Harmonic maps of Riemannian manifolds

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1 Harmonic maps

1.1 Variational approach: energy integral and tension field

Notation. Let M, M', M'' be Riemannian manifolds of dimension n, n' and n', n'' respectively. We will use $i, j, k, \dots, \alpha, \beta, \gamma, \dots, a, b, c$ for local coordinates of M, M', M''. Let $f : M \longrightarrow M', f' : M' \longrightarrow M''$ be a smooth maps, one denotes

$$f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}, \quad f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k f_k^\alpha$$

so that $\nabla h = h_i dx^i$ and $\nabla(\nabla h) = h_{ij} dx^i \otimes dx^j$ and $-\Delta h = \operatorname{Tr} \nabla(\nabla h) = g^{ij} h_{ij}$ for any smooth function h.

Definition 1. *The* ***energy desity*** *of f at $p \in m$ is defined by*

$$e(f)(p) = \frac{1}{2}\langle g, f^* g\rangle_p = \frac{1}{2} g^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$$

and the ***energy functional*** *of f is*

$$E(f) = \int_M e(f) dV = \frac{1}{2} \int_M g^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta} |\det(g_{ij})|^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$$

We recall that the inner product is between 2 tensors of type (p, q) $S = S_{j_1 \dots j_q}^{i_1 \dots i_p}, T = T_{l_1 \dots l_q}^{k_1 \dots k_p}$ is $\prod_{m,n} g_{i_m k_m} g^{j_n l_m} S_{j_1 \dots j_q}^{i_1 \dots i_p} T_{l_1 \dots l_q}^{k_1 \dots k_p}$

Remark 1. *The energy density is non-negative at every point. Hence $E(f) = 0$ if and only if $e(f) = 0$ at all points if and only if f is constant.*

Definition 2. *Let σ be a symmetric function of n variables and α be a symmetric (0,2) tensor field, one can define the* ***σ-energy desity*** *of α at $P \in M$ to be $\sigma(\beta_1, \dots, \beta_n)(P)$ where β_i are eigenvalues of the linear operator $(g^{ik}\alpha_{ij})_{k,j}$. The* ***$\sigma$-energy*** *of α is $I_\sigma(\alpha) := \int_M \sigma(\alpha) dV$*

Take $\alpha = f^* g'$, one calls $\sigma(\alpha)$ the ***σ-energy density*** *of f and $I_\sigma(\alpha)$ the σ-energy of f.*

Example 1. *For example, the energy functional $E(f)$ is $I_{\frac{\sigma_1}{2}}(f)$. $V(f) := I_{\sigma_n^{1/2}}(f)$ is called the* ***volume*** *of f.*

Lemma 1 (variation of the energy)**.** *Let $f_t : M \longrightarrow M'$ be a smooth family of smooth maps between Riemannian manifolds for $t \in (t_0, t_1)$. Then*

$$\frac{d}{dt} E(f_t) = -\int_M \left(-\Delta f_t^\gamma + g^{ij} \Gamma'^\gamma_{\alpha\beta} f_{t,i}^\alpha f_{t,j}^\beta \right) g'_{\gamma\nu} \frac{\partial f_t^\nu}{\partial t} dV, \qquad \forall t \in (t_0, t_1)$$

Proof. One has

$$\frac{dE}{dt}(f_t) = \frac{1}{2}\int \left[2g^{ij} f_i^\alpha \frac{\partial^2 f_t^\beta}{\partial x^j \partial t} g'_{\alpha\beta} + g^{ij} f_i^\alpha f_j^\beta \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} \frac{df_t^\nu}{dt} \right] dV(g)$$
$$= \frac{1}{2}\int \left[-\left(2g^{ij} f_i^\alpha g'_{\alpha\beta}\right)_j \frac{df_t^\beta}{dt} + g^{ij} f_i^\alpha f_j^\beta \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} \frac{df_t^\nu}{dt} \right] dV(g)$$

The first term is

$$-\left(2g^{ij} f_i^\alpha g'_{\alpha\beta}\right)_j = -2g^{ij} f_{ij}^\alpha \frac{df^\beta}{dt} g'_{\alpha\beta} - 2g^{ij} f_i^\alpha \frac{df^\beta}{dt} \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} f_j^\nu$$
$$= 2\Delta f^\alpha g'_{\alpha\beta} \frac{df_t^\beta}{dt} - 2g^{ij} f_i^\alpha f_j^\beta \frac{\partial g'_{\alpha\nu}}{\partial y^\beta} \frac{df_t^\nu}{dt}$$

It remains to check that

$$-2\frac{\partial g'_{\alpha\nu}}{\partial y^\beta} + \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} = -2\Gamma'^{\gamma}_{\alpha\beta} g'_{\gamma\nu}$$

when we are allowed to permute α, β, which is routine. □

Definition 3. *1. A* ***vector field along*** $f : M \longrightarrow M'$ *is a smooth application* $v : M \longrightarrow TM'$ *such that* $\pi \circ v = f$ *where* $\pi : TM' \longrightarrow M'$ *is the canonical projection. In other words, it is the association of each point* $P \in M$ *a tangent vector at* $f(P)$

2. *The* ***tension field*** *of* f *is the following vector field along* f *defined by*

$$\tau(f)^\gamma := -\Delta f^\gamma + g^{ij}\Gamma'^{\gamma}_{\alpha\beta} f_i^\alpha f_j^\beta$$

By the Lemma 1, $\tau(f)$ *is the unique vector field along* f *such that* $\frac{d}{dt}E(f_t) = -\int_M \langle \tau(f), \frac{df_t}{dt} \rangle$. *In particular, if* f_t *is the variation of* f *along a vector field* v *along* f, *i.e.* $f_t(P) = \exp_{f(P)}(tv(P))$ *then* $\nabla_v E(f) = -\langle \tau(f), v \rangle$ *along* f.

3. $f : M \longrightarrow M'$ *is called* ***harmonic*** *if* $\tau(f) = 0$, *or equivalently* f *is a critical point of* E.

In normal coordinates of M at P and M' at $f(P)$, the tension field of f is given by

$$\tau^\gamma(f)(P) = \sum_i \frac{\partial^2 f^\gamma}{\partial (x^i)^2}(P)$$

Remark 2. *1. If M' is flat, i.e. $R'_{\alpha\beta\gamma\delta} = 0$ then $\tau(f)^\gamma = -\Delta f^\gamma$ is linear in f. We refind the definition of harmonic function.*

2. Since $\tau(f)$ depends locally on f, isometries and covering maps are harmonic.

Proposition 1.1 (Holomorphicity implies harmonicity)**.** *Holomorphic maps between Kahler manifolds are harmonic.*

Proof. We recall that exponential function $\exp_P : T_P M \longrightarrow M'$ on a Kahler manifold M is holomorphic for any $P \in M$. In fact, let $v \in T_P M$ and $\delta v \in T_v(T_P M)$ be a tangent vector at v and denote abusively by J the complex structure of the complex vector space $T_P M$ and that of M, one needs to see that

$$D\exp_P(v).J\delta v = J(\exp_P(v))D\exp_P(v).\delta v \tag{1}$$

In fact, let Y_1, Y_2 be Jacobi fields along $U(t) = \exp_P(tv)$ the geodesics of M starting at P in direction v with $Y_1(0) = Y_2(0) = 0, \dot{Y}_1(0) = \delta v, \dot{Y}_2(0) = J\delta v$ then the LHS of (1) is $Y_2(1)$, and the RHS is $J(U(1))Y_1(1)$. Then one can see that $Y_2(t) - J(U(t))Y_1(t) = 0$ for every $t \in [0,1]$ since it is true at $t = 0$ and the derivative with respect to t vanishes as $\nabla_{\dot{U}} J = 0$.

Therefore, at a point P of a Kahler manifold M, there exist holomorphic coordinates $z^j = x^j + iy^j$ of M in a neighborhood of P such that $\{x_j, y_j : j = \overline{1, n/2}\}$ are normal coordinates centered in P. Using such coordinates for $P \in M$ and $f(P) \in M'$, one has $\Delta f^\gamma = 0$ since f^γ is holomorphic and $\Gamma'^\gamma_{\alpha\beta}(P) = 0$ by normality, it follows that $\tau(f) = 0$ at every point $P \in M$. $\square$

1.2 Formulation using connection on vector bundle

Setup and notation. Let E be a metric vector bundle over a Riemannian manifold M, i.e. each fiber of E is equiped with an inner product that we denote by $(g'_{\alpha\beta})$. The metric of M is denoted by (g_{ij}). Let n and m be the dimension of M of the fiber.

Covariant derivatives and exterior derivatives. We recall that a **covariant derivative** or a **connection** $\tilde{\nabla}$ of E is uniquely determined in local coordinates by an $m \times m$ matrix A of 1-forms, in other words, it is an 1-form on M with value in $Hom_M(E, E)$ which depends on the local frame of E (i.e. A is not a tensor with value in E). A is called the **connection form** of $\tilde{\nabla}$. Locally

$$\tilde{\nabla}_X(s^\alpha \tilde{e}_\alpha) = (\nabla_X s^\alpha)\tilde{e}_\alpha + A^\alpha_\beta(X)s^\beta \tilde{e}_\alpha.$$

When one prefers to work with forms rather than tensors with value in E, one uses an **exterior derivative**, a map $D : A^p(M, E) \longrightarrow A^{p+1}(M, E)$ which turns an p-form with value in E to an $p + 1$-form with value in E. Locally

$$D(s^\alpha \tilde{e}_\alpha) = (ds^\alpha)\tilde{e}_\alpha + A^\alpha_\beta \wedge s^\beta \tilde{e}_\alpha.$$

and

$$D^2(s^\alpha \tilde{e}_\alpha) = (dA + A \wedge A) \wedge s.$$

One notes $\Theta := dA + A \wedge A$, which is an $m \times m$ matrix of 2-forms of M. Unlike A, Θ, seen as an 2-form with value in $Hom_M(E, E)$ does not depend on the local frame of E, i.e. Θ transforms as a (0,2) tensor with value in E, called the **curvature form**.

The fibrewise metric structure of E and the metric tensor of M give rise to a pointwise inner product of (p, q) tensors of M with value in E, in particular a pointwise inner product $(s, s') \mapsto s \cdot s'$ from $A^p(M, E) \times A^p(M, E)$ to $C^\infty(M)$. Integrated over M, the pointwise inner product gives rise to a global inner product $\int_M \langle \cdot, \cdot \rangle$ of $A^p(M, E)$. One denotes by $\delta : A^{p+1}(M, E) \longrightarrow A^p(M, E)$ the adjoint operator of $D : A^p(M, E) \longrightarrow A^{p+1}(M, E)$ with respect to this inner product, i.e. $\int_M \langle Ds, s' \rangle_{A^{p+1}(M,E)} = \int_M \langle s, \delta s' \rangle_{A^p(M,E)}$ for all $s \in A^p(M, E), s' \in A^{p+1}(M, E)$.

Laplacian operator and harmonic forms. The **connection Laplacian** is defined as a endomorphism of $A^p(M, E)$ given by

$$\tilde{\Delta} = D\delta + \delta D$$

and a form $s \in A^p(M, E)$ is called **harmonic** if $\tilde{\Delta} s = 0$. Since the Laplacian operator represents the *Dirichlet integral*, i.e.

$$\int_M \langle Ds, Ds' \rangle + \int_M \langle \delta s, \delta s' \rangle = \int_M \langle \tilde{\Delta} s, s' \rangle,$$

one has $\tilde{\Delta} s = 0$ if and only if $Ds = \delta s = 0$.

Riemannian connected bundle. The metric vector bundle E over M is called a **Riemannian-connected bundle** if it is equipped with a connection $\tilde{\nabla}$ under which the metric g' of E is parallel, i.e. $\tilde{\nabla} g' = 0$, in other words, the matrix A in a orthonormal frame is anti-symmetric: $A + {}^{\mathrm{t}}A = 0$. Unless explicitly indicated, we always suppose that our metric vector bundle E is Riemannian-connected and the metric g' is parallel to the connection being used.

Example 2. *The case of our interest is when we have a smooth map* $f : M \longrightarrow M'$ *and* $E = f^*TM'$ *is a metric vector bundle over* M *under the metric* g' *induced from* M'. *Taking the connection* $\tilde{\nabla}$ *to be the Levi-Civita connection* ∇' *on* M', *meaning*

$$\tilde{\nabla}_X s = \nabla'_{f_* X} s,$$

for any vector field s *along* f, *one can see that* E *is a Riemannian-connected bundle over* M.

Lemma 2. *Let* E *be a Riemannian-connected bundle and* $s = s_i^\alpha dx^i \tilde{e}_\alpha \in A^1(M, E)$, *one has*

1. $\delta s = (\delta s)^\alpha \tilde{e}_\alpha \in A^0(M, E)$ *where*

$$(\delta s)^\alpha = -g^{ij} \left(\nabla_i s_j^\alpha + A_{\beta i}^\alpha s_j^\beta \right),$$

2. $\Delta s = (\Delta s)_i dx^i$ *where* $(\Delta s)_i$ *is an* $m \times m$ *matrix given by*

$$(\Delta s)_i = -\tilde{\nabla}^k \tilde{\nabla}_k s_i + {}^{\mathrm{t}}\!\left(\Theta_i^h - \mathrm{Ric}_i^h \right) s_h$$

   where:

   - *the indices* i, h, k *correspond to local coordinates of* M,
   - Θ_i^h *is the curvature form of* $\tilde{\nabla}$ *with its indices raised by the metric* g *of* M,
   - $\mathrm{Ric}_i^h = \mathrm{Ric}_i^h I_m$ *is the Ricci curvature tensor of* (M, g) *with indices raised by the metric* g, *multiplied by the identity* $m \times m$ *matrix,*
   - $\tilde{\nabla}^k = g^{hk} \tilde{\nabla}_h$.

3. *With* $s \cdot s'$ *denoting the pointwise inner product of* $A^1(M, E)$ *and* $\langle \cdot, \cdot \rangle_E$ *denoting the metric* g' *of* E, *one has*

$$-\frac{1}{2} \Delta(s \cdot s) = s \cdot \Delta s - \langle \tilde{\nabla}_i s_k, \tilde{\nabla}^i s^k \rangle_E - \left\langle {}^{\mathrm{t}}\!\left(\Theta_i^h - \mathrm{Ric}_i^h \right) s_h, s^i \right\rangle_E \quad (2)$$

   where the superscript i, h *are raised by the metric* g.

Proof. Computational in nature. □

Remark 3. *1. We note by $Q(s)$ the last term of (2), then Q is a (2,0) tensor on M with value in $E^* \otimes E^*$ where E^* is the dualised bundle of E. In practice, Q is an $mn \times mn$ matrix with coefficients*

$$Q^{hi}_{\alpha\beta} = g^{hk} g^{ij} \left[\left(g'_{\alpha\gamma} \Theta^{\gamma}_{\beta} \right)_{kj} - g'_{\alpha\beta} \mathrm{Ric}_{kj} \right]$$

.

2. Since $\int_M \Delta(s \cdot s) dV = 0$, if s is harmonic, one has

$$\begin{aligned} \int_M Q(s) dV &= - \int_M \langle \tilde{\nabla}_i s_k, \tilde{\nabla}^i s^k \rangle_E dV \\ &= - \int_M \| \nabla_i s^{\alpha}_k dx^i \otimes dx^k \otimes \tilde{e}_{\alpha} \|^2_{A^2(M,E)} dV \leq 0 \end{aligned} \tag{3}$$

1.3 The case of $E = f^* TM'$

1.3.1 Energy functional and tension field

Our interest will be the case of Example 2 where $E = f^* TM'$ for some smooth map $f : M \longrightarrow M'$ of Riemannian manifolds is a Riemannian-connected bundle over M with the connection $\tilde{\nabla}$ given by the Levi-Civita connection of M'.

In this section, the tangent map $Tf : TM \longrightarrow TM'$ can be interpreted as a form f_* in $A^1(M, E)$. The energy functional can be rewritten as

$$E(f) = \frac{1}{2} \int_M f^{\alpha}_i f^{\beta}_j g^{ij} g'_{\alpha\beta} dV = \frac{1}{2} \langle f_*, f_* \rangle_{A^1 M, E}.$$

Proposition 2.1. *Let $f : M \longrightarrow M'$ and $E = f^* TM'$ be the Riemannian-connected bundle over M. Then:*

1. *$A^{\beta}_{\alpha} = \Gamma'^{\beta}_{\gamma\alpha} f^{\gamma}_i dx^i$ where $\Gamma'^{\beta}_{\gamma,\alpha}$ are Christoffel symbols of (M', g').*

2. *$Df_* = 0$ where f_* is considered as an element of $A^1(M, E)$. Hence $\tilde{\Delta} f_* = D\delta f_*$.*

3. *The tension field of f is $\tau(f) = -\delta f_*$.*

Proof. 1. We will use the fact that $\tilde{\nabla} g' = 0$. Given two section $s = s^{\alpha} \tilde{e}_{\alpha}, t = t^{\beta} \tilde{e}_{\beta}$ of E, expanding $\nabla_i (s \cdot t) = (\tilde{\nabla}_i s) \cdot t + s \cdot \tilde{\nabla}_i t$, one has

$$s^{\alpha} t^{\beta} \frac{\partial g'_{\alpha\beta}}{\partial x^i} = s^{\alpha} t^{\beta} \left(A^{\gamma}_{\alpha i} g'_{\gamma\beta} + A^{\gamma}_{\beta i} g'_{\alpha\gamma} \right)$$

Taking s, t to be of small support, $\alpha = \beta$ and substituing $A^{\gamma}_{\alpha i} = \Gamma'^{\nu}_{\gamma\alpha} f^{\gamma}_i$, one obtains the first statement.

2. By direct computation:

$$Df_* = \left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \Gamma'^\alpha_{\gamma\beta} f_i^\gamma f_j^\beta \right) dx^j \wedge dx^i \otimes \tilde{e}_\alpha = 0$$

since it is the product of a symmetric quantity in (i,j) and an anti-symmetric one.

3. Using the first part of Lemma 2 for $s = f_* = f_i^\alpha dx^i \otimes \tilde{e}_\alpha$, one has $\delta f_* = -g^{ij} \left(\nabla_i \nabla_j f^\gamma + \Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta \right) \tilde{e}_\gamma = -\tau(f)$

□

It follows immediately that

Corollary 2.1. *$f : M \longrightarrow M'$ is a harmonic map of Riemannian manifolds if and only if f_* is harmonic as form in $A^1(M, f^*TM')$.*

1.3.2 Fundamental form, some results in case of signed curvature

Definition 4. *The **fundamental form** of a map $f : M \longrightarrow M'$ of Riemannian manifolds is the (0,2) symmetric tensor on M with value in $E = f^*TM'$ defined by*

$$\beta(f) := \tilde{\nabla} f_* = \left(f_{ij}^\gamma + \Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta \right) dx^i \otimes dx^j \otimes \tilde{e}_\gamma.$$

*The function f is called **totally geodesic** if $\beta(f) = 0$ identically on M.*

Remark 4. *1. The tension field $\tau(f) = g^{ij}\beta(f)_{ij}$ is the trace of the fundamental form.*

2. If f is totally geodesic then it is harmonic.

When $s = f_*$, Lemma 2 and Remark 3 become Lemma 3, with no more than direct computation. The appearance of the Riemann curvature tensor R' of (M', g') is due to the formula

$$R'^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma'^\rho{}_{\nu\sigma} - \partial_\nu \Gamma'^\rho{}_{\mu\sigma} + \Gamma'^\rho{}_{\mu\lambda}\Gamma'^\lambda{}_{\nu\sigma} - \Gamma'^\rho{}_{\nu\lambda}\Gamma'^\lambda{}_{\mu\sigma}.$$

Lemma 3. *1. $Q(f_*)$ is given by*

$$Q(f_*) = R'_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta g^{ik} g^{jl} - \mathrm{Ric}^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$$

and

$$Q(f_*)^{ij}_{\alpha\beta} = R'_{\alpha\beta\gamma\delta} f_k^\gamma f_l^\delta g^{ik} g^{jl} - \mathrm{Ric}^{ij} g'_{\alpha\beta}.$$

2. *If f is harmonic then*

$$-\Delta e(f) = |\beta(f)|^2 - Q(f_*)$$

where $|\beta(f)|$ is the pointwise norm of $\beta(f)$.

The previous computation of $Q(f_*)$ in term of Riemannian curvature of M' and Ricci curvature of M give the following result in the case where the curvature of M and M' are of definite sign.

Notation. Given a Riemannian manifold M, we will use the following notation:

1. $\mathrm{Ric} \geq 0$ (resp. $\mathrm{Ric} > 0$) if the Ricci curvature is positive semi-definite (resp. positive definite) as symmetric bilinear form.
2. $\mathrm{Riem} \leq 0$ (resp. $\mathrm{Riem} < 0$) if all sectional curvatures are negative (resp. strictly negative), i.e. $R_{ijhk}u^i v^j u^h v^k \leq 0$ (resp. $R_{ijhk}u^i v^j u^h v^k < 0$) for non-colinear vectors u, v.

Corollary 3.1. *Let $f : M \longrightarrow M'$ be a map of Riemannian manifolds.*

1. *If f is harmonic and $Q(f_*) \leq 0$ then f is totally geodesic and $e(f)$ is constant.*
2. *If $\mathrm{Ric}(M) \geq 0$ and $\mathrm{Riem}(M') \leq 0$ then f is harmonic if and only if f is totally geodesic.*
3. *Under the same condition as 2),*
   - *If $\mathrm{Ric}(M) > 0$ at one point of M then all harmonic maps are constant.*
   - *If $\mathrm{Riem}(M') < 0$ everywhere in the image of f and f is harmonic, then f is either constant or maps M onto a closed geodesic of M'.*

Proof. All the statements are consequence of 2) of Lemma 3 and the fact that $\int_M \Delta e(f) dV = 0$, noticing that

- $\mathrm{Ric}^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$ is $\mathrm{Ric} \otimes g'$ applied doubly to $f_i^\alpha dx^i \otimes \tilde{e}_\alpha$.
- $R'_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta g^{ik} g^{jl}$ is $(f^* R')_{ijhk} g^{ik} g^{jl}$. In a normal coordinate at P where $g^{ik} = \delta_{ik}, g^{jl} = \delta_{jl}$, it is the sum of sectional curvatures of tangent planes formed by $f_* e_i, f_* e_j$, and therefore negative.

For 3), if $\mathrm{Ric}(M) < 0$ at one point $P \in M$ then at that point $f_i^\alpha dx^i \tilde{e}_\alpha = 0$, meaning $f_* = 0$, hence $e(f)$ vanishes at P. Since $e(f)$ has to be constant, it vanishes identically, which implies that f is constant.

If $\mathrm{Riem}(M') < 0$, one sees that all $f_* e_i, f_* e_j$ are colinear, so the image of Tf is of one dimension, which leads to the conclusion, as we will see later that a totally geodesic map transforms geodesic to geodesic. $\square$

1.4 Example: Riemannian immersion

Let $f : M \longrightarrow M'$ be a Riemannian immersion, i.e. Tf is injective and $f^* g' = g$. We will see that the fundamental form $\beta(f)$ that we defined earlier is the same as usual definition in courses of Riemannian geometry.

1.4.1 Second fundamental form.

One defines the symmetric (0,2)-tensor II as the unique tangent vector of M' such that

$$\langle \mathrm{II}_{ij}, \xi_\sigma \rangle := -\langle \tilde{\nabla}_i \xi_\sigma, f_* e_j \rangle$$

for every vector field ξ_σ of M' orthogonal to M.

Lemma 4 (Second fundamental form)**.** *If f is a Riemannian immersion then $\beta(f)_{ij} = \mathrm{II}_{ij}$ and they are orthogonal to M. In particular, if f is totally geodesic than it maps geodesics of M to geodesics of M'*

Proof. One has

$$\begin{aligned}
\langle \tilde{\nabla}_i \xi_\sigma, f_* e_j \rangle &= \langle \xi_\sigma, \tilde{\nabla}_i (f_* e_j) \rangle = \langle \xi_\sigma, \tilde{\nabla}_i (f_l^\gamma dx^l \otimes \tilde{e}_\gamma) e_j + f_* \nabla_i e_j \rangle \\
&= \langle \xi_\sigma, (f_{il}^\gamma dx^l \tilde{e}_\gamma + f_l^\gamma dx^l \tilde{\nabla}_i \tilde{e}_\gamma) e_j \rangle \\
&= \langle \xi_\sigma, f_{ij}^\gamma \tilde{e}_\gamma + f_j^\gamma A_{\gamma_i}^\alpha \tilde{e}_\alpha \rangle = \left\langle \xi_\sigma, \left(f_{ij}^\gamma + \Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta \right) \tilde{e}_\gamma \right\rangle \\
&= \langle \xi_\sigma, \tilde{\nabla}_i (f_*).e_j \rangle = \langle \xi_\sigma, \beta_{ij}(f) \rangle
\end{aligned} \tag{4}$$

where we used $\xi_\sigma \perp f_* e_j$ in the first line and $\xi_\sigma \perp f_*([e_i, e_j])$ in the second line. Hence $\mathrm{II}_{ij} \equiv -\beta(f)_{ij}$ modulo an element in TM. In fact one has $\beta(f)_{ij} \perp M$ and therefore $\mathrm{II} = -\beta(f)$, since $\beta(f)_{ij} = \tilde{\nabla}_i (f_*).e_j$ and

$$\begin{aligned}
\langle \beta(f)_{ij}, f_* e_k \rangle &= \langle \tilde{\nabla}_i (f_*).e_j, f_* e_k \rangle = \tilde{\nabla}_i \langle f_* e_j, f_* e_k \rangle - \langle \nabla_i e_j, e_k \rangle - \langle f_* e_j, \tilde{\nabla}_i (f_* e_k) \rangle \\
&= -\langle \beta(f)_{ik}, f_* e_j \rangle + \nabla_i \langle e_j, e_k \rangle - \langle \nabla_i e_j, e_k \rangle - \langle e_j, \nabla_i e_k \rangle \\
&= -\langle \beta(f)_{ik}, f_* e_j \rangle
\end{aligned}$$

Then using the symmetric of $\beta(f)_{ij}$, one has $\langle \beta(f)_{ij}, f_* e_k \rangle = 0$.

Finally, if $\beta(f) = 0$ and X is a geodesic vector field of M, one needs to prove that $f_* X$ is a geodesic vector field of M'. In fact

$$\tilde{\nabla}_X(f_*X) = (\tilde{\nabla}_X f_*)X + f_*\nabla_X X = \beta(f)(X, X) = 0.$$

Hence f_*X is a geodesic field of M'. □

Example 3. *The inclusion $x \mapsto (x, y_0)$ of a Riemannian manifold M to the Riemannian product $M \times N$ is totally geodesic.*

Definition 5. *Given an orthonormal frame $(\xi_\sigma)_{1\leq\sigma\leq n'-n}$, the* ***mean normal curvature field*** *of M in M' at $P \in M$ is defined as*

$$\xi(P) := \sum_{\sigma=1}^{n'-n} g^{ij}\langle \mathrm{II}_{ij}, \xi_\sigma\rangle\xi_\sigma = -\sum_{\sigma=1}^{n'-n}\langle\tau(f), \xi_\sigma\rangle\xi_\sigma.$$

The immersion f is said to be ***minimal*** *if ξ vanishes identically on M.*

Remark 5. *1. Since $(\xi_\sigma)_{1\leq\sigma\leq n'-n}$ is an orthonormal frame, one also has*

$$\xi(P) = -g^{ij}\langle\tilde{\nabla}_i\xi_\sigma, f_*e_j\rangle\xi_\sigma(P) = -\sum_{\sigma=1}^{n'-n} \mathrm{div}\ (\xi_\sigma(P))\ \xi_\sigma(P)$$

2. The mean normal curvature field is the tension field of f, i.e. $\xi = -\tau(f)$. Minimal immersions are exactly harmonic immersion.

1.4.2 The case of signed curvature.

If $f : M \longrightarrow M'$ is a Riemannian immersion then the Ricci term of Lemma 3 is actually the scalar curvature of M, one has

Proposition 4.1. *Let $f : M \longrightarrow M'$ be a Riemannian immersion. Suppose that $\mathrm{Riem}(M') \leq 0$ and $r = g^{ij}\mathrm{Ric}_{ij} < 0$ at one point of M. If f is harmonic then it is constant.*

1.5 Example: Riemannian submersion

1.5.1 Results of Ehresmann and Hermann.

In this section, the function $f : M \longrightarrow M'$ will be a Riemannian submersion $\pi : M \longrightarrow B$, i.e. $T\pi$ is subjective and $\pi^*g' = g$. We will regard π as a fibration and calculate its tension field. We start with two theorems of Hermann with [**?**] as reference. A tangent vector of M lying in $\ker T_P\pi$ is said to be **vertical**. Since $\pi^* g' = g$, the plane $\mathcal{H}_P := \ker T_P\pi^{\perp}$ is isometric to $T_{\pi(P)}B$ and is said to be **horizontal**, such $\mathcal{H}_P$ form a distribution of planes as P varies in M.

Definition 6. *The plane distribution $\mathcal{H}$ is called **complete** if every curve γ in B **lifts** horizontally on M at each point P in $M_{\gamma(0)}$, i.e. there exists a curve $\hat{\gamma}$ in M such that $\pi \circ \hat{\gamma} = \gamma$ and $\hat{\gamma}(0) = P \in M$.*

*A vector field X of M is said to be **projectable** if $\pi_* X$ is well-defined, i.e. $\pi_* X$ does not change on each fibre. In that case, one says that X is π-**associated** to the vector field $\pi_* X$ of B.*

*X is said to be **basic** if it is projectable and horizontal.*

Remark 6. *If a vector field X on M is π-associated with a vector field $\check{X}$ on B, then*

1. *their flows are related by π: $\pi(\Phi^t_X) = \Phi^t_{\check{X}}$,*
2. *the Lie bracket satisfies: $[X, Y]$ is projectable and π-associated with $[\check{X}, \check{Y}]$.*

Theorem 5 (Ehresmann-Hermann)**.**

1. *If $\mathcal{H}$ is complete then the fibration $\pi : M \longrightarrow B$ is locally trivial.*
2. *If M is complete then $\mathcal{H}$ is a complete distribution and B is a complete manifold.*

Remark 7.

1. *The trivialising map $\phi : U_M \longrightarrow U_B \times F$, where U_M, U_B are open sets of M, B, is only a diffeomorphism and not a isometry, each fibre is equipped with different metric when identified with F.*
2. *The metric of M is not a Riemannian product of a (vertical) metric on F and the (horizontal) metric on B, but it is a product pointwise. To be precise, one has*

$$g_{(b,f)}(v_h^1 + v_v^1, v_h^2 + v_v^2) = g'_b(v_h^1, v_h^2) \times \hat{g}_{(b,f)}(v_v^1, v_v^2) \tag{5}$$

where $v^i = v_h^i + v_v^i$ is the decomposition of vector v^i to horizontal and vertical components, g' is the horizontal metric (the metric on M) and $\hat{g}_{(b,f)}$ is the restriction of g on the fibre M_b. However, when the fibration is of totally geodesic fibres, g is a Riemannian product $g_{(b,f)} = g'_b \times \hat{g}_f$, see Theorem 6.

Sketch of proof. The first part is due to Ehresmann, take a small geodesic ball center at P, and connect every point Q to P by a curve γ. Map every point $\hat{\gamma}(0) \in M_P$ to the point $\hat{\gamma}(1) \in M_Q$ where $\hat{\gamma}$ is the lift of γ starting from $\hat{\gamma}(0)$. One has a diffeomorphism $\theta_\gamma : M_{\gamma(0)} \longrightarrow M_{\gamma(1)}$.

The second part, due to Hermann, can be established in 2 steps:

First, by direct computation, one proves that if any geodesic field X on B lifts to a horizontal vector field $\hat{X}$ then $\hat{X}$ is a geodesic vector field. In fact, denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connection on M and B respectively and $\mathcal{V}, \mathcal{H}$ the vertical and horizontal projection of tangent vectors of M. Then $\nabla_{\hat{X}}\hat{X} = \mathcal{V}\nabla_{\hat{X}}\hat{X} + \mathcal{H}\nabla_{\hat{X}}\hat{X}$ in which $\mathcal{H}\nabla_{\hat{X}}\hat{X}$ is actually the horizontal lift of $\tilde{\nabla}_X X$ therefore vanishes. We claim that $\mathcal{V}\nabla_Y Y = \frac{1}{2}\mathcal{V}[Y, Y]$ hence also vanishes for every basic vector field Y. In fact let U be any vertical vector field then

$$\langle U, \mathcal{V}\nabla_Y Y\rangle = \langle U, \nabla_Y Y\rangle = -\langle \nabla_X U, X\rangle = \langle \nabla_U X, X\rangle = \frac{1}{2}\nabla_U\langle X, X\rangle = 0$$

where we used the fact that $\nabla_X U - \nabla_U X = [X, U] = [\widehat{\pi_* X, \pi_*} U] = 0$ and $\langle X, X\rangle$ is constant on each fibre (being $\langle \pi_* X, \pi_* X\rangle$), hence in every vertical direction U (Remark: this corresponds to the fact that g' only depend on b).

Now if M is complete then for every geodesic curve γ in B, let X be the velocity field of γ and $\hat{X}$ be the horizontal lift of X, which is now a horizontal, geodesic field of M, whose integral curves are lifts of γ. Therefore B is complete and every geodesic curve of B lifts horizontally to M.

For the general curve γ of M, the idea will be to approximate it by geodesics and lift part by part. $\square$

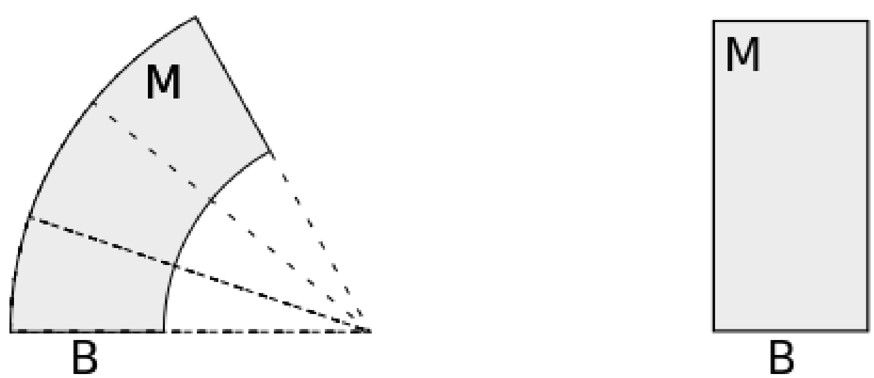

Figure 1: The trivialising map is only a diffeomorphism and not an isometry

Theorem 6 (Hermann). *If the fibration $\pi : M \longrightarrow B$ is of totally geodesic fibres then the diffeomorphisms $\hat{\gamma}(0) \longrightarrow \hat{\gamma}(1)$ are in fact isometries between fibres and M is then locally a Riemannian product of B and the fibre, now equipped with its unique metric induced by M.*

Proof. We need to prove that the metric on fibres $\hat{g}_{(b,f)}$ does not depend on the point f of the fibre, i.e. for every basic vector field X, one has $\mathcal{L}_X \hat{g} = 0$, where by $\hat{g}$, we mean the (0,2) symmetric tensor $(Y_1, Y_2) \mapsto \langle \mathcal{V}Y_1, \mathcal{V}Y_2 \rangle$. Let U, V be vertical vector fields of M then

$$\begin{aligned} X(\hat{g}(U,V)) &= (\mathcal{L}_X \hat{g})(U,V) + \hat{g}([X,U],V) + \hat{g}(U,[X,V]) \\ &= \langle \nabla_X U, V\rangle - \langle \nabla_U X, V\rangle + \langle U, \nabla_X V\rangle - \langle U, \nabla_V X\rangle + (\mathcal{L}_X \hat{g})(U,V) \end{aligned}$$

Hence $(\mathcal{L}_X \hat{g})(U,V) = \langle \nabla_U X, V\rangle + \langle U, \nabla_V X\rangle = -2\langle \mathrm{II}(U,V), X\rangle = 0$. Since the map $\hat{\gamma}(0) \mapsto \hat{\gamma}(1)$ is in the one-parameter group of differomorphism associate to a basic vector field X, it preserves $\hat{g}$. □

1.5.2 Tension fields and harmonic fibrations.

We will now calculate the tension field of a fibration map $\pi : M \longrightarrow B$.

Proposition 6.1. *Let $\pi : M^n \longrightarrow B^{n'}$ be a complete Riemannian fibration then*

1. *$e(\pi) = n/2$.*
2. *Let M_b be a fibre of π and $\iota_{M_b} : M_b \hookrightarrow M$ be the inclusion. Then $\tau(\pi) = -\pi_* \tau(\iota_{M_b})$ on M_b.*

In particular, π is harmonic if and only if its fibres are minimal submanifolds of M, i.e. the inclusions ι_{M_b} are harmonic.

Proof. 1. is obvious. For 2), note that

$$\tau(\iota_{M_b}) = -\sum_{\sigma=1}^{n'} \mathrm{div}\ (e_\sigma(P)) e_\sigma(P)$$

for any orthonormal frame $e_\sigma(P)$ of normal vectors of $M_{\pi(P)}$. Take e_σ to be $e_\sigma = \mathrm{grad}\ \pi^\sigma$ where π^σ is the σ-th component of π in a normal coordinate of V around $\pi(P)$. Note that e_σ are actually the horizontal lift of the basis vectors $\tilde{e}_\sigma$ of the frame at $\pi(P)$, and therefore are normal vectors of M_P.

Meanwhile, one has $\tau(\pi) = -\Delta \pi^\sigma \tilde{e}_\sigma$ at $\pi(P)$ since the Christoffel symbols vanish at P. Comparing the two vector fields, one has $\tau(\pi) = -\pi_* \tau(\iota_{M_b})$ at P. □

Example 4. *A complete Riemannian fibration* $\pi : M \longrightarrow B$ *with totally geodesic fibres are harmonic.*

1.6 Composition of maps

The following results come from direct computation of the second fundamental form and tension field of composition of maps between Riemannian manifolds. Again, we use indices $i, j, k, \dots$ for M, $\alpha, \beta, \gamma, \dots$ for M' and $a, b, c, \dots$ for M''.

Proposition 6.2. *Let* $f : M \longrightarrow M'$ *and* $f' : M' \longrightarrow M''$ *be smooth maps of Riemannian manifolds, then*

$$\beta(f' \circ f)^a_{ij} = \beta(f)^\gamma_{ij} f'^a_\gamma + \beta(f')^a_{\alpha\beta} f^\alpha_i f^\beta_j \tag{6}$$

and

$$\tau(f' \circ f)^a = \tau(f)^\gamma f'^a_\gamma + g^{ij} \beta(f')^a_{\alpha\beta} f^\alpha_i f^\beta_j \tag{7}$$

Therefore,

If f' is	*and f is*	*then $f' \circ f$ is*
totally geodesic	*totally geodesic*	*totally geodesic*
totally geodesic	*harmonic*	*harmonic*

and the inverse of a totally geodesic map is totally geodesic.

Proof. Direct computation. □

Remark 8. *It is not true in general that the composition of harmonic maps are harmonic.*

Proposition 6.3 (composition with immersion)**.** *If* $f' : M' \longrightarrow M''$ *is a Riemannian immersion and* $f : M \longrightarrow M'$ *then*

1. *Energy functionals:* $E(f) = E(f' \circ f)$.

2. *Tension fields:* $\tau(f)$ *is the projection of* $\tau(f' \circ f)$ *to* M'.

Proof. 1. One has $e(f) = \frac{1}{2}\langle g, f^* g' \rangle = \frac{1}{2}\langle g, (f' \circ f)^* g'' \rangle = e(f' \circ f)$.

2. One has $\tau(f' \circ f)^a = \tau(f)^a + g^{ij}\beta(f')^a_{\alpha\beta} f^\alpha_i f^\beta_j$ by (7). The second term being the restriction of the tension field of M' to the image of M, the conclusion follows.

□

The following immediate corollary of Proposition 6.3 is a generalization of the fact that a curve is geodesic if and only if it is perpendicular to its tension field.

Corollary 6.1. *A map $f : M \longrightarrow M'$ is harmonic if and only if $\tau(f' \circ f) \perp M'$*

Proposition 6.4 (composition with submersion)**.** *Let $f' : M' \longrightarrow M''$ be a Riemannian fibration with totally geodesic fibres and $f : M \longrightarrow M'$ then*

$$\tau(f' \circ f) = f'_*(\tau(f))$$

Proof. One can suppose that M' is a Riemannian product of M'' and its fibre, and f' is the projection to M'', since this is true locally and the proposition is local. Then the conclusion is that the tension field of the projection is the projection of the tension field, or equivalently the tension field of $f = f_1 \times f_2 : M \longrightarrow M'' \times F$ is $\tau(f) = (\tau f_1, \tau f_2)$. This follows from the explicit formula of $\tau(f)$, noting that the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ vanish except when the indice α, β, γ belong to the same tangent space (TM'' or TF). □

Example 5. *1. A map $f : M \longrightarrow M' \times M''$ is harmonic if $f = (f^1, f^2)$ with f^1, f^2 harmonic.*

2. Take $M'' = M$ in Proposition 6.4 and $f = s : M \longrightarrow M'$ a section of the fibration f', one sees that the tension field $\tau(s)$ is always vertical.

The following corollary is immediate.

Corollary 6.2. *Let $f' : M' \longrightarrow M''$ be a proper Riemannian embedding and N is a normal turbular neighborhood of M' which can be seen as a smooth fiber bundle over M'. Denote by $\pi : N \longrightarrow M'$ the projection. Then for all map $f : M \longrightarrow N$, $\pi \circ f$ is harmonic if and only if $\tau(f)$ is vertical.*

2 Nonlinear heat flow: Global equation and existence of harmonic maps.

2.1 Statement of the main results.

We want to prove in the next part the existence of harmonic map between manifolds M and M' by deforming any map $f : M \longrightarrow M'$ using the τ-flow, meaning solving the PDE:

$$\begin{cases} \frac{df_t}{dt} = \tau f_t, \quad t \in [\alpha, \omega] \\ f_\alpha = f, \end{cases} \tag{8}$$

The equation makes sense because both $\frac{df_t}{dt}$ and τf_t are vector fields along f_t. Since this is the gradient-descent equation for E, the energy of f_t decreases and we hope, under conditions, to obtain convergence of $\{f_t\}$ to a critical point f_∞ of E, this will prove that any homotopy class of $C^\infty(M, M')$ has at least a harmonic map.

It is proved by Eells and Sampson [**?**] that

Theorem 7 (Eells-Sampson)**.** *Let M and M' be compact Riemannian manifolds with* $\mathrm{Riem}(M') \leq 0$ *then there exists a harmonic map $f : M \longrightarrow M'$ in each homotopy class.*

Several boundary conditions, of Dirichlet, Neumann or mixed type, are also taken into account by Hamilton [**?**], as an example, we will state the Dirichlet problem:

Theorem 8 (Hamilton)**.** *Let M and M' be compact Riemannian manifolds possibly with boundary. Suppose that M' has* $\mathrm{Riem}(M') \leq 0$ *and $\partial M'$ is convex, then any relative homotopy class of $C^\infty(M, M')$ has a harmonic element.*

About the terminology, **relative homotopy class** means that we only deform f among maps with the same value on ∂M. The **convexity of** $\partial M'$ means that the geodesic at any point in $\partial M'$ with initial tangent vector paralell to the boundary does not enter the interior of M' in short time. This condition can be expressed using the Christoffel symbols of M' at the point in question.

It is easy to see that the convexity of $\partial M'$ is a necessary condition, as harmonic maps from $\mathbb{R}$ are geodesics: Suppose the condition does not hold at $x \in \partial M'$, meaning that upto time t the geodesic flow of M' initially tangent to $\partial M'$ remains in the interior. The geodesic of $\partial M'$ of length less than t with the same initial tangent therefore cannot be deformed into a geodesic of M' in relative homotopy class.

2.2 Strategy of the proof.

In order to have a global frame, we will embed M' into an Euclidean space V, but we will not use the Euclidean metric of V. In fact, let T be a tubular neighborhood of M' in V then T is diffeomorphic to $M' \times D$ where D is a sufficiently small of dimension being the codimension of M' in V, and we will equip T with the product metric of $M' \times D$.

Since $M' \equiv M' \times \{0\}$ is totally geodesic in T, one has for every smooth function $f : M \longrightarrow M'$:

$$\tau_V(f) = \tau_T(f) = \tau_{M'}(f)$$

The crucial property we expect for a global equation of (8), is that if the solution initially is in $M' \subset V$ then it remains in M' for all relevant time $t > \alpha$. Eells-Sampson [**?**] did this by using at the same time 2 different metrics on T, namely the product metric as tubular neighborhood and the Euclidean metric. I choose to present here the formulation of Hamilton, which is conceptually simpler with the only drawback being that we need to establish the uniqueness of solution of (8) first.

After having the global equation, we will prove the short time existence of solution by linearising the equation and using Implicit function theorem. The global formulation and the proof of short-time existence is independent of the negative curvature hypothesis, which will only be used later to establish energy estimates and assure the convergence of long-time solution and the vanishing of its tension field.

2.3 Global equation and Uniqueness of nonlinear heat equation.

Theorem 9 (Global equation)**.** *If the smooth function $F_t : M \times [\alpha, \beta] \longrightarrow V$ satisfies*

$$\frac{dF_t}{dt} = \tau_V(F_t) \tag{9}$$

and $F_t(M \times \{\alpha\}) \subset M'$ then $F_t(M \times [\alpha, \omega]) \subset M'$

Proof. Let ι be the isometry of T given by $(y, d) \mapsto (y, -d)$ for $(y, d) \in M' \times D \equiv T$ and pose $G_t = \iota F_t$ then G_t and F_t coincide initially since M' is fixed by ι. Moreover

$$\frac{dG_t}{dt} = d\iota.\frac{dF_t}{dt} = d\iota(\tau_V(F_t)) = \tau_V(\iota F_t) = \tau_V(G_t)$$

We conclude that $F_t = G_t = \iota F_t$, hence F_t remains in M' for all relevant t, using the following uniqueness of nonlinear heat equation. $\square$

Theorem 10 (Uniqueness of solution of nonlinear hear equation)**.** *Let $f_1, f_2 : M \times [\alpha, \omega] \longrightarrow M'$ be C^2 functions satisfying the non-linear heat equation $\frac{df_i}{dt} = \tau_{M'}(f_i)$, i.e.*

$$\frac{df_i}{dt} = -\Delta f^\gamma + g^{ij}\Gamma'^{\gamma}_{\alpha\beta} f_i^\alpha f_j^\beta$$

where $\Gamma'^{\gamma}_{\alpha\beta}$ are Christoffel symbols of M'. Suppose that f_1 and f_2 coincide on $M \times \{\alpha\}$. Then $f_1 = f_2$ on $M \times [\alpha, \omega]$.

Proof. It is sufficient to prove the theorem for ω very close to α, therefore by compactness of M, we can suppose that there exists a finite atlas $M = \bigcup_i U_i$ with $f_1(U_i \times [\alpha, \omega])$ and $f_2(U_i, [\alpha, \omega])$ being in the same chart V_i of M'. We consider the distance function $\sigma(a, b) = \frac{1}{2} d_{M'}(a, b)^2$ for $a, b \in M'$ to measure the difference between f_1 and f_2 by

$$\rho(x, t) = \sigma(f_1(x, t), f_2(x, t))$$

The strategy is to prove that there exists $C > 0$ such that $\frac{d\rho}{dt} \leq -\Delta + C\rho$, then by Maximum principle, one has $\rho = 0$.

Fix a chart U_i of M and the corresponding V_i of M', one has by straightforward calculation:

$$\begin{aligned}\frac{d\rho}{dt} &= -\Delta\rho - g^{ij}\left(\frac{\partial^2\sigma}{\partial f_1^\beta \partial f_1^\gamma} - \frac{\partial\sigma}{\partial f_1^\alpha}\Gamma'^{\alpha}_{\beta\gamma}(f_1)\right) {f_1}_i^\beta {f_1}_j^\gamma \\ &\quad - g^{ij}\left(\frac{\partial^2\sigma}{\partial f_2^\beta \partial f_2^\gamma} - \frac{\partial\sigma}{\partial f_2^\alpha}\Gamma'^{\alpha}_{\beta\gamma}(f_2)\right) {f_2}_i^\beta {f_2}_j^\gamma - 2g^{ij}\frac{\partial^2\sigma}{\partial f_1^\beta \partial f_2^\gamma} {f_1}_i^\beta {f_2}_j^\gamma\end{aligned}$$

where g^{ij} is the metric on M and $\Gamma'^{\alpha}_{\beta\gamma}$ are Christoffel symbols of M'.

Let c be a point in the chart V_i and choose the normal coordinates of M' at c. Then for $a, b \in M'$ near c, one has, since $\sigma(a, b) = \sigma(b, a)$ and $\sigma(a, b) = 0$ if $b^\gamma = ka^\gamma$ (the Euclidean straight line from a to ka viewed on M' is a geodesic):

$$\sigma(a, b) = \frac{1}{2} d_{M'}(a, b)^2 = \frac{1}{2} d_E(a, b)^2 + \lambda_{\beta\gamma,\delta}(a^\beta a^\gamma b^\delta + b^\beta b^\gamma a^\delta)$$

where d_E is the Euclidean distance, with $\lambda_{\beta\gamma,\delta} = \lambda_{\gamma\beta,\delta}$ and $\lambda_{\beta\gamma,\delta} + \lambda_{\gamma\delta,\beta} + \lambda_{\beta\delta,\gamma} = 0$. We then have the series development of σ at $(0, 0)$:

$$\sigma(a, b) = \frac{1}{2}\delta_{\beta\gamma}(a^\beta - b^\beta)(a^\gamma - b^\gamma) + \lambda_{\beta\gamma,\delta}(a^\beta a^\gamma b^\delta + b^\beta b^\gamma a^\delta) + O(|a| + |b|)^4 \quad (10)$$

and the development of its derivatives

$$
\begin{aligned}
\frac{\partial^2\sigma}{\partial a^\beta \partial b^\gamma}(a,b) &= -\delta_{\beta\gamma} + \lambda_{\beta\delta,\gamma}a^\delta + \lambda_{\gamma\delta,\beta}b^\delta + O(|a|+|b|)^2 \\
\frac{\partial^2\sigma}{\partial a^\beta \partial a^\gamma}(a,b) &= \delta_{\beta\gamma} + \lambda_{\beta\gamma,\delta}b^\delta + O(|a|+|b|)^2 \\
\frac{\partial^2\sigma}{\partial b^\beta \partial b^\gamma}(a,b) &= \delta_{\beta\gamma} + \lambda_{\beta\gamma,\delta}a^\delta + O(|a|+|b|)^2 \\
\frac{\partial\sigma}{\partial a^\alpha}(a,b) &= O(|a|+|b|), \quad {\Gamma'}^\alpha_{\beta\gamma}(a) = O(|a|)
\end{aligned}
$$

So choose c to be the midpoint of $f_1(x,t)$ and $f_2(x,t)$ and $(f_1(x,t), f_2(x,t)) = (w, -w)$ in the chart, one has:

$$
\frac{d\rho}{dt} = -\Delta\rho - \left(\delta_{\beta\gamma} - \lambda_{\beta\gamma,\delta}w^\delta + O(|w|^2)\right) {f_1}_i^\beta {f_1}_j^\gamma g^{ij} - \left(\delta_{\beta\gamma} + \lambda_{\beta\gamma,\delta}w^\delta + O(|w|^2)\right) {f_2}_i^\beta {f_2}_j^\gamma g^{ij} \tag{11}
$$

$$
- 2\left(-\delta_{\beta\gamma} + \lambda_{\beta\delta,\gamma}w^\delta - \lambda_{\gamma\delta,\beta}w^\delta + O(|w|^2)\right) {f_1}_i^\beta {f_2}_j^\gamma g^{ij} \tag{12}
$$

$$
= -\Delta\rho - |df_1 - df_2|^2 - w^\delta \lambda_{\beta\gamma,\delta} g^{ij}\left({f_2}_i^\beta {f_2}_j^\gamma - {f_1}_i^\beta {f_1}_j^\gamma\right) \tag{13}
$$

where we made a reduction of the term (12) by permuting β and γ to cancel the first order term w^δ. The last term of (13) can be bounded as follows:

$$
\begin{aligned}
\left| w^\delta \lambda_{\beta\gamma,\delta}\left({f_2}_i^\beta {f_2}_j^\gamma - {f_1}_i^\beta {f_1}_j^\gamma\right) g^{ij}\right| &= \left| w^\delta \lambda_{\beta\gamma,\delta}\left({f_2}_i^\beta ({f_2}_j^\gamma - {f_1}_j^\gamma) + {f_1}_j^\gamma ({f_2}_i^\beta - {f_1}_i^\beta)\right) g^{ij}\right| \\
&\leq 2|w^\delta \lambda_{\beta\gamma,\delta}||df_2 - df_1|(|df_1| + |df_2|) \\
&\leq |df_1 - df_2|^2 + O(|w|^2)
\end{aligned}
$$

where for the last inequality, we use $2uv \leq u^2 + v^2$ and the fact that $|df_1|$ and $|df_2|$ are bounded on M. The estimate (13) can be continued:

$$
\frac{d\rho}{dt} \leq -\Delta\rho + C(x,t)|w|^2 \leq -\Delta\rho + C\rho
$$

where $C > 0$ is a constant chosen to dominate all $C(x,t)$ for $x \in M$ in all charts and $t \in [\alpha, \omega]$. □

Remark 9. *The last proof was modified from [**?**]. The original proof made the reduction of the first order of w in* (12) *using the following development of σ:*

$$
\sigma = \frac{1}{2}\delta_{\beta\gamma}(a^\beta - b^\beta)(a^\gamma - b^\gamma) + \lambda_{\beta\gamma,\delta}(a^\beta - b^\beta)(a^\gamma - b^\gamma)(a^\delta + b^\delta) + O(|a|+|b|)^4
$$

which was justified by $\sigma(a,b)=\sigma(b,a)$ and $\sigma(a,a)=0$. Algebraically these symmetries of σ are not sufficient to justify the development, and one can also proved, using $\sigma(a,ka)=0$, that if such development was valid then all $\lambda_{\beta\gamma,\delta}$ would be zero, and there would be no third-order term. We made this reduction using the symmetry of $\frac{\partial^2\sigma}{\partial f_1^\beta \partial f_2^\gamma} {f_1}_i^\beta {f_2}_j^\gamma$ in (β,γ) and not just $\frac{\partial^2\sigma}{\partial f_1^\beta \partial f_2^\gamma}$ alone.

As a side note, if a,b,c are on $\mathbb{S}^2$ with $d(a,c)=d(b,c)=x\ll 1$ and the lines from a and b to c are orthogonal at c, then the geodesic distance $d(a,b)=\arccos(\cos^2(x))=x\sqrt{2}-\frac{1}{6\sqrt{2}}x^3+O(x^4)$. So $\sigma(a,b)=\frac{1}{2}d(a,b)^2$ has no third-order term.

3 A few energy estimates.

3.1 Estimate of density energies

We finish this part with a few straightforward computation concerning the **potential energy** $e(f_t)=\frac{1}{2}|\nabla f_t|^2$ and the **kinetic energy** $k(f_t)=\frac{1}{2}|\frac{\partial f_t}{\partial t}|$ of a nonlinear heat flow f_t satisfying (8).

Theorem 11 (Density of Potential energy)**.** *If f_t satisfies* (8) *then*

$$\frac{de(f_t)}{dt}=-\Delta e(f_t)-|\beta(f_t)|^2-\big\langle \mathrm{Ric}(M)\nabla_v f_t,\nabla_v f_t\rangle+\langle \mathrm{Riem}(M')(\nabla_v f_t,\nabla_w f_t)\nabla_v f_t,\nabla_w f_t\big\rangle$$

where $e(f_t)$ is the potential energy density and $\beta(f_t)$ is the fundamental form and in the curvature terms, the vectors v and w are contracted.

In particular, if $\mathrm{Riem}(M')\leq 0$ *and* $\mathrm{Ric}(M)\geq -C$ *then*

$$\frac{de}{dt}\leq -\Delta e+Ce-|\beta(f_t)|^2 \tag{14}$$

Proof. Apply Lemma 2 to $s=df_t$ and the Riemannian-connected bundle (F^*TM' over $M\times[\alpha,\omega]$ where $F(\cdot,t)=f_t$, the curvature terms cancel out and it remains to see that $\frac{de(f_t)}{dt}=-\langle df_t,\Delta df_t\rangle$, meaning that $\tilde{\nabla}_{\partial t} df_t=-\Delta df_t$. This can be easily justified:

$$\tilde{\nabla}_{\partial t} df_t=\tilde{\nabla}_{\partial t}\tilde{\nabla}^M F=\tilde{\nabla}^M\tilde{\nabla}_{\partial t}F=\tilde{\nabla}^M\ \tau(f_t)=-D\delta(df_t)=-\Delta df_t$$

where the last "$=$" is due to $Ddf_t=0$. Note that D and δ are the exterior derivative and its adjoint of the bundle $(f_t)^*TM'$ on M, where t can be fixed after the third "$=$" sign. $\square$

Theorem 12 (Density of Kinetic energy)**.** *If f_t satisfies* (8) *then*

$$\frac{dk(f_t)}{dt} = -\Delta k(f_t) - \left|\nabla \frac{\partial f_t}{\partial t}\right|^2 + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle$$

where $k(f_t)$ is the kinetic energy density and in the curvature terms, the vectors v is contracted,

In particular, if $\mathrm{Riem}(M') \leq 0$ *then*

$$\frac{dk}{dt} \leq -\Delta k - \left|\nabla \frac{\partial f_t}{\partial t}\right|^2 \tag{15}$$

Proof. Let $F : I \times M \longrightarrow M'$ be the total function with $F(t, \cdot) = f_t$ for $t \in I = [\alpha, \omega]$ and $E = F^*TM'$ is a Riemannian-connected bundle on $I \times M$ with curvature form Θ, then

$$\tilde{\nabla}_{\partial t}\tilde{\nabla}_v(dF.v) = \tilde{\nabla}_v\tilde{\nabla}_{\partial t}(dF.v) + \Theta(\partial t, v)dF.v \tag{16}$$

where dF is the exterior derivative of f_t on M. Note that $\tilde{\nabla}_v\tilde{\nabla}_{\partial t}(dF.v) = \tilde{\nabla}_v(\tilde{\nabla}_{\partial t}dF).v = \tilde{\nabla}_v(\tilde{\nabla}^M \frac{\partial f_t}{\partial t}).v$ since $\tilde{\nabla}^M \frac{\partial f_t}{\partial t} = \tilde{\nabla}^{I\times M}_{\partial t} dF = \tilde{\nabla}^I_{\partial_t} dF$ because $\tilde{\nabla}$ is torsionless on M'. Plugging this in (16) and taking contraction in v, one has

$$\tilde{\nabla}_{\partial t}\ \tau(f_t) = -\tilde{\Delta}\frac{\partial f_t}{\partial t} + \mathrm{Tr}\left(v \mapsto \Theta(\partial t, v)dF.v\right) \tag{17}$$

But $\Theta^\beta_\alpha = R'^\beta_{\alpha\nu\mu} F^\mu_i F^\nu_j dx^i \otimes dx^j$ where R' denotes the Riemannian curvature of M' and the indices i, j can be 0, with $x^0 \equiv t$. Hence

$$\Theta(\partial t, v)dF.v = R'^\beta_{\alpha\nu\mu} \frac{\partial f_t^\mu}{\partial t}\frac{\partial f_t^\nu}{\partial v}\frac{\partial f_t^\alpha}{\partial v}\tilde{e}_\beta = \mathrm{Riem}(M')\left(\nabla_v f_t, \frac{\partial f_t}{\partial t}\right)\nabla_v f_t$$

Plugging in (17) and taking inner product with $\frac{\partial f_t}{\partial t}$, one has

$$\begin{aligned}\frac{\partial k(f_t)}{\partial t} &= \left\langle \tilde{\nabla}_{\partial t}\ \tau(f_t), \frac{\partial f_t}{\partial t}\right\rangle = -\left\langle \tilde{\Delta}\frac{\partial f_t}{\partial t}, \frac{\partial f_t}{\partial t}\right\rangle + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle \\ &= -\Delta\left(\frac{1}{2}\left|\frac{\partial f_t}{\partial t}\right|^2\right) - \left|\tilde{\nabla}\frac{\partial f_t}{\partial t}\right|^2 + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle\end{aligned}$$

$\square$

3.2 Estimate of total energies

We will now work with the total energies, in particular the **total potential energy** $E(f_t) := \int_M e(f_t)$ and **total kinetic energy** $K(f_t) := \int_M k(f_t)$. Since tension field is the gradient of E, one has:

Theorem 13. *If $f_t : M \longrightarrow M'$ satisfies* (8) *then*

$$\frac{dE(f_t)}{dt} = -\int_M \left\langle \tau(f_t), \frac{\partial f_t}{\partial t} \right\rangle = -\int_M |\tau(f_t)|^2 = -2K(f_t)$$

and

$$\frac{d^2E(f_t)}{dt} = -2\int_M \left\langle \nabla_{\partial t} \frac{\partial f_t}{\partial t}, \tau(f_t) \right\rangle$$

Intergrating Theorem 12 on M, one obtains:

Theorem 14. *If f_t satisfies* (8) *and* $\mathrm{Riem}(M') \leq 0$ *then* $\frac{d}{dt}K(f_t) \leq 0$. *Together with Theorem 13, one has*

1. *The total potential energy $E(f_t)$ is ≥ 0, decreasing and convex.*
2. *The total kinetic energy $K(f_t)$ is ≥ 0, decreasing and if $\omega = +\infty$ then* $\lim_{t\to\infty} K(f_t) = 0$.

In particular, $\int_{M\times\{\tau\}} |\nabla f|^2$ and $\int_{M\times\{\tau\}} \left|\frac{\partial f_t}{\partial t}\right|^2$ are bounded above by a constant $C > 0$ independent of the time $\tau \in [\alpha, \omega]$.

Note that we ruled out the case where $K(f_t)$ decreases to a strictly positive number because $E(f_t)$ is bounded below and $\frac{d}{dt}E(f_t) = -2K(f_t)$.

Integrating Theorem 11 on M and use Theorem 14, one has:

Theorem 15. *If f_t satisfies* (8) *and* $\mathrm{Riem}(M') \leq 0$ *and* $\mathrm{Ric}(M)$ *is bounded below then*

$$\int_M |\beta(f_t)|^2 \leq C$$

for all time t where the constant C only depends on the curvature of M, M' and the initial total potential and kinetic energy, in particular, C does not depend on t.

This means that $\|f_t\|_{W^{2,2}(M)}$ is bounded by a constant C only depending on the curvatures and initial total energies.

Corollary 15.1 (Boundedness in $W^{2,2}(M)$). *If F_t satisfies* (9) *and* $\mathrm{Riem}(M') \leq 0$ *and* $\mathrm{Ric}(M)$ *is bounded below then*

$$\|F_t\|^2_{W^{2,2}(M)} := \int_M |\beta(F_t)|^2 + |\nabla F_t| + |F|^2 \leq C$$

for all time t where the constant C only depends on the curvature of M, M' and the initial total potential and kinetic energy, in particular, C does not depend on t.

Note that the term $|F|^2$ is trivially bounded since the image of F remains in an Euclidean ball B.