Symmetric spaces and Lie groups

darknmt

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Contents

1 Symmetric space **1**

2 Locally symmetric space **2**

3 Annex: Group of isometries as Lie group **3**

1 Symmetric space

By de Rham decomposition, we now focus more on the building blocks: Riemannian manifolds with irreducible holonomy. The theory of Lie groups allows us to understand a block if it is *symmetric*.

Definition 1. *A Riemannian manifold M is called* <u>*symmetric*</u> *if for every $x \in M$, there exists an isometry s_x of M such that x is an isolated fixed point and $s_x^2 = Id$.*

Let $x \in M$ and $v \in T_xM$, we note by $\exp_x(v)$ the point of distance $|v|$ in the geodesic starting in x with velocity $v/|v|$. We remark that any isometry s_x with $s_x^2 = Id$ and x as isolated fixed point satisfies

$$s_x(\exp_x(v)) = \exp_x(-v) \tag{1}$$

In fact the eigenvalues of T_xs_x have to be 1 or -1, but as x is an isolated fixed point one has $T_xs_x = -Id$. Then s_x as an isometry sends the geodesic starting at x with velocity v to one starting at $s_x(x) = x$ with velocity $(s_x)_*v = -v$ and we have (1).

Equation (1) tells us that s_x is a reflection of center x on every geodesic passing by x. We can compose two reflections s_x, s_y to form a translation on the geodesic connecting x and y. This shows that a symmetric space is complete and the group of isometries of the form $s_x \circ s_y$ acts transitively on M.

Theorem 1 (Symmetric space)**.** *Let M be a symmetric Riemannian manifold then*

1. *M is complete.*

2. *Fix $x_0 \in M$, let G be the group generated by the isometries of form $s_x \circ s_y$, $x, y \in M$ and H is the subgroup containing elements of G that fix x_0, then G is Lie subgroup of $Isom(M)$ connected by arc, H is a closed Lie subgroup of G and M is isometric to G/H. Moreover the holonomy group of M is H.*

Remark 1. *In general, for a Lie group G and a closed Lie subgroup H, if G has a metric left-invariant by G and right-invariant by H (i.e. the metric on $\mathfrak{g}$ is invariant by action of H by adjoint) then*

$$\mathfrak{g} = \mathfrak{h} \oplus^{\perp} \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

But if G/H is symmetric then one has the following extra information

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

It turns out that this condition is quite strong and allowed E. Cartan to classify all such pairs $(\mathfrak{g}, \mathfrak{h})$.

2 Locally symmetric space

The previous results can be extended to locally symmetric spaces.

Proposition-Definition 2. *Let M is a Riemannian manifold, the followings are equivalent*

1. *For every $x \in M$, there exists a neighborhood U of x and an isometry $s_x : U \longrightarrow U$ such that $s_x^2 = Id$ and x is the unique fixed point of s_x.*

2. *The curvature tensor R satisfies*

$$\nabla R = 0$$

If they are satisfied, M is called locally symmetric.

Theorem 2 (Locally symmetric space)**.** *Let M be a locally symmetric Riemannian manifold, then there exists a unique symmetric simply connected Riemannian manifold N such that M and N are locally isometric, i.e. for every $x \in M$ and $y \in N$, there exists neighborhoods U of x and V of y that are isometric.*

As a result, the reduced holonomy of M is the same as the holonomy of N.

3 Annex: Group of isometries as Lie group

We explain in this annex some subtle details: how can a group of isometries be a manifold. We state, with Montgomery-Zippin, *Transformation groups* as reference, the following general result:

Theorem 3 (faithful + locally compact $\implies$ Lie)**.** *Let G be a group acting faithfully on a connected manifold M of class C^k such that each action is C^1 and G is locally compact. Then G is a Lie group and the map $G \times M \longrightarrow M$ is C^1.*

Note that we equip a group of isometries with the **compact-open topology**, as M is locally compact and therefore second-countable (i.e. the topology admits a countable base), we see that a group of isometries is also second-countable. It suffices to prove the local compactness for the group of (all) isometries as this property is inherited by its closed subgroup. The detail can be found in Kobayashi-Nomizu's *Foundations of differential geometry* (Volume I, Theorem 4.7).

Theorem 4. *Let M be a connected, locally-compact metric space and G be the group of isometries of M, then*

1. *G is locally compact.*
2. *G_a the subset of isometries fixing a point $a \in M$ is compact.*
3. *If, in addition, M is compact then G is also compact.*