Minimal immersions of $\mathbb{S}^2$

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1 Brief view of Sacks and Uhlenbeck's strategy.

Let M and N be compact Riemannian manifolds (without boundary), M is a surface and N is isometrically embedded in $\mathbb{R}^k$. It was showed by Eells and Sampson [?] that if N is negatively curved than any map from M to N is homotopic to a harmonic map. The idea of Sacks and Uhlenbeck in [?] consists of (1) approximating the energy functional E by a family E_α satisfying Palais-Smale condition, whose *nontrivial* critical values can be more easily proved to exist and (2) trying to prove that the critical maps s_α of E_α converge in C^1-topology.

We will first review the general machinery of Morse-Palais-Smale theory and prove the existence of s_α. The convergence of s_α in the case of surface is due to the facts that energy functional E is a conformal invariant of M, in particular E is invariant by homotheties (i.e. E remains unchanged when we zoom in and out), which allows us to justify the C^1-convergence (under conditions of N) except at finitely many points using a local estimate and a suitable covering of M.

Sacks and Uhlenbeck used an extension result for harmonic map, in an elegant argument to prove that if the above sequence $\{s_\alpha\}$ fails to converge at a point, for a certain surface M, then one has a nontrivial harmonic map from $\mathbb{S}^2$ to N. Therefore if such sequence $\{s_\alpha\}$ from $\mathbb{S}^2$ to N exists, for example when $\pi_k(N)$ is nontrivial for a certain $k \geq 2$ then, whether s_α converges or not, there exists a nontrivial harmonic map from $\mathbb{S}^2$ to N.

Finally, the theory of branched immersion of surfaces by Gulliver-Osserman-Royden [**?**] can be applied to show that the harmonic map obtained this way is a conformal, branched, minimal immersion of $\mathbb{S}^2$ to N.

2 General machinery by Morse-Palais-Smale.

2.1 Perturbed functionals E_α.

Let $s : M \longrightarrow N \hookrightarrow \mathbb{R}^k$ be a map from a compact surface M to a compact Riemannian manifold N isometrically embedded into $\mathbb{R}^k$. Recall that the energy functional of s is given by $E(s) := \frac{1}{2}\int_M |ds|^2 dV_M = \frac{1}{2}\int_M \langle s^* g_N, g_M\rangle dV_M$. The perturbed energy functionals are

$$E_\alpha(s) := \int_M \left(1 + |ds|^2\right)^\alpha dV, \quad \alpha \geq 1$$

We will suppose, by rescaling the metric g_M of M that the volume of M is 1, so when $\alpha = 1$, $E_1 = 1 + 2E(s)$ is just the previously defined energy. Using $(a+b)^\alpha \geq a^\alpha + b^\alpha$ and Jensen's inequality, one has $E_\alpha(s) \geq 1 + (2E(s))^\alpha$ for all $\alpha \geq 1$. Also, since we only interest in the case α close to 1, let us also suppose that α from now on is smaller than 2.

By Sobolev embedding, one has $W^{1,2\alpha}(M, \mathbb{R}^k) \subset C^0(M, \mathbb{R}^k)$ compactly for all $\alpha > 1$. It then makes sense to talk about $W^{1,2\alpha}(M, N) \subset C^0(M, N)$ which consist of elements of $W^{1,2\alpha}(M, \mathbb{R}^k) \subset C^0(M, \mathbb{R}^k)$ whose image lies in N.

Theorem 1 (Palais)**.** *The spaces $C^\infty(M, N) \subset W^{1,2\alpha}(M, N) \subset C^0(M, N)$, where $\alpha > 1$, are of the same homotopy type and the inclusions are homotopy equivalences. In particular, their connected components are naturally in bijection.*

We will also need a version of Morse theory for Banach manifolds, also developed by R. Palais in [**?**]. For the terminologies, in the same way that a manifold is modeled by$\mathbb{R}^n$, a *Banach manifold* is modeled by Banach spaces. A *Finsler manifold* is a Banach manifold with a norm on its tangent space that is comparable with the norm of Banach charts.

Theorem 2 (Morse theory for Banach manifolds)**.** *1. If F is a C^2 functional on a complete C^2 Finsler manifold L, F is bounded below and F satisfies Palais-Smale condition (C) then*

(a) The functional F admits minimum on each connected component of L.

(b) If F has no critical value in $[a, b]$ then the sublevel $\{F \leq b\}$ retracts by deformation to the sublevel $\{F \leq a\}$.

2. The pair $(L, F) = (W^{1,2\alpha}(M, N), E_\alpha)$ with $\alpha > 1$ satisfies the condition of the first part.

The *Palais-Smale condition* is as follows:

(C): Let $S \subset L$ be a subset on which $|F|$ is bounded, but $|dF|$ is not bounded away from 0. Then there exists a critical point of F in $\bar{S}$.

The strategy to prove Theorem 2 is, as in finite dimensional case, to use a pseudo-gradient flow of F whose existence is due to a partition of unity of L (instead of a Riemannian metric on L). The role of Palais-Smale condition in the proof is as follows: Suppose that $\{x_n\}$ is a sequence in a connected component L_1 of L such that $F(x_n)$ tends to $\inf_{L_1} F$, then using the pseudo-gradient flow of F, we can suppose that x_n are critical points of F. Choose a sequence $\{y_n\}$ of regular points near x_n such that $F(y_n) \to \inf_{L_1} F$ and $|dF(y_n)| \to 0$ and use (C) for $S = \{y_n\}$, one obtains a limit point y_∞ of $\{y_n\}$, hence also of $\{x_n\}$, which minimises F.

As a consequence of Theorem 2, one has:

Corollary 2.1 (Component-wise minimum of E_α)**.** *The minimum of E_α in each connected component C of $W^{1,2\alpha}(M, N)$, $\alpha > 1$ is taken by some $s_\alpha \in C^\infty(M, N)$ and there exists $B > 0$ depending on the component C such that*

$$\min_C E_\alpha \leq (1 + B^2)^\alpha$$

Proof. By Theorem 2, E_α admits minimum at s_α on each component C of $W^{1,2\alpha}(M, N)$. By writing down the Euler-Lagrange equation of E_α and apply regularity estimates, one can prove that s_α is actually smooth. By

Theorem 1, the preimage of C by inclusion $C^\infty(M,N) \subset W^{1,2\alpha}(M,N)$ is a connected component C' of $C^\infty(M,N)$ over which s_α is the minimum of E_α. Take $B = \sup_M |du|$ for an arbitrary element $u \in C'$ and the conclusion follows. $\square$

Remark 1. *Corollary 2.1 is trivialised when $W^{1,2\alpha}(M,N)$ is connected (for one α or equivalently for all α). In this case, s_α is a constant map and $B = 0$.*

To establish a nontrivial analog of Corollary 2.1 in the case where the spaces of maps from M to N are connected, we will have to look at the submanifold $N_0 \cong N$ formed by constant maps.

2.2 Tubular neighborhood of the submanifold of trivial maps.

Fix $y \in N$, considered as a constant maps in N_0. We will summarise a few facts about the tangent space of $W^{1,2\alpha}(M,N)$ at y in the following Remark.

These facts come from the *differential structure* of the Banach manifold $W^{1,2\alpha}(M,N)$ that so far has not been introduced, since we only consider $W^{1,2\alpha}(M,N)$ as a closed subset of $W^{1,2\alpha}(M,\mathbb{R})^{\oplus k}$ (so only a topological structure was given). We summarise here, and refer to [**?**], how a differential structure is given to $W^{k,p}(M,N)$ with k,p such that $W^{k,p}(M) \hookrightarrow C^0(M)$:

- Let ξ be a finite dimensional vector bundle over a compact manifold M, then $W^{k,p}(\xi,M)$ can be defined as the Banach space of sections of ξ that are locally $W^{k,p}$. A norm of $W^{k,p}(\xi,M)$ can be given using a metric of ξ and a volume form of M, but by compactness of M, its equivalent class is independent of such choices.

- Let E be a fiber bundle over M, in our case, $E = N \times M$, and $s \in C^0(E)$ be a continuous section. It can be proved that there exists an open subset ξ of E containing s such that $\xi \to M$ has a vector bundle structure. We say that $s \in W^{k,p}(E,M)$ if $s \in W^{k,p}(\xi,M)$ and it turns out that this definition is independent of the choice of ξ. This defines $W^{k,p}(E,M)$ set-theoretically.

- The differential structure of $W^{k,p}(E,M)$ is given by the atlas $W^{k,p}(\xi,M)$.

Remark 2. *1. The tangent $T_yW^{1,2\alpha}(M,N)$ can be identified with $W^{1,2\alpha}(M,T_yN)$. The subspace T_yN_0 contains constant maps from M to T_yN.*

2. *The fiber $\mathcal{N}_y$ over y of the normal bundle $\mathcal{N}$ of N_0 can be identified with*

$$\mathcal{N}_y = \left\{ v \in W^{1,\alpha}(M, T_y N) : \int_M v dV = 0 \right\}$$

The exponential map on $TW^{1,2\alpha}(M,N)$ can be defined as follows:

$$\begin{aligned} e:\ TW^{1,2\alpha}(M,N) &\longrightarrow W^{1,2\alpha}(M,N) \\ (s,v) &\longmapsto \Big(x \mapsto \exp_{s(x)} v(x)\Big) \end{aligned}$$

where $s \in W^{1,2\alpha}(M,N)$ and $v \in T_s W^{1,2\alpha}(M,N)$ is a $W^{1,2\alpha}$ vector field along $s(x)$. With the representation of normal bundle $\mathcal{N}$ as Remark 2, the restriction of e on $\mathcal{N}$ is given by

$$\begin{aligned} e\big|_{\mathcal{N}}:\ \mathcal{N} &\longrightarrow W^{1,2\alpha}(M,N) \\ (y,v) &\longmapsto \big(x \mapsto \exp_y(v(x))\big) \end{aligned}$$

where $y \in N_0 \cong N$ and $v \in W^{1,2\alpha}(M, T_y N)$.

Lemma 3. *The restriction $e\big|_{\mathcal{N}}$ of e on $\mathcal{N}$ is a local diffeomorphism mapping a neighborhood of the zero-section of $\mathcal{N}$ onto a neighborhood of N_0 in $W^{1,2\alpha}(M,N)$.*

Proof. It can be calculated that

$$de_{(y,0)}(a,v) = (x \mapsto a + v(x)) \in T_y W^{1,2\alpha}(M,N) = W^{1,2\alpha}(M, T_y N)$$

for $a \in T_y N$ and $v \in \mathcal{N}_y \subset W^{1,2\alpha}(M, T_y N)$. It is invertible since a is tangential to N_0 and $v \in \mathcal{N}_y$ is in the normal component. The Inverse function theorem applies. $\square$

2.3 Critical values of E_α.

The exponential map previously defined on the normal bundle of N_0 in $W^{1,2\alpha}(M,N)$ allows us to retract by deformation a small neighborhood of N_0 to N_0. We will prove that if the energy $E_\alpha(s)$ is sufficiently close to $1 = E_\alpha(N_0)$ then s is sufficiently $W^{1,2\alpha}$-close to N_0 and hence can be retracted to N_0, in other words, $E_\alpha^{-1}[1, 1+\delta]$ retracts by deformation to $N_0 = E_\alpha^{-1}(1)$.

Proposition 3.1. *Given $\alpha > 1$, there exists $\delta > 0$ depending on α such that $E_\alpha^{-1}[1, 1+\delta]$ retracts by deformation to $E_\alpha^{-1}(1) = N_0$.*

Proof. Let $s \in E_\alpha^{-1}[1, 1+\delta]$, using $(a+b)^\alpha \geq a^\alpha + b^\alpha$, one has

$$1 + \delta > \int_M (1 + |ds|^2)^\alpha dV > 1 + \int_M |ds|^{2\alpha} dV$$

therefore $\|ds\|_{L^{2\alpha}} \leq \delta^{1/2\alpha}$. By Poincaré-Wirtinger inequality, $\|s - \int_M s\|_{W^{1,2\alpha}} \leq C\delta^{1/4}$ where C is the Poincaré-Wirtinger constant.

By Sobolev embedding, $\max_M |s - \int_M s| \leq C_\alpha \|s - \int_M s\|_{W^{1,2\alpha}}$ where the Sobolev constant C_α can no longer be chosen uniformly in $\alpha \to 1$. Fix an $x_0 \in M$, one has

$$d_{W^{1,2\alpha}}(s, N_0) \leq \|s - s(x_0)\|_{W^{1,2\alpha}} \leq \left\| s - \int_M s \right\|_{W^{1,2\alpha}} + \left| \int_M s - s(x_0) \right| \leq C_\alpha \delta^{1/4}$$

Now choose $\delta \ll 1$ depending on α such that s is in the neighborhood of N_0 given by Lemma 3, s can be written as

$$s(x) = e(y, v(x)) = \exp_y v(x)$$

where $y \in N_0$ and $v \in W^{1,2\alpha}(M, T_y N)$ depend continuously on $s \in W^{1,2\alpha}(M, N)$. We can define the deformation retraction by

$$\begin{aligned} \sigma: \ E_\alpha^{-1}[1, 1+\delta] \times [0,1] &\longrightarrow E_\alpha^{-1}[1, 1+\delta] \\ (s, t) &\longmapsto \big(x \mapsto \exp_y tv(x)\big) \end{aligned}$$

It is clear that σ is continuous and σ_0 is a retraction. The only thing to check is that the image of σ remains in $E_\alpha^{-1}[1, 1+\delta]$ at all time. This can be checked by showing that $\frac{d}{dt} E_\alpha(\sigma_t) \geq 0$, hence $E_\alpha(\sigma_t) \leq E_\alpha(\sigma_1) \leq 1 + \delta$ for all $0 \leq t \leq 1$. $\square$

We will now prove the existence of nontrivial critical value of E_α in an interval $(1, B)$ for a certain $B > 1$ sufficiently big independently of $\alpha > 1$.

Fix $z_0 \in M$ and consider the map

$$\begin{aligned} p: \ C^0(M, N) &\longrightarrow N \\ s &\longmapsto f(z_0) \end{aligned}$$

then p is a fiber bundle and therefore is a *Serre fibration.* In fact fix $q_0 \in N$ then for all $q \in N$ near q_0, there is a vector field v_q supported in a small ball centered at q_0 such that the flow of v_q from time 0 to 1 turns q_0 to q, i.e. $\Phi_{v_q}{}_0^1(q_0) = q$, and that v_q varies continuously in q. Then any fiber $p^{-1}(q)$ can be identified with $p^{-1}(q_0)$ using the flow of v_q. We will denote by $\Omega(M, N)$ the topological fiber of p.

We will use a few facts from algebraic topology, briefly summarised here.

Fact 1. *1. (Long exact sequence of homotopy) Let $p : E \longrightarrow B$ be a fiber bundle of fiber $F = p^{-1}(b_0) \ni f_0$, then one has the following long exact sequence*

$$\dots \xrightarrow{\partial} \pi_n(F) \xrightarrow{\iota_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \dots \longrightarrow \pi_0(E) \longrightarrow 0$$

where $\iota : F \longrightarrow E$ is the inclusion.

2. If p admits a global section s, then one has a retraction s_* of p_*:

$$\begin{tikzcd} \pi_n(E) \arrow[r, shift left, "p_*"] & \pi_n(B) \arrow[l, shift left, "s_*"] \end{tikzcd}$$

hence p_* is surjective and ∂ factors through 0, which gives us the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_n(F) \arrow[r, "\iota_*"] & \pi_n(E) \arrow[r, shift left, "p_*"] & \pi_n(B) \arrow[l, shift left, "s_*"] \arrow[r] & 0 \end{tikzcd}$$

where p_* admits a retraction s_*, so the short exact sequence splits and we have

$$\pi_n(E) \cong \pi_n(F) \oplus \pi_n(B).$$

Now apply this result to the fiber bundle $p : C^0(M, N) \longrightarrow N$ of fiber $\Omega(M, N)$, which has N_0 as a global section, one obtains

$$\pi_n(C^0(M, N)) \cong \pi_n(N) \oplus \pi_n(\Omega(M, N)).$$

Theorem 4 (Nontrivial critical value of E_α)**.** *If $C^0(M, N)$ is not connected, or if $\Omega(M, N)$ is not contractible, then there exists $B > 0$ such that for all $\alpha > 1$, E_α has critical values in the interval $(1, (1 + B^2)^\alpha)$.*

In particular, if $M = \mathbb{S}^2$ and if the universal covering $\tilde{N}$ of N is not contractible then E_α has critical values in $(1, (1 + B^2)^\alpha)$.

Proof. If $C^0(M, N)$ is not connected, one only needs to apply Corollary 2.1 to a connected component of $W^{1,2\alpha}(M, N)$ not containing N_0. We now suppose that $C^0(M, N)$ is connected and $\Omega(M, N)$ is not contractible.

In this case, there exists $n > 0$ such that $\pi_n(\Omega(M, N))$ is nontrivial and contains a nonzero element $\gamma : \mathbb{S}^n \longrightarrow \Omega(M, N)$ which is not homotopic to any $\tilde{\gamma} : \mathbb{S}^n \longrightarrow N_0$ in $\pi_n(C^0(M, N))$.

Choose $B := \max_{\theta \in \mathbb{S}^n, x \in M} |d\gamma(\theta)(x)|$ then by definition

$$E_\alpha(\gamma(\theta)) \le (1 + B^2)^\alpha \quad \forall \theta \in \mathbb{S}^n, \alpha > 1.$$

If E_α has no critical value in $[1+\frac{\delta_\alpha}{2},(1+B^2)^\alpha]$ where δ_α is given by Proposition 3.1, then by Theorem 2, $E_\alpha^{-1}[1,(1+B^2)^\alpha]$ retracts by deformation to $E_\alpha^{-1}[1,1+\delta_\alpha]$ which retracts by deformation to $E_\alpha^{-1}(1)=N_0$. But this means that γ is homotopic to a certain $\tilde{\gamma}\in\pi_n(N)$, which is a contradiction.

As an application, if $M=\mathbb{S}^2$ and the universal covering $\tilde{N}$ is not contractible then the long exact sequence of homotopy for the bundle $\tilde{N}\longrightarrow N$ with fiber of dimension 0, gives

$$\pi_n(\tilde{N})=\pi_n(N),\quad \forall n\geq 2.$$

Since $\tilde{N}$ is simply-connected and not contractible, there exists $n\geq 2$ such that $0\neq\pi_n(\tilde{N})=\pi_n(N)=\pi_{n-2}(\Omega(\mathbb{S}^2,N))$, where the last equality follows from definition of homotopy group. The general argument applies. $\square$

3 Local results: Estimates and extension.

We will say that the map $s:M\longrightarrow N$ is a critical point of E_α on a small disc $D(R)\subset M$ if s satisfies the Euler-Lagrange equation of E_α (as functional on $W^{1,2\alpha}(M,N)$) on $D(R)$.

Remark 3. *Rescaling $(D(R),g_M)$, where $R\ll 1$ and g_M is ϵ-close to the Euclidean metric, to the unit disc D one obtains a metric $\tilde{g}_M$ that is still ϵ-close to Euclidean metric. The curvature of $\tilde{g}_M$ is R^2 times smaller than that of g_M.*

If $s:\ D(R)\longrightarrow N$ is a critical map of E_α on $D(R)$, then the composition $\tilde{s}$ of s and the rescaling operator $D\longrightarrow D(R)$ satisfies the Euler-Lagrange equation of $\tilde{E}_\alpha=R^{2(1-\alpha)}\int_D(R^2+|d\tilde{s}|^2)^\alpha d\tilde{V}$ where $d\tilde{V}$ is the volume form of the rescaled metric $\tilde{g}_M$. We will abusively use the same notation for $\tilde{s}$ and s and regard s as a map on the unit disc D.

Lemma 5 (Sacks-Uhlenback's Main estimate)**.** *For all $p\in(1,+\infty)$, there exists $\epsilon>0$ and $\alpha_0>1$ depending on p such that if*

- *$s:\ (D,\tilde{g})\longrightarrow N$ is a critical map of E_α on $D(R)$*
- *$E(s)<\epsilon$, $1<\alpha<\alpha_0$*

then

$$\|ds\|_{W^{1,p}(D')}<C(p,D')\|ds\|_{L^2(D)},\quad \textit{for all disc } D'\Subset D$$

Remark 4. *In fact α_0,ϵ and $C(p,D')$ depend on the rescaled metric $\tilde{g}$ on D, but if $R\ll 1$ and $\tilde{g}$ is very close to Euclidean metric, then one can choose these parameters independently of $\tilde{g}$.*

A consequence of (the proof of) Lemma 5 is the following global result:

Theorem 6 (Critical maps of low energy are trivial)**.** *There exists* $\epsilon' > 0$ *and* $\alpha_0 > 1$ *such that if*

- $s : M \longrightarrow N$ *is critical map of* E_α
- $E(s) < \epsilon'$*,* $1 < \alpha < \alpha_0$

then $s \in N_0$ *and* $E(s) = 0$*.*

We proved in the last section that, under certain algebraic topological condition on N, E_α admits critical value $v_\alpha \in (1, (1 + B^2)^\alpha)$. We now can conclude that, by Theorem 6, the critical values v_α are bounded away from 1, i.e. $\inf_\alpha v_\alpha > 1$.

We will also need the following extension theorem:

Theorem 7 (Extension of harmonic maps)**.** *If* $s : D \setminus \{0\} \longrightarrow N$ *is a harmonic map with finite energy* $E(s) < \infty$*, then* s *extends to a smooth harmonic map* $\tilde{s} : D \longrightarrow N$*.*

4 Convergence of critical maps of E_α.

We proved in Theorem 4 that if $C^0(M, N)$ is not connected or if $\Omega(M, N)$ is not contractible, then there exists a family $\{s_\alpha\}$ of critical maps of E_α with bounded, nontrivial energy $E_\alpha(s_\alpha) < B$. Since

- $\int_M |ds_\alpha|^2 \leq (E_\alpha(s_\alpha) - 1)^{1/\alpha}$ is bounded uniformly on α
- $\|s_\alpha\|_{L^\infty}$ is bounded by compactness of N.

the $W^{1,2}(M, \mathbb{R}^k)$-norms of $\{s_\alpha\}$ are bounded. By reflexivity of Sobolev spaces, there exists a subsequence $\{s_\beta\}$ weakly converging to s in $W^{1,2}(M, \mathbb{R}^k)$ with

$$\|s\|_{W^{1,2}} \leq \liminf_{\beta \to 1} \|s_\beta\|_{W^{1,2}}$$

We do not know at this moment if the convergence is C^0, or if s is continuous, or even if the image of s remains in N. The following key lemma answer these questions on a small disc of M in the case the energy of s_α is small.

Lemma 8 (Key)**.** *There exists an* $\epsilon > 0$*, in fact given by the Main estimate Lemma 5 with* $p = 4$*, such that if*

- $s_\alpha : D(R) \longrightarrow N \subset \mathbb{R}^k$ *are critical maps of* E_α *in* $W^{1,2\alpha}(D(R), N)$*,*

- *$E(s_\alpha) < \epsilon$ and s_α converges weakly to s in $W^{1,2}(D(R), \mathbb{R}^k)$,*

then

- *the restriction of s on $\overline{D(R/2)}$ is smooth harmonic map with image in N,*
- *$s_\alpha \to s$ in $C^1(\overline{D(R/2)}, N))$.*

Remark 5. *There are two different ways to define convergence of a sequence s_n to s in $C^1(\Omega)$ on an open set Ω:*

1. *The sequence s_α and s extend to $C^1(\bar{\Omega})$ and have finite norm $\max_\Omega |s| + \max_\Omega |ds|$ and $\max_\Omega |s_\alpha| + \max_\Omega |ds_\alpha|$ and*
$$\max_\Omega |s_\alpha - s| + \max_\Omega |ds - ds_\alpha| \to 0.$$
In this case, we will say that s_α converges to s in $C^1(\bar{\Omega})$.
2. *$C^1(\Omega)$ is topologised by a family of seminorms $\Gamma_K : \ s \longmapsto \max_K |s| + \max_K |ds|$ for $K \Subset \Omega$. This makes $C^1(\Omega)$ a Fréchet topological vector space. If the sequence s_α converges to s under this topology then we will say that s_α converges uniformly to s on compacts of Ω.*

Proof. We consider s_α and s as maps from the unit disc D to $\mathbb{R}^k$, then by Main estimate Lemma 5 for $p = 4$, since $E(s_\alpha) < \epsilon$, one has:

$$\|ds_\alpha\|_{W^{1,4}(D(1/2),\mathbb{R}^k)} \leq C(4, D(1/2))\|ds_\alpha\|_{L^2(D)} = C(4, D(1/2))E(s_\alpha)^{1/2}$$

So $\{s_\alpha\}$ is bounded in $W^{1,4}(D(1/2), \mathbb{R}^k)$ which is embedded compactly into $C^1(\overline{D(1/2)}, \mathbb{R}^k)$.

We now can prove that s_α converges strongly to s in $C^1(\overline{D(1/2)}, \mathbb{R}^k)$: If there was a subsequence $\{s_\beta\}$ whose restriction to $\overline{D(1/2)}$ remains C^1-away from s, then by compactness of $W^{1,4}(D(1/2), \mathbb{R}^k) \hookrightarrow C^1(\overline{D(1/2)}, \mathbb{R}^k)$, we can suppose that $\{s_\beta\}$ converges in C^1 to a certain $\bar{s} \neq s$ on $\overline{D(1/2)}$. But as a subsequence of $\{s_\alpha\}$, $\{s_\beta\}$ converges weakly to s on D, hence on $\overline{D(1/2)}$, we than obtain a contradiction using the uniqueness of weak limit.

By considering the Euler-Lagrange equation and letting $\alpha \to 0$, one concludes that s is a harmonic map from $D(1/2)$ to N. □

The global convergence of $\{s_\alpha\}$ can be established by a well-chosen covering of M by small balls or radius R.

Proposition 8.1. *Let $s_\alpha : M \longrightarrow N \subset \mathbb{R}^k$ be critical maps of E_α on M such that s_α converges weakly to s in $W^{1,2}(M, \mathbb{R}^k)$ and $E(s_\alpha) < B$. Then there exists $l = l(B, N)$ such that given any $m > 0$, one can find a sequence $\{x_{m,1}, \ldots, x_{m,l}\} \subset M$ and a subsequence $\{s_{\alpha(m)}\}$ of $\{s_\alpha\}$ such that*

$$s_{\alpha(m)} \longrightarrow s \text{ in } C^1\left(M \setminus \bigcup_{i=1}^{l} D(x_{m,i}, 2^{-m+1}), N\right)$$

Proof. We cover M by finitely many balls $D(y_i, 2^{-m})$ such that each point is covered at most h times by the bigger balls $D(y_i, 2^{-m+1})$. By Lemma **??**, h can be chosen independently of m as $2^{-m} \to 0$.

Since $\sum_i \int_{D(y_i, 2^{-m+1})} |ds_\alpha|^2 < Bh$, choosing $l = \lceil \frac{Bh}{2\epsilon} \rceil$, we see that there are at most l balls $D(y_{\alpha,i}, 2^{-m+1})$ with centers depending on α, on which the energy $E(s_\alpha)$ is less than ϵ. Passing to a subsequence $\{s_{\alpha(m)}\}$ of $\{s_\alpha\}$, we can suppose that $\{y_{\alpha(m),i}\}$ converges to $x_{m,i}$ as $\{\alpha(m)\} \to 1$. But since the points $\{y_i\}$ are of finite number and separated, $y_{\alpha(m),i} \equiv x_{m,i}$ eventually and we can suppose that the bad balls $D(y_{\alpha(m),i})$ where energy of $s_{\alpha(m)}$ surpasses ϵ are the same for every $s_{\alpha(m)}$.

Now apply Lemma 8 to the sequence $\{s_{\alpha(m)}\}$ on all the other 2^{-m+1}-balls, one sees that $\{s_{\alpha(m)}\}$ converges in C^1 to s on all $\overline{D(y_i, 2^{-m})}$ except those centered at $x_{m,i}$. The conclusion follows. □

Using a diagonal argument, we can find a subsequence $\{s_\beta\}$ of $\{s_\alpha\}$ that converges to s uniformly on compacts of $M \setminus \{x_1, \ldots, x_l\}$.

Theorem 9 (Convergence of $\{s_\alpha\}$). *Let $s_\alpha : M \longrightarrow N \subset \mathbb{R}^k$ be critical maps of E_α on M such that s_α converges weakly to s in $W^{1,2}(M, \mathbb{R}^k)$ and $E(s_\alpha) < B$. Then there exist at most l points $x_1, \ldots, x_l$ in M, where l is given by Proposition 8.1, and a subsequence $\{s_\beta\}$ of $\{s_\alpha\}$ such that*

$$s_\beta \longrightarrow s \text{ in } C^1(M \setminus \{x_1, \ldots, x_l\}, \mathbb{R}^k) \text{ uniformly on compacts.}$$

Proof. By passing to a subsequence $\{m_k\}$ of $\{m\}$, we can suppose that $\{x_{m,i}\}$ converges to x_i in M. Choose the diagonal subsequence $\{s_\beta\}$ from $\{s_{\alpha(m)}\}$ that consists of $s_{\alpha(m)(a_m)}$ where a_m is sufficiently big such that $\alpha(m)(a_m)$ is increasing and $\|s_{\alpha(m)(b)} - s_{\alpha(m)(c)}\|_{C^1(M \setminus \cup_i D(x_{m,i}, 2^{-m+1})} < \frac{1}{m}$ for all $b, c \geq a_m$. Then the sequence $\{s_\beta\}$ converges uniformly on compacts of $M \setminus \{x_1, \ldots, x_l\}$ because $\{\bigcup_i D(x_{m,i}, 2^{-m+1})\}_m$ is an exhaustive family of compacts of $M \setminus \{x_1, \ldots, x_l\}$. □

Remark 6. *With the same notation as Theorem 9,*

1. *The image $s(M \backslash \{x_1, \ldots, x_l\})$ lies in N. Also, using the Euler-Lagrange equation, one sees that s is a (smooth) harmonic map from $M \backslash \{x_1, \ldots, x_l\}$ to N.*

2. *Since $E(s) \leq \|s\|^2_{W^{1,2}} \leq \liminf_{\alpha \to 1} \|s_\alpha\|^2 < +\infty$, $s\big|_{M \backslash \{x_1, \ldots, x_l\}}$ extends to a harmonic map $\tilde{s} : M \longrightarrow N$. We can therefore suppose that the limit s of Theorem 9 is smooth harmonic map on M and of image in N.*

5 Nontrivial harmonic maps from $\mathbb{S}^2$.

We will now prove the existence of nontrivial harmonic maps from $\mathbb{S}^2$ to a compact Riemannian manifold N satisfying the conditions of Theorem 4.

The following theorem does not suppose any condition on N.

Theorem 10. *Let M be a compact surface and s_α be critical maps of E_α. Suppose that*

- *s_α converges in C^1 to s uniformly on compacts of $M \setminus \{x_1, \ldots, x_l\}$ but not on $M \setminus \{x_2, \ldots, x_l\}$.*
- *$E(s_\alpha) < B$*

Then there exists a nontrivial harmonic map $s_* : \mathbb{S}^2 \longrightarrow N$.

Before proving the theorem, let us state its corollary.

Corollary 10.1 (Nontrivial harmonic map from $\mathbb{S}^2$)**.** *If the universal covering $\tilde{N}$ of N is not contractible then there exists a nontrivial harmonic map $s : \mathbb{S}^2 \longrightarrow N$.*

Proof. By Theorem 4 and Theorem 6, there exist critical maps $s_\alpha : \mathbb{S}^2 \longrightarrow N$ of E_α corresponding to critical values $E_\alpha(s_\alpha)$ in $(1+\delta, B)$. We claim that $\{s_\alpha\}$ cannot converge in $C^1(M)$ to a trivial harmonic map $s \in N_0$. In fact, if it did,

$$1 + \delta \leq \lim_{\alpha \to 1} \int_M (1 + |ds_\alpha|^2)^\alpha dV = \int_M (1 + |ds|^2) dV = 1$$

which is contradictory.

Therefore, we only have two possibilities:

- $\{s_\alpha\}$ does not converge in $C^1(M)$ to s, then by Theorem 10, there exists a nontrivial harmonic map $s_* : \mathbb{S}^2 \longrightarrow N$.

- If $\{s_\alpha\}$ converges in $C^1(M)$ to a certain $\tilde{s}$, then as argued above, $\tilde{s}$ is nontrivial.

In both cases, nontrivial harmonic map from $\mathbb{S}^2$ to N exists. $\square$

Let us now prove Theorem 10.

Proof of Theorem 10. If there is no C^1 convergence near x_1, we claim that:

Assertion 1. *For all $C > 0$ and $\delta > 0$, there exists $\alpha > 1$ arbitrarily close to 1 such that*

$$\max_{\overline{D}(x_1,2\delta)} |ds_\alpha| > C.$$

Moreover, we can suppose that $\max_{\overline{D}(x_1,2\delta)} |ds_\alpha| = \max_{D(x_1,\delta)} |ds_\alpha|$.

Suppose that was not the case, then there exist $C, \delta > 0$ such that $\max_{D(x_1,2\delta)} |ds_\alpha| \leq C$ for all $\alpha > 1$ sufficiently close to 1. Choose a radius $R \ll \delta$ such that

$$\int_{D(x_1,R)} |ds_\alpha|^2 \leq \pi R^2 C^2 < \epsilon$$

It suffices to apply Key lemma 8 to see that $s_\alpha \to s$ in $C^1(D(x_1, R/2))$, hence s_α converges to s in $C^1(M \setminus \{x_2, \ldots, x_l\})$ uniformly on compacts. Moreover, since $\{ds_\alpha\}$ converges uniformly to ds on $\overline{D}(x_1, 2\delta) \setminus D(x_1, \delta)$, we can suppose, with α sufficiently close to 1, that the maximum is actually attained in $D(x_1, \delta)$.

Therefore, we can choose a sequence $\{C_n\}$ increasing to $+\infty$ and $\{\delta_n\}$ decreasing to 0, such that $C_n\delta_n$ diverges to $+\infty$ and there exists a sequence $\{\alpha_n\}$ decreasing to 1 such that

$$|ds_{\alpha_n}(y_n)| := \max_{D(x_1,\delta_n)} |ds_{\alpha_n}| = \max_{D(x_1,2\delta_n)} |ds_{\alpha_n}| = C_n$$

We define

$$\begin{aligned} \tilde{s}_{\alpha_n} :\ D(\delta_n C_n) &\longrightarrow N \\ x &\longmapsto s_{\alpha_n}(y_n + C_n^{-1}x) \end{aligned}$$

then $|d\tilde{s}_{\alpha_n}(0)| = \max_{D(C_n\delta_n)} |d\tilde{s}_{\alpha_n}| = 1$.

Fix any large $R < +\infty$, since $C_n\delta_n \to +\infty$, $\tilde{s}_{\alpha_n}$ is eventually defined on $D(R)$ and is a critical point of E_{α_n} with respect to a metric $\tilde{g}_n$ on $D(R)$ converging to the Euclidean metric. The energy $E(\tilde{s}_{\alpha_n}\big|_{D(C_n\delta_n)}, \tilde{g}_n) = E(\tilde{s}_{\alpha_n}\big|_{D(y_n,\delta_n)}, g_M) \leq B$.

We claim that Proposition 8.1 and Theorem 9 remain correct when $M = D(R)$ and s_α are critical maps of E_α with respect to metrics $\tilde{g}_\alpha$ converging to the Euclidean metric. To be precise:

Assertion 2. *Let $\tilde{s}_\alpha : (D(R), \tilde{g}_\alpha) \longrightarrow N \subset \mathbb{R}^k$ be critical maps of E_α such that*

- *s_α converges weakly to s_* in $W^{1,2}(D(R), Euclid)$,*
- *$E(s_\alpha) < B$*

then there exists at most l points $\{x_1, \ldots, x_l\}$ in $\overline{D}(R)$ and a subsequence $\{s_\beta\}$ such that s_β converges to s_* in $C^1(\overline{D}(R/2) \setminus \{x_1, \ldots, x_l\}, \mathbb{R}^k)$ uniformly on compacts, and s_* is harmonic in $D(R/2)$.

The two ingredients of the proof of Proposition 8.1 and Theorem 9 to be investigated are the covering and the estimate from Lemma 5. For the estimates, we already remarked that the parameters $\alpha_0, \epsilon, C(p, D')$ of Lemma 5 can be chosen independent of the metric $\tilde{g}_\alpha$ if they are close to Euclidean. For the covering, the investigation is not on the constant h, which can be chosen to be $3^{\dim M}$, but on how small the radius of the covering balls must be, but Lemma **??** states that their size is dictated by the Ricci curvature and sectional curvature of $\tilde{g}_\alpha$, which are also uniformly bounded.

Using Assertion 2, passing to a subsequence of $\{\tilde{s}_{\alpha_n}\}$ if necessary, we can suppose that $\tilde{s}_{\alpha_n} \to s_*$ in $C^1(D(R), \mathbb{R}^k)$. Note that there is no singular point where $\{\tilde{s}_{\alpha_n}\}$ fails to converge because $|d\tilde{s}_{\alpha_n}|$ is bounded uniformly on $D(R)$ (hence cannot explode as in Assertion 1). We can also choose, by a diagonal argument, a subsequence of $\{\tilde{s}_{\alpha_n}\}$ that converges to s_* in $C^1(\mathbb{R}^2)$ uniformly on compacts.

It is clear that $s_* : \mathbb{R}^2 \longrightarrow N$ is harmonic and nontrivial because

$$|ds_*(0)|_{\text{Euclid}} = \lim_{\alpha_n \to 1} |d\tilde{s}_{\alpha_n}(0)|_{\tilde{g}_{\alpha_n}} = 1.$$

Also,

$$\int_{D(R)} |ds_*|^2 dE = \lim_{\alpha_n \to 1} \int_{D(R)} |d\tilde{s}_{\alpha_n}|^2 dV_{\tilde{g}_\alpha} \leq \limsup_{\alpha \to 1} 2E(s_\alpha\big|_{D(x_1, 2\delta_n)}) < 2B$$

which means the energy of s_* on $\mathbb{R}^2$ is bounded above by $2B$.

Now since $(\mathbb{R}^2, \text{Euclid})$ is conformal to $\mathbb{S}^2 \setminus \{p\}$, s_* can be seen as a harmonic map on $\mathbb{S}^2 \setminus \{p\}$ with the same (finite) energy. By Extension theorem 7, s_* extends to a nontrivial harmonic map from $\mathbb{S}^2$ to N. $\square$

Remark 7. *1. We can have a better estimate of $E(s_*)$. For any $R > 0$, one has*

$$E(s_*|_{D(R)}) + E(s|_{M\setminus\cup_{i=1}^{l} D(x_i,\delta_n)}) \le \limsup_{\alpha_n\to 1}\left[E(s_{\alpha_n}|_{D(x_1,\delta_n)}) + E(s_{\alpha_n}|_{M\setminus\cup_{i=1}^{l} D(x_i,\delta_n)})\right]$$

Let $\delta \to 0$ then $R \to +\infty$, one has

$$E(s_*) + E(s) \le \limsup_{\alpha\to 1} E(s_\alpha).$$

2. The proof of Theorem 10 also gives a constraint on the image of s_*: since $s_*(D(R)) \subset \overline{\cup_{1<\beta<\alpha} s_\beta(D(x_1, 2\delta))}$ for all α arbitrarily close to 1 and δ arbitrarily small, one has

$$s_*(\mathbb{S}^2) \subset \bigcap_{\delta\to 0}\bigcap_{\alpha\to 1}\overline{\bigcup_{1<\beta<\alpha} s_\beta(D(x_1,\delta))}$$

6 Minimal immersions of $\mathbb{S}^2$.

We use the following result:

Theorem 11 ([?], [?], [?]). *If $s : \mathbb{S}^2 \longrightarrow N$ is a nontrivial harmonic map and $\dim N \ge 3$, then s is a C^∞ conformal, branched, minimal immersion.*

The "minimal" part follows from [?], the "branched" part follows from [?] and the "conformal" part follows from [?] and the fact that there is no nontrivial holomorphic quadratic differential on $\mathbb{S}^2$. Theorem 10 gives:

Theorem 12. *If the universal covering $\tilde{N}$ of N is not contractible then there exists a C^∞ conformal, branched, minimal immersion $s : \mathbb{S}^2 \longrightarrow N$.*