

# Hodge decomposition and Kodaira embedding theorem

Tien

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## Contents

<b>1</b>	<b>Hodge theory</b>	<b>1</b>
1.1	Operators and their dual . . . . .	1
1.1.1	Scalar product on $\Omega^k(M)$ . . . . .	1
1.1.2	Hodge star and Hodge dual . . . . .	2
1.2	Elliptic theory and Hodge theorem for Riemannian manifolds	2
1.2.1	Symbol of a differential operator . . . . .	2
1.2.2	Elliptic operators . . . . .	3
1.3	Hodge decomposition for Kähler manifolds . . . . .	3
1.4	Hodge symmetries . . . . .	5
<b>2</b>	<b>Kodaira embedding theorem</b>	<b>5</b>

This is my review of lectures 15-19 of Denis Auroux course whose goal is to establish Hodge theory for compact Kähler varieties and present a proof of Donaldson for the Kodaira embedding theorem.

## 1 Hodge theory

### 1.1 Operators and their dual

#### 1.1.1 Scalar product on $\Omega^k(M)$

The scalar product on  $V$  induces one on  $\Omega^k(V)$  by setting  $\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle u_i, v_j \rangle)$ .

**Example 1.**  $\langle \sum \alpha_I dx^I, \beta_J dx^J \rangle = \sum \alpha_I \beta_I$  if  $\{\frac{\partial}{\partial x^i}\}$  form an orthonormal basis.

### 1.1.2 Hodge star and Hodge dual

**Definition 1.** The **Hodge star** is defined from  $\Omega^k(M) \longrightarrow \Omega^{n-k}(M)$  such that  $\alpha \wedge * \beta = \langle \alpha, \beta \rangle \text{vol}$  where  $\text{vol}$  is the volume form.

**Remark 1.** 1. An example:  $*dx^I = dx^{I^C}$  if  $\{\frac{\partial}{\partial x^i}\}$  form an orthonormal basis and the complement  $I^C$  is chosen so that  $\text{sgn}(I, I^C) = 1$ .

2. Note that  $** = (-1)^{k(n-k)}$

The **Hodge dual** of an operator  $P$  will be defined such that  $\langle P\alpha, \beta \rangle_{L^2} = \langle \alpha, P^*\beta \rangle_{L^2}$  where the  $\langle \cdot, \cdot \rangle_{L^2}$  is the integral of  $\langle \cdot, \cdot \rangle$  over  $M$ . For example,

**Definition 2.** Let  $d$  be the coboundary operator then  $d^* : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$  is defined by  $d^* = (-1)^{n(k-1)+1} * d *$

**Definition 3.** The **de Rham-Laplace** operator is defined by

$$\Delta = dd^* + d^*d = (d + d^*)^2$$

The space of **harmonic forms** is  $\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) : \Delta\alpha = 0\}$ .

**Remark 2.** 1.  $\Delta^* = \Delta$ .

2.  $\langle \Delta\alpha, \alpha \rangle = \|d^*\alpha\|^2 + \|d\alpha\|^2$

3. A harmonic form is closed and co-closed.

## 1.2 Elliptic theory and Hodge theorem for Riemannian manifolds

### 1.2.1 Symbol of a differential operator

**Definition 4.** A mapping  $L : \Gamma(E) \longrightarrow \Gamma(F)$  where  $E, F$  are vector bundles on a manifold  $M$  is called a **differential operator** of order  $k$  if in local coordinates,

$$L(s) = \sum_{|\alpha| \leq k} A_\alpha(x) \frac{\partial^{|\alpha|} s}{\partial x^\alpha}$$

where  $A_\alpha(x)$  is a matrix with  $C^\infty$  coefficients.

The **symbol** of  $L$  is  $\sigma_k(L, \xi) = \sum_\alpha A_\alpha \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} \in \text{Hom}(E_x, F_x)$  where  $\xi = \sum \xi_i dx^i \in T^*M$  in the same coordinate as  $A_\alpha$ .

**Remark 3.** 1.  $A_\alpha(x)$  depends on the local coordinates and does not transform naturally when one passes from one coordinates to another. In other words,  $A_\alpha(x)$  is not in  $\text{Hom}(E_x, F_x)$ .

2. However, the definition of differential operator does not depend on local coordinates.
3. The symbol transforms naturally (linearly) between coordinates.

From the third remark, one can define:

**Definition 5.** A differential operator  $L$  is called **elliptic** if its symbol  $L(x, \xi) : E_x \longrightarrow F_x$  is isomorphic.

### 1.2.2 Elliptic operators

**Theorem 1** (Elliptic operator). Every elliptic operator  $L : \Gamma(E) \longrightarrow \Gamma(F)$

1. has a pseudoinverse, i.e. there exists  $P : \Gamma(F) \longrightarrow \Gamma(E)$  such that  $L \circ P - id_{\Gamma(F)}$  and  $P \circ L - id_{\Gamma(E)}$  are smooth operators.
2. is extended to a Fredholm operator  $L_s : W^s(E) \longrightarrow W^{s-k}(F)$ , i.e.  $\ker L = \ker L_s$  and  $\text{coker } L_s$  are finite dimensional,  $\text{Im } L_s$  is closed.

Moreover, if  $L : \Gamma(E) \longrightarrow \Gamma(E)$  is elliptic and self-adjoint then there exists  $H_L, G_L : \Gamma(E) \longrightarrow \Gamma(E)$  such that

1.  $\text{Im } H_L \subset \ker L$ ,  $id_{\Gamma(E)} = H_L + L \circ G_L = H_L + G_L \circ L$ .
2.  $H_L, G_L$  extend to  $W^s(E) \longrightarrow W^s(E)$ .
3.  $\Gamma(E) = \ker L \oplus_{\perp L^2} \text{Im } L \circ G_L$ .

**Theorem 2** (Hodge). Let  $M$  be a compact, oriented Riemannian manifold, then

1.  $\Omega^k(M) = \mathcal{H}^k(M) \oplus_{\perp L^2} \text{Im } d \oplus_{\perp L^2} \text{Im } d^*$ .
2. The projection  $\mathcal{H}^k(M) \longrightarrow H_{dR}^k(M, \mathbb{R})$  is isomorphic. In other words, each class is uniquely represented by a harmonic form.

### 1.3 Hodge decomposition for Kähler manifolds

In case of Kähler manifolds, one has the Hodge decomposition of cohomology which comes from the following two remarks:

1. The Hodge star  $* : \Omega^{p,q} \longrightarrow \Omega^{n-q,n-p}$ . This is due to the compatible complex structure.

2. The auxiliary operator  $L : \alpha \longrightarrow \omega \wedge \alpha$  and its relation with  $d$ . This is due to the compatible symplectic structure.

We resume in the following table the definition, domain and Hodge dual of some operators.

Operator	Domain	Definition	Dual
$L$	$\Omega^{p,q} \longrightarrow \Omega^{p+1,q+1}$	$\alpha \mapsto \omega \wedge \alpha$	$L^* = (-1)^{p+q} * L *$
$d_c$	$\Omega^k \longrightarrow \Omega^{k+1}$	$J^{-1}dJ$	$d_c^* = (-1)^{k+1} J d^* J$
$\partial$	$\Omega^{p,q} \longrightarrow \Omega^{p+1,q}$		$\partial^* = - * \bar{\partial} *$
$\bar{\partial}$	$\Omega^{p,q} \longrightarrow \Omega^{p,q+1}$		$\bar{\partial}^* = - * \partial *$
$\square$	$\Omega^{p,q} \longrightarrow \Omega^{p,q}$	$\partial \partial^* + \partial^* \partial$	
$\bar{\square}$	$\Omega^{p,q} \longrightarrow \Omega^{p,q}$	$\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$	

In case of Kähler manifold, one has the following relation between these operators.

**Lemma 3.** *lem: In a compact Kähler manifold, one has*

1.  $[L, d] = [L^*, d^*] = 0$
2.  $[L, d^*] = d_c$
3.  $[L^*, d] = -d_c^*$
4.  $[L^*, d_c] = d^*$

Therefore,

1.  $\Delta_c = d_c d_c^* + d_c^* d_c = \Delta$
2.  $\partial^*$  is adjoint to  $\partial$  and  $\bar{\partial}^*$  to  $\bar{\partial}$ .
3.  $\Delta = 2\square = 2\bar{\square}$

One equip  $\Omega^k$  with the following Hermitian product

$$\langle \phi, \psi \rangle_{L^2} = \int_M \phi \wedge * \bar{\psi}$$

under which the  $\Omega^{p,q}$  are orthogonal.

One now applies the elliptic theory for  $\bar{\square} : \Omega^{p,q} \longrightarrow \Omega^{p,q}$  with  $\mathcal{H}_{\bar{\square}}^{p,q} = \ker \bar{\square}$  then one sees that

**Theorem 4** (Hodge decomposition). *1. Each class in the Dolbeault cohomology  $H_{\bar{\partial}}^{p,q}(M)$  contains exactly one harmonic form of  $\mathcal{H}_{\bar{\square}}^{p,q} = \ker \bar{\square}$*

2.  $H^k(M) = \mathcal{H}_{\Delta} = \bigoplus_{p+q=k} \mathcal{H}_{\bar{\square}}^{p,q} = \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(M)$ .

## 1.4 Hodge symmetries

Let  $h^{p,q} = \dim_{\mathbb{R}} H_{\bar{\partial}}^{p,q}(M)$  and  $h^k = \dim H_{dR}^k(M, \mathbb{R})$  then one has  $h^k = \sum_{p+q=k} h^{p,q}$ . The  $h^{p,q}$  are usually written down as Hodge's diamond

$$\begin{array}{cccc} h^{n,n} & h^{n,n-1} & \dots & h^{n,0} \\ h^{n-1,n} & h^{n-1,n-1} & \dots & h^{n-1,0} \\ \dots & \dots & \dots & \dots \\ h^{0,n} & h^{0,n-1} & \dots & h^{0,0} \end{array}$$

with the symmetries

1.  $h^{p,q} = h^{q,p}$  given by conjugation.
2.  $h^{p,q} = h^{n-q,n-p}$  given by the Hodge star.

## 2 Kodaira embedding theorem