Short-time existence and regularity for nonlinear heat equation

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We will establish in this part a regularity estimate for the quadratic term of nonlinear heat operator use it to setup a bootstrap scheme that eventually will prove that any sufficiently regular solution of nonlinear heat equation that is initially C^∞ will be always C^∞.

We will also prove short-time existence using well-known method of Inverse function theorem for Banach spaces. Since the solution is smooth, we can apply Theorem **??** to conclude that the it remains in $M' \subset \mathbb{R}^N$.

1 Review of Sobolev spaces and Linear equations.

The following results are well-known and their statements are written here in the case of our interest (linear heat equation on manifold). A more careful formulation with complete proofs can be found in the appendices.

1.1 Sobolev spaces.

Let M be a Riemannian manifold, the *Sobolev spaces* $W^{k,p}(M)$ on M can be defined as the completion of $C^\infty(M)$ with respect to the Sobolev norms

$$\|\varphi\|_{W^{k,p}} = \sum_{|\alpha|\le k} \|D^\alpha \varphi\|_{L^p}.$$

We will suppose that M is a compact manifold, then set-theoretically $W^{k,p}$ does not depend on the metric of M and their norm remains in the same equivalent class as the metric varies. The Sobolev spaces form a family of reflexive Banach spaces that is stable under holomorphic interpolation:

Theorem 1 (Interpolation of Sobolev spaces)**.** *Let $p, q \in (1, +\infty)$ and $k, l \in \mathbb{R}$ and M be a compact Riemannian manifold. Then the holomorphic interpolations of*

$$A_0 := W^{k,p}(M) \quad \textit{and} \quad A_1 := W^{l,q}(M)$$

are $A_\theta = W^{s,r}(M)$ where

$$\theta l + (1-\theta)k = s, \qquad \theta\frac{1}{q} + (1-\theta)\frac{1}{p} = \frac{1}{r}.$$

In particular, one has the Interpolation inequality

$$\|f\|_{W^{s,r}} \le 2\|f\|^{\theta}_{W^{l,q}} \|f\|^{1-\theta}_{W^{k,p}}.$$

Sobolev embeddings and Kondrachov theorem remain correct on manifold.

Theorem 2 (Sobolev embeddings)**.** *Given $k, l \in \mathbb{Z}$, $k > l \ge 0$ and $p, q \in \mathbb{R}$, $p > q \ge 1$. Then*

1. *If $\frac{1}{p} = \frac{1}{q} - \frac{k-l}{n}$ then*

$$W^{k,q}(M) \hookrightarrow W^{l,p}(M),$$

2. *If $\frac{k-r}{n} > \frac{1}{q}$ then*

$$W^{k,q}(M) \hookrightarrow C^r(M)$$

If $\frac{k-r-\alpha}{n} \le \frac{1}{q}$ then

$$W^{k,q}(M) \hookrightarrow C^{r,\alpha}(M)$$

where $C^r(M)$ denotes the space of C^r functions equipped with the norm $\|u\|_{C^r} = \max_{l\le r}\sup|\nabla^l u|$, and $C^{r,\alpha}$ is the subspace of C^r of functions whose r^{th}-derivative is α-Holder, equipped with the norm $\|u\|_{C^{r,\alpha}} = \|u\|_{C^r} + \sup_{P\neq Q}\{\frac{u(P)-u(Q)}{d(P,Q)^\alpha}\}$.

Theorem 3 (Kondrachov)**.** *Let $k \in \mathbb{Z}_{\ge 0}$ and $p, q \in \mathbb{R}_{>0}$ be such that $1 \ge \frac{1}{p} > \frac{1}{q} - \frac{k}{n} > 0$ then*

1. *The embedding $W^{k,q}(M) \hookrightarrow L^p(M)$ is compact,*
2. *The embedding $W^{k,q}(M) \hookrightarrow C^\alpha(M)$ is compact if $k - \alpha > \frac{n}{q}$ where $0 \le \alpha < 1$,*

It is also natural, for regularity results of parabolic equation, to use weighted Sobolev spaces because each derivative in time should be counted as twice as that in space. For example, the space $W^{2,p}(M \times [\alpha,\omega])$ is the completion of $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{W^{2,p}} := \|\varphi\|_{L^p} + \left\|\frac{d\varphi}{dt}\right\|_{L^p} + \sum_{i,j}\left\|\frac{\partial^2\varphi}{\partial x^i\partial x^j}\right\|_{L^p} + \sum_i\left\|\frac{\partial\varphi}{\partial x^i}\right\|_{L^p}$$

Similarly, one can define $W^{2k,p}(M \times [\alpha,\omega])$ using L^p-norm of derivatives $\partial_t^\beta\partial_x^\gamma\varphi$ of φ with $2\beta + \gamma \le 2k$.

We also want to be able to talk about $W^{k,p}$ when k is not an integer and not necessarily positive. This allows us to have a more flexible bootstrap scheme for nonlinear heat equation and to use Interpolation Theorem 1 more efficiently. We claim that these generalised Sobolev spaces (with weight and with non-integral regularity) can be defined on manifold and satisfy all the above properties (reflexivity, Interpolation theorem, Sobolev embedding and Kondrachov theorem) and refer to the appendices for all the details.

1.2 Trace theorem.

It is possible to avoid a discussion on Trace operator if we only want to make sense of the initial condition of nonlinear heat equation: one can consider only solutions with regularity greater than $W^{2,p}(M \times [\alpha,\omega])$ with $p \ge \dim M + 2$, which can be embedded in $C(M)$. It is however necessary to investigate regularity of Trace operator to have a complete proof of the bootstrap. We will review briefly some results.

The following two behaviors of trace are well-known:

1. If $-1+\frac{1}{p} < k < \frac{1}{p}$ then the natural map $W^{k,p}(M \times [\alpha, \omega]/\alpha) \hookrightarrow W^{k,p}(M \times [\alpha, \omega])$ is an isomorphism, where $W^{k,p}(M \times [\alpha, \omega]/\alpha)$ denotes the completion under $W^{k,p}$-norm of the space of smooth functions vanishing on a neighborhood of $M \times \{\alpha\}$. There is therefore no meaningful notion of trace in this case.

2. If $k > \frac{1}{p} + l,\ l \geq 0$, then the restriction map

$$B:\ C^\infty(M \times [\alpha, \omega]) \longrightarrow C^\infty(M):\ f(x,t) \longmapsto f(x, \alpha)$$

extends to a bounded operator $B:\ W^{k,p}(M \times [\alpha, \omega]) \longrightarrow W^{l,p}(M)$, called *Trace operator*.

We will topologise the space $\partial_\alpha W^{k,p}(M \times [\alpha, \omega])$ of restrictions to time $t = \alpha$ of functions in $W^{k,p}(M \times [\alpha, \omega])$, in case Trace operator is well defined, as cokernel of B, that is, as a quotient space of $W^{k,p}(M \times [\alpha, \omega])$. This makes $\partial_\alpha W^{k,p}(M \times [\alpha, \omega])$ a Banach space with stronger norm than any $W^{l,p}(M)$ for any $l < k - \frac{1}{p}$.

1.3 Linear equations on manifolds.

1.3.1 Existence and Regularity.

It can be easily verified that the linear heat operator $AF := \frac{d}{dt}F + \Delta F$ is a parabolic operator and therefore is also an elliptic operator. All of the following results holds for operator A.

Theorem 4 (Regularity for elliptic operator)**.** *Let M be a compact manifold and $AF := \frac{d}{dt}F + \Delta F$ be an elliptic operator of second order. Given $\frac{1}{p} < l < k < \infty$ and $F \in W^{l,p}(M \times [\alpha, \omega])$ and suppose that*

$$AF \in W^{k-2,p}(M \times [\alpha, \omega]), \quad f\big|_\alpha \in \partial_\alpha W^{k,p}(M \times [\alpha, \omega]), \quad f\big|_\omega \in \partial_\alpha W^{k,p}(M \times [\alpha, \omega]).$$

Then actually $F \in W^{k,p}(M \times [\alpha, \omega])$.

Theorem 5 (Causality of parabolic equation)**.** *Let M be a compact manifold and $AF := \frac{d}{dt}F + \Delta F + a\nabla F + bF$ be an parabolic operator. Then*

$$A:\ W^{k,p}(M \times [\alpha, \omega]/\alpha) \longrightarrow W^{k-2,p}(M \times [\alpha, \omega]/\alpha)$$

is an isomorphism of Banach spaces.

Theorem 6 (Gårding's Inequality and Regularity for parabolic operator)**.** *Let M be a compact manifold, $p \in (1, +\infty)$, $k > l > -\infty$ and $AF := \frac{d}{dt}F + \Delta F$ be a parabolic operator. We write $W^{k,p}([\beta, \gamma])$ shortly for $W^{k,p}(M \times [\beta, \gamma])$. Suppose that*

$$F \in W^{l,p}([\alpha, \omega]), \quad AF \in W^{k-2,p}([\alpha, \omega]).$$

Then $F \in W^{k,p}([\pi, \omega])$ for all $\pi \in (\alpha, \omega)$. Also, there exists a constant $C > 0$ such that

$$\|F\|_{W^{k,p}([\pi,\omega])} \leq C \left(\|AF\|_{W^{k-2,p}([\alpha,\omega])} + \|F\|_{W^{l,p}([\alpha,\pi])} \right).$$

In particular for homogeneous equation, the solution is C^∞ and an arbitrarily weak estimate in the past gives an arbitrarily strong estimate in the future.

1.3.2 Maximum principle and Comparison theorems.

Other than regularity results which are generally true for parabolic operators, the linear heat operator also enjoys the following versions of Maximum principle. See Appendices for their proofs.

Theorem 7 (Maximum principle)**.** *Let M be a compact manifold and $f : M \times [\alpha, \omega] \longrightarrow \mathbb{R}$ be a continuous function with $f\big|_\alpha \leq 0$. Suppose that whenever $f > 0$, f is smooth and*

$$\frac{\partial f}{\partial t} \leq -\Delta f + Cf.$$

Then in fact $f \leq 0$.

With the same proof as Theorem 7, one also has:

Theorem 8 (L^∞-Comparison theorem)**.** *Let $f : M \times [\alpha, \omega] \longrightarrow \mathbb{R}$ be a continuous function on M, smooth for all time $t > 0$ such that*

$$\frac{df}{dt} = -\Delta f + bf \text{ on } M \times (\alpha, \omega]$$

where b is a smooth function on M. Then there exists a constant B depending only on b such that

$$\|f\big|_\omega\|_{L^\infty} \leq e^{B(\omega-\alpha)} \|f\big|_\alpha\|_{L^\infty}.$$

Using backwards heat equation and Theorem 8, one can prove its version for L^1.

Theorem 9 (L^1-Comparison theorem)**.** *Let $f : M \times [\alpha, \omega] \longrightarrow \mathbb{R}$ be a continuous function on M, smooth for all time $t > 0$ such that*

$$\frac{df}{dt} = -\Delta f + bf \ \text{ on } M \times (\alpha, \omega]$$

where b is a smooth function on M. Then there exists a constant B depending only on b such that

$$\|f\big|_\omega\|_{L^1} \leq e^{B(\omega-\alpha)}\|f\big|_\alpha\|_{L^1}.$$

2 Regularity estimate of the quadratic term.

Theorem 10 (Regularity of the quadratic term)**.** *Let $F : M \times [\alpha, \omega] \longrightarrow B \subset \mathbb{R}^N$ be in $W^{s,q}(M \times [\alpha, \omega]) \cap C(M \times [\alpha, \omega])$ and*

$$PF := g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^\beta F_j^\gamma.$$

Suppose that

$$r \geq 0, \quad p, q \in (1, \infty), \quad r + 1 < s, \quad \frac{1}{p} > \frac{r+2}{s}\frac{1}{q}. \tag{1}$$

Then one has $PF \in W^{r,p}(X)$ and

$$\|PF\|_{W^{r,p}} \leq C\left(1 + \|F\|_{W^{s,q}}\right)^{q/p}.$$

where C is a constant independent of F.

Proof. We will suppose here that r, s are even integers so that the $W^{r,p}$ (respectively $W^{s,q}$) norm of PF (respectively F) can be written as sum of L^p (respectively L^q) norms of its derivatives. Also, we will use chain rule freely to differentiate the term $\Gamma'^{\alpha}_{\beta\gamma}(F)$ using weak derivatives of F. The general and rigorous proof, which involves non-integral Sobolev space to treat r, s and a detour to Besov spaces to justify chain rule, can be found in the appendices.

The derivatives of PF that appear in its $W^{r,p}$ norm are of form

$$C(x, F) \prod_i \partial_t^{b_i}\partial_x^{c_i}F^{\beta_i}$$

where $2\sum b_i + \sum c_i \leq r + 2$ and $\max\{2b_i + c_i\} \leq r + 1$ and $C(x, F)$ is bounded on M. Using Multiplication theorem for L^p-spaces, one has

$$\left\|C(x, F) \prod_i \partial_t^{b_i}\partial_x^{c_i}F^{\beta_i}\right\|_{L^p} \leq \|C(x, F)\|_{L^\infty} \prod_i \left\|\partial_t^{b_i}\partial_x^{c_i}F^{\beta_i}\right\|_{L^{p_i}} \leq \|C(x, F)\|_{L^\infty} \prod_i \|F\|_{W^{2b_i+c_i,p_i}}$$

as long as we choose $p_i \in (1,\infty)$ such that $\frac{1}{p} \geq \sum \frac{1}{p_i}$. The strategy is to choose $\frac{1}{p_i}$ big enough to have $W^{s,q} \hookrightarrow W^{2b_i+c_i,p_i}$ in order to bound $\|F\|_{W^{2b_i+c_i,p_i}}$ by $\|F\|_{W^{s,q}}$, then use the upper bound of $2b_i+c_i$ to justify that $\frac{1}{p} > \frac{r+2}{s}\frac{1}{q} \geq \sum \frac{1}{p_i}$, meaning that such choice of p_i are valid.

The straightforward way to have a sufficient condition of p_i such that $W^{s,q} \hookrightarrow W^{2b_i+c_i,p_i}$ is to use Sobolev embeddings but the result is sub-optimal because Sobolev embeddings do not take into account the L^∞-boundedness of F (its image lies in a compact of $\mathbb{R}^N$). A better way is to use Interpolation inequality, by remarking that $F \in W^{0,v}$ for all $v \in (1,+\infty)$ and writing $W^{2b_i+c_i,p_i}$ as an interpolation space of $W^{s,q}$ and $W^{0,v}$. It can be seen, by direct computation, that the sufficient condition for $W^{s,q} \hookrightarrow W^{2b_i+c_i,p_i}$ is $2b_i+c_i < s$ and

$$0 < \frac{1}{p_i} - \frac{2b_i+c_i}{s}\frac{1}{q} < 1 - \frac{2b_i+c_i}{s}.$$

Choose $\frac{1}{p_i}$ just a bit bigger than $\frac{2b_i+c_i}{s}\frac{1}{q}$, one still has

$$\sum \frac{1}{p_i} \simeq \sum \frac{2b_i+c_i}{s}\frac{1}{q} \leq \frac{r+2}{s}\frac{1}{q} < \frac{1}{p}.$$

The conclusion follows. $\square$

3 Regularity for nonlinear heat equation.

Let $p > \dim M + 2$, using the regularity estimate for the quadratic term, we now can prove:

Theorem 11 (Bootstrap for nonlinear heat equation)**.** *Let $F:\ M\times[\alpha,\omega] \longrightarrow B$ such that $F \in W^{2,p}(M\times[\alpha,\omega])$ and $\frac{dF_t}{dt} = \tau(F_t)$, i.e.*

$$\frac{dF^\alpha}{dt} = -\Delta F^\alpha + g^{ij}\Gamma'^\alpha_{\beta\gamma}(F)F^\beta_i F^\gamma_j$$

and $F\big|_{M\times\{\alpha\}}$ is smooth. Then F is smooth on $M\times[\alpha,\omega]$.

Remark 1. *Note that since $p > \dim M + 2 = \dim(M\times[\alpha,\omega]) + 1$, if $F \in W^{2,p}(M\times[\alpha,\omega])$ then F and $\frac{\partial F}{\partial x^i}$ are in $C(M\times[\alpha,\omega])$ by Sobolev embeddings. It makes sense then to talk about:*

1. *the restriction and boundary condition at time $t=\alpha$ (in fact, by Trace theorem, $p>1$ is enough).*
2. *the pointwise condition $F: M\times[\alpha,\omega] \longrightarrow B \subset V$.*

Proof. We define the operators $PF := g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^\beta F_j^\gamma$ and $AF := \frac{dF}{dt} + \Delta F$. We will abusively denote $W^{k,p}(M \times [\beta,\gamma])$ by $W^{k,p}([\beta,\gamma])$. Our bootstrap scheme consists of 3 steps:

1. Prove that $F \in W^{2,\tilde{p}}([\pi,\omega])$ for every $\pi > \alpha$ and $\tilde{p} \in (1,\infty)$. By compactness of M, it is sufficient to prove this for a sequence $\tilde{p} \to +\infty$.

2. Prove that F is C^∞ for all time $t > \alpha$.

3. Prove that F is C^∞ on $M \times [\alpha,\omega]$.

Step 1. By Theorem 10, $AF = PF \in W^{r,q}([\alpha,\omega])$ whenever $r < 1$ and $\frac{1}{q} > (\frac{r}{2}+1)\frac{1}{p}$. Apply Gårding inequality, for all $\pi > \alpha$, $F \in W^{r+2,q}([\pi,\omega]) \subset W^{2,\tilde{p}}([\pi,\omega])$ for $\frac{1}{\tilde{p}} = \frac{1}{q} - \frac{r}{\dim M+1}$. Choose $\frac{1}{q}$ very close to $(\frac{r}{2}+1)\frac{1}{p}$, one sees that the condition on $\tilde{p}$ is $\frac{1}{\tilde{p}} > (\frac{r}{2}+1)\frac{1}{p} - \frac{r}{p-1}$, which will be satisfied if $\frac{1}{\tilde{p}} > (1-\frac{r}{2})\frac{1}{p}$, i.e. for all $\tilde{p} < \frac{p}{1-r/2}$. It remains to repeat this result to finish the first step. We will say $F \in W^{2,*}([\pi,\omega])$ for $F \in W^{2,p}([\pi,\omega])$ for all $p \in (1,\infty)$.

Step 2. By Theorem 10, for all $r < 1$, one has $AF = PF \in W^{r,*}([\pi,\omega])$, therefore by Gårding inequality, $F \in W^{r+2,*}([\pi,\omega])$. Iterate this result and one has $F \in W^{k,*}([\pi,\omega])$ for all $k \in [2,\infty)$ and $\pi > \alpha$. So F is smooth for $t > \alpha$.

Step 3. We apply regularity result (Theorem 4) for elliptic operator A and boundary operators $B^0 : F \mapsto F\big|_{M\times\{\alpha\}}$ and $B^1 : F \mapsto F\big|_{M\times\{\omega\}}$. For q, r in Step 1, one has $AF = PF \in W^{r,q}([\alpha,\omega])$ and $B^jF \in \partial W^{r,q}$, therefore $F \in W^{r+2,q}([\alpha,\omega]) \subset W^{2,\tilde{p}}([\alpha,\omega])$ for the same $\tilde{p}$ as Step 1. This proves that $F \in W^{2,*}([\alpha,\omega])$, which also means that one has $F \in W^{r+2,q}([\alpha,\omega])$ with no additional condition on q except $q \in (1,\infty)$. Iterate and one obtains the regularity of F on $[\alpha,\omega]$. □

Remark 2. *The first 2 steps were to prove the regularity of $F\big|_{M\times\{\omega\}}$, which was then used as a boundary condition in order to apply regularity result for elliptic operator on manifold with boundary.*

4 Short-time existence for nonlinear heat equation.

We will choose as always $p > \dim M + 2$. As before, M is a compact Riemannian manifold and $B \subset \mathbb{R}^N$ is a large Euclidean ball.

Theorem 12 (Short-time existence)**.** *Let $F_\alpha : M \longrightarrow B$ be a smooth map, then there exist $\epsilon > 0$ depending on F_α and $F : M \times [\alpha, \alpha+\epsilon] \longrightarrow B$ such that $F \in W^{2,p}(M \times [\alpha, \alpha+\epsilon])$ with $F\big|_{M\times\{\alpha\}} = F_\alpha$ and*

$$\frac{dF_t}{dt} = \tau(F_t) \quad \text{on } M \times [\alpha, \alpha+\epsilon]$$

Proof. We find F as a sum $F = F_b + F_\#$ where $F_b \in C^\infty(M \times [\alpha, \omega])$ satisfies the initial condition and $F_\# \in W^{2,p}(M \times [\alpha, \alpha+\epsilon]/\alpha)$.

The nonlinear heat operator can be written as:

$$\begin{aligned} T : W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N} &\longrightarrow L^p(M \times [\alpha, \omega])^{\oplus N} \\ F_\# &\longmapsto \tau(F_b + F_\#) \end{aligned}$$

where $\tau(F)^\alpha = -\Delta F^\alpha + g^{ij}\Gamma'^\alpha_{\beta\gamma}(F)F_i^\beta F_j^\gamma$, which can be rewritten as $\tau(F) = -\Delta F + \Gamma(F)(\nabla F)^2$. The derivative of T at $F_\#$ in direction $k \in W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N}$ is

$$DT(F_\#)k = -\Delta k + D\Gamma(F)\cdot k.(\nabla F)^2 + 2\Gamma(F)\nabla F.\nabla k,$$

or in local coordinates:

$$DT(F_\#)^\alpha = g^{ij}\left(\frac{\partial^2 k^\alpha}{\partial x^i \partial x^j} - \Gamma^l_{ij}k^\alpha_l\right) + g^{ij}\frac{\partial \Gamma'^\alpha_{\beta\gamma}}{\partial y^\delta}k^\delta F_i^\beta F_j^\gamma + 2g^{ij}\Gamma'^\alpha_{\beta\gamma}(F)F_i^\beta F_j^\gamma$$

which is of form $DT(F_\#)k = -\Delta k - a(x,F)\nabla k - b(x,F)k$ where a, b are smooth.

Therefore if we note

$$\begin{aligned} H : W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N} &\longrightarrow L^p(M \times [\alpha, \omega])^{\oplus N} \\ F_\# &\longmapsto (\frac{d}{dt} - \tau)(F_b + F_\#) \end{aligned}$$

then the derivative of H at $F_\# = 0$ is

$$DH(0)\cdot k = \frac{dk}{dt} + \Delta k + a(x, F_b)\nabla k + b(x, F_b)k$$

which by Theorem 5 is an isomorphism from $W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N}$ to $W^{0,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N} = L^p(M \times [\alpha, \omega])^{\oplus N}$. This shows that H is a local isomorphism mapping a neighborhood of 0 to a neighborhood of $(\frac{d}{dt} - \tau)F_b$.

Define $g_\epsilon \in L^p(M \times [\alpha, \omega])^{\oplus N}$ by

$$g_\epsilon := \begin{cases} 0, & \text{if } t \in [\alpha, \alpha+\epsilon] \\ (\frac{d}{dt} - \tau)F_b, & \text{if } t > \alpha+\epsilon \end{cases}$$

which is arbitrarily $L^p(M \times [\alpha, \omega])$-close to $(\frac{d}{dt} - \tau)F_b$ for $0 < \epsilon \ll 1$. There exists therefore $F_\# \in W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N}$ such that $H(F_\#) = g_\epsilon$, meaning that the function $F = F_b + F_\# : M \longrightarrow V$ satisfies $F\big|_{M \times \{\alpha\}} = F_\alpha$ and $\frac{dF}{dt} - \tau(F_t) = 0$ for $t \in [\alpha, \alpha + \epsilon]$.

By Regularity Theorem 11, F is C^∞ for $t \in [\alpha, \alpha + \epsilon]$. Theorem **??** assures that the image of F is in M, hence in M' for $t \in [\alpha, \alpha + \epsilon]$. $\square$