Divisors, Picard group and Kodaira embedding theorem

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1 Divisors and Picard group

1.1 Holomorphic line bundles and first Chern class

A complex line bundle is a 2 dimensional vector bundle with a complex structure on each fiber, i.e. each change of coordinates $g_{ij} : U_j \cap U_i \times \mathbb{R}^2 \longrightarrow U_i \cap U_j \times \mathbb{R}^2$ is i-linear, i.e. g_{ij} can be represented by a function $U_i \cap U_j \longrightarrow \mathbb{C}$.

A holomorphic line bundle is a complex line bundle that is also a complex manifold with the projection being holomorphic. In the same notation, the g_{ij} are now holomorphic functions.

A hermitian metric on a line bundle L is a positive sesquilinear form on each fiber. To define the Chern form of L, let U be an open set of X over which L is trivialized and s_x is a holomorphic section of L over U that is non-vanishing, then one defines

$$\omega_{L,h} = \frac{1}{2\pi i}\partial\bar{\partial}\log|s|_h^2$$

which is independant of s since the ratio of two different s is in $\mathcal{O}^*(U)$.

Remark 1. *A Chern form is a real (1,1) form.*

Proposition 2. *The set of isomorphic class of holomorphic line bundle is in one-to-one correspondance to $H^1(X, \mathcal{O}_X^*)$*

The proof of this fact is straightforward, but it is worth to remark that this result convinces us that the natural mapping $\check{H}^1(\mathcal{U}, X) \longrightarrow \check{H}^1(\mathcal{V}, X)$ where $\mathcal{U}$ is a finer open covering than $\mathcal{V}$ is injective, since a line bundle is completely defined by *one* set of change of coordinates (g_{ij}).

Now the Chern class of a holomorphic line bundle is the class of $\omega_{L,h}$ in $H^2(X, \mathbb{Z})$, which turns out to be independent of h and is in fact lies in the image of $H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}) = H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. In fact, the class of $\omega_{L,h}$ can be defined using the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

where the injective arrow is the multiplication by $i2\pi$ and the surjective one is exponential. The Chern map is in fact $H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z})$. To prove this, one uses a double complex whose horizontal is the de Rham resolution and vertical is the Čech resolution and diagram chasing.

1.2 Divisors, line bundles and sheaves

Remark 3.
1. *A holomorphic line bundle is the same as a locally free $\mathcal{O}_X$-module of rank 1.*
2. *An isomorphic class of line bundles is the same as a locally free isomorphic sheaf of $\mathcal{O}_X$-module of rank 1.*

1.2.1 From divisors to Picard group

A divisor is a formal sum of irreducible hypersurface, which can also be intepreted as an element of $\check{H}^0(X, K_X^*/\mathcal{O}_X^*)$, which gives a mapping $Div(X) \longrightarrow \check{H}^0(X, K_X^*/\mathcal{O}_X^*)$ with principal divisors being exactly sent to elements of $K_X^*/\mathcal{O}_X^*$ coming from K_X^*. multiplicative).

Since the following sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow K_X^* \longrightarrow K_X^*/O_X^* \longrightarrow 0$$

is exact, one has an application $\mathcal{O}: Div(X) = H^0(X, K_X^*/\mathcal{O}_X^*) \longrightarrow H^1(X, \mathcal{O}_X^*) = Pic(X)$. The kernel of $\mathcal{O}$ corresponds to the the space of principal divisors. It is however worth having details of the application $\mathcal{O}$.

Let $D = (U_i, f_i) \in Div(X)$ where f_i are meromorphic function on U_i with $f_i/f_j \in \mathcal{O}_X^*$, then $\mathcal{O}(D)$ is defined as following:

$$\mathcal{O}(D)(U_i) = f_i^{-1}\mathcal{O}_X(U_i)$$

. Note that if D is effective, i.e. $f_i \in \mathcal{O}_X(U_i)$ then $\mathcal{O}(\mathcal{D})$ is the sheaf of holomorphic functions vanishing on D.

Remark 4. *To resume, here are some basic consequence of the above discussion: If D is effective then*

1. *If D is effective then* $H^0(X, \mathcal{O}(D) \neq 0$.
2. *If D is effective then* $\mathcal{O}(-D)$ *is the sheaf of holomorphic function vanishing on D. Therefore* $\mathcal{O}(-D)$ *can be viewed as a ideal subsheaf of* K_X *and one has the following exact sequence:*

   $$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_\mathcal{D} \longrightarrow 0$$

   where $\mathcal{O}_D$ *is the sheaf of "regular functions" on D.*
3. *If L is a holomorphic line bundle and* $0 \neq s \in H^0(X, L)$ *then* $\mathcal{O}(Z(s)) \equiv L$
4. *If D is effective then* $\mathcal{O}(D)$ *has a non-zero global section, for example section* $1 = (U_i, f_i)$

1.2.2 The corresponding line bundle of $\mathcal{O}(Y)$

Proposition 5. *Let Y be a hypersurface of X, then the line bundle* $\mathcal{O}(Y)$ *is isomorphic to* $\mathcal{N}_{Y,X}$ *the normal line bundle of Y in X.*

By consequence, $K_Y = (K_X \otimes \mathcal{O}(Y))\big|_Y$.

2 Example: Projective space

2.1 $\mathcal{O}(d)$ and its sections

Let's have some examples for the point of view discussed above, starting with the torsion sheaves $\mathcal{O}(d)$.

The sheaf $\mathcal{O}(-1)$, called tautological sheaf, is an invertible sheaf on $\mathbb{P}_\mathbb{C}^n$ such that the fiber over $l \in \mathbb{P}_\mathbb{C}^n$ of the corresponding line bundle is l itself. Let $l = [x_0 : \cdots : x_n]\ inU_i$

then a point in l is of form $t_i[\frac{x_0}{x_i} :, \cdots : \frac{x_n}{x_i}]$ with coordinates in U_i being t_i. So the change of coordinates from chart U_i to U_j is, since $\frac{t_j}{x_j} = \frac{t_i}{x_i}$:

$$g_{ji} = \frac{x_i}{x_j}$$

One notes by $\mathcal{O}(1)$ the dual of $\mathcal{O}(-1)$ and $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$ and $\mathcal{O}(-d) = \mathcal{O}(-1)^{\otimes d}$

Now if an invertible sheaf $\mathcal{L}$ with $\mathcal{L}(U_i) = \frac{1}{f_i}\mathcal{O}_X(U_i)$, the change of coordinates of the corresponding line bundle from chart U_i to U_j is $g_{ji} = \frac{f_j}{f_i}$. So for $\mathcal{O}(1)$, one has $\frac{f_j}{f_i} = \frac{x_j}{x_i}$, i.e. there exists a linear combination A of $x_0, \dots, x_n$ such that $f_i = \frac{A}{x_i}$ is a holomorphic function correponding to the 1-section viewed in chart U_i. As presented in the previous section, $\mathcal{O}(1)$ **is the associated line bundle of a hyperplane defined by the equation** $A = 0$.

Similarly, $\mathcal{O}(d)$ is the associated line bundle of a hypersurface A_d defined by a homogenous equation of degree d, and $\mathcal{O}(d)$ is the line bundle associated to the sheaf of holomorphic functions vanishing on A_d.

2.2 Line bundles and maps to projective space, Veronese embedding

A linearly independent family $s_0, \dots, s_N$ of global sections of a holomorphic line bundle L defines a holomorphic map $X \setminus Bs((s_i)) \longrightarrow \mathbb{P}^N_{\mathbb{C}}$ where $Bs((s_i))$ is the set of basepoints of (s_i) where all the sections s_i vanish.

The global sections of $\mathcal{O}(d)$ over $\mathbb{P}^n_{\mathbb{C}}$ are $H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d) = \mathbb{C}[z_0, \dots, z_n]_d$ the vector space of homogenous polynomial of degree d. The corresponding projective map is in fact a embedding, called Veronese embedding.

2.3 Canonical bundle and Euler sequence

$$K_{\mathbb{P}^n_{\mathbb{C}}=\mathcal{O}(-n-1)}$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus n} \longrightarrow \mathcal{T}_{\mathbb{P}^n_{\mathbb{C}}} \longrightarrow 0$$

3 Blowing-up

3.1 Blowing-up

Proposition 6 (Adjunction). *Let X be a complex manifold, $x \in X$ and $\pi : \hat{X} \longrightarrow X$ is the blow-up of X at x and E be the corresponding exceptional divisor, then*

$$K_{\hat{X}} = \pi^* K_X \otimes \mathcal{O}((n-1)E).$$

As a consequence, $\mathcal{O}(E)\big|_E = \mathcal{O}(-1)$ where $\mathcal{O}(-1)$ is the tautologic sheaf over $E = \mathbb{P}^{n-1}_{\mathbb{C}}$.

Proof. The appearance of the number $n-1$ is natural and can be explained as follow. First note that

1. in $\hat{X} \setminus E$, there is no difference between $K_{\hat{X}}$ and K_X.
2. π_* send every tangent vector of E to the tangent vector 0 at $x \in X$,
3. the pull-back $\pi^*\omega$ of an n-form on X always vanishes on E therefore cannot generate $K_{\hat{X}}$.

Our correction of this should be dividing $\pi^*\omega$ by f^k where f is the equation defining E in $\hat{X}$, i.e. tensoring π^*K_X by an appropriate multiple of $\mathcal{O}(E)$ depending on the order of vanishing of $\pi^*\omega$ at E, which we claim to be $n-1$.

Here is the argument I used to convince myself: this vanishing order is that of the ratio of $\pi^*\omega$ and a non-zero n-form (says the standard in the base formed by $n-1$ tangent vectors e_i^E of E and the normal vector v of E in X), i.e. the vanishing order of $\pi^*\omega(e_i^E, v)$. Each e_i^E plugged into $\pi^*\omega$ adds one order of vanishing resulting in $n-1$.

Here is the argument I would use to convince others: WLOG, suppose that $X = \mathbb{C}^n$ and $x = 0$, then $\hat{X}$ can be seen as a subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$ with the coordinates in each chart $U_i = \{(x_1, \ldots, x_n, [p_1 : \cdots : p_n]) : \ p_i \neq 0\}$ being $(\frac{p_1}{p_i}, \ldots, \frac{p_n}{p_i}, \zeta_i)$ with $z_k = \frac{p_k}{p_i}\zeta_i$. The map π is given in local coordinates as

$$(\frac{p_1}{p_i}, \ldots, \frac{p_n}{p_i}, \zeta_i) \mapsto (\frac{p_1}{p_i}\zeta_i, \ldots \zeta_i, \ldots \frac{p_n}{p_i}\zeta_i)$$

The pull-back of ω is

$$d(\frac{p_1}{p_i}\zeta_i) \wedge \cdots \wedge d\zeta_i \wedge \cdots \wedge d(\frac{p_n}{p_i}\zeta_i) = \zeta_i^{n-1} d(\frac{p_1}{p_i}) \wedge \cdots \wedge d\zeta_i \wedge \cdots \wedge d(\frac{p_n}{p_i})$$

which vanishes with order $n-1$.

For the consequence, note that $\mathcal{K}_E = \mathcal{O}(-n) = (K_{\hat{X}} \otimes \mathcal{O}(E))\big|_E = (\pi^*K_X \otimes \mathcal{O}(nE))\big|_E$, but π^*K_X is trivial over E, therefore $\mathcal{O}(E)\big|_E = \mathcal{O}(-1)$. □

4 Kodaira vanishing theorem

Theorem 7 (Kodaira vanishing)**.** *Let X be a compact complex manifold of dimension n and L is a positive holomorphic line bundle on X, i.e. there exists a hermitian metric h on L such that the Chern form $\omega_{L,h}$ is positive (i.e. a Kahler form). Then*

$$H^q(X, \Omega_X^p \otimes \mathcal{L}) = 0 \quad \forall p+q > n.$$

In particular,

$$H^i(X, K_X \otimes \mathcal{L}) = 0 \quad \forall i > 0.$$

Proof. Since $\mathcal{H}^{0,q}(E) \simeq H^q(X, \mathcal{E})$ for all hermitian holomorphic vector bundle E with $\mathcal{E}$ the corresponding sheaf of holomorphic sections. One needs to prove that all harmonic form in $\mathcal{A}_X^{p,q}(L)$ vanishes for $p + q > n$. This comes from the following two identities: Let ∇ be the Chern connection on L and $\nabla = \nabla' + \nabla''$ be its decomposition to $(1,0)$ and $(0,1)$ operators and Δ_L' be the Laplacian corresponding to ∇' then

1. $\Delta_L = \Delta_L' + 2\pi[L, \Lambda]$ where L and Λ are Lefshetz operators.
2. $[L, \Lambda] = (k - n)Id_{\mathcal{A}_X^k}$

Therefore $0 \leq (\alpha, \Delta' a)_{L^2} = 2\pi(n-k)(\alpha, \alpha)_{L^2} \leq 0$ □

5 The traditional proof of Kodaira embedding theorem

Theorem 8 (Kodaira embedding)**.** *Let X be a compact complex manifold with L a positive holomorphic line bundle on X. Then L is generated by finitely many of its global sections and X can be embedded in a projective space* $\mathbb{CP}^N$ *with N sufficiently large.*

Proof. The following approache is straight-forward: one shows that at every $x \in X$, there is a global (holomorphic) section s_x of $L^{\otimes m_x}$ such that $s_x(x) \neq 0$ then by compactness one can choose finitely many such sections and m_x which can be guaranteed to generate every germs at x of $L^{\otimes m}$ with $m = \max m_x$. That is one needs to prove that

$$H^0(X, \mathcal{L}^{\otimes m_x}) \twoheadrightarrow H^0(x, \mathcal{L}^{\otimes m_x}\big|_x)$$

is surjective. Let $\pi : \hat{X} \longrightarrow X$ be the blow-up of X at x and $E = \pi^{-1}(x)$ be the corresponding exceptional divisor then one has

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\begin{tikzcd}
  H^0(X,\mathcal{L}^{\otimes m_x}) \ar[r, twoheadrightarrow] \ar[d] & H^0({x}, \restr{\math
  H^0(\hat X,\pi^*\mathcal{L}^{\otimes m_x}) \ar[r, twoheadrightarrow] & H^0(E,\restr{\pi^*
\end{tikzcd}
\label{fig:kodaira-blowup}
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$$\begin{tikzcd}
H^0(X, \mathcal{L}^{\otimes m_x}) \arrow[r, twoheadrightarrow] \arrow[d] & H^0(x, \mathcal{L}^{\otimes m_x} x) \arrow[d] \\
H^0(\hat{X}, \pi^*\mathcal{L}^{\otimes m_x}) \arrow[r, twoheadrightarrow] & H^0(E, \pi^*\mathcal{L}^{\otimes m_x} E)
\end{tikzcd}$$

Figure 1: Insert caption [fig:kodaira-blowup]

where the vertical arrows are isomorphic (easy to see, maybe the right one needs Hartog). So one only needs to see that $H^0(\hat{X}, \pi^*\mathcal{L}^{\otimes m_x}) \twoheadrightarrow H^0(E, \pi^*\mathcal{L}^{\otimes m_x}\big|_E)$. Since E is a divisor of $\hat{X}$, one has \$ $0 \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{O}_{\hat{}X} \longrightarrow \mathcal{O}_E \longrightarrow 0$\$$hence 0 \longrightarrow \mathcal{O}(E) \otimes \pi^*\mathcal{L}^{\otimes m_x} \longrightarrow \pi^*\mathcal{L}^{\otimes m_x} \longrightarrow \pi^*\mathcal{L}^{\otimes m_x}\big|_E \longrightarrow 0$.

It remains to prove that $H^1(\hat{X}, \mathcal{O}(E) \otimes \pi^* \mathcal{L}^{\otimes m_x})$, or by 7, that $\mathcal{O}(E) \otimes \pi^* \mathcal{L}^{\otimes m_x} \otimes K_{\hat{X}}^{-1} = \mathcal{O}(-nE) \otimes \pi^*(\mathcal{L}^{\otimes m_x} \otimes K_X^{-1})$ is positive, where we used the fact that $K_{\hat{X}} = \pi^* K_X \otimes \mathcal{O}_{\hat{X}}((n-1)E)$.

Note that on $\hat{X} \setminus E$, one can choose m_x large enough such that $\mathcal{L}^{\otimes m_x} \otimes K_X^{-1} \times \mathcal{O}(-nE)$ is positive. It remains to observe $E \subset \hat{X}$ which is in fact $\mathbb{CP}^{n-1}$. But $\mathcal{O}(-E)\big|_E \equiv \mathcal{O}(1)$ is positive, which concludes the proof. $\square$

6 An analytic proof by Donaldson