# Moser's Isotopy method and Darboux theorem

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## 1 Symplectic geometry does not exist

We will prove a symplectic manifold, i.e. a smooth manifold equiped with a closed everywhere non-degenerate 2-form, <u>does not have local invariant</u>. This is a significant difference between symplectic manifold and riemannian manifold, whose local invariance is the <u>curvature</u>.

To do this, one uses a trick of Moser which in the compact case show that isotopic symplectic structures $\omega_0$ and $\omega_1$ are strongly isotopic, i.e. $\psi_t^* \omega_t = \omega_0$

## 2 Isotopy method

### 2.1 Moser's trick

Let $M$ be a closed manifold (compact, without boundary) and $\omega_t$ is a family of symplectic structures on $M$ such that

$$\frac{d}{dt}\omega_t = d\sigma_t$$

then there exists diffeomorphism $\psi_t$ of $M$ such that $\psi_t^* \omega_t = \omega_0$

**Construction of $\psi_t$.** One constructs $\psi_t$ by its flow $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ such that

$$0 = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) = \psi_t^*\left(d\sigma_t + X_t\neg d\omega_t + d(X_t\neg\omega_t)\right)$$

Since $\omega_t$ are closed and non-degenerate, it suffits to choose $X_t$, which exists uniquely, such that $X_t\neg\omega = \sigma_t$.

## 2.2 Application: Darboux theorem and Moser Stability

Using this trick, we can prove the following theorem of Darboux.

**Lemma 1.** *Let $M$ be a closed manifold with symplectic structures $\omega_0$ and $\omega_1$ such that they coincide on a fiber $T_qM$. Then there exists neighborhoods $\mathcal{N}_0, \mathcal{N}_1$ of $q$ and a diffeomorphism $\psi : \mathcal{N}_0 \longrightarrow \mathcal{N}_1$ such that $\psi^*\omega_1 = \omega_0$.*

*Proof.* We remark that it is enough to prove that there exists $\sigma$ locally defined near $q$ with $\omega_1 - \omega_0 = d\sigma$ where $\sigma = 0$ on $T_qM$. In fact, let $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ one then has a neighborhood $\mathcal{N}_0$ of $q$ such that $\omega_t$ are non-degenerate and the field $X_t$ constructed by Moser's technique ($X_t = 0$ at $q$) has its flow well-defined at time $t = 1$ when starting at $\mathcal{N}_0$. Then $\psi_1$ and $\mathcal{N}_1$ is what we want.

One then uses <u>another trick</u> to construct $\sigma$: Take any Riemannian metric on $M$ and let $\phi_t$ be constructed using the geodesic flow and retricting $\mathcal{N}_0$ to a geodesic ball such that $\phi_0|_{\mathcal{N}_0} = q$, $\phi_1 = Id$ and $d\phi_t(q) = Id_{T_qM}$. Then

$$\omega_1 - \omega_0 = \int_0^1 \frac{d}{dt}\phi_t^*(\omega_1 - \omega_0)dt = \int_0^1 \phi_t^* d(Y_t\neg(\omega_1 - \omega_0)) = d\left(\int_0^1 dt\,\phi_t^*(Y_t\neg(\omega_1 - \omega_0))\right).$$

It is straight-forward to see that the $\sigma$ constructed this way works. $\square$

The theorem of Darboux follows easily from Lemma 1.

**Theorem 2** (Darboux). *Every two symplectic form $\omega_0, \omega_1$ on a closed manifold $M$ are locally isomorphic.*