Bogomolov-Beauville classification

Tien

September 15, 2018

Contents

1 From the Riemannian results of de Rham and Berger

We will first prove a (conceptually) straightforward result of de Rham decomposition and Berger classification. The following theorem is taken from Beauville's article

Theorem 1 (Beauville)**.** *Let X be a compact Kähler manifold with flat Ricci curvature, then*

1. *The universal covering space $\tilde{X}$ of X decomposes isometrically and holomorphically as*

$$\tilde{X} = E \times \prod_i V_i \times \prod_j X_j$$

where $E = \mathbb{C}^k$, V_i and X_j are simply-connected compact manifolds of real dimension $2m_i$ and $4r_j$ with irreducible homonomy $SU(m_i)$ for V_i and $Sp(r_j)$ for X_j. One also has uniqueness in the strong sense as in de Rham decomposition.

2. *There exists a finite covering space X' of X such that*

$$X' = T \times \prod_i V_i \times \prod_j X_j$$

where T is a complex torus.

Proof. Note that the first point is obtained directly from Cheeger-Gromoll splitting and de Rham decomposition: The one-dimensional parallel subspaces (of trivial holonomy) are regrouped to E. By Cheeger-Gromoll splitting, $\tilde{X} = E \times M$ where M contains no line and is compact (note that we use compactness of X here). The irreducible factors in M are not symmetric spaces as Ricci curvature of symmetric spaces is non-degenerate. Holonomy of these factors are $SU(m_i)$ and $Sp(r_j)$ according to Berger list since they are Kähler manifolds and Ricci-flat. It remains to prove the second point.

We will regard each element of $\pi_1(X)$ by its isometric, free, proper action on $\tilde{X}$. As pointed out the arguments in our discussion of uniqueness of de Rham decomposition, every isometry of $\tilde{X}$ to itself preserves the components $T_{x_0}E$, $T_{x_i}V_i$ and $T_{x_j}X_j$ of $T_x\tilde{X}$, each isometry ϕ of $\tilde{X}$ is of form (ϕ_1, ϕ_2) where $\phi_1 \in Isom(E)$ and $\phi_2 \in Isom(M)$.

We will use here the fact that if M is a Kähler manifold, compact and Ricci-flat then $Isom(M)$ equipped with compact-open topology is discrete, therefore finite, which will be proved later (Lemma 3). We note $\Gamma := \{\phi = (\phi_1, \phi_2) \in \pi_1(X),\ \phi_2 = Id_M\}$ and sometime abusively regard Γ as a subgroup of $Isom(E)$. Note that Γ is a normal subgroup of $\pi_1(X)$ with finite index since the quotient is isomorphic to a subgroup of $Isom(M)$. Therefore $\tilde{X}/\Gamma = E/\Gamma \times M$ is compact as a finite cover of X.

We apply the following theorem of Bieberbach.

Theorem 2 (Bieberbach)**.** *Let $E = \mathbb{R}^n$ be an Euclidean space and Γ be a subgroup of $Isom(E)$ that satisfies*

1. *Γ is discrete under compact-open topology.*

2. *E/Γ is compact.*

Then the subgroup Γ' of translations in Γ is of finite index.

Suppose that the two conditions are satisfied then the theorem gives: $\tilde{X}/\Gamma' = E/\Gamma' \times M = T \times \prod_i V_i \times \prod_j X_j$ is a finite cover of $\tilde{X}/\Gamma$ as Γ' is a normal subgroup of Γ:

Fact. The subgroup of translations in $Isom(E)$, where $E = \mathbb{R}^n$ is an Euclidean space, is normal.

Therefore $X' = \tilde{X}/\Gamma'$ is a finite cover of X that we want to find.

It remains to prove that Γ is discrete, which is a consequence of

1. $\pi_1(X)$ is discrete, without limit point in $Isom(E) \times Isom(M)$ (obvious).

2. $Isom(M)$ is compact.

In fact given any $\phi = (\phi_1, \phi_2) \in Isom(E) \times Isom(M)$, there exists by (1.) a neighborhood $\mathcal{U}_1(\phi_1, \phi_2) \times \mathcal{U}_2(\phi_1, \phi_2)$ of ϕ in $Isom(E) \times Isom(M)$ such that all points of $\pi_1(X)$ lying in this region project to ϕ_1. By (2.) we can find a neighborhood $\mathcal{U}_1$ of ϕ_1 in $Isom(E)$ small enough that $\mathcal{U}_1(\phi_1) \times Isom(M) \subset \cup_{\phi_2 \in Isom(M)} \mathcal{U}_1(\phi_1, \phi_2) \times \mathcal{U}_2(\phi_1, \phi_2)$. Therefore the projection of $\pi_1(X)$ to $Isom(E)$ is discrete, by consequence Γ is discrete. $\square$

Lemma 3. *Let M be is a compact, simply-connected, Ricci-flat, Kähler manifold, then the group $Aut(M)$ of automorphism of M equipped with compact-open topology is discrete, therefore $Isom(M)$ is discrete, hence finite.*

Proof. The idea is that since $Aut(M)$ is a Lie group, it suffices to prove that its Lie algebra is of dimension 0. This is done using these facts.

1. The Lie algebra of $Aut(M)$ can be identified with the vector space of holomorphic vector fields on M.

2. *Bochner's principle*: All holomorphic tensor fields on a compact, Ricci-flat Kähler manifold are parallel.

3. The only invariant vector of the holonomy representation of M is 0 (obvious).

$\square$

Bochner principle for holomorphic vector fields comes from the following identity (called *Weitzenbock formula*):

$$\Delta(\frac{1}{2}\|X\|^2) = \|\Delta X\|^2 + g(X, \nabla \mathrm{div} X) + Ric(X, X)$$

for every vector field X. If X is holomorphic then it is harmonic and has $\mathrm{div} X = 0$. The fact that M is Ricci-flat gives $\Delta(\frac{1}{2}\|X\|^2) = \|\nabla X\|^2$ and the function $\|X\|^2$ is subharmonic, therefore constant since M is compact. We then have $\nabla X = 0$,i.e. X is parallel. The method of Bochner also works for tensor fields of any type in a Ricci-flat Kähler manifold and one also has $\Delta(\|\tau\|^2) = \|\nabla\tau\|^2$ and that every holomorphic tensor field is parallel. See P. Petersen, *Riemannian geometry* and A. Besse, *Einstein Manifolds* for more detail.

2 Towards a classification for complex manifold

To obtain a translation of Theorem 1 in a context of complex manifolds (without any preferred metric a priori), we study the 2 building blocks: manifolds with holonomy $SU(m)$ and $Sp(r)$. To be clear, recall that a complex manifold X is called of Kähler type if one can equip X with an Hermitian structure whose fundamental form ω satisfies $d\omega = 0$. When we say X is of Kähler type, we refer to X as a complex manifold without fixing a metric on X.

2.1 Special unitary manifolds (proper Calabi-Yau manifolds)

Remark 1. *Let X be a compact Kähler manifold with holonomy $SU(m)$ and complex dimension $m \geq 3$ then:*

1. $H^0(X, \Omega_X^p) = 0$ *for all* $0 < p < m$, *by consequence* $\chi(\mathcal{O}_X) = 1 + (-1)^m$.
2. X *is* projective, *that is X can be embedded into $\mathbb{P}^N$ as zero-locus of some (finitely) homogeneous polynomials.*
3. $\pi_1(X)$ *is finite and if m is even, X is simply connected.*

The first point is in fact algebraic in nature: it comes from the fact that the representation of $SU(m)$ over $\bigwedge^p T_x^* M$ is irreducible for all p et non-trivial for $0 < p < m$, therefore the action of $SU(m)$ on $\bigwedge^p T_x^* M$ for $0 < p < m$ has no invariant element, hence $H^0(X, \Omega_X^p) = 0$.

The second point follows the following facts:

1. (Kodaira's theorem) A compact Kähler manifold with $H^{2,0} = 0$ can be embedded in $\mathbb{P}^N$.
2. (Chow's theorem) A compact complex manifold embedded in $\mathbb{P}^N$ is algebraic, i.e. defined by a finite number of homogeneous polynomials.

The third point is a direct consequence of Riemann-Hurwitz formula. In fact, the universal cover $\tilde{X}$ of X is of holonomy $SU(m)$. This is due to the following remarks: $Hol(X) \supset Hol(X') \supset Hol_0(X') = Hol_0(X)$ and $Hol_0(X) = Hol(X) = SU(m)$ as $SU(n)$ is connected.

By Theorem 1, $\tilde{X}$ is compact by Lemma 3 a finite covering of X as $\pi_1(X)$ is finite. As $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\tilde{X}}) = 2$, one has $X = \tilde{X}$, hence X is simply-connected.

Theorem 4. *Given a compact manifold X of Kähler type and complex dimension m, the following properties are equivalent*

1. *There exists a compatible metric g over X such that $Hol(X,g) = SU(m)$.*

2. *K_X is trivial and $H^0(X', \Omega^p_{X'}) = 0$ for every $0 < p < m$ and X' a finite covering of X.*

Proof. (1) implies (2) as a finite covering space X' of a special unitary manifold X is still a special unitary.

For the implication (2) $\implies$ (1): by Yau's theorem we equip X with a Ricci-flat metric, by Theorem 1, there exists a finite cover $X' = T \times \prod_i V_i \times \prod_j X_j$ where T is a complex torus, $Hol(V_i) = SU(m_i), Hol(X_j) = Sp(r_j)$. But $H^0(X', \Omega^p_{X'}) = 0$ for $0 < p < m$, X' has to be one of the V_i as $H^0(X_j, \Omega^2_{X_j})$ and $H^0(V_i, \Omega^{m_i}_{V_i})$ do not vanish. Therefore $Hol(X') = SU(m)$, hence $Hol(X) = SU(m)$. □

Theorem 4 allows us to check if a manifold X is special unitary by looking at the $h^{0,p}(0 < p < m)$ coefficients of the Hodge diamond of X and its finite covers. We can see, by this criteria that the following examples are special unitary manifolds. All of them are algebraically constructed, since a construction by glueing local charts is difficult (or impossible).

Example 1 (Special unitary manifold)**.** 1. *Elliptic curves over $\mathbb{C}$ are special unitary, as any statement starting with "for every $0 < p < 1$" is formally true.*

2. *A K3 surface (simply-connected surface with trivial canonical bundle) is special unitary, its Hodge diamond is given below.*

3. *A quintic threefold (hypersurface of degree 5 in 4-dimensional projective space) is a special unitary manifold, the Hodge diamond of which is given is given below. In particular, the Fermat quintic defined by*

$$\{(z_0 : z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^4 : \sum z_i^5 = 0\}$$

4. *In general, any smooth hypersurface X of $\mathbb{CP}^{m+1}$ of degree $m + 2$ satisfies $h^{0,p} = 0$ for all $0 < p < m$. If X is simply-connected then it is a special unitary manifold.*

Table 1: Hodge diamond of a K3 surface.

		1		
	0		0	
1		20		1
	0		0	
		1		

Table 2: Hodge diamond of a quintic threefold.

			1			
		0		0		
	0		1		0	
1		101		101		1
	0		1		0	
		0		0		
			1			

2.2 Irreducible symplectic and hyperkähler manifolds

Remark 2. *Let X be a compact Kähler manifold with holonomy $Sp(r)$ and complex dimension $2r$ then:*

1. *There exists a holomorphic 2-form φ non-degenerate at every points.*
2. *$H^0(X, \Omega_X^{2l+1}) = 0, H^0(X, \Omega_X^{2l}) = \mathbb{C}\varphi^l$ for all $0 \leq l \leq r$. By consequence $\chi(\mathcal{O}_X) = r + 1$.*
3. *X is simply-connected.*

The first point of the remark follows directly from our discussion of Berger classification.

The second point is algebraic in nature: The representation of $Sp(r)$ on $\bigwedge^p T_x^* M$ splits into

$$\bigwedge^p T_x^* M = P_p \oplus P_{p-2}\varphi(x) \oplus P_{p-4}\varphi^2(x) \oplus \dots \qquad (1)$$

where $P_k, 0 \leq k \leq r$ are irreducible, non-trivial for $k > 0$ and $\varphi(x) \in \bigwedge^2 T_x^* M$ uniquely defined up to a constant. Therefore the only invariant elements are $c\varphi^{p/2}$ where c is a scalar.

For the last point, one uses the same arguments as Remark 1.

Theorem 5. *Given a compact manifold X of Kähler type and complex dimension $2r$, then:*

1. *The following properties are equivalent. X is called* <u>*hyperkähler*</u> *if it satisfies one of them.*
   - *(a) There exists a compatible metric g such that $Hol(X, g) \subset Sp(r)$.*
   - *(b) There exists a compatible symplectic structure: a 2-form that is closed, holomorphic and non-degenerate at every point.*
2. *The following properties are equivalent. X is called* <u>*irreducible symplectic*</u> *if it satisfies one of them.*
   - *(a) There exists a compatible metric g such that $Hol(X, g) = Sp(r)$*
   - *(b) X is simply-connected and there exists (uniquely up to a constant) a compatible symplectic structure on X.*

By "compatible", we mean "compatible with the complex structure".

Proof. 1. The fact that (a) implies (b) is obvious. For the other way: since K_X is trivial (existence of global non-null section) by Yau's theorem we equip X with a Ricci-flat metric, then the symplectic structure φ of X is parallel by Bochner's principle. Hence the holonomy is in $Sp(r)$.

2. For the implication (a) $\implies$ (b), it suffices to notice that the invariant elements φ in the decomposition (1) is unique. For the direction (b) $\implies$ (a), note that X can be equipped with a Calabi-Yau metric by the (b) $\implies$ (a) part of (1.), by Theorem 1, $X = \prod_{j=1}^{m} X_j$ where X_j are irreducible compact Kähler manifolds. The symplectique structure φ on X, restricted on each X_j, gives a symplectique structure φ_j of X_j. But any form $\sum_j \lambda_j pr_j^* \varphi_j$ is another symplectic structure of X, one must have $m = 1$ by uniqueness of φ.

□

Example 2. 1. *One can notice a trivial example: Every special unitary manifold of 2 complex dimensions is irreducible symplectic because $SU(2)$ is isomorphic to $Sp(1)$.*

2. *Let X be a smooth cubic hypersurface in $\mathbb{CP}^{n+1}$ and $F(X) = \{L \in Gr(1, \mathbb{CP}^{n+1}), L \subset X\} \subset Gr(1, \mathbb{CP}^{n+1})$ the manifold formed by lines in X. $F(X)$ is non-empty when $n > 1$, smooth if X is smooth and of dimension $2n - 4$. Beauville and Donagi proved that for $n = 4$, $F(X)$ is irreducible symplectic, therefore hyperkähler.*

2.3 Decomposition for complex manifold with vanishing Chern class

Theorem 1 can be translated to a decomposition for complex manifold in the following way:

Theorem 6 (Bogomolov-Beauville classification). *Let X be a compact manifold of Kähler type of vanishing first Chern class.*

1. *The universal covering space $\tilde{X}$ of X is isomorphic to a product $E \times \prod_i V_i \times \prod_j X_j$ where $E = \mathbb{C}^k$ and*
   - *(a) Each V_i is a projective simply-connected manifold of complex dimension $m_i \geq 3$, with trivial K_{V_i} and $H^0(V_i, \Omega^p_{V_i}) = 0$ for $0 < p < m_i$*
   - *(b) Each X_j is an hyperkähler manifold.*

   This decomposition is unique up to an order of i and j.

2. *There exists a finite cover X' of X isomorphic to the product $T \times \prod_i V_i \times \prod_j X_j$.*

The theorem follows directly from Theorem 1, the only point that needs proof is the uniqueness, which will be achieved in two steps:

1. Prove the uniqueness in the case that X is simply-connected.
2. Prove that every isomorphism $\phi : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z$ is splitted as $\phi = (\phi_1, \phi_2)$ where $\phi_1 : \mathbb{C}^k \longrightarrow \mathbb{C}^h$ and $\phi_2 : Y \longrightarrow Z$ are isomorphisms (by consequence $h = k$).

These two steps will be accomplished in the following two lemmas

Lemma 7. *Let $Y = \prod_j Y_j$ be a finite product of compact, simply-connected manifold of Kähler type with vanishing Chern class. The Calabi-Yau metrics of Y are then $g = \sum_l pr_j^* g_j$ where g_j are Calabi-Yau metrics of Y_j.*

Proof. Let g be a Calabi-Yau metric of Y and $[\omega]$ its class in $H^{1,1}(Y)$. Since Y_j are simply-connected, $[\omega] = \sum_j pr_j^*[\omega_j]$. By Yau's theorem, there exist unique Calabi-Yau metrics g_j of Y_j in each class $[\omega_j]$. The metric $g' = \sum_j pr_j^* g_j$ is in the same class ω of g and is also a Calabi-Yau metric, hence $g = g' = \sum_j pr_j^* g_j$. □

This lemma asserts that when our manifolds Y, Y_j are equipped with appropriate Calabi-Yau metrics, the decomposition map is also a (Riemannian) isometric, we therefore obtain uniqueness of V_i, X_j from uniqueness of Theorem 1.

Lemma 8. *Let Y, Z be compact, simply-connected manifold of Kähler type, then any isomorphism $u:\ \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z$ is splitted as $\phi = (\phi_1, \phi_2)$ where $\phi_1:\ \mathbb{C}^k \longrightarrow \mathbb{C}^h$ and $\phi_2:\ Y \longrightarrow Z$ are isomorphisms.*

Proof. It is clear that the composed function $u_1 : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z \longrightarrow \mathbb{C}^h$ is constant in Y, i.e. $u_1(t, y) = u_1(t)$ as holomorphic functions on Y are constant, therefore $u(t, y) = (u_1(t), u_2(t, y))$. As u is isomorphic, one has $h \leq k$ then by the same argument for u^{-1}, one has $h = k$, u_1 is an isomorphism and $u_2(t, \cdot)$ is an isomorphism from Y to Z. $u_2(0, \cdot)^{-1} \circ u_2(t, \cdot)$ is then a curve in $Aut(Y)$, which is discrete by Lemma 3. Therefore $u_2(t, \cdot) = u_2(0, \cdot)$ independent of t. □

‘