HARMONIC MAPS OF RIEMANNIAN MANIFOLDS

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Chapter 1

Summary

1.1 Summary

The goal of this part is to give a summary of what will be developed in the next chapters. In brief, we are interested in maps $f : M \longrightarrow M'$ between Riemannian manifolds (that to simplify, are supposed to be compact) that are critical points of the energy functional

$$E(f) = \frac{1}{2}\int_M |\nabla f|^2 dV.$$

By taking first order variation of E, these are maps whose **tension field** $\tau(f)$ vanishes.

1.1.1 Deformation using nonlinear heat equation.

The approach of [ES64] is to prove that, if the target space is negatively curved, then any smooth map $f_0 : M \longrightarrow M'$ can be deformed to a harmonic map using the gradient descent equation:

$$\begin{cases} \frac{df_t}{dt} = \tau(f_t) \\ f\Big|_{t=0} = f_0 \end{cases} \tag{1.1}$$

We will prove that if M' is negatively curved then this PDE admits a globally defined smooth solution f_t and that $f_\infty := \lim_{t\to\infty} f_t$ in C^∞ is a harmonic map.

The resolution of (1.1) can be organised in 3 steps:

1. Find the global equation. We will find a global frame of M' and express f in this frame, so that instead of solving for a map, we will have to solve for functions.

2. Study linear PDEs on manifolds. The equation, expressed in local coordinates, is a nonlinear heat equation, i.e. other than a heat operator, it has a quadratic term. Short-time existence and regularity for (1.1) follows from *standard* results of parabolic equation.

3. Prove long-time existence. In order to use continuity method, we will have to prove that $W^{k,p}$-norms of the solution f_t do not explode. This will be established first in the case $W^{2,2}$ using physical quantities, namely the potential energy E and the kinetic energy K. The general case is proved from the $W^{2,2}$ estimate using Gårding's inequality and Comparison theorem for parabolic equation.

The hypothesis of negative curvature is only used to establish the energy estimates. During deformation, the rate of potential energy can be calculated as:

$$\frac{de(f_t)}{dt} = -\Delta e(f_t) - |\beta(f_t)|^2 - \langle \mathrm{Ric}(M)\nabla_v f_t, \nabla_v f_t\rangle + \langle \mathrm{Riem}(M')(\nabla_v f_t, \nabla_w f_t)\nabla_v f_t, \nabla_w f_t\rangle$$

and the kinetic energy as:

$$\frac{dk(f_t)}{dt} = -\Delta k(f_t) - \left|\nabla \frac{\partial f_t}{\partial t}\right|^2 + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle$$

Therefore if all sectional curvatures of M' are negative, these rates can be controlled and the energies are guaranteed not to explode.

1.1.2 Existence using Morse-Palais-Smale theory.

We also give a less detailed review of the work by Sacks and Uhlenbeck [SU81]. This approach uses an approximating family E_α of the energy functional E whose critical functions in $W^{1,2\alpha}$ can be easily proved to exist using Morse-Palais-Smale theory. One then tries to prove that the critical sequence C^1-converges to a nontrivial limit.

As a concrete result, the authors proved, using an extension theorem for harmonic maps on surface and a suitable covering of M by small discs on which the energy E is sufficiently small, that if the fundamental group $\pi_k(M')$ is nontrivial for a certain $k \geq 2$, or equivalently, if the universal covering $\tilde{M}'$ of M' is not contractible, then there exists a nontrivial harmonic map from $\mathbb{S}^2$ to M'.

Part I

Harmonic maps: Introduction

Chapter 2

Harmonic maps of Riemannian manifolds

2.1 Harmonic maps

2.1.1 Variational approach: energy integral and tension field

Notation. Let M, M', M'' be Riemannian manifolds of dimension n, n' and n'' respectively. We will use indices $i, j, k, \ldots, \alpha, \beta, \gamma, \ldots, a, b, c$ to denote local coordinates of M, M', M''. Let $f : M \longrightarrow M', f' : M' \longrightarrow M''$ be a smooth maps, one denotes

$$f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}, \quad f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k f_k^\alpha$$

so that $\nabla h = h_i dx^i$ and $\nabla(\nabla h) = h_{ij} dx^i \otimes dx^j$ and $-\Delta h = \operatorname{Tr} \nabla(\nabla h) = g^{ij} h_{ij}$ for any smooth function h.

Definition 1. *The* ***energy desity*** *of f at $p \in m$ is defined by*

$$e(f)(p) = \frac{1}{2} \langle g, f^* g \rangle_p = \frac{1}{2} g^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$$

and the ***energy functional*** *of f is*

$$E(f) = \int_M e(f) dV = \frac{1}{2} \int_M g^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta} |\det(g_{ij})|^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$$

We recall that the inner product between 2 tensors of type (p, q) $S = S_{j_1 \ldots j_q}^{i_1 \ldots i_p}, T = T_{l_1 \ldots l_q}^{k_1 \ldots k_p}$ is $\prod_{m,n} g_{i_m k_m} g^{j_n l_m} S_{j_1 \ldots j_q}^{i_1 \ldots i_p} T_{l_1 \ldots l_q}^{k_1 \ldots k_p}$

Remark 1. *The energy density is non-negative at every point. And $E(f) = 0$ if and only if $e(f) = 0$ at all points if and only if f is constant.*

Definition 2. *Let σ be a symmetric function of n variables and α be a symmetric (0,2) tensor field, one can define the σ-**energy desity** of α at $P \in M$ to be $\sigma(\beta_1, \ldots, \beta_n)(P)$ where β_i are eigenvalues of the linear operator $(g^{ik}\alpha_{ij})_{k,j}$. The σ-**energy** of α is $I_\sigma(\alpha) := \int_M \sigma(\alpha)dV$*

*Take $\alpha = f^*g'$, one calls $\sigma(\alpha)$ the σ-**energy density** of f and $I_\sigma(\alpha)$ the σ-energy of f.*

Example 1. *The energy functional $E(f)$ is $I_{\frac{\sigma_1}{2}}(f)$. $V(f) := I_{\sigma_n^{1/2}}(f)$ is called the **volume** of f.*

Lemma 1 (variation of the energy)**.** *Let $f_t : M \longrightarrow M'$ be a smooth family of smooth maps between Riemannian manifolds for $t \in (t_0, t_1)$. Then*

$$\frac{d}{dt}E(f_t) = -\int_M \left(-\Delta f_t^\gamma + g^{ij}\Gamma'^\gamma_{\alpha\beta} f^\alpha_{t,i} f^\beta_{t,j}\right) g'_{\gamma\nu} \frac{\partial f_t^\nu}{\partial t} dV, \qquad \forall t \in (t_0, t_1)$$

Proof. One has

$$\begin{aligned}\frac{dE}{dt}(f_t) &= \frac{1}{2}\int \left[2g^{ij} f_i^\alpha \frac{\partial^2 f_t^\beta}{\partial x^j \partial t} g'_{\alpha\beta} + g^{ij} f_i^\alpha f_j^\beta \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} \frac{df_t^\nu}{dt}\right] dV(g) \\ &= \frac{1}{2}\int \left[-\left(2g^{ij} f_i^\alpha g'_{\alpha\beta}\right)_j \frac{df_t^\beta}{dt} + g^{ij} f_i^\alpha f_j^\beta \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} \frac{df_t^\nu}{dt}\right] dV(g)\end{aligned}$$

The first term is

$$\begin{aligned}-\left(2g^{ij} f_i^\alpha g'_{\alpha\beta}\right)_j \frac{df_t^\beta}{dt} &= -2g^{ij} f_{ij}^\alpha \frac{df^\beta}{dt} g'_{\alpha\beta} - 2g^{ij} f_i^\alpha \frac{df^\beta}{dt} \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} f_j^\nu \\ &= 2\Delta f^\alpha g'_{\alpha\beta} \frac{df_t^\beta}{dt} - 2g^{ij} f_i^\alpha f_j^\beta \frac{\partial g'_{\alpha\nu}}{\partial y^\beta} \frac{df_t^\nu}{dt}\end{aligned}$$

It remains to check that

$$-2\frac{\partial g'_{\alpha\nu}}{\partial y^\beta} + \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} = -2\Gamma'^\gamma_{\alpha\beta} g'_{\gamma\nu}$$

when we are allowed to permute α, β, which is routine. □

Definition 3. 1. *A **vector field along** $f : M \longrightarrow M'$ is a smooth application $v : M \longrightarrow TM'$ such that $\pi \circ v = f$ where $\pi : TM' \longrightarrow M'$ is the canonical projection. In other words, it is the association of each point $P \in M$ a tangent vector at $f(P)$*

2. *The **tension field** of f is the vector field along f defined by*

$$\tau(f)^\gamma := -\Delta f^\gamma + g^{ij}\Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta$$

By the Lemma 1, $\tau(f)$ is the unique vector field along f such that $\frac{d}{dt}E(f_t) = -\int_M \langle \tau(f), \frac{df_t}{dt}\rangle$. In particular, if f_t is the variation of f along a vector field v along f, i.e. $f_t(P) = \exp_{f(P)}(tv(P))$ then $\frac{d}{dt}E(f_t) = -\langle \tau(f), v\rangle$.

3. *$f : M \longrightarrow M'$ is called **harmonic** if $\tau(f) = 0$, or equivalently if f is a critical point of E.*

In normal coordinates of M at P and M' at $f(P)$, the tension field of f is given by

$$\tau^\gamma(f)(P) = \sum_i \frac{\partial^2 f^\gamma}{\partial (x^i)^2}(P)$$

Remark 2. *1. If M' is flat, i.e. $R'_{\alpha\beta\gamma\delta} = 0$ then $\tau(f)^\gamma = -\Delta f^\gamma$ is linear in f. We refind the definition of harmonic function.*

2. Since $\tau(f)$ depends locally on f, isometries and covering maps are harmonic.

Proposition 2 (Holomorphicity implies harmonicity)**.** *Holomorphic maps between Kahler manifolds are harmonic.*

Proof. We recall that exponential function $\exp_P : T_P M \longrightarrow M'$ on a Kahler manifold M is holomorphic for any $P \in M$. In fact, let $v \in T_P M$ and $\delta v \in T_v(T_P M)$ be a tangent vector at v and denote abusively by J the complex structure of the complex vector space $T_P M$ and that of M, one needs to see that

$$D\exp_P(v).J\delta v = J(\exp_P(v))D\exp_P(v).\delta v \tag{2.1}$$

In fact, let Y_1, Y_2 be Jacobi fields along $U(t) = \exp_P(tv)$ the geodesics of M starting at P in direction v with $Y_1(0) = Y_2(0) = 0, \dot{Y}_1(0) = \delta v, \dot{Y}_2(0) = J\delta v$ then the LHS of (2.1) is $Y_2(1)$, and the RHS is $J(U(1))Y_1(1)$. Then one can see that $Y_2(t) - J(U(t))Y_1(t) = 0$ for every $t \in [0,1]$ since it is true at $t = 0$ and the derivative with respect to t vanishes as $\nabla_{\dot{U}} J = 0$.

Therefore, at a point P of a Kahler manifold M, there exist holomorphic coordinates $z^j = x^j + iy^j$ of M in a neighborhood of P such that $\{x_j, y_j : j = \overline{1, n/2}\}$ are normal coordinates centered in P. Using such coordinates for $P \in M$ and $f(P) \in M'$, one has $\Delta f^\gamma = 0$ since f^γ is holomorphic and $\Gamma'^\gamma_{\alpha\beta}(P) = 0$ by normality, it follows that $\tau(f) = 0$ at every point $P \in M$. □

2.1.2 Formulation using connection on vector bundle

Setup and notation. Let E be a metric vector bundle over a Riemannian manifold M, i.e. each fiber of E is equiped with an inner product that we denote by $(g'_{\alpha\beta})$. The metric of M is denoted by (g_{ij}). Let n and m be the dimension of M of the fiber.

Covariant derivatives and exterior derivatives. We recall that a **covariant derivative** or a **connection** $\tilde{\nabla}$ of E is uniquely determined in local coordinates by an $m \times m$ matrix A of 1-forms, in other words, it is an 1-form on M with value in $Hom_M(E, E)$ which depends on the local frame of E (i.e. A is not a tensor with value in E). A is called the **connection form** of $\tilde{\nabla}$. Locally

$$\tilde{\nabla}_X(s^\alpha \tilde{e}_\alpha) = (\nabla_X s^\alpha)\tilde{e}_\alpha + A^\alpha_\beta(X) s^\beta \tilde{e}_\alpha.$$

When one prefers to work with forms rather than tensors with value in E, one uses an **exterior derivative**, a map $D : A^p(M,E) \longrightarrow A^{p+1}(M,E)$ which turns an p-form with value in E to an $p+1$-form with value in E. Locally

$$D(s^\alpha \tilde{e}_\alpha) = (ds^\alpha)\tilde{e}_\alpha + A^\alpha_\beta \wedge s^\beta \tilde{e}_\alpha.$$

and

$$D^2(s^\alpha \tilde{e}_\alpha) = (dA + A \wedge A) \wedge s.$$

One notes $\Theta := dA + A \wedge A$, which is an $m \times m$ matrix of 2-forms of M. Unlike A, Θ, seen as an 2-form with value in $Hom_M(E,E)$ does not depend on the local frame of E, i.e. Θ transforms as a (0,2) tensor with value in E, called the **curvature form**.

The fibrewise metric structure of E and the metric tensor of M give rise to a pointwise inner product of (p,q) tensors of M with value in E, in particular a pointwise inner product $(s,s') \mapsto s \cdot s'$ from $A^p(M,E) \times A^p(M,E)$ to $C^\infty(M)$. Integrated over M, the pointwise inner product gives rise to a global inner product $\int_M \langle \cdot, \cdot \rangle$ of $A^p(M,E)$. One denotes by $\delta : A^{p+1}(M,E) \longrightarrow A^p(M,E)$ the adjoint operator of $D : A^p(M,E) \longrightarrow A^{p+1}(M,E)$ with respect to this inner product, i.e. $\int_M \langle Ds, s' \rangle_{A^{p+1}(M,E)} = \int_M \langle s, \delta s' \rangle_{A^p(M,E)}$ for all $s \in A^p(M,E), s' \in A^{p+1}(M,E)$.

Laplacian operator and harmonic forms. The **Hodge Laplacian** is defined as a endomorphism of $A^p(M,E)$ given by

$$\tilde{\Delta} = D\delta + \delta D$$

and a form $s \in A^p(M,E)$ is called **harmonic** if $\tilde{\Delta} s = 0$. Since the Laplacian operator represents the *Dirichlet integral*, i.e.

$$\int_M \langle Ds, Ds' \rangle + \int_M \langle \delta s, \delta s' \rangle = \int_M \langle \tilde{\Delta} s, s' \rangle,$$

one has $\tilde{\Delta} s = 0$ if and only if $Ds = \delta s = 0$.

Riemannian connected bundle. The metric vector bundle E over M is called a **Riemannian-connected bundle** if it is equipped with a connection $\tilde{\nabla}$ under which the metric g' of E is parallel, i.e. $\tilde{\nabla} g' = 0$, in other words, the matrix A in an orthonormal frame is anti-symmetric: $A + {}^{t}A = 0$. Unless explicitly indicated, we always suppose that our metric vector bundle E is Riemannian-connected and the metric g' is parallel to the connection being used.

Example 2. *The case of our interest is when we have a smooth map $f : M \longrightarrow M'$ and $E = f^*TM'$ is a metric vector bundle over M under the metric g' induced from M'. Taking the connection $\tilde{\nabla}$ to be the Levi-Civita connection ∇' on M', meaning*

$$\tilde{\nabla}_X s = \nabla'_{f_* X} s,$$

for any vector field s along f, one can see that E is a Riemannian-connected bundle over M.

Lemma 3. *Let E be a Riemannian-connected bundle and $s = s_i^\alpha dx^i \tilde{e}_\alpha \in A^1(M, E)$, one has*

1. $\delta s = (\delta s)^\alpha \tilde{e}_\alpha \in A^0(M, E)$ *where*
$$(\delta s)^\alpha = -g^{ij}\left(\nabla_i s_j^\alpha + A_{\beta i}^\alpha s_j^\beta\right),$$
2. $\Delta s = (\Delta s)_i dx^i$ *where* $(\Delta s)_i$ *is an* $m \times m$ *matrix given by*
$$(\Delta s)_i = -\tilde{\nabla}^k \tilde{\nabla}_k s_i + {}^{\mathrm{t}}\left(\Theta_i^h - \mathrm{Ric}_i^h\right) s_h$$
where:
   - *the indices i, h, k correspond to local coordinates of M,*
   - Θ_i^h *is the curvature form of $\tilde{\nabla}$ with its indices raised by the metric g of M,*
   - $\mathrm{Ric}_i^h = \mathrm{Ric}_i^h I_m$ *is the Ricci curvature tensor of (M, g) with indices raised by the metric g, multiplied by the identity $m \times m$ matrix,*
   - $\tilde{\nabla}^k = g^{hk}\tilde{\nabla}_h$.
3. *With $s \cdot s'$ denoting the pointwise inner product of $A^1(M, E)$ and $\langle \cdot, \cdot \rangle_E$ denoting the metric g' of E, one has*
$$-\frac{1}{2}\Delta(s \cdot s) = -s \cdot \Delta s + \langle \tilde{\nabla}_i s_k, \tilde{\nabla}^i s^k \rangle_E - \left\langle {}^{\mathrm{t}}\left(\Theta_i^h - \mathrm{Ric}_i^h\right) s_h, s^i \right\rangle_E \tag{2.2}$$
where the superscript i, h are raised by the metric g.

Proof. Computational in nature. □

2.1.3 The case of $E = f^*TM'$

Energy functional and tension field

Our interest will be the case of Example 2 where $E = f^*TM'$ for a smooth map $f : M \longrightarrow M'$ of Riemannian manifolds is a Riemannian-connected bundle over M with the connection $\tilde{\nabla}$ given by the Levi-Civita connection of M'.

In this section, the tangent map $Tf : TM \longrightarrow TM'$ can be interpreted as a form f_* in $A^1(M, E)$. The energy functional can be rewritten as
$$E(f) = \frac{1}{2}\int_M f_i^\alpha f_j^\beta g^{ij} g'_{\alpha\beta} dV = \frac{1}{2}\langle f_*, f_* \rangle_{A^1 M, E}.$$

Proposition 4. *Let $f : M \longrightarrow M'$ and $E = f^*TM'$ be the Riemannian-connected bundle over M. Then:*

1. $A_\alpha^\beta = \Gamma'^\beta_{\gamma\alpha} f_i^\gamma dx^i$ *where* $\Gamma'^\beta_{\gamma,\alpha}$ *are Christoffel symbols of* (M', g').

2. $Df_* = 0$ where f_* is considered as an element of $A^1(M,E)$. Hence $\tilde{\Delta} f_* = D\delta f_*$.

3. The tension field of f is $\tau(f) = -\delta f_*$.

Proof. 1. One has

$$A_{\alpha i}^{\beta}\tilde{e}_\beta = \tilde{\nabla}_i \tilde{e}_\alpha = \tilde{\nabla}_{f_i^\gamma \tilde{e}_\gamma}\tilde{e}_\alpha = f_i^\gamma \Gamma'^{\beta}_{\gamma\alpha}\tilde{e}_\beta,$$

Therefore $A_\alpha^\beta = f_i^\gamma \Gamma'^{\beta}_{\gamma\alpha} dx^i$.

2. By direct computation:

$$Df_* = \left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \Gamma'^{\alpha}_{\gamma\beta} f_i^\gamma f_j^\beta\right) dx^j \wedge dx^i \otimes \tilde{e}_\alpha = 0$$

since it is the product of a symmetric quantity in (i,j) and an anti-symmetric one.

3. Using the first part of Lemma 3 for $s = f_* = f_i^\alpha dx^i \otimes \tilde{e}_\alpha$, one has $\delta f_* = -g^{ij}\left(\nabla_i \nabla_j f^\gamma + \Gamma'^{\gamma}_{\alpha\beta} f_i^\alpha f_j^\beta\right)\tilde{e}_\gamma = -\tau(f)$

□

It follows immediately that

Corollary 4.1. *$f : M \longrightarrow M'$ is a harmonic map of compact Riemannian manifolds if and only if f_* is harmonic as form in $A^1(M, f^*TM')$.*

Fundamental form, some results in case of signed curvature

Definition 4. *The **fundamental form** of a map $f : M \longrightarrow M'$ of Riemannian manifolds is the (0,2) symmetric tensor on M with value in $E = f^*TM'$ defined by*

$$\beta(f) := \tilde{\nabla} f_* = \left(f_{ij}^\gamma + \Gamma'^{\gamma}_{\alpha\beta} f_i^\alpha f_j^\beta\right) dx^i \otimes dx^j \otimes \tilde{e}_\gamma.$$

*The function f is called **totally geodesic** if $\beta(f) = 0$ identically on M.*

Remark 3. *1. The tension field $\tau(f) = g^{ij}\beta(f)_{ij}$ is the trace of the fundamental form.*

2. If f is totally geodesic then it is harmonic.

When $s = f_*$, Lemma 3 and Remark **??** become Lemma 5, with no more than direct computation. The appearance of Riemann curvature tensor R' of (M', g') is due to the formula

$$R'^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma'^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma'^{\rho}{}_{\mu\sigma} + \Gamma'^{\rho}{}_{\mu\lambda}\Gamma'^{\lambda}{}_{\nu\sigma} - \Gamma'^{\rho}{}_{\nu\lambda}\Gamma'^{\lambda}{}_{\mu\sigma}.$$

Lemma 5. *1. $Q(f_*)$ is given by*

$$Q(f_*) = R'_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta g^{ik} g^{jl} - \mathrm{Ric}^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$$

and

$$Q(f_*)^{ij}_{\alpha\beta} = R'_{\alpha\beta\gamma\delta} f_k^\gamma f_l^\delta g^{ik} g^{jl} - \mathrm{Ric}^{ij} g'_{\alpha\beta}.$$

2. If f is harmonic then

$$-\Delta e(f) = |\beta(f)|^2 - R'_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta g^{ik} g^{jl} + \mathrm{Ric}^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$$

where $|\beta(f)|$ is the pointwise norm of $\beta(f)$.

The previous computation of $Q(f_*)$ in term of Riemannian curvature of M' and Ricci curvature of M give the following result in case the curvature of M and M' are of definite sign.

Notation. Given a Riemannian manifold M, we will use the following notation:

1. $\mathrm{Ric} \geq 0$ (resp. $\mathrm{Ric} > 0$) if the Ricci curvature is positive semi-definite (resp. positive definite) as symmetric bilinear form.

2. $\mathrm{Riem} \leq 0$ (resp. $\mathrm{Riem} < 0$) if all sectional curvatures are negative (resp. strictly negative), i.e. $R_{ijhk} u^i v^j u^h v^k \leq 0$ (resp. $R_{ijhk} u^i v^j u^h v^k < 0$) for non-colinear vectors u, v.

Corollary 5.1. *Let $f : M \longrightarrow M'$ be a map of Riemannian manifolds.*

1. If f is harmonic and $Q(f_*) \leq 0$ then f is totally geodesic and $e(f)$ is constant.

2. If $\mathrm{Ric}(M) \geq 0$ and $\mathrm{Riem}(M') \leq 0$ then f is harmonic if and only if f is totally geodesic.

Proof. All the statements are consequence of 2) of Lemma 5 and the fact that $\int_M \Delta e(f) dV = 0$, noticing that

- $\mathrm{Ric}^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$ is $\mathrm{Ric} \otimes g'$ applied doubly to $f_i^\alpha dx^i \otimes \tilde{e}_\alpha$.

- $R'_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta g^{ik} g^{jl}$ is $(f^* R')_{ijhk} g^{ik} g^{jl}$. In a normal coordinate at P where $g^{ik} = \delta_{ik}, g^{jl} = \delta_{jl}$, it is the sum of sectional curvatures of tangent planes formed by $f_* e_i, f_* e_j$, and therefore negative.

□

2.1.4 Example: Riemannian immersion

Let $f : M \longrightarrow M'$ be a Riemannian immersion, i.e. Tf is injective and $f^* g' = g$. We will see that the fundamental form $\beta(f)$ that we defined earlier is the same as usual definition in courses of Riemannian geometry.

Second fundamental form.

One defines the symmetric (0,2)-tensor II of f^*TM' as the unique normal vector of M such that

$$\langle \mathrm{II}_{ij}, \xi_\sigma \rangle := -\langle \tilde{\nabla}_i \xi_\sigma, f_* e_j \rangle$$

for every vector field ξ_σ of M' orthogonal to M.

Lemma 6 (Second fundamental form)**.** *If f is a Riemannian immersion then $\beta(f)_{ij} = -\mathrm{II}_{ij}$ and they are orthogonal to M. In particular, if f is totally geodesic than it maps geodesics of M to geodesics of M'*

Proof. One has

$$\begin{aligned}\langle \tilde{\nabla}_i \xi_\sigma, f_* e_j \rangle &= \langle \xi_\sigma, \tilde{\nabla}_i (f_* e_j) \rangle = \langle \xi_\sigma, \tilde{\nabla}_i (f_l^\gamma dx^l \otimes \tilde{e}_\gamma) e_j + f_* \nabla_i e_j \rangle \\ &= \langle \xi_\sigma, (f_{il}^\gamma dx^l \tilde{e}_\gamma + f_l^\gamma dx^l \tilde{\nabla}_i \tilde{e}_\gamma) e_j \rangle \\ &= \langle \xi_\sigma, f_{ij}^\gamma \tilde{e}_\gamma + f_j^\gamma A_{\gamma_i}^\alpha \tilde{e}_\alpha \rangle = \left\langle \xi_\sigma, \left(f_{ij}^\gamma + \Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta \right) \tilde{e}_\gamma \right\rangle \\ &= \langle \xi_\sigma, \tilde{\nabla}_i (f_*).e_j \rangle = \langle \xi_\sigma, \beta(f)_{ij} \rangle\end{aligned} \tag{2.3}$$

where we used $\xi_\sigma \perp f_* e_j$ in the first line and $\xi_\sigma \perp f_*([e_i, e_j])$ in the second line. Hence $\mathrm{II}_{ij} \equiv -\beta(f)_{ij}$ modulo an element in TM. It remains to see that $\beta(f)_{ij} \perp M$ in order to conclude $\mathrm{II} = -\beta(f)$. By definition, one has $\beta(f)_{ij} = \tilde{\nabla}_i (f_*).e_j$ and

$$\begin{aligned}\langle \beta(f)_{ij}, f_* e_k \rangle &= \langle \tilde{\nabla}_i (f_*).e_j, f_* e_k \rangle = \tilde{\nabla}_i \langle f_* e_j, f_* e_k \rangle - \langle \nabla_i e_j, e_k \rangle - \langle f_* e_j, \tilde{\nabla}_i (f_* e_k) \rangle \\ &= \nabla_i \langle e_j, e_k \rangle - \langle \nabla_i e_j, e_k \rangle - \langle \beta(f)_{ik}, f_* e_j \rangle - \langle e_j, \nabla_i e_k \rangle \\ &= -\langle \beta(f)_{ik}, f_* e_j \rangle\end{aligned}$$

Then using the symmetric of $\beta(f)_{ij}$, one has $\langle \beta(f)_{ij}, f_* e_k \rangle = 0$.

Finally, if $\beta(f) = 0$ and X is a geodesic vector field of M, one needs to prove that $f_* X$ is a geodesic vector field of M'. In fact

$$\tilde{\nabla}_X (f_* X) = (\tilde{\nabla}_X f_*) X + f_* \nabla_X X = \beta(f)(X, X) = 0.$$

Hence $f_* X$ is a geodesic field of M'. □

Example 3. *The inclusion $x \mapsto (x, y_0)$ of a Riemannian manifold M to the Riemannian product $M \times N$ is totally geodesic.*

Definition 5. *Given an orthonormal frame $(\xi_\sigma)_{1 \le \sigma \le n'-n}$, the* ***mean normal curvature field*** *of M in M' at $P \in M$ is defined as*

$$\xi(P) := \sum_{\sigma=1}^{n'-n} g^{ij} \langle \mathrm{II}_{ij}, \xi_\sigma \rangle \xi_\sigma = -\sum_{\sigma=1}^{n'-n} \langle \tau(f), \xi_\sigma \rangle \xi_\sigma.$$

The immersion f is said to be ***minimal*** *if ξ vanishes identically on M.*

Remark 4. *1. Since $(\xi_\sigma)_{1\leq\sigma\leq n'-n}$ is an orthonormal frame, one also has*

$$\xi(P) = -g^{ij}\langle\tilde{\nabla}_i\xi_\sigma, f_*e_j\rangle\xi_\sigma(P) = -\sum_{\sigma=1}^{n'-n}\operatorname{div}\left(\xi_\sigma(P)\right)\xi_\sigma(P)$$

2. The mean normal curvature field is the tension field of f, i.e. $\xi = -\tau(f)$. Minimal immersions are exactly harmonic immersion.

The case of signed curvature.

If $f : M \longrightarrow M'$ is a Riemannian immersion then the Ricci term of Lemma 5 is actually the scalar curvature of M, one has

Proposition 7. *Let $f : M \longrightarrow M'$ be a Riemannian immersion. Suppose that* $\mathrm{Riem}(M') \leq 0$ *and $r = g^{ij}\mathrm{Ric}_{ij} < 0$ at one point of M. If f is harmonic then it is constant.*

2.1.5 Composition of maps

The following results come from direct computation of the second fundamental form and tension field of composition of maps between Riemannian manifolds. Again, we use indices $i, j, k, \ldots$ for M, $\alpha, \beta, \gamma, \ldots$ for M' and $a, b, c, \ldots$ for M''.

Proposition 8. *Let $f : M \longrightarrow M'$ and $f' : M' \longrightarrow M''$ be smooth maps of Riemannian manifolds, then*

$$\beta(f'\circ f)^a_{ij} = \beta(f)^\gamma_{ij}f'^a_\gamma + \beta(f')^a_{\alpha\beta}f^\alpha_i f^\beta_j \tag{2.4}$$

and

$$\tau(f'\circ f)^a = \tau(f)^\gamma f'^a_\gamma + g^{ij}\beta(f')^a_{\alpha\beta}f^\alpha_i f^\beta_j \tag{2.5}$$

Therefore,

If f' is	*and f is*	*then $f'\circ f$ is*
totally geodesic	*totally geodesic*	*totally geodesic*
totally geodesic	*harmonic*	*harmonic*

and the inverse of a totally geodesic map is totally geodesic.

Remark 5. *It is not true in general that the composition of harmonic maps are harmonic. For example, if one composes the harmonic maps $\mathbb{R} \longrightarrow \mathbb{R}^2 : x \mapsto (x, 2x)$ and $\mathbb{R}^2 \longrightarrow \mathbb{R} : (x, y) \mapsto x^2 - y^2$, the result is $\mathbb{R} \longrightarrow \mathbb{R} : x \mapsto -3x^2$, which is not harmonic.*

Proposition 9 (composition with immersion)**.** *If $f' : M' \longrightarrow M''$ is a Riemannian immersion and $f : M \longrightarrow M'$ then*

1. Energy functionals: $E(f) = E(f'\circ f)$.

2. *Tension fields: $\tau(f)$ is the projection of $\tau(f' \circ f)$ to M'.*

Proof. 1. One has $e(f) = \frac{1}{2}\langle g, f^*g' \rangle = \frac{1}{2}\langle g, (f' \circ f)^* g'' \rangle = e(f' \circ f)$.

2. One has $\tau(f' \circ f)^a = \tau(f)^a + g^{ij}\beta(f')^a_{\alpha\beta} f^\alpha_i f^\beta_j$ by (2.5). The conclusion follows since the second term is normal to M'.

□

The following immediate corollary of Proposition 9 is a generalization of the fact that a curve on a surface M' of $\mathbb{R}^3$ is geodesic of if and only if its curvature vector is orthogonal to M'.

Corollary 9.1. *If $f' : M' \longrightarrow M''$ is a Riemannian immersion, then a map $f : M \longrightarrow M'$ is harmonic if and only if $\tau(f' \circ f) \perp M'$.*

2.2 Nonlinear heat flow: Global equation and existence of harmonic maps.

2.2.1 Statement of the main results.

We want to prove in the next part existence of harmonic map between manifolds M and M' by deforming any map $f : M \longrightarrow M'$ using the τ-flow, meaning solving the PDE:

$$\begin{cases} \frac{df_t}{dt} = \tau(f_t), & t \in [\alpha, \omega] \\ f_\alpha = f, \end{cases} \tag{2.6}$$

The equation makes sense because both $\frac{df_t}{dt}$ and $\tau(f_t)$ are vector fields along f_t. Since this is the gradient-descent equation for E, the energy of f_t decreases and we hope, under conditions, to obtain convergence of $\{f_t\}$ to a critical point f_∞ of E, this will prove that any homotopy class of $C^\infty(M, M')$ has at least a harmonic map.

It is proved by Eells and Sampson [ES64] that

Theorem 10 (Eells-Sampson)**.** *Let M and M' be compact Riemannian manifolds with* $\mathrm{Riem}(M') \leq 0$ *then there exists a harmonic map $f : M \longrightarrow M'$ in each homotopy class.*

Several boundary conditions, of Dirichlet, Neumann or mixed type, are also taken into account by Hamilton [Ham75], as an example, we will state the Dirichlet problem:

Theorem 11 (Hamilton)**.** *Let M and M' be compact Riemannian manifolds possibly with boundary. Suppose that M' has* $\mathrm{Riem}(M') \leq 0$ *and $\partial M'$ is convex, then any relative homotopy class of $C^\infty(M, M')$ has a harmonic element.*

About the terminology, **relative homotopy class** means that we only deform f among maps with the same value on ∂M. The **convexity of** $\partial M'$ means that the geodesic at any point in $\partial M'$ with initial tangent vector parallel to the boundary does not enter the interior of M' in short time. This condition can be expressed using the Christoffel symbols of M' at the point in question: If M' is coordinated by $y^1, \dots, y^n$ with and $M' = \{y^n \geq 0\}$, then the convexity is translated as $\Gamma'^n_{\alpha\beta} \geq 0$ as a symmetric form $(1 \leq \alpha, \beta \leq n-1)$. This can be seen by the geometric intepretation of the second fundamental form of the embedding $s : \partial M' \hookrightarrow M'$, which is $\mathrm{II}(s) = -\Gamma'^n_{\alpha\beta}$.

It is easy to see that the convexity of $\partial M'$ is a necessary condition, as harmonic maps from $\mathbb{R}$ are geodesics: Suppose the condition does not hold at $x \in \partial M'$, meaning that upto time t the geodesic flow of M' initially pointing into M' remains in the interior. The geodesic of $\partial M'$ of length less than t with the same initial tangent therefore cannot be deformed into a geodesic of M' in relative homotopy class.

2.2.2 Strategy of the proof.

In order to have a global frame, we will embed M' into an Euclidean space V, but we will not use the Euclidean metric of V. In fact, let T be a tubular neighborhood of M' in V then if T is trivial, i.e. if it is diffeomorphic to $M' \times D$ where D is a sufficiently small ball of dimension being the codimension of M' in V, and we will equip T with the product metric of $M' \times D$.

If T is not trivial, using a partition of unity of M', one can construct a metric on T as linear combination of the product metrics on trivialised pieces so that the involution $\iota : T \longrightarrow T$ locally given by $(y, d) \mapsto (y, -d)$ for $y \in M', d \in D$ is an isometry. As a consequence, M' is totally geodesic in T.

Since $M' \equiv M' \times \{0\}$ is totally geodesic in T, one has for every smooth function $f : M \longrightarrow M'$:

$$\tau_T(f) = \tau_{M'}(f)$$

The crucial property we expect for a global equation of (2.6), is the following: if the solution initially is in $M' \subset V$ then it remains in M' for all relevant time $t > \alpha$. Eells-Sampson [ES64] did this by using at the same time 2 different metrics on T, namely the product metric as tubular neighborhood and the Euclidean metric. I choose to present here the formulation of Hamilton, which is conceptually simpler with the only drawback being that we need to establish the uniqueness of solution of (2.6) first.

After having the global equation, we will prove the short time existence of solution by linearising the equation and using Inverse function theorem. The global formulation and the proof of short-time existence are independent of the negative curvature hypothesis, which will only be used later to establish energy estimates and assure the convergence of long-time solution and the vanishing of its tension field.

2.2.3 Global equation and Uniqueness of nonlinear heat equation.

Theorem 12 (Global equation). *If the smooth function $F_t : M \times [\alpha, \beta] \longrightarrow V$ satisfies*

$$\frac{dF_t}{dt} = \tau_T(F_t) \tag{2.7}$$

and $F_t(M \times \{\alpha\}) \subset M'$ then $F_t(M \times [\alpha, \omega]) \subset M'$

Proof. Let ι be the isometry of T locally given by $(y, d) \mapsto (y, -d)$ for $(y, d) \in M' \times D \equiv T$ and pose $G_t = \iota F_t$ then G_t and F_t coincide initially since M' is fixed by ι. Moreover

$$\frac{dG_t}{dt} = d\iota . \frac{dF_t}{dt} = d\iota(\tau_T(F_t)) = \tau_T(\iota F_t) = \tau_T(G_t)$$

We conclude that $F_t = G_t = \iota F_t$, hence F_t remains in M' for all relevant t, using the following uniqueness of nonlinear heat equation. $\square$

Theorem 13 (Uniqueness of solution of nonlinear hear equation). *Let $f_1, f_2 : M \times [\alpha, \omega] \longrightarrow M'$ be C^2 functions satisfying the non-linear heat equation $\frac{df_i}{dt} = \tau_{M'}(f_i)$, i.e.*

$$\frac{df^\gamma}{dt} = -\Delta f^\gamma + g^{ij} \Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta$$

where $\Gamma'^\gamma_{\alpha\beta}$ are Christoffel symbols of M'. Suppose that f_1 and f_2 coincide on $M \times \{\alpha\}$. Then $f_1 = f_2$ on $M \times [\alpha, \omega]$.

Proof. It is sufficient to prove the theorem for ω very close to α, therefore by compactness of M, we can suppose that there exists a finite atlas $M = \bigcup_i U_i$ with $f_1(U_i \times [\alpha, \omega])$ and $f_2(U_i, [\alpha, \omega])$ being in the same chart V_i of M'. We consider the distance function $\sigma(a, b) = \frac{1}{2} d_{M'}(a, b)^2$ for $a, b \in M'$ to measure the difference between f_1 and f_2 by

$$\rho(x, t) = \sigma(f_1(x, t), f_2(x, t))$$

The strategy is to prove that there exists $C > 0$ such that $\frac{d\rho}{dt} \le -\Delta\rho + C\rho$, then by Maximum principle, one has $\rho = 0$.

Fix a chart U_i of M and the corresponding V_i of M', one has by straightforward calculation:

$$\begin{aligned}\frac{d\rho}{dt} &= -\Delta\rho - g^{ij}\left(\frac{\partial^2\sigma}{\partial f_1^\beta \partial f_1^\gamma} - \frac{\partial \sigma}{\partial f_1^\alpha}\Gamma'^\alpha_{\beta\gamma}(f_1)\right) f_{1i}^{\beta} f_{1j}^{\gamma} \\ &\quad - g^{ij}\left(\frac{\partial^2\sigma}{\partial f_2^\beta \partial f_2^\gamma} - \frac{\partial \sigma}{\partial f_2^\alpha}\Gamma'^\alpha_{\beta\gamma}(f_2)\right) f_{2i}^{\beta} f_{2j}^{\gamma} - 2g^{ij}\frac{\partial^2\sigma}{\partial f_1^\beta \partial f_2^\gamma} f_{1i}^{\beta} f_{2j}^{\gamma}\end{aligned} \tag{2.8}$$

where g^{ij} is the metric on M and $\Gamma'^\alpha_{\beta\gamma}$ are Christoffel symbols of M'.

Let c be a point in the chart V_i and choose the normal coordinates of M' at c. Then for $a, b \in M'$ near c, one has, since $\sigma(a,b) = \sigma(b,a)$ and $\sigma(a,b) = 0$ if $b^\gamma = ka^\gamma$ (the Euclidean straight line from a to ka viewed on M' is a geodesic):

$$\sigma(a,b) = \frac{1}{2}d_{M'}(a,b)^2 = \frac{1}{2}d_E(a,b)^2 + \lambda_{\beta\gamma,\delta}(a^\beta a^\gamma b^\delta + b^\beta b^\gamma a^\delta)$$

where d_E is the Euclidean distance, with $\lambda_{\beta\gamma,\delta} = \lambda_{\gamma\beta,\delta}$ and $\lambda_{\beta\gamma,\delta} + \lambda_{\gamma\delta,\beta} + \lambda_{\beta\delta,\gamma} = 0$. We then have the series development of σ at $(0,0)$:

$$\sigma(a,b) = \frac{1}{2}\delta_{\beta\gamma}(a^\beta - b^\beta)(a^\gamma - b^\gamma) + \lambda_{\beta\gamma,\delta}(a^\beta a^\gamma b^\delta + b^\beta b^\gamma a^\delta) + O(|a| + |b|)^4 \tag{2.9}$$

and the development of its derivatives

$$\begin{aligned}
\frac{\partial^2\sigma}{\partial a^\beta \partial b^\gamma}(a,b) &= -\delta_{\beta\gamma} + O(|a| + |b|)^2 \\
\frac{\partial^2\sigma}{\partial a^\beta \partial a^\gamma}(a,b) &= \delta_{\beta\gamma} + \lambda_{\beta\gamma,\delta}b^\delta + O(|a| + |b|)^2 \\
\frac{\partial^2\sigma}{\partial b^\beta \partial b^\gamma}(a,b) &= \delta_{\beta\gamma} + \lambda_{\beta\gamma,\delta}a^\delta + O(|a| + |b|)^2 \\
\frac{\partial\sigma}{\partial a^\alpha}(a,b) &= O(|a| + |b|), \quad \Gamma'^{\alpha}_{\beta\gamma}(a) = O(|a|)
\end{aligned}$$

So choose c to be the midpoint of $f_1(x,t)$ and $f_2(x,t)$ and $(f_1(x,t), f_2(x,t)) = (w, -w)$ in the chart, one has:

$$\frac{d\rho}{dt} = -\Delta\rho - \left(\delta_{\beta\gamma} - \lambda_{\beta\gamma,\delta}w^\delta + O(|w|^2)\right) f_{1}{}_i^\beta f_{1}{}_j^\gamma g^{ij} - \left(\delta_{\beta\gamma} + \lambda_{\beta\gamma,\delta}w^\delta + O(|w|^2)\right) f_{2}{}_i^\beta f_{2}{}_j^\gamma g^{ij} \tag{2.10}$$

$$- 2\left(-\delta_{\beta\gamma} + O(|w|^2)\right) f_{1}{}_i^\beta f_{2}{}_j^\gamma g^{ij} \tag{2.11}$$

$$= -\Delta\rho - |df_1 - df_2|^2 - w^\delta \lambda_{\beta\gamma,\delta} g^{ij}\left(f_{2}{}_i^\beta f_{2}{}_j^\gamma - f_{1}{}_i^\beta f_{1}{}_j^\gamma\right) + O(|\omega|^2) \tag{2.12}$$

The last term of (2.12) can be bounded as follows:

$$\begin{aligned}
\left|w^\delta \lambda_{\beta\gamma,\delta}\left(f_{2}{}_i^\beta f_{2}{}_j^\gamma - f_{1}{}_i^\beta f_{1}{}_j^\gamma\right) g^{ij}\right| &= \left|w^\delta \lambda_{\beta\gamma,\delta}\left(f_{2}{}_i^\beta (f_{2}{}_j^\gamma - f_{1}{}_j^\gamma) + f_{1}{}_j^\gamma (f_{2}{}_i^\beta - f_{1}{}_i^\beta)\right) g^{ij}\right| \\
&\le 2|w^\delta \lambda_{\beta\gamma,\delta}||df_2 - df_1|(|df_1| + |df_2|) \\
&\le |df_1 - df_2|^2 + O(|w|^2)
\end{aligned}$$

where for the last inequality, we use $2uv \le u^2 + v^2$ and the fact that $|df_1|$ and $|df_2|$ are bounded on M. The estimate (2.12) can be continued:

$$\frac{d\rho}{dt} \le -\Delta\rho + C(x,t)|w|^2 \le -\Delta\rho + C\rho$$

where $C > 0$ is a constant chosen to dominate all $C(x,t)$ for $x \in M$ in all charts and $t \in [\alpha, \omega]$. $\square$

Remark 6. *The original proof of [Ham75] made the reduction of the first order of w in* (2.11) *using the following development of σ:*

$$\sigma = \frac{1}{2}\delta_{\beta\gamma}(a^\beta - b^\beta)(a^\gamma - b^\gamma) + \lambda_{\beta\gamma,\delta}(a^\beta - b^\beta)(a^\gamma - b^\gamma)(a^\delta + b^\delta) + O(|a| + |b|)^4$$

which was justified by $\sigma(a,b) = \sigma(b,a)$ and $\sigma(a,a) = 0$. It can be proved that this is equivalent to (2.9) *and the symmetries $\lambda_{\beta\gamma,\delta} = \lambda_{\gamma\beta,\delta}$, $\lambda_{\beta\gamma,\delta} + \lambda_{\gamma\delta,\beta} + \lambda_{\beta\delta,\gamma} = 0$.*

As a side note, if a, b, c are on $\mathbb{S}^2$ with $d(a,c) = d(b,c) = x \ll 1$ and the lines from a and b to c are orthogonal at c, then the geodesic distance $d(a,b) = \arccos(\cos^2(x)) = x\sqrt{2} - \frac{1}{6\sqrt{2}}x^3 + O(x^4)$. So $\sigma(a,b) = \frac{1}{2}d(a,b)^2$ has no third-order term.

2.3 A few energy estimates.

2.3.1 Estimate of density energies

We finish this part with a few straightforward computation concerning the **potential energy** $e(f_t) = \frac{1}{2}|\nabla f_t|^2$ and the **kinetic energy** $k(f_t) = \frac{1}{2}|\frac{df_t}{dt}|^2$ of a nonlinear heat flow f_t satisfying (2.6).

Theorem 14 (Density of Potential energy)**.** *If f_t satisfies* (2.6) *then*

$$\frac{de(f_t)}{dt} = -\Delta e(f_t) - |\beta(f_t)|^2 - \langle \mathrm{Ric}(M)\nabla_v f_t, \nabla_v f_t\rangle + \langle \mathrm{Riem}(M')(\nabla_v f_t, \nabla_w f_t)\nabla_v f_t, \nabla_w f_t\rangle$$

where $e(f_t)$ is the potential energy density and $\beta(f_t)$ is the fundamental form and in the curvature terms, the vectors v and w are contracted.

In particular, if $\mathrm{Riem}(M') \leq 0$ and $\mathrm{Ric}(M) \geq -C$ then

$$\frac{de}{dt} \leq -\Delta e + Ce - |\beta(f_t)|^2 \tag{2.13}$$

Proof. Apply Lemma 3 to $s = df_t$ and the Riemannian-connected bundle F^*TM' over $M \times [\alpha, \omega]$ where $F(\cdot, t) = f_t$, the curvature terms cancel out and it remains to see that $\frac{de(f_t)}{dt} = -\langle df_t, \Delta df_t\rangle$, meaning that $\tilde{\nabla}_{\partial t} df_t = -\Delta df_t$. This can be easily justified:

$$\tilde{\nabla}_{\partial t} df_t = \tilde{\nabla}_{\partial t}\tilde{\nabla}^M F = \tilde{\nabla}^M \tilde{\nabla}_{\partial t} F = \tilde{\nabla}^M\ \tau(f_t) = -D\delta(df_t) = -\Delta df_t$$

where the last "=" is due to $Ddf_t = 0$. Note that D and δ are the exterior derivative and its adjoint of the bundle $(f_t)^*TM'$ on M, where t can be fixed after the third "=" sign. ☐

Theorem 15 (Density of Kinetic energy)**.** *If f_t satisfies* (2.6) *then*

$$\frac{dk(f_t)}{dt} = -\Delta k(f_t) - \left|\nabla \frac{\partial f_t}{\partial t}\right|^2 + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle$$

where $k(f_t)$ is the kinetic energy density and in the curvature terms, the vectors v is contracted, In particular, if $\mathrm{Riem}(M') \leq 0$ *then*

$$\frac{dk}{dt} \leq -\Delta k - \left|\nabla \frac{\partial f_t}{\partial t}\right|^2 \tag{2.14}$$

Proof. Let $F: \ I \times M \longrightarrow M'$ be the total function with $F(t,\cdot) = f_t$ for $t \in I = [\alpha, \omega]$ and $E = F^*TM'$ is a Riemannian-connected bundle on $I \times M$ with curvature form Θ, then

$$\tilde{\nabla}_{\partial t}\tilde{\nabla}_v(dF.v) = \tilde{\nabla}_v\tilde{\nabla}_{\partial t}(dF.v) + \Theta(\partial t, v)dF.v \tag{2.15}$$

where dF is the exterior derivative of f_t on M. Note that $\tilde{\nabla}_v\tilde{\nabla}_{\partial t}(dF.v) = \tilde{\nabla}_v(\tilde{\nabla}_{\partial t}dF).v = \tilde{\nabla}_v(\tilde{\nabla}^M\frac{\partial f_t}{\partial t}).v$ since $\tilde{\nabla}^M\frac{\partial f_t}{\partial t} = \tilde{\nabla}^{I\times M}_{\partial t}dF = \tilde{\nabla}^I_{\partial_t}dF$ because $\tilde{\nabla}$ is torsionless on M'. Plugging this in (2.15) and taking contraction in v, one has

$$\tilde{\nabla}_{\partial t}\ \tau(f_t) = -\tilde{\Delta}\frac{\partial f_t}{\partial t} + \mathrm{Tr}\left(v \mapsto \Theta(\partial t, v)dF.v\right) \tag{2.16}$$

But $\Theta^\beta_\alpha = R'^\beta_{\alpha\nu\mu}F^\mu_i F^\nu_j dx^i \otimes dx^j$ where R' denotes the Riemannian curvature of M' and the indices i, j can be 0, with $x^0 \equiv t$. Hence

$$\Theta(\partial t, v)dF.v = R'^\beta_{\alpha\nu\mu}\frac{\partial f^\mu_t}{\partial t}\frac{\partial f^\nu_t}{\partial v}\frac{\partial f^\alpha_t}{\partial v}\tilde{e}_\beta = \mathrm{Riem}(M')\left(\nabla_v f_t, \frac{\partial f_t}{\partial t}\right)\nabla_v f_t$$

Plugging in (2.16) and taking inner product with $\frac{\partial f_t}{\partial t}$, one has

$$\begin{aligned}\frac{\partial k(f_t)}{\partial t} &= \left\langle \tilde{\nabla}_{\partial t}\ \tau(f_t), \frac{\partial f_t}{\partial t}\right\rangle = -\left\langle \tilde{\Delta}\frac{\partial f_t}{\partial t}, \frac{\partial f_t}{\partial t}\right\rangle + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle \\ &= -\Delta\left(\frac{1}{2}\left|\frac{\partial f_t}{\partial t}\right|^2\right) - \left|\tilde{\nabla}\frac{\partial f_t}{\partial t}\right|^2 + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle\end{aligned}$$

□

2.3.2 Estimate of total energies

We will now work with the total energies, in particular the **total potential energy** $E(f_t) := \int_M e(f_t)$ and **total kinetic energy** $K(f_t) := \int_M k(f_t)$. Since tension field is the gradient of E, one has:

Theorem 16. *If $f_t : M \longrightarrow M'$ satisfies* (2.6) *then*

$$\frac{dE(f_t)}{dt} = -\int_M \left\langle \tau(f_t), \frac{\partial f_t}{\partial t}\right\rangle = -\int_M |\tau(f_t)|^2 = -2K(f_t) \leq 0.$$

Integrating Theorem 15 on M then using Theorem 16, one obtains:

Theorem 17. *If f_t satisfies* (2.6) *and* $\mathrm{Riem}(M') \leq 0$ *then* $\frac{d}{dt}K(f_t) \leq 0$ *and one has*

1. *The total potential energy $E(f_t)$ is ≥ 0, decreasing and convex.*
2. *The total kinetic energy $K(f_t)$ is ≥ 0, decreasing and if $\omega = +\infty$ then $\lim_{t\to\infty} K(f_t) = 0$.*

In particular, $\int_{M\times\{\tau\}} |\nabla f|^2$ and $\int_{M\times\{\tau\}} \left|\frac{\partial f_t}{\partial t}\right|^2$ are bounded above by a constant $C > 0$ independent of the time $\tau \in [\alpha, \omega]$.

Note that we ruled out the case $K(f_t)$ decreases to a strictly positive limit because $E(f_t)$ is bounded below and $\frac{d}{dt}E(f_t) = -2K(f_t)$.

Integrating Theorem 14 on M then using Theorem 17, one has:

Theorem 18. *If f_t satisfies* (2.6) *and* $\mathrm{Riem}(M') \leq 0$ *and* $\mathrm{Ric}(M)$ *is bounded below then*

$$\int_M |\beta(f_t)|^2 \leq C$$

for all time t where the constant C only depends on the curvature of M, M' and the initial total potential and kinetic energy, in particular, C does not depend on t.

This means that $\|f_t\|_{W^{2,2}(M)}$ is bounded by a constant C only depending on the curvatures and initial total energies.

Corollary 18.1 (Boundedness in $W^{2,2}(M)$)**.** *If F_t satisfies* (2.7) *and* $\mathrm{Riem}(M') \leq 0$ *and* $\mathrm{Ric}(M)$ *is bounded below then*

$$\|F_t\|^2_{W^{2,2}(M)} := \int_M |\beta(F_t)|^2 + |\nabla F_t|^2 + |F|^2 \leq C$$

for all time t where the constant C only depends on the curvature of M, M' and the initial total potential and kinetic energy, in particular, C does not depend on t.

Note that the term $|F|^2$ is trivially bounded since the image of F remains in an Euclidean ball B.

Part II

Resolution of nonlinear heat equation on manifold

Chapter 3

Short-time existence and regularity for nonlinear heat equation

We will establish in this part a regularity estimate for the quadratic term of nonlinear heat operator use it to setup a bootstrap scheme that eventually will prove that any sufficiently regular solution of nonlinear heat equation that is initially C^∞ will be always C^∞.

We will also prove short-time existence using well-known method of Inverse function theorem for Banach spaces. Since the solution is smooth, we can apply Theorem 12 to conclude that the it remains in $M' \subset \mathbb{R}^N$.

3.1 Review of Sobolev spaces and Linear equations.

The following results are well-known and their statements are written here in the case of our interest (linear heat equation on manifold). A more careful formulation with complete proofs can be found in the appendices.

3.1.1 Sobolev spaces.

Let M be a Riemannian manifold, the *Sobolev spaces* $W^{k,p}(M)$ on M can be defined as the completion of $C^\infty(M)$ with respect to the Sobolev norms

$$\|\varphi\|_{W^{k,p}} = \sum_{|\alpha| \le k} \|D^\alpha \varphi\|_{L^p}.$$

We will suppose that M is a compact manifold, then set-theoretically $W^{k,p}$ does not depend on the metric of M and their norm remains in the same equivalent class as the metric varies. The Sobolev spaces form a family of reflexive Banach spaces that is stable under holomorphic interpolation:

Theorem 19 (Interpolation of Sobolev spaces)**.** *Let $p, q \in (1, +\infty)$ and $k, l \in \mathbb{R}$ and M be a compact Riemannian manifold. Then the holomorphic interpolations of*

$$A_0 := W^{k,p}(M) \quad \text{and} \quad A_1 := W^{l,q}(M)$$

are $A_\theta = W^{s,r}(M)$ where

$$\theta l + (1-\theta)k = s, \qquad \theta \frac{1}{q} + (1-\theta)\frac{1}{p} = \frac{1}{r}.$$

In particular, one has the Interpolation inequality

$$\|f\|_{W^{s,r}} \leq 2\|f\|_{W^{l,q}}^{\theta}\|f\|_{W^{k,p}}^{1-\theta}.$$

Sobolev embeddings and Kondrachov theorem remain correct on manifold.

Theorem 20 (Sobolev embeddings)**.** *Given $k, l \in \mathbb{Z},\ k > l \geq 0$ and $p, q \in \mathbb{R},\ p > q \geq 1$. Then*

1. *If $\frac{1}{p} = \frac{1}{q} - \frac{k-l}{n}$ then*
$$W^{k,q}(M) \hookrightarrow W^{l,p}(M),$$
2. *If $\frac{k-r}{n} > \frac{1}{q}$ then*
$$W^{k,q}(M) \hookrightarrow C^r(M)$$
If $\frac{k-r-\alpha}{n} \leq \frac{1}{q}$ then
$$W^{k,q}(M) \hookrightarrow C^{r,\alpha}(M)$$

where $C^r(M)$ denotes the space of C^r functions equipped with the norm $\|u\|_{C^r} = \max_{l \leq r} \sup |\nabla^l u|$, and $C^{r,\alpha}$ is the subspace of C^r of functions whose r^{th}-derivative is α-Holder, equipped with the norm $\|u\|_{C^{r,\alpha}} = \|u\|_{C^r} + \sup_{P \neq Q}\{\frac{u(P)-u(Q)}{d(P,Q)^\alpha}\}$.

Theorem 21 (Kondrachov)**.** *Let $k \in \mathbb{Z}_{\geq 0}$ and $p, q \in \mathbb{R}_{>0}$ be such that $1 \geq \frac{1}{p} > \frac{1}{q} - \frac{k}{n} > 0$ then*

1. *The embedding $W^{k,q}(M) \hookrightarrow L^p(M)$ is compact,*
2. *The embedding $W^{k,q}(M) \hookrightarrow C^\alpha(M)$ is compact if $k - \alpha > \frac{n}{q}$ where $0 \leq \alpha < 1$,*

It is also natural, for regularity results of parabolic equation, to use weighted Sobolev spaces because each derivative in time should be counted as twice as that in space. For example, the space $W^{2,p}(M \times [\alpha, \omega])$ is the completion of $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{W^{2,p}} := \|\varphi\|_{L^p} + \left\|\frac{d\varphi}{dt}\right\|_{L^p} + \sum_{i,j}\left\|\frac{\partial^2\varphi}{\partial x^i \partial x^j}\right\|_{L^p} + \sum_i \left\|\frac{\partial \varphi}{\partial x^i}\right\|_{L^p}$$

Similarly, one can define $W^{2k,p}(M \times [\alpha, \omega])$ using L^p-norm of derivatives $\partial_t^\beta \partial_x^\gamma \varphi$ of φ with $2\beta + \gamma \leq 2k$.

We also want to be able to talk about $W^{k,p}$ when k is not an integer and not necessarily positive. This allows us to have a more flexible bootstrap scheme for nonlinear heat equation and to use Interpolation Theorem 19 more efficiently. We claim that these generalised Sobolev spaces (with weight and with non-integral regularity) can be defined on manifold and satisfy all the above properties (reflexivity, Interpolation theorem, Sobolev embedding and Kondrachov theorem) and refer to the appendices for all the details.

3.1.2 Trace theorem.

It is possible to avoid a discussion on Trace operator if we only want to make sense of the initial condition of nonlinear heat equation: one can consider only solutions with regularity greater than $W^{2,p}(M \times [\alpha, \omega])$ with $p \geq \dim M + 2$, which can be embedded in $C(M)$. It is however necessary to investigate regularity of Trace operator to have a complete proof of the bootstrap. We will review briefly some results.

The following two behaviors of trace are well-known:

1. If $-1 + \frac{1}{p} < k < \frac{1}{p}$ then the natural map $W^{k,p}(M \times [\alpha, \omega]/\alpha) \hookrightarrow W^{k,p}(M \times [\alpha, \omega])$ is an isomorphism, where $W^{k,p}(M \times [\alpha, \omega]/\alpha)$ denotes the completion under $W^{k,p}$-norm of the space of smooth functions vanishing on a neighborhood of $M \times \{\alpha\}$. There is therefore no meaningful notion of trace in this case.

2. If $k > \frac{1}{p} + l,\ l \geq 0$, then the restriction map
$$B: \ C^\infty(M \times [\alpha, \omega]) \longrightarrow C^\infty(M): \ f(x,t) \longmapsto f(x, \alpha)$$
extends to a bounded operator $B: \ W^{k,p}(M \times [\alpha, \omega]) \longrightarrow W^{l,p}(M)$, called *Trace operator*.

We will topologise the space $\partial_\alpha W^{k,p}(M \times [\alpha, \omega])$ of restrictions to time $t = \alpha$ of functions in $W^{k,p}(M \times [\alpha, \omega])$, in case Trace operator is well defined, as cokernel of B, that is, as a quotient space of $W^{k,p}(M \times [\alpha, \omega])$. This makes $\partial_\alpha W^{k,p}(M \times [\alpha, \omega])$ a Banach space with stronger norm than any $W^{l,p}(M)$ for any $l < k - \frac{1}{p}$.

3.1.3 Linear equations on manifolds.

Existence and Regularity.

It can be easily verified that the linear heat operator $AF := \frac{d}{dt}F + \Delta F$ is a parabolic operator and therefore is also an elliptic operator. All of the following results holds for operator A.

Theorem 22 (Regularity for elliptic operator)**.** *Let M be a compact manifold and $AF := \frac{d}{dt}F + \Delta F$ be an elliptic operator of second order. Given $\frac{1}{p} < l < k < \infty$ and $F \in W^{l,p}(M \times [\alpha, \omega])$ and suppose that*

$$AF \in W^{k-2,p}(M \times [\alpha, \omega]), \quad f\Big|_\alpha \in \partial_\alpha W^{k,p}(M \times [\alpha, \omega]), \quad f\Big|_\omega \in \partial_\alpha W^{k,p}(M \times [\alpha, \omega]).$$

Then actually $F \in W^{k,p}(M \times [\alpha, \omega])$*.*

Theorem 23 (Causality of parabolic equation)**.** *Let* M *be a compact manifold and* $AF := \frac{d}{dt}F + \Delta F + a\nabla F + bF$ *be an parabolic operator. Then*

$$A: \; W^{k,p}(M \times [\alpha, \omega]/\alpha) \longrightarrow W^{k-2,p}(M \times [\alpha, \omega]/\alpha)$$

is an isomorphism of Banach spaces.

Theorem 24 (Gårding's Inequality and Regularity for parabolic operator)**.** *Let* M *be a compact manifold,* $p \in (1, +\infty)$*,* $k > l > -\infty$ *and* $AF := \frac{d}{dt}F + \Delta F$ *be a parabolic operator. We write* $W^{k,p}([\beta, \gamma])$ *shortly for* $W^{k,p}(M \times [\beta, \gamma])$*. Suppose that*

$$F \in W^{l,p}([\alpha, \omega]), \quad AF \in W^{k-2,p}([\alpha, \omega]).$$

Then $F \in W^{k,p}([\pi, \omega])$ *for all* $\pi \in (\alpha, \omega)$*. Also, there exists a constant* $C > 0$ *such that*

$$\|F\|_{W^{k,p}([\pi,\omega])} \leq C \left(\|AF\|_{W^{k-2,p}([\alpha,\omega])} + \|F\|_{W^{l,p}([\alpha,\pi])} \right).$$

In particular for homogeneous equation, the solution is C^∞ *and an arbitrarily weak estimate in the past gives an arbitrarily strong estimate in the future.*

Maximum principle and Comparison theorems.

Other than regularity results which are generally true for parabolic operators, the linear heat operator also enjoys the following versions of Maximum principle. See Appendices for their proofs.

Theorem 25 (Maximum principle)**.** *Let* M *be a compact manifold and* $f : M \times [\alpha, \omega] \longrightarrow \mathbb{R}$ *be a continuous function with* $f\Big|_\alpha \leq 0$*. Suppose that whenever* $f > 0$*,* f *is smooth and*

$$\frac{\partial f}{\partial t} \leq -\Delta f + Cf.$$

Then in fact $f \leq 0$*.*

With the same proof as Theorem 25, one also has:

Theorem 26 (L^∞-Comparison theorem)**.** *Let* $f : M \times [\alpha, \omega] \longrightarrow \mathbb{R}$ *be a continuous function on* M*, smooth for all time* $t > 0$ *such that*

$$\frac{df}{dt} = -\Delta f + bf \text{ on } M \times (\alpha, \omega]$$

where b *is a smooth function on* M*. Then there exists a constant* B *depending only on* b *such that*

$$\|f\Big|_\omega\|_{L^\infty} \leq e^{B(\omega-\alpha)} \|f\Big|_\alpha\|_{L^\infty}.$$

Using backwards heat equation and Theorem 26, one can prove its version for L^1.

Theorem 27 (L^1-Comparison theorem)**.** *Let $f : M \times [\alpha, \omega] \longrightarrow \mathbb{R}$ be a continuous function on M, smooth for all time $t > 0$ such that*

$$\frac{df}{dt} = -\Delta f + bf \text{ on } M \times (\alpha, \omega]$$

where b is a smooth function on M. Then there exists a constant B depending only on b such that

$$\|f\big|_{\omega}\|_{L^1} \leq e^{B(\omega-\alpha)} \|f\big|_{\alpha}\|_{L^1}.$$

3.2 Regularity estimate of the quadratic term.

Theorem 28 (Regularity of the quadratic term)**.** *Let $F : M \times [\alpha, \omega] \longrightarrow B \subset \mathbb{R}^N$ be in $W^{s,q}(M \times [\alpha, \omega]) \cap C(M \times [\alpha, \omega])$ and*

$$PF := g^{ij} \Gamma'^{\alpha}_{\beta\gamma}(F) F_i^{\beta} F_j^{\gamma}.$$

Suppose that

$$r \geq 0, \quad p, q \in (1, \infty), \quad r + 1 < s, \quad \frac{1}{p} > \frac{r+2}{s}\frac{1}{q}. \tag{3.1}$$

Then one has $PF \in W^{r,p}(X)$ and

$$\|PF\|_{W^{r,p}} \leq C \left(1 + \|F\|_{W^{s,q}}\right)^{q/p}.$$

where C is a constant independent of F.

Proof. We will suppose here that r, s are even integers so that the $W^{r,p}$ (respectively $W^{s,q}$) norm of PF (respectively F) can be written as sum of L^p (respectively L^q) norms of its derivatives. Also, we will use chain rule freely to differentiate the term $\Gamma'^{\alpha}_{\beta\gamma}(F)$ using weak derivatives of F. The general and rigorous proof, which involves non-integral Sobolev space to treat r, s and a detour to Besov spaces to justify chain rule, can be found in the appendices.

The derivatives of PF that appear in its $W^{r,p}$ norm are of form

$$C(x, F) \prod_i \partial_t^{b_i} \partial_x^{c_i} F^{\beta_i}$$

where $2\sum b_i + \sum c_i \leq r + 2$ and $\max\{2b_i + c_i\} \leq r + 1$ and $C(x, F)$ is bounded on M. Using Multiplication theorem for L^p-spaces, one has

$$\left\| C(x, F) \prod_i \partial_t^{b_i} \partial_x^{c_i} F^{\beta_i} \right\|_{L^p} \leq \|C(x, F)\|_{L^\infty} \prod_i \left\| \partial_t^{b_i} \partial_x^{c_i} F^{\beta_i} \right\|_{L^{p_i}} \leq \|C(x, F)\|_{L^\infty} \prod_i \|F\|_{W^{2b_i + c_i, p_i}}$$

as long as we choose $p_i \in (1, \infty)$ such that $\frac{1}{p} \geq \sum \frac{1}{p_i}$. The strategy is to choose $\frac{1}{p_i}$ big enough to have $W^{s,q} \hookrightarrow W^{2b_i+c_i,p_i}$ in order to bound $\|F\|_{W^{2b_i+c_i,p_i}}$ by $\|F\|_{W^{s,q}}$, then use the upper bound of $2b_i + c_i$ to justify that $\frac{1}{p} > \frac{r+2}{s}\frac{1}{q} \geq \sum \frac{1}{p_i}$, meaning that such choice of p_i are valid.

One straightforward way to have a sufficient condition of p_i such that $W^{s,q} \hookrightarrow W^{2b_i+c_i,p_i}$ is to use Sobolev embeddings. Another way is to use Interpolation inequality, by remarking that $F \in W^{0,v}$ for all $v \in (1, +\infty)$ and writing $W^{2b_i+c_i,p_i}$ as an interpolation space of $W^{s,q}$ and $W^{0,v}$. It can be seen, by direct computation, that the sufficient condition for $W^{s,q} \hookrightarrow W^{2b_i+c_i,p_i}$ is $2b_i + c_i < s$ and

$$0 < \frac{1}{p_i} - \frac{2b_i + c_i}{s}\frac{1}{q} < 1 - \frac{2b_i + c_i}{s}.$$

Choose $\frac{1}{p_i}$ just a bit bigger than $\frac{2b_i+c_i}{s}\frac{1}{q}$, one still has

$$\sum \frac{1}{p_i} \simeq \sum \frac{2b_i + c_i}{s}\frac{1}{q} \leq \frac{r+2}{s}\frac{1}{q} < \frac{1}{p}.$$

The conclusion follows. □

3.3 Regularity for nonlinear heat equation.

Let $p > \dim M + 2$, using the regularity estimate for the quadratic term, we now can prove:

Theorem 29 (Bootstrap for nonlinear heat equation)**.** *Let $F: M \times [\alpha, \omega] \longrightarrow B$ such that $F \in W^{2,p}(M \times [\alpha, \omega])$ and $\frac{dF_t}{dt} = \tau(F_t)$, i.e.*

$$\frac{dF^\alpha}{dt} = -\Delta F^\alpha + g^{ij}\Gamma'^\alpha_{\beta\gamma}(F)F^\beta_i F^\gamma_j$$

and $F\Big|_{M\times\{\alpha\}}$ is smooth. Then F is smooth on $M \times [\alpha, \omega]$.

Remark 7. *Note that since $p > \dim M + 2 = \dim(M \times [\alpha, \omega]) + 1$, if $F \in W^{2,p}(M \times [\alpha, \omega])$ then F and $\frac{\partial F}{\partial x^i}$ are in $C(M \times [\alpha, \omega])$ by Sobolev embeddings. It makes sense then to talk about:*

1. *the restriction and boundary condition at time $t = \alpha$ (in fact, by Trace theorem, $p > 1$ is enough).*
2. *the pointwise condition $F : M \times [\alpha, \omega] \longrightarrow B \subset V$.*

Proof. We define the operators $PF := g^{ij}\Gamma'^\alpha_{\beta\gamma}(F)F^\beta_i F^\gamma_j$ and $AF := \frac{dF}{dt} + \Delta F$. We will abusively denote $W^{k,p}(M \times [\beta, \gamma])$ by $W^{k,p}([\beta, \gamma])$. Our bootstrap scheme consists of 3 steps:

1. Prove that $F \in W^{2,\tilde{p}}([\pi, \omega])$ for every $\pi > \alpha$ and $\tilde{p} \in (1, \infty)$. By compactness of M, it is sufficient to prove this for a sequence $\tilde{p} \to +\infty$.

2. Prove that F is C^∞ for all time $t > \alpha$.

3. Prove that F is C^∞ on $M \times [\alpha, \omega]$.

Step 1. By Theorem 28, $AF = PF \in W^{r,q}([\alpha,\omega])$ whenever $r < 1$ and $\frac{1}{q} > (\frac{r}{2}+1)\frac{1}{p}$. Apply Gårding inequality, for all $\pi > \alpha$, $F \in W^{r+2,q}([\pi,\omega]) \subset W^{2,\tilde{p}}([\pi,\omega])$ for $\frac{1}{\tilde{p}} = \frac{1}{q} - \frac{r}{\dim M+1}$. Choose $\frac{1}{q}$ very close to $(\frac{r}{2}+1)\frac{1}{p}$, one sees that the condition on $\tilde{p}$ is $\frac{1}{\tilde{p}} > (\frac{r}{2}+1)\frac{1}{p} - \frac{r}{p-1}$, which will be satisfied if $\frac{1}{\tilde{p}} > (1-\frac{r}{2})\frac{1}{p}$, i.e. for all $\tilde{p} < \frac{p}{1-r/2}$. It remains to repeat this result to finish the first step. We will say $F \in W^{2,*}([\pi,\omega])$ for $F \in W^{2,p}([\pi,\omega])$ for all $p \in (1,\infty)$.

Step 2. By Theorem 28, for all $r < 1$, one has $AF = PF \in W^{r,*}([\pi,\omega])$, therefore by Gårding inequality, $F \in W^{r+2,*}([\pi,\omega])$. Iterate this result and one has $F \in W^{k,*}([\pi,\omega])$ for all $k \in [2,\infty)$ and $\pi > \alpha$. So F is smooth for $t > \alpha$.

Step 3. We apply regularity result (Theorem 22) for elliptic operator A and boundary operators $B^0 : F \mapsto F\big|_{M\times\{\alpha\}}$ and $B^1 : F \mapsto F\big|_{M\times\{\omega\}}$: For q, r in Step 1, one has $AF = PF \in W^{r,q}([\alpha,\omega])$ and $B^j F \in \partial W^{r,q}$, therefore $F \in W^{r+2,q}([\alpha,\omega]) \subset W^{2,\tilde{p}}([\alpha,\omega])$ for the same $\tilde{p}$ as Step 1. This proves that $F \in W^{2,*}([\alpha,\omega])$, which also means that one has $F \in W^{r+2,q}([\alpha,\omega])$ with no additional condition on q except $q \in (1,\infty)$. Iterate and one obtains the regularity of F on $[\alpha,\omega]$. $\square$

Remark 8. *The first 2 steps were to prove the regularity of $F\big|_{M\times\{\omega\}}$, which was then used as a boundary condition in order to apply regularity result for elliptic operator on manifold with boundary.*

3.4 Short-time existence for nonlinear heat equation.

We will choose as always $p > \dim M + 2$. As before, M is a compact Riemannian manifold and $B \subset \mathbb{R}^N$ is a large Euclidean ball.

Theorem 30 (Short-time existence)**.** *Let $F_\alpha : M \longrightarrow B$ be a smooth map, then there exist $\epsilon > 0$ depending on F_α and $F : M \times [\alpha, \alpha+\epsilon] \longrightarrow B$ such that $F \in W^{2,p}(M \times [\alpha,\alpha+\epsilon])$ with $F\big|_{M\times\{\alpha\}} = F_\alpha$ and*

$$\frac{dF_t}{dt} = \tau(F_t) \quad \text{on } M \times [\alpha, \alpha+\epsilon]$$

Proof. We find F as a sum $F = F_b + F_\#$ where $F_b \in C^\infty(M \times [\alpha,\omega])$ satisfies the initial condition and $F_\# \in W^{2,p}(M \times [\alpha,\alpha+\epsilon]/\alpha)$.

The nonlinear heat operator can be written as:

$$\begin{aligned} T : W^{2,p}(M \times [\alpha,\omega]/\alpha)^{\oplus N} &\longrightarrow L^p(M \times [\alpha,\omega])^{\oplus N} \\ F_\# &\longmapsto \tau(F_b + F_\#) \end{aligned}$$

where $\tau(F)^\alpha = -\Delta F^\alpha + g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^\beta F_j^\gamma$, which can be rewritten as $\tau(F) = -\Delta F + \Gamma(F)(\nabla F)^2$. The derivative of T at $F_\#$ in direction $k \in W^{2,p}(M \times [\alpha,\omega]/\alpha)^{\oplus N}$ is

$$DT(F_\#)k = -\Delta k + D\Gamma(F)\cdot k.(\nabla F)^2 + 2\Gamma(F)\nabla F.\nabla k,$$

or in local coordinates:

$$DT(F_\#)^\alpha = g^{ij}\left(\frac{\partial^2 k^\alpha}{\partial x^i \partial x^j} - \Gamma^l_{ij}k^\alpha_l\right) + g^{ij}\frac{\partial \Gamma'^{\alpha}_{\beta\gamma}}{\partial y^\delta}k^\delta F_i^\beta F_j^\gamma + 2g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^\beta F_j^\gamma$$

which is of form $DT(F_\#)k = -\Delta k - a(x,F)\nabla k - b(x,F)k$ where a, b are smooth.

Therefore if we note

$$\begin{aligned} H : W^{2,p}(M \times [\alpha,\omega]/\alpha)^{\oplus N} &\longrightarrow L^p(M \times [\alpha,\omega])^{\oplus N} \\ F_\# &\longmapsto (\frac{d}{dt} - \tau)(F_b + F_\#) \end{aligned}$$

then the derivative of H at $F_\# = 0$ is

$$DH(0)\cdot k = \frac{dk}{dt} + \Delta k + a(x,F_b)\nabla k + b(x,F_b)k$$

which by Theorem 23 is an isomorphism from $W^{2,p}(M\times[\alpha,\omega]/\alpha)^{\oplus N}$ to $W^{0,p}(M\times[\alpha,\omega]/\alpha)^{\oplus N} = L^p(M \times [\alpha,\omega])^{\oplus N}$. This shows that H is a local isomorphism mapping a neighborhood of 0 to a neighborhood of $(\frac{d}{dt} - \tau)F_b$.

Define $g_\epsilon \in L^p(M \times [\alpha,\omega])^{\oplus N}$ by

$$g_\epsilon := \begin{cases} 0, & \text{if } t \in [\alpha, \alpha+\epsilon] \\ (\frac{d}{dt} - \tau)F_b, & \text{if } t > \alpha + \epsilon \end{cases}$$

which is arbitrarily $L^p(M \times [\alpha,\omega])$-close to $(\frac{d}{dt} - \tau)F_b$ for $0 < \epsilon \ll 1$. There exists therefore $F_\# \in W^{2,p}(M \times [\alpha,\omega]/\alpha)^{\oplus N}$ such that $H(F_\#) = g_\epsilon$, meaning that the function $F = F_b + F_\# : M \longrightarrow V$ satisfies $F\Big|_{M\times\{\alpha\}} = F_\alpha$ and $\frac{dF}{dt} - \tau(F_t) = 0$ for $t \in [\alpha, \alpha+\epsilon]$.

By Regularity Theorem 29, F is C^∞ for $t \in [\alpha, \alpha+\epsilon]$. Theorem 12 assures that the image of F is in M, hence in M' for $t \in [\alpha, \alpha+\epsilon]$. □

Chapter 4

Global existence for nonlinear heat equation and harmonic maps between Riemannian manifolds

Let M be a compact Riemannian manifold. We want to solve the following nonlinear heat equation where $F : M \longrightarrow M' \subset B \subset V = \mathbb{R}^N$:

$$\frac{dF_t}{dt} = -\Delta F_t + \Gamma(F_t)(\nabla F_t)^2$$

We have proved that the solution exists in short-time and is smooth whenever it exists. We will now establish long-time existence using continuity method: we will show that if the solution exists on $[\alpha, \omega_n]$ where ω_n is an increasing sequence to ω, then the solution exists on $[\alpha, \omega]$. We then apply short-time existence to gain a small open interval where solution still exists. We then conclude that the solution exists globally on $[\alpha, +\infty)$ since this interval is connected.

The crucial step to prove that the solution can be extended on $[\alpha, \omega]$ is to uniformly bound all of its derivatives in time of evolution $[\alpha, \omega)$. These estimates will also be useful to justify the convergence of F_t in $C^\infty(M)$ to a smooth function F_∞ which will eventually be a harmonic map from M to M'.

Recall that we proved in Corollary 18.1, under the hypothesis of negative curvature, the boundedness of $\|F_t\|_{W^{2,2}(M)}$ by a constant C depending only on curvatures of M, M' and the initial total energies. Since $\frac{dF_t}{dt}$ relates to spatial derivatives of F by the nonlinear heat equation, it is easy to see that $\|F_t\|_{W^{2,2}(M\times[\tau,\tau+\delta])}$ is bounded by a constant independent of τ. We will denote $W^{k,p}(M \times [\beta, \gamma])$ by $W^{k,p}([\beta, \gamma])$.

Theorem 31 ($W^{2,2}$-boundedness)**.** *Suppose* $\mathrm{Riem}(M') \leq 0$. *There exists a constant C depending only on δ, the metrics and initial total energies such that*

$$\|F\|_{W^{2,2}(\tau,\tau+\delta)} \leq C \quad \textit{for all } \alpha \leq \tau < \omega - \delta.$$

Proof. Since

$$\|F\|^2_{W^{2,2}([\tau,\tau+\delta])} \leq \int_\tau^{\tau+\delta} \|F_t\|^2_{W^{2,2}(M)} dt + 2\int_\tau^{\tau+\delta} \|\Delta F_t\|^2_{L^2} dt + 2\int_\tau^{\tau+\delta} \left\|\Gamma(F_t)(\nabla F_t)^2\right\|^2_{L^2} dt$$

The first term and the second term are bounded by $C^2\delta$, the third one, since $\Gamma(F_t)$ is bounded, by $C^2\delta$ where C is a constant only depending on the metrics and initial total energies. □

The estimates of higher derivatives of F will be established in the same strategy as the bootstrap: first in $W^{2,p}$ for all p then in $W^{k,p}$ for all k, p, then in C^∞.

4.1 Estimate of higher derivatives.

Lemma 32 ($W^{2,p}$-boundedness)**.** *Suppose* $\mathrm{Riem}(M') \leq 0$. *For all* $p \in (1, +\infty)$*, there exists a constant* $C > 0$ *depending only on* δ*,* p*, the metrics and initial energies such that for all* $\alpha + \delta \leq \tau \leq \omega - \delta$*:*

$$\|F\|_{W^{2,p}([\tau,\tau+\delta])} \leq C$$

Proof. Applying Gårding Inequality to the parabolic equation $AF = \Gamma(F)(\nabla F)^2$ where $A := \frac{\partial}{\partial t} + \Delta$ is the heat operator, one has

$$\|F\|_{W^{2,p}([\tau,\tau+\delta])} \leq C\left(\|\Gamma(F)(\nabla F)^2\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} + \|F\|_{W^{2,2}([\tau-\frac{\delta}{3},\tau+\delta])}\right)$$

The second term of RHS is already bounded by applying Theorem 31 to $\frac{4\delta}{3}$. For the first term:

$$\left\|\Gamma(F)(\nabla F)^2\right\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \leq C(M')\||\nabla F|^2\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} = C(M')\|e(F)\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])}.$$

Recall that by Theorem 14, the potential density satisfies $\frac{de}{dt} + \Delta e - Ce \leq 0$ for certain constant C depending only on the metric of M. By Maximum principle (Theorem 25), one has $e \leq \psi_\tau$ where ψ_τ is the solution of

$$\begin{cases} \frac{d}{dt}\psi_\tau + \Delta\psi_\tau - C\psi_\tau = 0 \\ \psi_\tau\Big|_{\tau-\frac{\delta}{2}} = e\Big|_{\tau-\frac{\delta}{2}} \end{cases}$$

We apply Gårding Inequality again for ψ_τ and obtain

$$\|e(F)\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \leq \|\psi_\tau\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \leq C\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2},\tau+\delta])}. \tag{4.1}$$

Now apply L^1-Comparison Theorem 27 to ψ_τ, one has

$$\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2},\tau+\delta])} \leq \int_0^{3\delta/2} \|\psi_\tau\Big|_{\tau-\frac{\delta}{2}}\|_{L^1} e^{Bt} dt \leq \int_0^{3\delta/2} e^{Bt} dt. \|e\Big|_{\tau-\frac{\delta}{2}}\|_{L^1} \leq C. \tag{4.2}$$

The lemma follows from (4.1) and (4.2). □

We can now estimate higher order derivatives.

Theorem 33 ($W^{k,p}$-boundedness). *Suppose* $\mathrm{Riem}(M') \leq 0$. *For all* $p \in (1,+\infty)$ *and* $k < +\infty$, *there exists* C *depending only on* k, p, *the metrics and initial energies such that*

$$\|F\|_{W^{k,p}([\tau,\tau+\delta])} \leq C$$

for all $\alpha + \delta \leq \tau \leq \omega - \delta$.

Proof. Applying Gårding Inequality to the equation $\frac{dF}{dt} + \Delta F_t = \Gamma(F)(\nabla F)^2$ then Regularity estimate for the quadratic term (Theorem 28), one has for $\epsilon \ll \delta$:

$$\begin{aligned}\|F\|_{W^{k,p}([\tau,\tau+\delta])} &\leq C_\epsilon \left(\|F\|_{W^{2,p}([\tau-\epsilon,\tau+\delta])} + \|\Gamma(F)(\nabla F)^2\|_{W^{k-2,p}([\tau-\epsilon,\tau+\delta])}\right) \\ &\leq C_\epsilon \left(1 + C\left(1 + \|F\|_{W^{s,q}([\tau-\epsilon,\tau+\delta])}\right)^{q/p}\right)\end{aligned}$$

as long as $k-1 < s$ and $\frac{1}{p} > \frac{k}{s}.\frac{1}{q}$. Therefore if $\|F\|_{W^{s,q}([\tau,\tau+\delta])} \leq C(\delta,s,q)$ for all $\beta \leq \tau \leq \omega - \delta$ and $q \in (1,+\infty)$, we just proved that

$$\|F\|_{W^{k,p}([\tau,\tau+\delta])} \leq C(\epsilon,k,p)$$

for all $\begin{cases} \beta + \epsilon \leq \tau \leq \omega - \delta \\ k < s+1, p \in (1,+\infty) \end{cases}$ since $\|F\|_{W^{s,q}([\tau-\epsilon,\tau+\delta])} \leq 2C(\delta,s,q)$.

One can then conclude by induction on k, with step $\frac{1}{2}$, starting with $k = 2$ and $\epsilon = \frac{\delta}{2}$ divided by 2 after each induction step. □

4.2 Global existence for nonlinear heat equation.

Theorem 34 (Global existence). *Suppose* $\mathrm{Riem}(M') \leq 0$. *The solution of nonlinear heat equation*

$$\frac{dF}{dt} = -\Delta F + \Gamma(F)(\nabla F)^2 \tag{4.3}$$

with smooth initial condition exists globally for all time $t > \alpha$.

Proof. Let F_n be a sequence of solution of (4.3) on $[\alpha,\omega_n]$ with ω_n increasing to ω then they coincide by uniqueness of the solution. As discussed in the beginning of this part, it is sufficient to prove that the solution extends to $[\alpha,\omega]$. Let F be the solution on $[\alpha,\omega)$ such that $F\Big|_{[\alpha,\omega_n]} = F_n$, then by Theorem 33, for all $\tau \in [\alpha, \omega-\delta)$:

$$\|D_t^u D_x^v F\|_{L^\infty(M\times[\tau,\tau+\delta])} \leq C_{\mathrm{Sobolev}} \|D_t^u D_x^v F\|_{W^{k,p}(M\times[\tau,\tau+\delta])} \leq C_{\mathrm{Sobolev}}.C(k,p,\delta)$$

where, if we choose k sufficiently large, C_{Sobolev} is the constant of Sobolev imbedding $W^{k,p}(M\times[0,\delta]) \hookrightarrow C(M\times[0,\delta])$ and $C(k,p,\delta)$ is the constant provided by Theorem 33.

So all partial derivatives of F are uniformly bounded on $[\alpha, \omega)$. This proves that F extends to a solution on $[\alpha, \omega]$. In fact $F\big|_\tau := F\big|_{M\times\{\tau\}}$ converges in $C^\infty(M)$ as $\tau \to \omega$, since

$$\|D^\alpha F\big|_\tau - D^\alpha F\big|_{\tau'}\|_{L^\infty} \leq \max_{\|\beta\|=\|\alpha\|+1} \|D^\beta F\|_{L^\infty}|\tau - \tau'|.$$

□

We have just proved the first part of the following theorem.

Theorem 35 (Eells-Sampson)**.** *1. Let M, M' be compact Riemannian manifolds with* $\mathrm{Riem}(M') \leq 0$*. Then for every smooth map $f_0 : M \longrightarrow M' \subset B \subset \mathbb{R}^N$, the nonlinear heat equation*

$$\begin{cases} \frac{df_t}{dt} = \tau(f_t), & \textit{for all } t \geq 0 \\ f\big|_{t=0} = f_0, \end{cases}$$

admits a globally defined smooth solution f_t. Moreover, all derivatives $D^\alpha f_t$ remain bounded as $t \to +\infty$.

2. For a suitable sequence $\{t_n\}$ increasing to $+\infty$ the sequence $\{f_{t_n}\}$ converges in $C^\infty(M)$ to a function f_∞ with $\tau(f_\infty) = 0$. Therefore any map $f_0 : M \longrightarrow M'$ is homotopic to a harmonic map.

Proof. For any sequence $\{t_n\}$, one can extract from $\{f_{t_n}\}$, since their derivatives are uniformly bounded, a subsequence $\{f_{t_{n_i}}\}$ converging in $C^k(M, \mathbb{R}^N)$. By a diagonal argument, one can extract from any sequence $\{f_{t_n}\}$ a subsequence converging in $C^\infty(M, \mathbb{R}^N)$ to f_∞. Abusively denote this subsequence by $\{f_{t_n}\}$, by Theorem 15

$$\lim_{n\to\infty} K(f_{t_n}) = \lim_{n\to\infty} \int_M |\tau(f_{t_n})|^2 = 0$$

Therefore $\tau(f_{t_n}) \to 0$ in $L^2(M)^{\oplus N}$. But also $\tau(f_{t_n}) \to \tau(f_\infty)$ in $C^\infty(M, \mathbb{R}^N)$, one has $\tau(f_\infty) = 0$. □

Part III

Existence using Morse-Palais-Smale theory

Chapter 5

Minimal immersions of $\mathbb{S}^2$

5.1 Brief view of Sacks and Uhlenbeck's strategy.

Let M and N be compact Riemannian manifolds (without boundary), M is a surface and N is isometrically embedded in $\mathbb{R}^k$. It was showed by Eells and Sampson [ES64] that if N is negatively curved than any map from M to N is homotopic to a harmonic map. The idea of Sacks and Uhlenbeck in [SU81] consists of (1) approximating the energy functional E by a family E_α satisfying Palais-Smale condition, whose *nontrivial* critical values can be more easily proved to exist and (2) trying to prove that the critical maps s_α of E_α converge in C^1-topology.

We will first review the general machinery of Morse-Palais-Smale theory and prove the existence of s_α. The convergence of s_α in the case of surface is due to the facts that energy functional E is a conformal invariant of M, in particular E is invariant by homotheties (i.e. E remains unchanged when we zoom in and out), which allows us to justify the C^1-convergence (under conditions of N) except at finitely many points using a local estimate and a suitable covering of M.

Sacks and Uhlenbeck used an extension result for harmonic map, in an elegant argument to prove that if the above sequence $\{s_\alpha\}$ fails to converge at a point, for a certain surface M, then one has a nontrivial harmonic map from $\mathbb{S}^2$ to N. Therefore if such sequence $\{s_\alpha\}$ from $\mathbb{S}^2$ to N exists, for example when $\pi_k(N)$ is nontrivial for a certain $k \geq 2$ then, whether s_α converges or not, there exists a nontrivial harmonic map from $\mathbb{S}^2$ to N.

Finally, the theory of branched immersion of surfaces by Gulliver-Osserman-Royden [GOR73] can be applied to show that the harmonic map obtained this way is a conformal, branched, minimal immersion of $\mathbb{S}^2$ to N.

5.2 General machinery by Morse-Palais-Smale.

5.2.1 Perturbed functionals E_α.

Let $s : M \longrightarrow N \hookrightarrow \mathbb{R}^k$ be a map from a compact surface M to a compact Riemannian manifold N isometrically embedded into $\mathbb{R}^k$. Recall that the energy functional of s is given by $E(s) := \frac{1}{2}\int_M |ds|^2 dV_M = \frac{1}{2}\int_M \langle s^* g_N, g_M\rangle dV_M$. The perturbed energy functionals are

$$E_\alpha(s) := \int_M \left(1 + |ds|^2\right)^\alpha dV, \quad \alpha \geq 1$$

We will suppose, by rescaling the metric g_M of M that the volume of M is 1, so when $\alpha = 1$, $E_1 = 1 + 2E(s)$ is just the previously defined energy. Using $(a+b)^\alpha \geq a^\alpha + b^\alpha$ and Jensen's inequality, one has $E_\alpha(s) \geq 1 + (2E(s))^\alpha$ for all $\alpha \geq 1$. Also, since we only interest in the case α close to 1, let us also suppose that α from now on is smaller than 2.

By Sobolev embedding, one has $W^{1,2\alpha}(M, \mathbb{R}^k) \subset C^0(M, \mathbb{R}^k)$ compactly for all $\alpha > 1$. It then makes sense to talk about $W^{1,2\alpha}(M, N) \subset C^0(M, N)$ which consist of elements of $W^{1,2\alpha}(M, \mathbb{R}^k) \subset C^0(M, \mathbb{R}^k)$ whose image lies in N.

Theorem 36 (Palais). *The spaces $C^\infty(M, N) \subset W^{1,2\alpha}(M, N) \subset C^0(M, N)$, where $\alpha > 1$, are of the same homotopy type and the inclusions are homotopy equivalences. In particular, their connected components are naturally in bijection.*

We will also need a version of Morse theory for Banach manifolds, also developed by R. Palais in [Pal66]. For the terminologies, in the same way that a manifold is modeled by$\mathbb{R}^n$, a *Banach manifold* is modeled by Banach spaces. A *Finsler manifold* is a Banach manifold with a norm on its tangent space that is comparable with the norm of Banach charts.

Theorem 37 (Morse theory for Banach manifolds).

1. *If F is a C^2 functional on a complete C^2 Finsler manifold L, F is bounded below and F satisfies Palais-Smale condition (C) then*
   (a) *The functional F admits minimum on each connected component of L.*
   (b) *If F has no critical value in $[a, b]$ then the sublevel $\{F \leq b\}$ retracts by deformation to the sublevel $\{F \leq a\}$.*
2. *The pair $(L, F) = (W^{1,2\alpha}(M, N), E_\alpha)$ with $\alpha > 1$ satisfies the condition of the first part.*

The *Palais-Smale condition* is as follows:

(C): Let $S \subset L$ be a subset on which $|F|$ is bounded, but $|dF|$ is not bounded away from 0. Then there exists a critical point of F in $\bar{S}$.

The strategy to prove Theorem 37 is, as in finite dimensional case, to use a pseudo-gradient flow of F whose existence is due to a partition of unity of L (instead of a Riemannian metric on L). The role of Palais-Smale condition in the proof is as follows: Suppose that $\{x_n\}$ is a sequence in a connected component L_1 of L such that $F(x_n)$ tends to $\inf_{L_1} F$, then using the pseudo-gradient flow of F, we can suppose that $|dF(x_n)|$ is arbitrarily small, in particular, we can suppose that $|dF(x_n)| \to 0$. Choose a sequence $\{y_n\}$ of regular points near x_n such that $F(y_n) \to \inf_{L_1} F$ and $|dF(y_n)| \to 0$ and use (C) for $S = \{y_n\}$, one obtains a limit point y_∞ of $\{y_n\}$, hence also of $\{x_n\}$, which minimises F.

As a consequence of Theorem 37, one has:

Corollary 37.1 (Component-wise minimum of E_α)**.** *The minimum of E_α in each connected component C of $W^{1,2\alpha}(M,N)$, $\alpha > 1$ is taken by some $s_\alpha \in C^\infty(M,N)$ and there exists $B > 0$ depending on the component C such that*

$$\min_C E_\alpha \leq (1 + B^2)^\alpha$$

Proof. By Theorem 37, E_α admits minimum at s_α on each component C of $W^{1,2\alpha}(M,N)$. By writing down the Euler-Lagrange equation of E_α and apply regularity estimates, one can prove that s_α is actually smooth. By Theorem 36, the preimage of C by inclusion $C^\infty(M,N) \subset W^{1,2\alpha}(M,N)$ is a connected component C' of $C^\infty(M,N)$ over which s_α is the minimum of E_α. Take $B = \sup_M |du|$ for an arbitrary element $u \in C'$ and the conclusion follows. ☐

Remark 9. *Corollary 37.1 is trivialised when $W^{1,2\alpha}(M,N)$ is connected (for one α or equivalently for all α). In this case, s_α is a constant map and $B = 0$.*

To establish a nontrivial analog of Corollary 37.1 in the case where the spaces of maps from M to N are connected, we will have to look at the submanifold $N_0 \cong N$ formed by constant maps.

5.2.2 Tubular neighborhood of the submanifold of trivial maps.

Fix $y \in N$, considered as a constant maps in N_0. We will summarise a few facts about the tangent space of $W^{1,2\alpha}(M,N)$ at y in the following Remark.

These facts come from the *differential structure* of the Banach manifold $W^{1,2\alpha}(M,N)$ that so far has not been introduced, since we only consider $W^{1,2\alpha}(M,N)$ as a closed subset of $W^{1,2\alpha}(M,\mathbb{R})^{\oplus k}$ (so only a topological structure was given). We summarise here, and refer to [Pal68], how a differential structure is given to $W^{k,p}(M,N)$ with k,p such that $W^{k,p}(M) \hookrightarrow C^0(M)$:

- Let ξ be a finite dimensional vector bundle over a compact manifold M, then $W^{k,p}(\xi,M)$ can be defined as the Banach space of sections of ξ that are locally $W^{k,p}$. A norm of $W^{k,p}(\xi,M)$ can be given using a metric of ξ and a volume form of M, but by compactness of M, its equivalent class is independent of such choices.

- Let E be a fiber bundle over M, in our case, $E = N \times M$, and $s \in C^0(E)$ be a continuous section. It can be proved that there exists an open subset ξ of E containing s such that $\xi \to M$ has a vector bundle structure. We say that $s \in W^{k,p}(E, M)$ if $s \in W^{k,p}(\xi, M)$ and it turns out that this definition is independent of the choice of ξ. This defines $W^{k,p}(E, M)$ set-theoretically.

- The differential structure of $W^{k,p}(E, M)$ is given by the atlas $W^{k,p}(\xi, M)$.

Remark 10. *1. The tangent $T_y W^{1,2\alpha}(M, N)$ can be identified with $W^{1,2\alpha}(M, T_y N)$. The subspace $T_y N_0$ contains constant maps from M to $T_y N$.*

2. The fiber $\mathcal{N}_y$ over y of the normal bundle $\mathcal{N}$ of N_0 can be identified with

$$\mathcal{N}_y = \left\{ v \in W^{1,\alpha}(M, T_y N) : \int_M v dV = 0 \right\}$$

The exponential map on $TW^{1,2\alpha}(M, N)$ can be defined as follows:

$$\begin{aligned} e : \ TW^{1,2\alpha}(M, N) &\longrightarrow W^{1,2\alpha}(M, N) \\ (s, v) &\longmapsto \left(x \mapsto \exp_{s(x)} v(x) \right) \end{aligned}$$

where $s \in W^{1,2\alpha}(M, N)$ and $v \in T_s W^{1,2\alpha}(M, N)$ is a $W^{1,2\alpha}$ vector field along $s(x)$. With the representation of normal bundle $\mathcal{N}$ as Remark 10, the restriction of e on $\mathcal{N}$ is given by

$$\begin{aligned} e\big|_{\mathcal{N}} : \ \mathcal{N} &\longrightarrow W^{1,2\alpha}(M, N) \\ (y, v) &\longmapsto \left(x \mapsto \exp_y(v(x)) \right) \end{aligned}$$

where $y \in N_0 \cong N$ and $v \in W^{1,2\alpha}(M, T_y N)$.

Lemma 38. *The restriction $e\big|_{\mathcal{N}}$ of e on $\mathcal{N}$ is a local diffeomorphism mapping a neighborhood of the zero-section of $\mathcal{N}$ onto a neighborhood of N_0 in $W^{1,2\alpha}(M, N)$.*

Proof. It can be calculated that

$$de_{(y,0)}(a, v) = (x \mapsto a + v(x)) \in T_y W^{1,2\alpha}(M, N) = W^{1,2\alpha}(M, T_y N)$$

for $a \in T_y N$ and $v \in \mathcal{N}_y \subset W^{1,2\alpha}(M, T_y N)$. It is invertible since a is tangential to N_0 and $v \in \mathcal{N}_y$ is in the normal component. The Inverse function theorem applies. $\square$

5.2.3 Critical values of E_α.

The exponential map previously defined on the normal bundle of N_0 in $W^{1,2\alpha}(M, N)$ allows us to retract by deformation a small neighborhood of N_0 to N_0. We will prove that if the energy $E_\alpha(s)$ is sufficiently close to $1 = E_\alpha(N_0)$ then s is sufficiently $W^{1,2\alpha}$-close to N_0 and hence can be retracted to N_0, in other words, $E_\alpha^{-1}[1, 1+\delta]$ retracts by deformation to $N_0 = E_\alpha^{-1}(1)$.

Proposition 39. *Given $\alpha > 1$, there exists $\delta > 0$ depending on α such that $E_\alpha^{-1}[1, 1+\delta]$ retracts by deformation to $E_\alpha^{-1}(1) = N_0$.*

Proof. Let $s \in E_\alpha^{-1}[1, 1+\delta]$, using $(a+b)^\alpha \geq a^\alpha + b^\alpha$, one has

$$1 + \delta > \int_M (1 + |ds|^2)^\alpha dV > 1 + \int_M |ds|^{2\alpha} dV$$

therefore $\|ds\|_{L^{2\alpha}} \leq \delta^{1/2\alpha}$. By Poincaré-Wirtinger inequality, $\|s - \fint_M s\|_{W^{1,2\alpha}} \leq C\delta^{1/4}$ where C is the Poincaré-Wirtinger constant.

By Sobolev embedding, $\max_M |s - \fint_M s| \leq C_\alpha \|s - \fint_M s\|_{W^{1,2\alpha}}$ where the Sobolev constant C_α can no longer be chosen uniformly in $\alpha \to 1$. Fix an $x_0 \in M$, one has

$$d_{W^{1,2\alpha}}(s, N_0) \leq \|s - s(x_0)\|_{W^{1,2\alpha}} \leq \left\| s - \fint_M s \right\|_{W^{1,2\alpha}} + \left| \fint_M s - s(x_0) \right| \leq C_\alpha \delta^{1/4}$$

Now choose $\delta \ll 1$ depending on α such that s is in the neighborhood of N_0 given by Lemma 38, s can be written as

$$s(x) = e(y, v(x)) = \exp_y v(x)$$

where $y \in N_0$ and $v \in W^{1,2\alpha}(M, T_yN)$ depend continuously on $s \in W^{1,2\alpha}(M, N)$. We can define the deformation retraction by

$$\begin{aligned} \sigma : \ E_\alpha^{-1}[1, 1+\delta] \times [0,1] &\longrightarrow E_\alpha^{-1}[1, 1+\delta] \\ (s, t) &\longmapsto \big(x \mapsto \exp_y tv(x) \big) \end{aligned}$$

It is clear that σ is continuous and σ_0 is a retraction. The only thing to check is that the image of σ remains in $E_\alpha^{-1}[1, 1+\delta]$ at all time. This can be checked by showing that $\frac{d}{dt} E_\alpha(\sigma_t) \geq 0$, hence $E_\alpha(\sigma_t) \leq E_\alpha(\sigma_1) \leq 1 + \delta$ for all $0 \leq t \leq 1$. $\square$

We will now prove the existence of nontrivial critical value of E_α in an interval $(1, B)$ for a certain $B > 1$ sufficiently big independently of $\alpha > 1$.

Fix $z_0 \in M$ and consider the map

$$\begin{aligned} p : \ C^0(M, N) &\longrightarrow N \\ s &\longmapsto f(z_0) \end{aligned}$$

then p is a fiber bundle and therefore is a *Serre fibration*. In fact fix $q_0 \in N$ then for all $q \in N$ near q_0, there is a vector field v_q supported in a small ball centered at q_0 such that the flow of v_q from time 0 to 1 turns q_0 to q, i.e. $\Phi_{v_q}{}_0^1(q_0) = q$, and that v_q varies continuously in q. Then any fiber $p^{-1}(q)$ can be identified with $p^{-1}(q_0)$ using the flow of v_q. We will denote by $\Omega(M, N)$ the topological fiber of p.

We will use a few facts from algebraic topology, briefly summarised here.

Fact 1. *1. (Long exact sequence of homotopy) Let $p : E \longrightarrow B$ be a fiber bundle of fiber $F = p^{-1}(b_0) \ni f_0$, then one has the following long exact sequence*

$$\dots \xrightarrow{\partial} \pi_n(F) \xrightarrow{\iota_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \dots \longrightarrow \pi_0(E) \longrightarrow 0$$

where $\iota : F \longrightarrow E$ is the inclusion.

2. If p admits a global section s, then one has a retraction s_* of p_*:

$$\begin{tikzcd} \pi_n(E) \arrow[r, shift left, "p_*"] & \pi_n(B) \arrow[l, shift left, "s_*"] \end{tikzcd}$$

hence p_* is surjective and ∂ factors through 0, which gives us the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_n(F) \arrow[r, "\iota_*"] & \pi_n(E) \arrow[r, shift left, "p_*"] & \pi_n(B) \arrow[l, shift left, "s_*"] \arrow[r] & 0 \end{tikzcd}$$

where p_* admits a retraction s_*, so the short exact sequence splits and we have

$$\pi_n(E) \cong \pi_n(F) \oplus \pi_n(B).$$

Now apply this result to the fiber bundle $p : C^0(M,N) \longrightarrow N$ of fiber $\Omega(M,N)$, which has N_0 as a global section, one obtains

$$\pi_n(C^0(M,N)) \cong \pi_n(N) \oplus \pi_n(\Omega(M,N)).$$

Theorem 40 (Nontrivial critical value of E_α)**.** *If $C^0(M,N)$ is not connected, or if $\Omega(M,N)$ is not contractible, then there exists $B > 0$ such that for all $\alpha > 1$, E_α has critical values in the interval $(1, (1+B^2)^\alpha)$.*

In particular, if $M = \mathbb{S}^2$ and if the universal covering $\tilde{N}$ of N is not contractible then E_α has critical values in $(1, (1+B^2)^\alpha)$.

Proof. If $C^0(M,N)$ is not connected, one only needs to apply Corollary 37.1 to a connected component of $W^{1,2\alpha}(M,N)$ not containing N_0. We now suppose that $C^0(M,N)$ is connected and $\Omega(M,N)$ is not contractible.

In this case, there exists $n > 0$ such that $\pi_n(\Omega(M,N))$ is nontrivial and contains a nonzero element $\gamma : \mathbb{S}^n \longrightarrow \Omega(M,N)$ which is not homotopic to any $\tilde{\gamma} : \mathbb{S}^n \longrightarrow N_0$ in $\pi_n(C^0(M,N))$.

Choose $B := \max_{\theta \in \mathbb{S}^n, x \in M} |d\gamma(\theta)(x)|$ then by definition

$$E_\alpha(\gamma(\theta)) \leq (1+B^2)^\alpha \quad \forall \theta \in \mathbb{S}^n, \alpha > 1.$$

If E_α has no critical value in $[1 + \frac{\delta_\alpha}{2}, (1+B^2)^\alpha]$ where δ_α is given by Proposition 39, then by Theorem 37, $E_\alpha^{-1}[1, (1+B^2)^\alpha]$ retracts by deformation to $E_\alpha^{-1}[1, 1+\delta_\alpha]$ which retracts by deformation to $E_\alpha^{-1}(1) = N_0$. But this means that γ is homotopic to a certain $\tilde{\gamma} \in \pi_n(N)$, which is a contradiction.

As an application, if $M = \mathbb{S}^2$ and the universal covering $\tilde{N}$ is not contractible then the long exact sequence of homotopy for the bundle $\tilde{N} \longrightarrow N$ with fiber of dimension 0, gives

$$\pi_n(\tilde{N}) = \pi_n(N), \quad \forall n \geq 2.$$

Since $\tilde{N}$ is simply-connected and not contractible, there exists $n \geq 2$ such that $0 \neq \pi_n(\tilde{N}) = \pi_n(N) = \pi_{n-2}(\Omega(\mathbb{S}^2, N))$, where the last equality follows from definition of homotopy group. The general argument applies. □

5.3 Local results: Estimates and extension.

We will say that the map $s : M \longrightarrow N$ is a critical point of E_α on a small disc $D(R) \subset M$ if s satisfies the Euler-Lagrange equation of E_α (as functional on $W^{1,2\alpha}(M, N)$) on $D(R)$.

Remark 11. *Rescaling $(D(R), g_M)$, where $R \ll 1$ and g_M is ϵ-close to the Euclidean metric, to the unit disc D one obtains a metric $\tilde{g}_M$ that is still ϵ-close to Euclidean metric. The curvature of $\tilde{g}_M$ is R^2 times smaller than that of g_M.*

If $s : D(R) \longrightarrow N$ is a critical map of E_α on $D(R)$, then the composition $\tilde{s}$ of s and the rescaling operator $D \longrightarrow D(R)$ satisfies the Euler-Lagrange equation of $\tilde{E}_\alpha = R^{2(1-\alpha)} \int_D (R^2 + |d\tilde{s}|^2)^\alpha d\tilde{V}$ where $d\tilde{V}$ is the volume form of the rescaled metric $\tilde{g}_M$. We will abusively use the same notation for $\tilde{s}$ and s and regard s as a map on the unit disc D.

Lemma 41 (Sacks-Uhlenback's Main estimate)**.** *For all $p \in (1, +\infty)$, there exists $\epsilon > 0$ and $\alpha_0 > 1$ depending on p such that if*

- *$s : (D, \tilde{g}) \longrightarrow N$ is a critical map of E_α on $D(R)$*
- *$E(s) < \epsilon$, $1 < \alpha < \alpha_0$*

then

$$\|ds\|_{W^{1,p}(D')} < C(p, D')\|ds\|_{L^2(D)}, \quad \textit{for all disc } D' \Subset D$$

Remark 12. *In fact α_0, ϵ and $C(p, D')$ depend on the rescaled metric $\tilde{g}$ on D, but if $R \ll 1$ and $\tilde{g}$ is very close to Euclidean metric, then one can choose these parameters independently of $\tilde{g}$.*

A consequence of (the proof of) Lemma 41 is the following global result:

Theorem 42 (Critical maps of low energy are trivial)**.** *There exists $\epsilon' > 0$ and $\alpha_0 > 1$ such that if*

- *$s : M \longrightarrow N$ is critical map of E_α*

- $E(s) < \epsilon'$, $1 < \alpha < \alpha_0$

then $s \in N_0$ *and* $E(s) = 0$.

We proved in the last section that, under certain algebraic topological condition on N, E_α admits critical value $v_\alpha \in (1, (1+B^2)^\alpha)$. We now can conclude that, by Theorem 42, the critical values v_α are bounded away from 1, i.e. $\inf_\alpha v_\alpha > 1$.

We will also need the following extension theorem:

Theorem 43 (Extension of harmonic maps)**.** *If* $s: \ D \setminus \{0\} \longrightarrow N$ *is a harmonic map with finite energy* $E(s) < \infty$*, then* s *extends to a smooth harmonic map* $\tilde{s}: \ D \longrightarrow N$*.*

5.4 Convergence of critical maps of E_α.

We proved in Theorem 40 that if $C^0(M, N)$ is not connected or if $\Omega(M, N)$ is not contractible, then there exists a family $\{s_\alpha\}$ of critical maps of E_α with bounded, nontrivial energy $E_\alpha(s_\alpha) < B$. Since

- $\int_M |ds_\alpha|^2 \leq (E_\alpha(s_\alpha) - 1)^{1/\alpha}$ is bounded uniformly on α
- $\|s_\alpha\|_{L^\infty}$ is bounded by compactness of N.

the $W^{1,2}(M, \mathbb{R}^k)$-norms of $\{s_\alpha\}$ are bounded. By reflexivity of Sobolev spaces, there exists a subsequence $\{s_\beta\}$ weakly converging to s in $W^{1,2}(M, \mathbb{R}^k)$ with

$$\|s\|_{W^{1,2}} \leq \liminf_{\beta \to 1} \|s_\beta\|_{W^{1,2}}$$

We do not know at this moment if the convergence is C^0, or if s is continuous, or even if the image of s remains in N. The following key lemma answer these questions on a small disc of M in the case the energy of s_α is small.

Lemma 44 (Key)**.** *There exists an* $\epsilon > 0$*, in fact given by the Main estimate Lemma 41 with* $p = 4$*, such that if*

- $s_\alpha: \ D(R) \longrightarrow N \subset \mathbb{R}^k$ *are critical maps of* E_α *in* $W^{1,2\alpha}(D(R), N)$*,*
- $E(s_\alpha) < \epsilon$ *and* s_α *converges weakly to* s *in* $W^{1,2}(D(R), \mathbb{R}^k)$*,*

then

- *the restriction of* s *on* $\overline{D(R/2)}$ *is smooth harmonic map with image in* N*,*
- $s_\alpha \to s$ *in* $C^1(\overline{D(R/2)}, N))$*.*

Remark 13. *There are two different ways to define convergence of a sequence s_n to s in $C^1(\Omega)$ on an open set Ω:*

1. *The sequence s_α and s extend to $C^1(\bar{\Omega})$ and have finite norm $\max_\Omega |s| + \max_\Omega |ds|$ and $\max_\Omega |s_\alpha| + \max_\Omega |ds_\alpha|$ and*

$$\max_\Omega |s_\alpha - s| + \max_\Omega |ds - ds_\alpha| \to 0.$$

In this case, we will say that s_α converges to s in $C^1(\bar{\Omega})$.

2. *$C^1(\Omega)$ is topologised by a family of seminorms $\Gamma_K : \ s \longmapsto \max_K |s| + \max_K |ds|$ for $K \Subset \Omega$. This makes $C^1(\Omega)$ a Fréchet topological vector space. If the sequence s_α converges to s under this topology then we will say that s_α converges uniformly to s on compacts of Ω.*

Proof. We consider s_α and s as maps from the unit disc D to $\mathbb{R}^k$, then by Main estimate Lemma 41 for $p = 4$, since $E(s_\alpha) < \epsilon$, one has:

$$\|ds_\alpha\|_{W^{1,4}(D(1/2),\mathbb{R}^k)} \leq C(4, D(1/2))\|ds_\alpha\|_{L^2(D)} = C(4, D(1/2))E(s_\alpha)^{1/2}$$

So $\{s_\alpha\}$ is bounded in $W^{1,4}(D(1/2), \mathbb{R}^k)$ which is embedded compactly into $C^1(\overline{D(1/2)}, \mathbb{R}^k)$.

We now can prove that s_α converges strongly to s in $C^1(\overline{D(1/2)}, \mathbb{R}^k)$: If there was a subsequence $\{s_\beta\}$ whose restriction to $\overline{D(1/2)}$ remains C^1-away from s, then by compactness of $W^{1,4}(D(1/2), \mathbb{R}^k) \hookrightarrow C^1(\overline{D(1/2)}, \mathbb{R}^k)$, we can suppose that $\{s_\beta\}$ converges in C^1 to a certain $\bar{s} \neq s$ on $\overline{D(1/2)}$. But as a subsequence of $\{s_\alpha\}$, $\{s_\beta\}$ converges weakly to s on D, hence on $\overline{D(1/2)}$, we than obtain a contradiction using the uniqueness of weak limit.

By considering the Euler-Lagrange equation and letting $\alpha \to 0$, one concludes that s is a harmonic map from $D(1/2)$ to N. □

The global convergence of $\{s_\alpha\}$ can be established by a well-chosen covering of M by small balls or radius R.

Proposition 45. *Let $s_\alpha : M \longrightarrow N \subset \mathbb{R}^k$ be critical maps of E_α on M such that s_α converges weakly to s in $W^{1,2}(M, \mathbb{R}^k)$ and $E(s_\alpha) < B$. Then there exists $l = l(B, N)$ such that given any $m > 0$, one can find a sequence $\{x_{m,1}, \ldots, x_{m,l}\} \subset M$ and a subsequence $\{s_{\alpha(m)}\}$ of $\{s_\alpha\}$ such that*

$$s_{\alpha(m)} \longrightarrow s \text{ in } C^1\left(M \setminus \bigcup_{i=1}^{l} D(x_{m,i}, 2^{-m+1}), N\right)$$

Proof. We cover M by finitely many balls $D(y_i, 2^{-m})$ such that each point is covered at most h times by the bigger balls $D(y_i, 2^{-m+1})$. By Lemma 110, h can be chosen independently of m as $2^{-m} \to 0$.

Since $\sum_i \int_{D(y_i,2^{-m+1})} |ds_\alpha|^2 < Bh$, choosing $l = \lceil \frac{Bh}{2\epsilon} \rceil$, we see that there are at most l balls $D(y_{\alpha,i}, 2^{-m+1})$ with centers depending on α, on which the energy $E(s_\alpha)$ is less than ϵ. Passing to a subsequence $\{s_{\alpha(m)}\}$ of $\{s_\alpha\}$, we can suppose that $\{y_{\alpha(m),i}\}$ converges to $x_{m,i}$ as $\{\alpha(m)\} \to 1$. But since the points $\{y_i\}$ are of finite number and separated, $y_{\alpha(m),i} \equiv x_{m,i}$ eventually and we can suppose that the bad balls $D(y_{\alpha(m),i})$ where energy of $s_{\alpha(m)}$ surpasses ϵ are the same for every $s_{\alpha(m)}$.

Now apply Lemma 44 to the sequence $\{s_{\alpha(m)}\}$ on all the other 2^{-m+1}-balls, one sees that $\{s_{\alpha(m)}\}$ converges in C^1 to s on all $\overline{D(y_i, 2^{-m})}$ except those centered at $x_{m,i}$. The conclusion follows. □

Using a diagonal argument, we can find a subsequence $\{s_\beta\}$ of $\{s_\alpha\}$ that converges to s uniformly on compacts of $M \setminus \{x_1, \dots, x_l\}$.

Theorem 46 (Convergence of $\{s_\alpha\}$)**.** *Let $s_\alpha : M \longrightarrow N \subset \mathbb{R}^k$ be critical maps of E_α on M such that s_α converges weakly to s in $W^{1,2}(M, \mathbb{R}^k)$ and $E(s_\alpha) < B$. Then there exist at most l points $x_1, \dots, x_l$ in M, where l is given by Proposition 45, and a subsequence $\{s_\beta\}$ of $\{s_\alpha\}$ such that*

$$s_\beta \longrightarrow s \text{ in } C^1(M \setminus \{x_1, \dots, x_l\}, \mathbb{R}^k) \text{ uniformly on compacts.}$$

Proof. By passing to a subsequence $\{m_k\}$ of $\{m\}$, we can suppose that $\{x_{m,i}\}$ converges to x_i in M. Choose the diagonal subsequence $\{s_\beta\}$ from $\{s_{\alpha(m)}\}$ that consists of $s_{\alpha(m)(a_m)}$ where a_m is sufficiently big such that $\alpha(m)(a_m)$ is increasing and $\|s_{\alpha(m)(b)} - s_{\alpha(m)(c)}\|_{C^1(M \setminus \cup_i D(x_{m,i}, 2^{-m+1})} < \frac{1}{m}$ for all $b, c \geq a_m$. Then the sequence $\{s_\beta\}$ converges uniformly on compacts of $M \setminus \{x_1, \dots, x_l\}$ because $\{\bigcup_i D(x_{m,i}, 2^{-m+1})\}_m$ is an exhaustive family of compacts of $M \setminus \{x_1, \dots, x_l\}$. □

Remark 14. *With the same notation as Theorem 46,*

1. *The image $s(M \setminus \{x_1, \dots, x_l\})$ lies in N. Also, using the Euler-Lagrange equation, one sees that s is a (smooth) harmonic map from $M \setminus \{x_1, \dots, x_l\}$ to N.*
2. *Since $E(s) \leq \|s\|^2_{W^{1,2}} \leq \liminf_{\alpha \to 1} \|s_\alpha\|^2 < +\infty$, $s\big|_{M \setminus \{x_1, \dots, x_l\}}$ extends to a harmonic map $\tilde{s} : M \longrightarrow N$. We can therefore suppose that the limit s of Theorem 46 is smooth harmonic map on M and of image in N.*

5.5 Nontrivial harmonic maps from $\mathbb{S}^2$.

We will now prove the existence of nontrivial harmonic maps from $\mathbb{S}^2$ to a compact Riemannian manifold N satisfying the conditions of Theorem 40.

The following theorem does not suppose any condition on N.

Theorem 47. *Let M be a compact surface and s_α be critical maps of E_α. Suppose that*

- *s_α converges in C^1 to s uniformly on compacts of $M \setminus \{x_1, \dots, x_l\}$ but not on $M \setminus \{x_2, \dots, x_l\}$.*

- $E(s_\alpha) < B$

Then there exists a nontrivial harmonic map $s_* : \mathbb{S}^2 \longrightarrow N$.

Before proving the theorem, let us state its corollary.

Corollary 47.1 (Nontrivial harmonic map from $\mathbb{S}^2$)**.** *If the universal covering $\tilde{N}$ of N is not contractible then there exists a nontrivial harmonic map $s : \mathbb{S}^2 \longrightarrow N$.*

Proof. By Theorem 40 and Theorem 42, there exist critical maps $s_\alpha : \mathbb{S}^2 \longrightarrow N$ of E_α corresponding to critical values $E_\alpha(s_\alpha)$ in $(1+\delta, B)$. We claim that $\{s_\alpha\}$ cannot converge in $C^1(M)$ to a trivial harmonic map $s \in N_0$. In fact, if it did,

$$1+\delta \leq \lim_{\alpha\to 1} \int_M (1+|ds_\alpha|^2)^\alpha dV = \int_M (1+|ds|^2) dV = 1$$

which is contradictory.

Therefore, we only have two possibilities:

- $\{s_\alpha\}$ does not converge in $C^1(M)$ to s, then by Theorem 47, there exists a nontrivial harmonic map $s_* : \mathbb{S}^2 \longrightarrow N$.
- If $\{s_\alpha\}$ converges in $C^1(M)$ to a certain $\tilde{s}$, then as argued above, $\tilde{s}$ is nontrivial.

In both cases, nontrivial harmonic map from $\mathbb{S}^2$ to N exists. □

Let us now prove Theorem 47.

Proof of Theorem 47. If there is no C^1 convergence near x_1, we claim that:

Assertion 1. *For all $C > 0$ and $\delta > 0$, there exists $\alpha > 1$ arbitrarily close to 1 such that*

$$\max_{\overline{D}(x_1, 2\delta)} |ds_\alpha| > C.$$

Moreover, we can suppose that $\max_{\overline{D}(x_1,2\delta)} |ds_\alpha| = \max_{D(x_1,\delta)} |ds_\alpha|$.

Suppose that was not the case, then there exist $C, \delta > 0$ such that $\max_{D(x_1,2\delta)} |ds_\alpha| \leq C$ for all $\alpha > 1$ sufficiently close to 1. Choose a radius $R \ll \delta$ such that

$$\int_{D(x_1,R)} |ds_\alpha|^2 \leq \pi R^2 C^2 < \epsilon$$

It suffices to apply Key lemma 44 to see that $s_\alpha \to s$ in $C^1(D(x_1, R/2))$, hence s_α converges to s in $C^1(M \setminus \{x_2, \ldots, x_l\})$ uniformly on compacts. Moreover, since $\{ds_\alpha\}$ converges uniformly to ds on $\overline{D}(x_1, 2\delta) \setminus D(x_1, \delta)$, we can suppose, with α sufficiently close to 1, that the maximum is actually attained in $D(x_1, \delta)$.

Therefore, we can choose a sequence $\{C_n\}$ increasing to $+\infty$ and $\{\delta_n\}$ decreasing to 0, such that $C_n\delta_n$ diverges to $+\infty$ and there exists a sequence $\{\alpha_n\}$ decreasing to 1 such that

$$|ds_{\alpha_n}(y_n)| := \max_{D(x_1,\delta_n)} |ds_{\alpha_n}| = \max_{D(x_1,2\delta_n)} |ds_{\alpha_n}| = C_n$$

We define

$$\begin{aligned} \tilde{s}_{\alpha_n}:\ D(\delta_n C_n) &\longrightarrow N \\ x &\longmapsto s_{\alpha_n}(y_n + C_n^{-1}x) \end{aligned}$$

then $|d\tilde{s}_{\alpha_n}(0)| = \max_{D(C_n\delta_n)} |d\tilde{s}_{\alpha_n}| = 1$.

Fix any large $R < +\infty$, since $C_n\delta_n \to +\infty$, $\tilde{s}_{\alpha_n}$ is eventually defined on $D(R)$ and is a critical point of E_{α_n} with respect to a metric $\tilde{g}_n$ on $D(R)$ converging to the Euclidean metric. The energy $E(\tilde{s}_{\alpha_n}\Big|_{D(C_n\delta_n)}, \tilde{g}_n) = E(\tilde{s}_{\alpha_n}\Big|_{D(y_n,\delta_n)}, g_M) \leq B$.

We claim that Proposition 45 and Theorem 46 remain correct when $M = D(R)$ and s_α are critical maps of E_α with respect to metrics $\tilde{g}_\alpha$ converging to the Euclidean metric. To be precise:

Assertion 2. *Let $\tilde{s}_\alpha :\ (D(R), \tilde{g}_\alpha) \longrightarrow N \subset \mathbb{R}^k$ be critical maps of E_α such that*

- *s_α converges weakly to s_* in $W^{1,2}(D(R), Euclid)$,*
- *$E(s_\alpha) < B$*

then there exists at most l points $\{x_1, \ldots, x_l\}$ in $\overline{D}(R)$ and a subsequence $\{s_\beta\}$ such that s_β converges to s_* in $C^1(\overline{D}(R/2) \setminus \{x_1, \ldots, x_l\}, \mathbb{R}^k)$ uniformly on compacts, and s_* is harmonic in $D(R/2)$.

The two ingredients of the proof of Proposition 45 and Theorem 46 to be investigated are the covering and the estimate from Lemma 41. For the estimates, we already remarked that the parameters $\alpha_0, \epsilon, C(p, D')$ of Lemma 41 can be chosen independent of the metric $\tilde{g}_\alpha$ if they are close to Euclidean. For the covering, the investigation is not on the constant h, which can be chosen to be $3^{\dim M}$, but on how small the radius of the covering balls must be, but Lemma 110 states that their size is dictated by the Ricci curvature and sectional curvature of $\tilde{g}_\alpha$, which are also uniformly bounded.

Using Assertion 2, passing to a subsequence of $\{\tilde{s}_{\alpha_n}\}$ if necessary, we can suppose that $\tilde{s}_{\alpha_n} \to s_*$ in $C^1(D(R), \mathbb{R}^k)$. Note that there is no singular point where $\{\tilde{s}_{\alpha_n}\}$ fails to converge because $|d\tilde{s}_{\alpha_n}|$ is bounded uniformly on $D(R)$ (hence cannot explode as in Assertion 1). We can also choose, by a diagonal argument, a subsequence of $\{\tilde{s}_{\alpha_n}\}$ that converges to s_* in $C^1(\mathbb{R}^2)$ uniformly on compacts.

It is clear that $s_* : \mathbb{R}^2 \longrightarrow N$ is harmonic and nontrivial because

$$|ds_*(0)|_{\text{Euclid}} = \lim_{\alpha_n \to 1} |d\tilde{s}_{\alpha_n}(0)|_{\tilde{g}_{\alpha_n}} = 1.$$

Also,

$$\int_{D(R)} |ds_*|^2 dE = \lim_{\alpha_n\to 1} \int_{D(R)} |d\tilde{s}_{\alpha_n}|^2 dV_{\tilde{g}_\alpha} \leq \limsup_{\alpha\to 1} 2E(s_\alpha\big|_{D(x_1,2\delta_n)}) < 2B$$

which means the energy of s_* on $\mathbb{R}^2$ is bounded above by $2B$.

Now since $(\mathbb{R}^2, \text{Euclid})$ is conformal to $\mathbb{S}^2 \setminus \{p\}$, s_* can be seen as a harmonic map on $\mathbb{S}^2 \setminus \{p\}$ with the same (finite) energy. By Extension theorem 43, s_* extends to a nontrivial harmonic map from $\mathbb{S}^2$ to N. □

Remark 15. 1. *We can have a better estimate of $E(s_*)$. For any $R > 0$, one has*

$$E(s_*\big|_{D(R)}) + E(s\big|_{M\setminus\cup_{i=1}^l D(x_i,\delta_n)}) \leq \limsup_{\alpha_n\to 1} \left[E(s_{\alpha_n}\big|_{D(x_1,\delta_n)}) + E(s_{\alpha_n}\big|_{M\setminus\cup_{i=1}^l D(x_i,\delta_n)})\right]$$

Let $\delta \to 0$ then $R \to +\infty$, one has

$$E(s_*) + E(s) \leq \limsup_{\alpha\to 1} E(s_\alpha).$$

2. *The proof of Theorem 47 also gives a constraint on the image of s_*: since $s_*(D(R)) \subset \overline{\cup_{1<\beta<\alpha} s_\beta(D(x_1, 2\delta))}$ for all α arbitrarily close to 1 and δ arbitrarily small, one has*

$$s_*(\mathbb{S}^2) \subset \bigcap_{\delta\to 0} \bigcap_{\alpha\to 1} \overline{\bigcup_{1<\beta<\alpha} s_\beta(D(x_1, \delta))}$$

5.6 Minimal immersions of $\mathbb{S}^2$.

We use the following result:

Theorem 48 ([CG75], [GOR73], [ES64])**.** *If $s : \mathbb{S}^2 \longrightarrow N$ is a nontrivial harmonic map and* $\dim N \geq 3$, *then s is a C^∞ conformal, branched, minimal immersion.*

The "minimal" part follows from [ES64], the "branched" part follows from [GOR73] and the "conformal" part follows from [CG75] and the fact that there is no nontrivial holomorphic quadratic differential on $\mathbb{S}^2$. Theorem 47 gives:

Theorem 49. *If the universal covering $\tilde{N}$ of N is not contractible then there exists a C^∞ conformal, branched, minimal immersion $s : \mathbb{S}^2 \longrightarrow N$.*

Part IV

Appendix 1: Resolution of linear equations on manifold

Chapter 6

Interpolation theory and Sobolev spaces on compact manifolds

6.1 Motivation

We will define a more general notion of Sobolev spaces on compact manifold than those in [Aub98] and [**?**], where Sobolev spaces on a (Riemannian) manifold $W^{k,p}(M)$ of dimension n are defined for $k \in \mathbb{Z}_{\geq 0}$ and for *uniform weight*, meaning that a function $f \in W^{k,p}(M)$ is supposed to be weakly differentiable up to order k in every variables $x_1, \dots, x_n$ in each smooth coordinates. The space $W^{k,p}(M)$ in this case can be defined by density with respect to a norm involving derivatives $\frac{\partial f}{\partial x^\alpha}$.

Meanwhile, the suitable function spaces to solve parabolic equations are those whose regularity in time is half of that in space, i.e. we will solve parabolic equations on the Sobolev spaces $W^{k,p}(M \times T)$ of functions k times regular in M and $k/2$ times regular in T. We cannot always, (for example when k is odd) find a simple norm involving derivatives of f to define $W^{k,p}$ by density. This generalisation will be done using Stein's multipliers.

Another generalisation will be made is to allow the manifold to have boundary. Even when we only want to solve parabolic equation on manifold M without boundary, the underlying space is $M \times [0, T]$ which has boundary. Moreover, we will have to discuss the notion of trace in order to use the initial condition at $t = 0$.

In this part, all manifolds will be compact, with no given metric. This is not really a generalisation since on compact manifolds, Sobolev spaces $W^{k,p}(M)$, as defined in [Aub98] and [**?**] set theoretically do not depend on the metric and (the equivalent class of) their norms also independent of the metric.

We will mainly follow the discussion in [Ham75], where the author also works on manifold with *corner*, i.e. irregular boundary. The corners, modeled by $\mathbb{R}^{n-k} \times \mathbb{R}^k_{\geq 0}$, appear naturally, for example at the boundary ∂M in $t = 0$. The extra effort to cover the case of corners is not much (see [Ham75, page 50]) and essentially algebraic.

6.2 Preparatory material

We will recall here basic elements of Fourier transform on the space of tempered distributions and then we will have a quick review of interpolation theory.

6.2.1 Stein's multiplier

Let $X = \mathbb{R}^n$ be the Euclidean space, coordinated by $x_1, \ldots, x_n$ and $\mathcal{E} = \mathbb{R}^n$, coordinated by $\xi_1, \ldots, \xi_n$ be the frequency domain of X. Recall that Fourier transform is an isomorphism in the following three levels

1. The Schwartz space of rapidly decreasing smooth functions $\mathcal{S}(X)$ whose elements are smooth and decrease more rapidly then any rational function. The Schwartz space are topologized by the family of semi-norms $|f|_{\alpha,\beta} = \sup_X |x^\alpha D_x^b f(x)|$.

2. The space $L^2(X)$ of doubly-integrable functions.

3. The space of tempered distributions, i.e. the dual space $\mathcal{S}^*(X)$ of $\mathcal{S}(X)$ under the weak-* topology given by $\mathcal{S}(X)$.

To simplify the notation, we use $D_x^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ and $P(D) = \sum_\alpha c_\alpha D^\alpha$ for any polynomial P.

Recall that for any $u \in \mathcal{S}(X)$ and for any polynomial P, one has $\widehat{P(D)u} = P(\xi)\hat{u}(\xi)$. This can be extended to non-polynomial function of $M(D)$ of D by

$$\widehat{M(D)u} := M(\xi)\hat{u}(\xi)$$

where M is a slowly growing function, i.e. $D^\alpha M(\xi)$ grows slower than certain polynomial as $|\xi| \to \infty$.

The following theorem give a criteria of the function M such that $M(D) : \mathcal{S}(X) \longrightarrow \mathcal{S}(X)$ extend to $L^p(X) \longrightarrow L^p(X)$.

Theorem 50 (Stein). *If for any* primitive *index $\alpha = (\alpha_1, \ldots, \alpha_n)$, i.e. each α_i being 0 or 1 (there are exactly 2^n primitive indices), one has*

$$|\xi^\alpha D^\alpha M(\xi)| \le C_\alpha$$

then $M(D)$ extend to a bounded linear operator on $L^p(X)$.

Definition 6. *1. A slowly growing function W on $\mathcal{E}$ with $W(\xi) > 0$ is called a **weight** if for all primitive index α, one has*

$$|\xi^\alpha D^\alpha W(\xi)| \le C_\alpha W(\xi).$$

2. *The* ***Sobolev space*** $W^{k,p}(X, W)$ *with respect to weight* W, $k \in \mathbb{R}, 1 < p < \infty$ *is the vector space*

$$W^{k,p}(X, W) = \left\{ u \in \mathcal{S}^*(X) : \; W(D)^k u \in L^p(X) \right\}$$

normed by $\|u\|_{W^{k,p}} = \|W(D)^k u\|_{L^p}$.

Example 4 (Weight given by $\Sigma = (\sigma_1, \ldots, \sigma_n)$)**.** *Note by* $\sigma := \operatorname{lcm}(\sigma_1, \ldots, \sigma_n)$ *then* $W_\Sigma(\xi) = \left(1 + \xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n}\right)^{1/2\sigma}$ *is a weight. We will only use weights of this type in our discussion. The index* $\Sigma = (\sigma_1, \ldots, \sigma_n)$ *is chosen according to the differential operator in the elliptic/parabolic equation. In particular, for Laplace equation, one chooses* $\Sigma = (1, \ldots, 1)$ *and for heat equation* $\Sigma = (1, 2, \ldots, 2)$ *where* 1 *is in the time component.*

Remark 16. 1. *If* W_1, W_2 *are weights then* $W_1 + sW_2, W_1 W_2, W_1^s (s > 0)$ *are also weights.*

2. *The operator* $W(D) : \; W^{k+r}, p(X, W) \longrightarrow W^{k,p}(X)$ *is bounded.*

3. *Given another weight* $V(\xi) \leq CW(\xi)$, *by Stein's criteria (Theorem 50) one has a bounded embedding* $W^{k,p}(X, W) \hookrightarrow W^{k,p}(X, V)$.

The Sobolev space $W^{k,p}(X, W_\Sigma$ has a simple definition by density when $\sigma \mid k$. Given an index $\alpha = (\alpha_1, \ldots, \alpha_n)$, note by $\|\alpha\| := \sum_{i=1}^n \alpha_i \frac{\sigma}{\sigma_i}$.

Theorem 51 (Equivalent norm when $\sigma \mid k$)**.** *If* $k > 0$ *and* $\sigma \mid k$ *and* $1 < p < \infty$, *then given* $u \in \mathcal{S}^*(X)$, *one has*

1. $u \in W^{k,p}(X)$ *if and only if* $D^\alpha u \in L^p(X)$ *for all* $\|\alpha\| \leq k$ *and the norm* $\sum_{\|\alpha \leq k\|} \|D^\alpha u\|_{L^p}$ *is equivalent to* $\|u\|_{W^{k,p}}$.

2. $u \in W^{-k,p}$ *if and only if there exists* $g_\alpha \in L^p$ *such that* $u = \sum_{\|\alpha\| \leq k} D^\alpha g_\alpha$ *and* $\|u\|_{W^{-k,p}}$ *is equivalent to*

$$\inf \left\{ \sum_{\|\alpha\| \leq k} \|g_\alpha\|_{L^p} : \; u = \sum_{\|\alpha\| \leq k} D^\alpha g_\alpha \right\}$$

Example 5. 1. *When* $\sigma_1 = \cdots = \sigma_n = 1$, *one has the familiar Sobolev spaces.*

2. *For (the weight of) heat equation,* $W^{2,p}$ *can be defined by density using the norm*

$$\|u(t, x)\| = \left\| \frac{\partial u}{\partial t} \right\|_{L^p} + \|Du\|_{L^p} + \|Du\|_{L^p}$$

where L^p *stands for* $L^p(X \times [0, T])$.

6.2.2 Holomorphic interpolation of Banach spaces

The Interpolation theory is based on the following Three-lines theorem whose proof follows from the classic Hadamard's three-lines theorem (the case $A = \mathbb{C}$) and the way we define complex Banach spaces and holomorphic maps taking value there.

Theorem 52 (Three-lines)**.** *Let A be a complex Banach space and $h: \ S = \{0 \leq \operatorname{Re} z \leq 1\} \subset \mathbb{C} \longrightarrow A$ be a holomorphic map, i.e. continuous and holomorphic in the interior such that h is bounded at infinity, i.e. $h(x+iy) \to 0$ as $y \to \infty$. Let $M(x) := \sup_y \|h(x+iy)\|$ then one has*

$$M(x) \leq M(1)^x M(0)^{1-x}$$

Let A_0, A_1 be complex Banach spaces such that

1. A_0, A_1 can be continuously embedded into a Hausdorff topological complex vector space E such that the complex structures are compatible with each others, i.e. the linear embeddings $A_i \hookrightarrow E$ preserve complex structures.

2. The intersection $A_0 \cap A_1$ in E is dense in $(A_i, \|\|_{A_i})$ for $i = 0, 1$.

such (A_0, A_1) is called an **interpolatable** pair.

The norms of $A_0 \cap A_1$ and $A_0 + A_1$ are defined such that the these spaces are Banach and the diagram

$$0 \longrightarrow A_0 \cap A_1 \longrightarrow A_0 \oplus A_1 \longrightarrow A_0 + A_1 \longrightarrow 0 \tag{6.1}$$

commutes and the arrows are continuous. By Open mapping theorem, this means that the norm on $A_0 \cap A_1$ is equivalent to $\|x\|_{A_0 \cap A_1} = \|x\|_{A_0} + \|x\|_{A_1}$ and the norm on $A_0 + A_1$ is equivalent to $\|x\|_{A_0+A_1} = \inf_{x = x_0 + x_1, x_i \in A_i} \{\|x_0\|_{A_0} + \|x\|_{A_1}\}$.

Remark 17. *A pair (A_0, A_1) of Banach spaces may give different interpolatable pairs depending how they are embedded into a common space E. It is not difficult to see that the data of* interpolatable pair *is uniquely determined by 2 complex Banach spaces U, V (which are eventually $A \cap B$ and $A + B$) and the diagram*

$$\begin{tikzcd}
& & 0 \arrow[d, bend right] & & \\
& & A_0 \arrow[u, bend right] \arrow[d, bend right] \arrow[dr] & & \\
0 \arrow[r] & U \arrow[ur] \arrow[r] \arrow[dr] & A_0 \oplus A_1 \arrow[u, bend right] \arrow[d, bend right] \arrow[r] & V \arrow[r] & 0 \\
& & A_1 \arrow[u, bend right] \arrow[d, bend right] \arrow[ur] & & \\
& & 0 \arrow[u, bend right] & &
\end{tikzcd} \tag{6.2}$$

in which

1. *All arrows are continuous and compatible with complex structures. The horizontal sequence is exact, the vertical sequence is exact and canonical.*

2. *The diagonal arrows from U to A_0, A_1 are injective and of dense image in A_0, A_1.*

3. *The maps composed by the diagonal arrows $U \to A_i \to V$ are injective for $i = 0, 1$. Since the two maps are additive inverse, it suffices to have injectivity for one of them.*

In the language that we will use to solve linear equation, these properties of diagram (6.2) *are equivalent to the square*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & V \end{tikzcd}$$

being exact.

The following construction will give a family of complex subspace A_θ of $A_0 + A_1$ containing $A_0 \cap A_1$ for $0 \le \theta \le 1$ that interpolates A_0 and A_1 that satisfies the following properties, called interpolation inequalities

Theorem 53 (Interpolation inequality for elements in the intersection)**.** *Let $a \in A_0 \cap A_1$ then $a \in A_\theta$ and*

$$\|a\|_{A_\theta} \le 2\|a\|_{A_1}^{\theta}\|a\|_{A_0}^{1-\theta}$$

Theorem 54 (Interpolation inequality for operators)**.** *Given interpolatable pairs (A_0, A_1) and $B_0, B_1)$, and T a bounded linear operator $T : A_0 \longrightarrow B_0$ and $T : A_1 \longrightarrow B_1$ such that T is well-defined on $A_0 \cap A_1$. Then T extends linearly and continuously to $T : \ A_0 + A_1 \longrightarrow B_0 + B_1$, that is*

$$\begin{tikzcd} 0 \arrow[r] & A_0 \cap A_1 \arrow[r] \arrow[d, "T"] & A_0 \oplus A_1 \arrow[r] \arrow[d, "T\oplus T"] & A_0 + A_1 \arrow[r] \arrow[d, "T"] & 0 \\ 0 \arrow[r] & B_0 \cap B_1 \arrow[r] & B_0 \oplus B_1 \arrow[r] & B_0 + B_1 \arrow[r] & 0 \end{tikzcd} \tag{6.3}$$

Also, T defines a bounded operator $T : \ A_\theta \longrightarrow B_\theta$ and

$$\|T\|_{L(A_\theta, B_\theta)} \le 2\|E\|_{L(A_1,B_1)}^{\theta}\|E\|_{L(A_0,B_0)}^{1-\theta}$$

To define A_θ, let

$$\mathcal{H}(A_0, A_1) := \Big\{h : S \longrightarrow A_0 + A_1 : \ h \text{ is holomorphic}, \ \lim_{|y|\to\infty} h(z) = 0, h(iy) \in A_0, \ h(1 + iy) \in A_1\}$$

where, as above, S denotes the strip $0 \le \operatorname{Re} z \le 1$. Then $\mathcal{H}(A_0, A_1)$ is a Banach space with the norm

$$\|h\|_{\mathcal{H}(A_0,A_1)} := \sup_y \|h(iy)\|_{A_0} + \sup_y \|h(1 + iy)\|_{A_1}$$

The space A_θ is defined set-theoretically as the space of all value in $A_0 + A_1$ that a function $h \in \mathcal{H}(A_0, A_1)$ can take at $\theta \in [0,1] \in S$. Therefore, set-theoretically A_θ coincides with A_0 and A_1 when $\theta = 0$ and $\theta = 1$. To define the norm on A_θ, let

$$\mathcal{K}_\theta(A_0, A_1) := \{h \in \mathcal{H}(A_0, A_1) : \ h(\theta) = 0\}$$

then $\mathcal{K}_\theta(A_0, A_1)$ is a closed complex subspace of the Banach space $\mathcal{H}(A_0, A_1)$. Then $A_\theta := \mathcal{H}(A_0, A_1)/\mathcal{K}_\theta(A_0, A_1)$ has the natural quotient norm inherited from $\mathcal{H}(A_0, A_1)$ and is still a Banach space.

It is not difficult to see that the norm on A_θ coincides with the norm $\|\cdot\|_{A_0}, \|\cdot\|_{A_1}$ when $\theta = 0$ or $\theta = 1$

Theorem 53 follows from the this lemma when one takes h to be a constant, and is in $A_0 \cap A_1$.

Lemma 55. *If $h \in \mathcal{H}(A_0, A_1)$ then $\|h(\theta)\|_{A_\theta} \leq 2M_1^\theta M_0^{1-\theta}$ where*

$$M_0 := \sup_y \|h(iy)\|_{A_0}, \quad M_1 := \sup_y \|h(1+iy)\|_{A_1}$$

Proof. The A_θ-norm of $h(\theta)$ only depends on the value of h at θ, one can therefore replace h by a function of form $h_{c,\epsilon}(z) = \exp(c(z-\theta) + \epsilon z^2)h(z)$, then let $\epsilon \to 0$ and choose the optimal c, which is $e^c = M_0/M_1$. □

Theorem 54 follows from Theorem 53 and the very definition of quotient norm.

Remark 18. *The optimal constant, as given by the proofs, is $\theta^{-\theta}(1-\theta)^{\theta-1} < 2$*

The interest of holomorphic interpolation theory comes from the fact that interpolation of Sobolev spaces are still Sobolev spaces, which, together with Theorem 54 and Theorem 53, gives a class of useful inequalities generally called interpolation inequalities.

Theorem 56 (Interpolation of Sobolev spaces)**.** *Let $p, q \in (1, +\infty)$ and $k, l \in \mathbb{R}$ and $X = \mathbb{R}^n$. Take*

$$A_0 := W^{k,p}(X), \quad A_1 := W^{l,q}(X)$$

then $A_\theta = W^{s,r}(X)$ where

$$\theta l + (1-\theta)k = s, \qquad \theta\frac{1}{q} + (1-\theta)\frac{1}{p} = \frac{1}{r}$$

The holomorphic interpolation behaves predictably with direct sum and compact operators

Theorem 57. *Let $(A_0, A_1), (B_0, B_1)$ be interpolatable pairs and denotes by $(A \oplus B)_\theta$ be the interpolation of $A_0 \oplus B_0$ and $A_1 \oplus B_1$ then one has $(A \oplus B)_\theta \cong A_\theta \oplus B_\theta$ by a canonical isomorphism.*

Proof. The set-theoretical bijection is easy to see: note that there is a natural inclusion $(A \oplus B)_\theta \hookrightarrow A_\theta \oplus B_\theta$, which is also a bijection because $\mathcal{H}(A_0 \oplus B_0, A_1 \oplus B_1) = \mathcal{H}(A_0, A_1) \oplus \mathcal{H}(B_0, B_1)$.

The most difficult part is to know what we mean by *isomorphism*. In fact the two norms (the interpolation norm and the direct-sum norm) do not coincide, but they are equivalent. One can prove, with basic sup-inf analysis that

$$\frac{1}{2}\| \cdot \|_{A_\theta \oplus B_\theta} \leq \| \cdot \|_{(A\oplus B)_\theta} \leq \| \cdot \|_{A_\theta \oplus B_\theta}$$

□

Theorem 57 can be generalised to the following result.

Theorem 58 (*)**.** *Let (X_0, X_1) and (Y_0, Y_1) be interpolatable pairs. Suppose that there are inclusion $X_0 \hookrightarrow Y_0$ and $X_1 \hookrightarrow Y_1$ with closed images in Y_0 and Y_1 respectively and the inclusions agree on $X_0 \cap X_1$ as mappings from $X_0 \cap X_1$ to $Y_0 + Y_1$. Moreover, suppose that the image of $X_0 + X_1$ in $Y_0 + Y_1$ is closed. Then there is a natural inclusion $X_\theta \hookrightarrow Y_\theta$ with closed image in Y_θ*

Remark 19.
1. *The condition $X_0 + X_1 \hookrightarrow Y_0 + Y_1$ being of closed image is redundant if $X_1 \hookrightarrow X_0$ and $Y_1 \hookrightarrow Y_0$, as in the case of interpolation of* certain *Sobolev spaces on manifolds. In general, one can also check that this condition holds for the maps $\iota_{k,p}$ and $\iota_{l,q}$ in Definition 7 of Sobolev spaces using the fact that they admit left-inverse given by $\{\tilde{\psi}_i\}$. See Remark 22.*

2. *If one has two exact sequences*

$$0 \longrightarrow X_i \longrightarrow Y_i \longrightarrow Z_i \longrightarrow 0, \quad i = 0, 1 \tag{6.4}$$

whose arrows commute with ones from the intersection and ambient spaces of interpolatable pairs $(X_0, X_1), (Y_0, Y_1), (Z_0, Z_1)$ then, since the images of $X_i \longrightarrow Y_i$ being kernel of $Y_i \longrightarrow Z_i$ are closed, one has the inclusion for interpolation spaces, also of closed image:

$$0 \longrightarrow X_\theta \longrightarrow Y_\theta, \quad 0 \leq \theta \leq 1.$$

3. *In particular, if the sequences in* (6.4) *split, meaning that one can find a retraction $0 \longrightarrow Z_i \longrightarrow Y_i$, then by applying the theorem for the retractions, one sees that the interpolation sequence extend to Z_θ, i.e.*

$$0 \longrightarrow X_\theta \longrightarrow Y_\theta \longrightarrow Z_\theta \longrightarrow 0$$

and also splits, meaning $Y_\theta \cong X_\theta \oplus Z_\theta$. Applying this results to the split-exact sequences

$$0 \longrightarrow A_i \longrightarrow A_i \oplus B_i \longrightarrow B_i \longrightarrow 0$$

one then obtains Theorem 57.

Proof. The inclusion $X_\theta \hookrightarrow Y_\theta$ is natural and due to the fact that $\mathcal{H}(X_0, X_1) \subset \mathcal{H}(Y_0, Y_1)$. The equivalence of the interpolation norm X_θ and the norm inherited from Y_θ on X_θ requires more than a simple sup-inf analysis as in the proof of Theorem 57 since $\mathcal{H}(X_0, X_1)$ is strictly included in $\mathcal{H}(Y_0, Y_1)$. What we can say is that the interpolation norm X_θ dominates the interpolation norm of Y_θ, since it involves the infimum on the smaller set. In other words, it means that the inclusion $X_\theta \hookrightarrow Y_\theta$ is continuous. It remains to check that the image of $X_\theta \hookrightarrow Y_\theta$ is closed.

Since

$$\begin{array}{ccc} X_\theta & \longrightarrow & Y_\theta \\ \| & & \| \\ \mathcal{H}(X_0, X_1)/\mathcal{K}_\theta(X_0, X_1) & & \mathcal{H}(Y_0, Y_1)/\mathcal{K}_\theta(Y_0, Y_1) \\ \uparrow & & \uparrow \\ \mathcal{H}(X_0, X_1) & \longrightarrow & \mathcal{H}(Y_0, Y_1) \end{array}$$

it suffices to show that the image $\mathcal{H}(X_0, X_1) \hookrightarrow \mathcal{H}(Y_0, Y_1)$ is closed, meaning if $\mathcal{H}(X_0, X_1) \ni h_n \to h$ in $\mathcal{H}(Y_0, Y_1)$, then h must take value in $X_0 + X_1$. This is easy to verify on ∂S: By the equivalence of the norm on X_i and the restricted norm from Y_i, $i = 0, 1$, one sees that $h(iy) \in X_0$ and $h(1 + iy) \in X_1$.

Since $X_0 + X_1$ is closed in $Y_0 + Y_1$, any holomorphic map $\mathcal{H}(Y_0, Y_1) \ni f : S \longrightarrow Y_0 + Y_1$ passes holomorphically to the quotient $S \longrightarrow (Y_0 + Y_1)/(X_0 + X_1)$. The fact that h takes value in $X_0 + X_1$ follows from Maximum modulus principle for holomorphic functions. □

Theorem 59 (Interpolation of compact embedding)**.** *If $A_1 \hookrightarrow A_0$ is a compact embedding, then $A_1 \cong A_\theta \cap A_1 \hookrightarrow A_\theta$ is a compact embedding where the first $\cong$ denotes the same space with equivalent norms.*

Proof. It follows from Theorem 53:

$$\|x_m - x_n\|_{A_\theta} \leq 2\|x_m - x_n\|_{A_0}^{1-\theta} \|x_m - x_n\|_{A_1}^{\theta}$$

Hence if $\{x_n\}$ is a bounded sequence in A_1, it converges in A_0 and therefore A_θ. □

The previous Theorem 53, together with Theorem 56 also gives a proof of Kondrachov's Theorem, that is the embedding $W^{k,p}(X) \hookrightarrow W^{l,p}(X)$ is compact if $k > h >\geq 0$. This follows from the following 2 remarks

1. The case $l = 0$ and $k \gg 1$ follows from the embedding $W^{k,p} \hookrightarrow C^1$ and Ascoli's theorem. Hence by Theorem 53, one has the compactness embedding if $k \gg 1$ and $l < k$.

2. For the case of small k, note that

$$W^{k+r,p}(X) \twoheadrightarrow W^{k,p}(X) : \ v \longmapsto W(D)^r u$$

is surjective and any $u \in W^{k,p}(X)$ can be lifted to an element $\tilde{u} \in W^{k+r,p}(X)$ of the same norm. In fact, if $W(\xi)^k \hat{u} \in L^p$ then choose $\tilde{u}$ such that $\widehat{\tilde{u}} = W(\xi)^{-r} \hat{u}$. Kondrachov's theorem follows from the diagram:

$$\begin{tikzcd} W^{k+r,p}(X) \arrow[r, two heads] \arrow[d, hook, "\text{compact}"'] & W^{k,p}(X) \arrow[d, hook] \\ W^{h+r,p}(X) \arrow[r, two heads] & W^{h,p}(X) \end{tikzcd}$$

Remark 20. *The advantage of this proof is that it is valid for weighted Sobolev spaces over manifolds.*

6.3 Sobolev spaces on compact manifold without boundary

Let M be a compact manifold without boundary. We fix a finite atlas of M by chart $\varphi_i : M \supset U_i \longrightarrow V_i \subset \mathbb{R}^n$ such that the transitions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : V_j \longrightarrow V_i$ are of strictly positive and bounded derivatives, i.e. $C(\alpha)^{-1} \leq D^\alpha \varphi_{ij} \leq C(\alpha)$ for all indices α. We will called such atlas a *good atlas*. One can always obtain such atlas by shrinking a bit each chart of a given atlas of M. Let ψ_i be a partition of unity subordinated to $\{U_i\}$

Definition 7. *1. The **Sobolev spaces** $W^{k,p}(M)$ is defined as*

$$W^{k,p}(M) := \left\{ f \in \mathcal{S}(M)^* : \ (\psi_i f) \circ \varphi_i^{-1} \in W^{k,p}(\mathbb{R}^n) \right\}$$

with the norm

$$\|f\|_{W^{k,p}} = \sum_i \|(\psi_i f) \circ \varphi_i^{-1}\|_{W^{k,p}(\mathbb{R}^n)}$$

2. *Weighted Sobolev spaces can be defined when M has a foliation structure, i.e. M is locally modeled by $0 \subsetneq F_1 \subsetneq \cdots \subset F_k \subsetneq \mathbb{R}^n$ where F_i are vector subspace of $\mathbb{R}^n$ of dimension $0 < n_1 < \cdots < n_k < n$ respectively and F_k are preserved by the transition maps φ_{ij}, for example when M is a product of manifolds of lower dimension. Then the above definition extends to weighted Sobolev spaces with weight $\sigma_1 = \cdots = \sigma_{n_1}$, $\sigma_{n_1+1} = \cdots = \sigma_{n_2}, \ldots$ $\sigma_{n_k+1} = \cdots = \sigma_n$.*

Remark 21. *1. One can define $\mathcal{S}(M)^*$ as the dual space of $\mathcal{S}(M) = C^\infty(M)$ under Schwartz topology with respect to any metric, because by compactness any two metrics on M are comparable. The distributions $\psi_i f$ are tempered because they are compactly supported.*

2. *One can identify $C^\infty(M)$ with a subspace of $\mathcal{S}^*(M)$ that is contained in any Sobolev space $W^{k,p}(M)$ by fixing a Riemannian metric g on M. The map $C^\infty(M) \hookrightarrow \mathcal{S}^*(M)$ may depend on g, but its image does not. Similarly, one can also identify an element of $W^{k,p}(\mathbb{R}^n)$ supported in V_i with an element in $W^{k,p}(M)$.*

3. *If one uses another good atlas U_i' or a different partition of unity, one obtains the same set $W^{k,p}(M)$ and an equivalent norm. To see this, let us call two good atlas* compatible *if their union is also a good atlas, then the statement holds for two compatible atlas by comparing their union. Moreover, for any two arbitrary good atlas $\{U_i\}, \{U_j'\}$, one can find a good atlas compatible with both of them by shrinking their union.*

By definition, one has an inclusion $\iota : W^{k,p}(M) \hookrightarrow \bigoplus_i W^{k,p}(\mathbb{R}^n)$. Also ι is of closed image because one can find a projection $\pi : \bigoplus_i W^{k,p}(\mathbb{R}^n) \longrightarrow W^{k,p}(M)$ with $\pi \circ \iota = \mathrm{Id}$. In fact, let $\tilde{\psi}_i$ be functions supported in U_i that equal 1 in the support of ψ_i, then

$$\pi : g \mapsto \sum \tilde{\psi}_i.(g \circ \varphi_i)$$

works. The continuity of π follows from straight-forward calculations.

The closedness of image of ι is equivalent to the fact that $W^{k,p}(M)$ is complete.

Remark 22. *Although ι preserves the norm of $W^{k,p}(M)$ and has a right-inverse, it is far from being an isomorphism (it is not surjective). Each summand of an element in the image of ι tends to 0 on the boundary of V_i (take $k \gg 1$ then everyone is continuous by Sobolev embedding, there is no subtlety in what we mean by "tends to 0"). [Ham75, page 54] seems to claim that ι is an isomorphism and apply Theorem 57 repeatedly to deduce Theorem 56 for Sobolev spaces on manifold, then the Sobolev embedding $W^{k,p} \hookrightarrow C^l(M)$ and Kondrachov's theorem.*

The above results are true and the correction is not difficult (use Theorem 58).

From the remark, one has

Theorem 60 (Interpolation of Sobolev spaces on manifold)**.** *Theorem 56 holds for Sobolev spaces $W^{k,p}(M)$ on compact manifold M.*

6.4 Sobolev spaces on compact manifold with boundary

In this part, we will define the Sobolev spaces $W^{k,p}(M/\mathcal{A})$ where $k \in \mathbb{R}, p \in (1, \infty)$ and M is a manifold with boundary and $\mathcal{A}$ is union of connected components of ∂M the boundary of M. These spaces contain $W^{k,p}(M)$ "functions" who vanish on $\mathcal{A}$. The motivation is that we will later take $M = M' \times [0, T]$ where M' is a manifold without boundary where we want to solve heat equation, and the natural $\mathcal{A}$ would be $M \times \{0\}$. We also want that the new definition coincides with the case of no boundary when $\mathcal{A} = \emptyset$

Suppose that we already define the Sobolev spaces on $X \times Y^+$ where $X = \mathbb{R}^n$ and $Y^+ = \mathbb{R}_{\geq 0}$, that is the space $W^{k,p}(X \times Y^+) = W^{k,p}(X \times Y^+/\emptyset)$ and $W^{k,p}(X \times Y^+, X \times \{0\})$. Then then we define the space $W^{k,p}(M/\mathcal{A})$ in analog of Definition 7 as follows

Definition 8. *1. The **Sobolev spaces** $W^{k,p}(M/\mathcal{A})$ where A is a connected component of ∂M is defined as*

$$W^{k,p}(M/\mathcal{A}) := \left\{ f \in \mathcal{S}(M)^* : \ (\psi_i f) \circ \varphi_i^{-1} \in W^{k,p}(R_i/\mathcal{A}_i) \right\}$$

where $\mathcal{A}_i = \varphi_i(U_i \cap \mathcal{A})$ and R_i is the Euclidean space containing $\varphi(U_i)$, that is either $\mathbb{R}^{n+1}$ when $\mathcal{A}_i = \emptyset$ or $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ when $\mathcal{A}_i \subset \mathbb{R}^n \times \{0\}$. The norm is given by

$$\|f\|_{W^{k,p}} = \sum_i \|(\psi_i f) \circ \varphi_i^{-1}\|_{W^{k,p}(R_i/\mathcal{A}_i)}$$

2. As before, weighted Sobolev spaces can be defined when M has a foliation structure compatible with its boundary.

The fact that different good atlas and different partition of unity defines the same space $W^{k,p}(M/\mathcal{A})$ (as a subset of $\mathcal{S}^*(M)$) with equivalents norm comes from the following lemma, which is just a formulation of arguments in the case of no boundary. For the proof, one reduces the lemma, by interpolation inequality, to the case k is a multiple of σ and use the criteria in Theorem 51 and the boundedness of derivative of the transition map.

Lemma 61. *Let $(U, \mathcal{A}_U)$ and $(V, \mathcal{A}_V)$ be subsets of $(X \times Y^+, X \times \{0\})$ and $\varphi_{VU} : \ (U, \mathcal{A}_U) \longrightarrow (V, \mathcal{A}_V)$ being a diffeomorphism between U and V mapping $\mathcal{A}_U \subset \partial U$ to $\mathcal{A}_V \subset \partial V$ bijectively and of bounded derivatives. Let $0 \leq \psi \leq 1$ be a smooth function compactly supported in V. Then the linear mapping $T : \mathcal{S}^*(X \times Y^+/X \times \{0\}) \longrightarrow \mathcal{S}^*(X \times Y^+/X \times \{0\}) : \ f \longrightarrow \psi.(f \circ \varphi_{VU}^{-1})$ extends to a bounded operator from $W^{k,p}(U/\mathcal{A}_U) \longrightarrow W^{k,p}(V, \mathcal{A}_V)$.*

We will sketch rapidly the (well known) ideas to define Sobolev spaces on half-plan and the trace operator in the next sections.

6.4.1 Sobolev spaces on half-plan

In this section, the Sobolev spaces on $X \times Y$ or $X \times Y^+$ are defined with weight $(\sigma_1, \ldots, \sigma_n, \rho)$ and $\sigma := \mathrm{lcm}(\sigma_1, \ldots, \sigma_n, \rho)$.

Smooth extensions

Let $\mathcal{S}(X \times Y^+)$ denote the space of smooth, rapidly decreasing functions (and all of their derivatives) on $X \times Y^+$ and $\mathcal{S}(X \times Y^+/0)$ denotes the subspace of functions who vanish, together with all their derivatives, at $X \times \{0\}$. Similar definition for $\mathcal{S}(X \times Y^-)$ and $\mathcal{S}(X \times Y^-/0)$. The following exact sequence is obvious and the arrows are continuous under Schwartz topology.

$$0 \longrightarrow \mathcal{S}(X \times Y^-/0) \xrightarrow{Z_-} \mathcal{S}(X \times Y) \xrightarrow{C_+} \mathcal{S}(X \times Y^+) \longrightarrow 0 \tag{6.5}$$

where Z_- be the extension by 0 and C_+ be the cut-off operator.

It is however not obvious that the sequence in (6.5) splits. Algebraically this is equivalent to the fact that C_+ admits a retraction, that we will note by E_+ since it is in fact an extension to the negative half-plan, which is continuous under Schwartz topology. The construction of E_+ is as follows

$$\begin{aligned} E_+ : \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y) \\ f &\longmapsto \left((x,y) \longmapsto \begin{cases} f(x,y), & \text{if } y \geq 0 \\ \int_0^\infty \varphi(\lambda) f(x, -\lambda y) d\lambda, & \text{if } y < 0 \end{cases} \right) \end{aligned}$$

where the difficult part is the choice of φ, which is resolved by the following lemma.

Lemma 62. *There exists a smooth function $\varphi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ such that $\int_0^{+\infty} x^n |\varphi(x)| dx < \infty \quad \forall n \in \mathbb{Z}$ and*

$$\int_0^{+\infty} x^n \varphi(x) dx = (-1)^n \quad \forall n \in \mathbb{Z} \setminus \{0\}$$

Moreover, $\varphi(\frac{1}{x}) = -x\varphi(x)$ for all $x > 0$.

In fact, the function

$$\varphi(x) = \frac{e^4}{\pi} . \frac{e^{-(x^{1/4} + x^{-1/4})} \sin(x^{1/4} - x^{-1/4})}{1+x}$$

works. The continuity of operator E_+ comes from these properties of φ and basic justification of Lebesgue's Dominated convergence. The projection R_- of Z_- in the sequence (6.5) is constructed algebraically:

$$\begin{aligned} R_- \ \mathcal{S}(X \times Y) &\longrightarrow \mathcal{S}(X \times Y^-/0) \\ f &\longmapsto f - E_+ C_+ f \end{aligned}$$

which is also continuous in Schwartz topology. To resume, one has the split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}(X \times Y^-/0) \arrow[r, shift left, "Z_-"] & \mathcal{S}(X \times Y) \arrow[l, shift left, "R_-"] \arrow[r, shift left, "C_+"] & \mathcal{S}(X \times Y^+) \arrow[l, shift left, "E_+"] \arrow[r] & 0 \end{tikzcd} \tag{6.6}$$

and a similar sequence for $\mathcal{S}(X \times Y^+/0$ and $\mathcal{S}(X \times Y^-)$ with operators Z_+, C_-, E_- and R_+.

Also, note that

$$\langle E_+ f, g \rangle = \langle f, R_+ g \rangle \tag{6.7}$$

where the first pairing is on $\mathcal{S}(X \times Y) \times \mathcal{S}(X \times Y)$ and the second is on $\mathcal{S}(X \times Y^+) \times \mathcal{S}(X \times Y^+/0)$.

Remark 23. 1. *The two pairings satisfy* $\langle D^\alpha u, v\rangle = (-1)^{|\alpha|}\langle u, D^\alpha v\rangle$.

2. *The second pairing gives two natural identifications*

$$\mathcal{S}(X\times Y^+/0)\hookrightarrow \mathcal{S}^*(X\times Y^+),\quad \mathcal{S}(X\times Y^+)\hookrightarrow \mathcal{S}^*(X\times Y^+/0)$$

while the first pairing gives $\mathcal{S}(X\times Y)\hookrightarrow \mathcal{S}^*(X\times Y)$.

3. (6.7) *shows that* E_+ *and* R_+ *are adjoint, strictly speaking* E_+ *is the restriction of* R_+^*, *that is*

$$\begin{tikzcd} \mathcal{S}(X\times Y^+) \arrow[r, "E_+"] \arrow[d, hook] & \mathcal{S}(X\times Y) \arrow[d, hook] \\ \mathcal{S}^*(X\times Y^+/0) \arrow[r, "R_+^*"] & \mathcal{S}^*(X\times Y) \end{tikzcd}$$

Similarly. since $\langle C_- f, g\rangle = \langle f, Z_- g\rangle$, one has

$$\begin{tikzcd} \mathcal{S}(X\times Y^-/0) \arrow[r, "Z_-"] \arrow[d, hook] & \mathcal{S}(X\times Y) \arrow[d, hook] \\ \mathcal{S}^*(X\times Y^-) \arrow[r, "C_-^*"] & \mathcal{S}^*(X\times Y) \end{tikzcd}$$

To resume, one can extend the sequence in (6.5) to the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}(X\times Y^-/0) \arrow[r, shift left, "Z_-"] \arrow[d, hook] & \mathcal{S}(X\times Y) \arrow[l, shift left, "R_-"] \arrow[r, shift left, "C_+"] \arrow[d, hook] & \mathcal{S}(X\times Y^+) \arrow[l, shift left, "E_+"] \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{S}^*(X\times Y^-) \arrow[r, shift left, "C_-^*"] & \mathcal{S}^*(X\times Y) \arrow[l, shift left, "E_-^*"] \arrow[r, shift left, "Z_+^*"] & \mathcal{S}^*(X\times Y^+/0) \arrow[l, shift left, "R_+^*"] \arrow[r] & 0 \end{tikzcd} \tag{6.8}$$

We will define Sobolev spaces $W^{k,p}(X\times Y^-/0)$ and $W^{k,p}(X\times Y^+)$ so that they form an intermediate row in diagram. Since the center cell $\mathcal{S}(X\times Y)\subset W^{k,p}(X\times Y)\subset \mathcal{S}^*(X\times Y)$ is already defined, there is only one natural way to do this.

Definition 9. 1. *The* ***Sobolev space upper on half-plan*** *is*

$$W^{k,p}(X\times Y^+) := \left\{ f\in \mathcal{S}^*(X\times Y^+/0):\ \exists g\in W^{k,p}(X\times Y), f = Z_+^* g\right\}$$

with norm $\|f\|_{W^{k,p}(X\times Y^+)} = \inf_g \|g\|_{W^{k,p}(X\times Y)}$.

2. *The* ***Sobolev space on lower half-plan with vanishing trace***

$$W^{k,p}(X\times Y^-/0) := \left\{ f\in \mathcal{S}^*(X\times Y^-):\ C_-^* f\in W^{k,p}(X\times Y)\right\}$$

with the induced norm $\|f\|_{W^{k,p}(X\times Y^-/0)} := \|C_-^* f\|_{W^{k,p}(X\times Y)}$.

Remark 24. *1. In other words, $W^{k,p}(X \times Y^-/0) = {C_-^*}^{-1}(W^{k,p}(X \times Y))$ and $W^{k,p}(X \times Y^+) = Z_+^*(W^{k,p}(X \times Y))$ and they are given by the induced norm and the quotient norm of $W^{k,p}(X \times Y)$ respectively. The operator C_-^* and Z_+^* are by definition bounded under Sobolev norm.*

2. The topology of $W^{k,p}(X\times Y)$ being finer than the induced of weak-* topology from $\mathcal{S}^*(X \times Y)$, the restricted operator $Z_+^*\big|_{W^{k,p}} : W^{k,p}(X\times Y) \longrightarrow \mathcal{S}^*(X\times Y^+/0)$ is continuous, hence $\ker Z_+^*\big|_{W^{k,p}} \subset W^{k,p}(X\times Y)$ is a closed subspace of the Banach space $W^{k,p}(X\times Y)$. But this is also the image by C_-^* of $W^{k,p}(X\times Y^-/0)$. Therefore $W^{k,p}(X\times Y^-/0)$ and $W^{k,p}(X\times Y^+)$ are Banach spaces.

3. Idem for $W^{k,p}(X \times Y^+/0)$ and $W^{k,p}(X \times Y^-)$.

Theorem 63. *1. For all $k \in \mathbb{R}$ and $p \in (1, \infty)$, the three lines of the following diagram are split-exact and the arrows of the second lines are bounded operators under Sobolev norms.*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}(X \times Y^-/0) \arrow[r, shift left, "Z_-"] \arrow[d, hook] & \mathcal{S}(X \times Y) \arrow[l, shift left, "R_-"] \arrow[r, shift left, "C_+"] \arrow[d, hook] & \mathcal{S}(X \times Y^+) \arrow[l, shift left, "E_+"] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & W^{k,p}(X \times Y^-/0) \arrow[r, shift left, "C_-^*"] \arrow[d, hook] & W^{k,p}(X \times Y) \arrow[l, shift left, "E_-^*"] \arrow[r, shift left, "Z_+^*"] \arrow[d, hook] & W^{k,p}(X \times Y^+) \arrow[l, shift left, "R_+^*"] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{S}^*(X \times Y^-) \arrow[r, shift left, "C_-^*"] & \mathcal{S}^*(X \times Y) \arrow[l, shift left, "E_-^*"] \arrow[r, shift left, "Z_+^*"] & \mathcal{S}^*(X \times Y^+/0) \arrow[l, shift left, "R_+^*"] \arrow[r] & 0
\end{tikzcd} \tag{6.9}$$

2. The subspaces $\mathcal{S}(X\times Y^-/0)$ and $\mathcal{S}(X\times Y^+)$ are dense in $W^{k,p}(X\times Y^0/0)$ and $W^{k,p}(X \times Y^+)$ respectively.

3. Interpolation theorem 56 holds for $W^{k,p}(X \times Y^-/0)$ and $W^{k,p}(X \times Y^+)$.

Proof. The commutativity of the diagram is purely algebraic. The continuity of C_-^* and Z_+^* in the $W^{k,p}$-row follows from the definition of norms in this row. The only non-trivial part is the continuity of E_-^* and R_+^* in the $W^{k,p}$-row, and it suffices to prove that $C_-^*E_-^*$ and $R_+^*Z_+^*$ are bounded as automorphism of $W^{k,p}(X \times Y)$. This follows from direct computation of these norm in the case $\sigma \mid k \in \mathbb{R}$ and interpolation inequality (Theorem 54) for intermediate k.

Once the continuity of E_-^* and R_+^* is established, the density of $\mathcal{S}(X \times Y^-/0)$ follows straight-forwardly and we see that $W^{k,p}(X \times Y^-/0)$ and $W^{l,p}(X \times Y^-/0)$ are interpolatable (the two spaces share a dense subspace). Theorem 58 applies and shows that Theorem 56 holds for $W^{k,p}(X \times Y^-/0)$.

Idem for the side of $\mathcal{S}(X \times Y^+) \subset W^{k,p}(X \times Y^+)$. □

Remark 25. *By dualising the diagram (6.9) and using the fact that the dual space of $W^{k,p}(X \times Y)$ is $W^{-k,p'}(X\times Y)$, one can prove that the dual space of $W^{k,p}(X\times Y^+)$ is $W^{-k,p'}(X\times Y^+/0)$.*

Functoriality of D_x and equivalent definitions

The following discussion appeared as 4 lemmas in [Ham75, page 38-42] in the proof of Vanishing trace theorem 65. I think these ideas can be presented without much computation.

Note that the weight $W(\xi,\eta) = \left(1+\xi_1^{2\sigma_1}+\cdots+\xi_n^{2\sigma_n}+\eta^{2\rho}\right)^{1/2\sigma}$ is comparable to $W(\xi)+W(\eta)$ where

$$W(\xi) = \left(1+\xi_1^{2\sigma_1}+\cdots+\xi_n^{2\sigma_n}\right)^{1/2\sigma}, \quad W(\eta) = \left(1+\eta^{2\rho}\right)^{1/2\sigma}$$

and also $W^k(\xi,\eta)$ is comparable to $W(\xi)^k+W(\eta)^k$. Hence $W(D_x)^l : W^{k,p}(X\times Y) \longrightarrow W^{k-l,p}(X\times Y)$ is a bounded operator.

The vertical arrows in the following diagram are the vertical arrows of (6.9). The dashed horizontal arrow indicates that it is established only in the center cells $W^{k,p}(X\times Y) \longrightarrow W^{k-l,p}(X\times Y)$.

$$\begin{tikzcd} & \mathcal{S}-\text{row} \arrow[dl] \arrow[dr] & \\ W^{k,p}-\text{row} \arrow[rr, dashed, "W(D_x)^l"] \arrow[dr] & & W^{k-l,p}-\text{row} \arrow[dl] \\ & \mathcal{S}^*-\text{row} & \end{tikzcd} \tag{6.10}$$

We will see that the dashed arrow can be extended to a full arrow, that is 3 arrows between the $W^{k,p}$-row and $W^{k-l,p}$-row that are compatible with the diagram (6.9).

One can construct $W(D_x)^l$ arrows from $W^{k,p}(X\times Y^-/0) \longrightarrow W^{k-l,p}(X\times Y^-/0)$ and $W^{k,p}(X\times Y^+) \longrightarrow W^{k-l,p}(X\times Y^+)$ as adjoint of $W(D_x)^l$ on $\mathcal{S}(X\times Y^+)$ and $\mathcal{S}(X\times Y^+/0)$. They are by definition continuous on the weak-* topology. It is easy to see that if we can prove that these two $W(D_x)^l$ arrows commute with C_-^*, E_-^* and Z_+^*, R_+^* on $W^{k,p}$-row and $W^{k-l,p}$-row, then by the continuity of the $W(D_x)^l$ arrow from $W^{k,p}(X\times Y) \longrightarrow W^{k-l,p}(X\times Y)$, these $W(D_x)^l$ arrows are bounded in $W^{k,p}$ norm.

The two new $W(D_x)^l$ arrows commute with all "$\longrightarrow$" arrows in the $W^{k,p}$-row of (6.9), i.e. C_-^* and Z_+^*, since for smooth functions, D_x commutes with Z_- (extension by 0) and C_+ (cut-off).

The fact that $W(D_x)^l$ commutes with the "$\longleftarrow$" arrows, i.e. E_-^* and R_+^* is due to:

$$\begin{tikzcd} \mathcal{S}(X\times Y^+) \arrow[r, "E_+"] \arrow[d, "W(D_x)^l"] & \mathcal{S}(X\times Y) \arrow[d, "W(D_x)^l"] \\ \mathcal{S}(X\times Y^+) \arrow[r, "E_+"] & \mathcal{S}(X\times Y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{S}(X\times Y) \arrow[r, "R_-"] \arrow[d, "W(D_x)^l"] & \mathcal{S}(X\times Y^-/0) \arrow[d, "W(D_x)^l"] \\ \mathcal{S}(X\times Y) \arrow[r, "R_-"] & \mathcal{S}(X\times Y^-/0) \end{tikzcd}$$

Remark 26. *There is no functoriality of D_y since for $y<0$*

$$D_y^l E_+ f(x,y) = \int_0^\infty (-\lambda)^l \varphi(\lambda) D_y^l f(x,-\lambda y)\,d\lambda \neq E_+ D_y^l f(x,y)$$

meaning that the D_y does not commute with E_+.

However $D_y^l E_+ f \in L^p(X\times Y)$ if and only if $E_+ D_y^l f \in L^p(X\times Y)$ if and only if $D_y^l f \in L^p(X\times Y^+)$. Moreover the 3 L^p norms are equivalent.

The density of $\mathcal{S}(X \times Y^-/0)$ and $\mathcal{S}(X \times Y^+)$ in the corresponding $W^{k,p}$ shows that the new $W^{k,p}$ spaces can also be defined by density using the $W^{k,p}$-norm of the extension (Z_- and E_+ respectively) from half-plan to the whole plan. By the continuity of R_+^* in the second row of (6.9) when $k = 0$, one sees that the L^p-norms of the extensions by Z_- and E_+ are equivalent to the L^p norm on the half-plan. Therefore, one has the following analog of Theorem 51.

Theorem 64. *Given $k > 0$ and $\sigma \mid k$,*

1. *If $f \in \mathcal{S}^*(X \times Y^+/0)$ then*
   (a) *$f \in W^{k,p}(X \times Y^+)$ if and only if $D_x^\alpha D_y^\beta f \in L^p(X \times Y^+)$ for $\|(\alpha, \beta)\| \leq k$.*
   (b) *$f \in W^{-k,p}(X \times Y^+)$ if and only if there exists $g_{\alpha\beta} \in L^p(X \times Y^+)$ such that $f = \sum_{\|(\alpha,\beta)\| \leq k} D_x^\alpha D_y^\beta g_{\alpha\beta}$.*
2. *If $f \in \mathcal{S}^*(X \times Y^+)$ then*
   (a) *$f \in W^{k,p}(X \times Y^+/0)$ if and only if $D_x^\alpha D_y^\beta f \in L^p(X \times Y^+)$ for $\|(\alpha, \beta)\| \leq k$.*
   (b) *$f \in W^{-k,p}(X \times Y^+/0)$ if and only if there exists $g_{\alpha\beta} \in L^p(X \times Y^+)$ such that $f = \sum_{\|(\alpha,\beta)\| \leq k} D_x^\alpha D_y^\beta g_{\alpha\beta}$.*

6.4.2 Trace theorems

To make the notation more intuitive, we abusively denote the horizontal arrows in the $W^{k,p}$-row and the $\mathcal{S}^*$-row by their corresponding arrows in the $\mathcal{S}$-row (i.e. their restriction on the space of smooth functions), that is we will use Z_-, C_+, R_-, E_+ instead of $C_-^*, Z_+^*, E_-^*, R_+^*$.

The goal of this section is to define the restriction of a function $f \in W^{k,p}(X \times Y^+)$ on $X \times \{0\}$. The pointwise restriction of f does not make sense because f is only defined up to a negligible set (i.e. of Lebesgue measure 0). The strategy is to take a sequence $f_n \in \mathcal{S}(X \times Y^+)$ that is $W^{k,p}$-converging to f and to see if $\{f_n\big|_{X \times \{0\}}\}$ converges in $L^p(X \times \{0\})$. If it does one calls the limit *trace* of f on $X \times \{0\}$. Theorem 65, Example 27 and Theorem 67 show that one should expect

- high regularity of f, i.e. k large enough, so that the limit exists,
- a drop of regularity of the restriction.

From diagram (6.9) and its opposite version (with all $+$ and $-$ signs interchanged), there is a natural inclusion $\iota :\ W^{k,p}(X \times Y^+/0)$ to $W^{k,p}(X \times Y^+)$, by first extending by zero, then cutting-off

$$\begin{tikzcd}
W^{k,p}(X \times Y^+/0) \arrow[rr, hook, "\iota"] \arrow[dr, "Z_+"'] & & W^{k,p}(X \times Y^+) \\
& W^{k,p}(X \times Y) \arrow[ur, "C_+"'] &
\end{tikzcd}$$

Theorem 65 (Vanishing trace). *If $p \in (1, +\infty)$ and $-1 + \frac{1}{p} < \rho\frac{k}{\sigma} < \frac{1}{p}$ then ι is an isomorphic*

Proof. Define

$$\begin{aligned} M_+(\lambda):\ \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y^+) \\ f(x,y) &\longmapsto f(x, \lambda y) \end{aligned}$$

Since $\langle M_+(\lambda) f, g\rangle = \langle f, N_+(\lambda) g\rangle$ for all $f \in \mathcal{S}(X \times Y^+), g \in \mathcal{S}(X \times Y^+/0)$ and $\lambda > 0$ where $N_+(\lambda) g(x,y) := \lambda^{-1} g(x, \lambda^{-1} y)$, one sees that $M_+(\lambda)$ extends to $\mathcal{S}^*(X \times Y^+/0) \longrightarrow \mathcal{S}^*(X \times Y^+/0)$ and that one extension of it is $N_+^*(\lambda)$ the adjoint of $N_+(\lambda)$:

$$\begin{tikzcd} \mathcal{S}(X \times Y^+) \arrow[r, "M_+(\lambda)"] \arrow[d, hook] & \mathcal{S}(X \times Y^+) \arrow[d, hook] \\ \mathcal{S}^*(X \times Y^+/0) \arrow[r, "N_+^*(\lambda)"] & \mathcal{S}^*(X \times Y^+/0) \end{tikzcd}$$

We abusively call $N_+^*(\lambda)$ by $M_+(\lambda)$. We will let $\lambda \to +\infty$, the operator $M_+(\lambda)$ intuitively "shrinks" to the boundary $X \times \{0\}$.

Lemma 66. *For $k \geq 0, \lambda \geq 1$, $M_+(\lambda) : W^{k,p}(X \times Y^+) \longrightarrow W^{k,p}(X \times Y^+)$ is bounded and*

$$\|M_+(\lambda) f\|_{W^{k,p}(X \times Y^+)} \leq C \lambda^{\frac{\rho k}{\sigma} - \frac{1}{p}} \|f\|_{W^{k,p}(X \times Y^+)}$$

where C does not depend on λ.

The proof of the Lemma 66 is straightforward: it suffices to prove the boundedness in the case $\sigma \mid k$ an use interpolation inequality 54, also one can suppose that $f \in \mathcal{S}(X \times Y^+)$. Note that $(\frac{\partial}{\partial y})^l M_+(\lambda) = \lambda^l M_+(\lambda) (\frac{\partial}{\partial y})^l$ while $\frac{\partial}{\partial x}$ commutes with $M_+(\lambda)$, hence in general $|D^\alpha_{(x,y)} M_+(\lambda) f| \leq \lambda^{k\rho/\sigma} |D^\alpha_{(x,y)} f|$ for all $\|\alpha\| \leq k, \lambda \geq 1$. The $-\frac{1}{p}$ in the exponent of λ is due to: $\|M_+(\lambda) g\|_{L^p} = \lambda^{-1/p} \|g\|_{L^p}$.

Back to Theorem 65, let $f \in \mathcal{S}(X \times Y^+)$ and define $\tilde{M}(\lambda) f$ to be f on $X \times Y^+$ and $M_-(\lambda) C_- E_+ f$ on $X \times Y^-$, then $\tilde{M}(\lambda) f \in W^{\sigma/\rho, p}(X \times Y)$. Note that $D_y \tilde{M}(\lambda) f$ is not continuous at $X \times \{0\}$ but is still in $L^p(X \times Y)$ because f and $M_-(\lambda) C_- E_+ f$ agrees on $X \times \{0\}$. Suppose we can prove that as $\lambda \to +\infty$ the sequence $\tilde{M}(\lambda) f$ converges to $\tilde{M} f$ in $W^{k,p}(X \times Y)$ then $C_- \tilde{M} f = \lim_{\lambda \to +\infty} M_-(\lambda) C_- E_+ f = 0$. One obtains, by exactness of the second row of diagram (6.9), existence of a $g \in W^{k,p}(X \times Y^+/0)$ such that $\tilde{M} f = Z_+ g$. Moreover, since $C_+ \tilde{M}(\lambda) f = f$ for all $\lambda > 0$, one has $C_+ \tilde{M} f = f$, hence $\iota g = C_+ Z_+ g = C_+ \tilde{M} f = f$.

It remains to prove the existence of such $\tilde{M} f$. By Lemma 66 and the fact that all $\tilde{M}(\lambda) f$ are the same on $X \times Y^+$, one has

$$\|\tilde{M}(\lambda) f - \tilde{M}(2\lambda) f\|_{W^{k,p}(X \times Y)} \leq 2C \lambda^{\frac{\rho k}{\sigma} - \frac{1}{p}} \|f\|_{W^{k,p}(X \times Y^+)}$$

Therefore if $\frac{\rho k}{\sigma} < \frac{1}{p}$, the sequence $\tilde{M}(2^n) f$ converge in $W^{k,p}(X \times Y)$ to $\tilde{M} f$. $\square$

Remark 27. *If $\rho = \sigma_i = 1$ then $\sigma = 1$, take $k = 0$ then the Theorem 65 claims that $\mathcal{S}(X \times Y^+/0)$ is dense in $L^P(X \times Y^+) \supset \mathcal{S}(X \times Y^+)$, or equivalently any smooth function $f \in \mathcal{S}(X \times Y^+)$ not necessarily vanishes on $X \times \{0\}$ can be L^p-approximated by smooth functions with all derivative vanishes on $X \times \{0\}$. This means that one cannot define any notion of trace on $X \times \{0\}$ that varies continuously under the L^p norm.*

In case of high regularity $\frac{\rho k}{\sigma} > \frac{1}{p}$, one can define a meaningful notion of trace.

Theorem 67 (Well-defined trace). *If $\frac{\rho k}{\sigma} > \frac{1}{p}$ then the restriction map*

$$\begin{aligned} B : \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X) \\ f(x,y) &\longmapsto f(x,0) \end{aligned}$$

extends to a bounded operator, abusively noted by $B:\ W^{k,p}(X \times Y^+) \longrightarrow L^p(X)$.

Definition 10. *We call $\partial W^{k,p}(X \times Y^+) := W^{k,p}(X \times Y^+)/\ker B$ the* ***space of boundary value*** *of function in $W^{k,p}(X \times Y^+)$.*

Theorem 67 can be strengthen by remarking that if $\sigma := \mathrm{lcm}(\sigma_1, \dots, \sigma_n, \rho) = \mathrm{lcm}(\sigma_1, \dots, \sigma_n)$ and if $W(\xi)$ denotes the weight $(1 + \xi_1^{2\sigma_1} + \dots + \xi_n^{2\sigma_n})^{1/2\sigma}$ then B and $W(D_x)$ commute, i.e.

$$\begin{tikzcd} W^{k,p}(X \times Y^+) \arrow[r, "B"] \arrow[d, "W(D_x)^l"'] & L^p(X) \subset \mathcal{S}^*(X) \arrow[d, "W(D_x)^l"] \\ W^{k-l,p}(X \times Y^+) \arrow[r, "B"] & L^p(X) \end{tikzcd}$$

as long as $\frac{\rho(k-l)}{\sigma} > \frac{1}{p}$. Therefore, one has

Theorem 68 (Regularity of trace). *If $0 \le l < k - \frac{\sigma}{\rho p}$ then the trace operator B in Theorem 67 actually of image in $W^{l,p}(X)$ and the operator*

$$B:\ W^{k,p}(X \times Y^+) \longrightarrow W^{l,p}(X)$$

is bounded.

Proof of Theorem 67. It suffices to prove that $\|Bf\|_{L^p(X)} \le C\|f\|_{W^{k,p}(X\times Y^+)}$ for all $f \in \mathcal{S}(X \times Y^+)$ and $1 \ge \frac{\rho k}{\sigma} > \frac{1}{p}$ (for higher k, embed in the $W^{k,p}$ smaller k). Define

$$\begin{aligned} T_v:\ \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y^+) \\ f &\longmapsto \left((x,y) \longmapsto \frac{1}{v}\int_0^v f(x, y+w)dw\right) \end{aligned}$$

for $v > 0$. One can check that T_v extends to a bounded operator $T_v :\ W^{k,p}(X \times Y^+) \longrightarrow W^{k,p}(X \times Y^+)$ for all $k \ge 0$ and that

$$\begin{cases} \|D_y T_v f\|_{L^p(X\times Y^+)} \le C v^{-1} \|f\|_{L^p(X\times Y^+)}, \\ \|D_y T_v f\|_{L^p(X\times Y^+)} \le C \|f\|_{W^{\sigma/\rho,p}(X\times Y^+)} \end{cases}$$

hence by Interpolation inequality Theorem 54, one obtains for all $0 \leq k \leq \sigma/\rho$: $\|D_y T_v f\|_{L^p(X\times Y^+)} \leq Cv^{\rho k/\sigma - 1}\|f\|_{W^{k,p}(X\times Y^+)}$ hence

$$\|D_y(T_{v/2} - T_v f)\|_{L^p(X\times Y^+)} \leq Cv^{\rho k/\sigma - 1}\|f\|_{W^{k,p}(X\times Y^+)} \tag{6.11}$$

Similarly, one can prove that for all $0 \leq k \leq \sigma/\rho$: $\|(\mathrm{Id} - T_v)f\|_{L^p(X\times Y^+)} \leq Cv^{\rho k/\sigma}\|f\|_{W^{k,p}(X\times Y^+)}$ therefore

$$\|(T_{v/2} - T_v)f\|_{L^p(X\times Y^+)} \leq Cv^{\rho k/\sigma}\|f\|_{W^{k,p}(X\times Y^+)} \tag{6.12}$$

Moreover, using Hölder inequality and Fundamental theorem of calculus, one has: if $g \in \mathcal{S}(X \times Y^+)$ then

$$\|Bg\|_{L^p(X)} \leq C\|g\|^{1/p'}_{L^p(X\times Y^+)}\|D_y g\|^{1/p}_{L^p(X\times Y^+)} \tag{6.13}$$

Substitute g by $(T_{v/2} - T_v)f$ in (6.13) then use apply (6.11) and (6.12), one has

$$\|B(T_{v/2} - T_v)\|_{L^p(X)} \leq Cv^{\frac{\rho k}{\sigma} - \frac{1}{p}}\|f\|_{W^{k,p}(X\times Y^+)}$$

Therefore if $\frac{1}{p} < \frac{\rho k}{\sigma} \leq 1$, the sequence $BT_{2^{-n}}f$ converges in $L^p(X)$ and the limit is of L^p-norm less than $C\|f\|_{W^{k,p}(X\times Y^+)}$. Since f is continuous, the limit is $f\Big|_{X\times\{0\}}$. The theorem follows. $\square$

Remark 28. *The fact that the condition on l in Theorem 68 is an open condition explains why we topologize the space of boundary value $\partial W^{k,p}(X \times Y^+)$ by the quotient $W^{k,p}$-norm instead of any $W^{l,p}$-norm. Also, we have completeness for free.*

In the proof of Theorem 65, we glue a function $f_+ \in \mathcal{S}(X \times Y^+)$ with $f_- \in \mathcal{S}(X \times Y^-)$ of the same value on $X \times \{0\}$ and the result is a function in $W^{\sigma/\rho,p}(X \times Y)$. This can be generalised as follow

Theorem 69 (Patching theorem)**.** *If $p \in (1, +\infty)$ and $\frac{1}{p} < \rho\frac{k}{\sigma} < 1 + \frac{1}{p}$, then given $f_+ \in W^{k,p}(X \times Y^+)$ and $f_- \in W^{k,p}(X \times Y^-)$ such that $Bf_+ = Bf_-$ in $L^p(X)$, one defines $f \in L^p(X\times Y)$ such that $f = f_+$ on $X\times Y^+$ and $f = f_-$ on $X\times Y^-$. Then actually $f \in W^{k,p}(X\times Y)$.*

6.4.3 Trace operator on manifold

The following paragraph does not appear in [Ham75] because of Remark 22.

To resume, we have defined Sobolev spaces on manifold with boundary as the space of currents whose cut-off restrictions on each chart are in $W^{k,p}$. Also we have defined trace operator of Sobolev spaces on half-plan in a vision to extend the notion to manifold.

Let $f \in W^{k,p}(M/\mathcal{A})$ and $\mathcal{B}$ be a connected component of ∂M. With the same notation as Definition 8, f gives the data of $f_i = (\psi_i f) \circ \varphi_i^{-1} \in W^{k,p}(R_i/\mathcal{A}_i)$ the cut-off restriction of f on each chart using a partition of unity $\{\psi_i\}_i$ subordinated to a good atlas $(U_i)_i$ of M, where R_i is an Euclidean space of the same dimension as M ($\mathcal{A}_i = \emptyset$), or a half-plan ($\mathcal{A}_i \subset \partial R_i$). Note that $U_i \cap \mathcal{B}$ is a good atlas of $\mathcal{B}$ and ψ_i is still a partition of unity subordinated to this atlas, therefore take $g_i \in W^{l,p}(\partial R_i)$ to be trace of f_i on the image of $\mathcal{B}$ of each chart. It remains to check that the data (g_i) corresponds to a unique element $g \in W^{l,p}(\mathcal{B})$. Recall that we have the following diagram:

$$0 \longrightarrow W^{l,p}(\mathcal{B}) \underset{\pi}{\overset{\iota}{\rightleftarrows}} \bigoplus_i W^{l,p}(\partial R_i)$$

where ι admits a projection π given by the cut-off functions $\tilde{\psi}_i$ that we choose to be the same ones used for M. Hence to see that $(g_i)_i$ is in the image of ι, it suffices to check that $\iota \circ \pi((g_i)_i) = (g_i)_i$ which should be straightforward, since $\sum_i \tilde{\psi}_i \psi_i = 1$.

Now that we defined a trace operator $B : W^{k,p}(M) \longrightarrow L^p(\partial M)$ that factor through $W^{k,p}(M) \longrightarrow W^{l,p}(\partial M)$ for all $0 \leq l < k - \frac{\sigma}{\rho p}$, we can define the space of boundary value of function in $W^{k,p}(M)$ by

$$\partial W^{k,p}(M) := W^{k,p}(M)/\ker B$$

which has a finer topology than its image in any $W^{l,p}(\partial M)$ for $0 \leq l < k - \frac{\sigma}{\rho p}$.

Chapter 7

Elliptic and parabolic equations on compact manifolds

7.1 Commutative diagram and linear PDE. Example: Semi-elliptic equation on $\mathbb{R}^n$

Fix a weight $(\sigma_1, \ldots, \sigma_n)$ on $X = \mathbb{R}^n$ and recall that for an index α, we note $\|\alpha\| := \sum_i \frac{\sigma}{\sigma_i}\alpha_i$.

We will consider in this section a partial differential operator A that is heterogeneous, of constant coefficient and of weight r, i.e.

$$A(D) = \sum_{\|\alpha\|=r} a_\alpha D^\alpha, \quad D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha$$

The **symbol** of A is $A(\xi) := \sum_{\|\alpha\|=r} a_\alpha \xi^\alpha$ and A is called **semi-elliptic** if $A(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus 0$

Remark 29. *If A is semi-elliptic then $\sigma \mid r$. In fact choose all $\xi_j = 0$ except $\xi_i \neq 0$, one sees that there must be a non-zero coefficient $a_{(0,\ldots,\frac{r\sigma_i}{\sigma},\ldots,0)}$, i.e. $\frac{r\sigma_i}{\sigma} \in \mathbb{Z}$ for all $i = \overline{1,n}$. Hence $\sigma \mid r$ ($\sigma = \operatorname{lcm}(\sigma_i)$ being a combination of σ_i, look at the same combination of $\frac{r\sigma_i}{\sigma}$).*

It is clear that the operator $A : W^{n,p}(X) \longrightarrow W^{n-r,p}(X)$ is bounded for all $n \in \mathbb{R}$ and the following diagram commutes for every real numbers $k < n$.

$$\begin{tikzcd} W^{n,p}(X) \arrow[r, "A(D)"] \arrow[d, hook, "i"] & W^{n-r,p}(X) \arrow[d, hook, "i"] \\ W^{k,p}(X) \arrow[r, "A(D)"] & W^{k-r,p}(X) \end{tikzcd} \tag{7.1}$$

Definition 11. *Let E, F, G, H be Banach spaces and l, m, p, q are bounded operator such that* *the following diagram* (diag:D) *commutes*

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "m"] & F \arrow[d, "p"] \\ G \arrow[r, "q"] & H \end{tikzcd} \tag{diag:D}$$

Then (diag:D) *is said to be an* ***exact square*** *if the following associated sequence is exact*

$$0 \longrightarrow E \xrightarrow{l \oplus m} F \oplus G \xrightarrow{p \ominus q} H \longrightarrow 0$$

Example 6. *If* (A, B) *is an interpolatable pair of Banach spaces then*

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A + B \end{tikzcd}$$

is exact, where arrows are natural inclusions.

The notion of exact square allows us to reformulate classical results of elliptic equation as

Theorem 70 (Elliptic equation with constant coefficients)**.** *The square* (7.1) *is exact for all* $k < n$ *in* $\mathbb{R}$*. This encodes the following 3 results:*

1. $W^{n,p}(X) \xhookrightarrow{A \oplus i} W^{n-r,p}(X) \oplus W^{k,p}(X)$ *is of closed image, i.e. there exists* $C > 0$ *such that*
$$\|f\|_{W^{n,p}(X)} \leq C \left(\|Af\|_{W^{n-r,p}(X)} + \|f\|_{W^{k,p}(X)} \right)$$
which is Gårding's inequality.

2. $\ker A \ominus i = \operatorname{Im} A \oplus i$*, i.e. if* $f \in W^{k,p}(X)$ *and* $Af \in W^{n-r}(X)$ *then actually* $f \in W^{n,p}(X)$*, which is regularity theorem.*

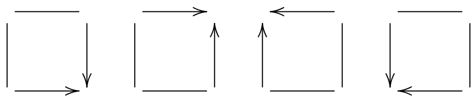
3. $\operatorname{Im} A \ominus i = W^{k-r,p}(X)$*, i.e. for all* $g \in W^{k-r,p}(X)$*, there exists* $f \in W^{k,p}(X)$ *such that* $Af - g \in W^{n-r,p}(X)$*, which is the existence of approximate solution (the idea behind* parametrix*).*

A way to prove that a square is exact is to show that it splits

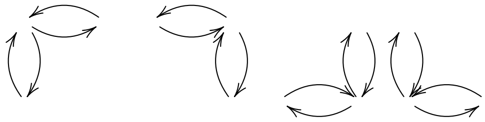
Definition 12. *The square* (diag:D) *is called* ***split*** *if there exists* l', m', p', q' *such that*

$$\begin{tikzcd} E \arrow[r, shift right, "l"'] \arrow[d, shift left, "m"] & F \arrow[l, shift right, "l'"'] \arrow[d, shift right, "p"'] \\ G \arrow[r, shift left, "q"] \arrow[u, shift left, "m'"] & H \arrow[l, shift left, "q'"] \arrow[u, shift right, "p'"'] \end{tikzcd} \tag{7.2}$$

commutes in 4 ways:



and splits in 4 ways



,

i.e. the sum of two circle in each diagram is the identities.

Theorem 71. *1. A split square is exact. In fact, if a square splits, then the associated short sequence splits.*

2. If E, F, G, H are Hilbert spaces then any exact square splits.

Proof of Theorem 70 . Since A is semi-elliptic, there exists $\epsilon > 0$ such that $|A(\xi)| \geq \epsilon \|\xi\|^r$ for $\|\xi\| := (\xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n})^{1/2\sigma}$. Let $\psi(\xi)$ be a radial function in ξ that is identically 1 for $\|\xi\| \leq 1$ and 0 for $\|\xi\| \geq 2$ and define

$$G(\xi) := \begin{cases} \frac{1-\psi(\xi)}{A(\xi)}, & \text{if } \|\xi\| \geq 1 \\ 0, & \text{if } \|\xi\| \leq 1 \end{cases}$$

Then by Stein's multiplier theorem,

$$\begin{aligned} G(D):\ W^{k-r,p}(X) &\longrightarrow W^{k,p}(X) \quad \forall k \in \mathbb{R}, \\ \psi(D):\ W^{l,p}(X) &\longrightarrow W^{k,p}(X) \quad \forall k, l \in \mathbb{R} \end{aligned}$$

are bounded operators. We say that $G(D)$ is an *approximate inverse* of $A(D)$ because $G(D)A(D) = A(D)G(D) = 1 - \psi(D)$. It is easy to check that (7.1) splits:

$$\begin{tikzcd} W^{k,p}(X) \arrow[r, shift right, "A(D)"'] \arrow[d, shift left, "i"] & W^{k-r,p}(X) \arrow[l, shift right, "G(D)"'] \arrow[d, shift right, "i"'] \\ W^{l,p}(X) \arrow[u, shift left, "\psi(D)"] \arrow[r, shift left, "A(D)"] & W^{l-r,p}(X) \arrow[l, shift left, "G(D)"] \arrow[u, shift right, "\psi(D)"'] \end{tikzcd}$$

□

The following abstract result shows that solutions of homogeneous equation $Af = 0$ are smooth (also proved in the second point of Theorem 70) and the solution space is of finite dimension.

Theorem 72. *Suppose that the square*

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "m"] & F \arrow[d, "p"] \\ G \arrow[r, "q"] & H \end{tikzcd}$$

is exact and m, p are compact operators. Then l and q have closed image, and their kernels and cokernels are isomorphic through m and p, and are of finite dimensional.

Proof. By basic diagram chasing, one can see that the restriction of m is an isomorphism $\ker l \longrightarrow \ker q$. But m is compact, $\ker l \cong \ker q$ are locally compact, hence of finite dimension.

It is easy to check (with sequential limit) that $\operatorname{Im} l$ is closed in F, since $\operatorname{Im}(l \oplus m) = \ker p \ominus q$ is closed and m is compact. So $\operatorname{coker} l$ is a Banach space.

Let $p'' : \operatorname{coker} l = F/l(E) \longrightarrow H/\overline{q(G)}$ be the map induced by p to the quotients, note that we have to take the closure of $q(G)$ to ensure that the quotient is Banach. Then p'' is obviously continuous and compact. Also p'' is surjective because $F \oplus G \xrightarrow{p \ominus q} H$.

We will prove that p'' is injective. If $f \in F \setminus l(E)$ then by Hahn-Banach theorem, there exists a linear functional $\lambda \in F^*$ such that $\lambda(f) = 1$ and $\lambda(l(E)) = 0$. One has

$$0 \longrightarrow H^* \xrightarrow{(p\ominus q)^*} F^* \oplus G^* \xrightarrow{(l\oplus m)^*} E^* \longrightarrow 0$$

and that $(l \oplus m)^*(\lambda \oplus 0) = 0$, hence there exists $\lambda' \in H^*$ such that $\lambda \oplus 0 = (p \ominus q)^* \lambda'$, i.e. $\lambda' \circ q = 0$ and $\lambda' \circ p = \lambda$, which means λ' vanishes on $q(G)$, hence $\overline{q}(G)$, and that $\lambda'(p(f)) = \lambda(f) = 1$. Hence $p(f) \notin \overline{q(G)}$ and p'' is injective.

The injectivity of p'' has 2 consequences. First, it means that $\operatorname{coker} l \cong H/\overline{q(G)}$ by a compact operator, hence the two are locally compact and of finite dimension.

Second, it proves that $q(G)$ is closed in H. In fact, given $h \in \overline{q(G)}$, by surjectivity of $p \ominus q$, one has $h = pf + qg$ for $f \in F$ and $g \in G$, this means $p''(\bar{f}) = \bar{0} \in H/\overline{q(G)}$, hence $\bar{f} = \bar{0} \in \operatorname{coker} l$, i.e. $f = l(e)$ for some $e \in E$. Therefore

$$h = p \circ l(e) + q(g) = q(m(e) + g) \in q(G)$$

and $p(G) = \overline{p(G)}$ is closed in H. □

Remark 30. *The proof of Theorem 72 is much simpler for split squares. We presented the version for exact squares because we will use it later. The advantage of using exact squares instead of split square is, as we will see, that among commutative squares, exact squares form a relatively open set, allowing us to "pertube" an exact square and extend the theory to cover the case A of variable coefficients.*

7.2 Elliptic equation on half-plan $X \times Y^+$. Boundary conditions.

We will quickly review in this part the ideas to solve elliptic equations with constant coefficients on half-plan. This does not require any more abstract (i.e. with diagram) results. The main tasks will be using suitable cut-off function on the frequent space (1) to define the approximate inverse of an elliptic operator on half-plan that is adapted to the boundary structure and (2) to approximately inverse the boundary operators.

We will solve elliptic equation on $X \times Y^+$ where the variables are $x_1, \dots, x_n$ and y, under weight $\Sigma = (\sigma_1, \dots, \sigma_n, \rho)$. Recall that $A(D) = \sum_{\|(\alpha,\beta)\|=r} a_{\alpha\beta} D_x^\alpha D_y^\beta$ with symbol $A(\xi, \eta) = \sum_{\|(\alpha,\beta)\|=r} \xi^\alpha \eta^\beta$.

If $A(D)$ is semi-elliptic then for all $\xi \neq 0$ the polynomial $\eta \mapsto A(\xi, \eta)$ has no real zeros, hence can be factorized to

$$A(\xi, \eta) = A^+(\xi, \eta) A^-(\xi, \eta)$$

where $A^+(\xi, \eta)$ (resp. $A^-(\xi, \eta)$) only has zeros η with $\operatorname{Im} \eta > 0$ (resp. $\operatorname{Im} \eta < 0$).

Remark 31.

1. *By semi-ellipticity, the monomial $a_{\alpha\beta}\xi^\alpha\eta^\beta$ with biggest β has index $\alpha = 0$. Hence we can suppose that the leading coefficients, as polynomials in η of A, A^+, A^- are 1.*

2. *As polynomial in η, $A^+(\xi, \eta) = \sum_{\beta=0}^m a_\beta^+(\xi)\eta^\beta$ where $m = r\rho/\sigma$ and $a_\beta^+(\xi)$ are Σ-heterogeneous of weight $(m - \beta)\rho$, i.e.*

$$a_\beta^+(t^{\sigma/\sigma_1}\xi_1, \dots, t^{\sigma/\sigma_n}\xi_n) = t^{(m-\beta)\rho} a_\beta^+(\xi)$$

Also, the coefficients a_β^+ are smooth in ξ.

We will solve the elliptic equation under some *suitable* boundary conditions. Let B^j, $1 \leq j \leq m$ be m Σ-heterogeneous *boundary operators* of weights r_j, i.e.

$$B^j(D) = \sum_{\|(\alpha,\beta)\|=r_j} b_{\alpha\beta}^j D_x^\alpha D_y^\beta$$

of symbol

$$B^j(\xi, \eta) = \sum_{\|(\alpha,\beta)\|=r_j} b_{\alpha\beta}^j \xi^\alpha \eta^\beta = \sum_{\|(\alpha,\beta)\|=r_j} b_\beta^j(\xi)\eta^\beta$$

where b_β^j are heterogeneous in ξ (actually polynomials) and of weight $r_j - \beta$.

As our discussion on trace operator, if $k > r_j + \frac{\sigma}{\rho p}$ then B^j extends to a bounded operator

$$\begin{tikzcd}
W^{k,p}(X \times Y^+) \arrow[rr, "B_j"] \arrow[dr] & & \partial W^{k-r_j,p}(X) \arrow[r, hook] & W^{l,p}(X) \\
& W^{k-r_j,p}(X \times Y^+) \arrow[ur] & &
\end{tikzcd}$$

for all $0 \leq l < n - r_j - \frac{\sigma}{\rho p}$.

Definition 13. *We will say that the operators $Bf = (B^1 f, \dots, B^m f)$ satisfy the* ***complementary boundary condition (CBC)*** *if the*

$$\det \left(c^j_\beta(\xi) \right)_{j,\beta} \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

where $c^j_\beta(\xi)$ are the cofficients of the remanders $C^j(\xi,\eta)$ when one divides $B^j(\xi,\eta)$ by $A^+(\xi,\eta)$ as polynomials in η, i.e.

$$B^j(\xi,\eta) \equiv C^j(\xi,\eta) = \sum_{\beta=0}^{m-1} c^j_\beta(\xi)\eta^\beta \qquad \mathrm{mod}\ A^+(\xi,\eta)$$

Approximate inverse of boundary operator B. The CBC condition allows us to approximately inverse boundary operator B.

Theorem 73 (Approximate inverse of B)**.** *Let $B: \ \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X)^{\oplus m}$ be a boundary operator that satisfies CBC condition, then there exists an operator*

$$\begin{aligned} H : \mathcal{S}(X)^{\oplus m} &\longrightarrow \mathcal{S}(X \times Y^+) \\ (h_1, \dots, h_m) &\longmapsto H_1 h_1 + \dots + H_m h_m \end{aligned}$$

such that

1. $(\mathrm{Id} - BH)h = \psi(D_x)h$ *for all* $h \in \mathcal{S}(X)^{\oplus m}$.
2. $(\mathrm{Id} - HB)f = \psi(D_x)f$ *for all* $f \in \ker A(D) : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$.

where $\psi(\xi)$ is the radial smooth cut-off function in ξ that equals 1 when $\|\xi\| \leq 1$ and 0 when $\|\xi\| \geq 2$.

Moreover, if $k > r_j + \frac{\sigma}{\rho p}$ then the operators $H_j : \mathcal{S}(X) \longrightarrow \mathcal{S}(X \times Y^+)$ extends to a bounded operator

$$H_j : \ \partial W^{n-r_j,p}(X) \longrightarrow W^{k,p}(X \times Y^+)$$

Sketch of proof. We define $H_j : \ \mathcal{S}(X) \longrightarrow \mathcal{S}(X \times Y^+)$ by its action on the frequent space of X, in particular, set

$$\tilde{H_j h}(\xi,\eta) := H_j(\xi,y)\tilde{h}(\xi)$$

where $\tilde{f}$ is the partial (in x) Fourier transform of f and $H_j(\xi,y)$ is given by

$$H_j(\xi,y) := (1 - \psi(\xi)) \int_\Gamma \sum_{\alpha=0}^{m-1} e^\alpha_j(\xi) \frac{A^+_\alpha(\xi,\eta)}{A^+(\xi,\eta)} e^{i\eta y} d\eta$$

where $\Gamma \subset \mathbb{C}$ is a curve enclosing all zeros of $A(\xi,\eta)$ with $\mathrm{Im}\,\eta > 0$, $(e^\alpha_j(\xi))_{\alpha,j}$ is the inverse matrix of $(c^j_\beta(\xi))_{j,\beta}$ and $A^+_\alpha(\xi,\eta) := \sum_{\beta=0}^{m-\alpha-1} a^+_{\alpha+\beta+1}(\xi)\eta^\beta$. $\square$

Some auxilary functions. We cannot use the operator G as in the case of whole plan as an inverse of A on the half-plan $X \times Y^+$, since we only have access to the frequent space of X. However we can modify the cut-off function to create an approximate inverse of A on the half-plan.

Let $\varphi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ be the function that we used in the definition of E_+, i.e.

$$\varphi(x) := \frac{e^4}{\pi} . \frac{e^{-(x^{1/4}+x^{-1/4})} \sin(x^{1/4} - x^{-1/4})}{1+x}, \quad x \geq 0$$

with the properties $\int_0^\infty x^n \varphi(x)dx = (-1)^n$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $\int_0^\infty \varphi(x)dx = 0$. Extending φ by 0 for $x < 0$, one still has a smooth function. Define $\chi(y) := -\varphi(-y-1)$, then $\chi \in \mathcal{S}(Y)$, with support in $(-\infty, -1]$ and

$$\int_{\mathbb{R}} y^n \chi(y)dy = \begin{cases} 0, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \end{cases}$$

In the frequent space of Y, this means $\hat{\chi}(0) = 1$ and $D_\eta^k \hat{\chi}(0) = 0$, i.e. $1 - \hat{\chi}(\eta)$ has a zero of infinite order at $\eta = 0$.

Also, since $\chi = 0$ when $y > -1$, the convolution

$$f \longmapsto \hat{\chi}(D_y)f = \chi * f$$

maps $\mathcal{S}(Y^-/0)$ to itself, hence induces a mapping from $\mathcal{S}(Y^+)$ to itself, since

$$0 \longrightarrow \mathcal{S}(Y^-/0) \longrightarrow \mathcal{S}(Y) \longrightarrow \mathcal{S}(Y^+) \longrightarrow 0$$

(given any $f \in \mathcal{S}(Y^+)$, any extension $\tilde{f}$ of f to $\mathcal{S}(Y)$ has the same restriction of $\hat{\chi}(D_y)\tilde{f}$ on Y^+).

Let $w(\xi, \eta) := \psi(\xi)\hat{\chi}(\eta)$ then w defines an operator

$$w(D) : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$$

In fact, for all $k, l \in \mathbb{R}$, there exists $C > 0$ such that

$$\|w(D)f\|_{W^{k,p}(X \times Y^+)} \leq C\|f\|_{W^{l,p}(X \times Y^+)}.$$

Approximate inverse of elliptic operator A on half-plan. The auxilary function w will play the role of ψ in the whole plan case.

Theorem 74 (Approximate inverse of A on $X \times Y^+$)**.** *There exists an operator $G : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$ such that:*

1. $(\mathrm{Id} - AG) = w(D)$

2. *For all $k, l \in \mathbb{R}$, there exists $C > 0$ such that for all $f \in \mathcal{S}(X \times Y^+)$:*

$$\|(\mathrm{Id} - GA)\psi(D_x)\|_{W^{k,p}(X\times Y^+)} \leq C\|f\|_{W^{l,p}(X\times Y^+)}$$

Also G extends to a bounded operator $G: \; W^{k-r,p}(X \times Y^+) \longrightarrow W^{k,p}(X \times Y^+)$ for all $k \in \mathbb{R}$.

Sketch of proof. In fact G is defined as follows:

$$G_0(\xi, \eta) := \frac{1 - w(\xi, \eta)}{A(\xi, \eta)}$$

which is smooth at $(0, 0)$, where $1 - w$ has a zero of infinite order. Then $G_0(D) : \mathcal{S}(X \times Y) \longrightarrow \mathcal{S}(X \times Y)$ extends to $W^{k-r,p}(X \times Y) \longrightarrow W^{k,p}(X \times Y)$. Finally, take $G = C_+GE_+$, which maps $\mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$ by first extending a function to the whole plan, applying G_0 and finally cutting-off. $\square$

Approximate inverse of the combined operator. Let $\mathcal{C}$ be the combined operator:

$$\begin{aligned} \mathcal{C} : \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y^+) \oplus \mathcal{S}(X)^{\oplus m} \\ f &\longmapsto (Af, Bf) \end{aligned}$$

and define the operator $\mathcal{J}$ as

$$\begin{aligned} \mathcal{J} : \mathcal{S}(X \times Y^+) \oplus \mathcal{S}(X)^{\oplus m} &\longrightarrow \mathcal{S}(X \times Y^+) \\ (g, h) &\longmapsto Gg + H(h - BGg) \end{aligned}$$

then one can prove with straightforward computation that $\mathcal{J}$ is an approximate inverse of $\mathcal{C}$.

Theorem 75 (Approximate inverse of C)**.** *For smooth functions $f \in \mathcal{S}(X \times Y^+)$ and $(g, h) \in \mathcal{S}(X \times Y^+) \oplus \mathcal{S}(X)^{\oplus m}$, one has*

1. $(\mathrm{Id} - \mathcal{C}\mathcal{J})(g, h) = (w(D)g, \psi(D_x)(h - BGg)) =: \lambda(g, h)$

2. $(\mathrm{Id} - \mathcal{J}\mathcal{C})f = \psi(D_x)\,(\mathrm{Id} - GA)f + (\mathrm{Id} - HB - \psi(D_x))\,w(D)f =: \mu(f)$

   Since G, H extend to Sobolev spaces, one also has

$$\mathcal{J}: \; W^{k-r,p}(X \times Y^+)\bigoplus_{j=1}^{m} \partial W^{k-r_j,p}(X) \longrightarrow W^{k,p}(X \times Y^+)$$

whenever $k > \frac{\sigma}{\rho p} + \max_j r_j$.

Theorem 76. *In analogue of Theorem 70, one has the following exact-split square*

$$\begin{tikzcd} W^{k,p}(X \times Y^+) \arrow[r, shift left, leftarrow, "\mathcal{J}"] \arrow[r, shift right, "\mathcal{C}"'] \arrow[d, shift left, "\iota"] & W^{k-r,p}(X \times Y^+) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(X) \arrow[d, shift right, "\iota"'] \\ W^{l,p}(X \times Y^+) \arrow[u, shift left, "\mu"] \arrow[r, shift left, "\mathcal{C}"] \arrow[r, shift right, leftarrow, "\mathcal{J}"'] & W^{l-r,p}(X \times Y^+) \bigoplus_{j=1}^m \partial W^{l-r_j,p}(X) \arrow[u, shift right, "\lambda"'] \end{tikzcd} \tag{7.3}$$

for all $k > l > \frac{\sigma}{\rho p} + \max_j r_j$.

7.3 From local to global.

7.3.1 Pertubation of exact squares and consequences.

We will extend the result of Theorem 70 (exactness of heterogeneous elliptic operator with constant coefficient on Euclidean plan) in 2 levels: (1) for general elliptic operators (non-heterogeneous and with variable coefficients) and (2) for such operators on compact manifold (with boundary if needed). These 2 generalizations will be done using the same technique: "cube by cube" approximating an exact square.

We topologize the space of commutative squares $\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "m"'] & F \arrow[d, "p"] \\ G \arrow[r, "q"] & H \end{tikzcd}$ as a closed subspace $SQ(E,F,G,H)$ of $L(E,F) \times L(F,H) \times L(E,G) \times L(G,H)$ defined by the equation $q \circ m = p \circ l$.

Theorem 77. *In $SQ(E,F,G,H)$, the exact squares form an open set.*

Instead of giving a proof (see [Ham75, page 75-77]), let us explain why Theorem 77 is true. The commutativity already tells us that the composition of any two consecutive arrows in

$$0 \longrightarrow E \xrightarrow{l \oplus m} F \oplus G \xrightarrow{p \ominus q} H \longrightarrow 0$$

is 0, and exactness is an extra condition of type "maximal rank", which is an open condition (For matrices, this means the derterminant does not vanish. The analogous phenomenon for Banach spaces is that a linear map sufficiently close to an invertible one is also invertible).

We will distinguish the following 2 types of cubes that we will use to cover a manifold. We will call the following set an *interior cube*

$$B_\epsilon := \left\{ (x_1, \dots, x_n) : |x_1| \le \epsilon^{\sigma/\sigma_1}, \dots |x_n| \le \epsilon^{\sigma/\sigma_n} \right\}$$

and the following an *boundary cube*

$$B_\epsilon^+ := \left\{ (x_1, \dots, x_n, y) : |x_1| \le \epsilon^{\sigma/\sigma_1}, \dots |x_n| \le \epsilon^{\sigma/\sigma_n}, 0 \le y \le \epsilon^{\sigma/\rho} \right\}$$

For the second type, we note by ∂_0 the part $y = 0$ of the boundary of B_ϵ^+, and by ∂_e the remaining part.

We will say that the $A := \sum_{\|\alpha\|\leq r} a_\alpha(x) D^\alpha$ is **semi-elliptic at** 0 if $A_0 := \sum_{\|\alpha\|=r} a_\alpha(0) D^\alpha$ is a semi-elliptic operator.

Proposition 78 (Approximate operator on interior cube). *Suppose that $A := \sum_{\|\alpha\|\leq r} a_\alpha(x) D^\alpha$ is defined in B_{ϵ_0} and A is semi-elliptic at $x = 0$. Fix $-\infty < l < k < +\infty$. Then there exists an $\epsilon > 0$ sufficiently small and an operator $A^\# = \sum_{\|\alpha\|\leq r} a_\alpha^\#(x) D^\alpha$ with smooth coefficients defined on X such that*

$$A^\# = \begin{cases} A, & \textit{inside } B_\epsilon \\ A_0, & \textit{outside } B_{2\epsilon} \end{cases}$$

and the "k, l" square corresponding to $A^\#$, i.e.

$$\begin{tikzcd} W^{k,p}(X) \arrow[r, "A^\#"] \arrow[d, hook, "\iota"] & W^{k-r,p}(X) \arrow[d, hook, "\iota"] \\ W^{l,p}(X) \arrow[r, "A^\#"] & W^{l-r,p}(X) \end{tikzcd}$$

is exact.

An analoguous result holds for boundary problem. The setup for boundary problem on half-plan $X \times Y^+$ is as follows.

$$\begin{aligned} A &:= \sum_{\|(\alpha,\beta)\|\leq r} a_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta \\ B^j &:= \sum_{\|(\alpha,\beta)\|\leq r_j} b_{\alpha,\beta}^j(x,y) D_x^\alpha D_y^\beta, \quad j = \overline{1,m} \end{aligned}$$

are operators with smooth coefficients on $B_{\epsilon_0}^+$ and

$$\begin{aligned} A_0 &:= \sum_{\|(\alpha,\beta)\|= r} a_{\alpha,\beta}(0,0) D_x^\alpha D_y^\beta \\ B_0^j &:= \sum_{\|(\alpha,\beta)\|= r_j} b_{\alpha,\beta}^j(0,0) D_x^\alpha D_y^\beta, \quad j = \overline{1,m} \end{aligned}$$

If A is semi-elliptic at 0 then we say that $\{B^j\}$ satisfy the CBC condition at 0 uf $\{B_0^j\}$ are CBC with respect to A_0. Note that this is an "open condition", i.e. if the condition is satisfied at $(0,0)$ then it is also satisfied in a neighborhood of $(0,0)$ in $X \times \{0\}$. The analoguous result for boundary problem can then be stated.

Proposition 79 (Approximate opearator on boundary cube). *Under the previous setup and with $\frac{\sigma}{\rho p} + \max_j r_j < l < k < +\infty$, for $\epsilon > 0$ sufficiently small, there exists operators $C^\# =$*

$(A^\#, B^\#)$ with smooth coefficient in $X \times Y^+$ agreeing with (A, B) in B^+_ϵ and with (A_0, B_0) outside of $B^+_{2\epsilon}$ such that the square

$$\begin{tikzcd} W^{k,p}(X \times Y^+) \arrow[r, "\mathcal{C}^\#"] \arrow[d, hook, "\iota"'] & W^{k-r,p}(X \times Y^+) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(X) \arrow[d, hook, "\iota"] \\ W^{l,p}(X \times Y^+) \arrow[r, "\mathcal{C}^\#"'] & W^{l-r,p}(X \times Y^+) \bigoplus_{j=1}^m \partial W^{l-r_j,p}(X) \end{tikzcd}$$

is exact.

We will prove Proposition 78 here to demonstrate how Theorem 77 is employed. Another reason is that the corresponding proof in [Ham75] is not very readable due to a notation/printing issue.

Proof of Proposition 78. We will use the change of coordinates $\tilde{x}_i = \lambda^{-\sigma/\sigma_i} x_i$, which gives a diffeomorphism h_λ from B_{ϵ_0} to $B_{\epsilon_0/\lambda}$, in which the derivative operators are

$$\left(\frac{\partial}{\partial \tilde{x}_i}\right)_i^\alpha = \lambda^{\alpha_i\sigma/\sigma_i} \left(\frac{\partial}{\partial x_i}\right)_i^\alpha, \quad \tilde{D}^\alpha = \lambda^{\|\alpha\|} D^\alpha$$

The operator A, viewed in h_λ, i.e. the operator $f \mapsto A(f \circ h_\lambda)$, is $\sum_{\|\alpha\| \le r} a_\alpha(\lambda^{\sigma/\sigma_i}\tilde{x}_i)\lambda^{-\|\alpha\|}\tilde{D}^\alpha$. We pose

$$\begin{aligned} \tilde{A}_\lambda &:= \sum_{\|\alpha\| \le r} \lambda^{r-\|\alpha\|} a_\alpha(\lambda^{\sigma/\sigma_i}\tilde{x}_i)\tilde{D}^\alpha \\ \tilde{A}_0 &:= \sum_{\|\alpha\| = r} a_\alpha(0)\tilde{D}^\alpha \\ \tilde{A}^*_\lambda &:= \varphi(\tilde{x})\tilde{A}_\lambda + (1 - \varphi(\tilde{x}))\tilde{A}_0 \end{aligned}$$

where φ is radial in $\tilde{x}$, equals 1 for $\|\tilde{x}\| \le 1$ and 0 for $\|\tilde{x}\| \ge 2$.

The coefficient before $\tilde{D}^\alpha$ of $\tilde{A}^*_\lambda$ is $\lambda^{r-\|\alpha\|}\left[\varphi(\tilde{x})a_\alpha(\lambda^{\sigma/\sigma_i}\tilde{x}_i) + (1 - \varphi(\tilde{x}))a_\alpha(0)\delta_{\|\alpha\|=r}\right]$ is the same as that of $\tilde{A}_0$ for $\tilde{x}$ outside of B_2 and C^0-converges to that of $\tilde{A}_0$ inside B_1. Hence for λ sufficiently small the corresponding "k, l" diagram of $\tilde{A}^*_\lambda$ is exact, hence so is the diagram of $\lambda^{-r}\tilde{A}^*_\lambda$. Choose $\epsilon = \lambda$ and $A^\#$ to be $\lambda^{-r}\tilde{A}^*_\lambda$ viewed in X through $h\lambda$. □

Remark 32. *To avoid making infinite intersection of open sets, we have to fix k and l first in Proposition 78 and Proposition 79. The approximate operators $A^\#, B^\#$ and the size ϵ of the cube therefore depend on k, l, but this dependence will not be a trouble when we pass from local to global situation.*

The exactness of semi-elliptic operator with variable coefficients on manifold will be establish analytically, meaning through the 3 statements similar to those of Theorem 70. Proposition 78 and 79 can be applied to prove the the local version of these statements.

Lemma 80. *With the same ϵ and k, l as Proposition 78 and the extra condition that $l \geq k-1$, one has for all $0 < \delta < \epsilon$*

1. $\|f\|_{W^{k,p}(B_\delta)} \leq C\left(\|Af\|_{W^{k-r,p}(B_\epsilon)} + \|f\|_{W^{l,p}(B_\epsilon)}\right)$ *for all* $f \in W^{k,p}(B_\epsilon)$.
2. *If* $f \in W^{l,p}(B_\epsilon)$ *and* $Af \in W^{k-r,p}(B_\epsilon)$ *then* $f \in W^{k,p}(B_\delta)$.
3. *If* $g \in W^{l-r,p}(B_\delta/\partial)$ *then there exists* $f \in W^{l,p}(B_\epsilon, \partial)$ *such that*
$$g - Af \in W^{k-r,p}(B_\epsilon, \partial B_\epsilon).$$

Proof. Let ψ be a cut-off function that equals 1 on B_δ and 0 outside of B_ϵ and $A^\#$ be the differential operator on X with exact "k, l" diagram given by Proposition 78 which equals A on B_ϵ. The idea of the remaining computation is to use the exactness of $A^\#$ on ψf and the reason for which the local-global passage is not trivial is that the operator $A^\#$ and the multiplication by ψ do not commute. The commutator $[A^\#, \psi]$, however, is of weight at least 1 less than A and with the choice $l \geq k-1$ the norm $\|[\psi, A^\#]f\|_{W^{k-r,p}(X)}$ is dominated by $\|f\|_{W^{l,p}}$.

1. If $f \in W^{k,p}(B_\epsilon)$ then $\psi f \in W^{k,p}(B_\epsilon, \partial)$ and
$$\begin{aligned}\|f\|_{W^{k,p}(B_\delta)} \leq \|\psi f\|_{W^{k,p}(X)} &\leq C\left(\|A^\#\psi f\|_{W^{k-r,p}(X)} + \|\psi f\|_{W^{l,p}(X)}\right)\\ &\leq C\left(\|\psi A^\# f\|_{W^{k-r,p}(X)} + \|[\psi, A^\#]f\|_{W^{k-r,p}(X)} + \|\psi f\|_{W^{l,p}(X)}\right)\\ &\leq C'\left(\|Af\|_{W^{k-r,p}(B_\epsilon)} + \|f\|_{W^{l,p}(B_\epsilon)}\right)\end{aligned}$$
2. Given $f \in W^{l,p}(B_\epsilon)$ and $Af \in W^{k-r,p}(B_\epsilon)$, one has $\psi f \in W^{l,p}(X)$. Also, $[A^\#, \psi]f \in W^{l-r+1,p}(X) \subset W^{k-r,p}(X)$ and $\psi A^\# f = \psi Af \in W^{k-r,p}(X)$, therefore $A^\#(\psi f) \in W^{k-r,p}(X)$. By exactness of $A^\#$, one has $\psi f \in W^{k,r}(X)$, so $f \in W^{k,r}(B_\delta)$.
3. If $g \in W^{l-r,p}(B_\delta/\partial) \subset W^{l-r,p}(X)$, by exactness of $A^\#$ we can find $\tilde{f} \in W^{l,p}(X)$ such that $g - A^\#\tilde{f} \in W^{k-r,p}(X)$. Choose $f = \psi\tilde{f} \in W^{l,p}(B_\epsilon/\partial)$ then
$$g - Af = g - A^\#(\psi\tilde{f}) = \psi(g - A^\#\tilde{f}) + [\psi, A^\#]\tilde{f} \in W^{k-r,p}(B_\epsilon)$$
since $\psi(g - A^\#\tilde{f}) \in W^{k-r,p}(B_\epsilon)$ and $[\psi, A^\#]\tilde{f} \in W^{l-r+1,p}(B_\epsilon) \subset W^{k-r,p}(B_\epsilon)$.

□

Lemma 81. *With (A, B) and ϵ, k, l as in Proposition 79 with the extra condition $l \geq k-1$, then for all $\delta < \epsilon$, one has*

1. $\|f\|_{W^{k,p}(B_\delta^+)} \leq C\left(\|Af\|_{W^{k-r,p}(B_\epsilon^+)} + \sum_{j=1}^m \|B^j f\|_{\partial W^{k-r_j,p}(\partial_0 B_\epsilon^+)} + \|f\|_{W^{l,p}(B_\epsilon^+)}\right)$ *for all* $f \in W^{k,p}(B_\epsilon^+)$.

2. *If $f \in W^{l,p}(B^+_\epsilon)$ and $Af \in W^{k-r,p}(B^+_\epsilon)$ and $B^j f \in \partial W^{k-r_j,p}(\partial_0 B^+_\epsilon)$ then actually $f \in W^{k,p}(B^+_\delta)$.*

3. *If $g \in W^{l-r,p}(B^+_\delta/\partial_e)$ and $h_j \in \partial W^{l-r_j,p}(\partial_0 B^+_\delta/\partial)$ then there exists $f \in W^{l,p}(B^+_\epsilon, \partial_e)$ with*

$$g - Af \in W^{k-r,p}(B^+_\epsilon, \partial_e), \quad h_j - B^j f \in \partial W^{k-r_j,p}(\partial_0 B^+_\epsilon/\partial).$$

The generalisation of Theorem 70 on manifold with variable coefficients is now straightforward. The only nontrivial issue is the definition of semi-elliptic operator A on manifold. This requires a Riemannian metric g and ellipticity is naturally defined at every point, viewed in a chart, as we did before. But this only defines the action of A on $C^\infty(M)$ (or $C^r(M)$ if regularity is important), but not on $W^{k,p}(M/\mathcal{A})$ where $\mathcal{A} \subset \partial M$ is a connected component.

The action of a differential operator A can be defined to be component-wise on $W^{k,p}(M/\mathcal{A}) \hookrightarrow \bigoplus_i W^{k,p}(\mathcal{R}_i/\mathcal{A}_i)$ where $\mathcal{R}_i$ is an Euclidean plan or a half-plan and $\mathcal{A}_i$ the corresponding boundary part, i.e.

$$\begin{tikzcd} W^{k,p}(M/\mathcal{A}) \arrow[r, "\iota"] \arrow[d, dashed, "A"'] & \bigoplus_i W^{k,p}(\mathcal{R}_i/\mathcal{A}_i) \arrow[d, "A"] \\ W^{l,p}(M/\mathcal{A}) \arrow[r, "\iota"] & \bigoplus_i W^{l,p}(\mathcal{R}_i, \mathcal{A}_i) \end{tikzcd}$$

It remains to check that the component-wise operation of A maps an element in the image on $W^{k,p}(M/\mathcal{A})$ to an element in the image of $W^{l,p}(M/\mathcal{A})$. This can be done using the projection as we did when defining trace operator on manifold, but the situation is much simpler here since we can differentiate directly an element in $\mathcal{S}^*(M)$.

Theorem 82 (Elliptic equation on manifold)**.** *Let M be a compact manifold possibly with boundary (and a compatible foliation if the weight is not uniform). Let A be a general semi-elliptic operator of weight r, of variable coefficients and $\{B^j\}_j$ be a set boundary operators of weight r_j satisfying CBC with respect to A. Then for all $\frac{\sigma}{\rho p} + \max_j r_j < l < k < +\infty$, the square*

$$\begin{tikzcd} W^{k,p}(M) \arrow[r, "\mathcal{C}"] \arrow[d, hook, "\iota"'] & W^{k-r,p}(M) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(\partial M) \arrow[d, hook, "\iota"] \\ W^{l,p}(M) \arrow[r, "\mathcal{C}"'] & W^{l-r,p}(M) \bigoplus_{j=1}^m \partial W^{l-r_j,p}(\partial M) \end{tikzcd}$$

is exact where $\mathcal{C} = (A, B^j)$.

Proof. We can suppose $l \geq k-1$, the general case follows using

Lemma 83. *If the two following squares are exact*

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "m"'] & F \arrow[d, "p"] \\ G \arrow[r, "q"'] & H \end{tikzcd}, \quad \begin{tikzcd} G \arrow[r, "q"] \arrow[d, "r"'] & H \arrow[d, "s"] \\ K \arrow[r, "t"'] & L \end{tikzcd}$$

then

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "rm"'] & F \arrow[d, "sp"] \\ K \arrow[r, "t"'] & L \end{tikzcd}$$

is exact.

Now covering M by finitely many charts of type $B_\delta \subset B_\epsilon$ and $B_\delta^+ \subset B_\epsilon^+$ such that the interior of B_δ and of B_δ^+ cover M. Also, choose a partition of unity $\sum \psi = 1$ subordinated to B_δ and B_δ^+. The exactness will be established if we can prove the analogue of the 2 last statements of Theorem 70

For the regularity statement: If $f \in W^{l,p}(M)$, $Af \in W^{k-r,p}(M)$ and $B^j f \in W^{k-r_j,p}(\partial M)$ then the same holds for ψf in B_ϵ and B_ϵ^+ since

$$[A, \psi]f \in W^{l-r+1,p} \subset W^{k-r,p}, \quad [B^j, \psi]f \in \partial W^{l-r_j+1,p} \subset \partial W^{k-r_j,p}$$

Therefore $\psi f \in W^{k,p}(B_\delta)$ or $W^{k,p}(B_\delta^+)$ hence $f \in W^{k,p}(M)$.

For the approximation: If $g \in W^{l-r,p}(M)$ and $h_j \in \partial W^{l-r_j,p}(\partial M)$ then $\psi g \in W^{l-r,p}(B_\delta/\partial)$ or $W^{l-r,p}(B_\delta^+/\partial_e)$ and $\psi h_j \in \partial W^{l-r_j,p}(\partial_0 B_\delta^+/\partial)$. Then by Lemma 81, we can find $\tilde{f} \in W^{l,p}(B_\epsilon/\partial)$ with $\psi g - A\tilde{f} \in W^{k-r,p}(B_\epsilon/\partial)$ or in a boundary cube $\tilde{f} \in W^{l,p}(B_\epsilon^+/\partial_e)$ with $\psi g - A\tilde{f} \in W^{k-r,p}(B_\epsilon^+/\partial_e)$ with $\psi h_j - B^j\tilde{f} \in \partial W^{k-r_j,p}(\partial_0 B_\epsilon^+/\partial)$. Then $f := \sum \tilde{f}$ makes sense and satisfies $\begin{cases} g - Af = \sum(\psi g - A\tilde{f}) & \text{is in } W^{k-r,p}(M) \\ h - B^j f = \sum(\psi h_j - B^j\tilde{f}) & \text{is in } \partial W^{k-r_j,p}(\partial M) \end{cases}$ □

7.3.2 Consequences of Theorem 72.

Under the same setup as Theorem 82, one has

Theorem 84 (Regularity of kernel and cokernel)**.** *The map $\mathcal{C} = (A, B) : W^{k,p}(M) \longrightarrow W^{k-r,p}(M) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(\partial M)$ has closed range, finite dimensional kernel and cokernel and the kernel and cokernel are independent of k in the sense of Theorem 72. In particular,* $\ker C \subset C^\infty(M)$

The analoguous regularity for cokernel is less straightforward. We resume here the result.

Theorem 85 (Regularity of cokernel)**.** *If $r > \max r_j$ then the image of $\mathcal{C}$ can be represented by finitely many linear relations: $(g, h) \in \operatorname{Im} \mathcal{C}$ if and only if it satisfies finitely many equations of type:*

$$\langle g, \gamma \rangle_M + \sum_{j=1}^m \langle h_j, \eta_j \rangle_{\partial M} = 0$$

with $\gamma \in C^\infty(M)$ and $\eta_j \in C^\infty(\partial M)$.

If $\max r_j - r = k \geq 0$ *then for all* $g \in W^{k-r,p}(M)$*, the normal derivatives* $\frac{\partial g}{\partial \nu^i}$ *are well defined if* $\frac{\sigma_i}{\rho} \leq k$*. The cokernel is then given by the relations*

$$\langle g, \gamma\rangle_M + \sum_{\sigma_i/\rho \leq k} \langle \frac{\partial g}{\partial \nu^i}, \chi_i\rangle_{\partial M} + \sum_{j=1}^{m} \langle h_j, \eta_j\rangle_{\partial M} = 0$$

with $\gamma \in C^\infty(M)$*,* $\chi_i \in C^\infty(\partial M)$*,* $\eta_j \in C^\infty(\partial M)$*.*

7.4 Parabolic equation on manifold.

7.4.1 Parabolicity and local results.

Definition 14. *The constant coefficient differential operator* $A(D_x, D_t) = \sum_{\|(\alpha,\beta)\| \leq r} a_{\alpha\beta} D_x^\alpha D_t^\beta$ *is called* ***parabolic*** *if its symbol* $A(\xi, \theta) := \sum_{\|(\alpha,\beta)\|=r} a_{\alpha\beta}\xi^\alpha\theta^\beta$ *has no zero when* $\xi \in \mathbb{R}$ *and* $\operatorname{Im}\theta \leq 0$ *except* $\xi = \theta = 0$*.*

Example 7. *Take* $A = \partial_t - \partial_x^2 - \partial_y^2 - \partial_z^2 = iD_t + D_x^2 + D_y^2 + D_z^2$*, the symbol is* $i\theta + \sum \xi_i^2$ *has no zero* $\xi \in \mathbb{R}^3, \operatorname{Im}\theta \leq 0$ *except* 0*. Generally, the operator* $\partial_t + A(D_{x^i})$ *is parabolic if* A *is an elliptic operator with the symbol* $A(\xi) \geq 0$ *for all* $\xi \in \mathbb{R}$ *with equality only at* $\xi = 0$*.*

Remark 33.
1. *If* $\sigma = \operatorname{lcm}(\sigma_1, \ldots, \sigma_n)$ *is the* lcm *of weights of variable* x_i *and* τ *is the weight of* t*, then parabolicity implies* $2\tau \mid \sigma$*. Therefore if the weights of* x_i *are uniform, one can suppose that* $\tau = 1$*.*

2. *Parabolicity implies ellipticity.*

Similarly to the elliptic case, we attempt to define an approximate inverse G of A, of the form

$$G(\xi, \theta) = (1 - \psi(\xi, \theta))/A(\xi, \theta)$$

such that $G(D_x, D_t) : W^{k-r,p}(X \times T^+/0) \longrightarrow W^{k,p}(X \times T^+/0)$ and $\psi(D_x, D_t) : W^{k,p}(X \times T^+/0) \longrightarrow W^{k,p}(X \times T^+/0)$ for all $k, l \in \mathbb{R}$.

The sufficient condition for this is that $\psi(\xi, \theta) = \psi(\xi)\hat{\chi}(\theta)$ where ψ is compactly support and $\hat{\chi} \in \mathcal{S}(T)$ with $\hat{\chi} - 1$ having a zero of infinite order at $\theta = 0$, and $\hat{\chi}$ extends to a holomorphic function in $\operatorname{Im}\theta \leq 0$. The function $\hat{\chi}$ used in section 7.2 suffices. We then have the following exact square

$$\begin{tikzcd}
W^{k,p}(X \times T^+/0) \arrow[r, shift left, "A"'] \arrow[d, shift left, "\iota"] & W^{k-r,p}(X \times T^+/0) \arrow[l, shift left, "G"'] \arrow[d, shift left, "\iota"'] \\
W^{l,p}(X \times T^+/0) \arrow[u, bend left, "\psi"] \arrow[r, shift left, "A"] & W^{l-r,p}(X \times T^+/0) \arrow[l, shift left, "G"] \arrow[u, bend right, "\psi"']
\end{tikzcd}$$

The theory in section 7.2 also allows us to treat spatial boundary condition, that is, to replace the Euclidean plan X by the half-plan $X \times Y^+$. The analog of **CBC condition** for boundary operators

$$B^j(D_x, D_y, D_t) = \sum_{\|(\alpha,\beta,\gamma)\| \le r_j} b^j_{\alpha\beta\gamma} D_x^\alpha D_y^\beta D_t^\gamma$$

is that the symbols

$$B^j(\xi, \eta, \theta) = \sum_{\|(\alpha,\beta,\gamma)\| = r_j} b^j_{\alpha\beta\gamma} \xi^\alpha \eta^\beta \theta^\gamma$$

are linearly independent modulo $A^+(\xi, \eta, \theta)$ as polynomial in η for all $\xi \in \mathbb{R}^n$ **and** for all $\operatorname{Im}\theta \le 0$ except when $\xi = \theta = 0$. In that case we have the exactness of

$$\begin{array}{ccc} W^{k,p}(X \times Y^+ \times T^+/0) & \xrightarrow{(A,B^j)} & W^{k-r,p}(X \times Y^+ \times T^+/0) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(X \times T^+/0) \\ \big\downarrow \iota & & \big\downarrow \iota \\ W^{l,p}(X \times Y^+ \times T^+/0) & \xrightarrow{(A,B^j)} & W^{l-r,p}(X \times Y^+ \times T^+/0) \bigoplus_{j=1}^m \partial W^{l-r_j,p}(X \times T^+/0) \end{array}$$

7.4.2 Global results and causality.

We will use the following setup. Let M be a compact manifold (possibly with boundary), of the form $N \times [\alpha, \omega] \ni (x, t)$. The global product gives a foliation that allows us to set the spatial weight to be uniformly σ and the temporal weight to be τ. The boundary of M has 3 parts: $\partial_\alpha M := N \times \alpha$, $\partial_\omega M := N \times \omega$ and $\partial_S M := \partial N \times [\alpha, \omega]$.

Let A be a parabolic operator, meaning that A is parabolic at every point and $B^j, j = \overline{1, m}$ be a set of boundary operator satisfying CBC condition at every point on $\partial_S M$. We take into account the initial condition by only considering the space $W^{k,p}(M/\partial_\alpha)$ of function vanishing before time $t = \alpha$. As before the operator

$$\mathcal{C} := (A, B^j) : \; W^{k,p}(M/\partial_\alpha) \longrightarrow W^{k-r,p}(M/\partial_\alpha) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(\partial_S M/\partial_\alpha)$$

has closed range, finite dimensional kernel and cokernel which are independent of $k > \frac{1}{p} + \max r_j$.

The same method allows us to conclude that $\ker \mathcal{C} \subset C^\infty(M)$ and the cokernel is given by finitely many linear relations of type

$$\langle g, \gamma \rangle_M + \sum_j \langle h_j, \eta_j \rangle_{\partial M} + \sum_i \langle \frac{\partial}{\partial \nu^i} g, \chi_i \rangle_{\partial_S M}$$

where $\gamma \in C^\infty(M/\partial_\omega)$, $\chi_i \in C^\infty(\partial_S M/\partial_\omega)$ and $\eta_j \in C^\infty(\partial_S M/\partial_\omega)$.

The difference with elliptic equation is that the kernel and cokernel of $\mathcal{C}$ are not only of finite dimension, but are zero.

Theorem 86 (Causality)**.** *With the previous setup, the operator* $\mathcal{C} = (A, B^j)$ *defines an isomorphism*

$$W^{k,p}(M/\partial_\alpha) \xrightarrow{\mathcal{C}} W^{k-r,p}(M/\partial_\alpha) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(\partial_S M/\partial_\alpha)$$

for all $k > \frac{1}{p} + \max_j r_j$*, and therefore an isomorphism*

$$C^\infty(M/\partial_\alpha) \xrightarrow{\mathcal{C}} C^\infty(M/\partial_\alpha) \bigoplus_{j=1}^m C^\infty(\partial_S M/\partial_\alpha)$$

Proof. Let $\beta \leq \gamma$ be real numbers in $[\alpha, \omega]$ and let $\ker(\beta,\gamma)$ and $\operatorname{coker}(\beta,\gamma)$ be the kernel and cokernel of operator $\mathcal{C}$ on $N \times [\beta, \gamma]$ with vanishing initial condition at β. Since $\dim\ker(\beta,\gamma)$ and $\dim\operatorname{coker}(\beta,\gamma)$ are integer-valued, using the fact that $\dim\ker(\beta,\omega)$ is decreasing in β and $\dim\operatorname{coker}(\alpha,\gamma)$ is increasing in γ, one can easily check that it suffices to show that the two functions are continuous in (β,γ) to prove that they are identically 0.

The following statements can be verified mechanically:

1. *Monotonicity:* $\dim\ker(\beta,\gamma)$ is decreasing in β, $\dim\operatorname{coker}(\beta,\gamma)$ is increasing in γ.

2. *One-sided continuity:* $\dim\ker(\beta,\gamma)$ is left-continuous in β, $\dim\operatorname{coker}(\beta,\gamma)$ is right-continuous in γ.

3. *One-sided semi-continuity:* $\dim\ker(\beta,\gamma)$ is left upper semi-continuous in γ, i.e.

$$\lim_{\gamma_1 \to \gamma_2^-} \inf \dim\ker(\beta,\gamma_1) \geq \dim\ker(\beta,\gamma_2)$$

   This is due to the left-continuity in first variable of $\dim\ker$ and the exact sequence

$$0 \longrightarrow \ker(\gamma_1,\gamma_2) \longrightarrow \ker(\beta,\gamma_2) \longrightarrow \ker(\beta,\gamma_1)$$

   where the last arrow is the restriction. Similar statement for coker:

$$\lim_{\beta_2 \to \beta_1^+} \dim\inf\operatorname{coker}(\beta_2,\gamma) \geq \dim\operatorname{coker}(\beta_1,\gamma)$$

This 3 statements suffice to finish the proof in the case where boundary conditions B^j on $\partial_S M$ are of constant coefficients since $\ker, \operatorname{coker}$ only depend on the difference $\gamma - \beta$, up to a translation in time of the solutions.

In case B^j are of variable coefficients, the idea of making translation in time can be formulated using Index theory for Fredholm maps:

We recall that Fredholm maps between Banach spaces E, F are those in $L(E,F)$ with closed image and finite dimensional kernel and cokernel. It is a classical result that

1. The set $\mathcal{F}$ of Fredholm maps are open in $L(E,F)$.

2. The index $i(l) := \dim\ker l - \dim\operatorname{coker} l$ is continuous in $\mathcal{F}$.

The difference $\dim\ker(\beta,\gamma) - \dim\operatorname{coker}(\beta,\gamma)$ can be regarded as the index of a continuous family $\mathcal{C}_{(\beta,\gamma)}$ of operators on the same space $N \times [0,1]$ using the diffeomorphism

$$N \times [0,1] \xrightarrow{\sim} N \times [\beta,\gamma].$$

Hence $\dim\ker(\beta,\gamma) - \dim\operatorname{coker}(\beta,\gamma)$ is constant. It follows that $\dim\ker(\beta,\gamma)$ is both increasing and one-sided semi-continuous in γ hence is right-continuous in γ, hence $\dim\operatorname{coker}(\beta,\gamma)$ is continuous in γ. Other continuities follows similarly. □

Remark 34. *To take into account the initial condition $f\Big|_\alpha = f_\alpha$ smooth, one looks for solution of the form $f = f_b + f_\#$ where f_b satisfies the initial condition and $f_\# \in W^{k,p}(N \times [\alpha,\omega]/\alpha)$ satisfying a parabolic equation $(Af_\#, B^j f) = (g,h)$ where g, h and the coefficients of A and B^j depend smoothly on f_b, and therefore still C^∞ in (x,t).*

7.4.3 Regularisation effect and Gårding inequality.

With the same technique used for elliptic equation, one can also prove regularity result for parabolic equation. There are 2 different points, in comparison with the elliptic case:

1. There is a regularisation effect of parabolic equation: An arbitrarily weak estimate in the past gives an arbitrarily strong estimate in the future. We will see that this is in fact a consequence of the causality of parabolic equation (Theorem 86) and Kondrachov's theorem.

2. The temporal boundary condition is thicken: We will look at the norm on $N \times [\alpha,\pi]$ rather than the restriction to $\partial_\alpha M$.

Theorem 87 (Regularity and Garding inequality)**.** *Under the same setup and notation as Section 7.4.2, let $p \in (1,+\infty)$ and $k > l > \frac{1}{p} + \max r_j$. We denote by $W^{k,p}([\beta,\gamma])$ the Sobolev space $W^{k,p}(N \times [\beta,\gamma])$. Suppose that*

$$f \in W^{l,p}([\alpha,\omega]), \quad Af \in W^{k-r,p}([\alpha,\omega]), \quad B^j f \in \partial W^{k-r_j,p}([\alpha,\omega])$$

then $f \in W^{k,p}([\pi,\omega])$ for all $\pi \in (\alpha,\omega)$. Also, for all $l' > -\infty$, there exists a constant $C > 0$ such that

$$\|f\|_{W^{k,p}([\pi,\omega])} \leq C\left(\|Af\|_{W^{k-r,p}([\alpha,\omega])} + \|B^j f\|_{\partial W^{k-r_j,p}([\alpha,\omega])} + \|f\|_{W^{l',p}([\alpha,\pi])}\right).$$

In particular, for homogeneous equation, i.e. $Af = 0, B^j f = 0$, the solution is C^∞ and an arbitrarily weak estimate in the past gives an arbitrarily strong estimate in the future.

Proof. Let us explain why the theorem is true in the case of no spatial boundary $\partial N = \emptyset$. In this case, there is no distinction between l and l'. Consider A as an elliptic operator on

$N \times [\tilde{\pi}, \omega]$ with $\tilde{\pi} = \frac{\alpha+\pi}{2}$ and with no boundary operator, one has the following exact diagram:

$$\begin{tikzcd} W^{k,p}([\tilde{\pi},\omega]) \arrow[r, "A"] \arrow[d, hook] & W^{k-r,p}([\tilde{\pi},\omega]) \arrow[d, hook] \\ W^{l,p}([\tilde{\pi},\omega]) \arrow[r, "A"] & W^{l-r,p}([\tilde{\pi},\omega]) \end{tikzcd}$$

Therefore the if $f \in W^{l,p}([\alpha,\omega])$ and $Af \in W^{k-r,p}([\alpha,\omega])$then $f \in W^{k,p}([\tilde{\pi},\omega]) \subset W^{k,p}([\pi,\omega])$ and

$$\|f\|_{W^{k,p}([\pi,\omega])} \leq \|f\|_{W^{k,p}([\tilde{\pi},\omega])} \leq C\left(\|Af\|_{W^{k-r,p}([\alpha,\omega])} + \|f\|_{W^{l,p}([\alpha,\omega])}\right) \tag{7.4}$$

$$\leq C\left(\|Af\|_{W^{k-r,p}([\alpha,\omega])} + \|f\|_{W^{l,p}([\alpha,\pi])} + \|f\|_{W^{l,p}([\tilde{\pi},\omega])}\right) \tag{7.5}$$

It remains to check that we can get rid of the $\|f\|_{W^{l,p}([\tilde{\pi},\omega])}$ term on the right hand side. Suppose not, then there exists a sequence $\{f_i\} \subset W^{l,p}([\alpha,\omega])$ such that $Af_i \to 0$ in $W^{k-r,p}([\alpha,\omega])$ and $f_i \to 0$ in $W^{l,p}([\alpha,\pi])$ but $\|f_i\|_{W^{l,p}([\tilde{\pi},\omega])} = 1$. Then by (7.5), $\{f_i\}$ is a bounded sequence in $W^{k,p}([\tilde{\pi},\omega])$ and, by Kondrachov's theorem, can be supposed to converge in $W^{l,p}([\tilde{\pi},\omega])$ to a function $\tilde{f}$ which has $\|\tilde{f}\|_{W^{l,p}([\tilde{\pi},\omega])} = 1$ and $A\tilde{f} = 0$ on $[\tilde{\pi},\omega]$ because A commutes with the restriction. Moreover, since $\|f_i\|_{W^{l,p}([\alpha,\pi])} \to 0$, one has $\tilde{f} \in W^{l,p}([\tilde{\pi},\omega]/\tilde{\pi})$ and the fact that $\tilde{f} \neq 0$ contradicts Theorem 86). □

Remark 35. *The proof of Theorem 87 in the general case, with spatial boundary taken into account requires the notion of bigraded Sobolev spaces on half-plan, see [Ham75, page 97-100]. This is also how the regularity result for cokernel of elliptic operator, Theorem 85, is proved.*

7.5 Example: Linear heat equation.

We use the same setup of M, N, α, ω as Section 7.4.2. Let Δ be the (geometer's) Laplacian

$$-\Delta f := g^{ij}(x)\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k(x)\frac{\partial f}{\partial x^k}\right)$$

It is easy to check that Δ is an elliptic operator with symbol $\Delta \geq 0$ (there is a factor i when passing from $\frac{\partial}{\partial x^i}$ to D_{x^i}). Hence on $M = N \times [\alpha,\omega]$ the operator $\frac{\partial}{\partial t} + \Delta$ is parabolic.

7.5.1 Linear system.

We will look at the linear parabolic system of equations for $F = (f^1, \ldots, f^n) : M \longrightarrow \mathbb{R}^n$:

$$\frac{\partial F}{\partial t} + \Delta F + a\nabla F + bF = G \tag{7.6}$$

where in local coordinates $(a\nabla F)^\alpha = a_\beta^{\alpha i}\frac{\partial f^\beta}{\partial x^i}$ and $(bF)^\alpha = b_\beta^\alpha f^\beta$ and $(\Delta F)^\alpha = \Delta f^\alpha$ and the coefficients $a_\beta^{\alpha i}$ and b_β^α are smooth.

We will say that a function $F = (f^1, \dots, f^n) : M \longrightarrow \mathbb{R}^n$ of class $W^{k,p}$ if it is $W^{k,p}$ component-wise. We also denote abusively by $W^{k,p}(M)$ the direct sum $W^{k,p}(M)^{\oplus n}$ where F belongs to.

Theorem 88 (Linear heat equation)**.** *Let $p > \dim M + 1 = \dim N + 2$ and $k \geq 0$, then for all $G \in W^{k,p}(N \times [\alpha, \omega]/\alpha)$, there exists a unique $F \in W^{k+2,p}(N \times [\alpha, \omega]/\alpha)$ that solves (7.6). Moreover, the operator*

$$F \longmapsto \frac{\partial F}{\partial t} + \Delta F + a\nabla F + bF$$

is an isomorphism between Banach spaces $W^{k+2,p}(N \times [\alpha, \omega]/\alpha) \longrightarrow W^{k,p}(N \times [\alpha, \omega]/\alpha)$.

Proof. Note that

$$\begin{aligned} H :\ W^{k+2,p}(N \times [\alpha, \omega]/\alpha) &\longrightarrow W^{k,p}(N \times [\alpha, \omega]/\alpha) \\ F &\longmapsto \frac{\partial F}{\partial t} + \Delta F \end{aligned}$$

is a direct sum of parabolic operators in each component, and hence an isomorphism, and

$$\begin{aligned} K :\ W^{k+2,p}(N \times [\alpha, \omega]/\alpha) &\longrightarrow W^{k,p}(N \times [\alpha, \omega]/\alpha) \\ F &\longmapsto a\nabla F + bF \end{aligned}$$

is a compact operator because it factors through $W^{k+1}(N \times [\alpha, \omega]/\alpha)$. Therefore $H + K$ is a Fredholm map with the same index as H, which is 0. It is sufficient to check that the kernel of $H + K$ is trivial.

Suppose that $F = (f^1, \dots, d^n) \in \ker(H + K)$ then $f^\alpha \in W^{2,p}(N \times [\alpha, \omega]/\alpha)$, so f^α and $\frac{\partial f^\alpha}{\partial x^i}$ are continuous function on $N \times [\alpha, \omega]$. Since

$$\frac{\partial f^\alpha}{\partial t} + \Delta f^\alpha = -a_\beta^{\alpha i}\frac{\partial f^\beta}{\partial x^i} - b_\beta^\alpha f^\beta,$$

by repeated use of Theorem 87 the f^α are smooth for $t > \alpha$.

Let $e := \frac{1}{2}|F|^2 := \frac{1}{2}\sum_\alpha |f_\alpha|^2$, then e is continuous on $N \times [\alpha, \omega]$, vanishes on $N \times \{\alpha\}$ and one has

$$\begin{aligned} \frac{de}{dt} &= -\Delta e - |\nabla F|^2 - a_\beta^{\alpha i} f^\alpha \frac{\partial f^\beta}{\partial x^i} - b_\beta^\alpha f^\alpha f^\beta \\ &\leq -\Delta e + \frac{1}{2}C|F|^2 = -\Delta e + Ce \end{aligned}$$

where we used the inequality $-u^2 - 2uv \leq v^2$ to bound the second and third terms. We conclude that $F = 0$ since $e = 0$ by the following Maximum principle. $\square$

7.5.2 Maximum principle and L^∞-Comparison theorem.

With the same proof as for open set in $\mathbb{R}^n$, one has the maximum principle for parabolic equation on manifolds. The constant C in the following Theorem 89 can depend on the point $x \in M$, but will be most of the time globally constant, since the manifold M is compact. The following statement of Maximum principle will be sufficient for most of our application.

Theorem 89 (Maximum principle). *Let $f : M \longrightarrow \mathbb{R}$ be a continuous function on $M = N \times [\alpha, \omega]$ with $f\Big|_{\partial_\alpha M} \leq 0$ and $f\Big|_{\partial_S M} \leq 0$. Suppose that whenever $f > 0$, f is smooth satisfies*

$$\frac{\partial f}{\partial t} \leq -\Delta f + Cf$$

Then in fact $f \leq 0$.

With the same proof as Theorem 89, one can prove the following L^∞ Comparison theorem.

Theorem 90 (L^∞-Comparison theorem). *Let $f : \; M = N \times [\alpha, \omega] \longrightarrow \mathbb{R}$ be a continuous function on M, smooth for time $t > 0$ such that*

$$\frac{df}{dt} = -\Delta f + a\nabla f + bf \; \text{ on } N \times (\alpha, \omega] \tag{7.7}$$

where a is a smooth vector field and b is a smooth function on N. Then there exists $B = B(a, b)$ depending only on a and b such that

$$\|f\Big|_\omega\|_{L^\infty} \leq e^{B(\omega-\alpha)}\|f\Big|_\alpha\|_{L^\infty}$$

Proof. We can suppose $b \leq -1$ and prove that $\|f\|_{L^\infty(\partial_\omega M)} \leq \|f\|_{L^\infty(\partial_\alpha M)}$. Intuitively, this means that since heat spreads out, the largest density must be attained at time $t = \alpha$. In fact, choose $B = \max_M b + 1$ and define $\tilde{f} = fe^{-B(t-\alpha)}$ then $\|\tilde{f}\Big|_\alpha\|_{L^\infty} = \|f\Big|_\alpha\|_{L^\infty}$ and $\|\tilde{f}\Big|_\omega\|_{L^\infty} = e^{-B(\omega-\alpha)}\|f\Big|_\omega\|_{l^\infty}$. The function $\tilde{f}$ satisfies the same heat equation (7.7) as f, with b replaced by $b - B \leq -1$.

Now let us prove that under this supposition, $|f|$ attains it maximum at time $t = \alpha$. Since we can replace the solution f of (7.7) by $-f$, we can suppose, for sake of contradiction, that $|f|$ attains it maximum on $N \times [\alpha, \omega]$ at (x^*, t^*) with $|f(x^*, t^*)| = f(x^*, t^*) > 0$ and $t^* > \alpha$. Then one has

$$\begin{cases} \nabla f(x^*, t^*) = 0, \\ \frac{df}{dt}(x^*, t^*) \geq 0, \; \text{(this is not true if } t^* = \alpha) \\ \Delta f(x^*, t^*) \geq 0, \\ f(x^*, t^*) > 0 \end{cases}$$

Plugging these in (7.7), one has a contradiction. $\square$

7.5.3 Backwards heat equation and L^1-Comparison theorem.

We will use backwards heat equation, which is just heat equation with the reversed sense of time (so with the reversed sign for Δ as well), in order to dualise the estimate of Theorem 90 and obtain a L^1 estimate of f at time $t=\omega$ in term of its L^1 norm at $t=\alpha$. In particular, we prove the following theorem.

Theorem 91 (L^1-comparison theorem)**.** *Let a be a smooth, divergence-free vector field on a Riemannian manifold N and b be a smooth function on N.Let $f: \ N\times[\alpha,\omega] \longrightarrow \mathbb{R}$ be a continuous function on M such that*

$$\frac{df}{dt} = -\Delta f + a\nabla f + bf \ \text{ on } N\times(\alpha,\omega]. \tag{7.8}$$

Then there exists $B=B(a,b)$ depending only on a and b such that

$$\|f\big|_{\omega}\|_{L^1} \leq e^{B(\omega-\alpha)}\|f\big|_{\alpha}\|_{L^1}$$

Proof. Since L^1 is the dual space of L^∞, it is sufficient to prove that for all $h\in C^\infty(N)$, one has

$$\int_{N\times\{\omega\}} fh \leq e^{B(\omega-\alpha)}\left\|f\big|_{\alpha}\right\|_{L^1}.\|h\|_{L^\infty}.$$

Consider the backwards heat equation $\begin{cases}\frac{dg}{dt} = \Delta g - \tilde{a}\nabla g - \tilde{b}g, & \text{on } N\times[\alpha,\omega]\\ g\big|_{\omega} = h, & \end{cases}$ which is just a heat equation on $N\times[\alpha,\omega]$ with initial condition at α if we pose $\tilde{g}(t) := g(\omega+\alpha-t)$. The solution g exists and is smooth on $N\times[\alpha,\omega]$. One has, at any time t

$$\int_N g\Delta f = \int_N g\left(-\frac{df}{dt} + a\nabla f + bf\right)$$
$$\int_N f\Delta g = \int_N f\left(\frac{dg}{dt} + \tilde{a}\nabla g + \tilde{b}g\right)$$

Therefore

$$\int_N f\frac{dg}{dt} + g\frac{df}{dt} = \int_N (a\nabla f)g - (\tilde{a}\nabla g)f + (b-\tilde{b})fg$$

Choose $b=\tilde{b}$ and $\tilde{a}=-a$ then the term $(b-\tilde{b})fg$ vanishes and the two first terms become $\int_N \nabla_a(fg) = -\int_N fg \,\mathrm{div}\, a = 0$ where $\mathrm{div}\, a := \frac{\partial}{\partial x^i}a^i$ is the divergence. Therefore one has $\frac{d}{dt}\int_N fg = 0$, meaning that

$$\int_N f\big|_{\omega}.h = \int_{N\times\omega} fg = \int_{N\times\alpha} fg \leq \|f\big|_{\alpha}\|_{L^1}.\|h\|_{L^\infty} \leq e^{B(\omega-\alpha)}\|f\big|_{\alpha}\|_{L^1}.\|h\|_{L^\infty}$$

where we applied Theorem 90 to g (strictly speaking, to $\tilde{g}$) and B only depends on $\tilde{a}=-a$ and $\tilde{b}=b$. ∎

Part V

Appendix 2: Besov spaces and Polynomial differential operators

Chapter 8

Regularity estimate of Polynomial differential operators

Definition 15. *We say that P is a* ***polynomial differential operator of type*** (n, k) *if P is of the form*

$$P(F) = \sum c_{\alpha_1,\ldots,\alpha_\nu}(x, F(x)) D^{\alpha_1} F^{a_1} \ldots D^{\alpha_\nu} F^{a_\nu}$$

where the coefficients $c_{\alpha_1,\ldots,\alpha_n u}$ depend smoothly and nonlinearly on x and F and $\alpha_i \in \mathbb{R}^N$ are indices with the weighted norm $\|\alpha_i\| \leq k$ and $\sum \|\alpha_i\| \leq n$.

Example 8. *On $M \times [\alpha, \omega]$ the tension field $\tau(F) := -\Delta F^\alpha + g^{ij} \Gamma'^\alpha_{\beta\gamma}(F) F_i^\beta F_j^\gamma$ is a polynomial differential operator of type (2,2). The quadratic term alone is of type (2,1).*

8.1 A regularity estimate for polynomial differential operator.

Our goal in this part is to prove the following estimate for polynomial differential operator, in which X will be $M \times [\alpha, \omega]$.

Theorem 92 (Regularity of polynomial differential operator)**.** *Let X be a compact Riemannian manifold, $B \subset \mathbb{R}^N$ is a large Euclidean ball and P be a polynomial differential operator of type (n, k) on X. Suppose that*

$$r \geq 0, \quad p, q \in (1, \infty), \quad r + k < s, \quad \frac{1}{p} > \frac{r+n}{s} \frac{1}{q}. \tag{8.1}$$

Then for all $F \in C(X, B) \cap W^{s,q}(X)$, one has $PF \in W^{r,p}(X)$ and

$$\|PF\|_{W^{r,p}} \leq C \left(1 + \|F\|_{W^{s,q}}\right)^{q/p}.$$

where C is a constant independent of F.

We will prove that the result is *local*, in a sense to be defined. Then we will prove the local statement using Besov spaces.

Proof (reduction of Theorem 92 to a local statement). Let $\{\varphi_i : U_i \longrightarrow V_i\}$ be an atlas of M. We denote a point in U_i by x and its coordinates in V_i by ξ. Let $\sum \psi_i = 1$ be a partition of unity subordinated to $\{U_i\}$ and $\tilde{\psi}_i$ be smooth functions supported in U_i with $0 \leq \tilde{\psi}_i \leq 1$ and $\tilde{\psi}_i = 1$ in the support of ψ_i, as in the definition of Sobolev spaces on manifold. We suppose the following local statement is true:

Lemma 93 (Local statement)**.** *Let P be a polynomial differential operator of type (n,k) and coefficients $c_{\alpha_1,\dots,\alpha_\nu}(x,F)$ are smooth and vanish when $x \in \mathbb{R}^{\dim X}$ is outside of a compact. Let $B \subset \mathbb{R}^N$ be a large Euclidean ball and r,p,q,s as in* (8.1)*. Then for all compactly supported $F \in C(\mathbb{R}^{\dim X}, B) \cap W^{s,q}(\mathbb{R}^{\dim X})$, one has*

$$\|PF\|_{W^{r,p}} \leq C\left(1 + \|F\|_{W^{s,q}}\right)^{q/p}$$

where the constant C depends only on B and the support of F, and not on F.

One has

$$\|PF\|_{W^{r,p}} := \sum_i \|\psi_i PF\|_{W^{r,p}}$$

where viewed in the chart U_i, each $\psi_i(x)PF(x)$ is $\sum_\alpha \psi_i(\xi).c_\alpha(\xi,g_i).D^\alpha g_i$ where $g_i = f_i \circ \varphi_i^{-1}$ is f_i viewed in the chart. Since $\tilde{\psi}_i = 1$ in the support of ψ_i, one has

$$\psi_i(\xi).c_\alpha(\xi,g_i).D^\alpha g_i = \psi_i(\xi).c_\alpha(\xi,\tilde{\psi}_i g_i)D^\alpha(\tilde{\psi}_i g_i)$$

hence by the local statement:

$$\|\psi_i(\xi).c_\alpha(\xi,g_i).D^\alpha g_i\|_{W^{r,p}} \leq C\left(1 + \|\tilde{\psi}_i g_i\|_{W^{s,q}}\right)^{q/p} \leq C\left(1 + \|F\|_{W^{s,q}}\right)^{q/p}.$$

Therefore $\|PF\|_{W^{r,p}} \leq mC\left(1 + \|F\|_{W^{s,q}}\right)^{q/p}$ where m is the number of charts we used to cover M. $\square$

Remark 36. *The use of partition of unity in the last proof is to decompose $PF = \sum \psi_i PF$ and not $F = \psi_i F$ since we no longer have linearity of the operator P in F.*

8.2 Review of Besov spaces $B^{s,p}$.

In this part, $X = \mathbb{R}^n$ coordinated by $(x_1,\dots,x_n)$ with weight $(\sigma_1,\dots,\sigma_n)$. We define

$$T_j^v f(x_1,\dots,x_n) := f(x_1,\dots,x_j+v,\dots,x_n), \quad \Delta_j^v := T_j^v - \mathrm{Id}$$

for $f \in \mathcal{S}(X)$.

For the notation, we will denote the Besov spaces by $B^{s,p}$ with $s \in \mathbb{R}_{>0} \setminus \mathbb{Z}$ and $p \in (1,\infty)$ so that they look similar to Sobolev space $W^{s,p}$. In a more standard notation, our spaces $B^{s,p}$ are denoted by $B^s_{p,p}$

Definition 16. *We define $B^{s,p}$ as the completion of $\mathcal{S}(X)$ under the norm*

$$\|f\|_{B^{s,p}} := \sum_{\|\gamma\|<s} \|D^\gamma f\|_{L^p} + \sum_{s-\frac{\sigma}{\sigma_j}<\|\gamma\|<s} \sup_v \frac{\|\Delta_j^v D^\gamma f\|_{L^p}}{|v|^{(s-\|\gamma\|)\sigma_j/\sigma}}$$

We cite here some well-known facts

1. While Sobolev spaces with non-integral regularity are complex interpolation of integral ones, Besov spaces are their real interpolation.
2. Besov spaces $B^{s,p}(X)$ are reflexive Banach spaces with their dual spaces being $B^{-s,p'}(X)$ where $\frac{1}{p}+\frac{1}{p'}=1$.

Theorem 94. *If $r<s$ then*

$$W^{s,p}(X) \subset B^{s,p}(X) \subset W^{r,p}(X).$$

Theorem 95 (Multiplication)**.** *For $f,g \in \mathcal{S}(X)$ and $\begin{cases} 0<\alpha<1, \tilde{p}\le p, \tilde{q}\le q, \tilde{r}\le r \\ \frac{1}{p}+\frac{1}{q}=\frac{1}{r}, \frac{1}{\tilde{p}}+\frac{1}{q}=\frac{1}{p}+\frac{1}{\tilde{q}}=\frac{1}{\tilde{r}} \end{cases}$, one has*

$$\|fg\|_{B^{\alpha,\tilde{r}}} \le C\left(\|f\|_{B^{\alpha,\tilde{p}}}\|g\|_{L^q} + \|f\|_{L^p}\|g\|_{B^{\alpha,\tilde{q}}}\right) \tag{8.2}$$

$$\|fg\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q} \tag{8.3}$$

Therefore by density (8.2) *is true for all $f \in L^p \cap B^{\alpha,\tilde{p}}, g \in L^q \cap B^{\alpha,\tilde{q}}$ and* (8.3) *is true for all $f \in L^p, g \in L^q$.*

The reason for which we use the Besov norm is the following estimate:

Theorem 96 (Composition)**.** *Let $\Gamma(x,y)$ be a continuous, nonlinear function of variables $x \in \mathbb{R}^n, y \in \mathbb{R}^N$. Suppose that Γ vanishes for all x outside of a compact in $\mathbb{R}^n$ and Γ is C-Lipschitz in y, and define*

$$\Gamma f := (x \longmapsto \Gamma(x, f(x))).$$

Then

$$\|\Gamma f\| \le C\left(1+\|f\|_{B^{\alpha,p}}\right)$$

8.3 Proof of the local estimate.

Since $B^{r+\epsilon,p}(X) \subset W^{r,p}(X)$, by increasing r a bit, we can suppose that $r \notin \mathbb{Z}$ and replace the $W^{r,p}$ norm in the statement by the $B^{r,p}$ norm, that is to estimate:

$$\|PF\|_{B^{r,p}} = \sum_{\|\gamma\|<r} \|D^\gamma(PF)\|_{L^p} + \sum_{r-\sigma/\sigma_j<\|\gamma\|<r} \frac{\|\Delta_j^v D^\gamma(PF)\|_{L^p}}{|v|^{(r-\|\gamma\|)\sigma_j/\sigma}}$$

where

$$D^\gamma(PF) = \sum c_{\beta_1,\ldots,\beta_\mu}(x,F) D^{\beta_1} f^{b_1} \ldots D^{\beta_\mu} f^{b_\mu} \tag{8.4}$$

with $\max \|\beta_i\| \le k + \|\gamma\|$ and $\sum \|\beta_i\| \le n + \|\gamma\|$.

Using $\Delta_j^v(fg) = \Delta_j^v f \; T_j^v g + f \Delta_j^v g$, one can see that $\Delta_j^v D^\gamma(PF)$ is a sum of terms of 2 types:

$$\Delta_j^v c_{\beta_1,\ldots,\beta_\mu} \; T_j^v(D^{\beta_1} f^{b_1}) \ldots T_j^v(D^{\beta_\mu} f^{b_\mu}) \tag{8.5}$$

and

$$c_{\beta_1,\ldots,\beta_\mu} \; D^{\beta_1} f^{b_1} \; \ldots \; D^{\beta_{i-1}} f^{b_{i-1}} \; \Delta_j^v(D^{\beta_i} f^{b_i}) \; T_j^v(D^{\beta_{i+1}} f^{b_{i+1}}) \; \ldots \; T_j^v(D^{\beta_\mu} f^{b_\mu}) \tag{8.6}$$

Our strategy is to use Theorem 95 to estimate the terms (8.4), (8.5) and (8.6) as follows, where we denote $\|g\|_p := \|g\|_{L^p}$

$$\left\| c_{\beta_1,\ldots,\beta_\mu}(x,F) \; D^{\beta_1} f^{b_1} \ldots D^{\beta_\mu} f^{b_\mu} \right\|_p \le \|c_{\beta_1,\ldots,\beta_\mu}\|_\infty . \|D^{\beta_1} f^{b_1}\|_{p_1} \ldots \|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \tag{8.7}$$

$$\left\| \Delta_j^v c_{\beta_1,\ldots,\beta_\mu} \; T_j^v(D^{\beta_1} f^{b_1}) \ldots T_j^v(D^{\beta_\mu} f^{b_\mu}) \right\|_p \le \|\Delta_j^v c_{\beta_1,\ldots,\beta_\mu}\|_{\tilde{p}_0} . \|D^{\beta_1} f^{b_1}\|_{p_1} \ldots \|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \tag{8.8}$$

$$\begin{aligned}&\left\| c_{\beta_1,\ldots,\beta_\mu} \; D^{\beta_1} f^{b_1} \; \ldots \; D^{\beta_{i-1}} f^{b_{i-1}} \; \Delta_j^v(D^{\beta_i} f^{b_i}) \; T_j^v(D^{\beta_{i+1}} f^{b_{i+1}}) \; \ldots \; T_j^v(D^{\beta_\mu} f^{b_\mu}) \right\|_p \le \\ &\|c_{\beta_1,\ldots,\beta_\mu}\|_\infty . \|D^{\beta_1} f^{b_1}\|_{p_1} \ldots \|D^{\beta_{i-1}} f^{b_{i-1}}\|_{p_{i-1}} . \|\Delta_j^v(D^{\beta_i} f^{b_i})\|_{\tilde{p}_i} . \|D^{\beta_{i+1}} f^{b_{i+1}}\|_{p_{i+1}} \ldots \|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu}\end{aligned} \tag{8.9}$$

Then continue by bounding the Δ_j^v terms:

$$\|\Delta_j^v c_{\beta_1,\ldots,\beta_\mu}\|_{\tilde{p}_0} \le |v|^{\theta\sigma_j/\sigma} C(1 + \|F\|_{B^{\theta,\tilde{p}_0}}) \le |v|^{\theta\sigma_j/\sigma} C(1 + \|F\|_{W^{\theta,\tilde{p}_0}}) \tag{8.10}$$

using Theorem 96, where C is the Lipschitz constant of $c_{\beta_1,\ldots,\beta_\mu}(x,F)$ in F, which exists because $c_{\beta_1,\ldots,\beta_\mu}$ is smooth and F always remains in a large Euclidean ball B. The next Δ_j^v term to bound is, using Theorem 94:

$$\|\Delta_j^v(D^{\beta_i} f^{b_i})\|_{\tilde{p}_i} \le |v|^{\theta\sigma_j/\sigma} \|f^{b_i}\|_{B^{\|\beta_i\|+\theta,\tilde{p}_i}} \le |v|^{\theta\sigma_j/\sigma} \|f^{b_i}\|_{W^{\|\beta_i\|+\theta,\tilde{p}_i}} \tag{8.11}$$

And finally plugging (8.10) and (8.11) in (8.8) and (8.9), and noting that $\|c_{\beta_1,\ldots,\beta_\mu}\|_\infty$ in (8.7) is bounded by a constant, it remains to estimate $\|f^{b_i}\|_{W^{\|\beta_i\|,p_i}}$, $\|f^{b_i}\|_{W^{\|\beta_i\|+\theta,\tilde{p}_i}}$ and $\|F\|_{W^{\theta,\tilde{p}_0}}$ in term of $\|F\|_{W^{s,q}}$, for which we will use the following consequence of Interpolation inequality.

Lemma 97. *Let $0 \le r \le s$ and $p, q \in (1, \infty)$ such that $0 < \frac{1}{p} - \frac{r}{s}\frac{1}{q} < 1 - \frac{r}{s}$. Then for all compactly supported $F \in C(X,B) \cap W^{s,q}$ where $B \subset \mathbb{R}^N$ is a large Euclidean ball, one has*

$$\|F\|_{W^{r,p}} \le C \|F\|_\infty^{1-r/s} \|F\|_{W^{s,q}}^{r/s} \le C' \|F\|_{W^{s,q}}^{r/s}$$

where C, C' depend only on B and the support of F, but not F.

Proof. Since F is bounded, $f^\alpha \in W^{s,q} \cap W^{0,v}$ for all $v > 1$. By Interpolation inequality

$$\|f^\alpha\|_{W^{r,p}} \leq 2\|f^\alpha\|_{W^{s,q}}^{r/s}\|f^\alpha\|_{W^{0,v}}^{1-r/s}$$

then choose v with $(1-\frac{r}{s})\frac{1}{v} = \frac{1}{p} - \frac{r}{s}\frac{1}{q}$. □

To apply Lemma 97, we have to choose $p_i, \tilde{p}_i, \tilde{p}_0, \theta$ such that $\begin{cases} 0 < \frac{1}{p_i} - \frac{\|\beta_i\|}{s}\frac{1}{q} < 1 - \frac{\|\beta_i\|}{s}, \\ 0 < \frac{1}{\tilde{p}_i} - \frac{\|\beta_i+\theta\|}{s}\frac{1}{q} < 1 - \frac{\|\beta_i+\theta\|}{s} \\ 0 < \frac{1}{\tilde{p}_0} - \frac{\theta}{s}\frac{1}{q} < 1 - \frac{\theta}{s} \end{cases}$
We choose $\frac{1}{p_i}$ just a bit bigger than $\frac{\|\beta_i\|}{s}\frac{1}{q}$, $\frac{1}{\tilde{p}_i}$ just a bit bigger than $\frac{\|\beta_i+\theta\|}{s}\frac{1}{q}$ and $\frac{1}{\tilde{p}_0}$ just a bit bigger than $\frac{\theta}{s}\frac{1}{q}$. We will now come back to justify the estimates (8.7), (8.8), (8.9). Since F is bounded in B and compactly supported in an open set V, we see that $\|f^\alpha\|_p \leq C(B,V)\|f^\alpha\|_q$ if $p \leq q$. Therefore,

1. For (8.7), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{p_1} + \cdots + \frac{1}{p_\mu}$$

   which is true because the RHS is is a bit bigger than $\frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|}{qs} < \frac{n+r}{qs} < \frac{1}{p}$.

2. For (8.8), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{\tilde{p}_0} + \frac{1}{p_1} + \cdots + \frac{1}{p_\mu}$$

   where the RHS is is a bit bigger than $\frac{\theta}{s}\frac{1}{q} + \frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|+\theta}{qs}$.

3. For (8.9), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{p_1} + \cdots + \frac{1}{\tilde{p}_i} + \cdots + \frac{1}{p_\mu}$$

   where the RHS is is a bit bigger than $\frac{\theta}{s}\frac{1}{q} + \frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|+\theta}{qs}$.

It is sufficient then to take $\theta = r - \|\gamma\|$. Now the estimates (8.7), (8.8), (8.9) can be continued as

$$RHS(8.7) \leq \prod_i \|f^{b_i}\|_{W^{s,q}}^{\|\beta_i\|/s} \leq \|F\|_{W^{s,q}}^{\frac{n+\|\gamma\|}{s}} \leq \|F\|_{W^{s,q}}^{q/p} \tag{8.12}$$

$$RHS(8.8) \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|F\|_{W^{s,q}}^{\theta/s}\right)\prod_i \|f^{b_i}\|_{W^{s,q}}^{\|\beta_i\|/s} \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|F\|_{W^{s,q}}^{\theta/s}\right)\|F\|_{W^{s,q}}^{q/p} \tag{8.13}$$

$$RHS(8.9) \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|f^{b_i}\|_{W^{s,q}}^{\frac{\|\beta_i\|+\theta}{s}}\right)\prod_{u\neq i}\|f^{b_u}\|_{W^{s,q}}^{\|\beta_u\|/s} \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|F\|_{W^{s,q}}^{\frac{\|\beta_i\|+\theta}{s}}\right)\|F\|_{W^{s,q}}^{q/p} \tag{8.14}$$

While (8.12) gives $\|D^\gamma(PF)\|_p \leq C\|F\|_{W^{s,q}}^{q/p}$, the last two (8.13) and (8.14) give

$$\sum_{s-\frac{\sigma}{\sigma_j}<\|\gamma\|<s} \sup_v \frac{\|\Delta_j^v D^\gamma(PF)\|_p}{|v|^{(r-\|\gamma\|)\sigma_j/\sigma}} \leq C\left(1+\|F\|_{W^{q,s}}^{(n+r)/s}\right)$$

We proved the local statement Lemma 93.

Part VI

Appendix 3: Parametrix and Linear equations

Chapter 9

A comparison theorem, Sobolev imbeddings and Konrachov theorem for Riemannian manifolds

In this part, we will first establish the Sobolev imbeddings theorem and the Kondrachov theorem for Riemannian manifolds from the Euclidean version of these theorems.

Theorem 98 (Sobolev Imbedding for $\mathbb{R}^n$). *Given $k, l \in \mathbb{Z},\ k > l \geq 0$ and $p, q \in \mathbb{R},\ p > q \geq 1$. Then*

1. *If $\frac{1}{p} = \frac{1}{q} - \frac{k-l}{n}$ then*
$$W^{k,q}(\mathbb{R}^n) \hookrightarrow W^{l,p}(\mathbb{R}^n)$$
is a continuous imbedding.

2. *If $\frac{k-r}{n} > \frac{1}{q}$ then*
$$W^{k,q}(\mathbb{R}^n) \hookrightarrow C^r_B(\mathbb{R}^n)$$
If $\frac{k-r-\alpha}{n} \leq \frac{1}{q}$ then
$$W^{k,q}(\mathbb{R}^n) \hookrightarrow C^{r,\alpha}(\mathbb{R}^n)$$

where $C^r_B(\mathbb{R}^\ltimes)$ denotes the space of C^r functions with bounded derivatives up to order n, equipped with the norm $\|u\|_{C^r_B} = \max_{l \leq r} \sup |\nabla^l u|$, and $C^{r,\alpha}$ is the subspace of C^r_B of functions whose r^{th}-derivative is α-Holder, equipped with the norm $\|u\|_{C^r,\alpha} = \|u\|_{C^r_B} + \sup_{P \neq Q} \{\frac{u(P)-u(Q)}{d(P,Q)^\alpha}\}$.

Theorem 99 (Kondrachov for $\Omega \subset \mathbb{R}^n$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with regular boundary and let $k \in \mathbb{Z}_{\geq 0}$ and $p, q \in \mathbb{R}_{>0}$ be such that $1 \geq \frac{1}{p} > \frac{1}{q} - \frac{k}{n} > 0$ then*

1. *The imbedding $W^{k,q}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

2. *The imbedding $W^{k,q}(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ is compact if $k - \alpha > \frac{n}{q}$ where $0 \leq \alpha < 1$.*

3. *The imbeddings $W_0^{k,q}(\Omega) \hookrightarrow L^p(\Omega)$ and $W_0^{k,q}(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ are compact, where $W_0^{k,q}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $W^{k,q}(\Omega)$, i.e. the subspace of functions whose trace vanishes on the boundary of Ω.*

Theorem 98 will be generalised for complete manifolds with bounded curvature and injectivity radius, while Theorem 99 holds for compact Riemannian manifolds.

The generalisation will be done in 2 steps

1. Compare the volume form of the Riemannian metric g near a point and that of the Euclidean metric on the tangent space at that point. Theorem 103 gives an equivalent between the integral under g and the integral under Euclidean metric via the exponential map.

2. Reasonably use partition of unity to establish global results from local results (the Euclidean case). We will need a covering lemma (Calabi's lemma), which essentially reduces to a combinatorial result (Vitali's covering lemma).

Finally, we will apply imbedding theorems to solve the equation $-\Delta u = f$ on a Riemannian manifold when f is square-integrable.

9.1 Quick recall of Jacobi fields, Index inequality

Definition 17. *A **Jacobi** field is a field Y defined along a geodesic $\gamma(t)$ such that*

$$\frac{D^2}{dt^2}Y(t) + R(Y(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0 \tag{9.1}$$

where R denotes the Riemann curvature tensor.

Remark 37. 1. *Since (9.1) is linear, a Jacobi field is uniquely defined given $Y(t_0)$ and $\dot{Y}(t_0)$.*

2. *If $Y(0) \perp \dot{\gamma}(0)$ and $\dot{Y}(0) \perp \dot{\gamma}(0)$ then $\dot{Y}(t) \perp \dot{\gamma}(t)$ for all t.*

3. *If Y, Z are Jacobi fields along γ then*

$$\langle Y, \dot{Z}\rangle - \langle \dot{Y}, Z\rangle = \text{const}$$

In particular, if Y, Z vanish at a same point p_0 in γ then $\langle Y, \dot{Z}\rangle = \langle \dot{Y}, Z\rangle$ on γ.

There are two ways to interpret Jacobi fields:

1. Jacobi fields are derivative of exponential maps

2. Jacobi fields are minimisers of Index form, i.e. the variation of second other of length.

The first interpretation is the content of the following Proposition.

Proposition 100. *Let* $Y(t) = D\exp_p(tu).t\xi$ *be a vector field defined on a geodesic* $\gamma(t) = \exp_p tu$*. Then* Y *satisfies*

$$\begin{cases} Y(0) = 0, \dot{Y}(0) = \xi, \\ \ddot{Y} + R(Y, \dot{\gamma})\dot{\gamma} = 0, \end{cases} \tag{9.2}$$

hence a Jacobi field.

In concrete term, denote by ψ the exponential function at $p \in M$ and $q = \gamma(r) = \exp_p r\dot{\gamma}(0)$, then Proposition 100 says that if the Jacobi field Y vanishes at $p = \gamma(0)$, i.e. $Y(0) = 0$ then $Y(r)$ at $\gamma(r)$ is defined as follow: pull-back $\dot{Y}(0)$ by ψ, transport parallelly, w.r.t to the Euclidean metric of T_pM, $\psi^*\dot{Y}(0)$ from 0 to $X_0 = \psi^{-1}(q)$, then push-forward by ψ, one obtains $Y(r)$. See Figure 9.1.

Figure 9.1: Jacobi fields and exponential maps.

Since Jacobi fields are derivatives of exponential maps, one can rephrase the phenomenon of cut-locus by Jacobi fields. Historically, a point q on a Riemannian manifold is said to be a **conjugate** point of p if there exists, along a geodesic connecting them, a Jacobi field vanishing on both p and q. This means that the exponential map with origin in p degenerates at a preimage of q. One can also prove that if q is in the cut-locus of p then at least one of the following situation occurs

1. q is a conjugate point of p.

2. There exists 2 minimising geodesic from p to q.

For another interpretation of Jacobi fields, note that given a geodesic γ and a vector field Z defined along γ, then the first variation of length when one varies γ by Z is 0 and the second variation can also be calculated without difficulty.

Proposition 101 (Second variation of length)**.** *Let* $\gamma : [0, r] \longrightarrow M$ *be a geodesic and* Z *be a vector field along* γ *that is orthogonal to* $\dot{\gamma}$ *at every point. Denote by* L_λ *length of the curve* $t \mapsto \exp_{\gamma(t)} \lambda Z$ *for* $\lambda \ll 1$*, then one has*

$$\frac{d^2}{d\lambda^2} L_\lambda \bigg|_{\lambda=0} = I(Z) := \int_0^r \left(\|Z(t)\|^2 + \langle R(\dot{\gamma}(t), Z(t))\dot{\gamma}(t), Z(t)\rangle \right) dt \tag{9.3}$$

Definition 18. *Let* $\gamma : [0, r] \longrightarrow M$ *be a geodesic and* Z *be a orthogonal vector field along* γ*. The* ***Index form*** $I(Z)$ *of* Z *is defined by the RHS of* (9.3)*.*

Remark 38. *The curvature term in* (9.3) *is* $K(\dot{\gamma}, Z)\|Z\|^2$ *where* K *denotes the sectional curvature of* M*.*

Jacobi fields can be seen as the unique minimiser of the Index form among vector fields defined on a geodesic $\gamma : [0, r] \longrightarrow M$ with the same value at $\gamma(0)$ and $\gamma(r)$.

Theorem 102 (Index inequality)**.** *Let $\gamma : [0, r] \longrightarrow M^n$ be a geodesic, $p = \gamma(0)$ and $q = \gamma(r)$ such that p has no conjugate point along γ, or equivalently the exponential map in direction $\dot\gamma(0)$ does not degenerate.*

- *Let Z be a (piecewise smooth) vector field along γ, orthogonal to $\dot\gamma$ with $Z(p) = 0$.*
- *Let Y be the Jacobi field along γ with $Y(0) = 0, Y(r) = Z(r)$ and Y is orthogonal to $\dot\gamma$.*

Then $I(Y) \le I(Z)$ and equality occurs if and only if $Y \equiv Z$.

Remark 39. *Note that such Jacobi field Y exists and is unique. Firstly, by the second point of Remark 37, one only need $Y(p) = 0$ and $\dot Y(0) \perp \gamma(0)$. The Jacobi fields satisfying these conditions form a vector space of dimension $n-1$ (by Cauchy problem, $\dot Y(0)$ is to be chosen in the orthogonal space of $\gamma(0)$). Since the exponential map does not degenerate on the preimage of γ, each $\dot Y(0)$ corresponds one-to-one with an $Y(r)$ by Proposition 100. The correspondence is linear, with source and target spaces of same dimension $(n-1)$, it follows that each $Z(r) \perp \gamma(r)$ gives uniquely a Jacobi field Y.*

More concretely, let $\dot V_i(0)$ be a basis of $\dot\gamma(0)$ in T_pM and V_i be the corresponding Jacobi fields with $V_i(0) = 0$, then

1. *$\{V_i(t)\}_{i=\overline{1,n-2}}$ is a basis of $\dot\gamma(t)$ in $T_{\gamma(t)}M$, where the orthogonal part follows from Remark 37 and the linear independence is by the non-degeneration of since $\exp_p$.*
2. *If $Z(t) = \sum f_i(t)V_i(t)$, where f_i are functions on $[0, r]$, then $Y(t) = \sum_i f_i(r)V_i(t)$.*

Proof. As Remark 39, let $Z = \sum_i f_iV_i$ and denote $W = \sum_i \dot f_iV_i$ then

$$I(Z) = \int_0^r \left(\|W\|^2 + 2\sum_i f_i\langle \dot V_i, W\rangle + \langle \sum_i f_i\dot V_i, \sum_j f_j\dot V_j\rangle + \langle R(\dot\gamma, \sum f_iV_i)\dot\gamma, \sum f_jV_j\rangle \right) dt$$

By definition of Jacobi field, $R(\dot\gamma, V_i)\dot\gamma = \ddot V_i$, hence the curvature term is

$$\begin{aligned}\int_0^r \left\langle R(\dot\gamma, \sum f_iV_i)\dot\gamma, \sum f_jV_j \right\rangle &= \sum_{i,j} \int_0^r f_if_j\langle \ddot V_i, V_j\rangle dt = \sum_{i,j} \int_0^r f_if_j \left(\frac{d}{dt}\langle \dot V_i, V_j\rangle - \langle \dot V_i, \dot V_j\rangle \right) dt \\ &= -\int_0^r \left\langle \sum_i f_i\dot V_i, \sum_j f_j\dot V_j \right\rangle dt + \langle \dot Y(r), Y(r)\rangle - 2\sum_{i,j} \int_0^r f_i\dot f_j\langle \dot V_i, V_j\rangle dt\end{aligned}$$

where for the second line, we integrated by part and used the fact that $\langle \dot V_i, V_j\rangle = \langle V_i, \dot V_j\rangle$ (point 3 of Remark 37). Therefore, one has

$$I(Z) = \int_0^r \|W\|^2 dt + \langle \dot Y(r), Y(r)\rangle.$$

In particular $I(Y) = \langle \dot Y(r), Y(r)\rangle \le I(Z)$. The equality occurs if and only if $W \equiv 0$, i.e. $Z \equiv Y$. $\square$

9.2 Local comparison with space forms

Our goal in this section is to prove the following Comparison Theorem. Before going to the precise statement, let us explain the notation.

Notation. Given M^n a Riemannian manifold and $B(p, r_0)$ be the geodesic ball centered in $p \in M$, of radius $r_0 < \delta_p$ the injectivity radius at p, equipped with the pullback metric of g via exponential map $\exp_p$, which can be expressed in polar geodesic coordinates as

$$(ds)^2 = (dr)^2 + r^2 g_{\theta^i\theta^j}(r,\theta) d\theta^i d\theta^j$$

where $\frac{\partial}{\partial\theta^1}, \ldots, \frac{\partial}{\partial\theta^{n-1}}$ is an Euclidean orthonormal frame of the sphere $r\mathbb{S}^{n-1}$. We note $|g_\theta| = \det(g_{\theta^i\theta^j})_{ij}$ and $g_{\theta\theta}$ be any component $g_{\theta^i\theta^i}$ for $i = 1, \ldots, n-1$.

Abusively, we say that $\frac{\sin \alpha r}{\alpha} = r$ if $\alpha = 0$ and $\sin \alpha r = \frac{1}{i} \sinh i\alpha r$ and $\cos \alpha r = \cosh i\alpha r$ if $\alpha \in i\mathbb{R}$.

Remark 40. *Note that the frame $\{\frac{\partial}{\partial\theta^i}\}_i$ may not be global, for example when n is odd (Hairy ball theorem). However the quantity $|g_\theta|$ is globally defined (except at p), in fact $|g_\theta| = r^{-2n+2}|g|$.*

Theorem 103 (comparison of volume forms)**.** *Let M^n be a Riemannian manifold with*

- *sectional curvature $-a^2 \le K \le b^2$*
- *Ricci curvature $\mathrm{Ric} \ge a' = (n-1)\alpha^2$ where α can be real or purely imaginary.*

Then with the notation of the last paragraph, for all $r \in (0, r_0)$,

1. *If $r < \frac{\pi}{b}$ then*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} &\ge \frac{\partial}{\partial r} \log \frac{\sin br}{r} \\ g_{\theta\theta} &\ge \left(\frac{\sin br}{br}\right)^2 \end{aligned} \tag{9.4}$$

2. *One has*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} &\le \frac{\partial}{\partial r} \log \frac{\sinh ar}{r} \\ g_{\theta\theta} &\le \left(\frac{\sinh ar}{ar}\right)^2 \end{aligned} \tag{9.5}$$

3. *One has*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_\theta} &\le (n-1)\frac{\partial}{\partial r} \log \frac{\sin \alpha r}{r} \le -a'\frac{r}{3} \\ \sqrt{|g_\theta|} &\le \left(\frac{\sin \alpha r}{\alpha r}\right)^{n-1} \end{aligned} \tag{9.6}$$

4. If $r < \frac{\pi}{b}$ then

$$\begin{aligned}\frac{\partial}{\partial r}\log\sqrt{g_\theta} &\geq (n-1)\frac{\partial}{\partial r}\log\frac{\sin br}{r}\\ \sqrt{|g_\theta|} &\geq \left(\frac{\sin br}{br}\right)^{n-1}\end{aligned} \tag{9.7}$$

Remark 41. *1. The moral of the estimates is that if $r \ll 1$ then the volume form of g, viewed in the tangent space at p, is equivalent to the Euclidean volume form of T_pM.*

2. One can always choose $\alpha \in i\mathbb{R}$ even when the Ricci curvature is positive, and RHS of (9.6) *will be a hyperbolic function and the estimate is not as sharp as if one choose $\alpha \in \mathbb{R}$, but it works to prove that the two volume forms are equivalent when $r \ll 1$.*

Remark 42. *A few consequences of Theorem 103:*

1. *For δ small, the metric volume form dV is* equivalent *to the Euclidean volume form of tangent space: there exists $C(\delta) > 0$ converging to 1 as $\delta \to 0$ such that $C(\delta)^{-1}dE \leq dV \leq C(\delta)dE$.*

2. *Let f be a smooth function defined on $B(p,\delta)$ then the gradient of f w.r.t the metric g is closed to the Euclidean gradient of f viewed in the chart (namely $f \circ \exp_p$):*

$$\begin{aligned}\|\nabla f\|_g &= \left|\frac{\partial f}{\partial r}\right|^2 + \sum_\theta \left|\frac{\partial f}{\partial \theta}(r,\theta)\right|^2 g_{\theta\theta}\\ \|\nabla(f\circ\exp_p)\|_E &= \left|\frac{\partial f}{\partial r}\right|^2 + \sum_\theta \left|\frac{\partial f}{\partial \theta}(r,\theta)\right|^2\end{aligned}$$

3. *Combining the last 2 points, one can see that if f is supported in a small geodesic ball $B(p,\delta)$, then the L^p-norm of ∇f is closed to the Euclidean L^p norm of $\nabla(f\circ\exp_p)$ if δ is sufficiently small.*

The ideal to prove Theorem 103 comes from Proposition 100 and Figure 9.1. Given a point $q \in M$ of distance $r < r_0$ from p, then denote by Y the Jacobi field along the unique geodesic connecting p and q such that Y vanishes at p and $Y(r) = \frac{\partial}{\partial\theta}$ at q, then with $\psi = \exp_p$ as in Figure 9.1,

$$\begin{aligned}\|Y(r)\|^2 = \|\psi_{X_0}^* Y(r)\|^2 &= \|\psi_0^*\dot{Y}(0)\|_{X_0}^2\\ &= r^2 g_{\theta\theta}\|\psi_0^*\dot{Y}(0)\|_0^2 = r^2 g_{\theta\theta}\|\dot{Y}(0)\|^2\end{aligned} \tag{9.8}$$

where we used the fact that

$$g\left(\left.\frac{\partial}{\partial\theta^i}\right|_{r\mathbb{S}^{n-1}}, \left.\frac{\partial}{\partial\theta^j}\right|_{r\mathbb{S}^{n-1}}\right) = r^2 g\left(\left.\frac{\partial}{\partial\theta^i}\right|_{\mathbb{S}^{n-1}}, \left.\frac{\partial}{\partial\theta^j}\right|_{\mathbb{S}^{n-1}}\right) = r^2 g_{\theta_i\theta_j}$$

.

Take logarithm and derive (9.8) w.r.t r, using the fact that $\|Y(r)\| = 1$, one obtains

$$\langle \dot{Y}(r), Y(r)\rangle = \frac{1}{r} + \frac{\partial}{\partial r}\log g_{\theta\theta} \tag{9.9}$$

It comes to estimate $\langle \dot{Y}(r), Y(r)\rangle$, which is in fact the Index form of Y. The following lemma give an estimate of the Index form in case of bounded sectional curvature, by comparing the it with the Index form under a metric with constant sectional curvature.

Lemma 104. *Suppose that the sectional curvature $K \leq b^2$, then for every Jacobi field Y defined a long a geodesic $\gamma : [0, r] \longrightarrow M$ with $r < \frac{\pi}{2b}$ such that $Y(0) = 0, Y \perp \dot{\gamma}$. Then*

$$I(Y) \geq I_b(Y) := \int_0^r \|\dot{Y}\|^2 - b\|Y\|^2 \geq b \cot br \|Y(r)\|^2$$

Proof. By the curvature bound, $I(T) \geq \int_0^r \|\dot{Y}\|^2 - b^2\|Y\|^2 =: I_b(Y)$. The quantity $I_b(Y)$ is exactly the Index form of Y along γ if the sectional curvature in constantly b. To be precise, we equip the tubular neighborhood of γ a metric g' of constant sectional curvature $K = b^2$ such that normal vectors of γ w.r.t the metric g remain normal under g'. Such g' is in fact easy to find since:

1. The tubular neighborhood is diffeomorphic to $[0, r] \times \mathbb{B}^{n-1}$ where the diffeomorphism (says ι_1) is actually isometry at points of γ, which are mapped to $[0, r] \times \{0\}$;

2. Also, there exists a diffeomorphism ι_2 mapping $[0, r] \times \mathbb{B}^{n-1}$ to a tubular neighborhood of an arc $\tilde{\gamma}$ of length r on the grand circle of $\mathbb{S}^n_{1/b}$ which is isometry on every point of $[0, r] \times \{0\}$. This is because $r < \frac{\pi}{2b} < 2\pi\frac{1}{b}$ the length of the grand circle.

3. One now can identify a tubular neighborhood of γ in M and that of $\tilde{\gamma}$ in $\mathbb{S}^n_{1/b}$ by $\iota = \iota_2 \circ \iota_1$. Take g' to be the pullback of the Eucidean metric on $\mathbb{S}^n_{1/b}$, which is of sectional curvature b^2.

Now under the metric g', Y is no longer a Jacobi field, but it is still orthogonal to γ, denote by $\tilde{Y}$ the Jacobi field (under g') on γ that vanishes at $\gamma(0)$ and has the same value as Y at $\gamma(r)$. By Theorem 102 (Index inequality), one has $I_b(Y) \geq I_b(\tilde{Y})$. The latter can be computed directly, as the field $\iota_*\tilde{Y}$ is given by

$$s \mapsto (s, \beta^1 \sin bs, \dots, \beta^{n-1} \sin bs), \quad s \in [0, r]$$

where $(\beta^1, \dots, \beta^{n-1})$ is the coordinates of $\iota_{1*}Y(r)$ in $[0, r] \times \mathbb{B}^{n-1}$, hence in this coordinates (also called *Fermi coordinates*), $\tilde{Y}(s) = \left(s, \frac{\sin bs}{\sin br}Y(r)\right)$. Hence $I_b(\tilde{Y}) = b \cot br \|Y(r)\|^2$. $\square$

Now the remaining part of the proof of Theorem 103 is straightforward.

Proof of Theorem 103. From (9.9) and Lemma 104, one has

$$\frac{\partial}{\partial r}\log\sqrt{g_{\theta\theta}} = I(Y) - \frac{1}{r} \geq b\cot br - \frac{1}{r}$$

This gives the estimates in (9.4).

For (9.5), the sign situation fits Theorem 102 better, and one does not need to explicitly evoke the space forms (as Lemma 104). It suffices to see that

$$\begin{aligned}\langle\dot{Y}(r), Y(r)\rangle = I(Y) &\leq I\left(\frac{\sinh at}{\sinh ar}Y(r)\right)\\ &\leq a^2\left(\int_0^r\left(\frac{\cosh at}{\sinh ar}\right)^2 + \int_0^r\left(\frac{\sinh at}{\sinh ar}\right)^2 dt\right)\|Y(r)\|^2\\ &= a\coth ar\|Y(r)\|^2\end{aligned}$$

The estimates in (9.6) comes from the comparison between Y and the field $t \mapsto \frac{\sin\alpha t}{\sin\alpha r}Y(r)$. Note that the field is well-defined even when $\alpha \in \mathbb{R}_{>0}$ (the hyperbolic case ($\alpha \in i\mathbb{R}_{>0}$ being obvious). This in fact comes from the following fact:

Theorem 105 (Myers)**.** *Let M^n be a connected, complete manifold with* $\mathrm{Ric} \geq (n-1)\alpha^2 > 0$ *then*

1. *M is compact.*
2. *The diameter of M is at most π/α.*

Taking sum of inequalities $I(Y_i) \leq I(\frac{\sin\alpha t}{\alpha r}Y_i(r)$ where Y_i are Jacobi fields vanishing at $\gamma(0)$ and whose values at $\gamma(r)$ are $\frac{\partial}{\partial\theta^i}$ respectively, one has

$$\begin{aligned}\sum_{i=1}^{n-1}\langle\dot{Y}_i(r), Y_i(r)\rangle &\leq (n-1)\alpha^2\int_0^r\left(\frac{\cos\alpha t}{\sin\alpha r}\right)^2 dt - \sum_{i=1}^{n-1}\int_0^r R_{r\theta^i r\theta^i}\left(\frac{\sin\alpha t}{\sin\alpha r}\right)^2 dt\\ &\leq (n-1)\alpha\cot\alpha r\end{aligned}$$

where for the second line, we used the fact that $\sum_i R_{r\theta^i r\theta^i} = \mathrm{Ric}_{rr} \geq (n-1)\alpha^2$. Hence

$$\begin{aligned}\frac{\partial}{\partial r}\log\sqrt{|g_\theta|} = \frac{\partial}{\partial r}\sum_i\log\sqrt{|g_{\theta^i\theta^i}|} &= \sum_i\langle\dot{Y}_{\theta^i}, Y_{\theta^i}\rangle - \frac{n-1}{r}\\ &\leq (n-1)\left(\alpha\cot\alpha r - \frac{1}{r}\right) = (n-1)\frac{\partial}{\partial r}\log\left(\frac{\sin\alpha r}{r}\right)\end{aligned}$$

The proof of (9.7) is essentially the same as (9.6) where one uses (9.4) for a lower bound of $I(Y_i) = \langle\dot{Y}_i(r), Y_i(r)\rangle$. □

As a side note, Lemma 104 can also be used to prove that a small geodeosic ball is geodesically convex.

Proposition 106. *Let M^n be a Riemannian manifold with sectional curvature $K \leq b^2$ and injectivity radius $\delta > 0$. Then for every $r < \min\{\frac{\delta}{2}, \frac{\pi}{4b}\}$, any geodesic ball $B(p,r)$ is geodesically convex, i.e. any two points is connected by a geodesic curve inside the ball.*

Proof. We first claim that

Lemma 107. *Given two point p, q of distance $d(p,q) = r < \frac{\pi}{2b}$ and $\Gamma_{p,q}$ the geodesic connecting the them. Let γ be a geodesic staring from q with a velocity vector perpendicular to $\Gamma_{p,q}$, then there exists a neighborhood of q inside of which the γ intersects $\Gamma_{p,q}$ only at q.*

First, let us prove that the Lemme implies Proposition 106. If r small as in the Proposition and $q_1, q_2 \in B(p,r)$ then

1. There exists a minimal geodesic Γ_{q_1,q_2} connecting q_1, q_2.

2. By triangle inequality, $\Gamma_{q_1,q_2} \subset B(p, 2r)$: every point $q \in \Gamma_{q_1,q_2}$ has to be $d(q_1,q_2)/2$-closed to one q_i, hence $d(p,q) \leq d(p,q_i) + d(q_i,q) \leq r + \frac{2r}{2} = 2r$.

Let $T \in \Gamma_{q_1,q_2}$ be the point minimising the distance to p. It suffices to show that T is one of the q_i. For the sake of contradiction, if T is strictly in the interior of Γ_{q_1,q_2} then

1. The geodesic $\Gamma_{p,T}$ connecting p and T is orthogonal to Γ_{q_1,q_2} at T. It is not difficult to prove that if the two are not orthogonal then there exist $T' \in \Gamma_{q_1,q_2}$ and $S \in \Gamma_{p,T}$, both being near to T, such that $d(p,T) > d(p,S) + d(S,T') \geq d(p,T')$.

2. The ball $B(p, d(p,T)) \cap \Gamma_{q_1,q_2} \supset \Gamma_{q_1,q_2}$.

These contradict the Lemma and prove that T does not lie in the interior.

It remains to prove the Lemma. Let Y be the Jacobi field which vanishes at p and whose value at q is $\dot{\gamma}$, then by Index inequality (Theorem 102), it suffices to prove that $I(Y) > 0$, because any variation of $\Gamma_{p,q}$ by orthogonal vector field Z along γ has $I(Z) > 0$ hence only increases the length, according to Proposition 101. But by Lemma 104 gives

$$I(Y) \geq I_b(Y) \geq b \cot br \|Y(q)\|^2 > 0 \text{ if } r < \frac{\pi}{2b}.$$

$\square$

9.3 Some covering lemmas

The goal of this section is to prove a covering lemma for Riemannian manifolds with injectivity radius $\delta_0 > 0$ and bounded curvature (Lemma 110). We start with a covering lemma that not yet requires curvature bound.

Lemma 108 (Calabi). *Let M^n be a Riemannian manifold with injectivity radius $\delta_0 > 0$, then for all $\delta \in (0, \delta_0)$, there exists $0 < \gamma < \beta \leq \delta$ and a partition of $M = \bigsqcup_{i \in I} \Omega_i$ and $p_i \in \Omega_i$ such that*

$$B(p_i, \gamma) \subset \Omega_i \subset B(p_i, \beta)$$

Moreover, one can choose $\gamma = \beta/10$ and $\beta = \delta$.

Proof. Note that it is enough to have

$$\begin{cases} \bigcup_i B(p_i, \beta) = M, \quad 2\gamma < \beta \\ B(p_i, 2\gamma) \text{ are disjoint} \end{cases} \tag{9.10}$$

In fact, let $\Omega_i' = B(p_i, \beta) \setminus \cup_{j \neq i} B(p_j, \gamma)$ then $\begin{cases} B(p_j, \gamma) \cap \Omega_i' = \emptyset, B(p_i, \gamma) \subset \Omega_i' \subset B(p_i, \beta) \\ \bigcup_i \Omega_i' = M \end{cases}$ (for $\bigcup_i \Omega_i' = M$: If $x \in M$ satisfies $x \in B(p_j, \gamma) \subset B(p_i, \beta)$ then there is no other $j' \neq j$ such that $x \in B(p_{j'}, \gamma)$, hence $x \in \Omega_j$. Now choose

$$\Omega_1 = \Omega_1', \Omega_2 = \Omega_2' \setminus \Omega_1, \ldots, \Omega_n = \Omega_n' \setminus \cup_{i=1}^{n-1} \Omega_i, \ldots$$

For the existence of (9.10), use the following Vitali covering lemma, whose proof is purely combinatorial in nature.

Lemma 109 (Vitali covering, Infinite version). *Let $\{B_j : j \in J\}$ be a collection of balls in a metric space such that*

$$\sup\{\mathrm{rad}(B_j) : j \in J\} < +\infty$$

where rad *denotes the radius, then there exists a countable subfamily $J' \subset J$ such that $\{B_j : j \in J'\}$ are disjoint and*

$$\cup_{j \in J} B_j \subset \cup_{j \in J'} 5B_j.$$

It remains to apply the lemma for the covering $M = \cup_{x \in M} B(x, 2\gamma)$, which also allows us to choose $\gamma = \beta/10$ and $\beta = \delta$. □

Lemma 110 (Uniformly locally finite covering). *Let M^n be a Riemannian manifold with injectivity radius $\delta_0 > 0$ and bounded curvature, then for all $\delta < \delta_0$ sufficiently small, there exists a **uniformly locally finite covering** of M by balls $\{B(p_i, \delta)\}_{i \in I}$, i.e. there exists $k(\delta) \in \mathbb{Z}_{>0}$ such that for all $q \in M$, there exists a neighborhood of q that intersects at most $k(\delta)$ balls. Moreover, one can also require that $\{B(p_i, \delta/2)\}_{i \in I}$ is still a covering.*

Proof. We will apply Lemma 108 with $\beta = \delta/2$ and $\gamma = \beta/10$, then for all $\delta \ll \delta_0$, the covering $\{B(p_i, 2\beta)\}$ satisfies. In fact, for every $q \in M$, take $B(q, \delta)$ as a neighborhood of q then $B(p_i, 2\beta) \cap B(q, \gamma) \neq \emptyset$ if and only if $p_i \in B(q, 2\beta + \gamma)$ Since the balls $B(p_i, \gamma)$ are disjoint, the number of p_i in $B(q, 2\beta + \gamma)$ is bounded by

$$k = \frac{\max \mathrm{vol}_g(B_{2\beta + 2\gamma})}{\min \mathrm{vol}_g(B_\gamma)} \leq C(\delta) \left(\frac{2\beta + 2\gamma}{\gamma} \right)^n$$

where $\max \mathrm{vol}_g(B_{2\beta+2\gamma})$ and $\min \mathrm{vol}_g(B_\gamma)$ denote the maximum and minimum volume of balls of radius $2\beta+2\gamma$ and γ, respectively. By Theorem 103, for $\delta < \epsilon(a', b)$ depending on the bound a' and b of Ricci curvature and sectional curvature, the volume of these balls are equivalent to that of Euclidean balls of the same radius. The constant of equivalence was denoted by $C(\delta)$. $\square$

9.4 Sobolev imbeddings for Riemannian manifolds

The goal of this section is to prove that Sobolev imbeddings are also available for complete Riemannian manifold with bounded curvature and strictly positive injectivity radius, that is, the following results.

Theorem 111 (Sobolev imbeddings)**.** *Theorem 98 holds when one replaces R^n by a complete Riemannian manifold of dimension n with bounded curvature (sectional and Ricci) and injectivity radius $\delta_0 > 0$.*

The definition of Sobolev spaces as completion of spaces of smooth functions, w.r.t the Sobolev norms generalises on Riemannian manifolds, namely, we denote by $W_0^{k,p}(M)$ the completion of $C_c^\infty(M)$ w.r.t the norm $\|\varphi\|_{W^{k,p}} = \|\varphi\|_{L^p} + \|\nabla\varphi\|_{L^p} + \cdots + \|\nabla^k\varphi\|_{L^p}$ where $\|\nabla^l\varphi\|_{L^p}$ are computed as follow: the metric g induces a fiberwise norm for l-covariant tensors, integrate that of $\nabla^l\varphi$, one obtains $\|\nabla^l\varphi\|_{L^p}$.

Similarly, the space $W^{1,p}(M)$ is defined as the completion of $C^\infty(M)$ w.r.t $\|\cdot\|_{W^{1,p}}$.

Remark 43.

1. *Unlike the Euclidean case, one does not define the derivatives term, e.g. $\nabla_v f$ for $f \in W^{1,p}(M)$ using integration by part and Riesz representation, that is, one does not expect a formular such as $\int_M (\nabla_v f)\varphi dV = -\int_M f \nabla_v \varphi dV$ since the "boundary term" $\int_M \nabla_v(f\varphi)dV$ does not vanish, even if $f\varphi \in C_c^\infty(M)$.*

2. *The exterior derivative df can be defined, which is in fact equivalent to de Rham's notion of current.*

3. *The term $\nabla^l f$ for $f \in W^{k,p}(M)$, when needed, can be defined as a L^p section of $(TM^*)^{\otimes l}$ giving by the L^p limit of smooth sections $\nabla^l\varphi_i$ for an equivalent class of Cauchy sequence φ_i representing f. The completeness of the space of L^p sections of a vector bundle follows from the result in each trivialising chart and the fact that restriction maps commute with the limit.*

Proposition 112 ($W^{1,p} = W_0^{1,p}$)**.** *If M is complete then $C_c^\infty(M)$ is dense in $W^{1,p}(M)$, equivalently $W^{1,p}(M) = W_0^{1,p}(M)$.*

Proof. It suffices to prove that given a function $\varphi \in C^\infty(M)$, one can approximate φ under the norm $\|\cdot\|_{W^{1,p}}$ by functions in $C_c^\infty(M)$. Fix $P \in M$, one uses a cut-off function χ_j which is 1 on $[0,j]$, 0 on $[j,\infty]$ and linear inside and defines $\varphi_j(Q) = \varphi(Q)\chi_j(d(Q,P))$. Note that the distance function is only Lipschitz and not necessarily smooth (so we did not mind taking a linear cut-off). However, since φ_j is compactly support and Lipschitz and we can approximate each φ_j by a sequence in $C_c^\infty(M)$: Let K_j be the support of φ_j and $\{\alpha_i\}_i$ be a finite partition of unity subordinating to an open coordinated cover of K. Since $\alpha_i\varphi_j$ is Lipschitz, viewed in a chart, it can be $W^{1,\infty}$-approximated by smooth functions, due to the following fact.

Fact. If $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\delta\Omega$ regular, then $\mathrm{Lip}(\Omega) = W^{1,\infty}(\Omega)$.

The approximation scheme looks like $\varphi \approx \varphi_j \approx \sum_i \alpha_{i,K_j}\varphi_j \approx \sum_i \psi_{i,j}$ where $\psi_{i,j}$ are smooth and compactly support. □

Remark 44. *The similar results for higher orders are complicated, for example, one can prove that $W_0^{2,p} = W^{2,p}$ under the hypothesis of bounded curvature and strictly positive injectivity radius. The third order requires extra conditions.*

The second part of the Theorem 111 is local in nature, and therefore easier. We will prove this second part by accepting the first one, which we will come back and prove eventually.

For the imbedding into $C_B^r(M)$, it suffices to establish the case $W^{1,q} \hookrightarrow C_B^0$, the higher order case then follows: If $\varphi \in W^{k,q}$ then $\nabla^r\varphi \in W^{k-r,q} \hookrightarrow W^{k-r,q} \hookrightarrow W^{1,\tilde{q}} \hookrightarrow C_B^0$ where $\frac{1}{n} \geq \frac{1}{\tilde{q}} \geq \frac{1}{q} - \frac{k-r-1}{n}$.

Similarly, for the imbedding into $C^{r,\alpha(M)}$, it suffices to establish the case $W^{1,q} \hookrightarrow C^{0,\alpha}$ for $\frac{1-\alpha}{n} \geq \frac{1}{q}$.

Since $W^{1,p}(M) = W_0^{1,p}(M)$, it suffices to prove the following Lemma 113 and Lemma 114.

Lemma 113 ($W^{1,q} \hookrightarrow C_B^0$). *Let M^n be a complete Riemannian manifold with injectivity radius $\delta_0 > 0$ and sectional curvature $K \leq b^2$, then for all $\varphi \in C_c^\infty(M)$, one has*

$$\sup_M |\varphi| \leq C(q)\|\varphi\|_{W^{1,q}}, \quad \forall q > n$$

Proof. Take $\delta < \min\{\delta_0, \frac{\pi}{2b}\}$ and let (r,θ) be the geodesic polar coordinate centered at $P \in M$, then by Theorem 103, the ratio of the metric volume form $dV := |g|dE$ and the Euclidean volume form dE of T_PM is $\sqrt{|g_\theta|} \geq \left(\frac{\sin br}{br}\right)^{n-1} \geq (\frac{2}{\pi})^{n-1}$.

let $\chi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ be a cut-off function which is constantly 1 near 0 and supported in $[0,\delta)$. Then

$$\varphi(P) = -\int_0^\delta \partial_r\left(\varphi(r,\theta)\chi(r)\right)dr, \quad \forall\theta \in \mathbb{S}^{n-1}$$

Integrate w.r.t $\theta \in \mathbb{S}^{n-1}$, recall that ω_n denotes the volume of $\mathbb{S}^{n-1}$:

$$
\begin{aligned}
|\varphi(p)| &\leq (\omega_{n-1})^{-1} \int_B |\nabla(\varphi(r,\theta)\chi(r))|\, r^{1-n} r^{n-1} dr d\theta \\
&\leq (\omega_{n-1})^{-1} \left(\int_B |\nabla(\varphi(r,\theta)\chi(r))|^q \, dE\right)^{1/q} \left(\omega_{n-1}\int_0^\delta r^{(n-1)(1-q)} dr\right)^{1/q'} \\
&\leq (\frac{\pi}{2})^{n-1} (\omega_{n-1})^{-1/q} \left(\|\nabla\varphi\|_{L^q} + \sup_{[0,\delta]} |\chi'| \|\varphi\|_{L^q}\right) \left(\frac{q-1}{q-n} \delta^{\frac{q-n}{q-1}}\right)^{1/q'}
\end{aligned}
$$

where q' denotes the Hölder conjugate of q and for we used Hölder inequality w.r.t dE for the second inequality and the comparison $dE \leq (\frac{\pi}{2})^{n-1} dV$ for the third. The conclusion follows. $\square$

Lemma 114 ($W^{1,q} \hookrightarrow C^{0,\alpha}$). *Let M^n be a complete Riemannian manifold with injectivity radius $\delta_0 > 0$ and bounded curvature, then for all $\varphi \in C_c^\infty(M)$, one has*

$$
\sup_M |\varphi| + \sup_{P\neq Q} |\varphi(P) - \varphi(Q)| \; d(P,Q)^{-\alpha} \leq C(\alpha, q)\|\varphi\|_{W^{1,q}}, \qquad \textit{for all } \frac{1-\alpha}{n} \geq \frac{1}{q}
$$

Proof. By Lemma 113, one can discard the term $\sup_M |\varphi|$ and only need to treat the second term of LHS. Let $\delta \leq \min\{\delta_0, \frac{\pi}{2b}\}$ as in the proof of Lemma 113 (b^2 being the upper bound of the sectional curvature). One only need to consider the case where $d = d(P,Q) < \delta/2$ because otherwise $|\varphi(P) - \varphi(Q)| \leq 2\|\varphi\|_{L^\infty} (\frac{\delta}{2})^{-\alpha} d(P,Q)^\alpha$.

Let O be the midpoint of P, Q, and denote by $h := \varphi \circ \exp_O$ defined on the Euclidean ball $B(0, 2d) \supset B_O := B(0, d/2)$. We also denote by P, Q the preimages of these points in B_O. See Figure 9.4.

Figure 9.2: Left: the picture viewed in normal polar coordinates at O. Right: the picture viewed in normal polar coordinates at Q.

Now place B_O in polar coordinate centered at Q:

$$
h(x) - h(Q) = \int_0^r \frac{\partial}{\partial r} h(r,\theta) dr = r \int_0^1 \frac{\partial}{\partial \rho} h(rt,\theta) dt
$$

Integrate on $B_O \ni x$ w.r.t to the measure dE_Q given by the normal polar coordinates at Q:

$$\begin{aligned}
\int_{B_O} |h(x)-\varphi(Q)| dE_Q &\le \int_{\theta\in\mathbb{S}^{n-1}} \int_{r=0}^{\rho(\theta)} r^{n-1} r \int_0^1 |\frac{\partial}{\partial\rho} h(rt,\theta)| dt dr d\theta \\
(u := rt, \rho(\theta)\le d) \qquad &\le \int_{\theta\in\mathbb{S}^{n-1}} \int_{t=0}^1 \int_{u=0}^{td} t^{-n-1} u^n \left|\frac{\partial}{\partial\rho} h(u,\theta)\right| dt du d\theta \\
&= \int_{t=0}^1 t^{-n-1} \left(\int_{u=0}^{td} \int_{\theta\in\mathbb{S}^{n-1}} \left|\frac{\partial}{\partial\rho} h(u,\theta)\right| u . dE_Q \right) dt \\
(\text{Holder w.r.t } dE_Q) \qquad &\le \int_{t=0}^1 t^{-n-1} \left(\int_{u=0}^{td} \int_{\theta\in\mathbb{S}^{n-1}} \left|\frac{\partial}{\partial\rho} h(u,\theta)\right|^q dE_Q \right)^{1/q} \left(\int_0^{td} \omega_{n-1} u^{q'} u^{n-1} du \right)^{1/q'} dt \\
(t\le 1) \qquad &\le \int_{t=0}^1 t^{-n-1} \left(\frac{1}{q'+n} (td)^{q'+n} \right)^{1/q'} \left(\int_{u=0}^{d} \int_{\theta\in\mathbb{S}^{n-1}} |\nabla\varphi|^q dE_Q \right)^{1/q} dt \\
&= C_1(q,n) d^{1+\frac{n}{q'}} \left(\int_{B(Q,d)} |\nabla\varphi|^q dE_Q \right)^{1/q}
\end{aligned} \tag{9.11}$$

Now using the fact that $\frac{1}{A} dV \le dE_Q \le A dV$ since the curvature is bounded, one has

$$\int_{B(O,d/2)} |\varphi(x)-\varphi(Q)| dV \le C_2(q,n) d^{1+\frac{n}{q'}} \|\nabla\varphi\|_{L^q}$$

Taking sum with the same computation for P, one has

$$|\varphi(P)-\varphi(Q)| \operatorname{vol}_g(B(O,d/2)) \le 2C_2(q,n) d^{1+\frac{n}{q'}} \|\nabla\varphi\|_{L^q}$$

since $\operatorname{vol}_g(B(O,d/2)) \ge A^{-1}\omega_{n-1} d^n$, one has

$$|\varphi(P)-\varphi(Q)| \le C_3(q,n) \|\nabla\varphi\|_{L^q} d^{1-n/q}$$

The conclusion follows since $1-\frac{n}{q} \ge \alpha$. □

For the first part of Theorem 111, it suffices to prove the case $k=l+1$, that is, there exists a constant $C_1, C_2 > 0$ such that $\|u\|_{L^p} \le C_1 \|\nabla u\|_{L^q} + C_2 \|u\|_{L^q}$ for $u \in W^{1,q}(M)$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.

The proof by [Aub98] tries to optimise the constant C_1, in an attempt to find the best inequality [Aub98, page 50]. We will follow their arguments, as the extra effort is not much. We will prove that

Proposition 115. *Given $p, q \in \mathbb{R}_{>0}$ such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{n} > 0$, for any $\epsilon > 0$, there exists $A_q(\epsilon)$ such that*

$$\|u\|_p \le (K(n,q)+\epsilon) \|\nabla u\|_{L^q} + A_q(\epsilon) \|u\|_{L^q}$$

The appearance of the constant $K(n,q)$, given by

$$K(n,q) := \begin{cases} \frac{q-1}{n(q-1)} \left[\frac{n-q}{n(q-1)}\right]^{1/q} \left[\frac{\Gamma(n+1)}{\Gamma(n/q)\Gamma(n+1-n/q)\omega_{n-1}}\right]^{1/n}, & \text{if } q > 1 \\ \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n}, & \text{if } q = 1 \end{cases}$$

is due to the following local result.

Theorem 116 (Aubin)**.** *Given $1 \le q < n$ and $u \in W^{1,q}(\mathbb{R}^n)$, with $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, one has*

$$\|u\|_{L^p} \le K(n,q)\|\nabla u\|_{L^q}.$$

In fact, $K(n,q)$ this the norm of the imbedding $W^{1,q}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$.

We will accept the local result and use the Covering Lemma 110 to prove Proposition 115, which implies Theorem 111.

Proof of Proposition 115. Note that given any smooth function f supported in a small geodesic ball $B(q,\delta)$, by applying theorem 116 to the f, viewed in the chart (that is, $f \circ \exp_q$) and use the fact that $C(\delta)^{-1}\|\nabla(f\circ\exp_q)\|_{L^q(dE)} \le \|\nabla f\|_{L^q(dV)} \le C(\delta)\|\nabla(f\circ\exp_q)\|_{L^q(dE)}$ (see remark 42), one has

$$\|f\|_{L^p} \le K_\delta(n,q)\|\nabla f\|_{L^q}$$

where $K_\delta(n,q)$ converges to $K(n,q)$ as $\delta \to 0$.

It suffice to cover M by geodesic ball $B(Q_i,\delta)$ such that there exists a partition of unity subordinated to $B(Q_i,\delta)$ such that $\|\nabla(h_i^{1/q})\| \le H = \text{const}$. In fact for $\varphi \in W^{1,q}(M)$, one has

$$\begin{aligned} \|\varphi\|_p^q &= \left(\int_M |\varphi|^p\right)^{q/p} = \left(\int_M \left(\sum_i |\varphi|^q h_i\right)^{p/q}\right)^{q/p} \\ \text{(since } p \ge q) \qquad &\le \sum_i \left(\int_M (|\varphi|^q h_i)^{p/q}\right)^{q/p} = \sum_i \left\|\varphi h_i^{1/q}\right\|_p^q \\ &\le K_\delta^q(n,q)\sum_i \left\|h_i^{1/q}\nabla\varphi + \varphi\nabla h_i^{1/q}\right\|_q^q \end{aligned}$$

Using the fact that there are at most $k(\delta)$ balls overlapping at a point and that $(a+b)^q = a^q\left(1+\frac{b}{a}\right)^q \le a^q(1 + 2^q\frac{b}{a} + 2^q(\frac{b}{a})^q) \le a^q + 2^q b a^{q-1} + 2^q b^q$, one has

$$\begin{aligned} \|\varphi\|_p^q &\le K_\delta^q(n,q)\left(\|\nabla\varphi\|_q^q + 2^q k(\delta)H^{q-1}\int_M |\varphi|^{q-1}|\nabla\varphi| + 2^q k(\delta)H^q\|\varphi\|_q^q\right) \\ &\le K_\delta^q(n,q)\left[\|\nabla\varphi\|_q^q + 2^q k(\delta)H^{q-1}\|\nabla\varphi\|_q\|\varphi\|_q^{q-1} + 2^q k(\delta)H^q\|\varphi\|_q^q\right] \end{aligned}$$

It is elementary to see that this implies $\|\varphi\|_p^q \le (1+\epsilon)^q K^q(n,q)\left[(1+\epsilon)\|\nabla\varphi\|_q^q + A(\epsilon)\|\varphi\|_q^q\right]$, from which the conclusion follows.

For the existence of such h_i, one cover M by balls $B(Q_i, \delta)$ using Lemma 110. Denote by $\varphi_i : B(Q_i, \delta) \longrightarrow B(0, \delta)$ the inverse of exponential maps and let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the smooth function, choose u to be a bell curve with maximal value 1 at 0, supported in $B(0, \delta)$ and $u \leq \frac{1}{2}$ in $B(0, \delta/2)$ and pose $u_i = u \circ \varphi_i$. Then

$$\|\nabla u_i\|_{g_M} \leq C_1(g_M, \delta)\|\nabla u\|_E = C_2(g_M, \delta)$$

Pose $h_i = \frac{u_i^m}{\sum u_j^m}$ with $m > q$ then

$$\begin{aligned}
\left|\nabla(h_i^{1/q})\right| &= \left|\frac{m}{q}\frac{u_i^{\frac{m}{q}-1}\nabla u_i}{(\sum u_j^m)^{1/q}} + u_i^{m/q}\left(\frac{-1}{q}\right)\frac{\sum \nabla(u_j^m)}{(\sum u_j^m)^{1+\frac{1}{q}}}\right| \\
&\leq \frac{m}{q.2^{-m/q}}|\nabla u_i| + \frac{1}{q}\sum m\frac{|\nabla u_j|}{(2^{-m})^{1+\frac{1}{q}}} \\
&\leq \left(\frac{m}{q}2^{m/q} + \frac{m}{q}2^{m(1+\frac{1}{q})}k(\delta)\right)C_2(g_M, \delta) = \text{const}
\end{aligned}$$

where $k(\delta)$, as in Lemma 110, is the upper bound of number of balls overlapping at the point in question. $\square$

9.5 Kondrachov's theorem

The generalised version of Kondrachov's theorem is much easier to prove

Theorem 117 (Kondrachov)**.** *Theorem 99 holds when one replaces Ω by a compact Riemannian manifolds of dimension n.*

Proof. Cover M by finitely many small geodesic ball $B(Q_i, \delta)$ subordinating a partition of unity $\sum_{i=1}^N \chi_i = 1$, then if a sequence $\{u_n\}_n \subset W^{k,q}$ is bounded then $\{\chi_i u_n\}_n$ is also bounded in $W^{k,q}$. The conclusion follows using Remark 42 and the Euclidean version of Kondrachov's theorem. $\square$

9.6 Solving $\Delta u = f$ on a Riemannian manifold.

With Kondrachov's theorem 117, one can uses the familiar "subsequence extracting" technique to find a minimiser of the quadratic functional $\psi \mapsto \frac{1}{2}\int_M \|\nabla\psi\|^2 dV$ in a suitable subspace of $W^{1,2}(M)$ (method of Lagrange multiplier), one can prove the following results.

Theorem 118 (Spectrum of Δ)**.** *Let M^n be a compact Riemannian manifold then*

1. *The eigenvalues of $\Delta - \nabla^\nu\nabla_\nu$ are ≥ 0.*

2. *The eigenfunctions of* $\delta_0 = 0$ *are constant functions.*

3. *The eigenvalue* λ_1 *is the minimum value of the functional*

$$\psi \mapsto \frac{1}{2}\int_M \|\nabla\psi\|^2 dV$$

on the subspace $\{\psi \in W^{1,2}(M) : \ \|\psi\|_2 = 1, \int \psi dV = 0\}$. *Moreover, first eigenfunctions are smooth.*

Theorem 119. *Given* M^n *be a compact Riemannian manifold, consider the Laplace equation on* M*:*

$$\Delta u = f \tag{9.12}$$

where $f \in L^2(M)$*, then:*

1. *There exists* $u \in W^{1,2}(M)$ *satisfying* (9.12) *in the weak sense if and only if* $\int_M f dV = 0$

2. u *is unique up to an additive constant.*

3. *If* $f \in C^{r,\alpha}$ *then* $u \in C^{r+2,\alpha}$.

Chapter 10

Parametrix and Green's function of Laplacian operator on Riemannian manifolds

Recall that in the Euclidean space $\mathbb{R}^n$, one obtains a representation of the solution u of equation $\Delta u = f$ by

- first solving for an explicit radial solution of $\Delta G = \delta_0$. In particular, $G = [(n-2)\omega_{n-1}]^{-1} r^{2-n}$ if $n > 2$ and $G = -(2\pi)^{-1}\log(r)$ if $n = 2$
- then tensoring G by f, one has the solution $u = G * f$ of $\Delta u = f$

To generalise this argument for Riemannian manifolds, there are a few points that have to be modified:

1. Since it does not make sense to add/substract points of a manifold, one will need to find different fundamental solutions for different points, so instead of fundamental solution, we will find the Green's function $G = G(p,q)(p, q \in M)$. The convolution will be replace by the following operation on functions X, Y defined on $(M \times M) \setminus \Delta_M$ where Δ_M denotes the diagonal:
$$(X * Y)(p,q) = \int_M X(p,r)Y(r,q)dV(r)$$
.
2. The distance function $q \mapsto d(p,q)$ is only smooth near p, outside of the cut-locus, the best one can says is that the function is Lipschitz. Since cut-loci are almost impossible to calculate or visualise (the cut-locus of an ellipsoid is still a conjecture, according to [Ber03]), one will cut-off the Euclidean solution, try to solve the equation near p and later add a correcting term. This inspires the definition of parametrix.

3. Another reason that we have to approximate the exact solution by parametrix, that also explain the iteration in Theorem 122, is that the expression of Laplacian, even in the geodesic polar coordinate and even near the origin, involves the metric, hence the Euclidean fundamental solution is not yet a solution even near the origin.

Remark 45. *To give a simplified analogy of what we will be doing, let us prove the existence of "Green's function" on Riemann surfaces (with boundary, so that we do not have to deal with the volume). The "Laplace equation" is*

$$-2i\partial\bar{\partial}g = \delta_0 \tag{10.1}$$

where the LHS is a 2-form and the RHS is a generalised 2-form in the sense of current. Contrary to the previous point 3, one knows the exact local solution of (10.1)*, namely* $z \mapsto -(2\pi)^{-1}\log(|z|)$*. Therefore, the argument will be simplified as:*

- *Given a holomorphic chart of a point* $0 \in M$*, pose* $h(z) := -(2\pi)^{-1}\log(|z|)\chi(|z|)$ *where* χ *is a cut-off function that is 1 on a neighborhood of* 0
- *The 2-form* $\alpha = -2i\partial\bar{\partial}h$ *is well-defined everywhere except* 0*, and vanishes on a neighborhood of* 0*. Denote by* α^{naiv} *its extension to* M*.*
- *Recall the fact that every smooth 2-form on a compact, connected, Riemann surface with boundary can be writen as* $\alpha^{\text{naiv}} = -2i\partial\bar{\partial}\varphi$*, pose* $g = h - \varphi$*.*

For Riemann surface without boundary, the equation is $-2i\partial\bar{\partial}g = \delta_0 - 2i\int_M \partial\bar{\partial}g$ *and the fact to evoke is that any smooth 2-form* α *with* $\int_M \alpha = 0$ *is of form* $\alpha = -2i\partial\bar{\partial}\varphi$

We will suppose that M^n is a Riemannian manifold with injectivity radius $\delta_0 > 0$, and of bounded curvature. Compact manifolds, for example, fall in this category.

10.1 Parametrix and the Green's formula

Definition 19. *A* ***Green's function*** $G(p,q)$ *of a compact Riemannian manifold is a function defined on* $(M \times M) \setminus \Delta_M$ *such that*

1. $\Delta_q^{\text{dist}} G(p,q) = \delta_p(q)$ *if* M *has boundary.*
2. $\Delta_q^{\text{dist}} G(p,q) = \delta_p(q) - V^{-1}$

where Δ_q^{dist} *concerns the distribution derivatives and* V *is the volume of* M*.*

Let $p, q \in M$ *be distinct points, the* ***parametrix*** H *is defined by*

$$H(p,q) = \begin{cases} [(n-2)\omega_{n-1}]^{-1} r^{2-n}\chi(r), & \text{if } n > 2 \\ -(2\pi)^{-1}\chi(r)\log r, & \text{if } n = 2 \end{cases}$$

where $r = d(p,q)$*,* $\chi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ *is smooth,* $\chi = 1$ *in a neighborhood of* 0 *and* $\chi(t) = 0$ *if* $t > \delta_0$*.*

Recall that in the geodesic polar coordinates, i.e. the polar coordinates on the tangent T_pM at $p \in M$, identified with a neighborhood of $p \in M$, the metric g is given by

$$g : ds^2 = dr^2 + r^2 g_{\theta_i\theta_j}(r,\theta)d\theta^i d\theta^j$$

and one denotes $|g_\theta| := \det(g_{\theta_i\theta_j})$, therefore $|g| = \det(g_{ij}) = r^{2(n-1)}|g_\theta|$

Lemma 120. *If a function $\varphi \in C^2$ defined locally around $p \in M$ and φ is radial, i.e. $\varphi = f(r)$ in a small geodesic ball $B(p,\delta)$ then*

$$-\Delta\varphi = f'' + \frac{n-1}{r}f' + f'\partial_r \log\sqrt{|g_\theta|}$$

Proof. One has

$$\begin{aligned}\Delta\varphi &= -\operatorname{Tr}\left(\nabla_i(g^{kj}\partial_j\varphi\ e_k)\right)_{i,k} = -\partial_i(g^{ij}\partial_j\varphi) - g^{kj}\partial_j\varphi\Gamma^i_{ik}\\ &= -|g|^{-1/2}\partial_i(g^{ij}|g|^{1/2}\partial_j\varphi)\end{aligned}$$

since $\Gamma^i_{ik} = \partial_k \log\sqrt{|g|} = \frac{\partial_i|g|}{2|g|}$. One concludes by substituting $|g| = r^{2n-2}|g_\theta|$ and noticing that $g^{r\theta_i} = g^{\theta_i\theta_j} = 0$ $(i \neq j)$. □

Remark 46. 1. *The Laplacian of the metric g, viewed in polar geodesic coordinates centered at p, i.e. in the tangent space T_pM is not the Euclidean Laplacian of T_pM, however the difference if $O(r)$ since $\partial_r \log\sqrt{|g_\theta|} \leq Ar$ where the bound A is given by Ricci curvature, see the Volume comparison theorem.*

2. *Applied the formula for $q \mapsto H(p,q)$, one has*

$$\Delta_q^{\text{naiv}} H(p,q) = [(n-2)\omega_{n-1}]^{-1} r^{1-n}\left((n-3)\chi' - r\chi'' + ((n-2)\chi - r\chi')\partial_r \log\sqrt{|g_\theta|}\right) \tag{10.2}$$

therefore $\Delta_q^{\text{naiv}} H(p,q) \leq Br^{2-n}$ where B does not depend on p.

3. *Unlike the case of Remark 45 where we know the exact fundamental solution and the form α^{naiv} has no singularity, there is no reason for that this holds true for $\Delta_q^{\text{naiv}} H(p,q)$. However, we proved that the order of singularity at $q = p$ can be controlled.*

Proposition 121 (Green's formula)**.** *For any function $\psi \in C^2(M)$, one has*

$$\psi(p) = \int_M H(p,q)\Delta\psi(q)dV(q) - \int_M \Delta_q^{\text{naiv}} H(p,q)\psi(q)dV(q) \tag{10.3}$$

where $\Delta_q^{\text{naiv}} H(p,q)$ denotes the pointwise derivative of $H(p,q)$, not the distribution derivative.

Remark 47. 1. *In other words, the theorem says that $\Delta_q^{\text{dist}} H(p,q) = \Delta_q^{\text{naiv}} H(p,q) + \delta_p(q)$ where Δ_q^{dist} is the distribution derivative. In particular, if there is no concern about regularity of the distance function $d(p,q)$ (as in the Euclidean case), allowing us to take the cut-off function $\chi = 1$ in the definition of parametrix, then $\Delta_q^{\text{naiv}} H(p,q) = 0$ and $\Delta_q^{\text{dist}} H(p,q) = \delta_p(q)$ which is not a surprise since $H(p,q)$ is also the Green's function.*

2. *Taking $\psi = 1$, one has*
$$\int_M \Delta_q^{\text{naiv}} H(p,q) = -1$$
3. *Multiplying* (10.3) *by $\phi(p)$ and integrate over M, one has*
$$\int_M \phi(q)\psi(q)dV(q) = \int_M \left(\int_M H(p,q)\phi(p)dV(p)\right)\Delta\psi(q)dV(q) - \int_M \left(\int_M \Delta_q^{\text{naiv}} H(p,q)\phi(p)\right)\psi(q)dV(q)$$
hence in distribution sense
$$\phi(q) = \Delta_q \int_M H(p,q)\phi(p)dV(p) - \int_M \Delta_q^{\text{naiv}} H(p,q)\phi(p) \tag{10.4}$$
The equation (10.4) *is called the transposition of equation* (10.3) *and what we have just done is a rigourous proof of the following heuristic justification of* (10.4)*: "Take the derivative Δ_q inside the integral, then use $\int_M \delta_p(q)\phi(p)dV(p)$ "=" $\phi(q)$".*

Proof. The intuition is clear:

- since one only modifies the fundamental solution at points q far from p, one only needs to recompense by $\Delta_q^{\text{naiv}} H(p,q)$
- there may be trouble near p caused by the difference between the Euclidean Laplacian and the metric Laplacian, however as explained by Remark 46, this difference is $O(r)$ as $r \to 0$.

For a rigorous proof, one calculates $\int_M H(p,q)\Delta\psi(q)dV(q)$ by decomposing M to $B(p,\epsilon)$ and $M \setminus B(p,\epsilon)$ with $0 < \epsilon < \delta_0$ tending to 0 eventually, then
$$\begin{aligned}\int_{M\setminus B(p,\epsilon)} H(p,q)\Delta\psi(q)dV(q) &= \int_{M\setminus B(p,\epsilon)} \left(\Delta_q^{\text{naiv}} H(p,q)\psi(q) + d(\psi \wedge *dH - H \wedge *d\psi)\right)dV(q)\\ &= \int_{M\setminus B(p,\epsilon)} \Delta_q^{\text{naiv}} H(p,q)\psi(q)dV(q) + \int_{\partial B(p,\epsilon)} (\psi \wedge *dH - H \wedge *d\psi)dV(q)\end{aligned}$$
by Stokes' theorem, where $*$ denotes the Hodge star. Therefore
$$\int_M H(p,q)\Delta\psi(q)dV(q) = \int_M \Delta_q^{\text{naiv}} H(p,q)\psi(q)dV(q) + I_1 + I_2$$
where $I_1 = \lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}(\psi \wedge *dH - H \wedge *d\psi)$ and $I_2 = \lim_{\epsilon\to 0}\int_{B(p,\epsilon)} H(p,q)\Delta\psi(q)dV(q)$.

Now $I_2 = \psi(p)$ since $(\frac{\sin(b\epsilon)}{b\epsilon})^{n-1} \le dV/dE \le (\frac{\sin(\alpha\epsilon)}{\alpha\epsilon})^{n-1}$ in $B(p,\epsilon)$ by Volume comparison theorem where b^2 is an upper bound of sectional curvature and $(n-1)\alpha^2$ is a lower bound of Ricci curvature ($\alpha \in \mathbb{C}$), and since $\Delta\psi(q) - \Delta_E\psi(q) = O(\varepsilon)$ in $B(p,\epsilon)$ where Δ_E is the Euclidean Laplacian.

For I_1, with ϵ small enough such that $\chi = 1$, one has $|H \wedge *d\psi| \le \text{const}\,\epsilon^{2-n}(*d\psi)$. By straightforward computation:
$$\begin{aligned}dH &= -\omega_{n-1}^{-1} r^{1-n} dr, \quad dV = r^{n-1}\sqrt{|g_\theta|}dr \wedge d\theta^1 \wedge \cdots \wedge d\theta^{n-1}\\ *dH &= -\omega_{n-1} r^{2n-2}\sqrt{|g_\theta|}d\theta^1 \wedge \cdots \wedge d\theta^{n-1}\end{aligned}$$

hence $\int_{\partial B(p,\epsilon)} H \wedge *d\psi = O(\epsilon)$ and $\int_{\partial B(p,\epsilon)} \psi \wedge *dH = O(\epsilon^{2n-2})$. Therefore $I_1 = 0$ and the conclusion follows. □

10.2 Existence of Green's function on compact Riemannian manifolds

Our goal is to prove the following theorem

Theorem 122 (Existence of Green's function)**.** *Let M^n be a compact Riemannian manifold without boundary, there exists a Green's function $G(p,q)$ of the Laplacian such that*

1. Green's function. *For all $\varphi \in C^2(M)$,*

$$\varphi(p) = V^{-1} \int_M \varphi(q) dV(q) + \int_M G(p,q) \Delta\varphi(q) dV(q) \tag{10.5}$$

2. Smooth. $G \in C^\infty((M \times M) \setminus \Delta_M)$.

3. Radial estimates. *There exists a constant k such that*

$$|G(p,q)| \leq \begin{cases} k(1 + |\log r|), & \text{if } n = 2 \\ kr^{2-n}, & \text{if } n > 2 \end{cases} \tag{10.6}$$

for $r = d(p,q)$. Moreover, one has the derivative estimates:

$$|\nabla_q G(p,q)| \leq kr^{1-n}, \qquad \left|\nabla_q^2 G(p,q)\right| \leq kr^{-n}, \tag{10.7}$$

4. *G is bounded below. Since G is defined upto a constant, one can choose the constant so that $G > 0$.*

5. Constant integral. *The integral $\int_M G(p,q) dV(p)$ is constant in q. Since G is defined upto a constant, one can choose the constant so that $\int_M G(p,q) dV(p) = 0$.*

6. Symmetric. *$G(p,q) = G(q,p)$ for $p \neq q$ in M.*

For a better notation, let us replace $\Delta_p U(p,q)$ by $\Delta_2 U(p,q)$. Recall that we already know how to solve the equation $\Delta u = f$ for $f \in L^2(M)$, this means we can solve $\Delta_2 U(p,q) = f_p(q)$ for double-integrable functions f_p, or briefly we can *solve* L^2 functions. Now, define

$$(X * Y)(p,q) := \int_M X(p,r) Y(r,q) dV(r)$$

if the integration is possible and if it commutes with derivation, one has

$$\Delta_2(F_1 * H) = F_1 * \Delta_2^{\text{dist}} H = F_1 + F_1 * \Delta_2^{\text{naiv}} H$$

So if one can *solve* $F_1 * \Delta^{\text{naiv}} H$, then one can *solve* F_1, i.e. if $\Delta_2 E_2 = F_1 * \Delta_2^{\text{naiv}} H$ then take $E_1 := F_1 * H - E_2$, one has $\Delta_2 E_1 = F_1$.

Now in order to prove that $\delta_\Delta - V^{-1}$ can be solved, it remains to check that

$$\delta_\Delta * (\Delta_2^{\text{naiv}} H)^{*k} \in L^2(M) \quad \text{for } k \gg 1. \tag{10.8}$$

This is the content of the following lemma.

Lemma 123. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $X, Y : (\Omega \times \Omega) \setminus \Delta_\Omega \longrightarrow \mathbb{R}$ be continuous functions such that*

$$|X(p,q)| \leq \text{const}\, d(p,q)^{\alpha-n}, \qquad |Y(p,q)| \leq \text{const}\, d(p,q)^{\beta-n}, \qquad \alpha, \beta \in (0,n)$$

then

$$Z(p,q) := \int_\Omega X(p,r) Y(r,q) dV(r)$$

is continuous in $(\Omega \times \Omega) \setminus \Delta_\Omega$ and

$$|Z(p,q)| \leq \begin{cases} \text{const}\, d(p,q)^{\alpha+\beta-n}, & \text{if } \alpha+\beta < n \\ \text{const}(1 + |\log d(p,q)|), & \text{if } \alpha+\beta = n \\ \text{const}, & \text{if } \alpha+\beta > n \end{cases}$$

In the case $\alpha + \beta > n$, Z admits a continuous extension to $\Omega \times \Omega$. The result also holds for compact Riemannian manifolds.

Proof. It suffices to consider p, q closed to each other. Let $d(p,q) = 2\rho$. Decompose $\Omega = (\Omega \cap B(p,\rho)) \cup (\omega \setminus B(q, 3\rho)) \cup \Omega \cap (B(q,3\rho) \setminus B(p,\rho))$, then

$$\left| \int_{\Omega \cap B(p,\rho)} X(p,r) Y(r,q) dV(r) \right| \leq C \rho^{\alpha+\beta-n}$$

$$\left| \int_{\Omega \cap B(q,3\rho) \setminus B(p,\rho)} X(p,r) Y(r,q) dV(r) \right| \leq C \rho^{\alpha+\beta-n}$$

$$\left| \int_{\Omega \setminus B(q,3\rho)} X(p,r) Y(r,q) dV(r) \right| \leq C \int_\rho^D \frac{dr}{r^{n-\alpha-\beta-1}}$$

where D is the diameter of Ω. For compact Riemannian manifold, take $\rho \ll \delta_0$, the injectivity radius and use Comparison theorem, one has the same estimates. □

Back to the proof of Theorem 122, one can see that it suffices to choose $k > \frac{n}{2}$ in (10.8). The rigorous proof is given below.

Proof of Theorem ref:thm:existence-green. Carefully do the algebraic part of the above argument, one poses

$$G(p,q) = H(p,q) + \sum_{i=1}^{k-1} (-\Delta_2^{\text{naiv}} H)^{*i} * H + F_k(p,q)$$

where $F_k(p,q)$ satisfies
$$\Delta_2 F_k(p,q) = (-\Delta_2^{\text{naiv}} H)^{*k} - V^{-1}$$
This is possible if one chooses $k > n/2$ since by repeated application of Lemma 123, $(-\Delta_2^{\text{naiv}} H)^{*k}$ is continuous. By regularity result of equation $\Delta u = f$, the function $q \mapsto F_k(p,q)$ is in $C^2(M \setminus \{p\})$. Each function $F_k(p,\cdot)$ is uniquely defined up to a constant, choose the constant such that $\int_M G(p,q)dV(q) = 0$, then the function $p \mapsto \int_M F_k(p,q)dV(q)$ is continuous. The condition 1) of the Theorem can be verified without difficulty. Moreover, since $\Delta_2 G(p,q) = 0$ if $q \neq p$, the function $q \mapsto G(p,q)$ is C^∞.

We will prove such $G(p,q)$ satisfies the statements 2-6, starting from a weaker form 2-) of 2), that is we will prove that $p \mapsto G(p,q)$ is continuous, then using this, we will prove 3-6, and eventually come back to prove 2 completely.

For 2-) we will use the following fact:

Fact. If $\Delta u = f$ and $f \in C^0(M)$ (hence $u \in C^2(M)$ and $\int_M u = 0$, then one has $\sup |u| \leq C \sup |f|$ where $C > 0$ is a constant.

Denote $\Gamma_i := (-\Delta_2^{\text{naiv}} H)^{*i}$ and apply the result for $u = F(p,\cdot) - V^{-1}\int_M F(p,q)dV(q)$ and $f = \Gamma_k(p,\cdot)$, one has
$$\sup \left| F(p,\cdot) - F(r,\cdot) - V^{-1}\int_M (F(p,\cdot) - F(r,\cdot)) \right| \leq C \sup_q |\Gamma_k(p,q) - \Gamma_k(r,q)|$$
Then the continuity of $p \mapsto F(p,\cdot)$ under C^0 topology is given by

- $p \mapsto \int_M F(p,\cdot)$ is continuous by the previous choice of constant.
- The uniform continuity of Γ_k on $M \times M$, which is the result of its continuity and the compactness of $M \times M$.

Hence $p \mapsto G(p,q)$ is continuous on $M \setminus \{q\}$ for all $q \in M$.

For 3), fix $p \in M$ and let $r = d(p,q)$ be small, then $H(p,q) = O(r^{2-n}$, $(\Gamma_i * H)(p,q) = O(r^{2i+2-n})$ by Lemma 123 and $F(p,q) = O(1)$ if $n > 2$. Hence $G(p,q) = O(r^{2-n})$, where here the constant in $O(r^{2-n})$, if checked carefully, does not depend on p. The case $n = 2$ can be treated similarly. For the derivative estimates, note that $\nabla_q G(p,q) = \nabla_q H(p,q) + \sum_{i=1}^{k-1}(\Gamma_i * \nabla_2 H)(p,q) + \nabla_q F(p,q)$ and $\nabla_q^2 G(p,q) = \nabla_q^2 H(p,q) + \sum_{i=1}^{k-1}(\Gamma_i * \nabla_2^2 H)(p,q) + \nabla_q^2 F(p,q)$ where the commutative of derivation and integration can be justified by Lebesgue's Dominated convergence. In both case, the dominant terms as $q \to p$ are $\nabla_q H(p,q)$ and $\nabla_q^2 H(p,q)$ respectively, which is $O(r^{1-n})$ and $O(r^{-n})$ where the constants in big-O do not depend on p.

For 4), note that $H(p,q)$ is the dominant term of $G(p,q)$ as $q \to p$ and $H(p,q) > 0$, one see that $G(p,q) > 0$ in a neighborhood of Δ_M. By the compactness of M and the continuity of G outside of Δ_M, one sees that G is bounded below.

To prove 5), take to transposition of (10.5), i.e. multiply by $\psi(p)$ and integrate, as in Remark 47, one obtains
$$\Delta_q \int_M G(p,q)\psi(p)dV(p) = \psi(q) - V^{-1}\int_M \psi(p)dV(p) \tag{10.9}$$

Substitute $\psi = 1$, one sees that $q \mapsto \int_M G(p,q)dV(p)$ is harmonic on M, hence is constant by compactness of M.

We will now prove 6). It follows from (10.5) that

$$\Delta_q \int_M G(p,q)\psi(q)dV(q) = \Delta_q \psi(q) \tag{10.10}$$

Also, from the transposition (10.9), replace ψ by $\Delta\psi$, one has

$$\Delta_q \int_M G(p,q)\Delta\psi(p)dV(p) = \Delta_q \psi(q)$$

Swap p and q and subtract to (10.10), one has

$$\Delta_p \int_M \left(G(p,q) - G(q,p)\right)\Delta\psi(q)dV(q) = 0$$

Hence $\int_M \left(G(p,q) - G(q,p)\right)\Delta\psi(q) = C$ const. Integrate by $p \in M$ and use the fact that we chose $\int_M G(q,p)dV(p) = 0$, one has $C = 0$, meaning that $\Delta_q \left(G(p,q) - G(q,p)\right) = C(p)$, being independent of q. By swapping p, q, one has $C(p) = -C(q)$ for all $p \neq q$. Since M contains more than 3 points, these constants are 0. Hence $G(p,q) = G(q,p)$.

Now coming back to 2), since $G(p,q) = G(q,p)$, we see that $p \mapsto G(p,q)$ is C^∞ for all $q \in M$. It remains to prove that $p \mapsto \nabla_q^h G(p,q)$ is continuous on $M \setminus \{q\}$, then Schwarz's lemma applies. For that, one may try the following argument:

$$\Delta_p \nabla_q^h G(p,q) = \nabla_q^h \Delta_p G(p,q) = 0, \quad p \in M \setminus [q\}$$

hence $p \mapsto \nabla_q^h G(p,q)$ is C^∞. It is however difficult to justify the commutativity of derivations, which is equivalent to

$$\int_M \nabla_q^h G(p,q)\Delta\varphi(p)dV(p) = \nabla_q^h \int_M G(p,q)\Delta\varphi(p)dV(p), \tag{10.11}$$

that is the ability to derive in the integral sign. A justification for this can be done in the case $h \leq 2$ using estimates of 3).

A simpler way is to note that it suffices to prove the continuity of $p \mapsto \nabla_q^h G(p,q)$ for p in a small open set V with $\bar{V}$ not containing q. Then claim that $\Delta_p \nabla_q^h G(p,q) = \nabla_q \Delta_p G(p,q) = 0$ as distributions on V, which is equivalent to (10.11) for all test functions φ with $\operatorname{supp}\varphi \in V$. Then Dominated convergence applies since $|\nabla_q^{h+1} G(p,q)| \leq C d(q, \bar{V})^{1-n-h}$ hence is bounded. $\square$

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