

Local results of several complex variables

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	$K(X) \xrightarrow{ch} H(X; \mathbb{Q});$	
	$\downarrow \qquad \qquad \downarrow$	
	$K(Y) \xrightarrow{ch} H(Y; \mathbb{Q});$	

1 de Rham currents

Let M be a differential m -dimensional manifold and $\mathcal{E}^p(M)$ be the vector space of smooth p -forms on M and $\mathcal{D}^p(M)$ be the space of those with compact support. Then $\mathcal{E}^p(M), \mathcal{D}^p(M)$ is a topological vector space with the pseudonorms $p_{K,\Omega}^s(\omega) = \max_{K, |\alpha| \leq s} |D^\alpha u_I|$ where $K \Subset \Omega$ an coordinated open set. The space of de Rham current with dimension p / degree $m - p$ is defined as the dual space of $\mathcal{D}^p(M)$, denoted by $\mathcal{D}'^{m-p}(M)$ or $D'_p(M)$

Remark 1. 1. We are still in \mathbb{R} , but the definition expands to the complex case, denoted by $\mathcal{D}'^{m-p,m-q}(M) = \mathcal{D}'_{p,q}(M)$ where m is the complex dimension of M .

2. The degree is defined such that the current $T_\omega : \eta \mapsto \int_M \omega \wedge \eta$ is of the same degree as ω . The dimension is defined so that the current $T_{[Z]} : \eta \mapsto \int_Z \eta$ is of the same dimension as Z .

Definition 1. One has the following operation on $\mathcal{D}'^{m-p}(M)$:

1. **Derivative:** $\langle dT, \omega \rangle = (-1)^{\deg T} \langle T, d\omega \rangle$
2. **Wedge product with a form:** $\langle T \wedge \eta, \omega \rangle = \langle T, \eta \wedge \omega \rangle$
3. **Pushforward:** If $F : X \rightarrow Y$ proper on $\text{supp } T$ then $\langle F_*T, \omega \rangle = \langle T, F^*\omega \rangle = \langle T, \chi F_*\omega \rangle$ where $\chi \in C^\infty(M)$ identically 1 on $\text{supp } T$. The proper condition is such that the pullback of ω is compactly support in $\text{supp } T$
4. **Pullback:** Let $F : X \rightarrow Y$ submersion then the pushforward of a form on X is well-defined by Fubini. One has $\langle F^*T, \omega \rangle = \langle T, F_*\omega \rangle$

Remark 2. 1. The sign of derivative is chosen so that $dT_\omega = T_{d\omega}$.

2. Pushforward keeps the dimension, as the arguments are of the same degree.
3. Pullback keeps the codimension, meaning the degree (think $F^*T_{[Z]} = T_{[F^{-1}(Z)]}$).
4. Locally a current is of form $T = \sum u_I dx^I$ where u_I are distribution. **Note:** Here distribution are indentified as a current of maximal degree and not zero degree as they naturally are. To be exact, the notation of u_I is contravariant and its action is $\varphi dx^1 \wedge \dots \wedge dx^N \mapsto \langle u_I, \varphi \rangle dx^1 \wedge \dots \wedge dx^n / \text{vol}$ where vol is a canonical volume form.

The last two remarks explain the sign in the following proposition.

Proposition 0.1 (Pushforward and Pullback). Let $F : M_1 \rightarrow M_2$, submersion if needed, then

1. $\text{supp } F_*T \subset F(\text{supp } T)$
2. $d(F_*T) = F_*dT$ (pushforward of a form is still that form)
3. $F_*(T \wedge F^*g) = (F_*T) \wedge g$

and

1. $F^*(dT) = (-1)^{m_1-m_2} d(F^*T)$
2. $F^*(T \wedge g) = (-1)^{m_1-m_2-\deg g} (F^*T) \wedge F^*g$

2 Subharmonic and Plurisubharmonic functions

Some properties of holomorphic functions that remain in several variables.

- Cauchy formula
- Analyticity: series development. Therefore its zeroes never form an open set (except for constant)
- Maximum modulus
- Cauchy inequality and Montel's theorem

2.1 Subharmonic functions

We are now in the context of \mathbb{R}^n .

Theorem 1 (Green kernel). *Let $\Omega \Subset \mathbb{R}^n$ be a smoothly bounded domain, then there exists uniquely a function $G_\Omega : \bar{\Omega} \times \bar{\Omega} \rightarrow [-\infty, 0]$, called the Green kernel of Ω , with the following properties:*

1. *Regular: G_Ω is C^∞ on $\bar{\Omega} \times \bar{\Omega} \setminus \Delta_\Omega$ where Δ_Ω denotes the diagonal,*
2. *Symetric: $G_\Omega(x, y) = G_\Omega(y, x)$,*
3. *Negative: $G_\Omega(x, y) < 0$ on $\Omega \times \Omega$ and $G_\Omega(x, y) = 0$ on $\partial\Omega \times \Omega$,*
4. *$\Delta_x G_\Omega(x, y) = \delta_y$ on Ω for every $y \in \Omega$.*

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Example 1 (case $\Omega = B(0, r)$). *One can take $G_r = N(x - y) - N(\frac{|y|}{r}(x - \frac{r^2}{|y|^2}y))$ where N is the Newton kernel (or Newtonian potential, the gravitational potential). Explicitly, one has*

$$G_r(x, y) = \frac{1}{4\pi} \log \frac{|x - y|^2}{r^2 - 2\langle x, y \rangle + \frac{1}{r^2}|x|^2|y|^2} \quad \text{if } n = 2 \quad (1)$$

$$G_r(x, y) = \frac{-1}{(m-2)\text{vol}(S^{m-1})} (|x - y|^{2-m} - (r^2 - 2\langle x, y \rangle + \frac{1}{r^2}|x|^2|y|^2)^{1-m/2}) \quad \text{if } n \geq 3 \quad (2)$$

Proposition 1.1 (Green-Riesz representation). *For $u \in C^2(\bar{\Omega}, \mathbb{R})$ one has*

$$u(x) = \int_{\Omega} G_{\Omega}(x, y) \Delta u(y) d\lambda(y) + \int_{\partial\Omega} u(y) \frac{\partial G_{\Omega}}{\partial \nu_y} d\sigma(y)$$

In particular, for $\Omega = B(0, r)$, one has

$$P_r(x, y) := \frac{\partial G}{\partial \nu_y} = \frac{1}{\text{vol}(S^{m-1})r} \frac{r^2 - |x|^2}{|x - y|^m}$$

called the Poisson kernel.

Proof. Use the Green-Riesz formula: $\int_{\Omega} u \Delta v - v \Delta u = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}$. \square

Definition 2. *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $u : \Omega \rightarrow [-\infty, \infty)$ a upper semi-continuous function:*

$$\limsup_{x \rightarrow x_0} u(x) \leq u(x_0)$$

One notes by $\mu_S(u, a, r)$ and $\mu_B(u, a, r)$ the average of u in the sphere and the disk centered in a of radius r . Then the following properties are equivalent and a function is called subharmonic if they are verified.

- 1) $u(x) \leq P_{a,r}[u](x) \quad \forall a, r, x \in B(a, r) \subset \Omega,$
- 2) $u(a) \leq \mu_S(u, a, r) \quad \forall B(a, r) \subset \Omega,$
- 2') $u(a) \leq \mu_S(u, a, r) \quad \text{for } B(a, r_n) \subset \Omega, r_n \rightarrow 0,$
- 3) $u(a) \leq \mu_B(u, a, r) \quad \forall B(a, r) \subset \Omega,$
- 3') $u(a) \leq \mu_B(u, a, r) \quad \text{for } B(a, r_n) \subset \Omega, r_n \rightarrow 0,$
- 4) *If $u \in C^2$, then $\Delta u \geq 0$.*

The convex cone of subharmonic functions on a domain Ω is denoted by $Sh(\Omega)$.

Proof. It is obvious that $(1) \rightarrow (2) \rightarrow (3) \rightarrow (3') \rightarrow (2')$. To prove $(2') \rightarrow (1)$ one needs the following 2 facts:

Lemma 2 (u.s.c function as limit of continuous functions). *Let u be a u.s.c. function on a compact metric space X , then there exists a sequence u_n continuous function on X that decreases to u pointwise.*

Proof. Let $\tilde{u}_k(x) = \max\{u(x), -k\}$ to exclude the $-\infty$ points. Then $v_k(x) = \sup_{y \in X} (u(y) - kd(x, y))$ works.

Lemma 3. (2') implies strict maximum principle (see 3.1).

Proof. By restriction to smaller neighborhood, one can suppose that u attains global maximum at x_0 in Ω . Then $W = \{x \in \Omega : u(x) < u(x_0)\}$ is an open set, and has a point y in its boundary if W nonempty. Then (2') is not satisfied at y since the measure of open arc is nonzero.

Note that if u is continue than (2') \rightarrow (1): Let $h = P_{a,r}[u]$ harmonic then $u - h$ satisfies (2'), therefore the maximum principle, hence $u - h \leq (u - h)|_{S(a,r)} = 0$.

If u is u.s.c, take a sequence v_k continuous that decreases to u and let $h_k = P_{a,r}[v_k]$ then $h_k \geq v_k \geq u$ and $h_k \rightarrow P_{a,r}[u]$ by monotone convergence. \square

Proposition 3.1. Let $u \in Sh(\Omega)$ then

(Strict) maximum principle. u cannot attain local maximum unless it is constant in the corresponding connected component,

Locally integrable. u is L^1_{loc} on each connected component where $u \neq -\infty$,

Pointwise decreasing limit The pointwise limit u of a decreasing sequence u_k of subharmonic functions is also subharmonic.

Regularisation. $\mu_S(u, a, \varepsilon), \mu_B(u, a, \varepsilon), \rho_\varepsilon * u$ increase in ε . Moreover, $\rho_\varepsilon * u \in Sh(\Omega)$ and decreases to u pointwise as $\varepsilon \rightarrow 0$.

Moreover, for $u \in \mathcal{D}'(\Omega)$

Positive measure. $u \in Sh(\Omega)$ iff $\Delta u \geq 0$ is a positive measure.

Proof. **Locally integrable.** To see that $u \in L^1_{loc}(\Omega)$ if Ω is connected and $u \neq -\infty$, let x be a point in the boundary of $W = \{y \in \Omega : u \text{ integrable in neighborhood of } y\}$, then apply mean value property in $a \in W$ such that $x \in B(a, r)$.

Pointwise decreasing limit. Infimum of a family of u.s.c functions is still u.s.c. The mean value property comes from monotone convergence.

Regularisation. Check first for C^2 functions, then regularise. One uses the following Gauss formula:

$$\mu_S(u, a, r) = u(a) + \frac{1}{n} \int_0^r \mu_B(\Delta u, a, t) t dt$$

to see that μ_S is increasing in r and

$$\mu_B(u, a, r) = m \int_0^1 t^{m-1} \mu_S(u, a, rt) dt$$

to see that μ_B is increasing. For the convolution, use

$$u * \rho_\varepsilon = \text{vol}(S^{n-1}) \int_0^1 \mu_S(u, a, \varepsilon t) \rho(t) t^{m-1} dt.$$

Positive measure. $\Delta u * \rho_\varepsilon \geq 0$ as function, therefore the limit ≥ 0 as measure (dominated convergence). □

Proposition 3.2 (new harmonic functions from old ones). *Let $u_k \in Sh(\Omega)$ then*

1. *If $\{u_k\}$ decrease to u then $u \in Sh(\Omega)$.*
2. *Let χ be a convex function, non-decreasing in each variable then $\chi(u_1, \dots, u_p) \in Sh(\Omega)$. Therefore, $\sum u_i$ and $\max\{u_i\}$ are subharmonic.*

Proposition 3.3 (Upper regularization). 1. *Let u be a real function on Ω then $u^*(x) = \lim_{\varepsilon \rightarrow 0} \sup_{x+\varepsilon B} u$, called the upper envelope of u is u.s.c and is in fact the smallest u.s.c function greater than u .*

2. **Choquet lemma.** *Let $\{u_\alpha\}$ be a family of real function, one defines the upper regularization of $\{u_\alpha\}$ by u^* where $u = \sup_\alpha u_\alpha$. Then from every such family, on can always find a countable subfamily $\{v_i\}$ such that $u^* = v^*$.*
3. *If $\{u_\alpha\} \subset Sh(\Omega)$ then $u^* = u$ a.e. and $u^* \in Sh(\Omega)$.*

Proof. 1. Obvious.

2. Let B_i be a countable base of the topology and $x_{i,j}$ be a sequence such that $u(x_{i,j}) \rightarrow \sup_{B_i} u$. Let $\{u_{i,j,k}\}$ be a countable subfamily such that $u_{i,j,k}(x_i) \rightarrow u(x_i)$ then it is a suitable subfamily.
3. WLOG, suppose that $\{u_\alpha\} = \{u_i\}$ countable then u satisfies the submean value property: $u(z) \leq \mu_B(u, z, r)$. By the continuity of $\mu_B(u, z, r)$ one has $u^*(z) \leq \mu_B(u, z, r) \leq \mu(u^*, z, r)$ therefore $u^* \in Sh(\Omega)$ and $u^*(z) = \lim_{r \rightarrow 0} \mu_B(u^*, z, r) = \lim_{r \rightarrow 0} \mu_B(u, z, r)$, from which $u = u^*$ □

2.2 Plurisubharmonic functions

The analog of harmonic functions over \mathbb{C} in multidimensional case $\Omega \subset \mathbb{C}^n$ is in fact *pluriharmonic functions* which is defined through the notion of plurisubharmonic functions

Definition 3. 1. A real function u is said to be plurisubharmonic if and only if its restriction to any complex line is subharmonic. One denotes by $Psh(\Omega)$ the space of plurisubharmonic function on Ω .

2. In case $u \in C^2$ on $\Omega \subset \mathbb{C}^n$, this is equivalent to

$$H(u)(\zeta) = \sum \frac{\partial^2 u}{\partial z^j \partial \bar{z}^k} \zeta^j \bar{\zeta}^k \geq 0 \quad \forall \zeta \in \mathbb{C}^n$$

where the notation $H(u)(\zeta)$ is invariant, i.e. if $f : M_1 \rightarrow M_2$ is holomorphic then $H(u \circ f)(\zeta) = H(u)df(\zeta)$.

3. In the general case, this is equivalent to $H(u)(\zeta) \geq 0 \quad \forall \zeta \in \mathbb{C}^n$ as a measure.

Remark 3. 1. The invariance can be noticed using $\zeta^j = \zeta^j d\zeta^j + \bar{\zeta}^j d\bar{\zeta}^j$ where LHS is interpreted as a vector in $T\mathbb{C}$. This allows us to extend the notion of $Psh(M)$ to any complex manifold M .

2. By consequence, $f^*u \in Psh(M_1)$ for all $u \in Psh(M_2)$ and $f : M_1 \rightarrow M_2$ holomorphic.

Proposition 3.4 (new Psh functions from old ones). *The construction of new plurisubharmonic function is the same as that of subharmonic function. Let $u_k \in Psh(\Omega)$ then*

1. If $\{u_k\}$ decrease to u then $u \in Psh(\Omega)$.
2. Let χ be a convex function, non-decreasing in each variable then $\chi(u_1, \dots, u_p) \in Psh(\Omega)$. Therefore, $\sum u_i$ and $\max\{u_i\}$ are plurisubharmonic.
3. The upper regularization u^* where $u = \sup_{\alpha} u_{\alpha}$ is also plurisubharmonic and $u = u^*$ almost everywhere.

Proof. The only nontrivial proof is the third one where upper envelop in \mathbb{C}^{\times} and in a line can be different. To fix this, use Choquet lemma 3.3 and dominated convergence, $u * \rho_{\varepsilon}$ satisfies the submean property on every complex line and decrease to u a.e. \square

2.3 Pluriharmonic functions

Definition 4. A function u is said to be pluriharmonic on Ω , denoted $u \in Ph(\Omega)$ if $u \in Psh(\Omega)$ and $-u \in Psh(\Omega) \setminus \setminus$ where (Ω) .

This is obviously equivalent to $H(u) = 0$, i.e. $\frac{\partial^2 u}{\partial z^j \partial \bar{z}^k} = 0 \quad \forall j, k$, i.e. $\partial \bar{\partial} u = 0$.

Remark 4. 1. By mean value property, $Ph(\Omega) \subset Harm(\Omega)$.

2. If $f \in \mathcal{O}(M)$ then $\Re f, \Im f \in Ph(M)$

Theorem 4 (analog of harmonic function). If M is a complex manifold such that $H_{dR}^1(X, \mathbb{R}) = 0$ then every pluriharmonic function u is a real part of a holomorphic function $f \in \mathcal{O}(M)$

Proof. Since $d(\bar{\partial}u) = 0$, and $H_{dR}^1 = 0$, one has $\bar{\partial}u = dg$. Therefore $d(u - 2\Re g) = (\bar{\partial}u - dg) + (\partial u - d\bar{g}) = 0$, hence one chooses $f = 2g + C$ on each connected component. \square

3 Resolution of $\bar{\partial}$, Dolbeault-Grothendieck lemma

The generalized Cauchy formula for several variables is the following (the formula in wikipedia is $K_{BM}^{0,0}$)

Theorem 5 (Bochner–Martinelli–Koppelman formula). The Bochner-Martinelli kernel is the following $(n, n-1)$ -form on \mathbb{C}^n

$$k_{BM} = (-1)^{n(n-1)/2} \frac{(n-1)!}{(2\pi i)^n} \sum_{1 \leq j \leq n} (-1)^j \frac{\bar{z}_j dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n}{|z|^{2n}}$$

then $\bar{\partial}k_{BM} = \delta_0$ on \mathbb{C}^n .

Let $K_{BM} = \pi^* k_{BM}$ where $\pi : (z, \zeta) \mapsto z - \zeta$ so that $\bar{\partial}K_{BM} = [\Delta]$, then: For any domain $\Omega \subset \mathbb{C}^n$ bounded with piecewise C^1 boundary and v a (p, q) -form of class C^1 on $\bar{\Omega}$ then

$$v(z) = \int_{\partial\Omega} K_{BM}^{p,q}(z, \zeta) \wedge v(\zeta) + \bar{\partial} \int_{\Omega} K_{BM}^{p,q-1}(z, \zeta) \wedge v(\zeta) + \int_{\Omega} K_{BM}^{p,q}(z, \zeta) \wedge \bar{\partial}v(\zeta)$$

where $K_{BM}^{p,q}$ denotes the component of K_{BM} type (p, q) in z and type $(n-p, n-q-1)$ in ζ

Another consequence of 5 is the *global* resolution of $\bar{\partial}$ in case of compact support.

Corollary 5.1. *If v is a (p, q) -form with $q \geq 1$ on \mathbb{C}^n , compactly supported, with regularity of class C^s such that $\bar{\partial}v = 0$ then there exists an $(p, q-1)$ -form u on \mathbb{C}^n with the same regularity as u such that $\bar{\partial}u = v$. In fact one can take*

$$u(z) = \int_{\mathbb{C}^n} K_{BM}^{p,q-1}(z, \zeta) \wedge v(\zeta)$$

In case $(p, q) = (0, 1)$ then u is compactly support. This means that the compact support $(0, 1)$ -Dolbeault cohomology $H_c^{0,1}(\mathbb{C}^n) = 0$.

Since $K_{BM} = O(|z|^{1-2n})$, one has $|u(z)| = O(|z|^{1-2n})$ at infinity. Therefore the compact support of u in case $(p, q) = (0, 1)$ is explained by Liouville theorem.

The Dolbeault-Grothendieck lemma solves the equation $\bar{\partial}u = v$ in a local scale if the compact support condition is dropped and gives regular result if v is a $(p, 0)$ -form.

Theorem 6 (Dolbeault-Grothendieck lemma). *Let $v \in \mathcal{D}'(p, q)(\Omega)$ such that $\bar{\partial}v = 0$.*

1. *If $q = 0$ then $v = \sum v_I dz^I$ where $v_I \in \mathcal{O}(\Omega)$.*
2. *If $q \geq 1$ then there exists $\omega \subset \Omega$ and $u \in \mathcal{D}'(p, q-1)(\Omega)$ such that $\bar{\partial}u = v$. Moreover, if $v \in \mathcal{E}^{p,q}(\Omega)$ then $u \in \mathcal{E}^{p,q-1}(\Omega)$*

Corollary 6.1 (Hypoellipticity in bidegree $(p, 0)$). *$\bar{\partial}$ is hypoellipticity in bidegree $(p, 0)$, i.e. if $\bar{\partial}u = v$, v of bidegree $(p, 1)$ and v is C^∞ then u is also C^∞ on the entire domain Ω .*

4 Extension theorems, Domain of holomorphy

Theorem 7 (Hartog extension). *Let $\Omega \subset \mathbb{C}^n$ be a domain and $K \Subset \Omega$ such that $\Omega \setminus K$ is connected. Then $\mathcal{O}(\omega)|_{\Omega \setminus K} = \mathcal{O}(\Omega \setminus K)$ every holomorphic function on $\Omega \setminus K$ extends to Ω*

Proof. Let $f \in \mathcal{O}(\Omega \setminus K)$ be the function we want to extend. Let φ be a function with support in a neighborhood of K and is identically 1 on K and $g = (1 - \varphi)f$ which coincides with f outside of $\text{supp } \varphi$. Then $v = \bar{\partial}g \in \mathcal{D}^{0,1}$ satisfies $\bar{\partial}v = 0$, therefore there exists $u \in C_c^\infty(\mathbb{C}^n)$ with $\text{supp } u \subset \text{supp } \varphi$ such that $\bar{\partial}u = v = \bar{\partial}g$, the holomorphic function $g - u$ is well-defined on Ω and coincides with f (and g) on $\Omega \setminus \text{supp } \varphi$, therefore coincides with f on $\Omega \setminus K$. \square

Note that although we do not need Ω to be small, this theorem counts as a local result due to the hypothesis that we are in \mathbb{C}^n .

A global result can be obtained using the Hartog figure, that is the union of an annulus $\{(z_1, z') : r < |z_1| < R\}$ and an open set in other dimension $\{(z_1, z') : z' \in \omega \text{ open}\}$. and use the interpolation $(z_1, z') \mapsto \int_{C_R} \frac{f(\zeta_1, z')}{\zeta_1 - z_1} d\zeta_1$ to extend f . The open set in z' -dimension is to show that the interpolation and f coincide on it. With one dimension z_1 to form the annulus and another dimension (says z_2 to form the open set, one can extend any holomorphic function to a submanifold of (complex) codimension at least 2.

Theorem 8 (Riemann extension). *Let M be a complex manifold and N a sub complex manifold of codimension ≥ 2 then any holomorphic function on $M \setminus N$ extends uniquely to M .*

4.1 Generalities

An approach to the extension problem on complex manifolds is through the notion of holomorphic hull and holomorphic convexity.

Definition 5. 1. *Let K be a compact in a complex manifold M . Then the holomorphic hull $\hat{K}_{\mathcal{O}(M)}$ is the set $\{z \in M : f(z) \leq \sup_K |f| \quad \forall f \in \mathcal{O}(M)\}$.*

2. *A complex manifold M is said to be holomorphically convex if $\hat{K}_{\mathcal{O}(X)}$ is compact for all compact $K \subset M$.*

Proposition 8.1 (holomorphic hull). *The following statements are obvious*

1. \hat{K} is a closed subset containing K and $\hat{\hat{K}} = \hat{K}$.
2. If $f : M_1 \rightarrow M_2$ is holomorphic then $f(\hat{K}) \subset \widehat{f(K)}$. (Think inclusion)
3. **Hole filling.** In particular, if $f : \bar{B} \rightarrow X$ and $f(\partial B) \subset K$ then $f(\bar{B}) \subset \hat{K}$.

Proposition 8.2 (holomorphically convex). *Let M be a holomorphically convex complex manifold then*

1. M admits a exhaustive sequence of compact K_ν , i.e. $K_\nu \Subset K_{\nu+1}$ and $\widehat{K_\nu} = K_\nu$.
2. M is weakly pseudoconvex, i.e. there exists $\psi \in Psh(M) \cap C^\infty(M)$ such that $\{\psi < c\}$ are relatively compact, i.e. $\lim_{K \rightarrow M} \psi|_{M \setminus K} = +\infty$

4.2 Case $\Omega \subset \mathbb{C}^n$

Definition 6. *Domain of holomorphy*

Proposition 8.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain then:*

1. *If Ω is a domain of holomorphy then $\hat{K}_{\mathcal{O}(\Omega)}$ is compact and $d(K, \partial\Omega) = d(\hat{K}, \partial\Omega)$.*
2. *The followings are equivalent:*
 - (a) *Ω is a domain of holomorphy.*
 - (b) *Ω is holomorphically convex.*
 - (c) *Let $\{z_k\}$ be a sequence in Ω without accumulation in Ω and $c_k \in \mathbb{C}$. There exists a function $f \in \mathcal{O}(\Omega)$ such that $f(z_k) = c_k$.*
 - (d) *There exists a function $F \in \mathcal{O}(\Omega)$ that is unbounded locally in any point on $\partial\Omega$.*