Local results of several complex variables

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Contents

1 de Rham currents

Let M be a differential m-dimensional manifold and $\mathcal{E}^p(M)$ be the vector space of smooth p-forms on M and $\mathcal{D}^p(M)$ be the space of those with compact support. Then $\mathcal{E}^p(M), \mathcal{D}^p(M)$ is a topological vector space with the pseudonorms $p^s_{K,\Omega}(\omega) = \max_{K,|\alpha|\leq s} |D^\alpha u_I|$ where $K \Subset \Omega$ an coordinated open set. The space of de Rham current with dimension p / degree $m-p$ is defined as the dual space of $\mathcal{D}^p(M)$, denoted by $\mathcal{D}'^{m-p}(M)$ or $D'_p(M)$

Remark 1. *1. We are still in $\mathbb{R}$, but the definition expands to the complex case, denoted by $\mathcal{D}'^{m-p,m-q}(M) = \mathcal{D}'_{p,q}(M)$ where m is the complex dimension of M.*

2. *The degree is defined such that the current* $T_\omega : \eta \mapsto \int_M \omega \wedge \eta$ *is of the same degree as* ω. *The dimension is defined so that the current* $T_{[Z]} : \eta \mapsto \int_Z \eta$ *is of the same dimension as* Z.

Definition 1. *One has the following operation on* $\mathcal{D}'^{m-p}(M)$*:*

1. ***Derivative:*** $\langle dT, \omega \rangle = (-1)^{\deg T} \langle T, d\omega \rangle$
2. ***Wedge product with a form:*** $\langle T \wedge \eta, \omega \rangle = \langle T, \eta \wedge \omega \rangle$
3. ***Pushforward:*** *If* $F : X \longrightarrow Y$ *proper on* $\operatorname{supp} T$ *then* $\langle F_* T, \omega \rangle = \langle T, F^* \omega \rangle = \langle T, \chi F_* \omega \rangle$ *where* $\chi \in C^\infty(M)$ *identically 1 on* $\operatorname{supp} T$. *The proper condition is such that the pullback of* ω *is compactly support in* $\operatorname{supp} T$
4. ***Pullback:*** *Let* $F : X \longrightarrow Y$ *submersion then the pushforward of a form on* X *is well-defined by Fubini. One has* $\langle F^* T, \omega \rangle = \langle T, F_* \omega \rangle$

Remark 2. 1. *The sign of derivative is chosen so that* $dT_\omega = T_{d\omega}$.

2. *Pushforward keeps the dimension, as the arguments are of the same degree.*
3. *Pullback keeps the codimension, meaning the degree (think* $F^* T_{[Z]} = T_{[F^{-1}(Z)]}$*).*
4. *Locally a current is of form* $T = \sum u_I dx^I$ *where* u_I *are distribution.* ***Note:*** *Here distribution are indentified as a current of maximal degree and not zero degree as they naturally are. To be exact, the notation of* u_I *is contravariant and its action is* $\varphi dx^1 \wedge \cdots \wedge dx^N \mapsto \langle u_I, \varphi \rangle dx^1 \wedge \cdots \wedge dx^n / vol$ *where vol is a canonical volume form.*

The last two remarks explain the sign in the following proposition.

Proposition 0.1 (Pushforward and Pullback)**.** *Let* $F : M_1 \longrightarrow M_2$*, submersion if needed, then*

1. $\operatorname{supp} F_* T \subset F(\operatorname{supp} T)$
2. $d(F_* T) = F_* dT$ *(pushforward of a form is still that form)*
3. $F_*(T \wedge F^* g) = (F_* T) \wedge g$

and

1. $F^*(dT) = (-1)^{m_1 - m_2} d(F^* T)$
2. $F^*(T \wedge g) = (-1)^{m_1 - m_2 - \deg g} (F^* T) \wedge F^* g$

2 Subharmonic and Plurisubharmonic functions

Some properties of holomorphic functions that remain in several variables.

- Cauchy formula
- Analyticity: series development. Therefore its zeroes never form an open set (except for constant)
- Maximum modulus
- Cauchy inequality and Montel's theorem

2.1 Subharmonic functions

We are now in the context of $\mathbb{R}^n$.

Theorem 1 (Green kernel). *Let $\Omega \Subset \mathbb{R}^n$ be a smoothly bounded domain, then there exists uniquely a function $G_\Omega : \bar{\Omega} \times \bar{\Omega} \longrightarrow [-\infty, 0]$, called the Green kernel of Ω, with the following properties:*

1. *Regular: G_Ω is C^∞ on $\bar{\Omega} \times \bar{\Omega} \setminus \Delta_\Omega$ where Δ_Ω denotes the diagonal,*
2. *Symetric: $G_\Omega(x, y) = G_\Omega(y, x)$,*
3. *Negative: $G_\Omega(x, y) < 0$ on $\Omega \times \Omega$ and $G_\Omega(x, y) = 0$ on $\partial\Omega \times \Omega$,*
4. *$\Delta_x G_\Omega(x, y) = \delta_y$ on Ω for every $y \in \Omega$.*

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Example 1 (case $\Omega = B(0, r)$). *One can take $G_r = N(x - y) - N(\frac{|y|}{r}(x - \frac{r^2}{|y|^2} y))$ where N is the Newton kernel (or Newtonian potential, the gravitational potential). Explicitly, one has*

$$G_r(x, y) = \frac{1}{4\pi} \log \frac{|x - y|^2}{r^2 - 2\langle x, y\rangle + \frac{1}{r^2}|x|^2|y|^2} \qquad \text{if } n = 2 \tag{1}$$

$$G_r(x, y) = \frac{-1}{(m - 2)\operatorname{vol}(S^{m-1})}(|x - y|^{2-m} - (r^2 - 2\langle x, y\rangle + \frac{1}{r^2}|x|^2|y|^2)^{1-m/2}) \qquad \text{if } n \geq 3 \tag{2}$$

Proposition 1.1 (Green-Riesz representation)**.** *For $u \in C^2(\bar{\Omega}, \mathbb{R})$ one has*

$$u(x) = \int_\Omega G_\Omega(x,y)\Delta u(y) d\lambda(y) + \int_{\partial\Omega} u(y) \frac{\partial G_\Omega}{\partial \nu_y} d\sigma(y)$$

In particular, for $\Omega = B(0,r)$, one has

$$P_r(x,y) := \frac{\partial G}{\partial \nu_y} = \frac{1}{\mathrm{vol}(S^{m-1})r} \frac{r^2 - |x|^2}{|x-y|^m}$$

called the Poisson kernel.

Proof. Use the Green-Riesz formula: $\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}$. □

Definition 2. *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $u : \Omega \longrightarrow [-\infty, \infty)$ a upper semi-continuous function:*

$$\limsup_{x \to x_0} u(x) \leq u(x_0)$$

One notes by $\mu_S(u,a,r)$ and $\mu_B(u,a,r)$ the average of u in the sphere and the disk centered in a of radius r. Then the following properties are equivalent and a function is called subharmonic if they are verified.

1) $u(x) \leq P_{a,r}[u](x) \quad \forall a, r, x \in B(a,r) \subset \Omega$,

2) $u(a) \leq \mu_S(u,a,r) \quad \forall B(a,r) \subset \Omega$,

2') $u(a) \leq \mu_S(u,a,r)$ *for* $B(a,r_n) \subset \Omega, r_n \to 0$,

3) $u(a) \leq \mu_B(u,a,r) \quad \forall B(a,r) \subset \Omega$,

3') $u(a) \leq \mu_B(u,a,r)$ *for* $B(a,r_n) \subset \Omega, r_n \to 0$,

4) *If $u \in C^2$, then $\Delta u \geq 0$.*

The convex cone of subharmonic functions on a domain Ω is denoted by $Sh(\Omega)$.

Proof. It is obvious that $(1) \to (2) \to (3) \to (3') \to (2')$. To prove $(2') \to (1)$ one needs the following 2 facts:

Lemma 2 (u.s.c function as limit of continuous functions)**.** *Let u be a u.s.c. function on a compact metric space X, then there exists a sequence u_n continuous function on X that decreases to u pointwise.*

Proof. Let $\tilde{u}_k(x) = \max\{u(x), -k\}$ to exclude the $-\infty$ points. Then $v_k(x) = \sup_{y \in X} (u(y) - kd(x,y))$ works.

Lemma 3. $(2')$ *implies strict maximum principle (see 3.1).*

Proof. By restriction to smaller neighborhood, one can suppose that u attains global maximum at x_0 in Ω. Then $W = \{x \in \Omega : \ u(x) < u(x_0)\}$ is an open set, and has a point y in its boundary if W nonempty. Then $(2')$ is not satisfied at y since the measure of open arc is nonzero.

Note that if u is continue than $(2') \to (1)$: Let $h = P_{a,r}[u]$ harmonic then $u - h$ satisfies $(2')$, therefore the maximum principle, hence $u - h \leq (u-h)\big|_{S(a,r)} = 0$.

If u is u.s.c, take a sequence v_k continuous that decreases to u and let $h_k = P_{a,r}[v_k]$ then $h_k \geq v_k \geq u$ and $h_k \to P_{a,r}[u]$ by monotone convergence. $\square$

Proposition 3.1. *Let $u \in Sh(\Omega)$ then*

(Strict) maximum principle. *u cannot attain local maximum unless it is constant in the corresponding connected component,*

Locally integrable. *u is L^1_{loc} on each connected component where $u \not\equiv -\infty$,*

Pointwise decreasing limit *The pointwise limit u of a decreasing sequence u_k of subharmonic functions is also subharmonic.*

Regularisation. *$\mu_S(u,a,\varepsilon), \mu_B(u,a,\varepsilon), \rho_\varepsilon * u$ increase in ε. Moreover, $\rho_\varepsilon * u \in Sh(\Omega)$ and decreases to u pointwise as $\varepsilon \to 0$.*

Moreover, for $u \in \mathcal{D}'(\Omega)$

Positive measure. *$u \in Sh(\Omega)$ iff $\Delta u \geq 0$ is a positive measure.*

Proof. **Locally integrable.** To see that $u \in L^1_{loc}(\Omega)$ if Ω is connected and $u \not\equiv -\infty$, let x be a point in the boundary of $W = \{y \in \Omega : u \text{ integrable in neighborhood of } y\}$, then apply mean value property in $a \in W$ such that $x \in B(a,r)$.

Pointwise decreasing limit. Infimum of a family of u.s.c functions is still u.s.c. The mean value property comes from monotone convergence.

Regularisation. Check first for C^2 functions, then regularise. One uses the following Gauss formula:

$$\mu_S(u,a,r) = u(a) + \frac{1}{n}\int_0^r \mu_B(\Delta u, a, t) t dt$$

to see that μ_S is increasing in r and

$$\mu_B(u, a, r) = m \int_0^1 t^{m-1} \mu_S(u, a, rt) dt$$

to see that μ_B is increasing. For the convolution, use

$$u * \rho_\varepsilon = \mathrm{vol}(S^{n-1}) \int_0^1 \mu_S(u, a, \varepsilon t) \rho(t) t^{m-1} dt.$$

Positive measure. $\Delta u * \rho_\varepsilon \geq 0$ as function, therefore the limit ≥ 0 as measure (dominated convergence).

$\square$

Proposition 3.2 (new harmonic functions from old ones)**.** *Let $u_k \in Sh(\Omega)$ then*

1. *If $\{u_k\}$ decrease to u then $u \in Sh(\Omega)$.*
2. *Let χ be a convex function, non-decreasing in each variable then $\chi(u_1, \ldots, u_p) \in Sh(\Omega)$. Therefore, $\sum u_i$ and $\max\{u_i\}$ are subharmonic.*

Proposition 3.3 (Upper regularization)**.** 1. *Let u be a real function on Ω then $u^*(x) = \lim_{\varepsilon \to 0} \sup_{x+\varepsilon B} u$, called the upper envelope of u is u.s.c and is in fact the smallest u.s.c function greater than u.*

2. ***Choquet lemma.*** *Let $\{u_\alpha\}$ be a family of real function, one defines the upper regularization of $\{u_\alpha\}$ by u^* where $u = \sup_\alpha u_\alpha$. Then from every such family, on can always find a countable subfamily $\{v_i\}$ such that $u^* = v^*$.*

3. *If $\{u_\alpha\} \subset Sh(\Omega)$ then $u^* = u$ a.e. and $u^* \in Sh(\Omega)$.*

Proof. 1. Obvious.

2. Let B_i be a countable base of the topology and $x_{i,j}$ be a sequence such that $u(x_{ij}) \to \sup_{B_i} u$. Let $\{u_{i,j,k}\}$ be a countable subfamily such that $u_{ijk}(x_i) \to u(x_i)$ then it is a suitable subfamily.

3. WLOG, suppose that $\{u_\alpha\} = \{u_i\}$ countable then u satisfies the submean value property: $u(z) \leq \mu_B(u, z, r)$. By the continuity of $\mu_B(u, z, r)$ one has $u^*(z) \leq \mu_B(u, z, r) \leq \mu(u^*, z, r)$ therefore $u^* \in Sh(\Omega)$ and $u^*(z) = \lim_{r \to 0} \mu_B(u^*, z, r) = \lim_{r \to 0} \mu_B(u, z, r)$, from which $u = u^*$

$\square$

2.2 Plurisubharmonic functions

The analog of harmonic functions over $\mathbb{C}$ in multidimensional case $\Omega \subset \mathbb{C}^n$ is in fact *pluriharmonic functions* which is defined through the notion of plurisubharmonic functions

Definition 3.
1. *A real function u is said to be* plurisubharmonic *if and only if its restriction to any complex line is subharmonic. One denotes by $Psh(\Omega)$ the space of plurisubharmonic function on Ω.*
2. *In case $u \in C^2$ on $\Omega \subset \mathbb{C}^n$, this is equivalent to*

$$H(u)(\zeta) = \sum \frac{\partial^2 u}{\partial z^j \partial \bar{z}^k} \zeta^j \bar{\zeta}^k \geq 0 \quad \forall \zeta \in \mathbb{C}^n$$

   where the notation $H(u)(\zeta)$ is invariant, i.e. if $f : M_1 \longrightarrow M_2$ is holomorphic then $H(u \circ f)(\zeta) = H(u) df(\zeta)$.
3. *In the general case, this is equivalent to $H(u)(\zeta) \geq 0 \quad \forall \zeta \in \mathbb{C}^n$ as a measure.*

Remark 3.
1. *The invariance can be noticed using $\zeta^j = \zeta^j d\zeta^j + \bar{\zeta}^j d\bar{\zeta}^j$ where LHS is intepreted as a vector in $T\mathbb{C}$. This allows us to extend the notion of $Psh(M)$ to any complex manifold M.*
2. *By consequence, $f^* u \in Psh(M_1)$ for all $u \in Psh(M_2)$ and $f : M_1 \longrightarrow M_2$ holomorphic.*

Proposition 3.4 (new Psh functions from old ones)**.** *The construction of new plurisubharmonic function is the same as that of subharmonic function. Let $u_k \in Psh(\Omega)$ then*

1. *If $\{u_k\}$ decrease to u then $u \in Psh(\Omega)$.*
2. *Let χ be a convex function, non-decreasing in each variable then $\chi(u_1, \ldots, u_p) \in Psh(\Omega)$. Therefore, $\sum u_i$ and $\max\{u_i\}$ are plurisubharmonic.*
3. *The upper regularization u^* where $u = \sup_\alpha u_\alpha$ is also plurisubharmonic and $u = u^*$ almost everywhere.*

Proof. The only nontrivial proof is the third one where upper envelop in $\mathbb{C}^\ltimes$ and in a line can be different. To fix this, use Choquet lemma 3.3 and dominated convergence, $u * \rho_\varepsilon$ satisfies the submean property on every complex line and decrease to u a.e. $\square$

2.3 Pluriharmonic functions

Definition 4. *A function u is said to be* <u>*pluriharmonic*</u> *on Ω, denoted $u \in Ph(\Omega)$ if $u \in Psh(\Omega) and -u \in Psh(\Omega) \setminus)$ where (Ω).*

This is obviously equivalent to $H(u) = 0$, i.e. $\frac{\partial^2 u}{\partial z^j \partial \bar{z}^k} = 0 \quad \forall j, k$, i.e. $\partial\bar{\partial} u = 0$.

Remark 4.

1. *By mean value property, $Ph(\Omega) \subset Harm(\Omega)$.*
2. *If $f \in \mathcal{O}(M)$ then $\Re f, \Im f \in Ph(M)$*

Theorem 4 (analog of harmonic function)**.** *If M is a complex manifold such that $H^1_{dR}(X, \mathbb{R}) = 0$ then every pluriharmonic function u is a real part of a holomorphic function $f \in \mathcal{O}(M)$*

Proof. Since $d(\bar{\partial} u) = 0$, and $H^1_{dR} = 0$, one has $\bar{\partial} u = dg$. Therefore $d(u - 2\Re g) = (\bar{\partial} u - dg) + (\partial u - d\bar{g}) = 0$, hence on chooses $f = 2g + C$ on each connected component. □

3 Resolution of $\bar{\partial}$, Dolbeault-Grothendieck lemma

The generalized Cauchy formula for several variables is the following (the formula in wikipedia is $K^{0,0}_{BM}$)

Theorem 5 (Bochner–Martinelli-Koppelman formula)**.** *The* <u>*Bochner-Martinelli kernel*</u> *is the following $(n, n-1)$-form on $\mathbb{C}^n$*

$$k_{BM} = (-1)^{n(n-1)/2} \frac{(n-1)!}{(2\pi i)^n} \sum_{1 \le j \le n} (-1)^j \frac{\bar{z}_j dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n}{|z|^{2n}}$$

then $\bar{\partial} k_{BM} = \delta_0$ on $\mathbb{C}^n$.

Let $K_{BM} = \pi^* k_{BM}$ where $\pi : (z, \zeta) \mapsto z - \zeta$ so that $\bar{\partial} K_{BM} = [\Delta]$, then: For any domain $\Omega \subset \mathbb{C}^n$ bounded with piecewise C^1 boundary and v a (p, q)-form of class C^1 on $\bar{\Omega}$ then

$$v(z) = \int_{\partial\Omega} K^{p,q}_{BM}(z, \zeta) \wedge v(\zeta) + \bar{\partial} \int_{\Omega} K^{p,q-1}_{BM}(z, \zeta) \wedge v(\zeta) + \int_{\Omega} K^{p,q}_{BM}(z, \zeta) \wedge \bar{\partial} v(\zeta)$$

where $K^{p,q}_{BM}$ denotes the component of K_{BM} type (p, q) in z and type $(n - p, n - q - 1)$ in ζ

Another consequence of 5 is the *global* resolution of $\bar{\partial}$ in case of compact support.

Corollary 5.1. *If v is a (p,q)-form with $q \geq 1$ on $\mathbb{C}^n$, compactly supported, with regularity of class C^s such that $\bar{\partial} v = 0$ then there exists an $(p, q-1)$-form u on $\mathbb{C}^n$ with the same regularity as u such that $\bar{\partial} u = v$. In fact one can take*

$$u(z) = \int_{\mathbb{C}^n} K_{BM}^{p,q-1}(z,\zeta) \wedge v(\zeta)$$

In case $(p,q) = (0,1)$ then u is compactly support. This means that the compact support $(0,1)$-Dolbeault cohomology $H_c^{0,1}(\mathbb{C}^n) = 0$.

Since $K_{BM} = O(|z|^{1-2n})$, one has $|u(z)| = O(|z|^{1-2n})$ at infinity. Therefore the compact support of u in case $(p,q) = (0,1)$ is explained by Liouville theorem.

The Dolbeault-Grothendieck lemma solves the equation $\bar{\partial} u = v$ in a local scale if the compact support condition is dropped and gives regular result if v is a $(p,0)$-form.

Theorem 6 (Dolbeault-Grothendieck lemma)**.** *Let $v \in \mathcal{D}'(p,q)(\Omega)$ such that $\bar{\partial} v = 0$.*

1. *If $q = 0$ then $v = \sum v_I dz^I$ where $v_I \in \mathcal{O}(\Omega)$.*
2. *If $q \geq 1$ then there exists $\omega \subset \Omega$ and $u \in \mathcal{D}'^{(}p, q-1)(\Omega)$ such that $\bar{\partial} u = v$. Moreover, if $v \in \mathcal{E}^{p,q}(\Omega)$ then $u \in \mathcal{E}^{p,q-1}(\Omega)$*

Corollary 6.1 (Hypoellipticity in bidegree $(p,0)$)**.** *$\bar{\partial}$ is hypoellipticity in bidegree $(p,0)$, i.e. if $\bar{\partial} u = v$, v of bidegree $(p,1)$ and v is C^∞ then u is also C^∞ on the entire domain Ω.*

4 Extension theorems, Domain of holomorphy

Theorem 7 (Hartog extension)**.** *Let $\Omega \subset \mathbb{C}^n$ be a domain and $K \Subset \Omega$ such that $\Omega \setminus K$ is connected. Then $\mathcal{O}(\omega)\big|_{\Omega \setminus K} = \mathcal{O}(\Omega \setminus K)$ every holomorphic function on $\Omega \setminus K$ extends to Ω*

Proof. Let $f \in \mathcal{O}(\Omega \setminus K$ be the function we want to extend. Let φ be a function with support in a neighborhood of K and is identically 1 on K and $g = (1-\varphi)f$ which coincides with f outside of $\operatorname{supp}\varphi$. Then $v = \bar{\partial} g \in \mathcal{D}^{0,1}$ satisfies $\bar{\partial} v = 0$, therefore there exists $u \in C_c^\infty(\mathbb{C}^n$ with $\operatorname{supp} u \subset \operatorname{supp}\varphi$ such that $\bar{\partial} u = v = \bar{\partial} g$, the holomorphic function $g - u$ is well-defined on Ω and coincides with f (and g) on $\Omega \setminus \operatorname{supp}\varphi$, therefore coincides with f on $\Omega \setminus K$. $\square$

Note that although we do not need Ω to be small, this theorem counts as a local result due to the hypothesis that we are in $\mathbb{C}^n$.

A global result can be obtained using the Hartog figure, that is the union of an anulus $\{(z_1, z') : \ r < |z_1| < R\}$ and an open set in other dimension $\{(z_1, z') : \ z' \in \omega \text{ open}\}$. and use the interpolation $(z_1, z') \mapsto \int_{C_R} \frac{f(\zeta_1, z')}{\zeta_1 - z_1} d\zeta_1$ to extend f. The open set in z'-dimension is to show that the interpolation and f coincide on it. With one dimension z_1 to form the annulus an another dimension (says z_2 to form the open set, one can extend any holomorphic function to a submanifold of (complex) codimension at least 2.

Theorem 8 (Riemann extension)**.** *Let M be a complex manifold and N a sub complex manifold of codimension ≥ 2 then any holomorphic function on $M \setminus N$ extends uniquely to M.*

4.1 Generalities

An approach to the extension problem on complex manifolds is through the notion of holomorphic hull and holomorphic convexity.

Definition 5.
1. *Let K be a compact in a complex manifold M. Then the holomorphic hull $\hat{K}_{\mathcal{O}(M)}$ is the set $\{z \in M : \ f(z) \leq \sup_K |f| \quad \forall f \in \mathcal{O}(M)\}$.*
2. *A complex manifold M is said to be holomorphically convex if $\hat{K}_{\mathcal{O}(X)}$ is compact for all compact $K \subset M$.*

Proposition 8.1 (holomorphic hull)**.** *The following statements are obvious*

1. *$\hat{K}$ is a closed subset containing K and $\hat{\hat{K}} = \hat{K}$.*
2. *If $f : \ M_1 \longrightarrow M_2$ is holomorphic then $f(\hat{K}) \subset \widehat{f(K)}$.* (Think inclusion)
3. ***Hole filling.*** *In particular, if $f : \ \bar{B} \longrightarrow X$ and $f(\partial B) \subset K$ then $f(\bar{B}) \subset \hat{K}$.*

Proposition 8.2 (holomorphically convex)**.** *Let M be a holomorphically convex complex manifold then*

1. *M admits a exhaustive sequence of compact K_ν, i.e. $K_\nu \Subset K_{\nu+1}$ and $\widehat{K_\nu} = K_\nu$.*
2. *M is weakly pseudoconvex, i.e. there exists $\psi \in Psh(M) \cap C^\infty(M)$ such that $\{\psi < c\}$ are relatively compact, i.e. $\lim_{K \to M} \psi\big|_{M \setminus K} = +\infty$*

4.2 Case $\Omega \subset \mathbb{C}^n$

Definition 6. *Domain of holomorphy*

Proposition 8.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain then:*

1. *If Ω is a domain of holomorphy then $\hat{K}_{\mathcal{O}(\Omega)}$ is compact and $d(K, \partial\Omega) = d(\hat{K}, \partial\Omega)$.*

2. *THe followings are equivalent:*

   (a) *Ω is a domain of holomorphy.*

   (b) *Ω is holomorphically convex.*

   (c) *Let $\{z_k\}$ be a sequence in Ω without accumulation in Ω and $c_k \in \mathbb{C}$. There exists a function $f \in \mathcal{O}(\Omega)$ such that $f(z_k) = c_k$.*

   (d) *There exists a function $F \in \mathcal{O}(\Omega)$ that is unbounded locally in any point on $\partial\Omega$.*

#+END$_{\text{theorem}}$

4.2.1 Different notion of pseudoconvexity

4.2.2 Richberg approximation theorem