Algebraisation theorems I: Chow's theorem

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We will prove in this post an algebraisation result of Chow. The arguments follow the course given by Jean-Benoît Bost at Institut de Mathématiques d'Orsay.

Theorem 1 (Chow)**.** *Every analytic compact subvariety V of $\mathbb{P}^N(\mathbb{C})$ is algebraic, i.e. V is the complex points of a subscheme of $\mathbb{P}^N_{\mathbb{C}}$.*

To fix the notation, $X(\mathbb{C})$ denotes the analytic space formed by $\mathbb{C}$-points of the scheme X over $\mathbb{C}$. $X(\mathbb{C})$ is equipped with the analytic topology, while X is equipped with the Zariski topology. If L is an algebraic line bundle (i.e. invertible sheaf) over X, L^{an} denotes the natural analytification of L, that is the holomorphic line bundle over $X(\mathbb{C})$ given by L and the fact that polynomials are holomorphic. $\Gamma(X, L)$ denotes the space of global algebraic sections of L and $\Gamma^{\mathrm{an}}(X(\mathbb{C}), L^{\mathrm{an}})$ denotes the space of global analytic (holomorphic) sections of L^{an}. Again, since polynomials are holomorphic functions, there is a natural map $\Gamma(X, L) \longrightarrow \Gamma^{\mathrm{an}}(X(\mathbb{C}, L^{\mathrm{an}})$ converting an algebraic section s to an analytic section s^{an}.

The first proof of Chow's theorem that I learned is from this lecture note. It starts with a remark that the Theorem is easy in codimension 1. In fact, varieties of codimension 1 are divisors and classes of divisors are elements of Picard group, i.e. line bundles of $\mathbb{P}^N_{\mathbb{C}}$ and $\mathbb{P}^N(\mathbb{C})$, and each divisor in a class L corresponds to a zero set of an (analytic or algebraic) of the line bundle L. Therefore the case of codimension 1 follows from the computational miracles:

1. All analytic line bundles of $\mathbb{P}^N(\mathbb{C})$ are of the form $\mathcal{O}(d)$ hence are algebraic.

2. All holomorphic global sections of $\mathcal{O}(d)$ are polynomial, hence algebraic.

In the higher codimension, one passes to the Grassmannians.

The proof that we will present is also done by counting global sections of line bundles, but with argument on dimension rather than divisors.

1 Algebraic foreplay

The strategy is to, given a projective analytic variety V, consider the Zariski closure $Z = \overline{V}^{\mathrm{Zar}}$ of V, and prove that the (algebraic) dimension of Z equals the (analytic) dimension of V then conclude that $V = Z$.

Reasonably, the direction $\dim Z \geq \dim V$ is easier and it suffices to show that there is a point which is both algebraically regular and analytically regular. The other direction $\dim Z \leq \dim V$ is less obvious and will be proved using Proposition 1.1 and Proposition 1.2 which together say that the dimension can be calculated as the order of the number of global sections of $L^{\otimes \eta}$ when $\eta \to \infty$, where L is a line bundle.

The correct statement of the Propositions and the proof of Theorem 1 are given below.

Proposition 1.1 (Algebraic sections)**.** *For all closed irreducible algebraic set $Z \neq \emptyset$ of $\mathbb{P}^N(\mathbb{C})$, abusively viewed as reduced $\mathbb{C}$-scheme, there exists a line bundle L over Z and $c \in \mathbb{R}_{>0}$ such that*

$$\forall \eta \in \mathbb{N}, \quad \dim_{\mathbb{C}} \Gamma(Z, L^{\otimes \eta}) \geq c\eta^{\dim Z}$$

where $\dim Z$ *is the dimension of the scheme Z.*

Proposition 1.2 (Analytic sections)**.** *Let M be a compact analytic variety of $\mathbb{P}^N(\mathbb{C})$ and L be an analytic line bundle over M. Then there exists $C \in \mathbb{R}_{>0}$ such that*

$$\forall \eta \in \mathbb{N}, \quad \dim_{\mathbb{C}} \Gamma^{\mathrm{an}}(M, L^{\otimes \eta}) \leq C\eta^{\dim M}$$

The first one is easy: Take $L := \mathcal{O}(1)\big|_Z$ then $\Gamma(Z, L^{\otimes \eta} = \Gamma(Z, \mathcal{O}(\eta)) = \mathbb{C}[X_0, \ldots, X_N]_\eta / I(Z)_\eta$, which is of dimension $\frac{1}{d!} \deg Z.\eta^d + O(\eta^{d-1})$ where $d = \dim Z$. This is usually taken to be the definition of $\dim Z$ and $\deg Z$ (definition by Hilbert's polynomial).

Proposition 1.1 also has an elementary proof as follow. By Noether's normalisation, there exists a linear projection $\pi : \mathbb{P}^N_{\mathbb{C}} \longrightarrow \mathbb{P}^N_{\mathbb{C}}$ whose restriction to Z is a finite morphism. Take $L := \pi^* \mathcal{O}(1)_{\mathbb{P}^d}$. Since the pullback

$\Gamma(\mathbb{P}^d_{\mathbb{C}}, \mathcal{O}(\eta)) \longrightarrow \Gamma(Z, L^{\otimes\eta})$ is injective (by surjectivity of $\pi\big|_Z$), the space of global sections of $L^{\otimes\eta}$ is of dimension at least $\dim_{\mathbb{C}} \Gamma(\mathbb{P}^d, \mathcal{O}(\eta)) = \binom{\eta+d}{d} = \frac{1}{d!}\eta^d + O(\eta^{d-1})$

The proof of Proposition 1.2 is deeper and will be proved later in the next section.

We now give a detailed proof of Chow's Theorem 1 from the 2 Propositions.

Proof of Theorem . With the definition of V and Z as above, we see that if V is connected then Z is irreducible. In fact, if $W \hookrightarrow \mathbb{A}^N(\mathbb{C})$ is a connected analytic affine variety then $\overline{W}^{\mathrm{Zar}}$ is irreducible, as $\Gamma(W, \mathcal{O}^{\mathrm{an}}_W)$ is an integral domain (this is where we use the connectivity) and

$$I(\overline{W}^{\mathrm{Zar}}) = \ker\left(\mathbb{C}[X_1, \dots X_N] \longrightarrow \Gamma(W, \mathcal{O}^{\mathrm{an}}_W)\right)$$

therefore is a prime ideal. Apply the result with W being the cone of Z in $\mathbb{A}^{N+1}(\mathbb{C})$, one sees that Z is irreducible.

We will prove that $\dim Z = \dim V$ where the LHS concerns the algebraic dimension of a scheme and the RHS concerns the analytic dimension, starting with the difficult direction $\dim Z \leq \dim V$. Denote by $\iota : V \hookrightarrow Z$ the inclusion. From the definition of $Z := \overline{V}^{\mathrm{Zar}}$, the map

$$\Gamma(Z, L^{\otimes\eta}) \longrightarrow \Gamma^{\mathrm{an}}(V, \iota^* L^{\mathrm{an}})$$

sending an algebraic section of on Z to the analytification of its restriction over V, is injective. Therefore $\dim_{\mathbb{C}} \Gamma(Z, L^{\otimes\eta}) \leq \dim_{\mathbb{C}} \Gamma(V, L^{\mathrm{an}\otimes\eta})$, take $L = \mathcal{O}(1)\big|_Z$ the inequality follows from two Propositions.

To see that $\dim V \leq \dim Z$, it suffices to prove that there exists a point in V which is also a regular point of Z since the 2 notions of dimension coincide at regular points. Recall that one can always write

$$Z = Z_{\mathrm{reg}} \sqcup Z_{\mathrm{sing}}$$

where Z_{reg} is Zariski-open and Z_{sing} is Zariski-closed and $Z_{\mathrm{reg}}(\mathbb{C})$ is a smooth analytic variety of dimension $\dim Z$. If $V \cap Z_{\mathrm{reg}} = \emptyset$ then $V \subset Z_{\mathrm{sing}}(\mathbb{C})$, taking Zariski closure, one see that $Z = Z_{\mathrm{sing}}$, which is contradictory. Hence $V \cap Z_{\mathrm{reg}}(\mathbb{C}) \neq \emptyset$ and one has $\dim V \leq \dim Z$. Hence $\dim V = \dim Z$.

To conclude, one needs the following two additional facts, the second of which is not easy to prove. Let X be a reduced quasi-projective scheme over $\mathbb{C}$. X is equipped with the Zariski topology and $X(\mathbb{C})$ is equipped with the analytic topology, then

Proposition 1.3. *For all Zariski open, dense subset $U \subset X$, $U(\mathbb{C})$ is dense in $X(\mathbb{C})$ in the analytic topology.*

Proposition 1.4. *If X is connected in Zariski topology (for example, if X is irreducible) then $X(\mathbb{C})$ is connected in analytic topology*

By Proposition 1.4, $Z_{\text{reg}}(\mathbb{C})$ is connected in analytic topology. Moreover, in analytic topology of $Z_{\text{reg}}(\mathbb{C})$, $V \cap Z_{\text{reg}}(\mathbb{C})$ is obviously closed and is also open by equality of dimension. One therefore has $Z_{\text{reg}}(\mathbb{C}) \subset V$. By Proposition 1.3, $Z_{\text{reg}}(\mathbb{C})$ is dense in $Z(\mathbb{C})$, one has $V = Z(\mathbb{C})$. □

2 Proof of the analytic result

We suppose that M is connected and of dimension n. By compactness, one covers M by finitely many charts $\varphi_\alpha : U_\alpha \subset M \xrightarrow{\sim} \mathbb{B}^n(1)$, $\alpha \in \overline{1, N}$, over which L is trivialised. Note by $p_\alpha := \varphi_\alpha^{-1}(0)$ the centers of these balls and by s_α the "unit" section (i.e. everywhere $\neq 0$) of $L\big|_{U_\alpha}$. Then Proposition 1.2 is straightforward consequence of the following

Proposition 1.5. *There exists $c \in \mathbb{R}_{>0}$ such that for all $\eta \in \mathbb{Z}_{>0}$, $s \in \Gamma(M, L^{\otimes \eta})$,*

$$\operatorname{mult}_{p_\alpha} s > c\eta \quad \forall \alpha \in \overline{1, N} \implies s \equiv 0$$

As claimed, proof that Proposition 1.5 implies Proposition 1.2 is not difficult.

Proof that Proposition 1.5 imples Proposition 1.2. By Proposition 1.5, a section s of $L^{\otimes \eta}$ is uniquely defined by the first $d := \lfloor c\eta \rfloor$ terms of its series development at p_α (viewed in charts φ_α), hence the (complex) dimension of $\Gamma(M, L^{\otimes \eta})$ is at most

$$N. \left(1 + \binom{n}{1} + \dots \binom{n+d}{d} \right) \leq C\eta^n$$

□

Remark 1. *Note that we adapt the* relative point of view *for analytic sections, that is we consider these sections locally as holomorphic functions, and not as meromorphic functions as usual in algebraic geometry.*

Proposition 1.5 is proved using Schwarz's Lemma:

Proof of Proposition 1.2. Fix a C^∞ metric $\|\cdot\|$ on L, which induces a metric, still noted by $\|\cdot\|$, on the tensor product $L^{\otimes\eta}$. Since s_α^η are still invertible sections of $L^{\otimes\eta}\big|_{U_\alpha}$ there exists holomorphic function $f_\alpha \in \mathcal{O}^{\mathrm{an}}(\mathbb{B}^n(1))$ such that $s\big|_{U_\alpha} = f_\alpha \circ \varphi_\alpha\ s_\alpha^{\otimes\eta}$.

We will need the following version of Schwarz's lemma

Lemma 2 (Schwarz)**.** *Let $R \in \mathbb{R}_{>0}$ and $i \in \mathbb{N}$. For any holomorphic function $f \in \mathcal{O}^{\mathrm{an}}(\mathbb{B}^n(R))$ such that $\mathrm{mult}_0\ f \geq i$, one has*

$$|f(x)| \leq \left(\frac{|z|}{R}\right)^i \sup_{\mathbb{B}^n(R)} |f|$$

By compactness of M, there exists $r < 1$ such that $M = \bigcup_\alpha \varphi_\alpha^{-1}(\mathbb{B}_i^n(r))$. Choose a $R \in (r, 1)$, one has

$$\begin{aligned}
\|s\|_\infty &= \sup_M \|s\| = \max_\alpha \sup_{r\mathbb{B}_\alpha^n} \|s\| \\
&= \max_\alpha \sup_{r\mathbb{B}_\alpha^n} \|s_\alpha\|^\eta \sup_{r\mathbb{B}_\alpha^n} |f_\alpha| \\
\text{(By Schwarz)} \qquad &\leq \max_\alpha \sup_{r\mathbb{B}_\alpha^n} \|s_\alpha\|^\eta \left(\frac{r}{R}\right)^{c\eta} \sup_{R\mathbb{B}_\alpha^n} |f_\alpha| \\
&\leq \left(\frac{r}{R}\right)^{c\eta} \max_\alpha \left(\sup_{r\mathbb{B}_\alpha^n} \|s_\alpha\|^\eta \sup_{R\mathbb{B}_\alpha^n} (\|s_\alpha^{\otimes\eta}\|^{-1}) \sup_{R\mathbb{B}_\alpha^n} \|s\| \right) \\
&\leq C^\eta \left(\frac{r}{R}\right)^{c\eta} \|s\|_\infty
\end{aligned}$$

where $C = \max_\alpha \left[\sup_{r\mathbb{B}_\alpha^n} \|s_\alpha\| \sup_{R\mathbb{B}_\alpha^n} (\|s_\alpha\|^{-1})\right]$. Now it suffices to take $c > \log C / \log(R/r)$. $\square$