Summary of my stage in Summer 2018

Tien NGUYEN MANH

May 20, 2018

Contents

1 Summary

The goal of this part is to give a summary of what will be developed in the next chapters. In brief, we are interested in maps $f : M \longrightarrow M'$ between Riemannian manifolds (that to simplify, are supposed to be compact) that are critical points of the energy functional

$$E(f) = \frac{1}{2}\int_M |\nabla f|^2 dV.$$

By taking first order variation of E, these are maps whose **tension field** $\tau(f)$ vanishes.

1.1 Deformation using nonlinear heat equation.

The approach of [**?**] is to prove that, if the target space is negatively curved, then any smooth map $f_0 : M \longrightarrow M'$ can be deformed to a harmonic map using the gradient descent equation, that is to show the equation

$$\begin{cases} \frac{df_t}{dt} = \tau(f_t) \\ f\big|_{t=0} = f_0 \end{cases} \tag{1}$$

We will prove that if M' is negatively curved then this PDE admits a globally defined smooth solution f_t and that $f_\infty := \lim_{t\to\infty} f_t$ in C^∞ is a harmonic map.

The resolution of (1) can be organised in 3 steps:

1. Find the global equation. We will find a global frame of M' and express f in this frame, so that instead of solving for a map, we will have to solve for functions.

2. Study linear PDEs on manifolds. The equation, expressed in local coordinates, is a nonlinear heat equation, i.e. other than a heat operator, it has a quadratic term. Short-time existence and regularity for (1) follows from *standard* results of parabolic equation.

3. Prove long-time existence. In order to use continuity method, we will have to prove that $W^{k,p}$-norms of the solution f_t do not explode. This will be established first in the case $W^{2,2}$ using physical quantities, namely the potential energy E and the kinetic energy K. The general case is proved from the $W^{2,2}$ estimate using Gårding's inequality and Comparison theorem for parabolic equation.

The hypothesis of negative curvature is only used to establish the energy estimates. During deformation, the rate of potential energy can be calculated as:

$$\frac{de(f_t)}{dt} = -\Delta e(f_t) - |\beta(f_t)|^2 - \langle \mathrm{Ric}(M)\nabla_v f_t, \nabla_v f_t\rangle + \langle \mathrm{Riem}(M')(\nabla_v f_t, \nabla_w f_t)\nabla_v f_t, \nabla_w f_t\rangle$$

and the kinetic energy as:

$$\frac{dk(f_t)}{dt} = -\Delta k(f_t) - \left|\nabla \frac{\partial f_t}{\partial t}\right|^2 + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle$$

Therefore if all sectional curvatures of M' are negative, these rates can be controlled and the energies are guaranteed not to explode.

1.2 Existence using Morse-Palais-Smale theory.

We also give a less detailed review of the work by Sacks and Uhlenbeck [**?**]. This approach uses an approximating family E_α of the energy functional E whose critical functions in $W^{1,2\alpha}$ can be easily proved to exist using Morse-Palais-Smale theory. We then try to prove that the critical sequence C^1-converges to a nontrivial limit.

As a concrete result, the authors proved, using an extension theorem for harmonic maps on surface and a suitable covering of M by small discs on which the energy E is sufficiently small, that if the fundamental group $\pi_k(M')$ is nontrivial for a certain $k \geq 2$, or equivalently, if the universal covering $\tilde{M}'$ of M' is not contractible, then there exists a nontrivial harmonic map from $\mathbb{S}^2$ to M'.