A comparison theorem, Sobolev imbeddings and Konrachov theorem for Riemannian manifolds

Tien NGUYEN MANH

Mar 22, 2018

Contents

This post is the first part of my reading note for [**?**]. The second part is here.

In this part, we will first establish the Sobolev imbeddings theorem and the Kondrachov theorem for Riemannian manifolds from the Euclidean version of these theorems.

Theorem 1 (Sobolev Imbedding for $\mathbb{R}^n$)**.** *Given $k, l \in \mathbb{Z},\ k > l \geq 0$ and $p, q \in \mathbb{R},\ p > q \geq 1$. Then*

1. *If $\frac{1}{p} = \frac{1}{q} - \frac{k-l}{n}$ then*

$$W^{k,q}(\mathbb{R}^n) \hookrightarrow W^{l,p}(\mathbb{R}^n)$$

is a continuous imbedding.

2. *If $\frac{k-r}{n} > \frac{1}{q}$ then*

$$W^{k,q}(\mathbb{R}^n) \hookrightarrow C^r_B(\mathbb{R}^n)$$

If $\frac{k-r-\alpha}{n} \le \frac{1}{q}$ *then*

$$W^{k,q}(\mathbb{R}^n) \hookrightarrow C^{r,\alpha}(\mathbb{R}^n)$$

where $C^r_B(\mathbb{R}^\ltimes)$ *denotes the space of* C^r *functions with bounded derivatives up to order* n*, equipped with the norm* $\|u\|_{C^r_B} = \max_{l\le r} \sup |\nabla^l u|$*, and* $C^{r,\alpha}$ *is the subspace of* C^r_B *of functions whose* r^{th}*-derivative is* α*-Holder, equipped with the norm* $\|u\|_{C^r,\alpha} = \|u\|_{C^r_B} + \sup_{P\neq Q}\{\frac{u(P)-u(Q)}{d(P,Q)^\alpha}\}$*.*

Theorem 2 (Kondrachov for $\Omega \subset \mathbb{R}^n$)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open subset with regular boundary and let* $k \in \mathbb{Z}_{\ge 0}$ *and* $p, q \in \mathbb{R}_{>0}$ *be such that* $1 \ge \frac{1}{p} > \frac{1}{q} - \frac{k}{n} > 0$ *then*

1. *The imbedding* $W^{k,q}(\Omega) \hookrightarrow L^p(\Omega)$ *is compact.*
2. *The imbedding* $W^{k,q}(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ *is compact if* $k - \alpha > \frac{n}{q}$ *where* $0 \le \alpha < 1$*.*
3. *The imbeddings* $W^{k,q}_0(\Omega) \hookrightarrow L^p(\Omega)$ *and* $W^{k,q}_0(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ *are compact, where* $W^{k,q}_0(\Omega)$ *denotes the closure of* $C^\infty_c(\Omega)$ *in* $W^{k,q}(\Omega)$*, i.e. the subspace of functions whose trace vanishes on the boundary of* Ω*.*

Theorem 1 will be generalised for complete manifolds with bounded curvature and injectivity radius, while Theorem 2 holds for compact Riemannian manifolds.

The generalisation will be done in 2 steps

1. Compare the volume form of the Riemannian metric g near a point and that of the Euclidean metric on the tangent space at that point. Theorem 4 gives an equivalent between the integral under g and the integral under Euclidean metric via the exponential map.
2. Reasonably use partition of unity to establish global results from local results (the Euclidean case). We will need a covering lemma (Calabi's lemma), which essentially reduces to a combinatorial result (Vitali's covering lemma).

Finally, we will apply imbedding theorems to solve the equation $-\Delta u = f$ on a Riemannian manifold when f is square-integrable.

1 Quick recall of Jacobi fields, Index inequality

Definition 1. *A **Jacobi** field is a field Y defined along a geodesic $\gamma(t)$ such that*

$$\frac{D^2}{dt^2}Y(t) + R(Y(t), \dot\gamma(t))\dot\gamma(t) = 0 \tag{1}$$

where R denotes the Riemann curvature tensor.

Remark 1.

1. *Since (1) is linear, a Jacobi field is uniquely defined given $Y(t_0)$ and $\dot Y(t_0)$.*
2. *If $Y(0) \perp \dot\gamma(0)$ and $\dot Y(0) \perp \dot\gamma(0)$ then $\dot Y(t) \perp \dot\gamma(t)$ for all t.*
3. *If Y, Z are Jacobi fields along γ then*

$$\langle Y, \dot Z\rangle - \langle \dot Y, Z\rangle = \text{const}$$

In particular, if Y, Z vanish at a same point p_0 in γ then $\langle Y, \dot Z\rangle = \langle \dot Y, Z\rangle$ on γ.

There are two ways to interpret Jacobi fields:

1. Jacobi fields are derivative of exponential maps
2. Jacobi fields are minimisers of Index form, i.e. the variation of second other of length.

The first interpretation is the content of the following Proposition.

Proposition 2.1. *Let $Y(t) = D\exp_p(tu).t\xi$ be a vector field defined on a geodesic $\gamma(t) = \exp_p tu$. Then Y satisfies*

$$\begin{cases} Y(0) = 0, \dot Y(0) = \xi, \\ \ddot Y + R(Y, \dot\gamma)\dot\gamma = 0, \end{cases} \tag{2}$$

hence a Jacobi field.

In concrete term, denote by ψ the exponential function at $p \in M$ and $q = \gamma(r) = \exp_p r\dot\gamma(0)$, then Proposition 2.1 says that if the Jacobi field Y vanishes at $p = \gamma(0)$, i.e. $Y(0) = 0$ then $Y(r)$ at $\gamma(r)$ is defined as follow: pull-back $\dot Y(0)$ by ψ, transport parallelly, w.r.t to the Euclidean metric of T_pM, $\psi^*\dot Y(0)$ from 0 to $X_0 = \psi^{-1}(q)$, then push-forward by ψ, one obtains $Y(r)$. See Figure 1.

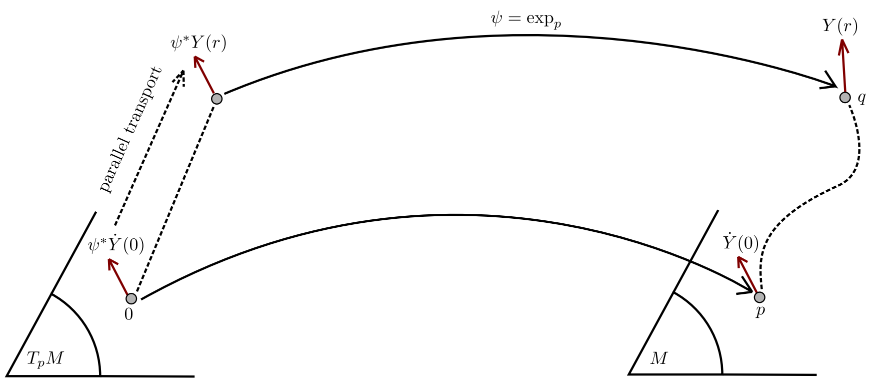

Figure 1: Jacobi fields and exponential maps.

Since Jacobi fields are derivatives of exponential maps, one can rephrase the phenomenon of cut-locus by Jacobi fields. Historically, a point q on a Riemannian manifold is said to be a **conjugate** point of p if there exists, along a geodesic connecting them, a Jacobi field vanishing on both p and q. This means that the exponential map with origin in p degenerates at a preimage of q. One can also prove that if q is in the cut-locus of p then at least one of the following situation occurs

1. q is a conjugate point of p.
2. There exists 2 minimising geodesic from p to q.

For another interpretation of Jacobi fields, note that given a geodesic γ and a vector field Z defined along γ, then the first variation of length when one varies γ by Z is 0 and the second variation can also be calculated without difficulty.

Proposition 2.2 (Second variation of length)**.** *Let $\gamma : [0, r] \longrightarrow M$ be a geodesic and Z be a vector field along γ that is orthogonal to $\dot{\gamma}$ at every point. Denote by L_λ length of the curve $t \mapsto \exp_{\gamma(t)} \lambda Z$ for $\lambda \ll 1$, then one has*

$$\frac{d^2}{d\lambda^2} L_\lambda \bigg|_{\lambda=0} = I(Z) := \int_0^r \left(\|Z(t)\|^2 + \langle R(\dot{\gamma}(t), Z(t))\dot{\gamma}(t), Z(t)\rangle \right) dt \qquad (3)$$

Definition 2. *Let $\gamma : [0, r] \longrightarrow M$ be a geodesic and Z be a orthogonal vector field along γ. The* ***Index form*** *$I(Z)$ of Z is defined by the RHS of* (3).

Remark 2. *The curvature term in* (3) *is* $K(\dot\gamma, Z)\|Z\|^2$ *where* K *denotes the sectional curvature of* M*.*

Jacobi fields can be seen as the unique minimiser of the Index form among vector fields defined on a geodesic $\gamma : [0, r] \longrightarrow M$ with the same value at $\gamma(0)$ and $\gamma(r)$.

Theorem 3 (Index inequality)**.** *Let* $\gamma : [0, r] \longrightarrow M^n$ *be a geodesic,* $p = \gamma(0)$ *and* $q = \gamma(r)$ *such that* p *has no conjugate point along* γ*, or equivalently the exponential map in direction* $\dot\gamma(0)$ *does not degenerate.*

- *Let* Z *be a (piecewise smooth) vector field along* γ*, orthogonal to* $\dot\gamma$ *with* $Z(p) = 0$*.*
- *Let* Y *be the Jacobi field along* γ *with* $Y(0) = 0, Y(r) = Z(r)$ *and* Y *is orthogonal to* $\dot\gamma$*.*

Then $I(Y) \le I(Z)$ *and equality occurs if and only if* $Y \equiv Z$*.*

Remark 3. *Note that such Jacobi field* Y *exists and is unique. Firstly, by the second point of Remark 1, one only need* $Y(p) = 0$ *and* $\dot Y(0) \perp \gamma(0)$*. The Jacobi fields satisfying these conditions form a vector space of dimension* $n-1$ *(by Cauchy problem,* $\dot Y(0)$ *is to be chosen in the orthogonal space of* $\gamma(0)$*). Since the exponential map does not degenerate on the preimage of* γ*, each* $\dot Y(0)$ *corresponds one-to-one with an* $Y(r)$ *by Proposition 2.1. The correspondence is linear, with source and target spaces of same dimension* $(n-1)$*, it follows that each* $Z(r) \perp \gamma(r)$ *gives uniquely a Jacobi field* Y*.*

More concretely, let $\dot V_i(0)$ *be a basis of* $\dot\gamma(0)$ *in* T_pM *and* V_i *be the corresponding Jacobi fields with* $V_i(0) = 0$*, then*

1. $\{V_i(t)\}_{i=\overline{1,n-2}}$ *is a basis of* $\dot\gamma(t)$ *in* $T_{\gamma(t)}M$*, where the orthogonal part follows from Remark 1 and the linear independence is by the non-degeneration of since* $\exp_p$*.*
2. *If* $Z(t) = \sum f_i(t)V_i(t)$*, where* f_i *are functions on* $[0, r]$*, then* $Y(t) = \sum_i f_i(r)V_i(t)$*.*

Proof. As Remark 3, let $Z = \sum_i f_iV_i$ and denote $W = \sum_i \dot f_iV_i$ then

$$I(Z) = \int_0^r \left(\|W\|^2 + 2\sum_i f_i\langle \dot V_i, W\rangle + \langle \sum_i f_i\dot V_i, \sum_j f_j\dot V_j\rangle + \langle R(\dot\gamma, \sum f_iV_i)\dot\gamma, \sum f_jV_j\rangle \right) dt$$

By definition of Jacobi field, $R(\dot{\gamma}, V_i)\dot{\gamma} = \ddot{V}_i$, hence the curvature term is

$$\int_0^r \left\langle R(\dot{\gamma}, \sum f_i V_i)\dot{\gamma}, \sum f_j V_j \right\rangle = \sum_{i,j} \int_0^r f_i f_j \langle \ddot{V}_i, V_j \rangle dt = \sum_{i,j} \int_0^r f_i f_j \left(\frac{d}{dt} \langle \dot{V}_i, V_j \rangle - \langle \dot{V}_i, \dot{V}_j \rangle \right) dt$$

$$= - \int_0^r \left\langle \sum_i f_i \dot{V}_i, \sum_j f_j \dot{V}_j \right\rangle dt + \langle \dot{Y}(r), Y(r) \rangle - 2 \sum_{i,j} \int_0^r f_i \dot{f}_j \langle \dot{V}_i, V_j \rangle dt$$

where for the second line, we integrated by part and used the fact that $\langle \dot{V}_i, V_j \rangle = \langle V_i, \dot{V}_j \rangle$ (point 3 of Remark 1). Therefore, one has

$$I(Z) = \int_0^r \|W\|^2 dt + \langle \dot{Y}(r), Y(r) \rangle.$$

In particular $I(Y) = \langle \dot{Y}(r), Y(r) \rangle \leq I(Z)$. The equality occurs if and only if $W \equiv 0$, i.e. $Z \equiv Y$. □

2 Local comparison with space forms

Our goal in this section is to prove the following Comparison Theorem. Before going to the precise statement, let us explain the notation.

Notation. Given M^n a Riemannian manifold and $B(p, r_0)$ be the geodesic ball centered in $p \in M$, of radius $r_0 < \delta_p$ the injectivity radius at p, equipped with the pullback metric of g via exponential map $\exp_p$, which can be expressed in polar geodesic coordinates as

$$(ds)^2 = (dr)^2 + r^2 g_{\theta^i \theta^j}(r, \theta) d\theta^i d\theta^j$$

where $\frac{\partial}{\partial \theta^1}, \ldots, \frac{\partial}{\partial \theta^{n-1}}$ is an Euclidean orthonormal frame of the sphere $r\mathbb{S}^{n-1}$. We note $|g_\theta| = \det(g_{\theta^i \theta^j})_{ij}$ and $g_{\theta\theta}$ be any component $g_{\theta^i \theta^i}$ for $i = 1, \ldots, n-1$.

Abusively, we say that $\frac{\sin \alpha r}{\alpha} = r$ if $\alpha = 0$ and $\sin \alpha r = \frac{1}{i} \sinh i\alpha r$ and $\cos \alpha r = \cosh i\alpha r$ if $\alpha \in i\mathbb{R}$.

Remark 4. *Note that the frame $\{\frac{\partial}{\partial \theta^i}\}_i$ may not be global, for example when n is odd (Hairy ball theorem). However the quantity $|g_\theta|$ is globally defined (except at p), in fact $|g_\theta| = r^{-2n+2}|g|$.*

Theorem 4 (comparison of volume forms)**.** *Let M^n be a Riemannian manifold with*

- *sectional curvature $-a^2 \leq K \leq b^2$*

- *Ricci curvature* $\mathrm{Ric} \geq a' = (n-1)\alpha^2$ *where* α *can be real or purely imaginary.*

Then with the notation of the last paragraph, for all $r \in (0, r_0)$*,*

1. *If* $r < \frac{\pi}{b}$ *then*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} &\geq \frac{\partial}{\partial r} \log \frac{\sin br}{r} \\ g_{\theta\theta} &\geq \left(\frac{\sin br}{br} \right)^2 \end{aligned} \tag{4}$$

2. *One has*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} &\leq \frac{\partial}{\partial r} \log \frac{\sinh ar}{r} \\ g_{\theta\theta} &\leq \left(\frac{\sinh ar}{ar} \right)^2 \end{aligned} \tag{5}$$

3. *One has*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_\theta} &\leq (n-1) \frac{\partial}{\partial r} \log \frac{\sin \alpha r}{r} \leq -a' \frac{r}{3} \\ \sqrt{|g_\theta|} &\leq \left(\frac{\sin \alpha r}{\alpha r} \right)^{n-1} \end{aligned} \tag{6}$$

4. *If* $r < \frac{\pi}{b}$ *then*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_\theta} &\geq (n-1) \frac{\partial}{\partial r} \log \frac{\sin br}{r} \\ \sqrt{|g_\theta|} &\geq \left(\frac{\sin br}{br} \right)^{n-1} \end{aligned} \tag{7}$$

Remark 5.

1. *The moral of the estimates is that if* $r \ll 1$ *then the volume form of* g*, viewed in the tangent space at* p*, is equivalent to the Euclidean volume form of* T_pM*.*

2. *One can always choose* $\alpha \in i\mathbb{R}$ *even when the Ricci curvature is positive, and RHS of* (6) *will be a hyperbolic function and the estimate is not as sharp as if one choose* $\alpha \in \mathbb{R}$*, but it works to prove that the two volume forms are equivalent when* $r \ll 1$ *.*

Remark 6. *A few consequences of Theorem 4:*

1. *For δ small, the metric volume form dV is* equivalent *to the Euclidean volume form of tangent space: there exists $C(\delta) > 0$ converging to 1 as $\delta \to 0$ such that $C(\delta)^{-1} dE \le dV \le C(\delta) dE$.*

2. *Let f be a smooth function defined on $B(p, \delta)$ then the gradient of f w.r.t the metric g is closed to the Euclidean gradient of f viewed in the chart (namely $f \circ \exp_p$):*

$$\|\nabla f\|_g = \left|\frac{\partial f}{\partial r}\right|^2 + \sum_\theta \left|\frac{\partial f}{\partial \theta}(r, \theta)\right|^2 g_{\theta\theta}$$

$$\|\nabla (f \circ \exp_p)\|_E = \left|\frac{\partial f}{\partial r}\right|^2 + \sum_\theta \left|\frac{\partial f}{\partial \theta}(r, \theta)\right|^2$$

3. *Combining the last 2 points, one can see that if f is supported in a small geodesic ball $B(p, \delta)$, then the L^p-norm of ∇f is closed to the Euclidean L^p norm of $\nabla(f \circ \exp_p)$ if δ is sufficiently small.*

The ideal to prove Theorem 4 comes from Proposition 2.1 and Figure 1. Given a point $q \in M$ of distance $r < r_0$ from p, then denote by Y the Jacobi field along the unique geodesic connecting p and q such that Y vanishes at p and $Y(r) = \frac{\partial}{\partial \theta}$ at q, then with $\psi = \exp_p$ as in Figure 1,

$$\begin{aligned} \|Y(r)\|^2 &= \|\psi^*_{X_0} Y(r)\|^2 = \|\psi^*_0 \dot{Y}(0)\|^2_{X_0} \\ &= r^2 g_{\theta\theta} \|\psi^*_0 \dot{Y}(0)\|^2_0 = r^2 g_{\theta\theta} \|\dot{Y}(0)\|^2 \end{aligned} \tag{8}$$

where we used the fact that

$$g\left(\left.\frac{\partial}{\partial \theta^i}\right|_{r\mathbb{S}^{n-1}}, \left.\frac{\partial}{\partial \theta^j}\right|_{r\mathbb{S}^{n-1}}\right) = r^2 g\left(\left.\frac{\partial}{\partial \theta^i}\right|_{\mathbb{S}^{n-1}}, \left.\frac{\partial}{\partial \theta^j}\right|_{\mathbb{S}^{n-1}}\right) = r^2 g_{\theta_i \theta_j}$$

.

Take logarithm and derive (8) w.r.t r, using the fact that $\|Y(r)\| = 1$, one obtains

$$\langle \dot{Y}(r), Y(r) \rangle = \frac{1}{r} + \frac{\partial}{\partial r} \log g_{\theta\theta} \tag{9}$$

It comes to estimate $\langle \dot{Y}(r), Y(r) \rangle$, which is in fact the Index form of Y. The following lemma give an estimate of the Index form in case of bounded sectional curvature, by comparing the it with the Index form under a metric with constant sectional curvature.

Lemma 5. *Suppose that the sectional curvature $K \leq b^2$, then for every Jacobi field Y defined a long a geodesic $\gamma : [0,r] \longrightarrow M$ with $r < \frac{\pi}{2b}$ such that $Y(0) = 0, Y \perp \dot{\gamma}$. Then*

$$I(Y) \geq I_b(Y) := \int_0^r \|\dot{Y}\|^2 - b\|Y\|^2 \geq b \cot br \|Y(r)\|^2$$

Proof. By the curvature bound, $I(T) \geq \int_0^r \|\dot{Y}\|^2 - b^2\|Y\|^2 =: I_b(Y)$. The quantity $I_b(Y)$ is exactly the Index form of Y along γ if the sectional curvature in constantly b. To be precise, we equip the tubular neighborhood of γ a metric g' of constant sectional curvature $K = b^2$ such that normal vectors of γ w.r.t the metric g remain normal under g'. Such g' is in fact easy to find since:

1. The tubular neighborhood is diffeomorphic to $[0,r] \times \mathbb{B}^{n-1}$ where the diffeomorphism (says ι_1) is actually isometry at points of γ, which are mapped to $[0,r] \times \{0\}$;

2. Also, there exists a diffeomorphism ι_2 mapping $[0,r] \times \mathbb{B}^{n-1}$ to a tubular neighborhood of an arc $\tilde{\gamma}$ of length r on the grand circle of $\mathbb{S}^n_{1/b}$ which is isometry on every point of $[0,r] \times \{0\}$. This is because $r < \frac{\pi}{2b} < 2\pi\frac{1}{b}$ the length of the grand circle.

3. One now can identify a tubular neighborhood of γ in M and that of $\tilde{\gamma}$ in $\mathbb{S}^n_{1/b}$ by $\iota = \iota_2 \circ \iota_1$. Take g' to be the pullback of the Eucidean metric on $\mathbb{S}^n_{1/b}$, which is of sectional curvature b^2.

Now under the metric g', Y is no longer a Jacobi field, but it is still orthogonal to γ, denote by $\tilde{Y}$ the Jacobi field (under g') on γ that vanishes at $\gamma(0)$ and has the same value as Y at $\gamma(r)$. By Theorem 3 (Index inequality), one has $I_b(Y) \geq I_b(\tilde{Y})$. The latter can be computed directly, as the field $\iota_*\tilde{Y}$ is given by

$$s \mapsto (s, \beta^1 \sin bs, \ldots, \beta^{n-1} \sin bs), \quad s \in [0,r]$$

where $(\beta^1, \ldots, \beta^{n-1})$ is the coordinates of $\iota_{1*}Y(r)$ in $[0,r] \times \mathbb{B}^{n-1}$, hence in this coordinates (also called *Fermi coordinates*), $\tilde{Y}(s) = \left(s, \frac{\sin bs}{\sin br} Y(r)\right)$. Hence $I_b(\tilde{Y}) = b \cot br \|Y(r)\|^2$. □

Now the remaining part of the proof of Theorem 4 is straightforward.

Proof of Theorem 4. From (9) and Lemma 5, one has

$$\frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} = I(Y) - \frac{1}{r} \geq b \cot br - \frac{1}{r}$$

This gives the estimates in (4).

For (5), the sign situation fits Theorem 3 better, and one does not need to explicitly evoke the space forms (as Lemma 5). It suffices to see that

$$\begin{aligned}
\langle \dot{Y}(r), Y(r)\rangle = I(Y) &\leq I\left(\frac{\sinh at}{\sinh ar} Y(r)\right) \\
&\leq a^2 \left(\int_0^r \left(\frac{\cosh at}{\sinh ar}\right)^2 + \int_0^r \left(\frac{\sinh at}{\sinh ar}\right)^2 dt\right) \|Y(r)\|^2 \\
&= a \coth ar \|Y(r)\|^2
\end{aligned}$$

The estimates in (6) comes from the comparison between Y and the field $t \mapsto \frac{\sin \alpha t}{\sin \alpha r} Y(r)$. Note that the field is well-defined even when $\alpha \in \mathbb{R}_{>0}$ (the hyperbolic case ($\alpha \in i\mathbb{R}_{>0}$ being obvious). This in fact comes from the following fact:

Theorem 6 (Myers)**.** *Let M^n be a connected, complete manifold with* $\mathrm{Ric} \geq (n-1)\alpha^2 > 0$ *then*

1. *M is compact.*
2. *The diameter of M is at most π/α.*

Taking sum of inequalities $I(Y_i) \leq I(\frac{\sin \alpha t}{\alpha r} Y_i(r)$ where Y_i are Jacobi fields vanishing at $\gamma(0)$ and whose values at $\gamma(r)$ are $\frac{\partial}{\partial \theta^i}$ respectively, one has

$$\begin{aligned}
\sum_{i=1}^{n-1} \langle \dot{Y}_i(r), Y_i(r)\rangle &\leq (n-1)\alpha^2 \int_0^r \left(\frac{\cos \alpha t}{\sin \alpha r}\right)^2 dt - \sum_{i=1}^{n-1} \int_0^r R_{r\theta^i r\theta^i} \left(\frac{\sin \alpha t}{\sin \alpha r}\right)^2 dt \\
&\leq (n-1)\alpha \cot \alpha r
\end{aligned}$$

where for the second line, we used the fact that $\sum_i R_{r\theta^i r\theta^i} = \mathrm{Ric}_{rr} \geq (n-1)\alpha^2$. Hence

$$\begin{aligned}
\frac{\partial}{\partial r} \log \sqrt{|g_\theta|} = \frac{\partial}{\partial r} \sum_i \log \sqrt{|g_{\theta^i\theta^i}|} &= \sum_i \langle \dot{Y}_{\theta^i}, Y_{\theta^i}\rangle - \frac{n-1}{r} \\
&\leq (n-1)\left(\alpha \cot \alpha r - \frac{1}{r}\right) = (n-1)\frac{\partial}{\partial r} \log\left(\frac{\sin \alpha r}{r}\right)
\end{aligned}$$

The proof of (7) is essentially the same as (6) where one uses (4) for a lower bound of $I(Y_i) = \langle \dot{Y}_i(r), Y_i(r)\rangle$. □

As a side note, Lemma 5 can also be used to prove that a small geodeosic ball is geodesically convex.

Proposition 6.1. *Let M^n be a Riemannian manifold with sectional curvature $K \leq b^2$ and injectivity radius $\delta > 0$. Then for every $r < \min\{\frac{\delta}{2}, \frac{\pi}{4b}\}$, any geodesic ball $B(p,r)$ is geodesically convex, i.e. any two points is connected by a geodesic curve inside the ball.*

Proof. We first claim that

Lemma 7. *Given two point p, q of distance $d(p,q) = r < \frac{\pi}{2b}$ and $\Gamma_{p,q}$ the geodesic connecting the them. Let γ be a geodesic staring from q with a velocity vector perpendicular to $\Gamma_{p,q}$, then there exists a neighborhood of q inside of which the γ intersects $\Gamma_{p,q}$ only at q.*

First, let us prove that the Lemme implies Proposition 6.1. If r small as in the Proposition and $q_1, q_2 \in B(p,r)$ then

1. There exists a minimal geodesic Γ_{q_1,q_2} connecting q_1, q_2.
2. By triangle inequality, $\Gamma_{q_1,q_2} \subset B(p, 2r)$: every point $q \in \Gamma_{q_1,q_2}$ has to be $d(q_1,q_2)/2$-closed to one q_i, hence $d(p,q) \leq d(p,q_i) + d(q_i,q) \leq r + \frac{2r}{2} = 2r$.

Let $T \in \Gamma_{q_1,q_2}$ be the point minimising the distance to p. It suffices to show that T is one of the q_i. For the sake of contradiction, if T is strictly in the interior of Γ_{q_1,q_2} then

1. The geodesic $\Gamma_{p,T}$ connecting p and T is orthogonal to Γ_{q_1,q_2} at T. It is not difficult to prove that if the two are not orthogonal then there exist $T' \in \Gamma_{q_1,q_2}$ and $S \in \Gamma_{p,T}$, both being near to T, such that $d(p,T) > d(p,S) + d(S,T') \geq d(p,T')$.
2. The ball $B(p, d(p,T)) \cap \Gamma_{q_1,q_2} \supset \Gamma_{q_1,q_2}$.

These contradict the Lemma and prove that T does not lie in the interior.

It remains to prove the Lemma. Let Y be the Jacobi field which vanishes at p and whose value at q is $\dot{\gamma}$, then by Index inequality (Theorem 3), it suffices to prove that $I(Y) > 0$, because any variation of $\Gamma_{p,q}$ by orthogonal vector field Z along γ has $I(Z) > 0$ hence only increases the length, according to Proposition 2.2. But by Lemma 5 gives

$$I(Y) \geq I_b(Y) \geq b \cot br \|Y(q)\|^2 > 0 \text{ if } r < \frac{\pi}{2b}.$$

□

3 Some covering lemmas

The goal of this section is to prove a covering lemma for Riemannian manifolds with injectivity radius $\delta_0 > 0$ and bounded curvature (Lemma 10). We start with a covering lemma that not yet requires curvature bound.

Lemma 8 (Calabi)**.** *Let M^n be a Riemannian manifold with injectivity radius $\delta_0 > 0$, then for all $\delta \in (0, \delta_0)$, there exists $0 < \gamma < \beta \le \delta$ and a partition of $M = \bigsqcup_{i \in I} \Omega_i$ and $p_i \in \Omega_i$ such that*

$$B(p_i, \gamma) \subset \Omega_i \subset B(p_i, \beta)$$

Moreover, one can choose $\gamma = \beta/10$ and $\beta = \delta$.

Proof. Note that it is enough to have

$$\begin{cases} \bigcup_i B(p_i, \beta) = M, \quad 2\gamma < \beta \\ B(p_i, 2\gamma) \text{ are disjoint} \end{cases} \tag{10}$$

In fact, let $\Omega_i' = B(p_i, \beta) \backslash \cup_{j \neq i} B(p_j, \gamma)$ then $\begin{cases} B(p_j, \gamma) \cap \Omega_i' = \emptyset, B(p_i, \gamma) \subset \Omega_i' \subset B(p_i, \beta) \\ \bigcup_i \Omega_i' = M \end{cases}$
(for $\bigcup_i \Omega_i' = M$: If $x \in M$ satisfies $x \in B(p_j, \gamma) \subset B(p_i, \beta)$ then there is no other $j' \neq j$ such that $x \in B(p_{j'}, \gamma)$, hence $x \in \Omega_j$. Now choose

$$\Omega_1 = \Omega_1', \Omega_2 = \Omega_2' \setminus \Omega_1, \ldots, \Omega_n = \Omega_n' \setminus \cup_{i=1}^{n-1} \Omega_i, \ldots$$

For the existence of (10), use the following Vitali covering lemma, whose proof is purely combinatorial in nature.

Lemma 9 (Vitali covering, Infinite version)**.** *Let $\{B_j : j \in J\}$ be a collection of balls in a metric space such that*

$$\sup\{\operatorname{rad}(B_j) : j \in J\} < +\infty$$

where rad *denotes the radius, then there exists a countable subfamily $J' \subset J$ such that $\{B_j : j \in J'\}$ are disjoint and*

$$\cup_{j \in J} B_j \subset \cup_{j \in J'} 5 B_j.$$

It remains to apply the lemma for the covering $M = \cup_{x \in M} B(x, 2\gamma)$, which also allows us to choose $\gamma = \beta/10$ and $\beta = \delta$. ☐

Lemma 10 (Uniformly locally finite covering)**.** *Let M^n be a Riemannian manifold with injectivity radius $\delta_0 > 0$ and bounded curvature, then for all $\delta < \delta_0$ sufficiently small, there exists a* ***uniformly locally finite covering*** *of M by balls $\{B(p_i, \delta)\}_{i \in I}$, i.e. there exists $k(\delta) \in \mathbb{Z}_{>0}$ such that for all $q \in M$, there exists a neighborhood of q that intersects at most $k(\delta)$ balls. Moreover, one can also require that $\{B(p_i, \delta/2)\}_{i \in I}$ is still a covering.*

Proof. We will apply Lemma 8 with $\beta = \delta/2$ and $\gamma = \beta/10$, then for all $\delta \ll \delta_0$, the covering $\{B(p_i, 2\beta)\}$ satisfies. In fact, for every $q \in M$, take $B(q, \delta)$ as a neighborhood of q then $B(p_i, 2\beta) \cap B(q, \gamma) \neq \emptyset$ if and only if $p_i \in B(q, 2\beta + \gamma)$ Since the balls $B(p_i, \gamma)$ are disjoint, the number of p_i in $B(q, 2\beta + \gamma)$ is bounded by

$$k = \frac{\max \mathrm{vol}_g(B_{2\beta+2\gamma})}{\min \mathrm{vol}_g(B_\gamma)} \leq C(\delta) \left(\frac{2\beta + 2\gamma}{\gamma} \right)^n$$

where $\max \mathrm{vol}_g(B_{2\beta+2\gamma})$ and $\min \mathrm{vol}_g(B_\gamma)$ denote the maximum and minimum volume of balls of radius $2\beta+2\gamma$ and γ, respectively. By Theorem 4, for $\delta < \epsilon(a', b)$ depending on the bound a' and b of Ricci curvature and sectional curvature, the volume of these balls are equivalent to that of Euclidean balls of the same radius. The constant of equivalence was denoted by $C(\delta)$. $\square$

4 Sobolev imbeddings for Riemannian manifolds

The goal of this section is to prove that Sobolev imbeddings are also available for complete Riemannian manifold with bounded curvature and strictly positive injectivity radius, that is, the following results.

Theorem 11 (Sobolev imbeddings)**.** *Theorem 1 holds when one replaces R^n by a complete Riemannian manifold of dimension n with bounded curvature (sectional and Ricci) and injectivity radius $\delta_0 > 0$.*

The definition of Sobolev spaces as completion of spaces of smooth functions, w.r.t the Sobolev norms generalises on Riemannian manifolds, namely, we denote by $W_0^{k,p}(M)$ the completion of $C_c^\infty(M)$ w.r.t the norm $\|\varphi\|_{W^{k,p}} = \|\varphi\|_{L^p} + \|\nabla\varphi\|_{L^p} + \cdots + \|\nabla^k\varphi\|_{L^p}$ where $\|\nabla^l\varphi\|_{L^p}$ are computed as follow: the metric g induces a fiberwise norm for l-covariant tensors, integrate that of $\nabla^l\varphi$, one obtains $\|\nabla^l\varphi\|_{L^p}$.

Similarly, the space $W^{1,p}(M)$ is defined as the completion of $C^\infty(M)$ w.r.t $\|\cdot\|_{W^{1,p}}$.

Remark 7. *1. Unlike the Euclidean case, one does not define the derivatives term, e.g. $\nabla_v f$ for $f \in W^{1,p}(M)$ using integration by part and Riesz representation, that is, one does not expect a formular such as $\int_M (\nabla_v f)\varphi dV = -\int_M f\nabla_v \varphi dV$ since the "boundary term" $\int_M \nabla_v(f\varphi)dV$ does not vanish, even if $f\varphi \in C_c^\infty(M)$.*

2. *The exterior derivative df can be defined, which is in fact equivalent to de Rham's notion of current.*

3. *The term $\nabla^l f$ for $f \in W^{k,p}(M)$, when needed, can be defined as a L^p section of $(TM^*)^{\otimes l}$ giving by the L^p limit of smooth sections $\nabla^l \varphi_i$ for an equivalent class of Cauchy sequence φ_i representing f. The completeness of the space of L^p sections of a vector bundle follows from the result in each trivialising chart and the fact that restriction maps commute with the limit.*

Proposition 11.1 ($W^{1,p} = W_0^{1,p}$)**.** *If M is complete then $C_c^\infty(M)$ is dense in $W^{1,p}(M)$, equivalently $W^{1,p}(M) = W_0^{1,p}(M)$.*

Proof. It suffices to prove that given a function $\varphi \in C^\infty(M)$, one can approximate φ under the norm $\|\cdot\|_{W^{1,p}}$ by functions in $C_c^\infty(M)$. Fix $P \in M$, one uses a cut-off function χ_j which is 1 on $[0,j]$, 0 on $[j,\infty]$ and linear inside and defines $\varphi_j(Q) = \varphi(Q)\chi_j(d(Q,P))$. Note that the distance function is only Lipschitz and not necessarily smooth (so we did not mind taking a linear cut-off). However, since φ_j is compactly support and Lipschitz and we can approximate each φ_j by a sequence in $C_c^\infty(M)$: Let K_j be the support of φ_j and $\{\alpha_i\}_i$ be a finite partition of unity subordinating to an open coordinated cover of K. Since $\alpha_i\varphi_j$ is Lipschitz, viewed in a chart, it can be $W^{1,\infty}$-approximated by smooth functions, due to the following fact.

Fact. If $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\delta\Omega$ regular, then $\mathrm{Lip}(\Omega) = W^{1,\infty}(\Omega)$.

The approximation scheme looks like $\varphi \approx \varphi_j \approx \sum_i \alpha_{i,K_j}\varphi_j \approx \sum_i \psi_{i,j}$ where $\psi_{i,j}$ are smooth and compactly support. □

Remark 8. *The similar results for higher orders are complicated, for example, one can prove that $W_0^{2,p} = W^{2,p}$ under the hypothesis of bounded curvature and strictly positive injectivity radius. The third order requires extra conditions.*

The second part of the Theorem 11 is local in nature, and therefore easier. We will prove this second part by accepting the first one, which we will come back and prove eventually.

For the imbedding into $C^r_B(M)$, it suffices to establish the case $W^{1,q} \hookrightarrow C^0_B$, the higher order case then follows: If $\varphi \in W^{k,q}$ then $\nabla^r\varphi \in W^{k-r,q} \hookrightarrow W^{k-r,q} \hookrightarrow W^{1,\tilde{q}} \hookrightarrow C^0_B$ where $\frac{1}{n} \geq \frac{1}{\tilde{q}} \geq \frac{1}{q} - \frac{k-r-1}{n}$.

Similarly, for the imbedding into $C^{r,\alpha(M)}$, it suffices to establish the case $W^{1,q} \hookrightarrow C^{0,\alpha}$ for $\frac{1-\alpha}{n} \geq \frac{1}{q}$.

Since $W^{1,p}(M) = W^{1,p}_0(M)$, it suffices to prove the following Lemma 12 and Lemma 13.

Lemma 12 ($W^{1,q} \hookrightarrow C^0_B$)**.** *Let M^n be a complete Riemannian manifold with injectivity radius $\delta_0 > 0$ and sectional curvature $K \leq b^2$, then for all $\varphi \in C^\infty_c(M)$, one has*

$$\sup_M |\varphi| \leq C(q)\|\varphi\|_{W^{1,q}}, \quad \forall q > n$$

Proof. Take $\delta < \min\{\delta_0, \frac{\pi}{2b}\}$ and let (r,θ) be the geodesic polar coordinate centered at $P \in M$, then by Theorem 4, the ratio of the metric volume form $dV := |g|dE$ and the Euclidean volume form dE of T_PM is $\sqrt{|g_\theta|} \geq \left(\frac{\sin br}{br}\right)^{n-1} \geq (\frac{2}{\pi})^{n-1}$.

let $\chi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ be a cut-off function which is constantly 1 near 0 and supported in $[0,\delta)$. Then

$$\varphi(P) = -\int_0^\delta \partial_r\left(\varphi(r,\theta)\chi(r)\right)dr, \quad \forall\theta \in \mathbb{S}^{n-1}$$

Integrate w.r.t $\theta \in \mathbb{S}^{n-1}$, recall that ω_n denotes the volume of $\mathbb{S}^{n-1}$:

$$\begin{aligned}
|\varphi(p)| &\leq (\omega_{n-1})^{-1}\int_B |\nabla(\varphi(r,\theta)\chi(r))|\, r^{1-n}r^{n-1}drd\theta \\
&\leq (\omega_{n-1})^{-1}\left(\int_B |\nabla(\varphi(r,\theta)\chi(r))|^q\, dE\right)^{1/q}\left(\omega_{n-1}\int_0^\delta r^{(n-1)(1-q)}dr\right)^{1/q'} \\
&\leq (\frac{\pi}{2})^{n-1}(\omega_{n-1})^{-1/q}\left(\|\nabla\varphi\|_{L^q} + \sup_{[0,\delta]}|\chi'|\|\varphi\|_{L^q}\right)\left(\frac{q-1}{q-n}\delta^{\frac{q-n}{q-1}}\right)^{1/q'}
\end{aligned}$$

where q' denotes the Hölder conjugate of q and for we used Hölder inequality w.r.t dE for the second inequality and the comparison $dE \leq (\frac{\pi}{2})^{n-1}dV$ for the third. The conclusion follows. $\square$

Lemma 13 ($W^{1,q} \hookrightarrow C^{0,\alpha}$)**.** *Let M^n be a complete Riemannian manifold with injectivity radius $\delta_0 > 0$ and bounded curvature, then for all $\varphi \in C^\infty_c(M)$, one has*

$$\sup_M |\varphi| + \sup_{P\neq Q}|\varphi(P)-\varphi(Q)|\, d(P,Q)^{-\alpha} \leq C(\alpha,q)\|\varphi\|_{W^{1,q}}, \quad \textit{for all } \frac{1-\alpha}{n} \geq \frac{1}{q}$$

Proof. By Lemma 12, one can discard the term $\sup_M |\varphi|$ and only need to treat the second term of LHS. Let $\delta \leq \min\{\delta_0, \frac{\pi}{2b}\}$ as in the proof of Lemma 12 (b^2 being the upper bound of the sectional curvature). One only need to consider the case where $d = d(P,Q) < \delta/2$ because otherwise $|\varphi(P) - \varphi(Q)| \leq 2\|\varphi\|_{L^\infty}(\frac{\delta}{2})^{-\alpha} d(P,Q)^\alpha$.

Let O be the midpoint of P, Q, and denote by $h := \varphi \circ \exp_O$ defined on the Euclidean ball $B(0,2d) \supset B_O := B(0,d/2)$. We also denote by P, Q the preimages of these points in B_O. See Figure 4.

Figure 2: Left: the picture viewed in normal polar coordinates at O. Right: the picture viewed in normal polar coordinates at Q.

Now place B_O in polar coordinate centered at Q:

$$h(x) - h(Q) = \int_0^r \frac{\partial}{\partial r} h(r,\theta)dr = r\int_0^1 \frac{\partial}{\partial \rho} h(rt,\theta)dt$$

Integrate on $B_O \ni x$ w.r.t to the measure dE_Q given by the normal polar coordinates at Q:

$$\begin{aligned}
\int_{B_O} |h(x) - \varphi(Q)| dE_Q &\leq \int_{\theta\in\mathbb{S}^{n-1}} \int_{r=0}^{\rho(\theta)} r^{n-1} r \int_0^1 |\frac{\partial}{\partial \rho} h(rt,\theta)| dt dr d\theta \\
(u := rt, \rho(\theta) \leq d) \qquad &\leq \int_{\theta\in\mathbb{S}^{n-1}} \int_{t=0}^{1} \int_{u=0}^{td} t^{-n-1} u^n \left|\frac{\partial}{\partial\rho} h(u,\theta)\right| dt du d\theta \\
&= \int_{t=0}^1 t^{-n-1} \left(\int_{u=0}^{td} \int_{\theta\in\mathbb{S}^{n-1}} \left|\frac{\partial}{\partial\rho} h(u,\theta)\right| u . dE_Q \right) dt \\
(\text{Holder w.r.t } dE_Q) \qquad &\leq \int_{t=0}^1 t^{-n-1} \left(\int_{u=0}^{td} \int_{\theta\in\mathbb{S}^{n-1}} \left|\frac{\partial}{\partial\rho} h(u,\theta)\right|^q dE_Q \right)^{1/q} \left(\int_0^{td} \omega_{n-1} u^{q'} u^{n-1} du \right)^{1/q'} dt \\
(t \leq 1) \qquad &\leq \int_{t=0}^1 t^{-n-1} \left(\frac{1}{q'+n} (td)^{q'+n} \right)^{1/q'} \left(\int_{u=0}^{d} \int_{\theta\in\mathbb{S}^{n-1}} |\nabla\varphi|^q dE_Q \right)^{1/q} dt \\
&= C_1(q,n) d^{1+\frac{n}{q'}} \left(\int_{B(Q,d)} |\nabla\varphi|^q dE_Q \right)^{1/q}
\end{aligned} \tag{11}$$

Now using the fact that $\frac{1}{A}dV \leq dE_Q \leq AdV$ since the curvature is bounded, one has

$$\int_{B(O,d/2)} |\varphi(x) - \varphi(Q)| dV \leq C_2(q,n) d^{1+\frac{n}{q'}} \|\nabla\varphi\|_{L^q}$$

Taking sum with the same computation for P, one has

$$|\varphi(P) - \varphi(Q)| \operatorname{vol}_g(B(O, d/2)) \leq 2C_2(q, n) d^{1+\frac{n}{q'}} \|\nabla\varphi\|_{L^q}$$

since $\operatorname{vol}_g(B(O, d/2)) \geq A^{-1}\omega_{n-1} d^n$, one has

$$|\varphi(P) - \varphi(Q)| \leq C_3(q, n) \|\nabla\varphi\|_{L^q} d^{1-n/q}$$

The conclusion follows since $1 - \frac{n}{q} \geq \alpha$. □

For the first part of Theorem 11, it suffices to prove the case $k = l+1$, that is, there exists a constant $C_1, C_2 > 0$ such that $\|u\|_{L^p} \leq C_1 \|\nabla u\|_{L^q} + C_2 \|u\|_{L^q}$ for $u \in W^{1,q}(M)$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.

The proof by [**?**] tries to optimise the constant C_1, in an attempt to find the best inequality [**?**, page 50]. We will follow their arguments, as the extra effort is not much. We will prove that

Proposition 13.1. *Given $p, q \in \mathbb{R}_{>0}$ such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{n} > 0$, for any $\epsilon > 0$, there exists $A_q(\epsilon)$ such that*

$$\|u\|_p \leq (K(n, q) + \epsilon) \|\nabla u\|_{L^q} + A_q(\epsilon) \|u\|_{L^q}$$

The appearance of the constant $K(n, q)$, given by

$$K(n, q) := \begin{cases} \frac{q-1}{n(q-1)} \left[\frac{n-q}{n(q-1)}\right]^{1/q} \left[\frac{\Gamma(n+1)}{\Gamma(n/q)\Gamma(n+1-n/q)\omega_{n-1}}\right]^{1/n}, & \text{if } q > 1 \\ \frac{1}{n} \left(\frac{n}{\omega_{n-1}}\right)^{1/n}, & \text{if } q = 1 \end{cases}$$

is due to the following local result.

Theorem 14 (Aubin)**.** *Given $1 \leq q < n$ and $u \in W^{1,q}(\mathbb{R}^n)$, with $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, one has*

$$\|u\|_{L^p} \leq K(n, q) \|\nabla u\|_{L^q}.$$

In fact, $K(n, q)$ this the norm of the imbedding $W^{1,q}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$.

We will accept the local result and use the Covering Lemma 10 to prove Proposition 13.1, which implies Theorem 11.

Proof of Proposition 13.1. Note that given any smooth function f supported in a small geodesic ball $B(q, \delta)$, by applying theorem 14 to the f, viewed in the chart (that is, $f \circ \exp_q$) and use the fact that $C(\delta)^{-1} \|\nabla(f \circ \exp_q)\|_{L^q(dE)} \leq \|\nabla f\|_{L^q(dV)} \leq C(\delta) \|\nabla(f \circ \exp_q)\|_{L^q(dE)}$ (see remark 6), one has

$$\|f\|_{L^p} \leq K_\delta(n, q) \|\nabla f\|_{L^q}$$

where $K_\delta(n,q)$ converges to $K(n,q)$ as $\delta \to 0$.

It suffice to cover M by geodesic ball $B(Q_i,\delta)$ such that there exists a partition of unity subordinated to $B(Q_i,\delta)$ such that $\|\nabla(h_i^{1/q})\| \le H =$ const. In fact for $\varphi \in W^{1,q}(M)$, one has

$$\begin{aligned}
\|\varphi\|_p^q = \left(\int_M |\varphi|^p\right)^{q/p} &= \left(\int_M \left(\sum_i |\varphi|^q h_i\right)^{p/q}\right)^{q/p} \\
\text{(since } p \ge q) \qquad &\le \sum_i \left(\int_M (|\varphi|^q h_i)^{p/q}\right)^{q/p} = \sum_i \left\|\varphi h_i^{1/q}\right\|_p^q \\
&\le K_\delta^q(n,q) \sum_i \left\|h_i^{1/q}\nabla\varphi + \varphi\nabla h_i^{1/q}\right\|_q^q
\end{aligned}$$

Using the fact that there are at most $k(\delta)$ balls overlapping at a point and that $(a+b)^q = a^q\left(1+\frac{b}{a}\right)^q \le a^q(1+2^q\frac{b}{a}+2^q(\frac{b}{a})^q) \le a^q + 2^q b a^{q-1} + 2^q b^q$, one has

$$\begin{aligned}
\|\varphi\|_p^q &\le K_\delta^q(n,q)\left(\|\nabla\varphi\|_q^q + 2^q k(\delta)H^{q-1}\int_M |\varphi|^{q-1}|\nabla\varphi| + 2^q k(\delta)H^q\|\varphi\|_q^q\right) \\
&\le K_\delta^q(n,q)\left[\|\nabla\varphi\|_q^q + 2^q k(\delta)H^{q-1}\|\nabla\varphi\|_q\|\varphi\|_q^{q-1} + 2^q k(\delta)H^q\|\varphi\|_q^q\right]
\end{aligned}$$

It is elementary to see that this implies $\|\varphi\|_p^q \le (1+\epsilon)^q K^q(n,q)\left[(1+\epsilon)\|\nabla\varphi\|_q^q + A(\epsilon)\|\varphi\|_q^q\right]$, from which the conclusion follows.

For the existence of such h_i, one cover M by balls $B(Q_i,\delta)$ using Lemma 10. Denote by $\varphi_i : B(Q_i,\delta) \longrightarrow B(0,\delta)$ the inverse of exponential maps and let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the smooth function, choose u to be a bell curve with maximal value 1 at 0, supported in $B(0,\delta)$ and $u \le \frac{1}{2}$ in $B(0,\delta/2)$ and pose $u_i = u \circ \varphi_i$. Then

$$\|\nabla u_i\|_{g_M} \le C_1(g_M,\delta)\|\nabla u\|_E = C_2(g_M,\delta)$$

Pose $h_i = \frac{u_i^m}{\sum u_j^m}$ with $m > q$ then

$$\begin{aligned}
\left|\nabla(h_i^{1/q})\right| &= \left|\frac{m}{q}\frac{u_i^{\frac{m}{q}-1}\nabla u_i}{(\sum u_j^m)^{1/q}} + u_i^{m/q}\left(\frac{-1}{q}\right)\frac{\sum\nabla(u_j^m)}{(\sum u_j^m)^{1+\frac{1}{q}}}\right| \\
&\le \frac{m}{q.2^{-m/q}}|\nabla u_i| + \frac{1}{q}\sum m\frac{|\nabla u_j|}{(2^{-m})^{1+\frac{1}{q}}} \\
&\le \left(\frac{m}{q}2^{m/q} + \frac{m}{q}2^{m(1+\frac{1}{q})}k(\delta)\right)C_2(g_M,\delta) = \text{const}
\end{aligned}$$

where $k(\delta)$, as in Lemma 10, is the upper bound of number of balls overlapping at the point in question. $\square$

5 Kondrachov's theorem

The generalised version of Kondrachov's theorem is much easier to prove

Theorem 15 (Kondrachov)**.** *Theorem 2 holds when one replaces Ω by a compact Riemannian manifolds of dimension n.*

Proof. Cover M by finitely many small geodesic ball $B(Q_i, \delta)$ subordinating a partition of unity $\sum_{i=1}^{N} \chi_i = 1$, then if a sequence $\{u_n\}_n \subset W^{k,q}$ is bounded then $\{\chi_i u_n\}_n$ is also bounded in $W^{k,q}$. The conclusion follows using Remark 6 and the Euclidean version of Kondrachov's theorem. $\square$

6 Solving $\Delta u = f$ on a Riemannian manifold.

With Kondrachov's theorem 15, one can uses the familiar "subsequence extracting" technique to find a minimiser of the quadratic functional $\psi \mapsto \frac{1}{2}\int_M \|\nabla\psi\|^2 dV$ in a suitable subspace of $W^{1,2}(M)$ (method of Lagrange multiplier), one can prove the following results.

Theorem 16 (Spectrum of Δ)**.** *Let M^n be a compact Riemannian manifold then*

1. *The eigenvalues of $\Delta - \nabla^\nu \nabla_\nu$ are ≥ 0.*
2. *The eigenfunctions of $\delta_0 = 0$ are constant functions.*
3. *The eigenvalue λ_1 is the minimum value of the functional*

$$\psi \mapsto \frac{1}{2}\int_M \|\nabla\psi\|^2 dV$$

on the subspace $\left\{\psi \in W^{1,2}(M) : \ \|\psi\|_2 = 1, \int \psi dV = 0\right\}$. Moreover, first eigenfunctions are smooth.

Theorem 17. *Given M^n be a compact Riemannian manifold, consider the Laplace equation on M:*

$$\Delta u = f \tag{12}$$

where $f \in L^2(M)$, then:

1. *There exists* $u \in W^{1,2}(M)$ *satisfying* (12) *in the weak sense if and only if* $\int_M f dV = 0$

2. u *is unique up to an additive constant.*

3. *If* $f \in C^{r,\alpha}$ *then* $u \in C^{r+2,\alpha}$.