Some results in one complex variable

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Theorem 1 (Runge)**.** *Let Ω be a domain of $\mathbb{C}$ and K a compact in Ω. Then the followings are equivalent*

1. K has no hole *in Ω, i.e. there is no connected component $\mathcal{C}$ of $\Omega \setminus K$ with $\bar{\mathcal{C}} \subset \Omega$.*

2. *Every holomorphic function in K can be $\|.\|_{K,\infty}$-approached by holomorphic functions in Ω.*

3. *For every $x \in \Omega \setminus K$, there exists a holomorphic function $f_x \in \mathcal{O}_\Omega$ such that $|f(x)| > \sup_K |f|$.*

Theorem 2 (Mittag-Leffler)**.** *Let (a_i) be a discrete sequence of points in Ω and f_i be meromorphic functions with pole only at a_i. Then there exists a meromorphic function f with poles only at (a_i) such that $f - f_i$ has removable singularity at a_i.*

Proof. Suppose that (f_i) are globally defined in Ω. Choose an exhaustive sequence $(\hat{K}_j = K_j)_j$ that increases slower than (a_i), i.e. $a_i \notin K_i$. By 1 for $K_i \subset \Omega$, there exist $g_i \in \mathcal{O}(\Omega)$ with $\|g_i - f_i\|_{K_i,\infty} < 2^{-i}$. Pose

$$f = \sum_i (f_i - g_i).$$

$\square$