[energy]^3$. The apparent discrepancy can be solved if one devides by the cube of $\hbar c$. In natural units $[\hbar c] = MeV \cdot fm$. Thus, if one now divides the number-density with the cube of $\hbar c$, then number density will have fm^{-3} as units. Which seems more understandable [4]. In what follows now, a trial has been made to use the results hitherto derived to match plots and data from [6]. In this report we will consider particle number-density at temperature $156.5 \pm 1.5\ MeV$, baryo-chemical potential is very close to zero, $\mu = 0.7 \pm 3.8\ MeV$. Giving these numbers as input-parameters to (41) along with particle-mass, one can estimate the theoretically-expected particle yield. What will be predicted thus, will purely be due to thermal yield of hadrons.

6.1 Number Density vs. Temperature

Note- images for this section can be accessed from here. The number density for each type of particle can be read-off at a particular temperature using the above plots. Take for instance the case of a pion+ (π_+): one wants to find out the number of π_+ that would be present at a given temperature and also the volume of the fireball at the same temperature. From experiments like **ALICE** at LHC, **STAR** at RHIC, it is possible to observe the number of particles registered in the detectors. So, out of these **high-energy heavy-ion collisions** the sole data one obtains are the particle-numbers, that formed due to hadronization of the QGP and due to resonance decays. This latter source we will ignore in our initial treatment of the subject. Maybe, later on we will include these contributions. One remark is due here, even after the inclusion of the decay product numbers, a marked difference will be observed only in the produce of light mesons. Heavy baryons and light nuclei numbers will not be affected much. The key step to out method is to observe that at a given temperature, baryo-chemical potential, mass and spin-degeneracy the ratio of particle number-density is fixed. Thus, having the real-time data of one particle, call this the reference particle, allows theoretical calculations of other particles due to (41). To find the volume one will have to convert into S.I. unit (fm^{-3}). Using the reference particle-number (N_1) and number-density (n_1) in MeV^3 we will fix the volume of the fireball (V_{exp}) thus:

$$\frac{n_1}{197.33^3} \cdot V_{exp} = N_1 \tag{42}$$

$$\implies V_{exp} = \frac{N_1 \cdot 197.33^3}{n_1} \tag{43}$$

[4] one can take a loot at *SUPPLEMENTARY NOTES ON NATURAL UNITS*, by Prof. Jaffe 1997 [5]

Also, to estimate the number of another particle (N_2) using its number-density (n_2):

$$N_2 = N_1 \cdot \frac{n_2}{n_1} \tag{44}$$

For this exercise, kaon+ (κ_+) was used as the reference particle. Its particle number (N_1) was taken to be 103.0. Temperature at which calculations were performed was 156.5 MeV. The baryo-chemical potentials are taken 0.0 MeV and 0.7 MeV for mesons and baryons respectively. The volume predicted using (42) is 8162.91 fm^3. The following table summarises the number of particles predicted by (44).

Particle	**Number Predicted**
pion+	368.15543169047936
pion-	368.15543169047936
kaon+	109.0
kaon-	109.0
kaon0	107.1613322070254
phi	26.38076390665611
proton	26.65530385942287
proton-bar	26.65530385942287
lamda	10.662570616959389
lamda-bar	10.662570616959389
cascade-	3.561206659260469
cascade+	3.698448728230954
omega-	1.0314189227795907

Table 1: Particle yield calculated

7 Scatter Plot of Particle-Yield

In this section a plot of Particle-yield against particle-name is shown. Take a look at this figure. The orange dots are calculated thermal yield points and the blue are experimental data points. One will notice that the **pion+** and **pion-** data points are above the calculated points. This is an effect of particle-yield that have non-thermal origin(thus not accounted in our model): one big factor is high-mass resonances' decays into lighter mesons. Because the model does not account for non-thermal effects, the light mesons numbers' mismatch to some extent.

8 Particle Yield vs. Particle Mass

[5]In the plot this figure logarithmic dependence of particle-yield is portrayed vs. particle-mass. This figure is in excellent agreement with the this figure that is taken from [6].

9 Varying Chemical Potential and Temperature

This figure is a plot that was produced by varying both chemical-potential and temperature simultaneously. Observe the trend of these 2-D surfaces. Observe the two tables below:

One observes that the baryon-to-baryon ratio at same temperatures but at different chemical potentials is different; not by a big fraction but different. However, if one looks at this figure then considerable effect of variation in μ can be seen towards the higher values of μ.

[5]See [6]

Temperature	Ratio of Λ to Ξ^- Baryons
50.0	42.7201
150.0	3.0457

Table 2: $\mu = 0.7\ MeV$

Temperature	Ratio of Λ to Ξ^- Baryons
50.0	42.2551
150.0	3.0460

Table 3: $\mu = 3.8\ MeV$

10 Example of Adding Decay Contributions

As an example of adding thermal decay contributions to thermal yield, consider a very basic and instructive example of π particles. Let N_{tot}, N_T and N_d denote the total, thermal and decay yields respectively. Then:

$$N_{tot} = N_T + N_d \tag{45}$$

The decay contribution is the total number of pions that are produced from the decay of parent particles. A crucial information for determining N_d is the **branching ratio** of parent particle. Branching ratio denotes the chance of a parent particle decaying into a particular channel. It is generally given in percentages. Let N_r denote the number of resonances of the r^{th} kind, and let n_r be the branching ratio of of resonance r decaying into particle h, then:

$$N_d = \sum_r N_r \cdot n_r \tag{46}$$

$$\implies N_{tot} = N_T + \sum_r N_r \cdot n_r \tag{47}$$

Consider the decay of Kaon, Λ and Ξ particles to pions. Their respective branching ratios are given in the following table:

Particle	Branching Ratio (%)
kaon	28.013
Λ	63.90
Ξ	99.89

Table 4: List of Particles that were taken as parent particles for pion and their corresponding branching ratios.

Look at this table for a comparison of the calculated particle yield before and after adding the decay contributions:

So, one observes that the particle yield has indeed increased and that the cited numbers in [6] can be obtained if one considers all the resonances that will decay via pion-channels.

Particle	Before	After
Pion+	368.155	418.085
Pion-	368.155	418.085

Table 5: Comparision of particle yield number before and after adding the decay contributions

11 Conclusion

In this report an introduction to Statistical Thermalisation model of HRG was given. The relations for pressure, energy density were derived in detail. Under Boltzmann condition the pressure was evaluated in two scenarios: i) zero mass case, ii) non-zero mass case. In the first case pressure is directly proportional to the fourth power of Temperature. In the latter case no analytic closed form of expression is obtained. So, numerical technique needs to be used. Using Gaussian-Lagrange Quadrature the integration was performed and plots for various masses were obtained.One milestone-observation is heavier particles exert lesser pressure. Same exercise was performed using Bessel-function of second kind.

Later, number density as a function of temperature is studied. An attempt is made to fix volume of fireball and fix numbers of various particles using the experimental number of Kaon particles. Then various plots were discussed of particle-yield vs. mass, chemical potential, temperature, etc. Also an example of resonance-decay contributions is worked out.

12 Acknowledgment

I would like to extend my heartfelt gratitude to Prof. Victor Roy, School of Physics, NISER for this opportunity of summer project, the warm and homely meetings and discussions on topics and doubts from the project. I would also like to thank Ankit Panda and Krishanu Sengupta bhaiya for their friendliness and insights into my work.

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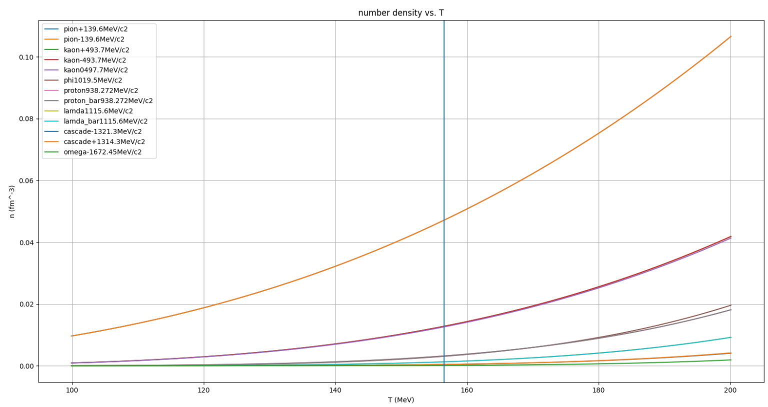
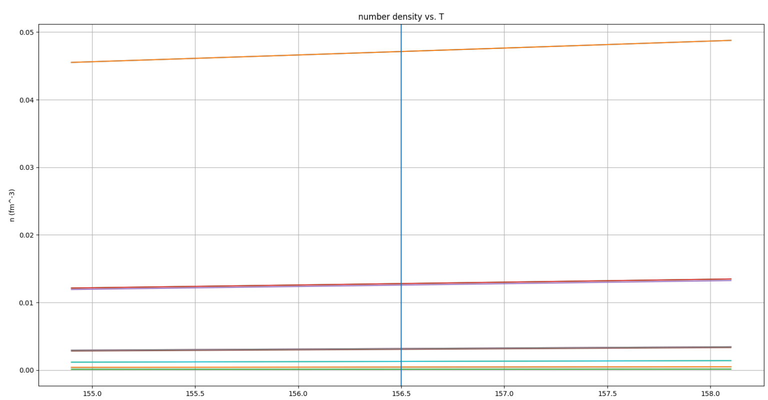


(a)



(b)

Figure 5: (a) Observe the legend in this sub-figure, it lists the names of particles. The trend of these plots is generally like T^3. A line $x = 156.5MeV$ is also drawn here for the representing the temperature at which the particle density is calculated. (b) This sub-figure is a more zoomed version of (a). The baryons are very close to each other and the pion and kaon (mesons) lines are higher than these.

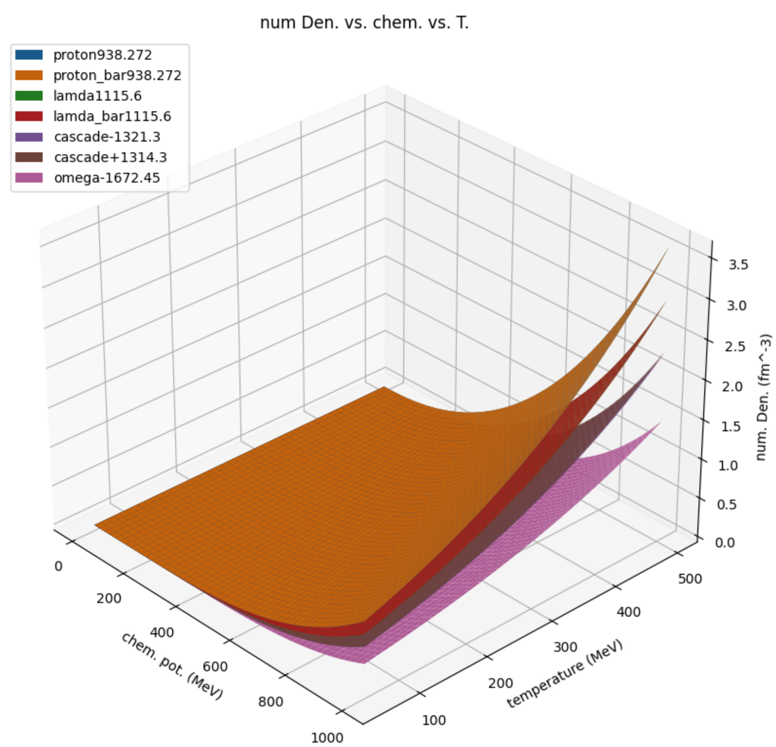


Figure 6: 3-D plot 1.

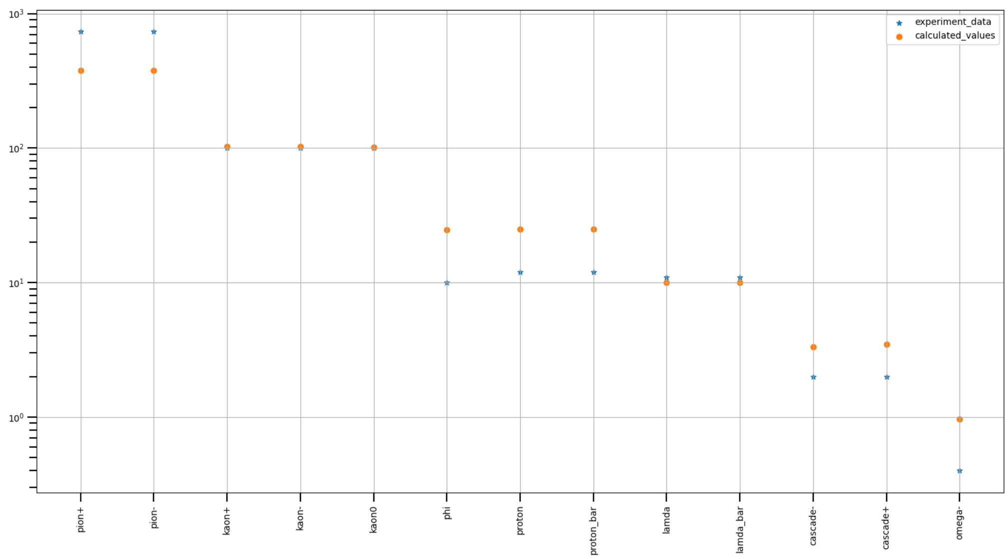


Figure 7: Scatter-plot comparing data points and predicted thermal yield points. The orange dots are thermal yield points and blue are experimental points. The data for the experimental dots are taken from [6].

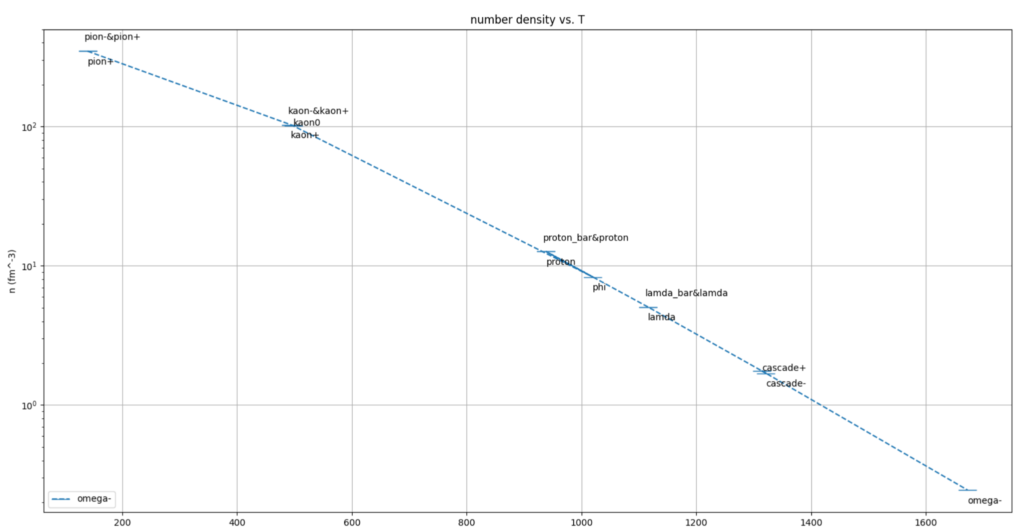


Figure 8: Log of particle yield vs. particle-mass.

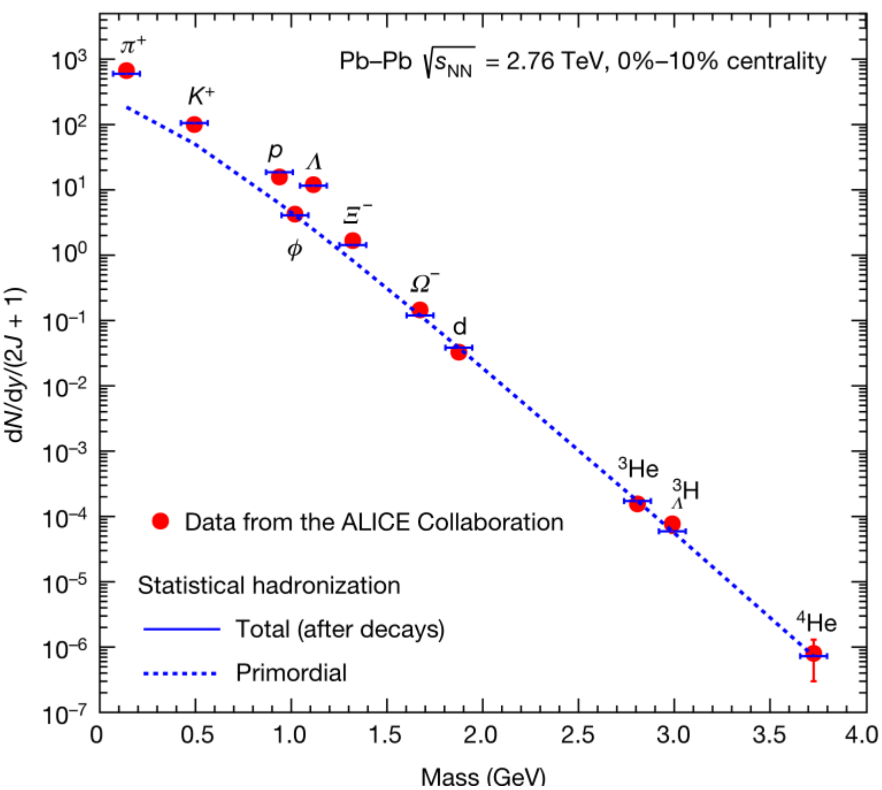


Figure 9: Figure from [6]. Observe its agreement upto Ω^- particle yield.