Local results of several complex variables

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1 Subharmonic and Plurisubharmonic functions

Some properties of holomorphic functions that remain in several variables.

- Cauchy formula
- Analyticity: series development. Therefore its zeroes never form an open set (except for constant)
- Maximum modulus
- Cauchy inequality and Montel's theorem

1.1 Subharmonic functions

We are now in the context of $\mathbb{R}^n$.

Theorem 1 (Green kernel). *Let $\Omega \Subset \mathbb{R}^n$ be a smoothly bounded domain, then there exists uniquely a function $G_\Omega : \bar{\Omega} \times \bar{\Omega} \longrightarrow [-\infty, 0]$, called <u>the Green kernel</u> of Ω, with the following properties:*

1. *Regular:* G_Ω *is* C^∞ *on* $\bar{\Omega} \times \bar{\Omega} \setminus \Delta_\Omega$ *where* Δ_Ω *denotes the diagonal,*

2. *Symetric:* $G_\Omega(x, y) = G_\Omega(y, x)$,

3. *Negative:* $G_\Omega(x, y) < 0$ *on* $\Omega \times \Omega$ *and* $G_\Omega(x, y) = 0$ *on* $\partial\Omega \times \Omega$,

4. $\Delta_x G_\Omega(x, y) = \delta_y$ *on* Ω *for every* $y \in \Omega$.

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Example 1 (case $\Omega = B(0, r)$)**.** *One can take* $G_r = N(x - y) - N(\frac{|y|}{r}(x - \frac{r^2}{|y|^2}y))$ *where* N *is the Newton kernel (or Newtonian potential, the gravitational potential). Explicitly, one has*

$$G_r(x, y) = \frac{1}{4\pi} \log \frac{|x - y|^2}{r^2 - 2\langle x, y\rangle + \frac{1}{r^2}|x|^2|y|^2} \qquad \text{if } n = 2 \tag{1}$$

$$G_r(x, y) = \frac{-1}{(m - 2)\,\mathrm{vol}(S^{m-1})}(|x - y|^{2-m} - (r^2 - 2\langle x, y\rangle + \frac{1}{r^2}|x|^2|y|^2)^{1-m/2}) \qquad \text{if } n \geq 3 \tag{2}$$

Proposition 1.1 (Green-Riesz representation)**.** *For* $u \in C^2(\bar{\Omega}, \mathbb{R})$ *one has*

$$u(x) = \int_\Omega G_\Omega(x, y)\Delta u(y)d\lambda(y) + \int_{\partial\Omega} u(y)\frac{\partial G_\Omega}{\partial \nu_y}d\sigma(y)$$

In particular, for $\Omega = B(0, r)$, *one has*

$$P_r(x, y) := \frac{\partial G}{\partial \nu_y} = \frac{1}{\mathrm{vol}(S^{m-1})r}\frac{r^2 - |x|^2}{|x - y|^m}$$

called the *<u>Poisson kernel</u>.*

Proof. Use the Green-Riesz formula: $\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}$. $\square$

Definition 1. *Let* $\Omega \subset \mathbb{R}^n$ *be an open subset and* $u : \Omega \longrightarrow [-\infty, \infty)$ *a upper semi-continuous function:*

$$\limsup_{x \to x_0} u(x) \leq u(x_0)$$

One notes by $\mu_S(u, a, r)$ *and* $\mu_B(u, a, r)$ *the average of* u *in the sphere and the disk centered in* a *of radius* r*. Then the following properties are equivalent and a function is called* *<u>subharmonic</u>* *if they are verified.*

1) $u(x) \leq P_{a,r}[u](x) \quad \forall a, r, x \in B(a,r) \subset \Omega$,

2) $u(a) \leq \mu_S(u, a, r) \quad \forall B(a,r) \subset \Omega$,

2') $u(a) \leq \mu_S(u, a, r)$ *for* $B(a, r_n) \subset \Omega, r_n \to 0$,

3) $u(a) \leq \mu_B(u, a, r) \quad \forall B(a,r) \subset \Omega$,

3') $u(a) \leq \mu_B(u, a, r)$ *for* $B(a, r_n) \subset \Omega, r_n \to 0$,

4) *If* $u \in C^2$, *then* $\Delta u \geq 0$.

The convex cone of subharmonic functions on a domain Ω *is denoted by* $Sh(\Omega)$.

Proof. It is obvious that $(1) \to (2) \to (3) \to (3') \to (2')$. To prove $(2') \to (1)$ one needs the following 2 facts:

Lemma 2 (u.s.c function as limit of continuous functions)**.** *Let* u *be a u.s.c. function on a compact metric space* X*, then there exists a sequence* u_n *continuous function on* X *that decreases to* u *pointwise.*

Proof. Let $\tilde{u}_k(x) = \max\{u(x), -k\}$ to exclude the $-\infty$ points. Then $v_k(x) = \sup_{y \in X} (u(y) - kd(x,y))$ works.

Lemma 3. $(2')$ *implies strict maximum principle (see 3.1).*

Proof. By restriction to smaller neighborhood, one can suppose that u attains global maximum at x_0 in Ω. Then $W = \{x \in \Omega : \; u(x) < u(x_0)\}$ is an open set, and has a point y in its boundary if W nonempty. Then $(2')$ is not satisfied at y since the measure of open arc is nonzero.

Note that if u is continue than $(2') \to (1)$: Let $h = P_{a,r}[u]$ harmonic then $u - h$ satisfies $(2')$, therefore the maximum principle, hence $u - h \leq (u-h)\big|_{S(a,r)} = 0$.

If u is u.s.c, take a sequence v_k continuous that decreases to u and let $h_k = P_{a,r}[v_k]$ then $h_k \geq v_k \geq u$ and $h_k \to P_{a,r}[u]$ by monotone convergence. □

Proposition 3.1. *Let* $u \in Sh(\Omega)$ *then*

(Strict) maximum principle. u *cannot attain local maximum unless it is constant in the corresponding connected component,*

Locally integrable. u *is* L^1_{loc} *on each connected component where* $u \not\equiv -\infty$,

Pointwise decreasing limit *The pointwise limit* u *of a decreasing sequence* u_k *of subharmonic functions is also subharmonic.*

Regularisation. *$\mu_S(u,a,\varepsilon), \mu_B(u,a,\varepsilon), \rho_\varepsilon * u$ increase in ε. Moreover, $\rho_\varepsilon * u \in Sh(\Omega)$ and decreases to u pointwise as $\varepsilon \to 0$.*

Moreover, for $u \in \mathcal{D}'(\Omega)$

Positive measure. *$u \in Sh(\Omega)$ iff $\Delta u \geq 0$ is a positive measure.*

Proof. **Locally integrable.** To see that $u \in L^1_{loc}(\Omega)$ if Ω is connected and $u \not\equiv -\infty$, let x be a point in the boundary of $W = \{y \in \Omega : u \text{ integrable in neighborhood of } y\}$, then apply mean value property in $a \in W$ such that $x \in B(a,r)$.

Pointwise decreasing limit. Infimum of a family of u.s.c functions is still u.s.c. The mean value property comes from monotone convergence.

Regularisation. Check first for C^2 functions, then regularise. One uses the following Gauss formula:

$$\mu_S(u,a,r) = u(a) + \frac{1}{n}\int_0^r \mu_B(\Delta u, a, t) t dt$$

to see that μ_S is increasing in r and

$$\mu_B(u,a,r) = m\int_0^1 t^{m-1}\mu_S(u,a,rt)dt$$

to see that μ_B is increasing. For the convolution, use

$$u * \rho_\varepsilon = \mathrm{vol}(S^{n-1})\int_0^1 \mu_S(u,a,\varepsilon t)\rho(t)t^{m-1}dt.$$

Positive measure. $\Delta u * \rho_\varepsilon \geq 0$ as function, therefore the limit ≥ 0 as measure (dominated convergence).

□

Proposition 3.2 (new harmonic functions from old ones)**.** *let $u_k \in Sh(\Omega)$ then*

1. *If $\{u_k\}$ decrease to u then $u \in Sh(\Omega)$.*
2. *Let χ be a convex function, non-decreasing in each variable then $\chi(u_1, \ldots, u_p) \in Sh(\Omega)$. Therefore, $\sum u_i$ and $\max\{u_i\}$ are subharmonic.*

Proposition 3.3 (Upper regularization)**.** *1. Let u be a real function on Ω then $u^*(x) = \lim_{\varepsilon\to 0}\sup_{x+\varepsilon B} u$, called the upper envelope of u is u.s.c and is in fact the smallest u.s.c function greater than u.*

2. ***Choquet lemma.*** *Let $\{u_\alpha\}$ be a family of real function, one defines the* <u>*upper regularization*</u> *of $\{u_\alpha\}$ by u^* where $u = \sup_\alpha u_\alpha$. Then from every such family, on can always find a countable subfamily $\{v_i\}$ such that $u^* = v^*$.*

3. *If $\{u_\alpha\} \subset Sh(\Omega)$ then $u^* = u$ a.e. and $u^* \in Sh(\Omega)$.*

Proof. 1. Obvious.

2. Let B_i be a countable base of the topology and $x_{i,j}$ be a sequence such that $u(x_{ij}) \to \sup_{B_i} u$. Let $\{u_{i,j,k}\}$ be a countable subfamily such that $u_{ijk}(x_i) \to u(x_i)$ then it is a suitable subfamily.

3. WLOG, suppose that $\{u_\alpha\} = \{u_i\}$ countable then u satisfies the submean value property: $u(z) \leq \mu_B(u, z, r)$. By the continuity of $\mu_B(u, z, r)$ one has $u^*(z) \leq \mu_B(u, z, r) \leq \mu(u^*, z, r)$ therefore $u^* \in Sh(\Omega)$ and $u^*(z) = \lim_{r\to 0} \mu_B(u^*, z, r) = \lim_{r\to 0} \mu_B(u, z, r)$, from which $u = u^*$ a.e.

□

2 de Rham currents

Let M be a differential m-dimensional manifold and $\mathcal{E}^p(M)$ be the vector space of smooth p-forms on M and $\mathcal{D}^p(M)$ be the space of those with compact support. Then $\mathcal{E}^p(M), \mathcal{D}^p(M)$ is a topological vector space with the pseudonorms $p^s_{K,\Omega}(\omega) = \max_{K,|\alpha|\leq s} |D^\alpha u_I|$ where $K \Subset \Omega$ an coordinated open set. The <u>space of de Rham current</u> with dimension p / <u>degree</u> $m - p$ is defined as the dual space of $\mathcal{D}^p(M)$, denoted by $\mathcal{D}'^{m-p}(M)$ or $D'_p(M)$

Remark 1. *1. We are still in $\mathbb{R}$, but the definition expands to the complex case, denoted by $\mathcal{D}'^{m-p,m-q}(M) = \mathcal{D}'_{p,q}(M)$ where m is the complex dimension of M.*

2. The degree is defined such that the current $T_\omega : \eta \mapsto \int_M \omega \wedge \eta$ is of the same degree as ω. The dimension is defined so that the current $T_{[Z]} : \eta \mapsto \int_Z \eta$ is of the same dimension as Z.

Definition 2. *One has the following operation on $\mathcal{D}'^{m-p}(M)$:*

1. ***Derivative:*** $\langle dT, \omega\rangle = (-1)^{\deg T}\langle T, d\omega\rangle$

2. ***Wedge product with a form:*** $\langle T \wedge \eta, \omega\rangle = \langle T, \eta \wedge \omega\rangle$

3. ***Pushforward:*** *If* $F : X \longrightarrow Y$ *proper on* $\operatorname{supp} T$ *then* $\langle F_* T, \omega\rangle = \langle T, F^*\omega\rangle = \langle T, \chi F_*\omega\rangle$ *where* $\chi \in C^\infty(M)$ *identically 1 on* $\operatorname{supp} T$*. The proper condition is such that the pullback of* ω *is compactly support in* $\operatorname{supp} T$

4. ***Pullback:*** *Let* $F : X \longrightarrow Y$ *submersion then the pushforward of a form on* X *is well-defined by Fubini. One has* $\langle F^* T, \omega\rangle = \langle T, F_*\omega\rangle$

Remark 2.

1. *The sign of derivative is chosen so that* $dT_\omega = T_{d\omega}$*.*

2. *Pushforward keeps the dimension, as the arguments are of the same degree.*

3. *Pullback keeps the codimension, meaning the degree (think* $F^* T_{[Z]} = T_{[F^{-1}(Z)]}$*).*

4. *Locally a current is of form* $T = \sum u_I dx^I$ *where* u_I *are distribution.* ***Note:*** *Here distribution are indentified as a current of maximal degree and* <u>*not*</u> *zero degree as they naturally are. To be exact, the notation of* u_I *is contravariant and its action is* $\varphi dx^1 \wedge \cdots \wedge dx^N \mapsto \langle u_I, \varphi\rangle dx^1 \wedge \cdots \wedge dx^n/vol$ *where vol is a canonical volume form.*

The last two remarks explain the sign in the following proposition.

Proposition 3.4 (Pushforward and Pullback)**.** *Let* $F : M_1 \longrightarrow M_2$*, submersion if needed, then*

1. $\operatorname{supp} F_* T \subset F(\operatorname{supp} T)$

2. $d(F_* T) = F_* dT$ *(pushforward of a form is still that form)*

3. $F_*(T \wedge F^* g) = (F_* T) \wedge g$

and

1. $F^*(dT) = (-1)^{m_1 - m_2} d(F^* T)$

2. $F^*(T \wedge g) = (-1)^{m_1 - m_2 - \deg g}(F^* T) \wedge F^* g$

3 Resolution of $\bar\partial$, Dolbeault-Grothendieck lemma

The generalized Cauchy formula for several variables is the following (the formula in wikipedia is $K^{0,0}_{BM}$)

Theorem 4 (Bochner–Martinelli-Koppelman formula)**.** *The* *Bochner-Martinelli kernel* *is the following* $(n, n-1)$*-form on* $\mathbb{C}^n$

$$k_{BM} = (-1)^{n(n-1)/2}\frac{(n-1)!}{(2\pi i)^n}\sum_{1\leq j\leq n}(-1)^j\frac{\bar{z}_j dz_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_1\wedge\cdots\wedge\widehat{d\bar{z}_j}\wedge\cdots\wedge d\bar{z}_n}{|z|^{2n}}$$

then $\bar\partial k_{BM} = \delta_0$ *on* $\mathbb{C}^n$*.*

Let $K_{BM} = \pi^* k_{BM}$ *where* $\pi : (z,\zeta)\mapsto z-\zeta$ *so that* $\bar\partial K_{BM} = [\Delta]$*, then: For any domain* $\Omega\subset\mathbb{C}^n$ *bounded with piecewise* C^1 *boundary and* v *a* (p,q)*-form of class* C^1 *on* $\bar\Omega$ *then*

$$v(z) = \int_{\partial\Omega} K_{BM}^{p,q}(z,\zeta)\wedge v(\zeta) + \bar\partial\int_\Omega K_{BM}^{p,q-1}(z,\zeta)\wedge v(\zeta) + \int_\Omega K_{BM}^{p,q}(z,\zeta)\wedge\bar\partial v(\zeta)$$

where $K_{BM}^{p,q}$ *denotes the component of* K_{BM} *type* (p,q) *in* z *and type* $(n-p, n-q-1)$ *in* ζ

Another consequence of 4 is the *global* resolution of $\bar\partial$ in case of compact support.

Corollary 4.1. *If* v *is a* (p,q)*-form with* $q\geq 1$ *on* $\mathbb{C}^n$*, compactly supported, with regularity of class* C^s *such that* $\bar\partial v = 0$ *then there exists an* $(p,q-1)$*-form* u *on* $\mathbb{C}^n$ *with the same regularity as* u *such that* $\bar\partial u = v$*. In fact one can take*

$$u(z) = \int_{\mathbb{C}^n} K_{BM}^{p,q-1}(z,\zeta)\wedge v(\zeta)$$

In case $(p,q) = (0,1)$ *then* u *is compactly support. This means that the compact support* $(0,1)$*-Dolbeault cohomology* $H_c^{0,1}(\mathbb{C}^n) = 0$*.*

Since $K_{BM} = O(|z|^{1-2n})$, one has $|u(z)| = O(|z|^{1-2n})$ at infinity. Therefore the compact support of u in case $(p,q) = (0,1)$ is explained by Liouville theorem.

The Dolbeault-Grothendieck lemma solves the equation $\bar\partial u = v$ in a local scale if the compact support condition is dropped and gives regular result if v is a $(p,0)$-form.

Theorem 5 (Dolbeault-Grothendieck lemma)**.** *Let* $v\in\mathcal{D}'(p,q)(\Omega)$ *such that* $\bar\partial v = 0$*.*

1. *If* $q = 0$ *then* $v = \sum v_I dz^I$ *where* $v_I\in\mathcal{O}(\Omega)$*.*
2. *If* $q\geq 1$ *then there exists* $\omega\subset\Omega$ *and* $u\in\mathcal{D}'^{(}p,q-1)(\Omega)$ *such that* $\bar\partial u = v$*. Moreover, if* $v\in\mathcal{E}^{p,q}(\Omega)$ *then* $u\in\mathcal{E}^{p,q-1}(\Omega)$

Corollary 5.1 (Hypoellipticity in bidegree $(p,0)$)**.** $\bar\partial$ *is hypoellipticity in bidegree* $(p,0)$*, i.e. if* $\bar\partial u = v$*,* v *of bidegree* $(p,1)$ *and* v *is* C^∞ *then* u *is also* C^∞ *on the entire domain* Ω*.*

4 Extension theorems, Domain of holomorphy

Theorem 6 (Hartog extension). *Let $\Omega \subset \mathbb{C}^n$ be a domain and $K \Subset \Omega$ such that $\Omega \setminus K$ is connected. Then $\mathcal{O}(\omega)\big|_{\Omega \setminus K} = \mathcal{O}(\Omega \setminus K)$ every holomorphic function on $\Omega \setminus K$ extends to Ω*

Proof. Let $f \in \mathcal{O}(\Omega \setminus K$ be the function we want to extend. Let φ be a function with support in a neighborhood of K and is identically 1 on K and $g = (1-\varphi)f$ which coincides with f outside of $\operatorname{supp}\varphi$. Then $v = \bar{\partial} g \in \mathcal{D}^{0,1}$ satisfies $\bar{\partial} v = 0$, therefore there exists $u \in C_c^\infty(\mathbb{C}^n$ with $\operatorname{supp} u \subset \operatorname{supp}\varphi$ such that $\bar{\partial} u = v = \bar{\partial} g$, the holomorphic function $g - u$ is well-defined on Ω and coincides with f (and g) on $\Omega \setminus \operatorname{supp}\varphi$, therefore coincides with f on $\Omega \setminus K$. □

Note that although we do not need Ω to be small, this theorem counts as a <u>local result due to the hypothesis that we are in $\mathbb{C}^n$</u>.

A global result can be obtained using the Hartog figure, that is the union of an anulus $\{(z_1, z') : \ r < |z_1| < R\}$ and an open set in other dimension $\{(z_1, z') : \ z' \in \omega \text{ open}\}$. and use the interpolation $(z_1, z') \mapsto \int_{C_R} \frac{f(\zeta_1, z')}{\zeta_1 - z_1} d\zeta_1$ to extend f. The open set in z'-dimension is to show that the interpolation and f coincide on it. With one dimension z_1 to form the anulus an another dimension (says z_2 to form the open set, one can extend any holomorphic function to a submanifold of (complex) codimension at least 2.

Theorem 7 (Riemann extension). *Let M be a complex manifold and N a sub complex manifold of codimension ≥ 2 then any holomorphic function on $M \setminus N$ extends uniquely to M.*

Emacs 25.3.1 (Org mode 9.0.5)