Harmonic maps of Riemannian manifolds

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Chapter 1

Summary

The goal of this part is to give a summary of what will be developed in the next chapters. In brief, we are interested in maps $f : M \longrightarrow M'$ between Riemannian manifolds (that to simplify, are supposed to be compact) that are critical points of the energy functional

$$E(f) = \frac{1}{2} \int_M |\nabla f|^2 dV$$

that is, by taking first order variation of E, those whose **tension field** $\tau(f)$ vanish.

We wish to prove that any smooth map $f_0 : M \longrightarrow M'$ can be deform to a harmonic map using the gradient descent equation, that is to show the equation

$$\begin{cases} \frac{df_t}{dt} = \tau(f_t) \\ f\big|_{t=0} = f_0 \end{cases} \tag{1.1}$$

We prove, in the rest of the memoire, that if M' is negatively curved then this PDE admits a globally defined smooth solution f_t and that $f_\infty := \lim_{t\to\infty} f_t$ in C^∞ is a harmonic map.

The resolution of (1.1) can be organised in 3 steps:

1. Find the global equation. We will find a global frame of M' and express f in this frame, so that in stead of solving for a map, we will have to solve for functions.

2. Study linear PDEs on manifolds. The equation, expressed in local coordinates is a nonlinear heat equation, i.e. other than a heat operator,

it has a nonlinear differential operator of strictly lower degree. Short-time existence and regularity for (1.1) follows from *standard* results of parabolic equation.

3. Prove long-time existence. This follows from several energy estimate.

Local form of (1.1) can be found using calculus on Riemannian-connected vector bundle. The relevant vector bundle here is f^*TM' over M in case of a single map $f : M \longrightarrow M'$, or F^*TM' over $M \times [\alpha, \omega]$ for a deformation $F_t : \ M \times [\alpha, \omega] \longrightarrow M'$.

1.1 Global equation.

We will explain here how the step 1 is done. We will embed M' in a Euclidean space $\mathbb{R}^N$, not necessarily isometric because we will not use the Euclidean metric on R^N anyway. We will equip a tubular neighborhood T of M', which is diffeomorphic to $M' \times D$ where D is an open disc of dimension equal the codimension of M', with the product of the metric of M' and the Euclidean metric of D. The global equation of (1.1), as we will prove, is the flow along the tension field of T, i.e.

$$\frac{df_t}{dt} = \tau_T(f_t). \tag{1.2}$$

What we want in a global equation is that locally it has to be the same as $\frac{df_t}{dt} = \tau_{M'}(f_t)$ and globally the image of f_t has to remain in M'. So in fact, if there is a global equation of (1.1), then it has to be $\frac{df_t}{dt} = \tau_T(f_t)$ because of the following fact:

Fact. If the inclusion $M' \longrightarrow T$ is totally geodesic and $f : M \longrightarrow M'$ be a smooth map. Then the tension field of f in M' is actually the tension field of f as a map to T.

It is, however just a necessary condition. To complete the argument, one needs to justify that the τ-flow in T always remain in M'. The following idea is due to Hamilton [Ham75]. The advantage, in comparison with Eells and Sampson [ES64] is its clarity in idea. The disadvantage, is that one needs to establish uniqueness of solution first. Let ι be the reflection in T around M' then if f_t is a solution of (1.2) then ιf_t is also a solution with the same initial condition, since ι preserves M'. Then by uniqueness of solution, $f_t = \iota f_t$ for all relevant time, meaning that f_t remains in M' since ι only preserves M'.

1.2 Linear PDEs on manifolds.

I passed more than half of my stage learning how to solve linear equations on manifold. I started with [Aub98] and [Jos08] as reference, where Sobolev spaces are defined by density and Laplace equation, heat equation are solved using parametrix. This approach has advantage of being quick, intuitive and clear in ideas. The disadvantage is that, while parametrix works perfectly for smooth functions, we have to remain fuzzy on function spaces while solving parabolic equation. I later discovered [Ham75] where a thorough treatment of Sobolev spaces on manifold was an important part of the paper. While the conceptual points behind are clear, the limit of this approach is that it only works for compact manifolds, unlike [Aub98] and [Jos08], whose manifolds are supposed to be complete, having strictly positive injectivity radius and bounded curvatures. This disadvantage cannot be remedied because for non-compact manifolds, Sobolev spaces, set-theoretically and as Banach spaces, depend on the metric. I choose to present here the second approach.

1.2.1 Sobolev spaces

Using a partition of unity $\{\psi_i\}$ subordinated to a finite atlas of a compact manifold M, the Sobolev space $W^{k,p}(M)$ is defined as the preimage of $\bigoplus_i W^{k,p}(\mathbb{R}^n)$ of:

$$\begin{aligned} \iota :\ \mathcal{S}^*(M) &\longrightarrow \bigoplus_i \mathcal{S}^*(\mathbb{R}^n) \\ f &\longmapsto \oplus_i \psi_i f \end{aligned} \tag{1.3}$$

meaning that we a natural inclusion

$$\iota : \mathcal{S}^*(M) \supset W^{k,p}(M) \hookrightarrow \bigoplus_i W^{k,p}(\mathbb{R}^n) \tag{1.4}$$

The definition of $W^{k,p}(M)$ as a subspace of a direct sum renders the task of generalising operators on $W^{k,p}(M)$ indirect.

For example, to define a differential operator A of order r on $W^{k,p}(M)$, we have to define it component-wise, that is $Af := \oplus_i A(\psi_i f_i)$ then verify that the RHS is actually in the image of $\iota :\ W^{k-r,p}(M) \hookrightarrow \bigoplus_i W^{k-r,p}(\mathbb{R}^n)$. This is straightforward for differential operators be cause we can differentiate elements of $\mathcal{S}^*(M)$: $\iota(Af) = \oplus_i A(\psi_i f_i)$.

A less straightforward example is the definition of trace operator of elements in $W^{k,p}(M)$ where M is a compact manifold with boundary. The

Sobolev space $W^{k,p}(M)$ when $\partial M \neq \emptyset$ is defined in the same spirit: so that we have the inclusion

$$\begin{aligned} \iota :\ W^{k,p}(M) &\longrightarrow \bigoplus_i W^{k,p}(R_i) \\ f &\longmapsto \oplus_i \psi_i f \end{aligned} \tag{1.5}$$

where R_i is $\mathbb{R}^n$ if the i^{th} chart does not intersect the boundary, or the upper half plan $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ if the intersection is not trivial. Then similarly, if k is sufficiently large, we can define a component-wise trace operator $W^{k,p}(\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}) \longrightarrow W^{l,p}(\mathbb{R}^{n-1})$. On ∂M, the restriction of the atlas of M is still a finite atlas, subordinated by the partition of unity $\{\psi_i\}$, one therefore has the following diagram, where the vertical arrow on the right was defined.

$$\begin{tikzcd} W^{k,p}(M) \arrow[r, hook] \arrow[d, dashed] & \bigoplus_i W^{k,p}(R_i) \arrow[d, "\mathrm{Tr}"] \\ W^{l,p}(\partial M) \arrow[r] & \bigoplus_i W^{l,p}(\partial R_i) \end{tikzcd}$$

The tricky part to define the dashed vertical arrow, in comparison with the case of differential operators, is that we cannot define trace operator on $\mathcal{S}^*(M)$.

The situation is resolved because the maps ι in (1.3), (1.4) and (1.5) admit a projection π (i.e. right-inverse) given by multiplication with a family of cut-off functions $\tilde{\psi}_i$ that are still supported in the chart, but are identically 1 on the supports of ψ_i. So to check whether an element is in $\operatorname{Im}\iota$, one only has to check if it is fixed by $\pi \circ \iota$, which is simple.

The existence of π also shows that $\operatorname{Im}\iota$ is closed, hence $W^{k,p}$ is a reflexive Banach space, and that we can extend interpolation theory for $W^{k,p}(M)$.

1.2.2 Elliptic and parabolic equations

We will encode classical results of linear equations in an exact diagram such as

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "m"] & F \arrow[d, "p"] \\ G \arrow[r, "q"] & H \end{tikzcd} \tag{1.6}$$

where exactness means that

$$0 \longrightarrow E \xrightarrow{l \oplus m} F \oplus G \xrightarrow{p \ominus q} H \longrightarrow 0$$

is a short exact sequence.

As a simplified example, for elliptic operator A with constant coefficient, one has the following exact diagram

$$\begin{tikzcd}
W^{n,p}(\mathbb{R}^n) \arrow[r, "A"] \arrow[d, hook, "i"] & W^{n-r,p}(\mathbb{R}^n) \arrow[d, hook, "i"] \\
W^{k,p}(\mathbb{R}^n) \arrow[r, "A"] & W^{k-r,p}(\mathbb{R}^n)
\end{tikzcd}$$

The closedness of $\operatorname{Im} A \oplus \iota = \ker A \ominus \iota$ implies, through Open mapping theorem, Gårding's inequality. The equality $\operatorname{Im} A \oplus \iota = \ker A \ominus \iota$ itself is the regularity result for elliptic equation. The subjectivity of $A \ominus \iota$ is the existence of approximate solution.

The exactness of the previous diagram comes from the fact that it slits, meaning that we can find compatible maps G and $\psi(D)$ such that

$$\begin{tikzcd}
W^{k,p}(\mathbb{R}^n) \arrow[r, shift right, "A"'] \arrow[d, shift left, "i"] & W^{k-r,p}(\mathbb{R}^n) \arrow[l, shift right, "G"'] \arrow[d, shift right, "i"'] \\
W^{l,p}(\mathbb{R}^n) \arrow[r, shift left, "A"] \arrow[u, shift left, "\psi(D)"] & W^{l-r,p}(\mathbb{R}^n) \arrow[l, shift left, "G"] \arrow[u, shift right, "\psi(D)"']
\end{tikzcd}$$

is a split diagram, where $\psi(D)$ is certain cut-off function on the frequency space. The splitness of the diagram in local (in $\mathbb{R}^n$) instead of just exactness reflects the fact that we have an algebraic formula of the solution/ of the Green kernel in $\mathbb{R}^n$.

The idea to go from local to global, naturally since the equation is linear, is to use a partition of unity and to remark that the commutator of a differential operator and the multiplication by a cut-off function is a differential operator of strictly lower order. This however is not the only ingredient. In the same spirit (but not the same technical reason) as the parametrix approach where we have to iterate to find the Green kernel, in the "diagram" approach, we lose algebraic control of the solution in an argument of the following type: a diagram sufficiently closed to an exact diagram (1.6) in $L(E,F) \times L(F,H) \times L(E,G) \times L(G,H)$ is still exact.

One also has a similar diagram for parabolic equation. The only difference is that one has *causality* in the parabolic case, meaning that the operator A is now an isomorphism. This is because when the initial condition is a vanishing condition $f\big|_{t=\alpha} = 0$ and when the boundary conditions on $\partial M \times [\alpha, \omega]$ is independent of time, we can make translation in time of the solution, and still have a solution. In the general case, we use Fredholm's Index theory.

1.3 Energy estimates.

The ingredients to bound higher order derivative of the solution include: (1) estimates of physical quantities, i.e. the total potential and kinetic energy, (2) Maximum principle and Gårding's inequality, (3) estimates of nonlinear differential operators using Besov spaces, and (4) L^1-comparison theorem for linear heat equation. Let us explain the last item. If f is a smooth solution of a linear heat equation

$$\frac{df}{dt} = -\Delta f + Cf \quad \text{on } [\alpha, \omega]$$

then by an argument similar to the proof of Maximum principle, one can estimate the L^∞-norm of $f\big|_{t=\omega}$ in term of $\|f\big|_\alpha\|_{L^\infty}$:

$$\|f\big|_\omega\|_{L^\infty} \leq e^{B(\omega-\alpha)}\|f\big|_\alpha\|_{L^\infty}.$$

We will need in certain moment to estimate $\|f\big|_\omega\|_{L^1}$ in term of $\|f\big|_\alpha\|_{L^1}$. Since L^1 is the dual space of L^∞, one can estimate $\|f\big|_\omega\|_{L^1}$ by finding an upper bound of $\int_{t=\omega} fh$ in term of $\|h\|_{L^\infty}$ where h is a smooth function on $M \times \{\omega\}$.

This can be done by considering the backwards heat equation propagating from time ω to α:

$$\begin{cases} \frac{dg}{dt} = \Delta g - Cg, & \text{on } N \times [\alpha, \omega] \\ g\big|_\omega = h, \end{cases}$$

This equation was chosen so that $\frac{d}{dt}\int_M fg = \int_M f\frac{d}{dt}g + g\frac{d}{dt}f = 0$, therefore $\int_{M\times\{\alpha\}} fg = \int_{M\times\{\omega\}} fg$. Apply L^∞-estimate to g and one obtains an L^1-estimate for f.

Part I

Harmonic maps: Introduction

Chapter 2

Harmonic maps of Riemannian manifolds

2.1 Harmonic maps

2.1.1 Variational approach: energy integral and tension field

Notation. Let M, M', M'' be Riemannian manifolds of dimension n, n' and n', n'' respectively. We will use $i, j, k, \ldots, \alpha, \beta, \gamma, \ldots, a, b, c$ for local coordinates of M, M', M''. Let $f : M \longrightarrow M', f' : M' \longrightarrow M''$ be a smooth maps, one denotes

$$f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}, \quad f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k f_k^\alpha$$

so that $\nabla h = h_i dx^i$ and $\nabla(\nabla h) = h_{ij} dx^i \otimes dx^j$ and $-\Delta h = \operatorname{Tr} \nabla(\nabla h) = g^{ij} h_{ij}$ for any smooth function h.

Definition 1. *The **energy desity** of f at $p \in m$ is defined by*

$$e(f)(p) = \frac{1}{2} \langle g, f^* g \rangle_p = \frac{1}{2} g^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$$

*and the **energy functional** of f is*

$$E(f) = \int_M e(f) dV = \frac{1}{2} \int_M g^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta} |\det(g_{ij})|^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$$

We recall that the inner product is between 2 tensors of type (p, q) $S = S_{j_1 \ldots j_q}^{i_1 \ldots i_p}, T = T_{l_1 \ldots l_q}^{k_1 \ldots k_p}$ is $\prod_{m,n} g_{i_m k_m} g^{j_n l_m} S_{j_1 \ldots j_q}^{i_1 \ldots i_p} T_{l_1 \ldots l_q}^{k_1 \ldots k_p}$

Remark 1. *The energy density is non-negative at every point. Hence $E(f) = 0$ if and only if $e(f) = 0$ at all points if and only if f is constant.*

Definition 2. *Let σ be a symmetric function of n variables and α be a symmetric (0,2) tensor field, one can define the **σ-energy desity** of α at $P \in M$ to be $\sigma(\beta_1, \ldots, \beta_n)(P)$ where β_i are eigenvalues of the linear operator $(g^{ik}\alpha_{ij})_{k,j}$. The **$\sigma$-energy** of α is $I_\sigma(\alpha) := \int_M \sigma(\alpha) dV$*

*Take $\alpha = f^* g'$, one calls $\sigma(\alpha)$ the **σ-energy density** of f and $I_\sigma(\alpha)$ the σ-energy of f.*

Example 1. *For example, the energy functional $E(f)$ is $I_{\frac{\sigma_1}{2}}(f)$. $V(f) := I_{\sigma_n^{1/2}}(f)$ is called the **volume** of f.*

Lemma 1 (variation of the energy)**.** *Let $f_t : M \longrightarrow M'$ be a smooth family of smooth maps between Riemannian manifolds for $t \in (t_0, t_1)$. Then*

$$\frac{d}{dt} E(f_t) = -\int_M \left(-\Delta f_t^\gamma + g^{ij} \Gamma'^\gamma_{\alpha\beta} f^\alpha_{t,i} f^\beta_{t,j} \right) g'_{\gamma\nu} \frac{\partial f_t^\nu}{\partial t} dV, \qquad \forall t \in (t_0, t_1)$$

Proof. One has

$$\begin{aligned} \frac{dE}{dt}(f_t) &= \frac{1}{2} \int \left[2 g^{ij} f_i^\alpha \frac{\partial^2 f_t^\beta}{\partial x^j \partial t} g'_{\alpha\beta} + g^{ij} f_i^\alpha f_j^\beta \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} \frac{df_t^\nu}{dt} \right] dV(g) \\ &= \frac{1}{2} \int \left[- \left(2 g^{ij} f_i^\alpha g'_{\alpha\beta} \right)_j \frac{df_t^\beta}{dt} + g^{ij} f_i^\alpha f_j^\beta \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} \frac{df_t^\nu}{dt} \right] dV(g) \end{aligned}$$

The first term is

$$\begin{aligned} - \left(2 g^{ij} f_i^\alpha g'_{\alpha\beta} \right)_j &= -2 g^{ij} f_{ij}^\alpha \frac{df^\beta}{dt} g'_{\alpha\beta} - 2 g^{ij} f_i^\alpha \frac{df^\beta}{dt} \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} f_j^\nu \\ &= 2 \Delta f^\alpha g'_{\alpha\beta} \frac{df_t^\beta}{dt} - 2 g^{ij} f_i^\alpha f_j^\beta \frac{\partial g'_{\alpha\nu}}{\partial y^\beta} \frac{df_t^\nu}{dt} \end{aligned}$$

It remains to check that

$$-2 \frac{\partial g'_{\alpha\nu}}{\partial y^\beta} + \frac{\partial g'_{\alpha\beta}}{\partial y^\nu} = -2 \Gamma'^\gamma_{\alpha\beta} g'_{\gamma\nu}$$

when we are allowed to permute α, β, which is routine. ☐

Definition 3. *1. A **vector field along** $f : M \longrightarrow M'$ is a smooth application $v : M \longrightarrow TM'$ such that $\pi \circ v = f$ where $\pi : TM' \longrightarrow M'$ is the canonical projection. In other words, it is the association of each point $P \in M$ a tangent vector at $f(P)$*

2. *The* ***tension field*** *of f is the following vector field along f defined by*

$$\tau(f)^\gamma := -\Delta f^\gamma + g^{ij}\Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta$$

By the Lemma 1, $\tau(f)$ is the unique vector field along f such that $\frac{d}{dt}E(f_t) = -\int_M \langle \tau(f), \frac{df_t}{dt}\rangle$. In particular, if f_t is the variation of f along a vector field v along f, i.e. $f_t(P) = \exp_{f(P)}(tv(P))$ then $\nabla_v E(f) = -\langle \tau(f), v\rangle$ along f.

3. *$f : M \longrightarrow M'$ is called* ***harmonic*** *if $\tau(f) = 0$, or equivalently f is a critical point of E.*

In normal coordinates of M at P and M' at $f(P)$, the tension field of f is given by

$$\tau^\gamma(f)(P) = \sum_i \frac{\partial^2 f^\gamma}{\partial (x^i)^2}(P)$$

Remark 2. 1. *If M' is flat, i.e. $R'_{\alpha\beta\gamma\delta} = 0$ then $\tau(f)^\gamma = -\Delta f^\gamma$ is linear in f. We refind the definition of harmonic function.*

2. *Since $\tau(f)$ depends locally on f, isometries and covering maps are harmonic.*

Proposition 1.1 (Holomorphicity implies harmonicity)**.** *Holomorphic maps between Kahler manifolds are harmonic.*

Proof. We recall that exponential function $\exp_P : T_P M \longrightarrow M'$ on a Kahler manifold M is holomorphic for any $P \in M$. In fact, let $v \in T_P M$ and $\delta v \in T_v(T_P M)$ be a tangent vector at v and denote abusively by J the complex structure of the complex vector space $T_P M$ and that of M, one needs to see that

$$D\exp_P(v).J\delta v = J(\exp_P(v))D\exp_P(v).\delta v \tag{2.1}$$

In fact, let Y_1, Y_2 be Jacobi fields along $U(t) = \exp_P(tv)$ the geodesics of M starting at P in direction v with $Y_1(0) = Y_2(0) = 0, \dot{Y}_1(0) = \delta v, \dot{Y}_2(0) = J\delta v$ then the LHS of (2.1) is $Y_2(1)$, and the RHS is $J(U(1))Y_1(1)$. Then one can see that $Y_2(t) - J(U(t))Y_1(t) = 0$ for every $t \in [0,1]$ since it is true at $t = 0$ and the derivative with respect to t vanishes as $\nabla_{\dot{U}} J = 0$.

Therefore, at a point P of a Kahler manifold M, there exist holomorphic coordinates $z^j = x^j + iy^j$ of M in a neighborhood of P such that $\{x_j, y_j : j = \overline{1, n/2}\}$ are normal coordinates centered in P. Using such coordinates for $P \in M$ and $f(P) \in M'$, one has $\Delta f^\gamma = 0$ since f^γ is holomorphic and $\Gamma'^\gamma_{\alpha\beta}(P) = 0$ by normality, it follows that $\tau(f) = 0$ at every point $P \in M$. $\square$

2.1.2 Formulation using connection on vector bundle

Setup and notation. Let E be a metric vector bundle over a Riemannian manifold M, i.e. each fiber of E is equiped with an inner product that we denote by $(g'_{\alpha\beta})$. The metric of M is denoted by (g_{ij}). Let n and m be the dimension of M of the fiber.

Covariant derivatives and exterior derivatives. We recall that a **covariant derivative** or a **connection** $\tilde{\nabla}$ of E is uniquely determined in local coordinates by an $m \times m$ matrix A of 1-forms, in other words, it is an 1-form on M with value in $Hom_M(E, E)$ which depends on the local frame of E (i.e. A is not a tensor with value in E). A is called the **connection form** of $\tilde{\nabla}$. Locally

$$\tilde{\nabla}_X(s^\alpha \tilde{e}_\alpha) = (\nabla_X s^\alpha)\tilde{e}_\alpha + A^\alpha_\beta(X) s^\beta \tilde{e}_\alpha.$$

When one prefers to work with forms rather than tensors with value in E, one uses an **exterior derivative**, a map $D : A^p(M, E) \longrightarrow A^{p+1}(M, E)$ which turns an p-form with value in E to an $p+1$-form with value in E. Locally

$$D(s^\alpha \tilde{e}_\alpha) = (ds^\alpha)\tilde{e}_\alpha + A^\alpha_\beta \wedge s^\beta \tilde{e}_\alpha.$$

and

$$D^2(s^\alpha \tilde{e}_\alpha) = (dA + A \wedge A) \wedge s.$$

One notes $\Theta := dA + A \wedge A$, which is an $m \times m$ matrix of 2-forms of M. Unlike A, Θ, seen as an 2-form with value in $Hom_M(E, E)$ does not depend on the local frame of E, i.e. Θ transforms as a (0,2) tensor with value in E, called the **curvature form**.

The fibrewise metric structure of E and the metric tensor of M give rise to a pointwise inner product of (p, q) tensors of M with value in E, in particular a pointwise inner product $(s, s') \mapsto s \cdot s'$ from $A^p(M, E) \times A^p(M, E)$ to $C^\infty(M)$. Integrated over M, the pointwise inner product gives rise to a global inner product $\int_M \langle \cdot, \cdot \rangle$ of $A^p(M, E)$. One denotes by $\delta : A^{p+1}(M, E) \longrightarrow A^p(M, E)$ the adjoint operator of $D : A^p(M, E) \longrightarrow A^{p+1}(M, E)$ with respect to this inner product, i.e. $\int_M \langle Ds, s' \rangle_{A^{p+1}(M,E)} = \int_M \langle s, \delta s' \rangle_{A^p(M,E)}$ for all $s \in A^p(M, E), s' \in A^{p+1}(M, E)$.

Laplacian operator and harmonic forms. The **connection Laplacian** is defined as a endomorphism of $A^p(M, E)$ given by

$$\tilde{\Delta} = D\delta + \delta D$$

and a form $s \in A^p(M, E)$ is called **harmonic** if $\tilde{\Delta} s = 0$. Since the Laplacian operator represents the *Dirichlet integral*, i.e.

$$\int_M \langle Ds, Ds' \rangle + \int_M \langle \delta s, \delta s' \rangle = \int_M \langle \tilde{\Delta} s, s' \rangle,$$

one has $\tilde{\Delta} s = 0$ if and only if $Ds = \delta s = 0$.

Riemannian connected bundle. The metric vector bundle E over M is called a **Riemannian-connected bundle** if it is equipped with a connection $\tilde{\nabla}$ under which the metric g' of E is parallel, i.e. $\tilde{\nabla} g' = 0$, in other words, the matrix A in a orthonormal frame is anti-symmetric: $A + {}^{\mathrm{t}}A = 0$. Unless explicitly indicated, we always suppose that our metric vector bundle E is Riemannian-connected and the metric g' is parallel to the connection being used.

Example 2. *The case of our interest is when we have a smooth map $f : M \longrightarrow M'$ and $E = f^*TM'$ is a metric vector bundle over M under the metric g' induced from M'. Taking the connection $\tilde{\nabla}$ to be the Levi-Civita connection ∇' on M', meaning*

$$\tilde{\nabla}_X s = \nabla'_{f_* X} s,$$

for any vector field s along f, one can see that E is a Riemannian-connected bundle over M.

Lemma 2. *Let E be a Riemannian-connected bundle and $s = s_i^\alpha dx^i \tilde{e}_\alpha \in A^1(M, E)$, one has*

1. *$\delta s = (\delta s)^\alpha \tilde{e}_\alpha \in A^0(M, E)$ where*

$$(\delta s)^\alpha = -g^{ij} \left(\nabla_i s_j^\alpha + A_{\beta i}^\alpha s_j^\beta \right),$$

2. *$\Delta s = (\Delta s)_i dx^i$ where $(\Delta s)_i$ is an $m \times m$ matrix given by*

$$(\Delta s)_i = -\tilde{\nabla}^k \tilde{\nabla}_k s_i + {}^{\mathrm{t}}\!\left(\Theta_i^h - \mathrm{Ric}_i^h \right) s_h$$

where:

- *the indices i, h, k correspond to local coordinates of M,*
- *Θ_i^h is the curvature form of $\tilde{\nabla}$ with its indices raised by the metric g of M,*

- $\mathrm{Ric}_i^h = \mathrm{Ric}_i^h I_m$ *is the Ricci curvature tensor of* (M, g) *with indices raised by the metric* g*, multiplied by the identity* $m \times m$ *matrix,*
- $\tilde{\nabla}^k = g^{hk}\tilde{\nabla}_h$.

3. *With* $s \cdot s'$ *denoting the pointwise inner product of* $A^1(M, E)$ *and* $\langle \cdot, \cdot \rangle_E$ *denoting the metric* g' *of* E*, one has*

$$-\frac{1}{2}\Delta(s \cdot s) = s \cdot \Delta s - \langle \tilde{\nabla}_i s_k, \tilde{\nabla}^i s^k \rangle_E - \left\langle {}^{\mathrm{t}}\left(\Theta_i^h - \mathrm{Ric}_i^h\right) s_h, s^i \right\rangle_E \quad (2.2)$$

where the superscript i, h *are raised by the metric* g*.*

Proof. Computational in nature. □

Remark 3. 1. *We note by* $Q(s)$ *the last term of* (2.2)*, then* Q *is a (2,0) tensor on* M *with value in* $E^* \otimes E^*$ *where* E^* *is the dualised bundle of* E*. In practice,* Q *is an* $mn \times mn$ *matrix with coefficients*

$$Q_{\alpha\beta}^{hi} = g^{hk} g^{ij} \left[\left(g'_{\alpha\gamma} \Theta_\beta^\gamma \right)_{kj} - g'_{\alpha\beta} \mathrm{Ric}_{kj} \right]$$

.

2. *Since* $\int_M \Delta(s \cdot s) dV = 0$*, if* s *is harmonic, one has*

$$\begin{aligned} \int_M Q(s) dV &= -\int_M \langle \tilde{\nabla}_i s_k, \tilde{\nabla}^i s^k \rangle_E dV \\ &= -\int_M \| \nabla_i s_k^\alpha dx^i \otimes dx^k \otimes \tilde{e}_\alpha \|_{A^2(M,E)}^2 dV \leq 0 \end{aligned} \quad (2.3)$$

2.1.3 The case of $E = f^*TM'$

Energy functional and tension field

Our interest will be the case of Example 2 where $E = f^*TM'$ for some smooth map $f : M \longrightarrow M'$ of Riemannian manifolds is a Riemannian-connected bundle over M with the connection $\tilde{\nabla}$ given by the Levi-Civita connection of M'.

In this section, the tangent map $Tf : TM \longrightarrow TM'$ can be interpreted as a form f_* in $A^1(M, E)$. The energy functional can be rewritten as

$$E(f) = \frac{1}{2}\int_M f_i^\alpha f_j^\beta g^{ij} g'_{\alpha\beta} dV = \frac{1}{2}\langle f_*, f_* \rangle_{A^1 M, E}.$$

Proposition 2.1. *Let* $f : M \longrightarrow M'$ *and* $E = f^*TM'$ *be the Riemannian-connected bundle over* M*. Then:*

1. *$A_\alpha^\beta = \Gamma'^\beta_{\gamma\alpha} f_i^\gamma dx^i$ where $\Gamma'^\beta_{\gamma,\alpha}$ are Christoffel symbols of (M', g').*

2. *$Df_* = 0$ where f_* is considered as an element of $A^1(M, E)$. Hence $\tilde{\Delta} f_* = D\delta f_*$.*

3. *The tension field of f is $\tau(f) = -\delta f_*$.*

Proof. 1. We will use the fact that $\tilde{\nabla} g' = 0$. Given two section $s = s^\alpha \tilde{e}_\alpha, t = t^\beta \tilde{e}_\beta$ of E, expanding $\nabla_i (s \cdot t) = (\tilde{\nabla}_i s) \cdot t + s \cdot \tilde{\nabla}_i t$, one has

$$s^\alpha t^\beta \frac{\partial g'_{\alpha\beta}}{\partial x^i} = s^\alpha t^\beta \left(A^\gamma_{\alpha i} g'_{\gamma\beta} + A^\gamma_{\beta i} g'_{\alpha\gamma} \right)$$

Taking s, t to be of small support, $\alpha = \beta$ and substituing $A^\gamma_{\alpha i} = \Gamma'^\nu_{\gamma\alpha} f_i^\gamma$, one obtains the first statement.

2. By direct computation:

$$Df_* = \left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \Gamma'^\alpha_{\gamma\beta} f_i^\gamma f_j^\beta \right) dx^j \wedge dx^i \otimes \tilde{e}_\alpha = 0$$

since it is the product of a symmetric quantity in (i, j) and an anti-symmetric one.

3. Using the first part of Lemma 2 for $s = f_* = f_i^\alpha dx^i \otimes \tilde{e}_\alpha$, one has $\delta f_* = -g^{ij} \left(\nabla_i \nabla_j f^\gamma + \Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta \right) \tilde{e}_\gamma = -\tau(f)$

□

It follows immediately that

Corollary 2.1. *$f : M \longrightarrow M'$ is a harmonic map of Riemannian manifolds if and only if f_* is harmonic as form in $A^1(M, f^*TM')$.*

Fundamental form, some results in case of signed curvature

Definition 4. *The **fundamental form** of a map $f : M \longrightarrow M'$ of Riemannian manifolds is the (0,2) symmetric tensor on M with value in $E = f^*TM'$ defined by*

$$\beta(f) := \tilde{\nabla} f_* = \left(f^\gamma_{ij} + \Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta \right) dx^i \otimes dx^j \otimes \tilde{e}_\gamma.$$

*The function f is called **totally geodesic** if $\beta(f) = 0$ identically on M.*

Remark 4. *1. The tension field $\tau(f) = g^{ij} \beta(f)_{ij}$ is the trace of the fundamental form.*

2. *If f is totally geodesic then it is harmonic.*

When $s = f_*$, Lemma 2 and Remark 3 become Lemma 3, with no more than direct computation. The appearance of the Riemann curvature tensor R' of (M', g') is due to the formula

$$R'^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma'^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma'^{\rho}{}_{\mu\sigma} + \Gamma'^{\rho}{}_{\mu\lambda}\Gamma'^{\lambda}{}_{\nu\sigma} - \Gamma'^{\rho}{}_{\nu\lambda}\Gamma'^{\lambda}{}_{\mu\sigma}.$$

Lemma 3. 1. *$Q(f_*)$ is given by*

$$Q(f_*) = R'_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta g^{ik} g^{jl} - \mathrm{Ric}^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$$

and

$$Q(f_*)^{ij}_{\alpha\beta} = R'_{\alpha\beta\gamma\delta} f_k^\gamma f_l^\delta g^{ik} g^{jl} - \mathrm{Ric}^{ij} g'_{\alpha\beta}.$$

2. *If f is harmonic then*

$$-\Delta e(f) = |\beta(f)|^2 - Q(f_*)$$

where $|\beta(f)|$ is the pointwise norm of $\beta(f)$.

The previous computation of $Q(f_*)$ in term of Riemannian curvature of M' and Ricci curvature of M give the following result in the case where the curvature of M and M' are of definite sign.

Notation. Given a Riemannian manifold M, we will use the following notation:

1. $\mathrm{Ric} \geq 0$ (resp. $\mathrm{Ric} > 0$) if the Ricci curvature is positive semi-definite (resp. positive definite) as symmetric bilinear form.

2. $\mathrm{Riem} \leq 0$ (resp. $\mathrm{Riem} < 0$) if all sectional curvatures are negative (resp. strictly negative), i.e. $R_{ijhk} u^i v^j u^h v^k \leq 0$ (resp. $R_{ijhk} u^i v^j u^h v^k < 0$) for non-colinear vectors u, v.

Corollary 3.1. *Let $f : M \longrightarrow M'$ be a map of Riemannian manifolds.*

1. *If f is harmonic and $Q(f_*) \leq 0$ then f is totally geodesic and $e(f)$ is constant.*

2. *If $\mathrm{Ric}(M) \geq 0$ and $\mathrm{Riem}(M') \leq 0$ then f is harmonic if and only if f is totally geodesic.*

3. *Under the same condition as 2),*

- *If* $\mathrm{Ric}(M) > 0$ *at one point of* M *then all harmonic maps are constant.*
- *If* $\mathrm{Riem}(M') < 0$ *everywhere in the image of* f *and* f *is harmonic, then* f *is either constant or maps* M *onto a closed geodesic of* M'.

Proof. All the statements are consequence of 2) of Lemma 3 and the fact that $\int_M \Delta e(f) dV = 0$, noticing that

- $\mathrm{Ric}^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$ is $\mathrm{Ric} \otimes g'$ applied doubly to $f_i^\alpha dx^i \otimes \tilde{e}_\alpha$.
- $R'_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta g^{ik} g^{jl}$ is $(f^*R')_{ijhk} g^{ik} g^{jl}$. In a normal coordinate at P where $g^{ik} = \delta_{ik}, g^{jl} = \delta_{jl}$, it is the sum of sectional curvatures of tangent planes formed by $f_* e_i, f_* e_j$, and therefore negative.

For 3), if $\mathrm{Ric}(M) < 0$ at one point $P \in M$ then at that point $f_i^\alpha dx^i \tilde{e}_\alpha = 0$, meaning $f_* = 0$, hence $e(f)$ vanishes at P. Since $e(f)$ has to be constant, it vanishes identically, which implies that f is constant.

If $\mathrm{Riem}(M') < 0$, one sees that all $f_* e_i, f_* e_j$ are colinear, so the image of Tf is of one dimension, which leads to the conclusion, as we will see later that a totally geodesic map transforms geodesic to geodesic. □

2.1.4 Example: Riemannian immersion

Let $f : M \longrightarrow M'$ be a Riemannian immersion, i.e. Tf is injective and $f^*g' = g$. We will see that the fundamental form $\beta(f)$ that we defined earlier is the same as usual definition in courses of Riemannian geometry.

Second fundamental form.

One defines the symmetric (0,2)-tensor II as the unique tangent vector of M' such that

$$\langle \mathrm{II}_{ij}, \xi_\sigma \rangle := -\langle \tilde{\nabla}_i \xi_\sigma, f_* e_j \rangle$$

for every vector field ξ_σ of M' orthogonal to M.

Lemma 4 (Second fundamental form)**.** *If* f *is a Riemannian immersion then* $\beta(f)_{ij} = \mathrm{II}_{ij}$ *and they are orthogonal to* M*. In particular, if* f *is totally geodesic than it maps geodesics of* M *to geodesics of* M'

Proof. One has

$$\begin{aligned}\langle\tilde{\nabla}_i\xi_\sigma, f_*e_j\rangle &= \langle\xi_\sigma, \tilde{\nabla}_i(f_*e_j)\rangle = \langle\xi_\sigma, \tilde{\nabla}_i(f^\gamma_l dx^l\otimes\tilde{e}_\gamma)e_j + f_*\nabla_i e_j\rangle \\ &= \langle\xi_\sigma, (f^\gamma_{il}dx^l\tilde{e}_\gamma + f^\gamma_l dx^l\tilde{\nabla}_i\tilde{e}_\gamma)e_j\rangle \\ &= \langle\xi_\sigma, f^\gamma_{ij}\tilde{e}_\gamma + f^\gamma_j A^\alpha_{\gamma i}\tilde{e}_\alpha\rangle = \left\langle\xi_\sigma, \left(f^\gamma_{ij} + \Gamma'^\gamma_{\alpha\beta}f^\alpha_i f^\beta_j\right)\tilde{e}_\gamma\right\rangle \\ &= \langle\xi_\sigma, \tilde{\nabla}_i(f_*).e_j\rangle = \langle\xi_\sigma, \beta_{ij}(f)\rangle\end{aligned} \tag{2.4}$$

where we used $\xi_\sigma \perp f_*e_j$ in the first line and $\xi_\sigma \perp f_*([e_i, e_j])$ in the second line. Hence $\mathrm{II}_{ij} \equiv -\beta(f)_{ij}$ modulo an element in TM. In fact one has $\beta(f)_{ij} \perp M$ and therefore $\mathrm{II} = -\beta(f)$, since $\beta(f)_{ij} = \tilde{\nabla}_i(f_*).e_j$ and

$$\begin{aligned}\langle\beta(f)_{ij}, f_*e_k\rangle &= \langle\tilde{\nabla}_i(f_*).e_j, f_*e_k\rangle = \tilde{\nabla}_i\langle f_*e_j, f_*e_k\rangle - \langle\nabla_i e_j, e_k\rangle - \langle f_*e_j, \tilde{\nabla}_i(f_*e_k)\rangle \\ &= -\langle\beta(f)_{ik}, f_*e_j\rangle + \nabla_i\langle e_j, e_k\rangle - \langle\nabla_i e_j, e_k\rangle - \langle e_j, \nabla_i e_k\rangle \\ &= -\langle\beta(f)_{ik}, f_*e_j\rangle\end{aligned}$$

Then using the symmetric of $\beta(f)_{ij}$, one has $\langle\beta(f)_{ij}, f_*e_k\rangle = 0$.

Finally, if $\beta(f) = 0$ and X is a geodesic vector field of M, one needs to prove that f_*X is a geodesic vector field of M'. In fact

$$\tilde{\nabla}_X(f_*X) = (\tilde{\nabla}_X f_*)X + f_*\nabla_X X = \beta(f)(X, X) = 0.$$

Hence f_*X is a geodesic field of M'. □

Example 3. *The inclusion $x \mapsto (x, y_0)$ of a Riemannian manifold M to the Riemannian product $M \times N$ is totally geodesic.*

Definition 5. *Given an orthonormal frame $(\xi_\sigma)_{1\le\sigma\le n'-n}$, the **mean normal curvature field** of M in M' at $P \in M$ is defined as*

$$\xi(P) := \sum_{\sigma=1}^{n'-n} g^{ij}\langle\mathrm{II}_{ij}, \xi_\sigma\rangle\xi_\sigma = -\sum_{\sigma=1}^{n'-n}\langle\tau(f), \xi_\sigma\rangle\xi_\sigma.$$

*The immersion f is said to be **minimal** if ξ vanishes identically on M.*

Remark 5.

1. *Since $(\xi_\sigma)_{1\le\sigma\le n'-n}$ is an orthonormal frame, one also has*

$$\xi(P) = -g^{ij}\langle\tilde{\nabla}_i\xi_\sigma, f_*e_j\rangle\xi_\sigma(P) = -\sum_{\sigma=1}^{n'-n}\mathrm{div}\ (\xi_\sigma(P))\ \xi_\sigma(P)$$

2. *The mean normal curvature field is the tension field of f, i.e. $\xi = -\tau(f)$. Minimal immersions are exactly harmonic immersion.*

The case of signed curvature.

If $f : M \longrightarrow M'$ is a Riemannian immersion then the Ricci term of Lemma 3 is actually the scalar curvature of M, one has

Proposition 4.1. *Let $f : M \longrightarrow M'$ be a Riemannian immersion. Suppose that $\mathrm{Riem}(M') \leq 0$ and $r = g^{ij}\mathrm{Ric}_{ij} < 0$ at one point of M. If f is harmonic then it is constant.*

2.1.5 Example: Riemannian submersion

Results of Ehresmann and Hermann.

In this section, the function $f : M \longrightarrow M'$ will be a Riemannian submersion $\pi : M \longrightarrow B$, i.e. $T\pi$ is subjective and $\pi^* g' = g$. We will regard π as a fibration and calculate its tension field. We start with two theorems of Hermann with [Bes07] as reference. A tangent vector of M lying in $\ker T_P\pi$ is said to be **vertical**. Since $\pi^* g' = g$, the plane $\mathcal{H}_P := \ker T_P\pi^{\perp}$ is isometric to $T_{\pi(P)}B$ and is said to be **horizontal**, such $\mathcal{H}_P$ form a distribution of planes as P varies in M.

Definition 6. *The plane distribution $\mathcal{H}$ is called **complete** if every curve γ in B **lifts** horizontally on M at each point P in $M_{\gamma(0)}$, i.e. there exists a curve $\hat{\gamma}$ in M such that $\pi \circ \hat{\gamma} = \gamma$ and $\hat{\gamma}(0) = P \in M$.*

*A vector field X of M is said to be **projectable** if $\pi_* X$ is well-defined, i.e. $\pi_* X$ does not change on each fibre. In that case, one says that X is π-**associated** to the vector field $\pi_* X$ of B.*

*X is said to be **basic** if it is projectable and horizontal.*

Remark 6. *If a vector field X on M is π-associated with a vector field $\check{X}$ on B, then*

1. *their flows are related by π: $\pi(\Phi^t_X) = \Phi^t_{\check{X}}$,*
2. *the Lie bracket satisfies: $[X, Y]$ is projectable and π-associated with $[\check{X}, \check{Y}]$.*

Theorem 5 (Ehresmann-Hermann)**.**

1. *If $\mathcal{H}$ is complete then the fibration $\pi : M \longrightarrow B$ is locally trivial.*
2. *If M is complete then $\mathcal{H}$ is a complete distribution and B is a complete manifold.*

Remark 7. 1. *The trivialising map $\phi : U_M \longrightarrow U_B \times F$, where U_M, U_B are open sets of M, B, is only a diffeomorphism and not a isometry, each fibre is equipped with different metric when identified with F.*

2. *The metric of M is not a Riemannian product of a (vertical) metric on F and the (horizontal) metric on B, but it is a product pointwise. To be precise, one has*

$$g_{(b,f)}(v_h^1 + v_v^1, v_h^2 + v_v^2) = g'_b(v_h^1, v_h^2) \times \hat{g}_{(b,f)}(v_v^1, v_v^2) \tag{2.5}$$

where $v^i = v_h^i + v_v^i$ is the decomposition of vector v^i to horizontal and vertical components, g' is the horizontal metric (the metric on M) and $\hat{g}_{(b,f)}$ is the restriction of g on the fibre M_b. However, when the fibration is of totally geodesic fibres, g is a Riemannian product $g_{(b,f)} = g'_b \times \hat{g}_f$, see Theorem 6.

Sketch of proof. The first part is due to Ehresmann, take a small geodesic ball center at P, and connect every point Q to P by a curve γ. Map every point $\hat{\gamma}(0) \in M_P$ to the point $\hat{\gamma}(1) \in M_Q$ where $\hat{\gamma}$ is the lift of γ starting from $\hat{\gamma}(0)$. One has a diffeomorphism $\theta_\gamma : M_{\gamma(0)} \longrightarrow M_{\gamma(1)}$.

The second part, due to Hermann, can be established in 2 steps:

First, by direct computation, one proves that if any geodesic field X on B lifts to a horizontal vector field $\hat{X}$ then $\hat{X}$ is a geodesic vector field. In fact, denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connection on M and B respectively and $\mathcal{V}, \mathcal{H}$ the vertical and horizontal projection of tangent vectors of M. Then $\nabla_{\hat{X}}\hat{X} = \mathcal{V}\nabla_{\hat{X}}\hat{X} + \mathcal{H}\nabla_{\hat{X}}\hat{X}$ in which $\mathcal{H}\nabla_{\hat{X}}\hat{X}$ is actually the horizontal lift of $\tilde{\nabla}_X X$ therefore vanishes. We claim that $\mathcal{V}\nabla_Y Y = \frac{1}{2}\mathcal{V}[Y,Y]$ hence also vanishes for every basic vector field Y. In fact let U be any vertical vector field then

$$\langle U, \mathcal{V}\nabla_Y Y\rangle = \langle U, \nabla_Y Y\rangle = -\langle \nabla_X U, X\rangle = \langle \nabla_U X, X\rangle = \frac{1}{2}\nabla_U \langle X, X\rangle = 0$$

where we used the fact that $\nabla_X U - \nabla_U X = [X,U] = \widehat{[\pi_* X, \pi_* U]} = 0$ and $\langle X, X\rangle$ is constant on each fibre (being $\langle \pi_* X, \pi_* X\rangle$), hence in every vertical direction U (Remark: this corresponds to the fact that g' only depend on b).

Now if M is complete then for every geodesic curve γ in B, let X be the velocity field of γ and $\hat{X}$ be the horizontal lift of X, which is now a horizontal, geodesic field of M, whose integral curves are lifts of γ. Therefore B is complete and every geodesic curve of B lifts horizontally to M.

For the general curve γ of M, the idea will be to approximate it by geodesics and lift part by part. $\square$

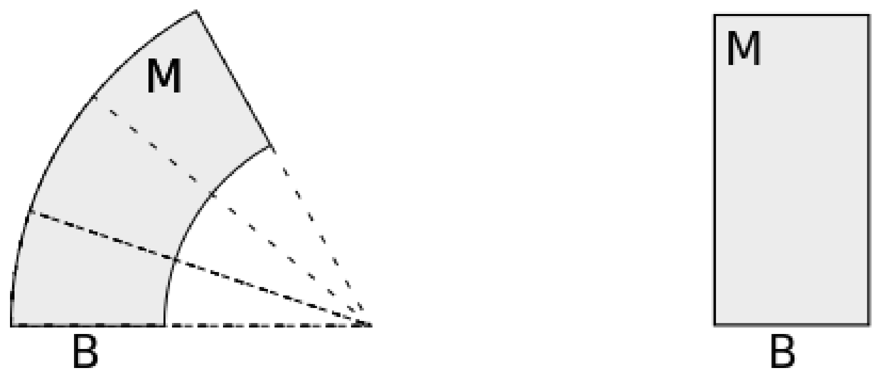


Figure 2.1: The trivialising map is only a diffeomorphism and not an isometry

Theorem 6 (Hermann)**.** *If the fibration $\pi : M \longrightarrow B$ is of totally geodesic fibres then the diffeomorphisms $\hat{\gamma}(0) \longrightarrow \hat{\gamma}(1)$ are in fact isometries between fibres and M is then locally a Riemannian product of B and the fibre, now equipped with its unique metric induced by M.*

Proof. We need to prove that the metric on fibres $\hat{g}_{(b,f)}$ does not depend on the point f of the fibre, i.e. for every basic vector field X, one has $\mathcal{L}_X \hat{g} = 0$, where by $\hat{g}$, we mean the (0,2) symmetric tensor $(Y_1, Y_2) \mapsto \langle \mathcal{V}Y_1, \mathcal{V}Y_2 \rangle$. Let U, V be vertical vector fields of M then

$$\begin{aligned} X(\hat{g}(U,V)) &= (\mathcal{L}_X \hat{g})(U,V) + \hat{g}([X,U],V) + \hat{g}(U,[X,V]) \\ &= \langle \nabla_X U, V \rangle - \langle \nabla_U X, V \rangle + \langle U, \nabla_X V \rangle - \langle U, \nabla_V X \rangle + (\mathcal{L}_X \hat{g})(U,V) \end{aligned}$$

Hence $(\mathcal{L}_X \hat{g})(U,V) = \langle \nabla_U X, V \rangle + \langle U, \nabla_V X \rangle = -2\langle \mathrm{II}(U,V), X \rangle = 0$. Since the map $\hat{\gamma}(0) \mapsto \hat{\gamma}(1)$ is in the one-parameter group of differomorphism associate to a basic vector field X, it preserves $\hat{g}$. □

Tension fields and harmonic fibrations.

We will now calculate the tension field of a fibration map $\pi : M \longrightarrow B$.

Proposition 6.1. *Let $\pi : M^n \longrightarrow B^{n'}$ be a complete Riemannian fibration then*

1. $e(\pi) = n/2$.
2. *Let M_b be a fibre of π and $\iota_{M_b} : M_b \hookrightarrow M$ be the inclusion. Then $\tau(\pi) = -\pi_* \tau(\iota_{M_b})$ on M_b.*

In particular, π is harmonic if and only if its fibres are minimal submanifolds of M, i.e. the inclusions ι_{M_b} are harmonic.

Proof. 1. is obvious. For 2), note that

$$\tau(\iota_{M_b}) = -\sum_{\sigma=1}^{n'} \text{div } (e_\sigma(P))e_\sigma(P)$$

for any orthonormal frame $e_\sigma(P)$ of normal vectors of $M_{\pi(P)}$. Take e_σ to be $e_\sigma = \text{grad } \pi^\sigma$ where π^σ is the σ-th component of π in a normal coordinate of V around $\pi(P)$. Note that e_σ are actually the horizontal lift of the basis vectors $\tilde{e}_\sigma$ of the frame at $\pi(P)$, and therefore are normal vectors of M_P.

Meanwhile, one has $\tau(\pi) = -\Delta\pi^\sigma \tilde{e}_\sigma$ at $\pi(P)$ since the Christoffel symbols vanish at P. Comparing the two vector fields, one has $\tau(\pi) = -\pi_* \tau(\iota_{M_b})$ at P. □

Example 4. *A complete Riemannian fibration $\pi : M \longrightarrow B$ with totally geodesic fibres are harmonic.*

2.1.6 Composition of maps

The following results come from direct computation of the second fundamental form and tension field of composition of maps between Riemannian manifolds. Again, we use indices $i, j, k, \ldots$ for M, $\alpha, \beta, \gamma, \ldots$ for M' and $a, b, c, \ldots$ for M''.

Proposition 6.2. *Let $f : M \longrightarrow M'$ and $f' : M' \longrightarrow M''$ be smooth maps of Riemannian manifolds, then*

$$\beta(f' \circ f)^a_{ij} = \beta(f)^\gamma_{ij} f'^a_\gamma + \beta(f')^a_{\alpha\beta} f^\alpha_i f^\beta_j \tag{2.6}$$

and

$$\tau(f' \circ f)^a = \tau(f)^\gamma f'^a_\gamma + g^{ij} \beta(f')^a_{\alpha\beta} f^\alpha_i f^\beta_j \tag{2.7}$$

Therefore,

If f' is	*and f is*	*then $f' \circ f$ is*
totally geodesic	*totally geodesic*	*totally geodesic*
totally geodesic	*harmonic*	*harmonic*

and the inverse of a totally geodesic map is totally geodesic.

Proof. Direct computation. □

Remark 8. *It is not true in general that the composition of harmonic maps are harmonic.*

Proposition 6.3 (composition with immersion)**.** *If $f' : M' \longrightarrow M''$ is a Riemannian immersion and $f : M \longrightarrow M'$ then*

1. *Energy functionals: $E(f) = E(f' \circ f)$.*
2. *Tension fields: $\tau(f)$ is the projection of $\tau(f' \circ f)$ to M'.*

Proof.

1. One has $e(f) = \frac{1}{2}\langle g, f^* g' \rangle = \frac{1}{2}\langle g, (f' \circ f)^* g'' \rangle = e(f' \circ f)$.
2. One has $\tau(f' \circ f)^a = \tau(f)^a + g^{ij}\beta(f')^a_{\alpha\beta} f^\alpha_i f^\beta_j$ by (2.7). The second term being the restriction of the tension field of M' to the image of M, the conclusion follows.

□

The following immediate corollary of Proposition 6.3 is a generalization of the fact that a curve is geodesic if and only if it is perpendicular to its tension field.

Corollary 6.1. *A map $f : M \longrightarrow M'$ is harmonic if and only if $\tau(f' \circ f) \perp M'$*

Proposition 6.4 (composition with submersion)**.** *Let $f' : M' \longrightarrow M''$ be a Riemannian fibration with totally geodesic fibres and $f : M \longrightarrow M'$ then*

$$\tau(f' \circ f) = f'_*(\tau(f))$$

Proof. One can suppose that M' is a Riemannian product of M'' and its fibre, and f' is the projection to M'', since this is true locally and the proposition is local. Then the conclusion is that the tension field of the projection is the projection of the tension field, or equivalently the tension field of $f = f_1 \times f_2 : M \longrightarrow M'' \times F$ is $\tau(f) = (\tau f_1, \tau f_2)$. This follows from the explicit formula of $\tau(f)$, noting that the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ vanish except when the indice α, β, γ belong to the same tangent space (TM'' or TF). □

Example 5.

1. *A map $f : M \longrightarrow M' \times M''$ is harmonic if $f = (f^1, f^2)$ with f^1, f^2 harmonic.*
2. *Take $M'' = M$ in Proposition 6.4 and $f = s : M \longrightarrow M'$ a section of the fibration f', one sees that the tension field $\tau(s)$ is always vertical.*

The following corollary is immediate.

Corollary 6.2. *Let $f' : M' \longrightarrow M''$ be a proper Riemannian embedding and N is a normal turbular neighborhood of M' which can be seen as a smooth fiber bundle over M'. Denote by $\pi : N \longrightarrow M'$ the projection. Then for all map $f : M \longrightarrow N$, $\pi \circ f$ is harmonic if and only if $\tau(f)$ is vertical.*

2.2 Nonlinear heat flow: Global equation and existence of harmonic maps.

2.2.1 Statement of the main results.

We want to prove in the next part the existence of harmonic map between manifolds M and M' by deforming any map $f : M \longrightarrow M'$ using the τ-flow, meaning solving the PDE:

$$\begin{cases} \frac{df_t}{dt} = \tau f_t, \quad t \in [\alpha, \omega] \\ f_\alpha = f, \end{cases} \tag{2.8}$$

The equation makes sense because both $\frac{df_t}{dt}$ and τf_t are vector fields along f_t. Since this is the gradient-descent equation for E, the energy of f_t decreases and we hope, under conditions, to obtain convergence of $\{f_t\}$ to a critical point f_∞ of E, this will prove that any homotopy class of $C^\infty(M, M')$ has at least a harmonic map.

It is proved by Eells and Sampson [ES64] that

Theorem 7 (Eells-Sampson)**.** *Let M and M' be compact Riemannian manifolds with* $\mathrm{Riem}(M') \leq 0$ *then there exists a harmonic map $f : M \longrightarrow M'$ in each homotopy class.*

Several boundary conditions, of Dirichlet, Neumann or mixed type, are also taken into account by Hamilton [Ham75], as an example, we will state the Dirichlet problem:

Theorem 8 (Hamilton)**.** *Let M and M' be compact Riemannian manifolds possibly with boundary. Suppose that M' has* $\mathrm{Riem}(M') \leq 0$ *and $\partial M'$ is convex, then any relative homotopy class of $C^\infty(M, M')$ has a harmonic element.*

About the terminology, **relative homotopy class** means that we only deform f among maps with the same value on ∂M. The **convexity of $\partial M'$** means that the geodesic at any point in $\partial M'$ with initial tangent vector paralell to the boundary does not enter the interior of M' in short time.

This condition can be expressed using the Christoffel symbols of M' at the point in question.

It is easy to see that the convexity of $\partial M'$ is a necessary condition, as harmonic maps from $\mathbb{R}$ are geodesics: Suppose the condition does not hold at $x \in \partial M'$, meaning that upto time t the geodesic flow of M' initially tangent to $\partial M'$ remains in the interior. The geodesic of $\partial M'$ of length less than t with the same initial tangent therefore cannot be deformed into a geodesic of M' in relative homotopy class.

2.2.2 Strategy of the proof.

In order to have a global frame, we will embed M' into an Euclidean space V, but we will not use the Euclidean metric of V. In fact, let T be a tubular neighborhood of M' in V then T is diffeomorphic to $M' \times D$ where D is a sufficiently small of dimension being the codimension of M' in V, and we will equip T with the product metric of $M' \times D$.

Since $M' \equiv M' \times \{0\}$ is totally geodesic in T, one has for every smooth function $f : M \longrightarrow M'$:

$$\tau_V(f) = \tau_T(f) = \tau_{M'}(f)$$

The crucial property we expect for a global equation of (2.8), is that if the solution initially is in $M' \subset V$ then it remains in M' for all relevant time $t > \alpha$. Eells-Sampson [ES64] did this by using at the same time 2 different metrics on T, namely the product metric as tubular neighborhood and the Euclidean metric. I choose to present here the formulation of Hamilton, which is conceptually simpler with the only drawback being that we need to establish the uniqueness of solution of (2.8) first.

After having the global equation, we will prove the short time existence of solution by linearising the equation and using Implicit function theorem. The global formulation and the proof of short-time existence is independent of the negative curvature hypothesis, which will only be used later to establish energy estimates and assure the convergence of long-time solution and the vanishing of its tension field.

2.2.3 Global equation and Uniqueness of nonlinear heat equation.

Theorem 9 (Global equation). *If the smooth function $F_t : M \times [\alpha, \beta] \longrightarrow V$ satisfies*

$$\frac{dF_t}{dt} = \tau_V(F_t) \tag{2.9}$$

and $F_t(M \times \{\alpha\}) \subset M'$ *then* $F_t(M \times [\alpha, \omega]) \subset M'$

Proof. Let ι be the isometry of T given by $(y,d) \mapsto (y,-d)$ for $(y,d) \in M' \times D \equiv T$ and pose $G_t = \iota F_t$ then G_t and F_t coincide initially since M' is fixed by ι. Moreover

$$\frac{dG_t}{dt} = d\iota.\frac{dF_t}{dt} = d\iota(\tau_V(F_t)) = \tau_V(\iota F_t) = \tau_V(G_t)$$

We conclude that $F_t = G_t = \iota F_t$, hence F_t remains in M' for all relevant t, using the following uniqueness of nonlinear heat equation. $\square$

Theorem 10 (Uniqueness of solution of nonlinear hear equation)**.** *Let* $f_1, f_2 : M \times [\alpha, \omega] \longrightarrow M'$ *be* C^2 *functions satisfying the non-linear heat equation* $\frac{df_i}{dt} = \tau_{M'}(f_i)$, *i.e.*

$$\frac{df_i}{dt} = -\Delta f^\gamma + g^{ij}\Gamma'^{\gamma}_{\alpha\beta} f_i^\alpha f_j^\beta$$

where $\Gamma'^{\gamma}_{\alpha\beta}$ *are Christoffel symbols of* M'. *Suppose that* f_1 *and* f_2 *coincide on* $M \times \{\alpha\}$. *Then* $f_1 = f_2$ *on* $M \times [\alpha, \omega]$.

Proof. It is sufficient to prove the theorem for ω very close to α, therefore by compactness of M, we can suppose that there exists a finite atlas $M = \bigcup_i U_i$ with $f_1(U_i \times [\alpha, \omega])$ and $f_2(U_i, [\alpha, \omega])$ being in the same chart V_i of M'. We consider the distance function $\sigma(a,b) = \frac{1}{2} d_{M'}(a,b)^2$ for $a, b \in M'$ to measure the difference between f_1 and f_2 by

$$\rho(x,t) = \sigma(f_1(x,t), f_2(x,t))$$

The strategy is to prove that there exists $C > 0$ such that $\frac{d\rho}{dt} \leq -\Delta + C\rho$, then by Maximum principle, one has $\rho = 0$.

Fix a chart U_i of M and the corresponding V_i of M', one has by straightforward calculation:

$$\begin{aligned}
\frac{d\rho}{dt} = -\Delta\rho - g^{ij} & \left(\frac{\partial^2 \sigma}{\partial f_1^\beta \partial f_1^\gamma} - \frac{\partial \sigma}{\partial f_1^\alpha} \Gamma'^{\alpha}_{\beta\gamma}(f_1) \right) f_{1i}^{\beta} f_{1j}^{\gamma} \\
& - g^{ij} \left(\frac{\partial^2 \sigma}{\partial f_2^\beta \partial f_2^\gamma} - \frac{\partial \sigma}{\partial f_2^\alpha} \Gamma'^{\alpha}_{\beta\gamma}(f_2) \right) f_{2i}^{\beta} f_{2j}^{\gamma} - 2g^{ij} \frac{\partial^2 \sigma}{\partial f_1^\beta \partial f_2^\gamma} f_{1i}^{\beta} f_{2j}^{\gamma}
\end{aligned}$$

where g^{ij} is the metric on M and $\Gamma'^{\alpha}_{\beta\gamma}$ are Christoffel symbols of M'.

Let c be a point in the chart V_i and choose the normal coordinates of M' at c. Then for $a, b \in M'$ near c, one has, since $\sigma(a,b) = \sigma(b,a)$ and

$\sigma(a,b)=0$ if $b^\gamma = ka^\gamma$ (the Euclidean straight line from a to ka viewed on M' is a geodesic):

$$\sigma(a,b)=\frac{1}{2}d_{M'}(a,b)^2=\frac{1}{2}d_E(a,b)^2+\lambda_{\beta\gamma,\delta}(a^\beta a^\gamma b^\delta + b^\beta b^\gamma a^\delta)$$

where d_E is the Euclidean distance, with $\lambda_{\beta\gamma,\delta}=\lambda_{\gamma\beta,\delta}$ and $\lambda_{\beta\gamma,\delta}+\lambda_{\gamma\delta,\beta}+\lambda_{\beta\delta,\gamma}=0$. We then have the series development of σ at $(0,0)$:

$$\sigma(a,b)=\frac{1}{2}\delta_{\beta\gamma}(a^\beta-b^\beta)(a^\gamma-b^\gamma)+\lambda_{\beta\gamma,\delta}(a^\beta a^\gamma b^\delta+b^\beta b^\gamma a^\delta)+O(|a|+|b|)^4 \quad (2.10)$$

and the development of its derivatives

$$\begin{aligned}
\frac{\partial^2\sigma}{\partial a^\beta \partial b^\gamma}(a,b) &= -\delta_{\beta\gamma}+\lambda_{\beta\delta,\gamma}a^\delta+\lambda_{\gamma\delta,\beta}b^\delta+O(|a|+|b|)^2\\
\frac{\partial^2\sigma}{\partial a^\beta \partial a^\gamma}(a,b) &= \delta_{\beta\gamma}+\lambda_{\beta\gamma,\delta}b^\delta+O(|a|+|b|)^2\\
\frac{\partial^2\sigma}{\partial b^\beta \partial b^\gamma}(a,b) &= \delta_{\beta\gamma}+\lambda_{\beta\gamma,\delta}a^\delta+O(|a|+|b|)^2\\
\frac{\partial\sigma}{\partial a^\alpha}(a,b) &= O(|a|+|b|), \quad \Gamma'^\alpha_{\beta\gamma}(a)=O(|a|)
\end{aligned}$$

So choose c to be the midpoint of $f_1(x,t)$ and $f_2(x,t)$ and $(f_1(x,t),f_2(x,t))=(w,-w)$ in the chart, one has:

$$\frac{d\rho}{dt}=-\Delta\rho-\left(\delta_{\beta\gamma}-\lambda_{\beta\gamma,\delta}w^\delta+O(|w|^2)\right)f_1{}_i^\beta f_1{}_j^\gamma g^{ij}-\left(\delta_{\beta\gamma}+\lambda_{\beta\gamma,\delta}w^\delta+O(|w|^2)\right)f_2{}_i^\beta f_2{}_j^\gamma g^{ij} \quad (2.11)$$

$$-2\left(-\delta_{\beta\gamma}+\lambda_{\beta\delta,\gamma}w^\delta-\lambda_{\gamma\delta,\beta}w^\delta+O(|w|^2)\right)f_1{}_i^\beta f_2{}_j^\gamma g^{ij} \quad (2.12)$$

$$=-\Delta\rho-|df_1-df_2|^2-w^\delta\lambda_{\beta\gamma,\delta}g^{ij}\left(f_2{}_i^\beta f_2{}_j^\gamma-f_1{}_i^\beta f_1{}_j^\gamma\right) \quad (2.13)$$

where we made a reduction of the term (2.12) by permuting β and γ to cancel the first order term w^δ. The last term of (2.13) can be bounded as follows:

$$\begin{aligned}
\left|w^\delta\lambda_{\beta\gamma,\delta}\left(f_2{}_i^\beta f_2{}_j^\gamma-f_1{}_i^\beta f_1{}_j^\gamma\right)g^{ij}\right| &= \left|w^\delta\lambda_{\beta\gamma,\delta}\left(f_2{}_i^\beta(f_2{}_j^\gamma-f_1{}_j^\gamma)+f_1{}_j^\gamma(f_2{}_i^\beta-f_1{}_i^\beta)\right)g^{ij}\right|\\
&\le 2|w^\delta\lambda_{\beta\gamma,\delta}||df_2-df_1|(|df_1|+|df_2|)\\
&\le |df_1-df_2|^2+O(|w|^2)
\end{aligned}$$

where for the last inequality, we use $2uv\le u^2+v^2$ and the fact that $|df_1|$ and $|df_2|$ are bounded on M. The estimate (2.13) can be continued:

$$\frac{d\rho}{dt}\le-\Delta\rho+C(x,t)|w|^2\le-\Delta\rho+C\rho$$

where $C > 0$ is a constant chosen to dominate all $C(x,t)$ for $x \in M$ in all charts and $t \in [\alpha, \omega]$. $\square$

Remark 9. *The last proof was modified from [Ham75]. The original proof made the reduction of the first order of w in (2.12) using the following development of σ:*

$$\sigma = \frac{1}{2}\delta_{\beta\gamma}(a^\beta - b^\beta)(a^\gamma - b^\gamma) + \lambda_{\beta\gamma,\delta}(a^\beta - b^\beta)(a^\gamma - b^\gamma)(a^\delta + b^\delta) + O(|a| + |b|)^4$$

which was justified by $\sigma(a,b) = \sigma(b,a)$ and $\sigma(a,a) = 0$. Algebraically these symmetries of σ are not sufficient to justify the development, and one can also proved, using $\sigma(a,ka) = 0$, that if such development was valid then all $\lambda_{\beta\gamma,\delta}$ would be zero, and there would be no third-order term. We made this reduction using the symmetry of $\frac{\partial^2\sigma}{\partial f_1^\beta \partial f_2^\gamma} f_1{}_i^\beta f_2{}_j^\gamma$ in (β,γ) and not just $\frac{\partial^2\sigma}{\partial f_1^\beta \partial f_2^\gamma}$ alone.

As a side note, if a,b,c are on $\mathbb{S}^2$ with $d(a,c) = d(b,c) = x \ll 1$ and the lines from a and b to c are orthogonal at c, then the geodesic distance $d(a,b) = \arccos(\cos^2(x)) = x\sqrt{2} - \frac{1}{6\sqrt{2}}x^3 + O(x^4)$. So $\sigma(a,b) = \frac{1}{2}d(a,b)^2$ has no third-order term.

2.3 A few energy estimates.

2.3.1 Estimate of density energies

We finish this part with a few straightforward computation concerning the **potential energy** $e(f_t) = \frac{1}{2}|\nabla f_t|^2$ and the **kinetic energy** $k(f_t) = \frac{1}{2}|\frac{\partial f_t}{\partial t}|$ of a nonlinear heat flow f_t satisfying (2.8).

Theorem 11 (Density of Potential energy)**.** *If f_t satisfies* (2.8) *then*

$$\frac{de(f_t)}{dt} = -\Delta e(f_t) - |\beta(f_t)|^2 - \langle \mathrm{Ric}(M)\nabla_v f_t, \nabla_v f_t\rangle + \langle \mathrm{Riem}(M')(\nabla_v f_t, \nabla_w f_t)\nabla_v f_t, \nabla_w f_t\rangle$$

where $e(f_t)$ is the potential energy density and $\beta(f_t)$ is the fundamental form and in the curvature terms, the vectors v and w are contracted.

In particular, if $\mathrm{Riem}(M') \leq 0$ *and* $\mathrm{Ric}(M) \geq -C$ *then*

$$\frac{de}{dt} \leq -\Delta e + Ce - |\beta(f_t)|^2 \tag{2.14}$$

Proof. Apply Lemma 2 to $s = df_t$ and the Riemannian-connected bundle $(F^*TM'$ over $M \times [\alpha, \omega]$ where $F(\cdot, t) = f_t$, the curvature terms cancel out and it remains to see that $\frac{de(f_t)}{dt} = -\langle df_t, \Delta df_t\rangle$, meaning that $\tilde{\nabla}_{\partial t} df_t = -\Delta df_t$. This can be easily justified:

$$\tilde{\nabla}_{\partial t} df_t = \tilde{\nabla}_{\partial t}\tilde{\nabla}^M F = \tilde{\nabla}^M \tilde{\nabla}_{\partial t} F = \tilde{\nabla}^M \ \tau(f_t) = -D\delta(df_t) = -\Delta df_t$$

where the last "=" is due to $Ddf_t = 0$. Note that D and δ are the exterior derivative and its adjoint of the bundle $(f_t)^*TM'$ on M, where t can be fixed after the third "=" sign. $\square$

Theorem 12 (Density of Kinetic energy)**.** *If f_t satisfies (2.8) then*

$$\frac{dk(f_t)}{dt} = -\Delta k(f_t) - \left|\nabla \frac{\partial f_t}{\partial t}\right|^2 + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle$$

where $k(f_t)$ is the kinetic energy density and in the curvature terms, the vectors v is contracted,

In particular, if $\mathrm{Riem}(M') \leq 0$ *then*

$$\frac{dk}{dt} \leq -\Delta k - \left|\nabla \frac{\partial f_t}{\partial t}\right|^2 \tag{2.15}$$

Proof. Let $F: \ I \times M \longrightarrow M'$ be the total function with $F(t, \cdot) = f_t$ for $t \in I = [\alpha, \omega]$ and $E = F^*TM'$ is a Riemannian-connected bundle on $I \times M$ with curvature form Θ, then

$$\tilde{\nabla}_{\partial t}\tilde{\nabla}_v(dF.v) = \tilde{\nabla}_v\tilde{\nabla}_{\partial t}(dF.v) + \Theta(\partial t, v)dF.v \tag{2.16}$$

where dF is the exterior derivative of f_t on M. Note that $\tilde{\nabla}_v\tilde{\nabla}_{\partial t}(dF.v) = \tilde{\nabla}_v(\tilde{\nabla}_{\partial t} dF).v = \tilde{\nabla}_v(\tilde{\nabla}^M \frac{\partial f_t}{\partial t}).v$ since $\tilde{\nabla}^M \frac{\partial f_t}{\partial t} = \tilde{\nabla}^{I\times M}_{\partial t} dF = \tilde{\nabla}^I_{\partial_t} dF$ because $\tilde{\nabla}$ is torsionless on M'. Plugging this in (2.16) and taking contraction in v, one has

$$\tilde{\nabla}_{\partial t} \ \tau(f_t) = -\tilde{\Delta}\frac{\partial f_t}{\partial t} + \mathrm{Tr}\,(v \mapsto \Theta(\partial t, v)dF.v) \tag{2.17}$$

But $\Theta^\beta_\alpha = R'^\beta_{\alpha\nu\mu} F^\mu_i F^\nu_j dx^i \otimes dx^j$ where R' denotes the Riemannian curvature of M' and the indices i, j can be 0, with $x^0 \equiv t$. Hence

$$\Theta(\partial t, v)dF.v = R'^\beta_{\alpha\nu\mu}\frac{\partial f^\mu_t}{\partial t}\frac{\partial f^\nu_t}{\partial v}\frac{\partial f^\alpha_t}{\partial v}\tilde{e}_\beta = \mathrm{Riem}(M')\left(\nabla_v f_t, \frac{\partial f_t}{\partial t}\right)\nabla_v f_t$$

Plugging in (2.17) and taking inner product with $\frac{\partial f_t}{\partial t}$, one has

$$\begin{aligned}\frac{\partial k(f_t)}{\partial t} &= \left\langle \tilde{\nabla}_{\partial t} \ \tau(f_t), \frac{\partial f_t}{\partial t}\right\rangle = -\left\langle \tilde{\Delta}\frac{\partial f_t}{\partial t}, \frac{\partial f_t}{\partial t}\right\rangle + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle \\ &= -\Delta\left(\frac{1}{2}\left|\frac{\partial f_t}{\partial t}\right|^2\right) - \left|\tilde{\nabla}\frac{\partial f_t}{\partial t}\right|^2 + \left\langle \mathrm{Riem}(M')(\nabla_v f_t, \frac{\partial f_t}{\partial t})\nabla_v f_t, \frac{\partial f_t}{\partial t}\right\rangle\end{aligned}$$

$\square$

2.3.2 Estimate of total energies

We will now work with the total energies, in particular the **total potential energy** $E(f_t) := \int_M e(f_t)$ and **total kinetic energy** $K(f_t) := \int_M k(f_t)$. Since tension field is the gradient of E, one has:

Theorem 13. *If $f_t : M \longrightarrow M'$ satisfies* (2.8) *then*

$$\frac{dE(f_t)}{dt} = -\int_M \left\langle \tau(f_t), \frac{\partial f_t}{\partial t} \right\rangle = -\int_M |\tau(f_t)|^2 = -2K(f_t)$$

and

$$\frac{d^2E(f_t)}{dt} = -2\int_M \left\langle \nabla_{\partial t} \frac{\partial f_t}{\partial t}, \tau(f_t) \right\rangle$$

Intergrating Theorem 12 on M, one obtains:

Theorem 14. *If f_t satisfies* (2.8) *and* $\mathrm{Riem}(M') \leq 0$ *then* $\frac{d}{dt}K(f_t) \leq 0$. *Together with Theorem 13, one has*

1. *The total potential energy $E(f_t)$ is ≥ 0, decreasing and convex.*

2. *The total kinetic energy $K(f_t)$ is ≥ 0, decreasing and if $\omega = +\infty$ then* $\lim_{t\to\infty} K(f_t) = 0$.

In particular, $\int_{M\times\{\tau\}} |\nabla f|^2$ and $\int_{M\times\{\tau\}} \left|\frac{\partial f_t}{\partial t}\right|^2$ are bounded above by a constant $C > 0$ independent of the time $\tau \in [\alpha, \omega]$.

Note that we ruled out the case where $K(f_t)$ decreases to a strictly positive number because $E(f_t)$ is bounded below and $\frac{d}{dt}E(f_t) = -2K(f_t)$.

Integrating Theorem 11 on M and use Theorem 14, one has:

Theorem 15. *If f_t satisfies* (2.8) *and* $\mathrm{Riem}(M') \leq 0$ *and* $\mathrm{Ric}(M)$ *is bounded below then*

$$\int_M |\beta(f_t)|^2 \leq C$$

for all time t where the constant C only depends on the curvature of M, M' and the initial total potential and kinetic energy, in particular, C does not depend on t.

This means that $\|f_t\|_{W^{2,2}(M)}$ is bounded by a constant C only depending on the curvatures and initial total energies.

Corollary 15.1 (Boundedness in $W^{2,2}(M)$). *If F_t satisfies* (2.9) *and* $\mathrm{Riem}(M') \leq 0$ *and* $\mathrm{Ric}(M)$ *is bounded below then*

$$\|F_t\|^2_{W^{2,2}(M)} := \int_M |\beta(F_t)|^2 + |\nabla F_t| + |F|^2 \leq C$$

for all time t where the constant C only depends on the curvature of M, M' and the initial total potential and kinetic energy, in particular, C does not depend on t.

Note that the term $|F|^2$ is trivially bounded since the image of F remains in an Euclidean ball B.

Part II

Resolution of linear equations on manifold

Chapter 3

Interpolation theory and Sobolev spaces on compact manifolds

3.1 Motivation

We will define a more general notion of Sobolev spaces on compact manifold than those in [Aub98] and [Jos08], where Sobolev spaces on a (Riemannian) manifold $W^{k,p}(M)$ of dimension n are defined for $k \in \mathbb{Z}_{\geq 0}$ and for *uniform weight*, meaning that a function $f \in W^{k,p}(M)$ is supposed to be weakly differentiable up to order k in every variables $x_1, \ldots, x_n$ in each smooth coordinates. The space $W^{k,p}(M)$ in this case can be defined by density with respect to a norm involving derivatives $\frac{\partial f}{\partial x^\alpha}$.

Meanwhile, the suitable function spaces to solve parabolic equations are those whose regularity in time is half of that in space, i.e. we will solve parabolic equations on the Sobolev spaces $W^{k,p}(M \times T)$ of functions k times regular in M and $k/2$ times regular in T. We cannot always, (for example when k is odd) find a simple norm involving derivatives of f to define $W^{k,p}$ by density. This generalisation will be done using Stein's multipliers.

Another generalisation will be made is to allow the manifold to have boundary. Even when we only want to solve parabolic equation on manifold M without boundary, the underlying space is $M \times [0, T]$ which has boundary. Moreover, we will have to discuss the notion of trace in order to use the initial condition at $t = 0$.

In this part, all manifolds will be compact, with no given metric. This is not really a generalisation since on compact manifolds, Sobolev spaces

$W^{k,p}(M)$, as defined in [Aub98] and [Jos08] set theoretically do not depend on the metric and (the equivalent class of) their norms also independent of the metric.

We will mainly follow the discussion in [Ham75], where the author also works on manifold with *corner*, i.e. irregular boundary. The corners, modeled by $\mathbb{R}^{n-k} \times \mathbb{R}^k_{\geq 0}$, appear naturally, for example at the boundary ∂M in $t = 0$. The extra effort to cover the case of corners is not much (see [Ham75, page 50]) and essentially algebraic.

3.2 Preparatory material

We will recall here basic elements of Fourier transform on the space of tempered distributions and then we will have a quick review of interpolation theory.

3.2.1 Stein's multiplier

Let $X = \mathbb{R}^n$ be the Euclidean space, coordinated by $x_1, \ldots, x_n$ and $\mathcal{E} = \mathbb{R}^n$, coordinated by $\xi_1, \ldots, \xi_n$ be the frequency domain of X. Recall that Fourier transform is an isomorphism in the following three levels

1. The Schwartz space of rapidly decreasing smooth functions $\mathcal{S}(X)$ whose elements are smooth and decrease more rapidly then any rational function. The Schwartz space are topologized by the family of semi-norms $|f|_{\alpha,\beta} = \sup_X |x^\alpha D_x^b f(x)|$.
2. The space $L^2(X)$ of doubly-integrable functions.
3. The space of tempered distributions, i.e. the dual space $\mathcal{S}^*(X)$ of $\mathcal{S}(X)$ under the weak-* topology given by $\mathcal{S}(X)$.

To simplify the notation, we use $D_x^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ and $P(D) = \sum_\alpha c_\alpha D^\alpha$ for any polynomial P.

Recall that for any $u \in \mathcal{S}(X)$ and for any polynomial P, one has $\widehat{P(D)u} = P(\xi)\hat{u}(\xi)$. This can be extended to non-polynomial function of $M(D)$ of D by

$$\widehat{M(D)u} := M(\xi)\hat{u}(\xi)$$

where M is a slowly growing function, i.e. $D^\alpha M(\xi)$ grows slower than certain polynomial as $|\xi| \to \infty$.

The following theorem give a criteria of the function M such that $M(D) : \mathcal{S}(X) \longrightarrow \mathcal{S}(X)$ extend to $L^p(X) \longrightarrow L^p(X)$.

Theorem 16 (Stein). *If for any* primitive *index $\alpha = (\alpha_1, \ldots, \alpha_n)$, i.e. each α_i being* 0 *or* 1 *(there are exactly 2^n primitive indices), one has*

$$|\xi^\alpha D^\alpha M(\xi)| \leq C_\alpha$$

then $M(D)$ extend to a bounded linear operator on $L^p(X)$.

Definition 7.

1. *A slowly growing function W on $\mathcal{E}$ with $W(\xi) > 0$ is called a* ***weight*** *if for all primitive index α, one has*

$$|\xi^\alpha D^\alpha W(\xi)| \leq C_\alpha W(\xi).$$

2. *The* ***Sobolev space*** *$W^{k,p}(X, W)$ with respect to weight W, $k \in \mathbb{R}, 1 < p < \infty$ is the vector space*

$$W^{k,p}(X, W) = \Big\{ u \in \mathcal{S}^*(X) : \ W(D)^k u \in L^p(X) \Big\}$$

normed by $\|u\|_{W^{k,p}} = \|W(D)^k u\|_{L^p}$.

Example 6 (Weight given by $\Sigma = (\sigma_1, \ldots, \sigma_n)$). *Note by $\sigma := \mathrm{lcm}(\sigma_1, \ldots, \sigma_n)$ then $W_\Sigma(\xi) = \Big(1 + \xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n}\Big)^{1/2\sigma}$ is a weight. We will only use weights of this type in our discussion. The index $\Sigma = (\sigma_1, \ldots, \sigma_n)$ is chosen according to the differential operator in the elliptic/parabolic equation. In particular, for Laplace equation, one chooses $\Sigma = (1, \ldots, 1)$ and for heat equation $\Sigma = (1, 2, \ldots, 2)$ where* 1 *is in the time component.*

Remark 10.

1. *If W_1, W_2 are weights then $W_1 + sW_2, W_1 W_2, W_1^s (s > 0)$ are also weights.*

2. *The operator $W(D) : \ W^{k+r}, p(X, W) \longrightarrow W^{k,p}(X)$ is bounded.*

3. *Given another weight $V(\xi) \leq CW(\xi)$, by Stein's criteria (Theorem 16) one has a bounded embedding $W^{k,p}(X, W) \hookrightarrow W^{k,p}(X, V)$.*

The Sobolev space $W^{k,p}(X, W_\Sigma$ has a simple definition by density when $\sigma \mid k$. Given an index $\alpha = (\alpha_1, \ldots, \alpha_n)$, note by $\|\alpha\| := \sum_{i=1}^n \alpha_i \frac{\sigma}{\sigma_i}$.

Theorem 17 (Equivalent norm when $\sigma \mid k$). *If $k > 0$ and $\sigma \mid k$ and $1 < p < \infty$, then given $u \in \mathcal{S}^*(X)$, one has*

1. *$u \in W^{k,p}(X)$ if and only if $D^\alpha u \in L^p(X)$ for all $\|\alpha\| \leq k$ and the norm $\sum_{\|\alpha \leq k\|} \|D^\alpha u\|_{L^p}$ is equivalent to $\|u\|_{W^{k,p}}$.*

2. $u \in W^{-k,p}$ *if and only if there exists* $g_\alpha \in L^p$ *such that* $u = \sum_{\|\alpha\| \le k} D^\alpha g_\alpha$ *and* $\|u\|_{W^{-k,p}}$ *is equivalent to*

$$\inf \left\{ \sum_{\|\alpha\| \le k} \|g_\alpha\|_{L^p} : \ u = \sum_{\|\alpha\| \le k} D^\alpha g_\alpha \right\}$$

Example 7. 1. *When* $\sigma_1 = \cdots = \sigma_n = 1$*, one has the familiar Sobolev spaces.*

2. *For (the weight of) heat equation,* $W^{2,p}$ *can be defined by density using the norm*

$$\|u(t,x)\| = \left\| \frac{\partial u}{\partial t} \right\|_{L^p} + \|Du\|_{L^p} + \|Du\|_{L^p}$$

where L^p *stands for* $L^p(X \times [0,T])$.

3.2.2 Holomorphic interpolation of Banach spaces

The Interpolation theory is based on the following Three-lines theorem whose proof follows from the classic Hadamard's three-lines theorem (the case $A = \mathbb{C}$) and the way we define complex Banach spaces and holomorphic maps taking value there.

Theorem 18 (Three-lines)**.** *Let* A *be a complex Banach space and* $h : S = \{0 \le \operatorname{Re} z \le 1\} \subset \mathbb{C} \longrightarrow A$ *be a holomorphic map, i.e. continuous and holomorphic in the interior such that* h *is bounded at infinity, i.e.* $h(x+iy) \to 0$ *as* $y \to \infty$*. Let* $M(x) := \sup_y \|h(x+iy)\|$ *then one has*

$$M(x) \le M(1)^x M(0)^{1-x}$$

Let A_0, A_1 be complex Banach spaces such that

1. A_0, A_1 can be continuously embedded into a Hausdorff topological complex vector space E such that the complex structures are compatible with each others, i.e. the linear embeddings $A_i \hookrightarrow E$ preserve complex structures.

2. The intersection $A_0 \cap A_1$ in E is dense in $(A_i, \|\|_{A_i})$ for $i = 0, 1$.

such (A_0, A_1) is called an **interpolatable** pair.

The norms of $A_0 \cap A_1$ and $A_0 + A_1$ are defined such that the these spaces are Banach and the diagram

$$0 \longrightarrow A_0 \cap A_1 \longrightarrow A_0 \oplus A_1 \longrightarrow A_0 + A_1 \longrightarrow 0 \qquad (3.1)$$

commutes and the arrows are continuous. By Open mapping theorem, this means that the norm on $A_0 \cap A_1$ is equivalent to $\|x\|_{A_0 \cap A_1} = \|x\|_{A_0} + \|x\|_{A_1}$ and the norm on A_0+A_1 is equivalent to $\|x\|_{A_0+A_1} = \inf_{x=x_0+x_1, x_i \in A_i} \{\|x_0\|_{A_0} + \|x\|_{A_1}\}$.

Remark 11. *A pair (A_0, A_1) of Banach spaces may give different interpolatable pairs depending how they are embedded into a common space E. It is not difficult to see that the data of* interpolatable pair *is uniquely determined by 2 complex Banach spaces U, V (which are eventually $A \cap B$ and $A + B$) and the diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d, bend right] & & \\
 & & A_0 \arrow[u, bend right] \arrow[d, bend right] \arrow[dr] & & \\
0 \arrow[r] & U \arrow[r] \arrow[ur] \arrow[dr] & A_0 \oplus A_1 \arrow[r] \arrow[u, bend right] \arrow[d, bend right] & V \arrow[r] & 0 \\
 & & A_1 \arrow[u, bend right] \arrow[d, bend right] \arrow[ur] & & \\
 & & 0 \arrow[u, bend right] & &
\end{tikzcd} \tag{3.2}$$

in which

1. *All arrows are continuous and compatible with complex structures. The horizontal sequence is exact, the vertical sequence is exact and canonical.*

2. *The diagonal arrows from U to A_0, A_1 are injective and of dense image in A_0, A_1.*

3. *The maps composed by the diagonal arrows $U \to A_i \to V$ are injective for $i = 0, 1$. Since the two maps are additive inverse, it suffices to have injectivity for one of them.*

In the language that we will use to solve linear equation, these properties of diagram (3.2) *are equivalent to the square*

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & V
\end{tikzcd}$$

being exact.

The following construction will give a family of complex subspace A_θ of $A_0 + A_1$ containing $A_0 \cap A_1$ for $0 \leq \theta \leq 1$ that interpolates A_0 and A_1 that satisfies the following properties, called interpolation inequalities

Theorem 19 (Interpolation inequality for elements in the intersection)**.** *Let $a \in A_0 \cap A_1$ then $a \in A_\theta$ and*

$$\|a\|_{A_\theta} \leq 2\|a\|_{A_1}^{\theta}\|a\|_{A_0}^{1-\theta}$$

Theorem 20 (Interpolation inequality for operators)**.** *Given interpolatable pairs (A_0, A_1) and $B_0, B_1)$, and T a bounded linear operator $T : A_0 \longrightarrow B_0$ and $T : A_1 \longrightarrow B_1$ such that T is well-defined on $A_0 \cap A_1$. Then T extends linearly and continuously to $T : \ A_0 + A_1 \longrightarrow B_0 + B_1$, that is*

$$\begin{tikzcd}
0 \arrow[r] & A_0 \cap A_1 \arrow[r] \arrow[d, "T"] & A_0 \oplus A_1 \arrow[r] \arrow[d, "T\oplus T"] & A_0 + A_1 \arrow[r] \arrow[d, "T"] & 0 \\
0 \arrow[r] & B_0 \cap B_1 \arrow[r] & B_0 \oplus B_1 \arrow[r] & B_0 + B_1 \arrow[r] & 0
\end{tikzcd} \tag{3.3}$$

Also, T defines a bounded operator $T : \ A_\theta \longrightarrow B_\theta$ and

$$\|T\|_{L(A_\theta,B_\theta)} \leq 2\|E\|_{L(A_1,B_1)}^{\theta}\|E\|_{L(A_0,B_0)}^{1-\theta}$$

To define A_θ, let

$$\mathcal{H}(A_0, A_1) := \Big\{h : S \longrightarrow A_0 + A_1 : \ h \text{ is holomorphic}, \ \lim_{|y|\to\infty} h(z) = 0, h(iy) \in A_0, \ h(1 + iy) \in A_1$$

where, as above, S denotes the strip $0 \leq \operatorname{Re} z \leq 1$. Then $\mathcal{H}(A_0, A_1)$ is a Banach space with the norm

$$\|h\|_{\mathcal{H}(A_0,A_1)} := \sup_y \|h(iy)\|_{A_0} + \sup_y \|h(1 + iy)\|_{A_1}$$

The space A_θ is defined set-theoretically as the space of all value in $A_0 + A_1$ that a function $h \in \mathcal{H}(A_0, A_1)$ can take at $\theta \in [0, 1] \in S$. Therefore, set-theoretically A_θ coincides with A_0 and A_1 when $\theta = 0$ and $\theta = 1$. To define the norm on A_θ, let

$$\mathcal{K}_\theta(A_0, A_1) := \{h \in \mathcal{H}(A_0, A_1) : \ h(\theta) = 0\}$$

then $\mathcal{K}_\theta(A_0, A_1)$ is a closed complex subspace of the Banach space $\mathcal{H}(A_0, A_1)$. Then $A_\theta := \mathcal{H}(A_0, A_1)/\mathcal{K}_\theta(A_0, A_1)$ has the natural quotient norm inherited from $\mathcal{H}(A_0, A_1)$ and is still a Banach space.

It is not difficult to see that the norm on A_θ coincides with the norm $\|\cdot\|_{A_0}, \|\cdot\|_{A_1}$ when $\theta = 0$ or $\theta = 1$

Theorem 19 follows from the this lemma when one takes h to be a constant, and is in $A_0 \cap A_1$.

Lemma 21. *If $h \in \mathcal{H}(A_0, A_1)$ then $\|h(\theta)\|_{A_\theta} \le 2M_1^\theta M_0^{1-\theta}$ where*

$$M_0 := \sup_y \|h(iy)\|_{A_0}, \quad M_1 := \sup_y \|h(1+iy)\|_{A_1}$$

Proof. The A_θ-norm of $h(\theta)$ only depends on the value of h at θ, one can therefore replace h by a function of form $h_{c,\epsilon}(z) = \exp(c(z-\theta) + \epsilon z^2)h(z)$, then let $\epsilon \to 0$ and choose the optimal c, which is $e^c = M_0/M_1$. $\square$

Theorem 20 follows from Theorem 19 and the very definition of quotient norm.

Remark 12. *The optimal constant, as given by the proofs, is $\theta^{-\theta}(1-\theta)^{\theta-1} < 2$*

The interest of holomorphic interpolation theory comes from the fact that interpolation of Sobolev spaces are still Sobolev spaces, which, together with Theorem 20 and Theorem 19, gives a class of useful inequalities generally called interpolation inequalities.

Theorem 22 (Interpolation of Sobolev spaces)**.** *Let $p, q \in (1, +\infty)$ and $k, l \in \mathbb{R}$ and $X = \mathbb{R}^n$. Take*

$$A_0 := W^{k,p}(X), \quad A_1 := W^{l,q}(X)$$

then $A_\theta = W^{s,r}(X)$ where

$$\theta l + (1-\theta)k = s, \qquad \theta\frac{1}{q} + (1-\theta)\frac{1}{p} = \frac{1}{r}$$

The holomorphic interpolation behaves predictably with direct sum and compact operators

Theorem 23. *Let $(A_0, A_1), (B_0, B_1)$ be interpolatable pairs and denotes by $(A \oplus B)_\theta$ be the interpolation of $A_0 \oplus B_0$ and $A_1 \oplus B_1$ then one has $(A \oplus B)_\theta \cong A_\theta \oplus B_\theta$ by a canonical isomorphism.*

Proof. The set-theoretical bijection is easy to see: note that there is a natural inclusion $(A \oplus B)_\theta \hookrightarrow A_\theta \oplus B_\theta$, which is also a bijection because $\mathcal{H}(A_0 \oplus B_0, A_1 \oplus B_1) = \mathcal{H}(A_0, A_1) \oplus \mathcal{H}(B_0, B_1)$.

The most difficult part is to know what we mean by *isomorphism*. In fact the two norms (the interpolation norm and the direct-sum norm) do not coincide, but they are equivalent. One can prove, with basic sup-inf analysis that

$$\frac{1}{2}\|\cdot\|_{A_\theta \oplus B_\theta} \leq \|\cdot\|_{(A\oplus B)_\theta} \leq \|\cdot\|_{A_\theta \oplus B_\theta}$$

□

Theorem 23 can be generalised to the following result.

Theorem 24 (*). *Let (X_0, X_1) and (Y_0, Y_1) be interpolatable pairs. Suppose that there are inclusion $X_0 \hookrightarrow Y_0$ and $X_1 \hookrightarrow Y_1$ with closed images in Y_0 and Y_1 respectively and the inclusions agree on $X_0 \cap X_1$ as mappings from $X_0 \cap X_1$ to $Y_0 + Y_1$. Moreover, suppose that the image of $X_0 + X_1$ in $Y_0 + Y_1$ is closed. Then there is a natural inclusion $X_\theta \hookrightarrow Y_\theta$ with closed image in Y_θ*

Remark 13.

1. *The condition $X_0 + X_1 \hookrightarrow Y_0 + Y_1$ being of closed image is redundant if $X_1 \hookrightarrow X_0$ and $Y_1 \hookrightarrow Y_0$, as in the case of interpolation of* certain *Sobolev spaces on manifolds. In general, one can also check that this condition holds for the maps $\iota_{k,p}$ and $\iota_{l,q}$ in Definition 8 of Sobolev spaces using the fact that they admit left-inverse given by $\{\tilde{\psi}_i\}$. See Remark 16.*

2. *If one has two exact sequences*

$$0 \longrightarrow X_i \longrightarrow Y_i \longrightarrow Z_i \longrightarrow 0, \quad i = 0, 1 \tag{3.4}$$

whose arrows commute with ones from the intersection and ambient spaces of interpolatable pairs $(X_0, X_1), (Y_0, Y_1), (Z_0, Z_1)$ then, since the images of $X_i \longrightarrow Y_i$ being kernel of $Y_i \longrightarrow Z_i$ are closed, one has the inclusion for interpolation spaces, also of closed image:

$$0 \longrightarrow X_\theta \longrightarrow Y_\theta, \quad 0 \leq \theta \leq 1.$$

3. *In particular, if the sequences in (3.4) split, meaning that one can find a retraction $0 \longrightarrow Z_i \longrightarrow Y_i$, then by applying the theorem for the retractions, one sees that the interpolation sequence extend to Z_θ, i.e.*

$$0 \longrightarrow X_\theta \longrightarrow Y_\theta \longrightarrow Z_\theta \longrightarrow 0$$

and also splits, meaning $Y_\theta \cong X_\theta \oplus Z_\theta$. Applying this results to the split-exact sequences

$$0 \longrightarrow A_i \longrightarrow A_i \oplus B_i \longrightarrow B_i \longrightarrow 0$$

one then obtains Theorem 23.

Proof. The inclusion $X_\theta \hookrightarrow Y_\theta$ is natural and due to the fact that $\mathcal{H}(X_0, X_1) \subset \mathcal{H}(Y_0, Y_1)$. The equivalence of the interpolation norm X_θ and the norm inherited from Y_θ on X_θ requires more than a simple sup-inf analysis as in the proof of Theorem 23 since $\mathcal{H}(X_0, X_1)$ is strictly included in $\mathcal{H}(Y_0, Y_1)$. What we can say is that the interpolation norm X_θ dominates the interpolation norm of Y_θ, since it involves the infimum on the smaller set. In other words, it means that the inclusion $X_\theta \hookrightarrow Y_\theta$ is continuous. It remains to check that the image of $X_\theta \hookrightarrow Y_\theta$ is closed.

Since

$$\begin{tikzcd}
X_\theta \arrow[rr] \arrow[d, equal] & & Y_\theta \arrow[d, equal] \\
\mathcal{H}(X_0, X_1)/\mathcal{K}_\theta(X_0, X_1) & & \mathcal{H}(Y_0, Y_1)/\mathcal{K}_\theta(Y_0, Y_1) \\
\mathcal{H}(X_0, X_1) \arrow[u, two heads] \arrow[rr] & & \mathcal{H}(Y_0, Y_1) \arrow[u, two heads]
\end{tikzcd}$$

it suffices to show that the image $\mathcal{H}(X_0, X_1) \hookrightarrow \mathcal{H}(Y_0, Y_1)$ is closed, meaning if $\mathcal{H}(X_0, X_1) \ni h_n \to h$ in $\mathcal{H}(Y_0, Y_1)$, then h must take value in $X_0 + X_1$. This is easy to verify on ∂S: By the equivalence of the norm on X_i and the restricted norm from Y_i, $i = 0, 1$, one sees that $h(iy) \in X_0$ and $h(1+iy) \in X_1$.

Since $X_0 + X_1$ is closed in $Y_0 + Y_1$, any holomorphic map $\mathcal{H}(Y_0, Y_1) \ni f : S \longrightarrow Y_0 + Y_1$ passes holomorphically to the quotient $S \longrightarrow (Y_0 + Y_1)/(X_0 + X_1)$. The fact that h takes value in $X_0 + X_1$ follows from Maximum modulus principle for holomorphic functions. □

Theorem 25 (Interpolation of compact embedding)**.** *If $A_1 \hookrightarrow A_0$ is a compact embedding, then $A_1 \cong A_\theta \cap A_1 \hookrightarrow A_\theta$ is a compact embedding where the first $\cong$ denotes the same space with equivalent norms.*

Proof. It follows from Theorem 19:

$$\|x_m - x_n\|_{A_\theta} \leq 2\|x_m - x_n\|_{A_0}^{1-\theta}\|x_m - x_n\|_{A_1}^{\theta}$$

Hence if $\{x_n\}$ is a bounded sequence in A_1, it converges in A_0 and therefore A_θ. □

The previous Theorem 19, together with Theorem 22 also gives a proof of Kondrachov's Theorem, that is the embedding $W^{k,p}(X) \hookrightarrow W^{l,p}(X)$ is compact if $k > h >\geq 0$. This follows from the following 2 remarks

1. The case $l = 0$ and $k \gg 1$ follows from the embedding $W^{k,p} \hookrightarrow C^1$ and Ascoli's theorem. Hence by Theorem 19, one has the compactness embedding if $k \gg 1$ and $l < k$.

2. For the case of small k, note that

$$W^{k+r,p}(X) \twoheadrightarrow W^{k,p}(X): \ v \longmapsto W(D)^r u$$

   is surjective and any $u \in W^{k,p}(X)$ can be lifted to an element $\tilde{u} \in W^{k+r,p}(X)$ of the same norm. In fact, if $W(\xi)^k \hat{u} \in L^p$ then choose $\tilde{u}$ such that $\widehat{\tilde{u}} = W(\xi)^{-r}\hat{u}$. Kondrachov's theorem follows from the diagram:

$$\begin{tikzcd} W^{k+r,p}(X) \arrow[r, two heads] \arrow[d, hook, "\text{compact}"'] & W^{k,p}(X) \arrow[d, hook] \\ W^{h+r,p}(X) \arrow[r, two heads] & W^{h,p}(X) \end{tikzcd}$$

Remark 14. *The advantage of this proof is that it is valid for weighted Sobolev spaces over manifolds.*

3.3 Sobolev spaces on compact manifold without boundary

Let M be a compact manifold without boundary. We fix a finite atlas of M by chart $\varphi_i : \ M \supset U_i \longrightarrow V_i \subset \mathbb{R}^n$ such that the transitions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : V_j \longrightarrow V_i$ are of strictly positive and bounded derivatives, i.e. $C(\alpha)^{-1} \le D^\alpha \varphi_{ij} \le C(\alpha)$ for all indices α. We will called such atlas a *good atlas*. One can always obtain such atlas by shrinking a bit each chart of a given atlas of M. Let ψ_i be a partition of unity subordinated to $\{U_i\}$

Definition 8. *1. The **Sobolev spaces** $W^{k,p}(M)$ is defined as*

$$W^{k,p}(M) := \Big\{ f \in \mathcal{S}(M)^* : \ (\psi_i f) \circ \varphi_i^{-1} \in W^{k,p}(\mathbb{R}^n) \Big\}$$

with the norm

$$\|f\|_{W^{k,p}} = \sum_i \|(\psi_i f) \circ \varphi_i^{-1}\|_{W^{k,p}(\mathbb{R}^n)}$$

2. Weighted Sobolev spaces can be defined when M has a foliation structure, i.e. M is locally modeled by $0 \subsetneq F_1 \subsetneq \cdots \subset F_k \subsetneq \mathbb{R}^n$ where F_i are *vector subspace of* $\mathbb{R}^n$ *of dimension* $0 < n_1 < \cdots < n_k < n$ *respectively and* F_k *are preserved by the transition maps* φ_{ij}, *for example when* M *is a product of manifolds of lower dimension. Then the above definition extends to weighted Sobolev spaces with weight* $\sigma_1 = \cdots = \sigma_{n_1}$, $\sigma_{n_1+1} = \cdots = \sigma_{n_2}, \ldots \sigma_{n_k+1} = \cdots = \sigma_n$.

Remark 15.

1. *One can define* $\mathcal{S}(M)^*$ *as the dual space of* $\mathcal{S}(M) = C^\infty(M)$ *under Schwartz topology with respect to any metric, because by compactness any two metrics on* M *are comparable. The distributions* $\psi_i f$ *are tempered because they are compactly supported.*

2. *One can identify* $C^\infty(M)$ *with a subspace of* $\mathcal{S}^*(M)$ *that is contained in any Sobolev space* $W^{k,p}(M)$ *by fixing a Riemannian metric* g *on* M. *The map* $C^\infty(M) \hookrightarrow \mathcal{S}^*(M)$ *may depend on* g, *but its image does not. Similarly, one can also identify an element of* $W^{k,p}(\mathbb{R}^n)$ *supported in* V_i *with an element in* $W^{k,p}(M)$.

3. *If one uses another good atlas* U'_i *or a different partition of unity, one obtains the same set* $W^{k,p}(M)$ *and an equivalent norm. To see this, let us call two good atlas* compatible *if their union is also a good atlas, then the statement holds for two compatible atlas by comparing their union. Moreover, for any two arbitrary good atlas* $\{U_i\}, \{U'_j\}$, *one can find a good atlas compatible with both of them by shrinking their union.*

By definition, one has an inclusion $\iota : W^{k,p}(M) \hookrightarrow \bigoplus_i W^{k,p}(\mathbb{R}^n)$. Also ι is of closed image because one can find a projection $\pi : \bigoplus_i W^{k,p}(\mathbb{R}^n) \longrightarrow W^{k,p}(M)$ with $\pi \circ \iota = \mathrm{Id}$. In fact, let $\tilde{\psi}_i$ be functions supported in U_i that equal 1 in the support of ψ_i, then

$$\pi : g \mapsto \sum \tilde{\psi}_i.(g \circ \varphi_i)$$

works. The continuity of π follows from straight-forward calculations.

The closedness of image of ι is equivalent to the fact that $W^{k,p}(M)$ is complete.

Remark 16. *Although* ι *preserves the norm of* $W^{k,p}(M)$ *and has a right-inverse, it is far from being an isomorphism (it is not surjective). Each summand of an element in the image of* ι *tends to 0 on the boundary of* V_i *(take* $k \gg 1$ *then everyone is continuous by Sobolev embedding, there is no subtlety in what we mean by "tends to 0"). [Ham75, page 54] seems to claim that* ι *is an isomorphism and apply Theorem 23 repeatedly to deduce Theorem 22 for Sobolev spaces on manifold, then the Sobolev embedding* $W^{k,p} \hookrightarrow C^l(M)$ *and Kondrachov's theorem.*

The above results are true and the correction is not difficult (use Theorem 24).

From the remark, one has

Theorem 26 (Interpolation of Sobolev spaces on manifold)**.** *Theorem 22 holds for Sobolev spaces $W^{k,p}(M)$ on compact manifold M.*

3.4 Sobolev spaces on compact manifold with boundary

In this part, we will define the Sobolev spaces $W^{k,p}(M/\mathcal{A})$ where $k \in \mathbb{R}, p \in (1, \infty)$ and M is a manifold with boundary and $\mathcal{A}$ is union of connected components of ∂M the boundary of M. These spaces contain $W^{k,p}(M)$ "functions" who vanish on $\mathcal{A}$. The motivation is that we will later take $M = M' \times [0, T]$ where M' is a manifold without boundary where we want to solve heat equation, and the natural $\mathcal{A}$ would be $M \times \{0\}$. We also want that the new definition coincides with the case of no boundary when $\mathcal{A} = \emptyset$

Suppose that we already define the Sobolev spaces on $X \times Y^+$ where $X = \mathbb{R}^n$ and $Y^+ = \mathbb{R}_{\geq 0}$, that is the space $W^{k,p}(X \times Y^+) = W^{k,p}(X \times Y^+/\emptyset)$ and $W^{k,p}(X \times Y^+, X \times \{0\})$. Then then we define the space $W^{k,p}(M/\mathcal{A})$ in analog of Definition 8 as follows

Definition 9. 1. *The* ***Sobolev spaces*** $W^{k,p}(M/\mathcal{A})$ *where A is a connected component of ∂M is defined as*

$$W^{k,p}(M/\mathcal{A}) := \left\{ f \in \mathcal{S}(M)^* : \ (\psi_i f) \circ \varphi_i^{-1} \in W^{k,p}(R_i/\mathcal{A}_i) \right\}$$

where $\mathcal{A}_i = \varphi_i(U_i \cap \mathcal{A})$ and R_i is the Euclidean space containing $\varphi(U_i)$, that is either $\mathbb{R}^{n+1}$ when $\mathcal{A}_i = \emptyset$ or $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ when $\mathcal{A}_i \subset \mathbb{R}^n \times \{0\}$. The norm is given by

$$\|f\|_{W^{k,p}} = \sum_i \|(\psi_i f) \circ \varphi_i^{-1}\|_{W^{k,p}(R_i/\mathcal{A}_i)}$$

2. *As before, weighted Sobolev spaces can be defined when M has a foliation structure compatible with its boundary.*

The fact that different good atlas and different partition of unity defines the same space $W^{k,p}(M/\mathcal{A})$ (as a subset of $\mathcal{S}^*(M)$) with equivalents norm comes from the following lemma, which is just a formulation of arguments in the case of no boundary. For the proof, one reduces the lemma, by interpolation inequality, to the case k is a multiple of σ and use the criteria in Theorem 17 and the boundedness of derivative of the transition map.

Lemma 27. *Let $(U, \mathcal{A}_U)$ and $(V, \mathcal{A}_V)$ be subsets of $(X \times Y^+, X \times \{0\})$ and $\varphi_{VU} : (U, \mathcal{A}_U) \longrightarrow (V, \mathcal{A}_V)$ being a diffeomorphism between U and V mapping $\mathcal{A}_U \subset \partial U$ to $\mathcal{A}_V \subset \partial V$ bijectively and of bounded derivatives. Let $0 \le \psi \le 1$ be a smooth function compactly supported in V. Then the linear mapping $T : \mathcal{S}^*(X \times Y^+/X \times \{0\}) \longrightarrow \mathcal{S}^*(X \times Y^+/X \times \{0\}) : f \longrightarrow \psi.(f \circ \varphi_{VU}^{-1})$ extends to a bounded operator from $W^{k,p}(U/\mathcal{A}_U) \longrightarrow W^{k,p}(V, \mathcal{A}_V)$.*

We will sketch rapidly the (well known) ideas to define Sobolev spaces on half-plan and the trace operator in the next sections.

3.4.1 Sobolev spaces on half-plan

In this section, the Sobolev spaces on $X \times Y$ or $X \times Y^+$ are defined with weight $(\sigma_1, \ldots, \sigma_n, \rho)$ and $\sigma := \mathrm{lcm}(\sigma_1, \ldots, \sigma_n, \rho)$.

Smooth extensions

Let $\mathcal{S}(X \times Y^+)$ denote the space of smooth, rapidly decreasing functions (and all of their derivatives) on $X \times Y^+$ and $\mathcal{S}(X \times Y^+/0)$ denotes the subspace of functions who vanish, together with all their derivatives, at $X \times \{0\}$. Similar definition for $\mathcal{S}(X \times Y^-)$ and $\mathcal{S}(X \times Y^-/0)$. The following exact sequence is obvious and the arrows are continuous under Schwartz topology.

$$0 \longrightarrow \mathcal{S}(X \times Y^-/0) \xrightarrow{Z_-} \mathcal{S}(X \times Y) \xrightarrow{C_+} \mathcal{S}(X \times Y^+) \longrightarrow 0 \tag{3.5}$$

where Z_- be the extension by 0 and C_+ be the cut-off operator.

It is however not obvious that the sequence in (3.5) splits. Algebraically this is equivalent to the fact that C_+ admits a retraction, that we will note by E_+ since it is in fact an extension to the negative half-plan, which is continuous under Schwartz topology. The construction of E_+ is as follows

$$\begin{aligned} E_+ : \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y) \\ f &\longmapsto \left((x, y) \longmapsto \begin{cases} f(x, y), & \text{if } y \ge 0 \\ \int_0^\infty \varphi(\lambda) f(x, -\lambda y) d\lambda, & \text{if } y < 0 \end{cases} \right) \end{aligned}$$

where the difficult part is the choice of φ, which is resolved by the following lemma.

Lemma 28. *There exists a smooth function $\varphi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ such that $\int_0^{+\infty} x^n |\varphi(x)| dx < \infty \quad \forall n \in \mathbb{Z}$ and*

$$\int_0^{+\infty} x^n \varphi(x) dx = (-1)^n \quad \forall n \in \mathbb{Z} \setminus \{0\}$$

Moreover, $\varphi(\frac{1}{x}) = -x\varphi(x)$ for all $x > 0$.

In fact, the function

$$\varphi(x) = \frac{e^4}{\pi} . \frac{e^{-(x^{1/4}+x^{-1/4})} \sin(x^{1/4} - x^{-1/4})}{1+x}$$

works. The continuity of operator E_+ comes from these properties of φ and basic justification of Lebesgue's Dominated convergence. The projection R_- of Z_- in the sequence (3.5) is constructed algebraically:

$$\begin{aligned} R_- \ \mathcal{S}(X \times Y) &\longrightarrow \mathcal{S}(X \times Y^-/0) \\ f &\longmapsto f - E_+ C_+ f \end{aligned}$$

which is also continuous in Schwartz topology. To resume, one has the split exact sequence

$$0 \longrightarrow \mathcal{S}(X \times Y^-/0) \underset{R_-}{\overset{Z_-}{\rightleftarrows}} \mathcal{S}(X \times Y) \underset{E_+}{\overset{C_+}{\rightleftarrows}} \mathcal{S}(X \times Y^+) \longrightarrow 0 \tag{3.6}$$

and a similar sequence for $\mathcal{S}(X \times Y^+/0$ and $\mathcal{S}(X \times Y^-)$ with operators Z_+, C_-, E_- and R_+.

Also, note that

$$\langle E_+ f, g \rangle = \langle f, R_+ g \rangle \tag{3.7}$$

where the first pairing is on $\mathcal{S}(X \times Y) \times \mathcal{S}(X \times Y)$ and the second is on $\mathcal{S}(X \times Y^+) \times \mathcal{S}(X \times Y^+/0)$.

Remark 17. 1. *The two pairings satisfy $\langle D^\alpha u, v \rangle = (-1)^{|\alpha|} \langle u, D^\alpha v \rangle$.*

2. *The second pairing gives two natural identifications*

$$\mathcal{S}(X \times Y^+/0) \hookrightarrow \mathcal{S}^*(X \times Y^+), \quad \mathcal{S}(X \times Y^+) \hookrightarrow \mathcal{S}^*(X \times Y^+/0)$$

while the first pairing gives $\mathcal{S}(X \times Y) \hookrightarrow \mathcal{S}^*(X \times Y)$.

3. (3.7) *shows that* E_+ *and* R_+ *are adjoint, strictly speaking* E_+ *is the restriction of* R_+^*, *that is*

$$\begin{tikzcd} \mathcal{S}(X \times Y^+) \arrow[r, "E_+"] \arrow[d, hook] & \mathcal{S}(X \times Y) \arrow[d, hook] \\ \mathcal{S}^*(X \times Y^+/0) \arrow[r, "R_+^*"] & \mathcal{S}^*(X \times Y) \end{tikzcd}$$

Similarly. since $\langle C_- f, g\rangle = \langle f, Z_- g\rangle$, one has

$$\begin{tikzcd} \mathcal{S}(X \times Y^-/0) \arrow[r, "Z_-"] \arrow[d, hook] & \mathcal{S}(X \times Y) \arrow[d, hook] \\ \mathcal{S}^*(X \times Y^-) \arrow[r, "C_-^*"] & \mathcal{S}^*(X \times Y) \end{tikzcd}$$

To resume, one can extend the sequence in (3.5) to the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}(X \times Y^-/0) \arrow[r, shift left, "Z_-"] \arrow[d, hook] & \mathcal{S}(X \times Y) \arrow[l, shift left, "R_-"] \arrow[r, shift left, "C_+"] \arrow[d, hook] & \mathcal{S}(X \times Y^+) \arrow[l, shift left, "E_+"] \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{S}^*(X \times Y^-) \arrow[r, shift left, "C_-^*"] & \mathcal{S}^*(X \times Y) \arrow[l, shift left, "E_-^*"] \arrow[r, shift left, "Z_+^*"] & \mathcal{S}^*(X \times Y^+/0) \arrow[l, shift left, "R_+^*"] \arrow[r] & 0 \end{tikzcd} \tag{3.8}$$

We will define Sobolev spaces $W^{k,p}(X\times Y^-/0)$ and $W^{k,p}(X\times Y^+)$ so that they form an intermediate row in diagram. Since the center cell $\mathcal{S}(X\times Y) \subset W^{k,p}(X \times Y) \subset \mathcal{S}^*(X \times Y)$ is already defined, there is only one natural way to do this.

Definition 10. 1. *The* ***Sobolev space upper on half-plan*** *is*

$$W^{k,p}(X\times Y^+) := \left\{ f \in \mathcal{S}^*(X \times Y^+/0) : \ \exists g \in W^{k,p}(X \times Y), f = Z_+^* g \right\}$$

with norm $\|f\|_{W^{k,p}(X\times Y^+)} = \inf_g \|g\|_{W^{k,p}(X\times Y)}$.

2. *The* ***Sobolev space on lower half-plan with vanishing trace***

$$W^{k,p}(X \times Y^-/0) := \left\{ f \in \mathcal{S}^*(X \times Y^-) : \ C_-^* f \in W^{k,p}(X \times Y) \right\}$$

with the induced norm $\|f\|_{W^{k,p}(X\times Y^-/0)} := \|C_-^* f\|_{W^{k,p}(X\times Y)}$.

Remark 18. *1. In other words,* $W^{k,p}(X\times Y^-/0) = C_-^{*\,-1}(W^{k,p}(X\times Y))$ *and* $W^{k,p}(X\times Y^+) = Z_+^*(W^{k,p}(X\times Y))$ *and they are given by the induced norm and the quotient norm of* $W^{k,p}(X\times Y)$ *respectively. The operator* C_-^* *and* Z_+^* *are by definition bounded under Sobolev norm.*

2. The topology of $W^{k,p}(X\times Y)$ *being finer than the induced of weak-* topology from* $\mathcal{S}^*(X\times Y)$*, the restricted operator* $Z_+^*\big|_{W^{k,p}} : W^{k,p}(X\times Y) \longrightarrow \mathcal{S}^*(X\times Y^+/0)$ *is continuous, hence* $\ker Z_+^*\big|_{W^{k,p}} \subset W^{k,p}(X\times Y)$ *is a closed subspace of the Banach space* $W^{k,p}(X\times Y)$*. But this is also the image by* C_-^* *of* $W^{k,p}(X\times Y^-/0)$*. Therefore* $W^{k,p}(X\times Y^-/0)$ *and* $W^{k,p}(X\times Y^+)$ *are Banach spaces.*

3. Idem for $W^{k,p}(X\times Y^+/0)$ *and* $W^{k,p}(X\times Y^-)$*.*

Theorem 29. *1. For all* $k\in\mathbb{R}$ *and* $p\in(1,\infty)$*, the three lines of the following diagram are split-exact and the arrows of the second lines are bounded operators under Sobolev norms.*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}(X\times Y^-/0) \arrow[r, shift left, "Z_-"] \arrow[d, hook] & \mathcal{S}(X\times Y) \arrow[l, shift left, "R_-"] \arrow[r, shift left, "C_+"] \arrow[d, hook] & \mathcal{S}(X\times Y^+) \arrow[l, shift left, "E_+"] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & W^{k,p}(X\times Y^-/0) \arrow[r, shift left, "C_-^*"] \arrow[d, hook] & W^{k,p}(X\times Y) \arrow[l, shift left, "E_-^*"] \arrow[r, shift left, "Z_+^*"] \arrow[d, hook] & W^{k,p}(X\times Y^+) \arrow[l, shift left, "R_+^*"] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{S}^*(X\times Y^-) \arrow[r, shift left, "C_-^*"] & \mathcal{S}^*(X\times Y) \arrow[l, shift left, "E_-^*"] \arrow[r, shift left, "Z_+^*"] & \mathcal{S}^*(X\times Y^+/0) \arrow[l, shift left, "R_+^*"] \arrow[r] & 0
\end{tikzcd} \tag{3.9}$$

2. The subspaces $\mathcal{S}(X\times Y^-/0)$ *and* $\mathcal{S}(X\times Y^+)$ *are dense in* $W^{k,p}(X\times Y^0/0)$ *and* $W^{k,p}(X\times Y^+)$ *respectively.*

3. Interpolation theorem 22 holds for $W^{k,p}(X\times Y^-/0)$ *and* $W^{k,p}(X\times Y^+)$*.*

Proof. The commutativity of the diagram is purely algebraic. The continuity of C_-^* and Z_+^* in the $W^{k,p}$-row follows from the definition of norms in this row. The only non-trivial part is the continuity of E_-^* and R_+^* in the $W^{k,p}$-row, and it suffices to prove that $C_-^*E_-^*$ and $R_+^*Z_+^*$ are bounded as automorphism of $W^{k,p}(X\times Y)$. This follows from direct computation of these norm in the case $\sigma \mid k\in\mathbb{R}$ and interpolation inequality (Theorem 20) for intermediate k.

Once the continuity of E_-^* and R_+^* is established, the density of $\mathcal{S}(X\times Y^-/0)$ follows straight-forwardly and we see that $W^{k,p}(X\times Y^-/0)$ and

$W^{l,p}(X \times Y^-/0)$ are interpolatable (the two spaces share a dense subspace). Theorem 24 applies and shows that Theorem 22 holds for $W^{k,p}(X \times Y^-/0)$.

Idem for the side of $\mathcal{S}(X \times Y^+) \subset W^{k,p}(X \times Y^+)$. $\square$

Remark 19. *By dualising the diagram* (3.9) *and using the fact that the dual space of $W^{k,p}(X \times Y)$ is $W^{-k,p'}(X \times Y)$, one can prove that the dual space of $W^{k,p}(X \times Y^+)$ is $W^{-k,p'}(X \times Y^+/0)$.*

Functoriality of D_x and equivalent definitions

The following discussion appeared as 4 lemmas in [Ham75, page 38-42] in the proof of Vanishing trace theorem 31. I think these ideas can be presented without much computation.

Note that the weight $W(\xi, \eta) = \left(1 + \xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n} + \eta^{2\rho}\right)^{1/2\sigma}$ is comparable to $W(\xi) + W(\eta)$ where

$$W(\xi) = \left(1 + \xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n}\right)^{1/2\sigma}, \quad W(\eta) = \left(1 + \eta^{2\rho}\right)^{1/2\sigma}$$

and also $W^k(\xi, \eta)$ is comparable to $W(\xi)^k + W(\eta)^k$. Hence $W(D_x)^l : W^{k,p}(X \times Y) \longrightarrow W^{k-l,p}(X \times Y)$ is a bounded operator.

The vertical arrows in the following diagram are the vertical arrows of (3.9). The dashed horizontal arrow indicates that it is established only in the center cells $W^{k,p}(X \times Y) \longrightarrow W^{k-l,p}(X \times Y)$.

$$\begin{tikzcd}
& \mathcal{S} - \text{row} \arrow[dl] \arrow[dr] & \\
W^{k,p} - \text{row} \arrow[rr, dashed, "W(D_x)^l"] \arrow[dr] & & W^{k-l,p} - \text{row} \arrow[dl] \\
& \mathcal{S}^* - \text{row} &
\end{tikzcd} \tag{3.10}$$

We will see that the dashed arrow can be extended to a full arrow, that is 3 arrows between the $W^{k,p}$-row and $W^{k-l,p}$-row that are compatible with the diagram (3.9).

One can construct $W(D_x)^l$ arrows from $W^{k,p}(X \times Y^-/0) \longrightarrow W^{k-l,p}(X \times Y^-/0)$ and $W^{k,p}(X \times Y^+) \longrightarrow W^{k-l,p}(X \times Y^+)$ as adjoint of $W(D_x)^l$ on $\mathcal{S}(X \times Y^+)$ and $\mathcal{S}(X \times Y^+/0)$. They are by definition continuous on the weak-* topology. It is easy to see that if we can prove that these two $W(D_x)^l$ arrows commute with C_-^*, E_-^* and Z_+^*, R_+^* on $W^{k,p}$-row and $W^{k-l,p}$-row, then by the continuity of the $W(D_x)^l$ arrow from $W^{k,p}(X\times Y) \longrightarrow W^{k-l,p}(X\times Y)$, these $W(D_x)^l$ arrows are bounded in $W^{k,p}$ norm.

The two new $W(D_x)^l$ arrows commute with all "$\longrightarrow$" arrows in the $W^{k,p}$-row of (3.9), i.e. C_-^* and Z_+^*, since for smooth functions, D_x commutes with Z_- (extension by 0) and C_+ (cut-off).

The fact that $W(D_x)^l$ commutes with the "$\longleftarrow$" arrows, i.e. E_-^* and R_+^* is due to:

$$\begin{array}{ccc} \mathcal{S}(X\times Y^+) & \xrightarrow{E_+} & \mathcal{S}(X\times Y) \\ \big\downarrow{\scriptstyle W(D_x)^l} & & \big\downarrow{\scriptstyle W(D_x)^l} \\ \mathcal{S}(X\times Y^+) & \xrightarrow{E_+} & \mathcal{S}(X\times Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{S}(X\times Y) & \xrightarrow{R_-} & \mathcal{S}(X\times Y^-/0) \\ \big\downarrow{\scriptstyle W(D_x)^l} & & \big\downarrow{\scriptstyle W(D_x)^l} \\ \mathcal{S}(X\times Y) & \xrightarrow{R_-} & \mathcal{S}(X\times Y^-/0) \end{array}$$

Remark 20. *There is no functoriality of D_y since for $y<0$*

$$D_y^l E_+ f(x,y) = \int_0^\infty (-\lambda)^l \varphi(\lambda) D_y^l f(x,-\lambda y) d\lambda \neq E_+ D_y^l f(x,y)$$

meaning that the D_y does not commute with E_+.

However $D_y^l E_+ f \in L^p(X\times Y)$ if and only if $E_+ D_y^l f \in L^p(X\times Y)$ if and only if $D_y^l f \in L^p(X\times Y^+)$. Moreover the 3 L^p norms are equivalent.

The density of $\mathcal{S}(X\times Y^-/0)$ and $\mathcal{S}(X\times Y^+)$ in the corresponding $W^{k,p}$ shows that the new $W^{k,p}$ spaces can also be defined by density using the $W^{k,p}$-norm of the extension (Z_- and E_+ respectively) from half-plan to the whole plan. By the continuity of R_+^* in the second row of (3.9) when $k=0$, one sees that the L^p-norms of the extensions by Z_- and E_+ are equivalent to the L^p norm on the half-plan. Therefore, one has the following analog of Theorem 17.

Theorem 30. *Given $k>0$ and $\sigma \mid k$,*

1. *If $f\in\mathcal{S}^*(X\times Y^+/0)$ then*
   (a) *$f\in W^{k,p}(X\times Y^+)$ if and only if $D_x^\alpha D_y^\beta f\in L^p(X\times Y^+)$ for $\|(\alpha,\beta)\|\le k$.*
   (b) *$f\in W^{-k,p}(X\times Y^+)$ if and only if there exists $g_{\alpha\beta}\in L^p(X\times Y^+)$ such that $f=\sum_{\|(\alpha,\beta)\|\le k} D_x^\alpha D_y^\beta g_{\alpha\beta}$.*
2. *If $f\in\mathcal{S}^*(X\times Y^+)$ then*
   (a) *$f\in W^{k,p}(X\times Y^+/0)$ if and only if $D_x^\alpha D_y^\beta f\in L^p(X\times Y^+)$ for $\|(\alpha,\beta)\|\le k$.*
   (b) *$f\in W^{-k,p}(X\times Y^+/0)$ if and only if there exists $g_{\alpha\beta}\in L^p(X\times Y^+)$ such that $f=\sum_{\|(\alpha,\beta)\|\le k} D_x^\alpha D_y^\beta g_{\alpha\beta}$.*

3.4.2 Trace theorems

To make the notation more intuitive, we abusively denote the horizontal arrows in the $W^{k,p}$-row and the $\mathcal{S}^*$-row by their corresponding arrows in the $\mathcal{S}$-row (i.e. their restriction on the space of smooth functions), that is we will use Z_-, C_+, R_-, E_+ instead of $C_-^*, Z_+^*, E_-^*, R_+^*$.

The goal of this section is to define the restriction of a function $f \in W^{k,p}(X \times Y^+)$ on $X \times \{0\}$. The pointwise restriction of f does not make sense because f is only defined up to a negligible set (i.e. of Lebesgue measure 0). The strategy is to take a sequence $f_n \in \mathcal{S}(X \times Y^+)$ that is $W^{k,p}$-converging to f and to see if $\{f_n\big|_{X\times\{0\}}\}$ converges in $L^p(X \times \{0\})$. If it does one calls the limit *trace* of f on $X \times \{0\}$. Theorem 31, Example 21 and Theorem 33 show that one should expect

- high regularity of f, i.e. k large enough, so that the limit exists,
- a drop of regularity of the restriction.

From diagram (3.9) and its opposite version (with all $+$ and $-$ signs interchanged), there is a natural inclusion $\iota :\ W^{k,p}(X \times Y^+/0)$ to $W^{k,p}(X \times Y^+)$, by first extending by zero, then cutting-off

$$\begin{tikzcd} W^{k,p}(X \times Y^+/0) \arrow[rr, hook, "\iota"] \arrow[dr, "Z_+"'] & & W^{k,p}(X \times Y^+) \\ & W^{k,p}(X \times Y) \arrow[ur, "C_+"'] & \end{tikzcd}$$

Theorem 31 (Vanishing trace)**.** *If $p \in (1, +\infty)$ and $-1 + \frac{1}{p} < \rho\frac{k}{\sigma} < \frac{1}{p}$ then ι is an isomorphic*

Proof. Define

$$\begin{aligned} M_+(\lambda) :\ \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y^+) \\ f(x,y) &\longmapsto f(x, \lambda y) \end{aligned}$$

Since $\langle M_+(\lambda)f, g\rangle = \langle f, N_+(\lambda)g\rangle$ for all $f \in \mathcal{S}(X \times Y^+), g \in \mathcal{S}(X \times Y^+/0)$ and $\lambda > 0$ where $N_+(\lambda)g(x,y) := \lambda^{-1}g(x, \lambda^{-1}y)$, one sees that $M_+(\lambda)$ extends to $\mathcal{S}^*(X \times Y^+/0) \longrightarrow \mathcal{S}^*(X \times Y^+/0)$ and that one extension of it is $N_+^*(\lambda)$ the adjoint of $N_+(\lambda)$:

$$\begin{tikzcd} \mathcal{S}(X \times Y^+) \arrow[r, "M_+(\lambda)"] \arrow[d, hook] & \mathcal{S}(X \times Y^+) \arrow[d, hook] \\ \mathcal{S}^*(X \times Y^+/0) \arrow[r, "N_+^*(\lambda)"] & \mathcal{S}^*(X \times Y^+/0) \end{tikzcd}$$

We abusively call $N_+^*(\lambda)$ by $M_+(\lambda)$. We will let $\lambda \to +\infty$, the operator $M_+(\lambda)$ intuitively "shrinks" to the boundary $X \times \{0\}$.

Lemma 32. *For $k \geq 0, \lambda \geq 1$, $M_+(\lambda) : W^{k,p}(X \times Y^+) \longrightarrow W^{k,p}(X \times Y^+)$ is bounded and*

$$\|M_+(\lambda)f\|_{W^{k,p}(X\times Y^+)} \leq C\lambda^{\frac{\rho k}{\sigma}-\frac{1}{p}}\|f\|_{W^{k,p}(X\times Y^+)}$$

where C does not depend on λ.

The proof of the Lemma 32 is straightforward: it suffices to prove the boundedness in the case $\sigma \mid k$ an use interpolation inequality 20, also one can suppose that $f \in \mathcal{S}(X \times Y^+)$. Note that $(\frac{\partial}{\partial y})^l M_+(\lambda) = \lambda^l M_+(\lambda)(\frac{\partial}{\partial y})^l$ while $\frac{\partial}{\partial x}$ commutes with $M_+(\lambda)$, hence in general $|D^\alpha_{(x,y)} M_+(\lambda)f| \leq \lambda^{k\rho/\sigma}|D^\alpha_{(x,y)}f|$ for all $\|\alpha\| \leq k, \lambda \geq 1$. The $-\frac{1}{p}$ in the exponent of λ is due to: $\|M_+(\lambda)g\|_{L^p} = \lambda^{-1/p}\|g\|_{L^p}$.

Back to Theorem 31, let $f \in \mathcal{S}(X \times Y^+)$ and define $\tilde{M}(\lambda)f$ to be f on $X \times Y^+$ and $M_-(\lambda)C_-E_+f$ on $X \times Y^-$, then $\tilde{M}(\lambda)f \in W^{\sigma/\rho,p}(X \times Y)$. Note that $D_y\tilde{M}(\lambda)f$ is not continuous at $X \times \{0\}$ but is still in $L^p(X \times Y)$ because f and $M_-(\lambda)C_-E_+f$ agrees on $X \times \{0\}$. Suppose we can prove that as $\lambda \to +\infty$ the sequence $\tilde{M}(\lambda)f$ converges to $\tilde{M}f$ in $W^{k,p}(X \times Y)$ then $C_-\tilde{M}f = \lim_{\lambda\to+\infty} M_-(\lambda)C_-E_+f = 0$. One obtains, by exactness of the second row of diagram (3.9), existence of a $g \in W^{k,p}(X \times Y^+/0)$ such that $\tilde{M}f = Z_+g$. Moreover, since $C_+\tilde{M}(\lambda)f = f$ for all $\lambda > 0$, one has $C_+\tilde{M}f = f$, hence $\iota g = C_+Z_+g = C_+\tilde{M}f = f$.

It remains to prove the existence of such $\tilde{M}f$. By Lemma 32 and the fact that all $\tilde{M}(\lambda)f$ are the same on $X \times Y^+$, one has

$$\|\tilde{M}(\lambda)f - \tilde{M}(2\lambda)f\|_{W^{k,p}(X\times Y)} \leq 2C\lambda^{\frac{\rho k}{\sigma}-\frac{1}{p}}\|f\|_{W^{k,p}(X\times Y^+)}$$

Therefore if $\frac{\rho k}{\sigma} < \frac{1}{p}$, the sequence $\tilde{M}(2^n)f$ converge in $W^{k,p}(X \times Y)$ to $\tilde{M}f$. $\square$

Remark 21. *If $\rho = \sigma_i = 1$ then $\sigma = 1$, take $k = 0$ then the Theorem 31 claims that $\mathcal{S}(X \times Y^+/0)$ is dense in $L^P(X \times Y^+) \supset \mathcal{S}(X \times Y^+)$, or equivalently any smooth function $f \in \mathcal{S}(X \times Y^+)$ not necessarily vanishes on $X \times \{0\}$ can be L^p-approximated by smooth functions with all derivative vanishes on $X \times \{0\}$. This means that one cannot define any notion of trace on $X \times \{0\}$ that varies continuously under the L^p norm.*

In case of high regularity $\frac{\rho k}{\sigma} > \frac{1}{p}$, one can define a meaningful notion of trace.

Theorem 33 (Well-defined trace). *If $\frac{\rho k}{\sigma} > \frac{1}{p}$ then the restriction map*

$$\begin{aligned} B : \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X) \\ f(x,y) &\longmapsto f(x,0) \end{aligned}$$

extends to a bounded operator, abusively noted by $B :\ W^{k,p}(X \times Y^+) \longrightarrow L^p(X)$.

Definition 11. *We call $\partial W^{k,p}(X \times Y^+) := W^{k,p}(X \times Y^+)/\ker B$ the **space of boundary value** of function in $W^{k,p}(X \times Y^+)$.*

Theorem 33 can be strengthen by remarking that if $\sigma := \mathrm{lcm}(\sigma_1, \ldots, \sigma_n, \rho) = \mathrm{lcm}(\sigma_1, \ldots, \sigma_n)$ and if $W(\xi)$ denotes the weight $(1 + \xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n})^{1/2\sigma}$ then B and $W(D_x)$ commute, i.e.

$$\begin{tikzcd} W^{k,p}(X \times Y^+) \arrow[r, "B"] \arrow[d, "W(D_x)^l"'] & L^p(X) \subset \mathcal{S}^*(X) \arrow[d, "W(D_x)^l"] \\ W^{k-l,p}(X \times Y^+) \arrow[r, "B"] & L^p(X) \end{tikzcd}$$

as long as $\frac{\rho(k-l)}{\sigma} > \frac{1}{p}$. Therefore, one has

Theorem 34 (Regularity of trace). *If $0 \leq l < k - \frac{\sigma}{\rho p}$ then the trace operator B in Theorem 33 actually of image in $W^{l,p}(X)$ and the operator*

$$B :\ W^{k,p}(X \times Y^+) \longrightarrow W^{l,p}(X)$$

is bounded.

Proof of Theorem 33. It suffices to prove that $\|Bf\|_{L^p(X)} \leq C\|f\|_{W^{k,p}(X \times Y^+)}$ for all $f \in \mathcal{S}(X \times Y^+)$ and $1 \geq \frac{\rho k}{\sigma} > \frac{1}{p}$ (for higher k, embed in the $W^{k,p}$ smaller k). Define

$$\begin{aligned} T_v :\ \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y^+) \\ f &\longmapsto \left((x,y) \longmapsto \frac{1}{v} \int_0^v f(x, y+w)dw \right) \end{aligned}$$

for $v > 0$. One can check that T_v extends to a bounded operator $T_v : W^{k,p}(X \times Y^+) \longrightarrow W^{k,p}(X \times Y^+)$ for all $k \geq 0$ and that

$$\begin{cases} \|D_y T_v f\|_{L^p(X \times Y^+)} \leq C v^{-1} \|f\|_{L^p(X \times Y^+)}, \\ \|D_y T_v f\|_{L^p(X \times Y^+)} \leq C \|f\|_{W^{\sigma/\rho,p}(X \times Y^+)} \end{cases}$$

hence by Interpolation inequality Theorem 20, one obtains for all $0 \leq k \leq \sigma/\rho$: $\|D_y T_v f\|_{L^p(X\times Y^+)} \leq C v^{\rho k/\sigma - 1}\|f\|_{W^{k,p}(X\times Y^+)}$ hence

$$\|D_y(T_{v/2} - T_v f)\|_{L^p(X\times Y^+)} \leq C v^{\rho k/\sigma-1}\|f\|_{W^{k,p}(X\times Y^+)} \tag{3.11}$$

Similarly, one can prove that for all $0 \leq k \leq \sigma/\rho$: $\|(\mathrm{Id}-T_v)f\|_{L^p(X\times Y^+)} \leq C v^{\rho k/\sigma}\|f\|_{W^{k,p}(X\times Y^+)}$ therefore

$$\|(T_{v/2} - T_v)f\|_{L^p(X\times Y^+)} \leq C v^{\rho k/\sigma}\|f\|_{W^{k,p}(X\times Y^+)} \tag{3.12}$$

Moreover, using Hölder inequality and Fundamental theorem of calculus, one has: if $g \in \mathcal{S}(X\times Y^+)$ then

$$\|Bg\|_{L^p(X)} \leq C\|g\|^{1/p'}_{L^p(X\times Y^+)}\|D_y g\|^{1/p}_{L^p(X\times Y^+)} \tag{3.13}$$

Substitute g by $(T_{v/2} - T_v)f$ in (3.13) then use apply (3.11) and (3.12), one has

$$\|B(T_{v/2} - T_v)\|_{L^p(X)} \leq C v^{\frac{\rho k}{\sigma} - \frac{1}{p}}\|f\|_{W^{k,p}(X\times Y^+)}$$

Therefore if $\frac{1}{p} < \frac{\rho k}{\sigma} \leq 1$, the sequence $BT_{2^-n}f$ converges in $L^p(X)$ and the limit is of L^p-norm less than $C\|f\|_{W^{k,p}(X\times Y^+)}$. Since f is continuous, the limit is $f\big|_{X\times\{0\}}$. The theorem follows. □

Remark 22. *The fact that the condition on l in Theorem 34 is an open condition explains why we topologize the space of boundary value $\partial W^{k,p}(X\times Y^+)$ by the quotient $W^{k,p}$-norm instead of any $W^{l,p}$-norm. Also, we have completeness for free.*

In the proof of Theorem 31, we glue a function $f_+ \in \mathcal{S}(X\times Y^+)$ with $f_- \in \mathcal{S}(X\times Y^-)$ of the same value on $X\times\{0\}$ and the result is a function in $W^{\sigma/\rho,p}(X\times Y)$. This can be generalised as follow

Theorem 35 (Patching theorem)**.** *If $p \in (1,+\infty)$ and $\frac{1}{p} < \rho\frac{k}{\sigma} < 1+\frac{1}{p}$, then given $f_+ \in W^{k,p}(X\times Y^+)$ and $f_- \in W^{k,p}(X\times Y^-)$ such that $Bf_+ = Bf_-$ in $L^p(X)$, one defines $f \in L^p(X\times Y)$ such that $f = f_+$ on $X\times Y^+$ and $f = f_-$ on $X\times Y^-$. Then actually $f \in W^{k,p}(X\times Y)$.*

3.4.3 Trace operator on manifold

The following paragraph does not appear in [Ham75] because of Remark 16.

To resume, we have defined Sobolev spaces on manifold with boundary as the space of currents whose cut-off restrictions on each chart are in $W^{k,p}$.

Also we have defined trace operator of Sobolev spaces on half-plan in a vision to extend the notion to manifold.

Let $f \in W^{k,p}(M/\mathcal{A})$ and $\mathcal{B}$ be a connected component of ∂M. With the same notation as Definition 9, f gives the data of $f_i = (\psi_i f) \circ \varphi_i^{-1} \in W^{k,p}(R_i/\mathcal{A}_i)$ the cut-off restriction of f on each chart using a partition of unity $\{\psi_i\}_i$ subordinated to a good atlas $(U_i)_i$ of M, where R_i is an Euclidean space of the same dimension as M ($\mathcal{A}_i = \emptyset$), or a half-plan ($\mathcal{A}_i \subset \partial R_i$). Note that $U_i \cap \mathcal{B}$ is a good atlas of $\mathcal{B}$ and ψ_i is still a partition of unity subordinated to this atlas, therefore take $g_i \in W^{l,p}(\partial R_i)$ to be trace of f_i on the image of $\mathcal{B}$ of each chart. It remains to check that the data (g_i) corresponds to a unique element $g \in W^{l,p}(\mathcal{B})$. Recall that we have the following diagram:

$$0 \longrightarrow W^{l,p}(\mathcal{B}) \underset{\pi}{\overset{\iota}{\rightleftarrows}} \bigoplus_i W^{l,p}(\partial R_i)$$

where ι admits a projection π given by the cut-off functions $\tilde{\psi}_i$ that we choose to be the same ones used for M. Hence to see that $(g_i)_i$ is in the image of ι, it suffices to check that $\iota \circ \pi((g_i)_i) = (g_i)_i$ which should be straightforward, since $\sum_i \tilde{\psi}_i \psi_i = 1$.

Now that we defined a trace operator $B: W^{k,p}(M) \longrightarrow L^p(\partial M)$ that factor through $W^{k,p}(M) \longrightarrow W^{l,p}(\partial M)$ for all $0 \leq l < k - \frac{\sigma}{\rho p}$, we can define the space of boundary value of function in $W^{k,p}(M)$ by

$$\partial W^{k,p}(M) := W^{k,p}(M)/\ker B$$

which has a finer topology than its image in any $W^{l,p}(\partial M)$ for $0 \leq l < k - \frac{\sigma}{\rho p}$.

Chapter 4

Elliptic and parabolic equations on compact manifolds

4.1 Commutative diagram and linear PDE. Example: Semi-elliptic equation on $\mathbb{R}^n$

Fix a weight $(\sigma_1, \ldots, \sigma_n)$ on $X = \mathbb{R}^n$ and recall that for an index α, we note $\|\alpha\| := \sum_i \frac{\sigma}{\sigma_i}\alpha_i$.

We will consider in this section a partial differential operator A that is heterogeneous, of constant coefficient and of weight r, i.e.

$$A(D) = \sum_{\|\alpha\|=r} a_\alpha D^\alpha, \quad D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha$$

The **symbol** of A is $A(\xi) := \sum_{\|\alpha\|=r} a_\alpha \xi^\alpha$ and A is called **semi-elliptic** if $A(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus 0$

Remark 23. *If A is semi-elliptic then $\sigma \mid r$. In fact choose all $\xi_j = 0$ except $\xi_i \neq 0$, one sees that there must be a non-zero coefficient $a_{(0,\ldots,\frac{r\sigma_i}{\sigma},\ldots,0)}$, i.e. $\frac{r\sigma_i}{\sigma} \in \mathbb{Z}$ for all $i = \overline{1,n}$. Hence $\sigma \mid r$ ($\sigma = \mathrm{lcm}(\sigma_i)$ being a combination of σ_i, look at the same combination of $\frac{r\sigma_i}{\sigma}$).*

It is clear that the operator $A : W^{n,p}(X) \longrightarrow W^{n-r,p}(X)$ is bounded for all $n \in \mathbb{R}$ and the following diagram commutes for every real numbers $k < n$.

$$\begin{tikzcd} W^{n,p}(X) \arrow[r, "A(D)"] \arrow[d, hook, "i"] & W^{n-r,p}(X) \arrow[d, hook, "i"] \\ W^{k,p}(X) \arrow[r, "A(D)"] & W^{k-r,p}(X) \end{tikzcd} \tag{4.1}$$

Definition 12. *Let E, F, G, H be Banach spaces and l, m, p, q are bounded operator such that the following diagram* (diag:D) *commutes*

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "m"] & F \arrow[d, "p"] \\ G \arrow[r, "q"] & H \end{tikzcd} \tag{diag:D}$$

Then (diag:D) *is said to be an* ***exact square*** *if the following associated sequence is exact*

$$0 \longrightarrow E \xrightarrow{l \oplus m} F \oplus G \xrightarrow{p \ominus q} H \longrightarrow 0$$

Example 8. *If (A, B) is an interpolatable pair of Banach spaces then*

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A + B \end{tikzcd}$$

is exact, where arrows are natural inclusions.

The notion of exact square allows us to reformulate classical results of elliptic equation as

Theorem 36 (Elliptic equation with constant coefficients)**.** *The square* (4.1) *is exact for all $k < n$ in $\mathbb{R}$. This encodes the following 3 results:*

1. $W^{n,p}(X) \xhookrightarrow{A \oplus i} W^{n-r,p}(X) \oplus W^{k,p}(X)$ *is of closed image, i.e. there exists $C > 0$ such that*

$$\|f\|_{W^{n,p}(X)} \leq C \left(\|Af\|_{W^{n-r,p}(X)} + \|f\|_{W^{k,p}(X)} \right)$$

which is Gårding's inequality.

2. $\ker A \ominus i = \operatorname{Im} A \oplus i$, *i.e. if* $f \in W^{k,p}(X)$ *and* $Af \in W^{n-r}(X)$ *then actually* $f \in W^{n,p}(X)$, *which is regularity theorem.*

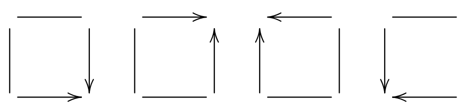
3. $\operatorname{Im} A \ominus i = W^{k-r,p}(X)$, *i.e. for all* $g \in W^{k-r,p}(X)$, *there exists* $f \in W^{k,p}(X)$ *such that* $Af - g \in W^{n-r,p}(X)$, *which is the existence of approximate solution (the idea behind* parametrix*).*

A way to prove that a square is exact is to show that it splits

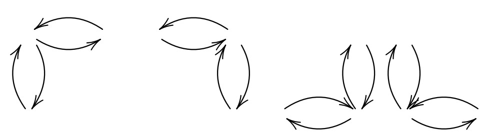
Definition 13. *The square* (diag:D) *is called* ***split*** *if there exists* l', m', p', q' *such that*

$$\begin{tikzcd} E \arrow[r, "l"'] \arrow[d, "m"] & F \arrow[l, bend right, "l'"'] \arrow[d, "p"'] \\ G \arrow[u, bend left, "m'"] \arrow[r, "q"] & H \arrow[u, bend right, "p'"'] \arrow[l, bend left, "q'"] \end{tikzcd} \tag{4.2}$$

commutes in 4 ways:



and splits in 4 ways



,

i.e. the sum of two circle in each diagram is the identities.

Theorem 37. *1. A split square is exact. In fact, if a square splits, then the associated short sequence splits.*

2. If E, F, G, H *are Hilbert spaces then any exact square splits.*

Proof of Theorem 36 . Since A is semi-elliptic, there exists $\epsilon > 0$ such that $|A(\xi)| \geq \epsilon \|\xi\|^r$ for $\|\xi\| := (\xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n})^{1/2\sigma}$. Let $\psi(\xi)$ be a radial function in ξ that is identically 1 for $\|\xi\| \leq 1$ and 0 for $\|\xi\| \geq 2$ and define

$$G(\xi) := \begin{cases} \frac{1-\psi(\xi)}{A(\xi)}, & \text{if } \|\xi\| \geq 1 \\ 0, & \text{if } \|\xi\| \leq 1 \end{cases}$$

Then by Stein's multiplier theorem,

$$\begin{aligned} G(D):\ W^{k-r,p}(X) &\longrightarrow W^{k,p}(X) \quad \forall k \in \mathbb{R}, \\ \psi(D):\ W^{l,p}(X) &\longrightarrow W^{k,p}(X) \quad \forall k, l \in \mathbb{R} \end{aligned}$$

are bounded operators. We say that $G(D)$ is an *approximate inverse* of $A(D)$ because $G(D)A(D) = A(D)G(D) = 1 - \psi(D)$. It is easy to check that (4.1) splits:

$$\begin{tikzcd} W^{k,p}(X) \arrow[r, shift left, "A(D)"'] \arrow[d, shift left, "i"] & W^{k-r,p}(X) \arrow[l, shift left, "G(D)"'] \arrow[d, shift right, "i"'] \\ W^{l,p}(X) \arrow[u, bend left, "\psi(D)"] \arrow[r, shift left, "A(D)"] & W^{l-r,p}(X) \arrow[l, shift left, "G(D)"] \arrow[u, bend right, "\psi(D)"'] \end{tikzcd}$$

$\square$

The following abstract result shows that solutions of homogeneous equation $Af = 0$ are smooth (also proved in the second point of Theorem 36) and the solution space is of finite dimension.

Theorem 38. *Suppose that the square*

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "m"] & F \arrow[d, "p"] \\ G \arrow[r, "q"] & H \end{tikzcd}$$

is exact and m, p are compact operators. Then l and q have closed image, and their kernels and cokernels are isomorphic through m and p, and are of finite dimensional.

Proof. By basic diagram chasing, one can see that the restriction of m is an isomorphism $\ker l \longrightarrow \ker q$. But m is compact, $\ker l \cong \ker q$ are locally compact, hence of finite dimension.

It is easy to check (with sequential limit) that $\operatorname{Im} l$ is closed in F, since $\operatorname{Im}(l \oplus m) = \ker p \ominus q$ is closed and m is compact. So $\operatorname{coker} l$ is a Banach space.

Let $p'' : \operatorname{coker} l = F/l(E) \longrightarrow H/\overline{q(G)}$ be the map induced by p to the quotients, note that we have to take the closure of $q(G)$ to ensure that the quotient is Banach. Then p'' is obviously continuous and compact. Also p'' is surjective because $F \oplus G \xrightarrow{p \ominus q} H$.

We will prove that p'' is injective. If $f \in F \setminus l(E)$ then by Hahn-Banach theorem, there exists a linear functional $\lambda \in F^*$ such that $\lambda(f) = 1$ and $\lambda(l(E)) = 0$. One has

$$0 \longrightarrow H^* \xrightarrow{(p\ominus q)^*} F^* \oplus G^* \xrightarrow{(l\oplus m)^*} E^* \longrightarrow 0$$

and that $(l \oplus m)^*(\lambda \oplus 0) = 0$, hence there exists $\lambda' \in H^*$ such that $\lambda \oplus 0 = (p \ominus q)^*\lambda'$, i.e. $\lambda' \circ q = 0$ and $\lambda' \circ p = \lambda$, which means λ' vanishes on $q(G)$, hence $\overline{q}(G)$, and that $\lambda'(p(f)) = \lambda(f) = 1$. Hence $p(f) \notin \overline{q(G)}$ and p'' is injective.

The injectivity of p'' has 2 consequences. First, it means that $\operatorname{coker} l \cong H/\overline{q(G)}$ by a compact operator, hence the two are locally compact and of finite dimension.

Second, it proves that $q(G)$ is closed in H. In fact, given $h \in \overline{q(G)}$, by surjectivity of $p \ominus q$, one has $h = pf + qg$ for $f \in F$ and $g \in G$, this means $p''(\bar{f}) = \bar{0} \in H/\overline{q(G)}$, hence $\bar{f} = \bar{0} \in \operatorname{coker} l$, i.e. $f = l(e)$ for some $e \in E$. Therefore

$$h = p \circ l(e) + q(g) = q(m(e) + g) \in q(G)$$

and $p(G) = \overline{p(G)}$ is closed in H. $\square$

Remark 24. *The proof of Theorem 38 is much simpler for split squares. We presented the version for exact squares because we will use it later. The advantage of using exact squares instead of split square is, as we will see, that among commutative squares, exact squares form a relatively open set, allowing us to "pertube" an exact square and extend the theory to cover the case A of variable coefficients.*

4.2 Elliptic equation on half-plan $X \times Y^+$. Boundary conditions.

We will quickly review in this part the ideas to solve elliptic equations with constant coefficients on half-plan. This does not require any more abstract (i.e. with diagram) results. The main tasks will be using suitable cut-off function on the frequent space (1) to define the approximate inverse of an elliptic operator on half-plan that is adapted to the boundary structure and (2) to approximately inverse the boundary operators.

We will solve elliptic equation on $X \times Y^+$ where the variables are $x_1, \dots, x_n$ and y, under weight $\Sigma = (\sigma_1, \dots, \sigma_n, \rho)$. Recall that $A(D) = \sum_{\|(\alpha,\beta)\|=r} a_{\alpha\beta} D_x^\alpha D_y^\beta$ with symbol $A(\xi, \eta) = \sum_{\|(\alpha,\beta)\|=r} \xi^\alpha \eta^\beta$.

If $A(D)$ is semi-elliptic then for all $\xi \neq 0$ the polynomial $\eta \mapsto A(\xi,\eta)$ has no real zeros, hence can be factorized to

$$A(\xi,\eta) = A^+(\xi,\eta)A^-(\xi,\eta)$$

where $A^+(\xi,\eta)$ (resp. $A^-(\xi,\eta)$) only has zeros η with $\operatorname{Im}\eta > 0$ (resp. $\operatorname{Im}\eta < 0$).

Remark 25. *1. By semi-ellipticity, the monomial $a_{\alpha\beta}\xi^\alpha\eta^\beta$ with biggest β has index $\alpha = 0$. Hence we can suppose that the leading coefficients, as polynomials in η of A, A^+, A^- are 1.*

2. As polynomial in η, $A^+(\xi,\eta) = \sum_{\beta=0}^m a_\beta^+(\xi)\eta^\beta$ where $m = r\rho/\sigma$ and $a_\beta^+(\xi)$ are Σ-heterogeneous of weight $(m-\beta)\rho$, i.e.

$$a_\beta^+(t^{\sigma/\sigma_1}\xi_1,\dots,t^{\sigma/\sigma_n}\xi_n) = t^{(m-\beta)\rho}a_\beta^+(\xi)$$

Also, the coefficients a_β^+ are smooth in ξ.

We will solve the elliptic equation under some *suitable* boundary conditions. Let B^j, $1 \leq j \leq m$ be m Σ-heterogeneous *boundary operators* of weights r_j, i.e.

$$B^j(D) = \sum_{\|(\alpha,\beta)\|=r_j} b^j_{\alpha\beta}D_x^\alpha D_y^\beta$$

of symbol

$$B^j(\xi,\eta) = \sum_{\|(\alpha,\beta)\|=r_j} b^j_{\alpha\beta}\xi^\alpha\eta^\beta = \sum_{\|(\alpha,\beta)\|=r_j} b^j_\beta(\xi)\eta^\beta$$

where b^j_β are heterogeneous in ξ (actually polynomials) and of weight $r_j - \beta$.

As our discussion on trace operator, if $k > r_j + \frac{\sigma}{\rho p}$ then B^j extends to a bounded operator

$$\begin{tikzcd}
W^{k,p}(X\times Y^+) \arrow[rr, "B_j"] \arrow[dr] & & \partial W^{k-r_j,p}(X) \arrow[r, hook] & W^{l,p}(X) \\
& W^{k-r_j,p}(X\times Y^+) \arrow[ur] & &
\end{tikzcd}$$

for all $0 \leq l < n - r_j - \frac{\sigma}{\rho p}$.

Definition 14. *We will say that the operators $Bf = (B^1 f, \ldots, B^m f)$ satisfy the* ***complementary boundary condition (CBC)*** *if the*

$$\det \left(c^j_\beta(\xi) \right)_{j,\beta} \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

where $c^j_\beta(\xi)$ are the cofficients of the remanders $C^j(\xi,\eta)$ when one divides $B^j(\xi,\eta)$ by $A^+(\xi,\eta)$ as polynomials in η, i.e.

$$B^j(\xi,\eta) \equiv C^j(\xi,\eta) = \sum_{\beta=0}^{m-1} c^j_\beta(\xi)\eta^\beta \qquad \mod A^+(\xi,\eta)$$

Approximate inverse of boundary operator B. The CBC condition allows us to approximately inverse boundary operator B.

Theorem 39 (Approximate inverse of B)**.** *Let $B: \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X)^{\oplus m}$ be a boundary operator that satisfies CBC condition, then there exists an operator*

$$\begin{aligned} H : \mathcal{S}(X)^{\oplus m} &\longrightarrow \mathcal{S}(X \times Y^+) \\ (h_1, \ldots, h_m) &\longmapsto H_1 h_1 + \cdots + H_m h_m \end{aligned}$$

such that

1. $(\mathrm{Id} - BH)h = \psi(D_x)h$ *for all* $h \in \mathcal{S}(X)^{\oplus m}$.
2. $(\mathrm{Id} - HB)f = \psi(D_x)f$ *for all* $f \in \ker A(D) : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$.

where $\psi(\xi)$ is the radial smooth cut-off function in ξ that equals 1 when $\|\xi\| \leq 1$ and 0 when $\|\xi\| \geq 2$.

Moreover, if $k > r_j + \frac{\sigma}{\rho p}$ then the operators $H_j : \mathcal{S}(X) \longrightarrow \mathcal{S}(X \times Y^+)$ extends to a bounded operator

$$H_j : \partial W^{n-r_j,p}(X) \longrightarrow W^{k,p}(X \times Y^+)$$

Sketch of proof. We define $H_j : \mathcal{S}(X) \longrightarrow \mathcal{S}(X \times Y^+)$ by its action on the frequent space of X, in particular, set

$$\tilde{H_j h}(\xi,\eta) := H_j(\xi,y)\tilde{h}(\xi)$$

where $\tilde{f}$ is the partial (in x) Fourier transform of f and $H_j(\xi,y)$ is given by

$$H_j(\xi,y) := (1-\psi(\xi)) \int_\Gamma \sum_{\alpha=0}^{m-1} e^\alpha_j(\xi) \frac{A^+_\alpha(\xi,\eta)}{A^+(\xi,\eta)} e^{i\eta y} d\eta$$

where $\Gamma \subset \mathbb{C}$ is a curve enclosing all zeros of $A(\xi,\eta)$ with $\operatorname{Im}\eta > 0$, $(e^\alpha_j(\xi))_{\alpha,j}$ is the inverse matrix of $(c^j_\beta(\xi))_{j,\beta}$ and $A^+_\alpha(\xi,\eta) := \sum_{\beta=0}^{m-\alpha-1} a^+_{\alpha+\beta+1}(\xi)\eta^\beta$. $\square$

Some auxilary functions. We cannot use the operator G as in the case of whole plan as an inverse of A on the half-plan $X \times Y^+$, since we only have access to the frequent space of X. However we can modify the cut-off function to create an approximate inverse of A on the half-plan.

Let $\varphi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ be the function that we used in the definition of E_+, i.e.

$$\varphi(x) := \frac{e^4}{\pi} . \frac{e^{-(x^{1/4}+x^{-1/4})} \sin(x^{1/4} - x^{-1/4})}{1+x}, \quad x \geq 0$$

with the properties $\int_0^\infty x^n \varphi(x)dx = (-1)^n$ for all $n \in \mathbb{Z}\backslash\{0\}$ and $\int_0^\infty \varphi(x)dx = 0$. Extending φ by 0 for $x < 0$, one still has a smooth function. Define $\chi(y) := -\varphi(-y-1)$, then $\chi \in \mathcal{S}(Y)$, with support in $(-\infty, -1]$ and

$$\int_{\mathbb{R}} y^n \chi(y)dy = \begin{cases} 0, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \end{cases}$$

In the frequent space of Y, this means $\hat{\chi}(0) = 1$ and $D_\eta^k \hat{\chi}(0) = 0$, i.e. $1 - \hat{\chi}(\eta)$ has a zero of infinite order at $\eta = 0$.

Also, since $\chi = 0$ when $y > -1$, the convolution

$$f \longmapsto \hat{\chi}(D_y)f = \chi * f$$

maps $\mathcal{S}(Y^-/0)$ to itself, hence induces a mapping from $\mathcal{S}(Y^+)$ to itself, since

$$0 \longrightarrow \mathcal{S}(Y^-/0) \longrightarrow \mathcal{S}(Y) \longrightarrow \mathcal{S}(Y^+) \longrightarrow 0$$

(given any $f \in \mathcal{S}(Y^+)$, any extension $\tilde{f}$ of f to $\mathcal{S}(Y)$ has the same restriction of $\hat{\chi}(D_y)\tilde{f}$ on Y^+).

Let $w(\xi, \eta) := \psi(\xi)\hat{\chi}(\eta)$ then w defines an operator

$$w(D) : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$$

In fact, for all $k, l \in \mathbb{R}$, there exists $C > 0$ such that

$$\|w(D)f\|_{W^{k,p}(X \times Y^+)} \leq C\|f\|_{W^{l,p}(X \times Y^+)}.$$

Approximate inverse of elliptic operator A on half-plan. The auxilary function w will play the role of ψ in the whole plan case.

Theorem 40 (Approximate inverse of A on $X \times Y^+$)**.** *There exists an operator $G : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$ such that:*

1. $(\mathrm{Id} - AG) = w(D)$

2. *For all $k, l \in \mathbb{R}$, there exists $C > 0$ such that for all $f \in \mathcal{S}(X \times Y^+)$:*

$$\|(\mathrm{Id} - GA)\psi(D_x)\|_{W^{k,p}(X\times Y^+)} \leq C\|f\|_{W^{l,p}(X\times Y^+)}$$

Also G extends to a bounded operator $G: W^{k-r,p}(X \times Y^+) \longrightarrow W^{k,p}(X \times Y^+)$ for all $k \in \mathbb{R}$.

Sketch of proof. In fact G is defined as follows:

$$G_0(\xi, \eta) := \frac{1 - w(\xi, \eta)}{A(\xi, \eta)}$$

which is smooth at $(0, 0)$, where $1 - w$ has a zero of infinite order. Then $G_0(D) : \mathcal{S}(X\times Y) \longrightarrow \mathcal{S}(X\times Y)$ extends to $W^{k-r,p}(X\times Y) \longrightarrow W^{k,p}(X\times Y)$. Finally, take $G = C_+GE_+$, which maps $\mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$ by first extending a function to the whole plan, applying G_0 and finally cutting-off. □

Approximate inverse of the combined operator. Let $\mathcal{C}$ be the combined operator:

$$\begin{aligned} \mathcal{C} : \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y^+) \oplus \mathcal{S}(X)^{\oplus m} \\ f &\longmapsto (Af, Bf) \end{aligned}$$

and define the operator $\mathcal{J}$ as

$$\begin{aligned} \mathcal{J} : \mathcal{S}(X \times Y^+) \oplus \mathcal{S}(X)^{\oplus m} &\longrightarrow \mathcal{S}(X \times Y^+) \\ (g, h) &\longmapsto Gg + H(h - BGg) \end{aligned}$$

then one can prove with straightforward computation that $\mathcal{J}$ is an approximate inverse of $\mathcal{C}$.

Theorem 41 (Approximate inverse of C)**.** *For smooth functions $f \in \mathcal{S}(X \times Y^+)$ and $(g, h) \in \mathcal{S}(X \times Y^+) \oplus \mathcal{S}(X)^{\oplus m}$, one has*

1. $(\mathrm{Id} - \mathcal{C}\mathcal{J})(g, h) = (w(D)g, \psi(D_x)(h - BGg)) =: \lambda(g, h)$
2. $(\mathrm{Id} - \mathcal{J}\mathcal{C})f = \psi(D_x)\,(\mathrm{Id} - GA)f + (\mathrm{Id} - HB - \psi(D_x))\,w(D)f =: \mu(f)$

Since G, H extend to Sobolev spaces, one also has

$$\mathcal{J} : \; W^{k-r,p}(X \times Y^+) \bigoplus_{j=1}^{m} \partial W^{k-r_j,p}(X) \longrightarrow W^{k,p}(X \times Y^+)$$

whenever $k > \frac{\sigma}{\rho p} + \max_j r_j$.

Theorem 42. *In analogue of Theorem 36, one has the following exact-split square*

$$\begin{tikzcd} W^{k,p}(X \times Y^+) \arrow[r, shift right, "\mathcal{C}"'] \arrow[d, shift left, "\iota"] & W^{k-r,p}(X \times Y^+) \bigoplus_{j=1}^{m} \partial W^{k-r_j,p}(X) \arrow[l, shift right, "\mathcal{J}"'] \arrow[d, shift right, "\iota"'] \\ W^{l,p}(X \times Y^+) \arrow[r, shift left, "\mathcal{C}"] \arrow[u, shift left, "\mu"] & W^{l-r,p}(X \times Y^+) \bigoplus_{j=1}^{m} \partial W^{l-r_j,p}(X) \arrow[l, shift left, "\mathcal{J}"] \arrow[u, shift right, "\lambda"'] \end{tikzcd} \tag{4.3}$$

for all $k > l > \frac{\sigma}{\rho p} + \max_j r_j$.

4.3 From local to global.

4.3.1 Pertubation of exact squares and consequences.

We will extend the result of Theorem 36 (exactness of heterogeneous elliptic operator with constant coefficient on Euclidean plan) in 2 levels: (1) for general elliptic operators (non-heterogeneous and with variable coefficients) and (2) for such operators on compact manifold (with boundary if needed). These 2 generalizations will be done using the same technique: "cube by cube" approximating an exact square.

We topologize the space of commutative squares $\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "m"'] & F \arrow[d, "p"] \\ G \arrow[r, "q"] & H \end{tikzcd}$ as a closed subspace $SQ(E, F, G, H)$ of $L(E, F) \times L(F, H) \times L(E, G) \times L(G, H)$ defined by the equation $q \circ m = p \circ l$.

Theorem 43. *In* $SQ(E, F, G, H)$*, the exact squares form an open set.*

Instead of giving a proof (see [Ham75, page 75-77]), let us explain why Theorem 43 is true. The commutativity already tells us that the composition of any two consecutive arrows in

$$0 \longrightarrow E \xrightarrow{l \oplus m} F \oplus G \xrightarrow{p \ominus q} H \longrightarrow 0$$

is 0, and exactness is an extra condition of type "maximal rank", which is an open condition (For matrices, this means the derterminant does not vanish. The analogous phenomenon for Banach spaces is that a linear map sufficiently close to an invertible one is also invertible).

We will distinguish the following 2 types of cubes that we will use to cover a manifold. We will call the following set an *interior cube*

$$B_\epsilon := \left\{(x_1,\dots,x_n) : |x_1| \le \epsilon^{\sigma/\sigma_1}, \dots |x_n| \le \epsilon^{\sigma/\sigma_n}\right\}$$

and the following an *boundary cube*

$$B_\epsilon^+ := \left\{(x_1,\dots,x_n,y) : |x_1| \le \epsilon^{\sigma/\sigma_1}, \dots |x_n| \le \epsilon^{\sigma/\sigma_n}, 0 \le y \le \epsilon^{\sigma/\rho}\right\}$$

For the second type, we note by ∂_0 the part $y = 0$ of the boundary of B_ϵ^+, and by ∂_e the remaining part.

We will say that the $A := \sum_{\|\alpha\|\le r} a_\alpha(x) D^\alpha$ is **semi-elliptic at** 0 if $A_0 := \sum_{\|\alpha\|=r} a_\alpha(0) D^\alpha$ is a semi-elliptic operator.

Proposition 43.1 (Approximate operator on interior cube)**.** *Suppose that $A := \sum_{\|\alpha\|\le r} a_\alpha(x) D^\alpha$ is defined in B_{ϵ_0} and A is semi-elliptic at $x = 0$. Fix $-\infty < l < k < +\infty$. Then there exists an $\epsilon > 0$ sufficiently small and an operator $A^\# = \sum_{\|\alpha\|\le r} a^\#_\alpha(x) D^\alpha$ with smooth coefficients defined on X such that*

$$A^\# = \begin{cases} A, & \textit{inside } B_\epsilon \\ A_0, & \textit{outside } B_{2\epsilon} \end{cases}$$

and the "k, l" square corresponding to $A^\#$, i.e.

$$\begin{array}{ccc} W^{k,p}(X) & \xrightarrow{A^\#} & W^{k-r,p}(X) \\ \Big\downarrow \iota & & \Big\downarrow \iota \\ W^{l,p}(X) & \xrightarrow{A^\#} & W^{l-r,p}(X) \end{array}$$

is exact.

An analoguous result holds for boundary problem. The setup for boundary problem on half-plan $X \times Y^+$ is as follows.

$$A := \sum_{\|(\alpha,\beta)\|\le r} a_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta$$
$$B^j := \sum_{\|(\alpha,\beta)\|\le r_j} b^j_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta, \quad j = \overline{1,m}$$

are operators with smooth coefficients on $B^+_{\epsilon_0}$ and

$$A_0 := \sum_{\|(\alpha,\beta)\|=r} a_{\alpha,\beta}(0,0) D^\alpha_x D^\beta_y$$
$$B^j_0 := \sum_{\|(\alpha,\beta)\|=r_j} b^j_{\alpha,\beta}(0,0) D^\alpha_x D^\beta_y, \quad j = \overline{1,m}$$

If A is semi-elliptic at 0 then we say that $\{B^j\}$ satisfy the CBC condition at 0 uf $\{B^j_0\}$ are CBC with respect to A_0. Note that this is an "open condition", i.e. if the condition is satisfied at $(0,0)$ then it is also satisfied in a neighborhood of $(0,0)$ in $X \times \{0\}$. The analoguous result for boundary problem can then be stated.

Proposition 43.2 (Approximate opearator on boundary cube). *Under the previous setup and with $\frac{\sigma}{\rho p} + \max_j r_j < l < k < +\infty$, for $\epsilon > 0$ sufficiently small, there exists operators $C^\# = (A^\#, B^\#)$ with smooth coefficient in $X \times Y^+$ agreeing with (A, B) in B^+_ϵ and with (A_0, B_0) outside of $B^+_{2\epsilon}$ such that the square*

$$\begin{tikzcd}
W^{k,p}(X \times Y^+) \arrow[r, "\mathcal{C}^\#"] \arrow[d, hook, "\iota"'] & W^{k-r,p}(X \times Y^+) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(X) \arrow[d, hook, "\iota"] \\
W^{l,p}(X \times Y^+) \arrow[r, "\mathcal{C}^\#"'] & W^{l-r,p}(X \times Y^+) \bigoplus_{j=1}^m \partial W^{l-r_j,p}(X)
\end{tikzcd}$$

is exact.

We will prove Proposition 43.1 here to demonstrate how Theorem 43 is employed. Another reason is that the corresponding proof in [Ham75] is not very readable due to a notation/printing issue.

Proof of Proposition 43.1. We will use the change of coordinates $\tilde{x}_i = \lambda^{-\sigma/\sigma_i} x_i$, which gives a diffeomorphism h_λ from B_{ϵ_0} to $B_{\epsilon_0/\lambda}$, in which the derivative operators are

$$\left(\frac{\partial}{\partial \tilde{x}_i}\right)^\alpha_i = \lambda^{\alpha_i \sigma/\sigma_i} \left(\frac{\partial}{\partial x_i}\right)^\alpha_i, \quad \tilde{D}^\alpha = \lambda^{\|\alpha\|} D^\alpha$$

The operator A, viewed in h_λ, i.e. the operator $f \mapsto A(f \circ h_\lambda)$, is

$\sum_{\|\alpha\|\leq r} a_\alpha(\lambda^{\sigma/\sigma_i}\tilde{x}_i)\lambda^{-\|\alpha\|}\tilde{D}^\alpha$. We pose

$$\begin{aligned}
\tilde{A}_\lambda &:= \sum_{\|\alpha\|\leq r} \lambda^{r-\|\alpha\|} a_\alpha(\lambda^{\sigma/\sigma_i}\tilde{x}_i)\tilde{D}^\alpha \\
\tilde{A}_0 &:= \sum_{\|\alpha\|=r} a_\alpha(0)\tilde{D}^\alpha \\
\tilde{A}^*_\lambda &:= \varphi(\tilde{x})\tilde{A}_\lambda + (1-\varphi(\tilde{x}))\tilde{A}_0
\end{aligned}$$

where φ is radial in $\tilde{x}$, equals 1 for $\|\tilde{x}\| \leq 1$ and 0 for $\|\tilde{x}\| \geq 2$.

The coefficient before $\tilde{D}^\alpha$ of $\tilde{A}^*_\lambda$ is $\lambda^{r-\|\alpha\|}\left[\varphi(\tilde{x})a_\alpha(\lambda^{\sigma/\sigma_i}\tilde{x}_i) + (1-\varphi(\tilde{x}))a_\alpha(0)\delta_{\|\alpha\|=r}\right]$ is the same as that of $\tilde{A}_0$ for $\tilde{x}$ outside of B_2 and C^0-converges to that of $\tilde{A}_0$ inside B_1. Hence for λ sufficiently small the corresponding "k,l" diagram of $\tilde{A}^*_\lambda$ is exact, hence so is the diagram of $\lambda^{-r}\tilde{A}^*_\lambda$. Choose $\epsilon = \lambda$ and $A^\#$ to be $\lambda^{-r}\tilde{A}^*_\lambda$ viewed in X through $h\lambda$. $\square$

Remark 26. *To avoid making infinite intersection of open sets, we have to fix k and l first in Proposition 43.1 and Proposition 43.2. The approximate operators $A^\#, B^\#$ and the size ϵ of the cube therefore depend on k, l, but this dependence will not be a trouble when we pass from local to global situation.*

The exactness of semi-elliptic operator with variable coefficients on manifold will be establish analytically, meaning through the 3 statements similar to those of Theorem 36. Proposition 43.1 and 43.2 can be applied to prove the the local version of these statements.

Lemma 44. *With the same ϵ and k, l as Proposition 43.1 and the extra condition that $l \geq k-1$, one has for all $0 < \delta < \epsilon$*

1. $\|f\|_{W^{k,p}(B_\delta)} \leq C\left(\|Af\|_{W^{k-r,p}(B_\epsilon)} + \|f\|_{W^{l,p}(B_\epsilon)}\right)$ *for all* $f \in W^{k,p}(B_\epsilon)$.
2. *If* $f \in W^{l,p}(B_\epsilon)$ *and* $Af \in W^{k-r,p}(B_\epsilon)$ *then* $f \in W^{k,p}(B_\delta)$.
3. *If* $g \in W^{l-r,p}(B_\delta/\partial)$ *then there exists* $f \in W^{l,p}(B_\epsilon, \partial)$ *such that*
$$g - Af \in W^{k-r,p}(B_\epsilon, \partial B_\epsilon).$$

Proof. Let ψ be a cut-off function that equals 1 on B_δ and 0 outside of B_ϵ and $A^\#$ be the differential operator on X with exact "k,l" diagram given by Proposition 43.1 which equals A on B_ϵ. The idea of the remaining computation is to use the exactness of $A^\#$ on ψf and the reason for which the local-global passage is not trivial is that the operator $A^{\#}$ and the multiplication by ψ do not commute. The commutator $[A^{\#}, \psi]$, however, is of weight at least 1 less than A and with the choice $l \geq k-1$ the norm $\|[\psi, A^{\#}]f\|_{W^{k-r,p}(X)}$ is dominated by $\|f\|_{W^{l,p}}$.

1. If $f \in W^{k,p}(B_\epsilon)$ then $\psi f \in W^{k,p}(B_\epsilon, \partial)$ and

$$\begin{aligned} \|f\|_{W^{k,p}(B_\delta)} &\leq \|\psi f\|_{W^{k,p}(X)} \leq C\left(\|A^{\#}\psi f\|_{W^{k-r,p}(X)} + \|\psi f\|_{W^{l,p}(X)}\right) \\ &\leq C\left(\|\psi A^{\#} f\|_{W^{k-r,p}(X)} + \|[\psi, A^{\#}]f\|_{W^{k-r,p}(X)} + \|\psi f\|_{W^{l,p}(X)}\right) \\ &\leq C'\left(\|Af\|_{W^{k-r,p}(B_\epsilon)} + \|f\|_{W^{l,p}(B_\epsilon)}\right) \end{aligned}$$

2. Given $f \in W^{l,p}(B_\epsilon)$ and $Af \in W^{k-r,p}(B_\epsilon)$, one has $\psi f \in W^{l,p}(X)$. Also, $[A^{\#}, \psi]f \in W^{l-r+1,p}(X) \subset W^{k-r,p}(X)$ and $\psi A^{\#} f = \psi A f \in W^{k-r,p}(X)$, therefore $A^{\#}(\psi f) \in W^{k-r,p}(X)$. By exactness of $A^{\#}$, one has $\psi f \in W^{k,r}(X)$, so $f \in W^{k,r}(B_\delta)$.

3. If $g \in W^{l-r,p}(B_\delta/\partial) \subset W^{l-r,p}(X)$, by exactness of $A^{\#}$ we can find $\tilde{f} \in W^{l,p}(X)$ such that $g - A^{\#}\tilde{f} \in W^{k-r,p}(X)$. Choose $f = \psi\tilde{f} \in W^{l,p}(B_\epsilon/\partial)$ then

$$g - Af = g - A^{\#}(\psi\tilde{f}) = \psi(g - A^{\#}\tilde{f}) + [\psi, A^{\#}]\tilde{f} \in W^{k-r,p}(B_\epsilon)$$

since $\psi(g - A^{\#}\tilde{f}) \in W^{k-r,p}(B_\epsilon)$ and $[\psi, A^{\#}]\tilde{f} \in W^{l-r+1,p}(B_\epsilon) \subset W^{k-r,p}(B_\epsilon)$.

$\square$

Lemma 45. *With (A, B) and ϵ, k, l as in Proposition 43.2 with the extra condition $l \geq k-1$, then for all $\delta < \epsilon$, one has*

1. $\|f\|_{W^{k,p}(B_\delta^+)} \leq C\left(\|Af\|_{W^{k-r,p}(B_\epsilon^+)} + \sum_{j=1}^m \|B^j f\|_{\partial W^{k-r_j,p}(\partial_0 B_\epsilon^+)} + \|f\|_{W^{l,p}(B_\epsilon^+)}\right)$ *for all $f \in W^{k,p}(B_\epsilon^+)$.*

2. *If $f \in W^{l,p}(B_\epsilon^+)$ and $Af \in W^{k-r,p}(B_\epsilon^+)$ and $B^j f \in \partial W^{k-r_j,p}(\partial_0 B_\epsilon^+)$ then actually $f \in W^{k,p}(B_\delta^+)$.*

3. *If $g \in W^{l-r,p}(B_\delta^+/\partial_e)$ and $h_j \in \partial W^{l-r_j,p}(\partial_0 B_\delta^+/\partial)$ then there exists $f \in W^{l,p}(B_\epsilon^+, \partial_e)$ with*

$$g - Af \in W^{k-r,p}(B_\epsilon^+, \partial_e), \quad h_j - B^j f \in \partial W^{k-r_j,p}(\partial_0 B_\epsilon^+/\partial).$$

The generalisation of Theorem 36 on manifold with variable coefficients is now straightforward. The only nontrivial issue is the definition of semi-elliptic operator A on manifold. This requires a Riemannian metric g and ellipticity is naturally defined at every point, viewed in a chart, as we did before. But this only defines the action of A on $C^\infty(M)$ (or $C^r(M)$ if regularity is important), but not on $W^{k,p}(M/\mathcal{A})$ where $\mathcal{A} \subset \partial M$ is a connected component.

The action of a differential operator A can be defined to be component-wise on $W^{k,p}(M/\mathcal{A}) \hookrightarrow \bigoplus_i W^{k,p}(\mathcal{R}_i/\mathcal{A}_i)$ where $\mathcal{R}_i$ is an Euclidean plan or a half-plan and $\mathcal{A}_i$ the corresponding boundary part, i.e.

$$\begin{tikzcd} W^{k,p}(M/\mathcal{A}) \arrow[r, "\iota"] \arrow[d, dashed, "A"'] & \bigoplus_i W^{k,p}(\mathcal{R}_i/\mathcal{A}_i) \arrow[d, "A"] \\ W^{l,p}(M/\mathcal{A}) \arrow[r, "\iota"] & \bigoplus_i W^{l,p}(\mathcal{R}_i, \mathcal{A}_i) \end{tikzcd}$$

It remains to check that the component-wise operation of A maps an element in the image on $W^{k,p}(M/\mathcal{A})$ to an element in the image of $W^{l,p}(M/\mathcal{A})$. This can be done using the projection as we did when defining trace operator on manifold, but the situation is much simpler here since we can differentiate directly an element in $\mathcal{S}^*(M)$.

Theorem 46 (Elliptic equation on manifold)**.** *Let M be a compact manifold possibly with boundary (and a compatible foliation if the weight is not uniform). Let A be a general semi-elliptic operator of weight r, of variable coefficients and $\{B^j\}_j$ be a set boundary operators of weight r_j satisfying CBC with respect to A. Then for all $\frac{\sigma}{\rho p} + \max_j r_j < l < k < +\infty$, the square*

$$\begin{tikzcd} W^{k,p}(M) \arrow[r, "\mathcal{C}"] \arrow[d, hook, "\iota"'] & W^{k-r,p}(M) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(\partial M) \arrow[d, hook, "\iota"] \\ W^{l,p}(M) \arrow[r, "\mathcal{C}"'] & W^{l-r,p}(M) \bigoplus_{j=1}^m \partial W^{l-r_j,p}(\partial M) \end{tikzcd}$$

is exact where $\mathcal{C} = (A, B^j)$.

Proof. We can suppose $l \geq k-1$, the general case follows using

Lemma 47. *If the two following squares are exact*

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "m"'] & F \arrow[d, "p"] \\ G \arrow[r, "q"'] & H \end{tikzcd} , \quad \begin{tikzcd} G \arrow[r, "q"] \arrow[d, "r"'] & H \arrow[d, "s"] \\ K \arrow[r, "t"'] & L \end{tikzcd}$$

then

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "rm"'] & F \arrow[d, "sp"] \\ K \arrow[r, "t"'] & L \end{tikzcd}$$

is exact.

Now covering M by finitely many charts of type $B_\delta \subset B_\epsilon$ and $B_\delta^+ \subset B_\epsilon^+$ such that the interior of B_δ and of B_δ^+ cover M. Also, choose a partition of unity $\sum \psi = 1$ subordinated to B_δ and B_δ^+. The exactness will be established if we can prove the analogue of the 2 last statements of Theorem 36

For the regularity statement: If $f \in W^{l,p}(M)$, $Af \in W^{k-r,p}(M)$ and $B^j f \in W^{k-r_j,p}(\partial M)$ then the same holds for ψf in B_ϵ and B_ϵ^+ since

$$[A,\psi]f \in W^{l-r+1,p} \subset W^{k-r,p}, \quad [B^j,\psi]f \in \partial W^{l-r_j+1,p} \subset \partial W^{k-r_j,p}$$

Therefore $\psi f \in W^{k,p}(B_\delta)$ or $W^{k,p}(B_\delta^+)$ hence $f \in W^{k,p}(M)$.

For the approximation: If $g \in W^{l-r,p}(M)$ and $h_j \in \partial W^{l-r_j,p}(\partial M)$ then $\psi g \in W^{l-r,p}(B_\delta/\partial)$ or $W^{l-r,p}(B_\delta^+/\partial_e)$ and $\psi h_j \in \partial W^{l-r_j,p}(\partial_0 B_\delta^+/\partial)$. Then by Lemma 45, we can find $\tilde f \in W^{l,p}(B_\epsilon/\partial)$ with $\psi g - A\tilde f \in W^{k-r,p}(B_\epsilon/\partial)$ or in a boundary cube $\tilde f \in W^{l,p}(B_\epsilon^+/\partial_e)$ with $\psi g - A\tilde f \in W^{k-r,p}(B_\epsilon^+/\partial_e)$ with $\psi h_j - B^j \tilde f \in \partial W^{k-r_j,p}(\partial_0 B_\epsilon^+/\partial)$. Then $f := \sum \tilde f$ makes sense and satisfies

$$\begin{cases} g - Af = \sum(\psi g - A\tilde f) & \text{is in } W^{k-r,p}(M) \\ h - B^j f = \sum(\psi h_j - B^j \tilde f) & \text{is in } \partial W^{k-r_j,p}(\partial M) \end{cases}$$

□

4.3.2 Consequences of Theorem 38.

Under the same setup as Theorem 46, one has

Theorem 48 (Regularity of kernel and cokernel)**.** *The map $\mathcal{C} = (A,B)$: $W^{k,p}(M) \longrightarrow W^{k-r,p}(M) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(\partial M)$ has closed range, finite dimensional kernel and cokernel and the kernel and cokernel are independent of k in the sense of Theorem 38. In particular, $\ker C \subset C^\infty(M)$*

The analoguous regularity for cokernel is less straightforward. We resume here the result.

Theorem 49 (Regularity of cokernel)**.** *If $r > \max r_j$ then the image of $\mathcal{C}$ can be represented by finitely many linear relations: $(g,h) \in \operatorname{Im}\mathcal{C}$ if and only if it satisfies finitely many equations of type:*

$$\langle g,\gamma\rangle_M + \sum_{j=1}^m \langle h_j,\eta_j\rangle_{\partial M} = 0$$

with $\gamma \in C^\infty(M)$ and $\eta_j \in C^\infty(\partial M)$.

If $\max r_j - r = k \geq 0$ then for all $g \in W^{k-r,p}(M)$, the normal derivatives $\frac{\partial g}{\partial \nu^i}$ are well defined if $\frac{\sigma_i}{\rho} \leq k$. The cokernel is then given by the relations

$$\langle g, \gamma \rangle_M + \sum_{\sigma_i/\rho \leq k} \langle \frac{\partial g}{\partial \nu^i}, \chi_i \rangle_{\partial M} + \sum_{j=1}^{m} \langle h_j, \eta_j \rangle_{\partial M} = 0$$

with $\gamma \in C^\infty(M)$, $\chi_i \in C^\infty(\partial M)$, $\eta_j \in C^\infty(\partial M)$.

4.4 Parabolic equation on manifold.

4.4.1 Parabolicity and local results.

Definition 15. *The constant coefficient differential operator $A(D_x, D_t) = \sum_{\|(\alpha,\beta)\| \leq r} a_{\alpha\beta} D_x^\alpha D_t^\beta$ is called **parabolic** if its symbol $A(\xi, \theta) := \sum_{\|(\alpha,\beta)\| = r} a_{\alpha\beta} \xi^\alpha \theta^\beta$ has no zero when $\xi \in \mathbb{R}$ and $\operatorname{Im} \theta \leq 0$ except $\xi = \theta = 0$.*

Example 9. *Take $A = \partial_t - \partial_x^2 - \partial_y^2 - \partial_z^2 = iD_t + D_x^2 + D_y^2 + D_z^2$, the symbol is $i\theta + \sum \xi_i^2$ has no zero $\xi \in \mathbb{R}^3, \operatorname{Im} \theta \leq 0$ except 0. Generally, the operator $\partial_t + A(D_{x^i})$ is parabolic if A is an elliptic operator with the symbol $A(\xi) \geq 0$ for all $\xi \in \mathbb{R}$ with equality only at $\xi = 0$.*

Remark 27.

1. *If $\sigma = \operatorname{lcm}(\sigma_1, \ldots, \sigma_n)$ is the lcm of weights of variable x_i and τ is the weight of t, then parabolicity implies $2\tau \mid \sigma$. Therefore if the weights of x_i are uniform, one can suppose that $\tau = 1$.*

2. *Parabolicity implies ellipticity.*

Similarly to the elliptic case, we attempt to define an approximate inverse G of A, of the form

$$G(\xi, \theta) = (1 - \psi(\xi, \theta))/A(\xi, \theta)$$

such that $G(D_x, D_t) : W^{k-r,p}(X \times T^+/0) \longrightarrow W^{k,p}(X \times T^+/0)$ and $\psi(D_x, D_t) : W^{k,p}(X \times T^+/0) \longrightarrow W^{k,p}(X \times T^+/0)$ for all $k, l \in \mathbb{R}$.

The sufficient condition for this is that $\psi(\xi, \theta) = \psi(\xi)\hat{\chi}(\theta)$ where ψ is compactly support and $\hat{\chi} \in \mathcal{S}(T)$ with $\hat{\chi} - 1$ having a zero of infinite order at $\theta = 0$, and $\hat{\chi}$ extends to a holomorphic function in $\operatorname{Im} \theta \leq 0$. The function

$\hat{\chi}$ used in section 4.2 suffices. We then have the following exact square

$$\begin{tikzcd} W^{k,p}(X \times T^+/0) \arrow[r, shift right, "A"'] \arrow[d, shift left, hook, "\iota"] & W^{k-r,p}(X \times T^+/0) \arrow[l, shift right, "G"'] \arrow[d, shift right, hook, "\iota"'] \\ W^{l,p}(X \times T^+/0) \arrow[r, shift left, "A"] \arrow[u, shift left, "\psi"] & W^{l-r,p}(X \times T^+/0) \arrow[l, shift left, "G"] \arrow[u, shift right, "\psi"'] \end{tikzcd}$$

The theory in section 4.2 also allows us to treat spatial boundary condition, that is, to replace the Euclidean plan X by the half-plan $X \times Y^+$. The analog of **CBC condition** for boundary operators

$$B^j(D_x, D_y, D_t) = \sum_{\|(\alpha,\beta,\gamma)\| \le r_j} b^j_{\alpha\beta\gamma} D^\alpha_x D^\beta_y D^\gamma_t$$

is that the symbols

$$B^j(\xi, \eta, \theta) = \sum_{\|(\alpha,\beta,\gamma)\| = r_j} b^j_{\alpha\beta\gamma} \xi^\alpha \eta^\beta \theta^\gamma$$

are linearly independent modulo $A^+(\xi, \eta, \theta)$ as polynomial in η for all $\xi \in \mathbb{R}^n$ **and** for all $\operatorname{Im}\theta \le 0$ except when $\xi = \theta = 0$. In that case we have the exactness of

$$\begin{tikzcd} W^{k,p}(X \times Y^+ \times T^+/0) \arrow[r, "(A,B^j)"] \arrow[d, hook, "\iota"] & W^{k-r,p}(X \times Y^+ \times T^+/0) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(X \times T^+/0) \arrow[d, hook, "\iota"'] \\ W^{l,p}(X \times Y^+ \times T^+/0) \arrow[r, "(A,B^j)"] & W^{l-r,p}(X \times Y^+ \times T^+/0) \bigoplus_{j=1}^m \partial W^{l-r_j,p}(X \times T^+/0) \end{tikzcd}$$

4.4.2 Global results and causality.

We will use the following setup. Let M be a compact manifold (possibly with boundary), of the form $N \times [\alpha, \omega] \ni (x, t)$. The global product gives a foliation that allows us to set the spatial weight to be uniformly σ and the temporal weight to be τ. The boundary of M has 3 parts: $\partial_\alpha M := N \times \alpha$, $\partial_\omega M := N \times \omega$ and $\partial_S M := \partial N \times [\alpha, \omega]$.

Let A be a parabolic operator, meaning that A is parabolic at every point and $B^j, j = \overline{1,m}$ be a set of boundary operator satisfying CBC condition at every point on $\partial_S M$. We take into account the initial condition by only considering the space $W^{k,p}(M/\partial_\alpha)$ of function vanishing before time $t = \alpha$. As before the operator

$$\mathcal{C} := (A, B^j) : \; W^{k,p}(M/\partial_\alpha) \longrightarrow W^{k-r,p}(M/\partial_\alpha) \bigoplus_{j=1}^{m} \partial W^{k-r_j,p}(\partial_S M/\partial_\alpha)$$

has closed range, finite dimensional kernel and cokernel which are independent of $k > \frac{1}{p} + \max r_j$.

The same method allows us to conclude that $\ker \mathcal{C} \subset C^\infty(M)$ and the cokernel is given by finitely many linear relations of type

$$\langle g, \gamma \rangle_M + \sum_j \langle h_j, \eta_j \rangle_{\partial M} + \sum_i \langle \frac{\partial}{\partial \nu^i} g, \chi_i \rangle_{\partial_S M}$$

where $\gamma \in C^\infty(M/\partial_\omega)$, $\chi_i \in C^\infty(\partial_S M/\partial_\omega)$ and $\eta_j \in C^\infty(\partial_S M/\partial_\omega)$.

The difference with elliptic equation is that the kernel and cokernel of $\mathcal{C}$ are not only of finite dimension, but are zero.

Theorem 50 (Causality)**.** *With the previous setup, the operator $\mathcal{C} = (A, B^j)$ defines an isomorphism*

$$W^{k,p}(M/\partial_\alpha) \xrightarrow{\mathcal{C}} W^{k-r,p}(M/\partial_\alpha) \textstyle\bigoplus_{j=1}^{m} \partial W^{k-r_j,p}(\partial_S M/\partial_\alpha)$$

for all $k > \frac{1}{p} + \max_j r_j$, and therefore an isomorphism

$$C^\infty(M/\partial_\alpha) \xrightarrow{\mathcal{C}} C^\infty(M/\partial_\alpha) \textstyle\bigoplus_{j=1}^{m} C^\infty(\partial_S M/\partial_\alpha)$$

Proof. Let $\beta \leq \gamma$ be real numbers in $[\alpha, \omega]$ and let $\ker(\beta, \gamma)$ and $\operatorname{coker}(\beta, \gamma)$ be the kernel and cokernel of operator $\mathcal{C}$ on $N \times [\beta, \gamma]$ with vanishing initial condition at β. Since $\dim \ker(\beta, \gamma)$ and $\dim \operatorname{coker}(\beta, \gamma)$ are integer-valued, using the fact that $\dim \ker(\beta, \omega)$ is decreasing in β and $\dim \operatorname{coker}(\alpha, \gamma)$ is increasing in γ, one can easily check that it suffices to show that the two functions are continuous in (β, γ) to prove that they are identically 0.

The following statements can be verified mechanically:

1. *Monotonicity:* $\dim \ker(\beta, \gamma)$ is decreasing in β, $\dim \operatorname{coker}(\beta, \gamma)$ is increasing in γ.

2. *One-sided continuity:* $\dim \ker(\beta, \gamma)$ is left-continuous in β, $\dim \operatorname{coker}(\beta, \gamma)$ is right-continuous in γ.

3. *One-sided semi-continuity:* $\dim\ker(\beta,\gamma)$ is left upper semi-continuous in γ, i.e.
$$\lim_{\gamma_1\to\gamma_2^-}\inf\dim\ker(\beta,\gamma_1)\geq\dim\ker(\beta,\gamma_2)$$
This is due to the left-continuity in first variable of $\dim\ker$ and the exact sequence
$$0\longrightarrow\ker(\gamma_1,\gamma_2)\longrightarrow\ker(\beta,\gamma_2)\longrightarrow\ker(\beta,\gamma_1)$$
where the last arrow is the restriction. Similar statement for coker:
$$\lim_{\beta_2\to\beta_1^+}\dim\inf\operatorname{coker}(\beta_2,\gamma)\geq\dim\operatorname{coker}(\beta_1,\gamma)$$

This 3 statements suffice to finish the proof in the case where boundary conditions B^j on $\partial_S M$ are of constant coefficients since $\ker, \operatorname{coker}$ only depend on the difference $\gamma-\beta$, up to a translation in time of the solutions.

In case B^j are of variable coefficients, the idea of making translation in time can be formulated using Index theory for Fredholm maps:

We recall that Fredholm maps between Banach spaces E, F are those in $L(E,F)$ with closed image and finite dimensional kernel and cokernel. It is a classical result that

1. The set $\mathcal{F}$ of Fredholm maps are open in $L(E,F)$.
2. The index $i(l):=\dim\ker l-\dim\operatorname{coker} l$ is continuous in $\mathcal{F}$.

The difference $\dim\ker(\beta,\gamma)-\dim\operatorname{coker}(\beta,\gamma)$ can be regarded as the index of a continuous family $\mathcal{C}_{(\beta,\gamma)}$ of operators on the same space $N\times[0,1]$ using the diffeomorphism
$$N\times[0,1]\xrightarrow{\sim}N\times[\beta,\gamma].$$

Hence $\dim\ker(\beta,\gamma)-\dim\operatorname{coker}(\beta,\gamma)$ is constant. It follows that $\dim\ker(\beta,\gamma)$ is both increasing and one-sided semi-continuous in γ hence is right-continuous in γ, hence $\dim\operatorname{coker}(\beta,\gamma)$ is continuous in γ. Other continuities follows similarly. $\square$

Remark 28. *To take into account the initial condition $f\big|_\alpha=f_\alpha$ smooth, one looks for solution of the form $f=f_b+f_\#$ where f_b satisfies the initial condition and $f_\#\in W^{k,p}(N\times[\alpha,\omega]/\alpha)$ satisfying a parabolic equation $(Af_\#, B^jf)=(g,h)$ where g,h and the coefficients of A and B^j depend smoothly on f_b, and therefore still C^∞ in (x,t).*

4.4.3 Regularisation effect and Gårding inequality.

With the same technique used for elliptic equation, one can also prove regularity result for parabolic equation. There are 2 different points, in comparison with the elliptic case:

1. There is a regularisation effect of parabolic equation: An arbitrarily weak estimate in the past gives an arbitrarily strong estimate in the future. We will see that this is in fact a consequence of the causality of parabolic equation (Theorem 50) and Kondrachov's theorem.

2. The temporal boundary condition is thicken: We will look at the norm on $N \times [\alpha, \pi]$ rather than the restriction to $\partial_\alpha M$.

Theorem 51 (Regularity and Garding inequality)**.** *Under the same setup and notation as Section 4.4.2, let $p \in (1, +\infty)$ and $k > l > \frac{1}{p} + \max r_j$. We denote by $W^{k,p}([\beta, \gamma])$ the Sobolev space $W^{k,p}(N \times [\beta, \gamma])$. Suppose that*

$$f \in W^{l,p}([\alpha, \omega]), \quad Af \in W^{k-r,p}([\alpha, \omega]), \quad B^j f \in \partial W^{k-r_j,p}([\alpha, \omega])$$

then $f \in W^{k,p}([\pi, \omega])$ for all $\pi \in (\alpha, \omega)$. Also, for all $l' > -\infty$, there exists a constant $C > 0$ such that

$$\|f\|_{W^{k,p}([\pi,\omega])} \leq C \left(\|Af\|_{W^{k-r,p}([\alpha,\omega])} + \|B^j f\|_{\partial W^{k-r_j,p}([\alpha,\omega])} + \|f\|_{W^{l',p}([\alpha,\pi])} \right).$$

In particular, for homogeneous equation, i.e. $Af = 0, B^j f = 0$, the solution is C^∞ and an arbitrarily weak estimate in the past gives an arbitrarily strong estimate in the future.

Proof. Let us explain why the theorem is true in the case of no spatial boundary $\partial N = \emptyset$. In this case, there is no distinction between l and l'. Consider A as an elliptic operator on $N \times [\tilde{\pi}, \omega]$ with $\tilde{\pi} = \frac{\alpha+\pi}{2}$ and with no boundary operator, one has the following exact diagram:

$$\begin{tikzcd}
W^{k,p}([\tilde{\pi}, \omega]) \arrow[r, "A"] \arrow[d, hook] & W^{k-r,p}([\tilde{\pi}, \omega]) \arrow[d, hook] \\
W^{l,p}([\tilde{\pi}, \omega]) \arrow[r, "A"] & W^{l-r,p}([\tilde{\pi}, \omega])
\end{tikzcd}$$

Therefore the if $f \in W^{l,p}([\alpha, \omega])$ and $Af \in W^{k-r,p}([\alpha, \omega])$then $f \in W^{k,p}([\tilde{\pi}, \omega]) \subset$

$W^{k,p}([\pi,\omega])$ and

$$\|f\|_{W^{k,p}([\pi,\omega])} \leq \|f\|_{W^{k,p}([\tilde{\pi},\omega])} \leq C\left(\|Af\|_{W^{k-r,p}([\alpha,\omega])} + \|f\|_{W^{l,p}([\alpha,\omega])}\right) \tag{4.4}$$

$$\leq C\left(\|Af\|_{W^{k-r,p}([\alpha,\omega])} + \|f\|_{W^{l,p}([\alpha,\pi])} + \|f\|_{W^{l,p}([\tilde{\pi},\omega])}\right) \tag{4.5}$$

It remains to check that we can get rid of the $\|f\|_{W^{l,p}([\tilde{\pi},\omega])}$ term on the right hand side. Suppose not, then there exists a sequence $\{f_i\} \subset W^{l,p}([\alpha,\omega])$ such that $Af_i \to 0$ in $W^{k-r,p}([\alpha,\omega])$ and $f_i \to 0$ in $W^{l,p}([\alpha,\pi])$ but $\|f_i\|_{W^{l,p}([\tilde{\pi},\omega])} = 1$. Then by (4.5), $\{f_i\}$ is a bounded sequence in $W^{k,p}([\tilde{\pi},\omega])$ and, by Kondrachov's theorem, can be supposed to converge in $W^{l,p}([\tilde{\pi},\omega])$ to a function $\tilde{f}$ which has $\|\tilde{f}\|_{W^{l,p}([\tilde{\pi},\omega])} = 1$ and $A\tilde{f} = 0$ on $[\tilde{\pi},\omega]$ because A commutes with the restriction. Moreover, since $\|f_i\|_{W^{l,p}([\alpha,\pi])} \to 0$, one has $\tilde{f} \in W^{l,p}([\tilde{\pi},\omega]/\tilde{\pi})$ and the fact that $\tilde{f} \neq 0$ contradicts Theorem 50). □

Remark 29. *The proof of Theorem 51 in the general case, with spatial boundary taken into account requires the notion of bigraded Sobolev spaces on half-plan, see [Ham75, page 97-100]. This is also how the regularity result for cokernel of elliptic operator, Theorem 49, is proved.*

4.5 Example: Linear heat equation.

We use the same setup of M, N, α, ω as Section 4.4.2. Let Δ be the (geometer's) Laplacian

$$-\Delta f := g^{ij}(x)\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k(x)\frac{\partial f}{\partial x^k}\right)$$

It is easy to check that Δ is an elliptic operator with symbol $\Delta \geq 0$ (there is a factor i when passing from $\frac{\partial}{\partial x^i}$ to D_{x^i}). Hence on $M = N \times [\alpha,\omega]$ the operator $\frac{\partial}{\partial t} + \Delta$ is parabolic.

4.5.1 Linear system.

We will look at the linear parabolic system of equations for $F = (f^1, \ldots, f^n) : M \longrightarrow \mathbb{R}^n$:

$$\frac{\partial F}{\partial t} + \Delta F + a\nabla F + bF = G \tag{4.6}$$

where in local coordinates $(a\nabla F)^\alpha = a^{\alpha i}_\beta \frac{\partial f^\beta}{\partial x^i}$ and $(bF)^\alpha = b^\alpha_\beta f^\beta$ and $(\Delta F)^\alpha = \Delta f^\alpha$ and the coefficients $a^{\alpha i}_\beta$ and b^α_β are smooth.

We will say that a function $F = (f^1, \dots, f^n) : M \longrightarrow \mathbb{R}^n$ of class $W^{k,p}$ if it is $W^{k,p}$ component-wise. We also denote abusively by $W^{k,p}(M)$ the direct sum $W^{k,p}(M)^{\oplus n}$ where F belongs to.

Theorem 52 (Linear heat equation)**.** *Let $p > \dim M + 1 = \dim N + 2$ and $k \geq 0$, then for all $G \in W^{k,p}(N \times [\alpha, \omega]/\alpha)$, there exists a unique $F \in W^{k+2,p}(N \times [\alpha,\omega]/\alpha)$ that solves (4.6). Moreover, the operator*

$$F \longmapsto \frac{\partial F}{\partial t} + \Delta F + a\nabla F + bF$$

is an isomorphism between Banach spaces $W^{k+2,p}(N\times[\alpha,\omega]/\alpha) \longrightarrow W^{k,p}(N\times [\alpha,\omega]/\alpha)$.

Proof. Note that

$$\begin{aligned} H :\ W^{k+2,p}(N \times [\alpha,\omega]/\alpha) &\longrightarrow W^{k,p}(N \times [\alpha,\omega]/\alpha) \\ F &\longmapsto \frac{\partial F}{\partial t} + \Delta F \end{aligned}$$

is a direct sum of parabolic operators in each component, and hence an isomorphism, and

$$\begin{aligned} K :\ W^{k+2,p}(N \times [\alpha,\omega]/\alpha) &\longrightarrow W^{k,p}(N \times [\alpha,\omega]/\alpha) \\ F &\longmapsto a\nabla F + bF \end{aligned}$$

is a compact operator because it factors through $W^{k+1}(N\times[\alpha,\omega]/\alpha)$. Therefore $H + K$ is a Fredholm map with the same index as H, which is 0. It is sufficient to check that the kernel of $H + K$ is trivial.

Suppose that $F = (f^1, \dots, d^n) \in \ker(H + K)$ then $f^\alpha \in W^{2,p}(N \times [\alpha,\omega]/\alpha)$, so f^α and $\frac{\partial f^\alpha}{\partial x^i}$ are continuous function on $N \times [\alpha, \omega]$. Since

$$\frac{\partial f^\alpha}{\partial t} + \Delta f^\alpha = -a^{\alpha i}_\beta \frac{\partial f^\beta}{\partial x^i} - b^\alpha_\beta f^\beta,$$

by repeated use of Theorem 51 the f^α are smooth for $t > \alpha$.

Let $e := \frac{1}{2}|F|^2 := \frac{1}{2}\sum_\alpha |f_\alpha|^2$, then e is continuous on $N\times[\alpha,\omega]$, vanishes on $N \times \{\alpha\}$ and one has

$$\begin{aligned} \frac{de}{dt} &= -\Delta e - |\nabla F|^2 - a^{\alpha i}_\beta f^\alpha \frac{\partial f^\beta}{\partial x^i} - b^\alpha_\beta f^\alpha f^\beta \\ &\leq -\Delta e + \frac{1}{2} C|F|^2 = -\Delta e + Ce \end{aligned}$$

where we used the inequality $-u^2 - 2uv \leq v^2$ to bound the second and third terms. We conclude that $F = 0$ since $e = 0$ by the following Maximum principle. $\square$

4.5.2 Maximum principle and L^∞-Comparison theorem.

With the same proof as for open set in $\mathbb{R}^n$, one has the maximum principle for parabolic equation on manifolds. The constant C in the following Theorem 53 can depend on the point $x \in M$, but will be most of the time globally constant, since the manifold M is compact. The following statement of Maximum principle will be sufficient for most of our application.

Theorem 53 (Maximum principle)**.** *Let $f : M \longrightarrow \mathbb{R}$ be a continuous function on $M = N \times [\alpha, \omega]$ with $f\big|_{\partial_\alpha M} \leq 0$ and $f\big|_{\partial_S M} \leq 0$. Suppose that whenever $f > 0$, f is smooth satisfies*

$$\frac{\partial f}{\partial t} \leq -\Delta f + Cf$$

Then in fact $f \leq 0$.

With the same proof as Theorem 53, one can prove the following L^∞ Comparison theorem.

Theorem 54 (L^∞-Comparison theorem)**.** *Let $f : \; M = N \times [\alpha, \omega] \longrightarrow \mathbb{R}$ be a continuous function on M, smooth for time $t > 0$ such that*

$$\frac{df}{dt} = -\Delta f + a\nabla f + bf \text{ on } N \times (\alpha, \omega] \tag{4.7}$$

where a is a smooth vector field and b is a smooth function on N. Then there exists $B = B(a, b)$ depending only on a and b such that

$$\|f\big|_\omega\|_{L^\infty} \leq e^{B(\omega-\alpha)}\|f\big|_\alpha\|_{L^\infty}$$

Proof. We can suppose $b \leq -1$ and prove that $\|f\|_{L^\infty(\partial_\omega M)} \leq \|f\|_{L^\infty(\partial_\alpha M)}$. Intuitively, this means that since heat spreads out, the largest density must be attained at time $t = \alpha$. In fact, choose $B = \max_M b + 1$ and define $\tilde{f} = fe^{-B(t-\alpha)}$ then $\|\tilde{f}\big|_\alpha\|_{L^\infty} = \|f\big|_\alpha\|_{L^\infty}$ and $\|\tilde{f}\big|_\omega\|_{L^\infty} = e^{-B(\omega-\alpha)}\|f\big|_\omega\|_{l^\infty}$. The function $\tilde{f}$ satisfies the same heat equation (4.7) as f, with b replaced by $b - B \leq -1$.

Now let us prove that under this supposition, $|f|$ attains it maximum at time $t = \alpha$. Since we can replace the solution f of (4.7) by $-f$, we can suppose, for sake of contradiction, that $|f|$ attains it maximum on $N\times[\alpha,\omega]$ at (x^*,t^*) with $|f(x^*,t^*)|=f(x^*,t^*)>0$ and $t^*>\alpha$. Then one has

$$\begin{cases}\nabla f(x^*,t^*)=0,\\ \frac{df}{dt}(x^*,t^*)\geq 0, \text{ (this is not true if } t^*=\alpha)\\ \Delta f(x^*,t^*)\geq 0,\\ f(x^*,t^*)>0\end{cases}$$

Plugging these in (4.7), one has a contradiction. □

4.5.3 Backwards heat equation and L^1-Comparison theorem.

We will use backwards heat equation, which is just heat equation with the reversed sense of time (so with the reversed sign for Δ as well), in order to dualise the estimate of Theorem 54 and obtain a L^1 estimate of f at time $t=\omega$ in term of its L^1 norm at $t=\alpha$. In particular, we prove the following theorem.

Theorem 55 (L^1-comparison theorem)**.** *Let a be a smooth, divergence-free vector field on a Riemannian manifold N and b be a smooth function on N.Let $f:\ N\times[\alpha,\omega]\longrightarrow\mathbb{R}$ be a continuous function on M such that*

$$\frac{df}{dt}=-\Delta f+a\nabla f+bf \text{ on } N\times(\alpha,\omega]. \tag{4.8}$$

Then there exists $B=B(a,b)$ depending only on a and b such that

$$\|f\big|_\omega\|_{L^1}\leq e^{B(\omega-\alpha)}\|f\big|_\alpha\|_{L^1}$$

Proof. Since L^1 is the dual space of L^∞, it is sufficient to prove that for all $h\in C^\infty(N)$, one has

$$\int_{N\times\{\omega\}}fh\leq e^{B(\omega-\alpha)}\left\|f\big|_\alpha\right\|_{L^1}.\|h\|_{L^\infty}.$$

Consider the backwards heat equation $\begin{cases}\frac{dg}{dt}=\Delta g-\tilde{a}\nabla g-\tilde{b}g, & \text{on } N\times[\alpha,\omega]\\ g\big|_\omega=h,\end{cases}$

which is just a heat equation on $N\times[\alpha,\omega]$ with initial condition at α if we pose $\tilde{g}(t):=g(\omega+\alpha-t)$. The solution g exists and is smooth on $N\times[\alpha,\omega]$. One has, at any time t

$$\int_N g\Delta f=\int_N g\left(-\frac{df}{dt}+a\nabla f+bf\right)$$
$$\int_N f\Delta g=\int_N f\left(\frac{dg}{dt}+\tilde{a}\nabla g+\tilde{b}g\right)$$

Therefore

$$\int_N f\frac{dg}{dt} + g\frac{df}{dt} = \int_N (a\nabla f)g - (\tilde{a}\nabla g)f + (b - \tilde{b})fg$$

Choose $b = \tilde{b}$ and $\tilde{a} = -a$ then the term $(b - \tilde{b})fg$ vanishes and the two first terms become $\int_N \nabla_a(fg) = -\int_N fg \text{ div } a = 0$ where $\text{div } a := \frac{\partial}{\partial x^i}a^i$ is the divergence. Therefore one has $\frac{d}{dt}\int_N fg = 0$, meaning that

$$\int_N f\big|_\omega.h = \int_{N\times\omega} fg = \int_{N\times\alpha} fg \leq \|f\big|_\alpha\|_{L^1}.\|h\|_{L^\infty} \leq e^{B(\omega-\alpha)}\|f\big|_\alpha\|_{L^1}.\|h\|_{L^\infty}$$

where we applied Theorem 54 to g (strictly speaking, to $\tilde{g}$) and B only depends on $\tilde{a} = -a$ and $\tilde{b} = b$. $\square$

Part III

Resolution of nonlinear heat equation on manifold

Chapter 5

Short-time existence and regularity for nonlinear heat equation: Polynomial differential operators and Besov spaces

We will establish in this part a regularity estimate for differential operator with coefficient depending nonlinearly in x and $f(x)$. Although the result can be stated using only Sobolev spaces, it is natural for the proof to make a detour to Besov space where we can use Theorem 60.

We will then apply the regularity estimate for the nonlinear part of the heat operator in order to setup a bootstrap scheme that eventually will prove that any $W^{2,p}$ solution of nonlinear heat equation that is initially C^∞ will be always C^∞.

We will also prove short-time existence using well-known method of Implicit function theorem for Banach spaces. Since the solution is smooth, we can apply Theorem 9 to conclude that the it remains in $M' \subset \mathbb{R}^N$.

5.1 Polynomial differential operator.

Definition 16. *We say that P is a* ***polynomial differential operator of type*** (n, k) *if P is of the form*

$$P(F) = \sum c_{\alpha_1,\ldots,\alpha_\nu}(x, F(x)) D^{\alpha_1} F^{a_1} \ldots D^{\alpha_\nu} F^{a_\nu}$$

where the coefficients $c_{\alpha_1,\ldots,\alpha_n u}$ *depend smoothly and nonlinearly on* x *and* F *and* $\alpha_i \in \mathbb{R}^N$ *are indices with the weighted norm* $\|\alpha_i\| \leq k$ *and* $\sum \|\alpha_i\| \leq n$.

Example 10. *On* $M \times [\alpha, \omega]$ *the nonlinear heat operator* $PF := \frac{dF}{dt} - \tau(F_t)$ *is a polynomial differential operator of type (2,2). The tension field alone is of type (2,1).*

5.1.1 A regularity estimate for polynomial differential operator.

Our goal in this part is to prove the following estimate for polynomial differential operator, in which X will be $M \times [\alpha, \omega]$.

Theorem 56 (Regularity of polynomial differential operator)**.** *Let* X *be a compact Riemannian manifold,* $B \subset \mathbb{R}^N$ *is a large Euclidean ball and* P *be a polynomial differential operator of type* (n, k) *on* X*. Suppose that*

$$r \geq 0, \quad p, q \in (1, \infty), \quad r + k < s, \quad \frac{1}{p} > \frac{r+n}{s}\frac{1}{q}. \tag{5.1}$$

Then for all $F \in C(X, B) \cap W^{s,q}(X)$*, one has* $PF \in W^{r,p}(X)$ *and*

$$\|PF\|_{W^{r,p}} \leq C\left(1 + \|F\|_{W^{s,q}}\right)^{q/p}.$$

where C *is a constant independent of* F*.*

We will prove that the result is *local*, in a sense to be defined. Then we will prove the local statement using Besov spaces.

Proof (reduction of Theorem 56 to a local statement). Let $\{\varphi_i : U_i \longrightarrow V_i\}$ be an atlas of M. We denote a point in U_i by x and its coordinates in V_i by ξ. Let $\sum \psi_i = 1$ be a partition of unity subordinated to $\{U_i\}$ and $\tilde{\psi}_i$ be smooth functions supported in U_i with $0 \leq \tilde{\psi}_i \leq 1$ and $\tilde{\psi}_i = 1$ in the support of ψ_i, as in the definition of Sobolev spaces on manifold. We suppose the following local statement is true:

Lemma 57 (Local statement)**.** *Let* P *be a polynomial differential operator of type* (n, k) *and coefficients* $c_{\alpha_1,\ldots,\alpha_\nu}(x, F)$ *are smooth and vanish when* $x \in \mathbb{R}^{\dim X}$ *is outside of a compact. Let* $B \subset \mathbb{R}^N$ *be a large Euclidean ball and* r, p, q, s *as in* (5.1)*. Then for all compactly supported* $F \in C(\mathbb{R}^{\dim X}, B) \cap W^{s,q}(\mathbb{R}^{\dim X})$*, one has*

$$\|PF\|_{W^{r,p}} \leq C\left(1 + \|F\|_{W^{s,q}}\right)^{q/p}$$

where the constant C *depends only on* B *and the support of* F*, and not on* F*.*

One has
$$\|PF\|_{W^{r,p}} := \sum_i \|\psi_i PF\|_{W^{r,p}}$$
where viewed in the chart U_i, each $\psi_i(x)PF(x)$ is $\sum_\alpha \psi_i(\xi).c_\alpha(\xi, g_i).D^\alpha g_i$ where $g_i = f_i \circ \varphi_i^{-1}$ is f_i viewed in the chart. Since $\tilde{\psi}_i = 1$ in the support of ψ_i, one has

$$\psi_i(\xi).c_\alpha(\xi, g_i).D^\alpha g_i = \psi_i(\xi).c_\alpha(\xi, \tilde{\psi}_i g_i)D^\alpha(\tilde{\psi}_i g_i)$$

hence by the local statement:

$$\|\psi_i(\xi).c_\alpha(\xi, g_i).D^\alpha g_i\|_{W^{r,p}} \leq C\left(1 + \|\tilde{\psi}_i g_i\|_{W^{s,q}}\right)^{q/p} \leq C\left(1 + \|F\|_{W^{s,q}}\right)^{q/p}.$$

Therefore $\|PF\|_{W^{r,p}} \leq mC\left(1 + \|F\|_{W^{s,q}}\right)^{q/p}$ where m is the number of charts we used to cover M. □

Remark 30. *The use of partition of unity in the last proof is to decompose $PF = \sum \psi_i PF$ and not $F = \psi_i F$ since we no longer have linearity of the operator P in F.*

5.1.2 Review of Besov spaces $B^{s,p}$.

In this part, $X = \mathbb{R}^n$ coordinated by $(x_1, \ldots, x_n)$ with weight $(\sigma_1, \ldots, \sigma_n)$. We define

$$T_j^v f(x_1, \ldots, x_n) := f(x_1, \ldots, x_j + v, \ldots, x_n), \quad \Delta_j^v := T_j^v - \mathrm{Id}$$

for $f \in \mathcal{S}(X)$.

For the notation, we will denote the Besov spaces by $B^{s,p}$ with $s \in \mathbb{R}_{>0} \backslash \mathbb{Z}$ and $p \in (1, \infty)$ so that they look similar to Sobolev space $W^{s,p}$. In a more standard notation, our spaces $B^{s,p}$ are denoted by $B^s_{p,p}$

Definition 17. *We define $B^{s,p}$ as the completion of $\mathcal{S}(X)$ under the norm*

$$\|f\|_{B^{s,p}} := \sum_{\|\gamma\| < s} \|D^\gamma f\|_{L^p} + \sum_{s - \frac{\sigma}{\sigma_j} < \|\gamma\| < s} \sup_v \frac{\|\Delta_j^v D^\gamma f\|_{L^p}}{|v|^{(s - \|\gamma\|)\sigma_j/\sigma}}$$

We cite here some well-known facts

1. While Sobolev spaces with non-integral regularity are complex interpolation of integral ones, Besov spaces are their real interpolation.

2. Besov spaces $B^{s,p}(X)$ are reflexive Banach spaces with their dual spaces being $B^{-s,p'}(X)$ where $\frac{1}{p}+\frac{1}{p'}=1$.

Theorem 58. *If $r<s$ then*

$$W^{s,p}(X)\subset B^{s,p}(X)\subset W^{r,p}(X).$$

Theorem 59 (Multiplication)**.** *For $f,g\in\mathcal{S}(X)$ and $\begin{cases}0<\alpha<1,\tilde{p}\le p,\tilde{q}\le q,\tilde{r}\le r\\ \frac{1}{p}+\frac{1}{q}=\frac{1}{r},\frac{1}{\tilde{p}}+\frac{1}{q}=\frac{1}{p}+\frac{1}{\tilde{q}}=\frac{1}{\tilde{r}}\end{cases}$, one has*

$$\|fg\|_{B^{\alpha,\tilde{r}}}\le C\left(\|f\|_{B^{\alpha,\tilde{p}}}\|g\|_{L^q}+\|f\|_{L^p}\|g\|_{B^{\alpha,\tilde{q}}}\right) \tag{5.2}$$

$$\|fg\|_{L^r}\le\|f\|_{L^p}\|g\|_{L^q} \tag{5.3}$$

Therefore by density (5.2) *is true for all $f\in L^p\cap B^{\alpha,\tilde{p}},g\in L^q\cap B^{\alpha,\tilde{q}}$ and* (5.3) *is true for all $f\in L^p,g\in L^q$.*

The reason for which we use the Besov norm is the following estimate:

Theorem 60 (Composition)**.** *Let $\Gamma(x,y)$ be a continuous, nonlinear function of variables $x\in\mathbb{R}^n,y\in\mathbb{R}^N$. Suppose that Γ vanishes for all x outside of a compact in $\mathbb{R}^n$ and Γ is C-Lipschitz in y, and define*

$$\Gamma f:=\left(x\longmapsto\Gamma(x,f(x))\right).$$

Then

$$\|\Gamma f\|\le C\left(1+\|f\|_{B^{\alpha,p}}\right)$$

5.1.3 Proof of the local estimate.

Since $B^{r+\epsilon,p}(X)\subset W^{r,p}(X)$, by increasing r a bit, we can suppose that $r\notin\mathbb{Z}$ and replace the $W^{r,p}$ norm in the statement by the $B^{r,p}$ norm, that is to estimate:

$$\|PF\|_{B^{r,p}}=\sum_{\|\gamma\|<r}\|D^\gamma(PF)\|_{L^p}+\sum_{r-\sigma/\sigma_j<\|\gamma\|<r}\frac{\|\Delta_j^v D^\gamma(PF)\|_{L^p}}{|v|^{(r-\|\gamma\|)\sigma_j/\sigma}}$$

where

$$D^\gamma(PF)=\sum c_{\beta_1,\ldots,\beta_\mu}(x,F)D^{\beta_1}f^{b_1}\ldots D^{\beta_\mu}f^{b_\mu} \tag{5.4}$$

with $\max\|\beta_i\|\le k+\|\gamma\|$ and $\sum\|\beta_i\|\le n+\|\gamma\|$.

Using $\Delta_j^v(fg) = \Delta_j^v f\ T_j^v g + f\Delta_j^v g$, one can see that $\Delta_j^v D^\gamma(PF)$ is a sum of terms of 2 types:

$$\Delta_j^v c_{\beta_1,\ldots,\beta_\mu}\ T_j^v(D^{\beta_1} f^{b_1})\ldots T_j^v(D^{\beta_\mu} f^{b_\mu}) \tag{5.5}$$

and

$$c_{\beta_1,\ldots,\beta_\mu}\ D^{\beta_1} f^{b_1}\ \ldots\ D^{\beta_{i-1}} f^{b_{i-1}}\ \Delta_j^v(D^{\beta_i} f^{b_i})\ T_j^v(D^{\beta_{i+1}} f^{b_{i+1}})\ \ldots\ T_j^v(D^{\beta_\mu} f^{b_\mu}) \tag{5.6}$$

Our strategy is to use Theorem 59 to estimate the terms (5.4), (5.5) and (5.6) as follows, where we denote $\|g\|_p := \|g\|_{L^p}$

$$\left\| c_{\beta_1,\ldots,\beta_\mu}(x,F)\ D^{\beta_1} f^{b_1}\ldots D^{\beta_\mu} f^{b_\mu} \right\|_p \le \|c_{\beta_1,\ldots,\beta_\mu}\|_\infty . \|D^{\beta_1} f^{b_1}\|_{p_1}\ldots\|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \tag{5.7}$$

$$\left\| \Delta_j^v c_{\beta_1,\ldots,\beta_\mu}\ T_j^v(D^{\beta_1} f^{b_1})\ldots T_j^v(D^{\beta_\mu} f^{b_\mu}) \right\|_p \le \|\Delta_j^v c_{\beta_1,\ldots,\beta_\mu}\|_{\tilde p_0} . \|D^{\beta_1} f^{b_1}\|_{p_1}\ldots\|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \tag{5.8}$$

$$\begin{aligned} &\left\| c_{\beta_1,\ldots,\beta_\mu}\ D^{\beta_1} f^{b_1}\ \ldots\ D^{\beta_{i-1}} f^{b_{i-1}}\ \Delta_j^v(D^{\beta_i} f^{b_i})\ T_j^v(D^{\beta_{i+1}} f^{b_{i+1}})\ \ldots\ T_j^v(D^{\beta_\mu} f^{b_\mu}) \right\|_p \le \\ &\|c_{\beta_1,\ldots,\beta_\mu}\|_\infty . \|D^{\beta_1} f^{b_1}\|_{p_1}\ldots\|D^{\beta_{i-1}} f^{b_{i-1}}\|_{p_{i-1}} . \|\Delta_j^v(D^{\beta_i} f^{b_i})\|_{\tilde p_i} . \|D^{\beta_{i+1}} f^{b_{i+1}}\|_{p_{i+1}}\ldots\|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \end{aligned} \tag{5.9}$$

Then continue by bounding the Δ_j^v terms:

$$\|\Delta_j^v c_{\beta_1,\ldots,\beta_\mu}\|_{\tilde p_0} \le |v|^{\theta\sigma_j/\sigma} C(1 + \|F\|_{B^{\theta,\tilde p_0}}) \le |v|^{\theta\sigma_j/\sigma} C(1 + \|F\|_{W^{\theta,\tilde p_0}}) \tag{5.10}$$

using Theorem 60, where C is the Lipschitz constant of $c_{\beta_1,\ldots,\beta_\mu}(x,F)$ in F, which exists because $c_{\beta_1,\ldots,\beta_\mu}$ is smooth and F always remains in a large Euclidean ball B. The next Δ_j^v term to bound is, using Theorem 58:

$$\|\Delta_j^v(D^{\beta_i} f^{b_i})\|_{\tilde p_i} \le |v|^{\theta\sigma_j/\sigma} \|f^{b_i}\|_{B^{\|\beta_i\|+\theta,\tilde p_i}} \le |v|^{\theta\sigma_j/\sigma} \|f^{b_i}\|_{W^{\|\beta_i\|+\theta,\tilde p_i}} \tag{5.11}$$

And finally plugging (5.10) and (5.11) in (5.8) and (5.9), and noting that $\|c_{\beta_1,\ldots,\beta_\mu}\|_\infty$ in (5.7) is bounded by a constant, it remains to estimate $\|f^{b_i}\|_{W^{\|\beta_i\|,p_i}}$, $\|f^{b_i}\|_{W^{\|\beta_i\|+\theta,\tilde p_i}}$ and $\|F\|_{W^{\theta,\tilde p_0}}$ in term of $\|F\|_{W^{s,q}}$, for which we will use the following consequence of Interpolation inequality.

Lemma 61. *Let $0 \leq r \leq s$ and $p, q \in (1, \infty)$ such that $0 < \frac{1}{p} - \frac{r}{s}\frac{1}{q} < 1 - \frac{r}{s}$. Then for all compactly supported $F \in C(X, B) \cap W^{s,q}$ where $B \subset \mathbb{R}^N$ is a large Euclidean ball, one has*

$$\|F\|_{W^{r,p}} \leq C\|F\|_{\infty}^{1-r/s}\|F\|_{W^{s,q}}^{r/s} \leq C'\|F\|_{W^{s,q}}^{r/s}$$

where C, C' depend only on B and the support of F, but not F.

Proof. Since F is bounded, $f^\alpha \in W^{s,q} \cap W^{0,v}$ for all $v > 1$. By Interpolation inequality

$$\|f^\alpha\|_{W^{r,p}} \leq 2\|f^\alpha\|_{W^{s,q}}^{r/s}\|f^\alpha\|_{W^{0,v}}^{1-r/s}$$

then choose v with $(1 - \frac{r}{s})\frac{1}{v} = \frac{1}{p} - \frac{r}{s}\frac{1}{q}$. ☐

To apply Lemma 61, we have to choose $p_i, \tilde{p}_i, \tilde{p}_0, \theta$ such that $\begin{cases} 0 < \frac{1}{p_i} - \frac{\|\beta_i\|}{s}\frac{1}{q} < 1 - \frac{\|\beta_i\|}{s}, \\ 0 < \frac{1}{\tilde{p}_i} - \frac{\|\beta_i+\theta\|}{s}\frac{1}{q} < 1 - \frac{\|\beta_i+\theta\|}{s} \\ 0 < \frac{1}{\tilde{p}_0} - \frac{\theta}{s}\frac{1}{q} < 1 - \frac{\theta}{s} \end{cases}$

We choose $\frac{1}{p_i}$ just a bit bigger than $\frac{\|\beta_i\|}{s}\frac{1}{q}$, $\frac{1}{\tilde{p}_i}$ just a bit bigger than $\frac{\|\beta_i+\theta\|}{s}\frac{1}{q}$ and $\frac{1}{\tilde{p}_0}$ just a bit bigger than $\frac{\theta}{s}\frac{1}{q}$. We will now come back to justify the estimates (5.7), (5.8), (5.9). Since F is bounded in B and compactly supported in an open set V, we see that $\|f^\alpha\|_p \leq C(B, V)\|f^\alpha\|_q$ if $p \leq q$. Therefore,

1. For (5.7), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{p_1} + \cdots + \frac{1}{p_\mu}$$

which is true because the RHS is is a bit bigger than $\frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|}{qs} < \frac{n+r}{qs} < \frac{1}{p}$.

2. For (5.8), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{\tilde{p}_0} + \frac{1}{p_1} + \cdots + \frac{1}{p_\mu}$$

where the RHS is is a bit bigger than $\frac{\theta}{s}\frac{1}{q} + \frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|+\theta}{qs}$.

3. For (5.9), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{p_1} + \cdots + \frac{1}{\tilde{p}_i} + \cdots + \frac{1}{p_\mu}$$

where the RHS is is a bit bigger than $\frac{\theta}{s}\frac{1}{q} + \frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|+\theta}{qs}$.

It is sufficient then to take $\theta = r - \|\gamma\|$. Now the estimates (5.7), (5.8), (5.9) can be continued as

$$RHS(5.7) \leq \prod_i \|f^{b_i}\|_{W^{s,q}}^{\|\beta_i\|/s} \leq \|F\|_{W^{s,q}}^{\frac{n+\|\gamma\|}{s}} \leq \|F\|_{W^{s,q}}^{q/p} \tag{5.12}$$

$$RHS(5.8) \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|F\|_{W^{s,q}}^{\theta/s}\right)\prod_i \|f^{b_i}\|_{W^{s,q}}^{\|\beta_i\|/s} \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|F\|_{W^{s,q}}^{\theta/s}\right)\|F\|_{W^{s,q}}^{q/p} \tag{5.13}$$

$$RHS(5.9) \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|f^{b_i}\|_{W^{s,q}}^{\frac{\|\beta_i\|+\theta}{s}}\right)\prod_{u\neq i} \|f^{b_u}\|_{W^{s,q}}^{\|\beta_u\|/s} \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|F\|_{W^{s,q}}^{\frac{\|\beta_i\|+\theta}{s}}\right)\|F\|_{W^{s,q}}^{q/p} \tag{5.14}$$

While (5.12) gives $\|D^\gamma(PF)\|_p \leq C\|F\|_{W^{s,q}}^{q/p}$, the last two (5.13) and (5.14) give

$$\sum_{s-\frac{\sigma}{\sigma_j}<\|\gamma\|<s} \sup_v \frac{\|\Delta_j^v D^\gamma(PF)\|_p}{|v|^{(r-\|\gamma\|)\sigma_j/\sigma}} \leq C\left(1+\|F\|_{W^{q,s}}^{(n+r)/s}\right)$$

We proved the local statement Lemma 57.

5.2 Regularity for nonlinear heat equation.

Let $p > \dim M + 2$, using the regularity estimate for polynomial differential operator, we now can prove:

Theorem 62 (Bootstrap for nonlinear heat equation)**.** *Let $F:\ M\times[\alpha,\omega] \longrightarrow B$ such that $F \in W^{2,p}(M\times[\alpha,\omega])$ and $\frac{dF_t}{dt} = \tau(F_t)$, i.e.*

$$\frac{dF^\alpha}{dt} = -\Delta F^\alpha + g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^\beta F_j^\gamma$$

and $F\big|_{M\times\{\alpha\}}$ is smooth. Then F is smooth on $M\times[\alpha,\omega]$.

Remark 31. *Note that since $p > \dim M + 2 = \dim(M\times[\alpha,\omega]) + 1$, if $F \in W^{2,p}(M\times[\alpha,\omega])$ then F and $\frac{\partial F}{\partial x^i}$ are in $C(M\times[\alpha,\omega])$ by Sobolev imbeddings. It makes sense then to talk about:*

1. *the restriction and boundary condition at time $t=\alpha$ (in fact, by Trace theorem, $p>1$ is enough).*

2. *the pointwise condition $F : M\times[\alpha,\omega] \longrightarrow B \subset V$.*

Proof. We define the operators $PF := g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^{\beta}F_j^{\gamma}$ of type (2,1) and $AF := \frac{dF}{dt} + \Delta F$ of type (2,2). As in Theorem 51, we will abusively denote $W^{k,p}(M \times [\beta,\gamma])$ by $W^{k,p}([\beta,\gamma])$. Our bootstrap scheme consists of 3 steps:

1. Prove that $F \in W^{2,\tilde{p}}([\pi,\omega])$ for every $\pi > \alpha$ and $\tilde{p} \in (1,\infty)$. By compactness of M, it is sufficient to prove this for a sequence $\tilde{p} \to +\infty$.

2. Prove that F is C^∞ for all time $t > \alpha$.

3. Prove that F is C^∞ on $M \times [\alpha,\omega]$.

Step 1. By Theorem 56, $AF = PF \in W^{r,q}([\alpha,\omega])$ whenever $r < 1$ and $\frac{1}{q} > (\frac{r}{2}+1)\frac{1}{p}$. Apply Gårding inequality, for all $\pi > \alpha$, $F \in W^{r+2,q}([\pi,\omega]) \subset W^{2,\tilde{p}}([\pi,\omega])$ for $\frac{1}{\tilde{p}} = \frac{1}{q} - \frac{r}{\dim M+1}$. Choose $\frac{1}{q}$ very close to $(\frac{r}{2}+1)\frac{1}{p}$, one sees that the condition on $\tilde{p}$ is $\frac{1}{\tilde{p}} > (\frac{r}{2}+1)\frac{1}{p} - \frac{r}{p-1}$, which will be satisfied if $\frac{1}{\tilde{p}} > (1-\frac{r}{2})\frac{1}{p}$, i.e. for all $\tilde{p} < \frac{p}{1-r/2}$. It remains to repeat this result to finish the first step. We will say $F \in W^{2,*}([\pi,\omega])$ for $F \in W^{2,p}([\pi,\omega])$ for all $p \in (1,\infty)$.

Step 2. By Theorem 56, for all $r < 1$, one has $AF = PF \in W^{r,*}([\pi,\omega])$, therefore by Gårding inequality, $F \in W^{r+2,*}([\pi,\omega])$. Iterate this result and one has $F \in W^{k,*}([\pi,\omega])$ for all $k \in [2,\infty)$ and $\pi > \alpha$. So F is smooth for $t > \alpha$.

Step 3. We apply regularity result (Theorem 46) for elliptic operator A and boundary operators $B^0 : F \mapsto F\big|_{M\times\{\alpha\}}$ and $B^1 : F \mapsto F\big|_{M\times\{\omega\}}$, both are of weight 0: For q, r in Step 1, one has $AF = PF \in W^{r,q}([\alpha,\omega])$ and $B^jF \in \partial W^{r,q}$, therefore $F \in W^{r+2,q}([\alpha,\omega]) \subset W^{2,\tilde{p}}([\alpha,\omega])$ for the same $\tilde{p}$ as Step 1. This proves that $F \in W^{2,*}([\alpha,\omega])$, which also means that one has $F \in W^{r+2,q}([\alpha,\omega])$ with no additional condition on q except $q \in (1,\infty)$. Iterate and one obtains the regularity of F on $[\alpha,\omega]$. $\square$

Remark 32. *The first 2 steps were to prove the regularity of $F\big|_{M\times\{\omega\}}$, which was then used as a boundary condition in order to apply regularity result for elliptic operator on manifold with boundary.*

5.3 Short-time existence for nonlinear heat equation.

We will choose as always $p > \dim M + 2$. As before, M is a compact Riemannian manifold and $f : M \longrightarrow B \subset V = \mathbb{R}^N$ where B is a large Euclidean ball.

Theorem 63 (Short-time existence). *Let $F_\alpha : M \longrightarrow B$ be a smooth map, then there exists $\epsilon > 0$ depending on F_α and $F : M \times [\alpha, \alpha+\epsilon] \longrightarrow B$ such that $F \in W^{2,p}(M \times [\alpha, \alpha+\epsilon])$ with $F\big|_{M\times\{\alpha\}} = F_\alpha$ and*

$$\frac{dF_t}{dt} = \tau(F_t) \quad \textit{on } M \times [\alpha, \alpha+\epsilon]$$

Proof. We find F as a sum $F = F_b + F_\#$ where $F_b \in C^\infty(M \times [\alpha, \omega])$ satisfies the initial condition and $F_\# \in W^{2,p}(M \times [\alpha, \alpha+\epsilon]/\alpha)$.

The nonlinear heat operator is

$$\begin{aligned} T : W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N} &\longrightarrow L^p(M \times [\alpha, \omega])^{\oplus N} \\ F_\# &\longmapsto \tau(F_b + F_\#) \end{aligned}$$

where $\tau(F)^\alpha = \Delta F^\alpha + g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^\beta F_j^\gamma$, which can be rewritten as $\tau(F) = -\Delta F + \Gamma(F)(\nabla F)^2$. The derivative of T at $F_\#$ in direction $k \in W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N}$ is

$$DT(F_\#)k = \Delta k + D\Gamma(F)\cdot k.(\nabla F)^2 + 2\Gamma(F)\nabla F.\nabla k,$$

or in local coordinates:

$$DT(F_\#)^\alpha = g^{ij}\left(\frac{\partial^2 k^\alpha}{\partial x^i \partial x^j} - k_l^\alpha \Gamma_{ij}^l\right) + g^{ij}\frac{\partial \Gamma'^{\alpha}_{\beta\gamma}}{\partial y^\delta} k^\delta F_i^\beta F_j^\gamma + 2g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^\beta F_j^\gamma$$

which is of form $DT(F_\#)k = -\Delta k - a(x, F)\nabla k - b(x, F)k$ where a, b are smooth.

Therefore if we note by

$$\begin{aligned} H : W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N} &\longrightarrow L^p(M \times [\alpha, \omega])^{\oplus N} \\ F_\# &\longmapsto (\frac{d}{dt} - \tau)(F_b + F_\#) \end{aligned}$$

then the derivative of H at $F_\# = 0$ is

$$DH(0)\cdot k = \frac{dk}{dt} + \Delta k + a(x, F_b)\nabla k + b(x, F_b)k$$

which by Theorem 52 is an isomorphism from $W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N}$ to $W^{0,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N} = L^p(M \times [\alpha, \omega])^{\oplus N}$. This shows that H is a local isomorphism mapping a neighborhood of 0 to a neighborhood of $(\frac{d}{dt} - \tau)F_b$.

Define $g_\epsilon \in L^p(M \times [\alpha, \omega])^{\oplus N}$ by

$$g_\epsilon := \begin{cases} 0, & \text{if } t \in [\alpha, \alpha+\epsilon] \\ (\frac{d}{dt} - \tau)F_b, & \text{if } t > \alpha+\epsilon \end{cases}$$

which is arbitrarily $L^p(M \times [\alpha, \omega])$-close to $(\frac{d}{dt} - \tau)F_b$ for $0 < \epsilon \ll 1$. There exists therefore $F_\# \in W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N}$ such that $H(F_\#) = g_\epsilon$, meaning that the function $F = F_b + F_\# : M \longrightarrow V$ satisfies $F\big|_{M \times \{\alpha\}} = F_\alpha$ and $\frac{dF}{dt} - \tau(F_t) = 0$ for $t \in [\alpha, \alpha + \epsilon]$.

By Regularity Theorem 62, F is C^∞ for $t \in [\alpha, \alpha + \epsilon]$. Theorem 9 assures that the image of F is in B, hence in M' for $t \in [\alpha, \alpha + \epsilon]$. $\square$

Chapter 6

Global existence for nonlinear heat equation and harmonic maps between Riemannian manifolds

Let M be a compact Riemannian manifold. We want to solve the following nonlinear heat equation where $F : M \longrightarrow M' \subset B \subset V = \mathbb{R}^N$:

$$\frac{dF_t}{dt} = -\Delta F_t + \Gamma(F_t)(\nabla F_t)^2$$

We have proved that the solution exists in short-time and is smooth whenever it exists. We will now establish long-time existence using continuity method, that is we will show that if the solution exists on $[\alpha, \omega_n]$ where ω_n is an increasing sequence to ω, then the solution exists on $[\alpha, \omega]$. We then apply short-time existence to gain a small open interval where solution still exists. We then conclude that the solution exists globally on $[\alpha, +\infty)$ since this interval is connected.

The crucial step to prove that the solution can be extended on $[\alpha, \omega]$ is to uniformly bound all of its derivatives in time of evolution $[\alpha, \omega)$. These estimates will also be useful to justify the convergence of F_t in $C^\infty(M)$ to a smooth function F_∞ which will eventually be a harmonic map from M to M'.

Recall that we proved in Corollary 15.1 the boundedness of $\|F_t\|_{W^{2,2}(M)}$ by a constant C depending only on curvatures of M, M' and the initial total energies. Since $\frac{dF_t}{dt}$ relates to spatial derivatives of F by the nonlinear heat equation, it is easy to see that $\|F_t\|_{W^{2,2}(M\times[\tau,\tau+\delta])}$ is bounded by a constant independent of τ. Again, we will denote $W^{k,p}(M\times[\beta,\gamma])$ by $W^{k,p}([\beta,\gamma])$.

Theorem 64 ($W^{2,2}$-boundedness)**.** *There exist a constant C depending only on δ, the metrics and initial total energies such that*

$$\|F\|_{W^{2,2}(\tau,\tau+\delta)} \leq C \quad \textit{for all } \alpha \leq \tau < \omega - \delta.$$

Proof. Since

$$\|F\|^2_{W^{2,2}([\tau,\tau+\delta])} \leq \int_\tau^{\tau+\delta} \|F_t\|^2_{W^{2,2}(M)}dt+2\int_\tau^{\tau+\delta}\|\Delta F_t\|^2_{L^2}\,dt+2\int_\tau^{\tau+\delta}\left\|\Gamma(F_t)(\nabla F_t)^2\right\|^2_{L^2}dt$$

The first term and the second term are bounded by $C^2\delta$, the third one, since $\Gamma(F_t)$ is bounded, by $C^2\delta$ where C is a constant only depending on the metrics and initial total energies. $\square$

The estimates of higher derivatives of F will be established in the following order: first in $W^{2,p}$ for all p norm then in $W^{k,p}$ for all k,p, then in C^∞.

6.1 Estimate of higher derivatives.

Lemma 65 ($W^{2,p}$-boundedness)**.** *For all $p\in(1,+\infty)$, there exists a constant $C>0$ depending only on δ, p, the metrics and initial energies such that for all $\alpha+\delta\leq\tau\leq\omega-\delta$:*

$$\|F\|_{W^{2,p}([\tau,\tau+\delta])} \leq C$$

Proof. Applying Gårding Inequality to the parabolic equation $AF=\Gamma(F)(\nabla F)^2$ where $A := \frac{\partial}{\partial t}+\Delta$ is the heat operator, one has

$$\|F\|_{W^{2,p}([\tau,\tau+\delta])} \leq C\left(\|\Gamma(F)(\nabla F)^2\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])}+\|F\|_{W^{2,2}([\tau-\frac{\delta}{3},\tau+\delta])}\right)$$

The second term of RHS is already bounded by applying Theorem 64 to $\frac{4\delta}{3}$. For the first term:

$$\left\|\Gamma(F)(\nabla F)^2\right\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \leq C(M')\||\nabla F|^2\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} = C(M')\|e(F)\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])}.$$

Recall that, by Theorem 11, the potential density satisfies $\frac{de}{dt}+\Delta e - Ce\leq 0$ for certain constant C depending only on the metric of M, by Maximum principle (Theorem 53), one has $e\leq\psi_\tau$ where ψ_τ is the solution of $\begin{cases} \frac{d}{dt}\psi_\tau + \Delta\psi_\tau - C\psi_\tau = 0 \\ \psi_\tau\big|_{\tau-\frac{\delta}{2}} = e\big|_{\tau-\frac{\delta}{2}} \end{cases}$ We apply Gårding Inequality again for ψ_τ and obtain

$$\|e(F)\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \leq \|\psi_\tau\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \leq C\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2},\tau+\delta])}. \tag{6.1}$$

Now apply L^1-Comparison Theorem 55 to ψ_τ, one has

$$\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2},\tau+\delta])} \leq \int_0^{3\delta/2} \|\psi_\tau\big|_{\tau-\frac{\delta}{2}}\|_{L^1} e^{Bt} dt \leq \int_0^{3\delta/2} e^{Bt} dt.\|e\big|_{\tau-\frac{\delta}{2}}\|_{L^1} \leq C. \tag{6.2}$$

The lemma follows from (6.1) and (6.2). □

We can now estimate higher order derivatives.

Theorem 66 ($W^{k,p}$-boundedness)**.** *For all $p \in (1,+\infty)$ and $k < +\infty$, there exists C depending only on k, p, the metrics and initial energies such that*

$$\|F\|_{W^{k,p}([\tau,\tau+\delta])} \leq C$$

for all $\alpha + \delta \leq \tau \leq \omega - \delta$.

Proof. Applying Gårding Inequality to the equation $\frac{dF}{dt} + \Delta F_t = \Gamma(F)(\nabla F)^2$ then Regularity Theorem 56 for polynomial differential operator, one has for $\epsilon \ll \delta$:

$$\begin{aligned} \|F\|_{W^{k,p}([\tau,\tau+\delta])} &\leq C_\epsilon \left(\|F\|_{W^{2,p}([\tau-\epsilon,\tau+\delta])} + \|\Gamma(F)(\nabla F)^2\|_{W^{k-2,p}([\tau-\epsilon,\tau+\delta])} \right) \\ &\leq C_\epsilon \left(1 + C\left(1 + \|F\|_{W^{s,q}([\tau-\epsilon,\tau+\delta])}\right)^{q/p} \right) \end{aligned}$$

as long as $k-1 < s$ and $\frac{1}{p} > \frac{k}{s}.\frac{1}{q}$. Therefore if $\|F\|_{W^{s,q}([\tau,\tau+\delta])} \leq C(\delta,s,q)$ for all $\beta \leq \tau \leq \omega - \delta$ and $q \in (1,+\infty)$, we just proved that

$$\|F\|_{W^{k,p}([\tau,\tau+\delta])} \leq C(\epsilon,k,p)$$

for all $\begin{cases} \beta + \epsilon \leq \tau \leq \omega - \delta \\ k < s+1, p \in (1,+\infty) \end{cases}$ since $\|F\|_{W^{s,q}([\tau-\epsilon,\tau+\delta])} \leq 2C(\delta,s,q)$.

One can then conclude by induction on k, with step $\frac{1}{2}$, starting with $k = 2$ and $\epsilon = \frac{\delta}{2}$ and each time dividing ϵ by 2. □

6.2 Global existence for nonlinear heat equation.

Theorem 67 (Global existence)**.** *The solution of nonlinear heat equation*

$$\frac{dF}{dt} = -\Delta F + \Gamma(F)(\nabla F)^2 \tag{6.3}$$

with smooth initial condition exists globally for all time $t > \alpha$.

Proof. Let F_n be a sequence of solution of (6.3) on $[\alpha, \omega_n]$ with ω_n increasing to ω then they coincide by uniqueness of solution the equation. As discussed in the beginning of this part, it is sufficient to prove that the solution extends to $[\alpha, \omega]$. Let F be the solution on $[\alpha, \omega)$ such that $F\big|_{[\alpha,\omega_n]} = F_n$, then by Theorem 66, for all $\tau \in [\alpha, \omega - \delta)$:

$$\|D_t^u D_x^v F\|_{L^\infty(M\times[\tau,\tau+\delta])} \leq C_{\text{Sobolev}} \|D_t^u D_x^v F\|_{W^{k,p}(M\times[\tau,\tau+\delta])} \leq C_{\text{Sobolev}}.C(k,p,\delta)$$

where, if we choose k sufficiently large, C_{Sobolev} is the constant of Sobolev imbedding $W^{k,p}(M \times [0,\delta]) \hookrightarrow C(M \times [0,\delta])$ and $C(k,p,\delta)$ is the constant provided by Theorem 66.

So all partial derivatives of F is uniformly bounded on $[\alpha + \delta, \omega)$. This proves that F extends to a solution on $[\alpha, \omega]$. In fact $F\big|_\tau := F\big|_{M\times\{\tau\}}$ converges in $C^\infty(M)$ as $\tau \to \omega$, since $\|D^\alpha F\big|_\tau - D^\alpha F\big|_{\tau'}\|_{L^\infty} \leq \max_{\|\beta\|=\|\alpha\|+1} \|D^\beta F\|_{L^\infty} |\tau - \tau'|$. $\square$

We have just proved the first part of the following theorem. The second part is a reformulation of Theorem 7 of Eells and Sampson.

Theorem 68.

1. *Let M, M' be compact Riemannian manifolds with* $\operatorname{Riem}(M') \leq 0$. *Then for every smooth map $f_0 : M \longrightarrow M' \subset B \subset \mathbb{R}^N$, the nonlinear heat equation*

$$\begin{cases} \frac{df_t}{dt} = \tau(f_t), & \text{for all } t \geq 0 \\ f\big|_{t=0} = f_0, \end{cases}$$

admit a globally defined smooth solution f_t. Moreover, all derivatives $D^\alpha f_t$ remains uniformly bounded as $t \to +\infty$.

2. *For a suitable sequence t_n increasing to $+\infty$ the sequence f_{t_n} converges in $C^\infty(M)$ to a function f_∞ with $\tau(f_\infty) = 0$. Therefore any map $f_0 : M \longrightarrow M'$ is homotopic to a harmonic map.*

Proof. For any sequence t_n, one can extract from $\{f_{t_n}\}$, since their derivatives are uniformly bounded, a subsequence $\{f_{t_{n_i}}\}$ converging in $C^k(M, \mathbb{R}^N)$. By a diagonalisation argument, one can extract from any sequence $\{f_{t_n}\}$ a subsequence converging in $C^\infty(M, \mathbb{R}^N)$ to f_∞. Abusively denote this subsequence by $\{f_{t_n}\}$, by Theorem 12

$$\lim_{n\to\infty} K(f_{t_n}) = \lim_{n\to\infty} \int_M |\tau(f_{t_n})|^2 = 0$$

Therefore $\tau(f_{t_n}) \to 0$ in $L^2(M)^{\oplus N}$. But also $\tau(f_{t_n}) \to \tau(f_\infty)$ in $C^\infty(M, \mathbb{R}^N)$, one has $\tau(f_\infty) = 0$. The homotopic conlusion follows by rescaling the deformation time between f_{t_n} and $f_{t_{n+1}}$ to $\frac{1}{2^n}$. $\square$

Part IV

Appendices: Parametrix and Linear equations

Chapter 7

A comparison theorem, Sobolev imbeddings and Konrachov theorem for Riemannian manifolds

In this part, we will first establish the Sobolev imbeddings theorem and the Kondrachov theorem for Riemannian manifolds from the Euclidean version of these theorems.

Theorem 69 (Sobolev Imbedding for $\mathbb{R}^n$)**.** *Given $k, l \in \mathbb{Z},\ k > l \geq 0$ and $p, q \in \mathbb{R},\ p > q \geq 1$. Then*

1. *If $\frac{1}{p} = \frac{1}{q} - \frac{k-l}{n}$ then*
$$W^{k,q}(\mathbb{R}^n) \hookrightarrow W^{l,p}(\mathbb{R}^n)$$
is a continuous imbedding.

2. *If $\frac{k-r}{n} > \frac{1}{q}$ then*
$$W^{k,q}(\mathbb{R}^n) \hookrightarrow C^r_B(\mathbb{R}^n)$$
If $\frac{k-r-\alpha}{n} \leq \frac{1}{q}$ then
$$W^{k,q}(\mathbb{R}^n) \hookrightarrow C^{r,\alpha}(\mathbb{R}^n)$$

where $C^r_B(\mathbb{R}^\ltimes)$ denotes the space of C^r functions with bounded derivatives up to order n, equipped with the norm $\|u\|_{C^r_B} = \max_{l\leq r} \sup |\nabla^l u|$, and $C^{r,\alpha}$ is the subspace of C^r_B of functions whose r^{th}-derivative is α-Holder, equipped with the norm $\|u\|_{C^r,\alpha} = \|u\|_{C^r_B} + \sup_{P\neq Q}\{\frac{u(P)-u(Q)}{d(P,Q)^\alpha}\}$.

Theorem 70 (Kondrachov for $\Omega \subset \mathbb{R}^n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with regular boundary and let $k \in \mathbb{Z}_{\geq 0}$ and $p, q \in \mathbb{R}_{>0}$ be such that $1 \geq \frac{1}{p} > \frac{1}{q} - \frac{k}{n} > 0$ then*

1. *The imbedding $W^{k,q}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*
2. *The imbedding $W^{k,q}(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ is compact if $k - \alpha > \frac{n}{q}$ where $0 \leq \alpha < 1$.*
3. *The imbeddings $W_0^{k,q}(\Omega) \hookrightarrow L^p(\Omega)$ and $W_0^{k,q}(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ are compact, where $W_0^{k,q}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $W^{k,q}(\Omega)$, i.e. the subspace of functions whose trace vanishes on the boundary of Ω.*

Theorem 69 will be generalised for complete manifolds with bounded curvature and injectivity radius, while Theorem 70 holds for compact Riemannian manifolds.

The generalisation will be done in 2 steps

1. Compare the volume form of the Riemannian metric g near a point and that of the Euclidean metric on the tangent space at that point. Theorem 72 gives an equivalent between the integral under g and the integral under Euclidean metric via the exponential map.
2. Reasonably use partition of unity to establish global results from local results (the Euclidean case). We will need a covering lemma (Calabi's lemma), which essentially reduces to a combinatorial result (Vitali's covering lemma).

Finally, we will apply imbedding theorems to solve the equation $-\Delta u = f$ on a Riemannian manifold when f is square-integrable.

7.1 Quick recall of Jacobi fields, Index inequality

Definition 18. *A **Jacobi** field is a field Y defined along a geodesic $\gamma(t)$ such that*

$$\frac{D^2}{dt^2}Y(t) + R(Y(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0 \tag{7.1}$$

where R denotes the Riemann curvature tensor.

Remark 33.

1. *Since (7.1) is linear, a Jacobi field is uniquely defined given $Y(t_0)$ and $\dot{Y}(t_0)$.*
2. *If $Y(0) \perp \dot{\gamma}(0)$ and $\dot{Y}(0) \perp \dot{\gamma}(0)$ then $\dot{Y}(t) \perp \dot{\gamma}(t)$ for all t.*

3. *If Y, Z are Jacobi fields along γ then*

$$\langle Y, \dot{Z}\rangle - \langle \dot{Y}, Z\rangle = \mathrm{const}$$

In particular, if Y, Z vanish at a same point p_0 in γ then $\langle Y, \dot{Z}\rangle = \langle \dot{Y}, Z\rangle$ on γ.

There are two ways to interpret Jacobi fields:

1. Jacobi fields are derivative of exponential maps

2. Jacobi fields are minimisers of Index form, i.e. the variation of second other of length.

The first interpretation is the content of the following Proposition.

Proposition 70.1. *Let $Y(t) = D\exp_p(tu).t\xi$ be a vector field defined on a geodesic $\gamma(t) = \exp_p tu$. Then Y satisfies*

$$\begin{cases} Y(0) = 0, \dot{Y}(0) = \xi, \\ \ddot{Y} + R(Y, \dot{\gamma})\dot{\gamma} = 0, \end{cases} \tag{7.2}$$

hence a Jacobi field.

In concrete term, denote by ψ the exponential function at $p \in M$ and $q = \gamma(r) = \exp_p r\dot{\gamma}(0)$, then Proposition 70.1 says that if the Jacobi field Y vanishes at $p = \gamma(0)$, i.e. $Y(0) = 0$ then $Y(r)$ at $\gamma(r)$ is defined as follow: pull-back $\dot{Y}(0)$ by ψ, transport parallelly, w.r.t to the Euclidean metric of T_pM, $\psi^*\dot{Y}(0)$ from 0 to $X_0 = \psi^{-1}(q)$, then push-forward by ψ, one obtains $Y(r)$. See Figure 7.1.

Figure 7.1: Jacobi fields and exponential maps.

Since Jacobi fields are derivatives of exponential maps, one can rephrase the phenomenon of cut-locus by Jacobi fields. Historically, a point q on a Riemannian manifold is said to be a **conjugate** point of p if there exists, along a geodesic connecting them, a Jacobi field vanishing on both p and q. This means that the exponential map with origin in p degenerates at a preimage of q. One can also prove that if q is in the cut-locus of p then at least one of the following situation occurs

1. q is a conjugate point of p.

2. There exists 2 minimising geodesic from p to q.

For another interpretation of Jacobi fields, note that given a geodesic γ and a vector field Z defined along γ, then the first variation of length when one varies γ by Z is 0 and the second variation can also be calculated without difficulty.

Proposition 70.2 (Second variation of length). *Let $\gamma : [0, r] \longrightarrow M$ be a geodesic and Z be a vector field along γ that is orthogonal to $\dot{\gamma}$ at every point. Denote by L_λ length of the curve $t \mapsto \exp_{\gamma(t)} \lambda Z$ for $\lambda \ll 1$, then one has*

$$\frac{d^2}{d\lambda^2} L_\lambda \bigg|_{\lambda=0} = I(Z) := \int_0^r \left(\|Z(t)\|^2 + \langle R(\dot{\gamma}(t), Z(t))\dot{\gamma}(t), Z(t) \rangle \right) dt \tag{7.3}$$

Definition 19. *Let $\gamma : [0, r] \longrightarrow M$ be a geodesic and Z be a orthogonal vector field along γ. The **Index form** $I(Z)$ of Z is defined by the RHS of (7.3).*

Remark 34. *The curvature term in (7.3) is $K(\dot{\gamma}, Z)\|Z\|^2$ where K denotes the sectional curvature of M.*

Jacobi fields can be seen as the unique minimiser of the Index form among vector fields defined on a geodesic $\gamma : [0, r] \longrightarrow M$ with the same value at $\gamma(0)$ and $\gamma(r)$.

Theorem 71 (Index inequality). *Let $\gamma : [0, r] \longrightarrow M^n$ be a geodesic, $p = \gamma(0)$ and $q = \gamma(r)$ such that p has no conjugate point along γ, or equivalently the exponential map in direction $\dot{\gamma}(0)$ does not degenerate.*

- *Let Z be a (piecewise smooth) vector field along γ, orthogonal to $\dot{\gamma}$ with $Z(p) = 0$.*
- *Let Y be the Jacobi field along γ with $Y(0) = 0, Y(r) = Z(r)$ and Y is orthogonal to $\dot{\gamma}$.*

Then $I(Y) \leq I(Z)$ and equality occurs if and only if $Y \equiv Z$.

Remark 35. *Note that such Jacobi field Y exists and is unique. Firstly, by the second point of Remark 33, one only need $Y(p) = 0$ and $\dot{Y}(0) \perp \gamma(0)$. The Jacobi fields satisfying these conditions form a vector space of dimension $n - 1$ (by Cauchy problem, $\dot{Y}(0)$ is to be chosen in the orthogonal space of $\gamma(0)$). Since the exponential map does not degenerate on the preimage of γ, each $\dot{Y}(0)$ corresponds one-to-one with an $Y(r)$ by Proposition 70.1. The* *correspondence is linear, with source and target spaces of same dimension $(n-1)$, it follows that each $Z(r) \perp \gamma(r)$ gives uniquely a Jacobi field Y.*

More concretely, let $\dot{V}_i(0)$ be a basis of $\dot{\gamma}(0)$ in T_pM and V_i be the corresponding Jacobi fields with $V_i(0) = 0$, then

1. *$\{V_i(t)\}_{i=\overline{1,n-2}}$ is a basis of $\dot{\gamma}(t)$ in $T_{\gamma(t)}M$, where the orthogonal part follows from Remark 33 and the linear independence is by the non-degeneration of since* $\exp_p$.

2. *If $Z(t) = \sum f_i(t)V_i(t)$, where f_i are functions on $[0,r]$, then $Y(t) = \sum_i f_i(r)V_i(t)$.*

Proof. As Remark 35, let $Z = \sum_i f_i V_i$ and denote $W = \sum_i \dot{f}_i V_i$ then

$$I(Z) = \int_0^r \left(\|W\|^2 + 2\sum_i f_i \langle \dot{V}_i, W\rangle + \langle \sum_i f_i \dot{V}_i, \sum_j f_j \dot{V}_j\rangle + \langle R(\dot{\gamma}, \sum f_i V_i)\dot{\gamma}, \sum f_j V_j\rangle \right) dt$$

By definition of Jacobi field, $R(\dot{\gamma}, V_i)\dot{\gamma} = \ddot{V}_i$, hence the curvature term is

$$\begin{aligned}\int_0^r \left\langle R(\dot{\gamma}, \sum f_i V_i)\dot{\gamma}, \sum f_j V_j \right\rangle &= \sum_{i,j} \int_0^r f_i f_j \langle \ddot{V}_i, V_j\rangle dt = \sum_{i,j} \int_0^r f_i f_j \left(\frac{d}{dt}\langle \dot{V}_i, V_j\rangle - \langle \dot{V}_i, \dot{V}_j\rangle \right) dt \\ &= -\int_0^r \left\langle \sum_i f_i \dot{V}_i, \sum_j f_j \dot{V}_j \right\rangle dt + \langle \dot{Y}(r), Y(r)\rangle - 2\sum_{i,j} \int_0^r f_i \dot{f}_j \langle \dot{V}_i, V_j\rangle dt\end{aligned}$$

where for the second line, we integrated by part and used the fact that $\langle \dot{V}_i, V_j\rangle = \langle V_i, \dot{V}_j\rangle$ (point 3 of Remark 33). Therefore, one has

$$I(Z) = \int_0^r \|W\|^2 dt + \langle \dot{Y}(r), Y(r)\rangle.$$

In particular $I(Y) = \langle \dot{Y}(r), Y(r)\rangle \leq I(Z)$. The equality occurs if and only if $W \equiv 0$, i.e. $Z \equiv Y$. □

7.2 Local comparison with space forms

Our goal in this section is to prove the following Comparison Theorem. Before going to the precise statement, let us explain the notation.

Notation. Given M^n a Riemannian manifold and $B(p, r_0)$ be the geodesic ball centered in $p \in M$, of radius $r_0 < \delta_p$ the injectivity radius at p, equipped with the pullback metric of g via exponential map $\exp_p$, which can be expressed in polar geodesic coordinates as

$$(ds)^2 = (dr)^2 + r^2 g_{\theta^i\theta^j}(r,\theta) d\theta^i d\theta^j$$

where $\frac{\partial}{\partial\theta^1}, \dots, \frac{\partial}{\partial\theta^{n-1}}$ is an Euclidean orthonormal frame of the sphere $r\mathbb{S}^{n-1}$. We note $|g_\theta| = \det(g_{\theta^i\theta^j})_{ij}$ and $g_{\theta\theta}$ be any component $g_{\theta^i\theta^i}$ for $i = 1, \dots, n-1$.

Abusively, we say that $\frac{\sin \alpha r}{\alpha} = r$ if $\alpha = 0$ and $\sin \alpha r = \frac{1}{i} \sinh i\alpha r$ and $\cos \alpha r = \cosh i\alpha r$ if $\alpha \in i\mathbb{R}$.

Remark 36. *Note that the frame $\{\frac{\partial}{\partial\theta^i}\}_i$ may not be global, for example when n is odd (Hairy ball theorem). However the quantity $|g_\theta|$ is globally defined (except at p), in fact $|g_\theta| = r^{-2n+2}|g|$.*

Theorem 72 (comparison of volume forms)**.** *Let M^n be a Riemannian manifold with*

- *sectional curvature $-a^2 \leq K \leq b^2$*
- *Ricci curvature $\mathrm{Ric} \geq a' = (n-1)\alpha^2$ where α can be real or purely imaginary.*

Then with the notation of the last paragraph, for all $r \in (0, r_0)$,

1. *If $r < \frac{\pi}{b}$ then*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} &\geq \frac{\partial}{\partial r} \log \frac{\sin br}{r} \\ g_{\theta\theta} &\geq \left(\frac{\sin br}{br}\right)^2 \end{aligned} \tag{7.4}$$

2. *One has*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} &\leq \frac{\partial}{\partial r} \log \frac{\sinh ar}{r} \\ g_{\theta\theta} &\leq \left(\frac{\sinh ar}{ar}\right)^2 \end{aligned} \tag{7.5}$$

3. *One has*

$$\begin{aligned} \frac{\partial}{\partial r} \log \sqrt{g_\theta} &\leq (n-1)\frac{\partial}{\partial r} \log \frac{\sin \alpha r}{r} \leq -a'\frac{r}{3} \\ \sqrt{|g_\theta|} &\leq \left(\frac{\sin \alpha r}{\alpha r}\right)^{n-1} \end{aligned} \tag{7.6}$$

4. If $r < \frac{\pi}{b}$ then

$$\begin{aligned} \frac{\partial}{\partial r}\log\sqrt{g_\theta} &\geq (n-1)\frac{\partial}{\partial r}\log\frac{\sin br}{r} \\ \sqrt{|g_\theta|} &\geq \left(\frac{\sin br}{br}\right)^{n-1} \end{aligned} \tag{7.7}$$

Remark 37. *1. The moral of the estimates is that if $r \ll 1$ then the volume form of g, viewed in the tangent space at p, is equivalent to the Euclidean volume form of T_pM.*

2. One can always choose $\alpha \in i\mathbb{R}$ even when the Ricci curvature is positive, and RHS of (7.6) will be a hyperbolic function and the estimate is not as sharp as if one choose $\alpha \in \mathbb{R}$, but it works to prove that the two volume forms are equivalent when $r \ll 1$.

Remark 38. *A few consequences of Theorem 72:*

1. *For δ small, the metric volume form dV is* equivalent *to the Euclidean volume form of tangent space: there exists $C(\delta) > 0$ converging to 1 as $\delta \to 0$ such that $C(\delta)^{-1}dE \leq dV \leq C(\delta)dE$.*

2. *Let f be a smooth function defined on $B(p,\delta)$ then the gradient of f w.r.t the metric g is closed to the Euclidean gradient of f viewed in the chart (namely $f \circ \exp_p$):*

$$\begin{aligned} \|\nabla f\|_g &= \left|\frac{\partial f}{\partial r}\right|^2 + \sum_\theta \left|\frac{\partial f}{\partial \theta}(r,\theta)\right|^2 g_{\theta\theta} \\ \|\nabla (f\circ \exp_p)\|_E &= \left|\frac{\partial f}{\partial r}\right|^2 + \sum_\theta \left|\frac{\partial f}{\partial \theta}(r,\theta)\right|^2 \end{aligned}$$

3. *Combining the last 2 points, one can see that if f is supported in a small geodesic ball $B(p,\delta)$, then the L^p-norm of ∇f is closed to the Euclidean L^p norm of $\nabla(f\circ\exp_p)$ if δ is sufficiently small.*

The ideal to prove Theorem 72 comes from Proposition 70.1 and Figure 7.1. Given a point $q \in M$ of distance $r < r_0$ from p, then denote by Y the Jacobi field along the unique geodesic connecting p and q such that Y vanishes at p and $Y(r) = \frac{\partial}{\partial\theta}$ at q, then with $\psi = \exp_p$ as in Figure 7.1,

$$\begin{aligned} \|Y(r)\|^2 &= \|\psi^*_{X_0} Y(r)\|^2 = \|\psi^*_0 \dot{Y}(0)\|^2_{X_0} \\ &= r^2 g_{\theta\theta}\|\psi^*_0\dot{Y}(0)\|^2_0 = r^2 g_{\theta\theta}\|\dot{Y}(0)\|^2 \end{aligned} \tag{7.8}$$

where we used the fact that

$$g\left(\frac{\partial}{\partial\theta^i}\bigg|_{r\mathbb{S}^{n-1}}, \frac{\partial}{\partial\theta^j}\bigg|_{r\mathbb{S}^{n-1}}\right) = r^2 g\left(\frac{\partial}{\partial\theta^i}\bigg|_{\mathbb{S}^{n-1}}, \frac{\partial}{\partial\theta^j}\bigg|_{\mathbb{S}^{n-1}}\right) = r^2 g_{\theta_i\theta_j}$$

.

Take logarithm and derive (7.8) w.r.t r, using the fact that $\|Y(r)\| = 1$, one obtains

$$\langle \dot{Y}(r), Y(r)\rangle = \frac{1}{r} + \frac{\partial}{\partial r}\log g_{\theta\theta} \tag{7.9}$$

It comes to estimate $\langle \dot{Y}(r), Y(r)\rangle$, which is in fact the Index form of Y. The following lemma give an estimate of the Index form in case of bounded sectional curvature, by comparing the it with the Index form under a metric with constant sectional curvature.

Lemma 73. *Suppose that the sectional curvature $K \leq b^2$, then for every Jacobi field Y defined a long a geodesic $\gamma : [0, r] \longrightarrow M$ with $r < \frac{\pi}{2b}$ such that $Y(0) = 0, Y \perp \dot{\gamma}$. Then*

$$I(Y) \geq I_b(Y) := \int_0^r \|\dot{Y}\|^2 - b\|Y\|^2 \geq b\cot br\|Y(r)\|^2$$

Proof. By the curvature bound, $I(T) \geq \int_0^r \|\dot{Y}\|^2 - b^2\|Y\|^2 =: I_b(Y)$. The quantity $I_b(Y)$ is exactly the Index form of Y along γ if the sectional curvature in constantly b. To be precise, we equip the tubular neighborhood of γ a metric g' of constant sectional curvature $K = b^2$ such that normal vectors of γ w.r.t the metric g remain normal under g'. Such g' is in fact easy to find since:

1. The tubular neighborhood is diffeomorphic to $[0, r] \times \mathbb{B}^{n-1}$ where the diffeomorphism (says ι_1) is actually isometry at points of γ, which are mapped to $[0, r] \times \{0\}$;

2. Also, there exists a diffeomorphism ι_2 mapping $[0, r]\times\mathbb{B}^{n-1}$ to a tubular neighborhood of an arc $\tilde{\gamma}$ of length r on the grand circle of $\mathbb{S}^n_{1/b}$ which is isometry on every point of $[0, r] \times \{0\}$. This is because $r < \frac{\pi}{2b} < 2\pi\frac{1}{b}$ the length of the grand circle.

3. One now can identify a tubular neighborhood of γ in M and that of $\tilde{\gamma}$ in $\mathbb{S}^n_{1/b}$ by $\iota = \iota_2 \circ \iota_1$. Take g' to be the pullback of the Eucidean metric on $\mathbb{S}^n_{1/b}$, which is of sectional curvature b^2.

Now under the metric g', Y is no longer a Jacobi field, but it is still orthogonal to γ, denote by $\tilde{Y}$ the Jacobi field (under g') on γ that vanishes at $\gamma(0)$ and has the same value as Y at $\gamma(r)$. By Theorem 71 (Index inequality), one has $I_b(Y) \geq I_b(\tilde{Y})$. The latter can be computed directly, as the field $\iota_* \tilde{Y}$ is given by

$$s \mapsto (s, \beta^1 \sin bs, \ldots, \beta^{n-1} \sin bs), \quad s \in [0, r]$$

where $(\beta^1, \ldots, \beta^{n-1})$ is the coordinates of $\iota_{1*} Y(r)$ in $[0, r] \times \mathbb{B}^{n-1}$, hence in this coordinates (also called *Fermi coordinates*), $\tilde{Y}(s) = \left(s, \frac{\sin bs}{\sin br} Y(r)\right)$. Hence $I_b(\tilde{Y}) = b \cot br \|Y(r)\|^2$. □

Now the remaining part of the proof of Theorem 72 is straightforward.

Proof of Theorem 72. From (7.9) and Lemma 73, one has

$$\frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} = I(Y) - \frac{1}{r} \geq b \cot br - \frac{1}{r}$$

This gives the estimates in (7.4).

For (7.5), the sign situation fits Theorem 71 better, and one does not need to explicitly evoke the space forms (as Lemma 73). It suffices to see that

$$\begin{aligned}
\langle \dot{Y}(r), Y(r) \rangle = I(Y) &\leq I\left(\frac{\sinh at}{\sinh ar} Y(r)\right) \\
&\leq a^2 \left(\int_0^r \left(\frac{\cosh at}{\sinh ar}\right)^2 + \int_0^r \left(\frac{\sinh at}{\sinh ar}\right)^2 dt\right) \|Y(r)\|^2 \\
&= a \coth ar \|Y(r)\|^2
\end{aligned}$$

The estimates in (7.6) comes from the comparison between Y and the field $t \mapsto \frac{\sin \alpha t}{\sin \alpha r} Y(r)$. Note that the field is well-defined even when $\alpha \in \mathbb{R}_{>0}$ (the hyperbolic case ($\alpha \in i\mathbb{R}_{>0}$ being obvious). This in fact comes from the following fact:

Theorem 74 (Myers). *Let M^n be a connected, complete manifold with* $\mathrm{Ric} \geq (n-1)\alpha^2 > 0$ *then*

1. *M is compact.*

2. *The diameter of M is at most π/α.*

Taking sum of inequalities $I(Y_i) \leq I(\frac{\sin \alpha t}{\alpha r} Y_i(r)$ where Y_i are Jacobi fields vanishing at $\gamma(0)$ and whose values at $\gamma(r)$ are $\frac{\partial}{\partial \theta^i}$ respectively, one has

$$\begin{aligned}
\sum_{i=1}^{n-1} \langle \dot{Y}_i(r), Y_i(r) \rangle &\leq (n-1)\alpha^2 \int_0^r \left(\frac{\cos \alpha t}{\sin \alpha r} \right)^2 dt - \sum_{i=1}^{n-1} \int_0^r R_{r\theta^i r\theta^i} \left(\frac{\sin \alpha t}{\sin \alpha r} \right)^2 dt \\
&\leq (n-1)\alpha \cot \alpha r
\end{aligned}$$

where for the second line, we used the fact that $\sum_i R_{r\theta^i r\theta^i} = \mathrm{Ric}_{rr} \geq (n-1)\alpha^2$. Hence

$$\begin{aligned}
\frac{\partial}{\partial r} \log \sqrt{|g_\theta|} &= \frac{\partial}{\partial r} \sum_i \log \sqrt{|g_{\theta^i \theta^i}|} = \sum_i \langle \dot{Y}_{\theta^i}, Y_{\theta^i} \rangle - \frac{n-1}{r} \\
&\leq (n-1) \left(\alpha \cot \alpha r - \frac{1}{r} \right) = (n-1) \frac{\partial}{\partial r} \log \left(\frac{\sin \alpha r}{r} \right)
\end{aligned}$$

The proof of (7.7) is essentially the same as (7.6) where one uses (7.4) for a lower bound of $I(Y_i) = \langle \dot{Y}_i(r), Y_i(r) \rangle$. ☐

As a side note, Lemma 73 can also be used to prove that a small geodeosic ball is geodesically convex.

Proposition 74.1. *Let M^n be a Riemannian manifold with sectional curvature $K \leq b^2$ and injectivity radius $\delta > 0$. Then for every $r < \min\{\frac{\delta}{2}, \frac{\pi}{4b}\}$, any geodesic ball $B(p,r)$ is geodesically convex, i.e. any two points is connected by a geodesic curve inside the ball.*

Proof. We first claim that

Lemma 75. *Given two point p, q of distance $d(p,q) = r < \frac{\pi}{2b}$ and $\Gamma_{p,q}$ the geodesic connecting the them. Let γ be a geodesic staring from q with a velocity vector perpendicular to $\Gamma_{p,q}$, then there exists a neighborhood of q inside of which the γ intersects $\Gamma_{p,q}$ only at q.*

First, let us prove that the Lemme implies Proposition 74.1. If r small as in the Proposition and $q_1, q_2 \in B(p,r)$ then

1. There exists a minimal geodesic Γ_{q_1,q_2} connecting q_1, q_2.

2. By triangle inequality, $\Gamma_{q_1,q_2} \subset B(p, 2r)$: every point $q \in \Gamma_{q_1,q_2}$ has to be $d(q_1,q_2)/2$-closed to one q_i, hence $d(p,q) \leq d(p,q_i) + d(q_i,q) \leq r + \frac{2r}{2} = 2r$.

Let $T \in \Gamma_{q_1,q_2}$ be the point minimising the distance to p. It suffices to show that T is one of the q_i. For the sake of contradiction, if T is strictly in the interior of Γ_{q_1,q_2} then

1. The geodesic $\Gamma_{p,T}$ connecting p and T is orthogonal to Γ_{q_1,q_2} at T. It is not difficult to prove that if the two are not orthogonal then there exist $T' \in \Gamma_{q_1,q_2}$ and $S \in \Gamma_{p,T}$, both being near to T, such that $d(p,T) > d(p,S) + d(S,T') \geq d(p,T')$.

2. The ball $B(p, d(p,T)) \cap \Gamma_{q_1,q_2} \supset \Gamma_{q_1,q_2}$.

These contradict the Lemma and prove that T does not lie in the interior.

It remains to prove the Lemma. Let Y be the Jacobi field which vanishes at p and whose value at q is $\dot{\gamma}$, then by Index inequality (Theorem 71), it suffices to prove that $I(Y) > 0$, because any variation of $\Gamma_{p,q}$ by orthogonal vector field Z along γ has $I(Z) > 0$ hence only increases the length, according to Proposition 70.2. But by Lemma 73 gives

$$I(Y) \geq I_b(Y) \geq b \cot br \|Y(q)\|^2 > 0 \text{ if } r < \frac{\pi}{2b}.$$

□

7.3 Some covering lemmas

The goal of this section is to prove a covering lemma for Riemannian manifolds with injectivity radius $\delta_0 > 0$ and bounded curvature (Lemma 78). We start with a covering lemma that not yet requires curvature bound.

Lemma 76 (Calabi). *Let M^n be a Riemannian manifold with injectivity radius $\delta_0 > 0$, then for all $\delta \in (0, \delta_0)$, there exists $0 < \gamma < \beta \leq \delta$ and a partition of $M = \bigsqcup_{i \in I} \Omega_i$ and $p_i \in \Omega_i$ such that*

$$B(p_i, \gamma) \subset \Omega_i \subset B(p_i, \beta)$$

Moreover, one can choose $\gamma = \beta/10$ and $\beta = \delta$.

Proof. Note that it is enough to have

$$\begin{cases} \bigcup_i B(p_i, \beta) = M, \quad 2\gamma < \beta \\ B(p_i, 2\gamma) \text{ are disjoint} \end{cases} \tag{7.10}$$

In fact, let $\Omega_i' = B(p_i,\beta)\backslash\cup_{j\neq i}B(p_j,\gamma)$ then $\begin{cases} B(p_j,\gamma)\cap\Omega_i' = \emptyset, B(p_i,\gamma)\subset\Omega_i'\subset B(p_i,\beta) \\ \bigcup_i \Omega_i' = M \end{cases}$
(for $\bigcup_i \Omega_i' = M$: If $x\in M$ satisfies $x\in B(p_j,\gamma)\subset B(p_i,\beta)$ then there is no other $j'\neq j$ such that $x\in B(p_{j'},\gamma)$, hence $x\in\Omega_j$. Now choose

$$\Omega_1 = \Omega_1', \Omega_2 = \Omega_2'\setminus\Omega_1, \ldots, \Omega_n = \Omega_n'\setminus\cup_{i=1}^{n-1}\Omega_i, \ldots$$

For the existence of (7.10), use the following Vitali covering lemma, whose proof is purely combinatorial in nature.

Lemma 77 (Vitali covering, Infinite version)**.** *Let $\{B_j : j\in J\}$ be a collection of balls in a metric space such that*

$$\sup\{\mathrm{rad}(B_j) : j\in J\} < +\infty$$

where rad *denotes the radius, then there exists a countable subfamily $J'\subset J$ such that $\{B_j : j\in J'\}$ are disjoint and*

$$\cup_{j\in J}B_j \subset \cup_{j\in J'}5B_j.$$

It remains to apply the lemma for the covering $M = \cup_{x\in M}B(x,2\gamma)$, which also allows us to choose $\gamma = \beta/10$ and $\beta = \delta$. □

Lemma 78 (Uniformly locally finite covering)**.** *Let M^n be a Riemannian manifold with injectivity radius $\delta_0 > 0$ and bounded curvature, then for all $\delta < \delta_0$ sufficiently small, there exists a* ***uniformly locally finite covering*** *of M by balls $\{B(p_i,\delta)\}_{i\in I}$, i.e. there exists $k(\delta)\in\mathbb{Z}_{>0}$ such that for all $q\in M$, there exists a neighborhood of q that intersects at most $k(\delta)$ balls. Moreover, one can also require that $\{B(p_i,\delta/2)\}_{i\in I}$ is still a covering.*

Proof. We will apply Lemma 76 with $\beta = \delta/2$ and $\gamma = \beta/10$, then for all $\delta \ll \delta_0$, the covering $\{B(p_i,2\beta)\}$ satisfies. In fact, for every $q\in M$, take $B(q,\delta)$ as a neighborhood of q then $B(p_i,2\beta)\cap B(q,\gamma)\neq\emptyset$ if and only if $p_i\in B(q,2\beta+\gamma)$ Since the balls $B(p_i,\gamma)$ are disjoint, the number of p_i in $B(q,2\beta+\gamma)$ is bounded by

$$k = \frac{\max\mathrm{vol}_g(B_{2\beta+2\gamma})}{\min\mathrm{vol}_g(B_\gamma)} \leq C(\delta)\left(\frac{2\beta+2\gamma}{\gamma}\right)^n$$

where $\max\mathrm{vol}_g(B_{2\beta+2\gamma})$ and $\min\mathrm{vol}_g(B_\gamma)$ denote the maximum and minimum volume of balls of radius $2\beta+2\gamma$ and γ, respectively. By Theorem 72, for $\delta < \epsilon(a',b)$ depending on the bound a' and b of Ricci curvature and sectional curvature, the volume of these balls are equivalent to that of Euclidean balls of the same radius. The constant of equivalence was denoted by $C(\delta)$. □

7.4 Sobolev imbeddings for Riemannian manifolds

The goal of this section is to prove that Sobolev imbeddings are also available for complete Riemannian manifold with bounded curvature and strictly positive injectivity radius, that is, the following results.

Theorem 79 (Sobolev imbeddings)**.** *Theorem 69 holds when one replaces R^n by a complete Riemannian manifold of dimension n with bounded curvature (sectional and Ricci) and injectivity radius $\delta_0 > 0$.*

The definition of Sobolev spaces as completion of spaces of smooth functions, w.r.t the Sobolev norms generalises on Riemannian manifolds, namely, we denote by $W_0^{k,p}(M)$ the completion of $C_c^\infty(M)$ w.r.t the norm $\|\varphi\|_{W^{k,p}} = \|\varphi\|_{L^p} + \|\nabla\varphi\|_{L^p} + \cdots + \|\nabla^k\varphi\|_{L^p}$ where $\|\nabla^l\varphi\|_{L^p}$ are computed as follow: the metric g induces a fiberwise norm for l-covariant tensors, integrate that of $\nabla^l\varphi$, one obtains $\|\nabla^l\varphi\|_{L^p}$.

Similarly, the space $W^{1,p}(M)$ is defined as the completion of $C^\infty(M)$ w.r.t $\|\cdot\|_{W^{1,p}}$.

Remark 39. 1. *Unlike the Euclidean case, one does not define the derivatives term, e.g. $\nabla_v f$ for $f \in W^{1,p}(M)$ using integration by part and Riesz representation, that is, one does not expect a formular such as $\int_M (\nabla_v f)\varphi dV = -\int_M f\nabla_v\varphi dV$ since the "boundary term" $\int_M \nabla_v(f\varphi)dV$ does not vanish, even if $f\varphi \in C_c^\infty(M)$.*

2. *The exterior derivative df can be defined, which is in fact equivalent to de Rham's notion of current.*

3. *The term $\nabla^l f$ for $f \in W^{k,p}(M)$, when needed, can be defined as a L^p section of $(TM^*)^{\otimes l}$ giving by the L^p limit of smooth sections $\nabla^l\varphi_i$ for an equivalent class of Cauchy sequence φ_i representing f. The completeness of the space of L^p sections of a vector bundle follows from the result in each trivialising chart and the fact that restriction maps commute with the limit.*

Proposition 79.1 ($W^{1,p} = W_0^{1,p}$)**.** *If M is complete then $C_c^\infty(M)$ is dense in $W^{1,p}(M)$, equivalently $W^{1,p}(M) = W_0^{1,p}(M)$.*

Proof. It suffices to prove that given a function $\varphi \in C^\infty(M)$, one can approximate φ under the norm $\|\cdot\|_{W^{1,p}}$ by functions in $C_c^\infty(M)$. Fix $P \in M$, one uses a cut-off function χ_j which is 1 on $[0, j]$, 0 on $[j, \infty]$ and linear inside and defines $\varphi_j(Q) = \varphi(Q)\chi_j(d(Q, P))$. Note that the distance function is only Lipschitz and not necessarily smooth (so we did not mind taking a linear cut-off). However, since φ_j is compactly support and Lipschitz and we can approximate each φ_j by a sequence in $C_c^\infty(M)$: Let K_j be the support of φ_j and $\{\alpha_i\}_i$ be a finite partition of unity subordinating to an open coordinated cover of K. Since $\alpha_i\varphi_j$ is Lipschitz, viewed in a chart, it can be $W^{1,\infty}$-approximated by smooth functions, due to the following fact.

Fact. If $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\delta\Omega$ regular, then $\mathrm{Lip}(\Omega) = W^{1,\infty}(\Omega)$.

The approximation scheme looks like $\varphi \approx \varphi_j \approx \sum_i \alpha_{i,K_j}\varphi_j \approx \sum_i \psi_{i,j}$ where $\psi_{i,j}$ are smooth and compactly support. $\square$

Remark 40. *The similar results for higher orders are complicated, for example, one can prove that $W_0^{2,p} = W^{2,p}$ under the hypothesis of bounded curvature and strictly positive injectivity radius. The third order requires extra conditions.*

The second part of the Theorem 79 is local in nature, and therefore easier. We will prove this second part by accepting the first one, which we will come back and prove eventually.

For the imbedding into $C_B^r(M)$, it suffices to establish the case $W^{1,q} \hookrightarrow C_B^0$, the higher order case then follows: If $\varphi \in W^{k,q}$ then $\nabla^r\varphi \in W^{k-r,q} \hookrightarrow W^{k-r,q} \hookrightarrow W^{1,\tilde{q}} \hookrightarrow C_B^0$ where $\frac{1}{n} \geq \frac{1}{\tilde{q}} \geq \frac{1}{q} - \frac{k-r-1}{n}$.

Similarly, for the imbedding into $C^{r,\alpha(M)}$, it suffices to establish the case $W^{1,q} \hookrightarrow C^{0,\alpha}$ for $\frac{1-\alpha}{n} \geq \frac{1}{q}$.

Since $W^{1,p}(M) = W_0^{1,p}(M)$, it suffices to prove the following Lemma 80 and Lemma 81.

Lemma 80 ($W^{1,q} \hookrightarrow C_B^0$)**.** *Let M^n be a complete Riemannian manifold with injectivity radius $\delta_0 > 0$ and sectional curvature $K \leq b^2$, then for all $\varphi \in C_c^\infty(M)$, one has*

$$\sup_M |\varphi| \leq C(q)\|\varphi\|_{W^{1,q}}, \quad \forall q > n$$

Proof. Take $\delta < \min\{\delta_0, \frac{\pi}{2b}\}$ and let (r,θ) be the geodesic polar coordinate centered at $P \in M$, then by Theorem 72, the ratio of the metric volume form $dV := |g|dE$ and the Euclidean volume form dE of T_PM is $\sqrt{|g_\theta|} \geq \left(\frac{\sin br}{br}\right)^{n-1} \geq (\frac{2}{\pi})^{n-1}$.

let $\chi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ be a cut-off function which is constantly 1 near 0 and supported in $[0,\delta)$. Then

$$\varphi(P) = -\int_0^\delta \partial_r\left(\varphi(r,\theta)\chi(r)\right)dr, \quad \forall\theta \in \mathbb{S}^{n-1}$$

Integrate w.r.t $\theta \in \mathbb{S}^{n-1}$, recall that ω_n denotes the volume of $\mathbb{S}^{n-1}$:

$$\begin{aligned}
|\varphi(p)| &\leq (\omega_{n-1})^{-1} \int_B |\nabla(\varphi(r,\theta)\chi(r))|\, r^{1-n} r^{n-1} dr d\theta \\
&\leq (\omega_{n-1})^{-1} \left(\int_B |\nabla(\varphi(r,\theta)\chi(r))|^q \, dE \right)^{1/q} \left(\omega_{n-1} \int_0^\delta r^{(n-1)(1-q)} dr \right)^{1/q'} \\
&\leq (\frac{\pi}{2})^{n-1} (\omega_{n-1})^{-1/q} \left(\|\nabla\varphi\|_{L^q} + \sup_{[0,\delta]} |\chi'| \|\varphi\|_{L^q} \right) \left(\frac{q-1}{q-n} \delta^{\frac{q-n}{q-1}} \right)^{1/q'}
\end{aligned}$$

where q' denotes the Hölder conjugate of q and for we used Hölder inequality w.r.t dE for the second inequality and the comparison $dE \leq (\frac{\pi}{2})^{n-1} dV$ for the third. The conclusion follows. $\square$

Lemma 81 ($W^{1,q} \hookrightarrow C^{0,\alpha}$)**.** *Let M^n be a complete Riemannian manifold with injectivity radius $\delta_0 > 0$ and bounded curvature, then for all $\varphi \in C_c^\infty(M)$, one has*

$$\sup_M |\varphi| + \sup_{P \neq Q} |\varphi(P) - \varphi(Q)| \, d(P,Q)^{-\alpha} \leq C(\alpha, q) \|\varphi\|_{W^{1,q}}, \quad \textit{for all } \frac{1-\alpha}{n} \geq \frac{1}{q}$$

Proof. By Lemma 80, one can discard the term $\sup_M |\varphi|$ and only need to treat the second term of LHS. Let $\delta \leq \min\{\delta_0, \frac{\pi}{2b}\}$ as in the proof of Lemma 80 (b^2 being the upper bound of the sectional curvature). One only need to consider the case where $d = d(P,Q) < \delta/2$ because otherwise $|\varphi(P) - \varphi(Q)| \leq 2\|\varphi\|_{L^\infty} (\frac{\delta}{2})^{-\alpha} d(P,Q)^\alpha$.

Let O be the midpoint of P, Q, and denote by $h := \varphi \circ \exp_O$ defined on the Euclidean ball $B(0, 2d) \supset B_O := B(0, d/2)$. We also denote by P, Q the preimages of these points in B_O. See Figure 7.4.

Figure 7.2: Left: the picture viewed in normal polar coordinates at O. Right: the picture viewed in normal polar coordinates at Q.

Now place B_O in polar coordinate centered at Q:

$$h(x) - h(Q) = \int_0^r \frac{\partial}{\partial r} h(r,\theta) dr = r \int_0^1 \frac{\partial}{\partial \rho} h(rt, \theta) dt$$

Integrate on $B_O \ni x$ w.r.t to the measure dE_Q given by the normal polar coordinates at Q:

$$
\begin{aligned}
\int_{B_O} |h(x) - \varphi(Q)| dE_Q &\leq \int_{\theta\in\mathbb{S}^{n-1}} \int_{r=0}^{\rho(\theta)} r^{n-1} r \int_0^1 |\frac{\partial}{\partial\rho} h(rt,\theta)| dt dr d\theta \\
(u := rt, \rho(\theta) \leq d) \qquad &\leq \int_{\theta\in\mathbb{S}^{n-1}} \int_{t=0}^{1} \int_{u=0}^{td} t^{-n-1} u^n \left| \frac{\partial}{\partial\rho} h(u,\theta) \right| dt du d\theta \\
&= \int_{t=0}^{1} t^{-n-1} \left(\int_{u=0}^{td} \int_{\theta\in\mathbb{S}^{n-1}} \left| \frac{\partial}{\partial\rho} h(u,\theta) \right| u . dE_Q \right) dt \\
(\text{Holder w.r.t } dE_Q) \qquad &\leq \int_{t=0}^{1} t^{-n-1} \left(\int_{u=0}^{td} \int_{\theta\in\mathbb{S}^{n-1}} \left| \frac{\partial}{\partial\rho} h(u,\theta) \right|^q dE_Q \right)^{1/q} \left(\int_0^{td} \omega_{n-1} u^{q'} u^{n-1} du \right) \\
(t \leq 1) \qquad &\leq \int_{t=0}^{1} t^{-n-1} \left(\frac{1}{q'+n} (td)^{q'+n} \right)^{1/q'} \left(\int_{u=0}^{d} \int_{\theta\in\mathbb{S}^{n-1}} |\nabla\varphi|^q dE_Q \right)^{1/q} dt \\
&= C_1(q,n) d^{1+\frac{n}{q'}} \left(\int_{B(Q,d)} |\nabla\varphi|^q dE_Q \right)^{1/q}
\end{aligned}
\tag{7.11}
$$

Now using the fact that $\frac{1}{A} dV \leq dE_Q \leq A dV$ since the curvature is bounded, one has

$$
\int_{B(O,d/2)} |\varphi(x) - \varphi(Q)| dV \leq C_2(q,n) d^{1+\frac{n}{q'}} \|\nabla\varphi\|_{L^q}
$$

Taking sum with the same computation for P, one has

$$
|\varphi(P) - \varphi(Q)| \operatorname{vol}_g(B(O,d/2)) \leq 2C_2(q,n) d^{1+\frac{n}{q'}} \|\nabla\varphi\|_{L^q}
$$

since $\operatorname{vol}_g(B(O,d/2)) \geq A^{-1}\omega_{n-1} d^n$, one has

$$
|\varphi(P) - \varphi(Q)| \leq C_3(q,n) \|\nabla\varphi\|_{L^q} d^{1-n/q}
$$

The conclusion follows since $1 - \frac{n}{q} \geq \alpha$. $\square$

For the first part of Theorem 79, it suffices to prove the case $k = l+1$, that is, there exists a constant $C_1, C_2 > 0$ such that $\|u\|_{L^p} \leq C_1 \|\nabla u\|_{L^q} + C_2 \|u\|_{L^q}$ for $u \in W^{1,q}(M)$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.

The proof by [Aub98] tries to optimise the constant C_1, in an attempt to find the best inequality [Aub98, page 50]. We will follow their arguments, as the extra effort is not much. We will prove that

Proposition 81.1. *Given $p, q \in \mathbb{R}_{>0}$ such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{n} > 0$, for any $\epsilon > 0$, there exists $A_q(\epsilon)$ such that*

$$\|u\|_p \leq (K(n,q) + \epsilon)\|\nabla u\|_{L^q} + A_q(\epsilon)\|u\|_{L^q}$$

The appearance of the constant $K(n,q)$, given by

$$K(n,q) := \begin{cases} \frac{q-1}{n(q-1)} \left[\frac{n-q}{n(q-1)}\right]^{1/q} \left[\frac{\Gamma(n+1)}{\Gamma(n/q)\Gamma(n+1-n/q)\omega_{n-1}}\right]^{1/n}, & \text{if } q > 1 \\ \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n}, & \text{if } q = 1 \end{cases}$$

is due to the following local result.

Theorem 82 (Aubin)**.** *Given $1 \leq q < n$ and $u \in W^{1,q}(\mathbb{R}^n)$, with $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, one has*

$$\|u\|_{L^p} \leq K(n,q)\|\nabla u\|_{L^q}.$$

In fact, $K(n,q)$ this the norm of the imbedding $W^{1,q}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$.

We will accept the local result and use the Covering Lemma 78 to prove Proposition 81.1, which implies Theorem 79.

Proof of Proposition 81.1. Note that given any smooth function f supported in a small geodesic ball $B(q,\delta)$, by applying theorem 82 to the f, viewed in the chart (that is, $f \circ \exp_q$) and use the fact that $C(\delta)^{-1}\|\nabla(f \circ \exp_q)\|_{L^q(dE)} \leq \|\nabla f\|_{L^q(dV)} \leq C(\delta)\|\nabla(f \circ \exp_q)\|_{L^q(dE)}$ (see remark 38), one has

$$\|f\|_{L^p} \leq K_\delta(n,q)\|\nabla f\|_{L^q}$$

where $K_\delta(n,q)$ converges to $K(n,q)$ as $\delta \to 0$.

It suffice to cover M by geodesic ball $B(Q_i,\delta)$ such that there exists a partition of unity subordinated to $B(Q_i,\delta)$ such that $\|\nabla(h_i^{1/q})\| \leq H =$ const. In fact for $\varphi \in W^{1,q}(M)$, one has

$$\begin{aligned} \|\varphi\|_p^q &= \left(\int_M |\varphi|^p\right)^{q/p} = \left(\int_M \left(\sum_i |\varphi|^q h_i\right)^{p/q}\right)^{q/p} \\ \text{(since } p \geq q) \qquad &\leq \sum_i \left(\int_M (|\varphi|^q h_i)^{p/q}\right)^{q/p} = \sum_i \left\|\varphi h_i^{1/q}\right\|_p^q \\ &\leq K_\delta^q(n,q) \sum_i \left\|h_i^{1/q}\nabla\varphi + \varphi\nabla h_i^{1/q}\right\|_q^q \end{aligned}$$

Using the fact that there are at most $k(\delta)$ balls overlapping at a point and that $(a+b)^q = a^q\left(1+\frac{b}{a}\right)^q \le a^q(1+2^q\frac{b}{a}+2^q(\frac{b}{a})^q) \le a^q + 2^q b a^{q-1} + 2^q b^q$, one has

$$\begin{aligned}\|\varphi\|_p^q &\le K_\delta^q(n,q)\left(\|\nabla\varphi\|_q^q + 2^q k(\delta)H^{q-1}\int_M |\varphi|^{q-1}|\nabla\varphi| + 2^q k(\delta)H^q\|\varphi\|_q^q\right)\\ &\le K_\delta^q(n,q)\left[\|\nabla\varphi\|_q^q + 2^q k(\delta)H^{q-1}\|\nabla\varphi\|_q\|\varphi\|_q^{q-1} + 2^q k(\delta)H^q\|\varphi\|_q^q\right]\end{aligned}$$

It is elementary to see that this implies $\|\varphi\|_p^q \le (1+\epsilon)^q K^q(n,q)\left[(1+\epsilon)\|\nabla\varphi\|_q^q + A(\epsilon)\|\varphi\|_q^q\right]$, from which the conclusion follows.

For the existence of such h_i, one cover M by balls $B(Q_i,\delta)$ using Lemma 78. Denote by $\varphi_i : B(Q_i,\delta) \longrightarrow B(0,\delta)$ the inverse of exponential maps and let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the smooth function, choose u to be a bell curve with maximal value 1 at 0, supported in $B(0,\delta)$ and $u \le \frac{1}{2}$ in $B(0,\delta/2)$ and pose $u_i = u \circ \varphi_i$. Then

$$\|\nabla u_i\|_{g_M} \le C_1(g_M,\delta)\|\nabla u\|_E = C_2(g_M,\delta)$$

Pose $h_i = \frac{u_i^m}{\sum u_j^m}$ with $m > q$ then

$$\begin{aligned}\left|\nabla(h_i^{1/q})\right| &= \left|\frac{m}{q}\frac{u_i^{\frac{m}{q}-1}\nabla u_i}{(\sum u_j^m)^{1/q}} + u_i^{m/q}\left(\frac{-1}{q}\right)\frac{\sum\nabla(u_j^m)}{(\sum u_j^m)^{1+\frac{1}{q}}}\right|\\ &\le \frac{m}{q.2^{-m/q}}|\nabla u_i| + \frac{1}{q}\sum m\frac{|\nabla u_j|}{(2^{-m})^{1+\frac{1}{q}}}\\ &\le \left(\frac{m}{q}2^{m/q} + \frac{m}{q}2^{m(1+\frac{1}{q})}k(\delta)\right)C_2(g_M,\delta) = \text{const}\end{aligned}$$

where $k(\delta)$, as in Lemma 78, is the upper bound of number of balls overlapping at the point in question. $\square$

7.5 Kondrachov's theorem

The generalised version of Kondrachov's theorem is much easier to prove

Theorem 83 (Kondrachov)**.** *Theorem 70 holds when one replaces Ω by a compact Riemannian manifolds of dimension n.*

Proof. Cover M by finitely many small geodesic ball $B(Q_i,\delta)$ subordinating a partition of unity $\sum_{i=1}^N \chi_i = 1$, then if a sequence $\{u_n\}_n \subset W^{k,q}$ is bounded then $\{\chi_i u_n\}_n$ is also bounded in $W^{k,q}$. The conclusion follows using Remark 38 and the Euclidean version of Kondrachov's theorem. $\square$

7.6 Solving $\Delta u = f$ on a Riemannian manifold.

With Kondrachov's theorem 83, one can uses the familiar "subsequence extracting" technique to find a minimiser of the quadratic functional $\psi \mapsto \frac{1}{2}\int_M \|\nabla\psi\|^2 dV$ in a suitable subspace of $W^{1,2}(M)$ (method of Lagrange multiplier), one can prove the following results.

Theorem 84 (Spectrum of Δ)**.** *Let M^n be a compact Riemannian manifold then*

1. *The eigenvalues of $\Delta - \nabla^\nu\nabla_\nu$ are ≥ 0.*
2. *The eigenfunctions of $\delta_0 = 0$ are constant functions.*
3. *The eigenvalue λ_1 is the minimum value of the functional*

$$\psi \mapsto \frac{1}{2}\int_M \|\nabla\psi\|^2 dV$$

on the subspace $\left\{\psi \in W^{1,2}(M) : \ \|\psi\|_2 = 1, \int \psi dV = 0\right\}$. Moreover, first eigenfunctions are smooth.

Theorem 85. *Given M^n be a compact Riemannian manifold, consider the Laplace equation on M:*

$$\Delta u = f \tag{7.12}$$

where $f \in L^2(M)$, then:

1. *There exists $u \in W^{1,2}(M)$ satisfying (7.12) in the weak sense if and only if $\int_M f dV = 0$*
2. *u is unique up to an additive constant.*
3. *If $f \in C^{r,\alpha}$ then $u \in C^{r+2,\alpha}$.*

Chapter 8

Parametrix and Green's function of Laplacian operator on Riemannian manifolds

Recall that in the Euclidean space $\mathbb{R}^n$, one obtains a representation of the solution u of equation $\Delta u = f$ by

- first solving for an explicit radial solution of $\Delta G = \delta_0$. In particular, $G = [(n-2)\omega_{n-1}]^{-1} r^{2-n}$ if $n > 2$ and $G = -(2\pi)^{-1} \log(r)$ if $n = 2$
- then tensoring G by f, one has the solution $u = G * f$ of $\Delta u = f$

To generalise this argument for Riemannian manifolds, there are a few points that have to be modified:

1. Since it does not make sense to add/substract points of a manifold, one will need to find different fundamental solutions for different points, so instead of fundamental solution, we will find the Green's function $G = G(p,q)(p,q \in M)$. The convolution will be replace by the following operation on functions X, Y defined on $(M \times M) \setminus \Delta_M$ where Δ_M denotes the diagonal:

$$(X * Y)(p,q) = \int_M X(p,r)Y(r,q)dV(r)$$

.

2. The distance function $q \mapsto d(p,q)$ is only smooth near p, outside of the cut-locus, the best one can says is that the function is Lipschitz. Since

cut-loci are almost impossible to calculate or visualise (the cut-locus of an ellipsoid is still a conjecture, according to [Ber03]), one will cut-off the Euclidean solution, try to solve the equation near p and later add a correcting term. This inspires the definition of parametrix.

3. Another reason that we have to approximate the exact solution by parametrix, that also explain the iteration in Theorem 87, is that the expression of Laplacian, even in the geodesic polar coordinate and even near the origin, involves the metric, hence the Euclidean fundamental solution is not yet a solution even near the origin.

Remark 41. *To give a simplified analogy of what we will be doing, let us prove the existence of "Green's function" on Riemann surfaces (with boundary, so that we do not have to deal with the volume). The "Laplace equation" is*

$$-2i\partial\bar{\partial}g = \delta_0 \tag{8.1}$$

where the LHS is a 2-form and the RHS is a generalised 2-form in the sense of current. Contrary to the previous point 3, one knows the exact local solution of (8.1)*, namely* $z \mapsto -(2\pi)^{-1}\log(|z|)$*. Therefore, the argument will be simplified as:*

- *Given a holomorphic chart of a point* $0 \in M$*, pose* $h(z) := -(2\pi)^{-1}\log(|z|)\chi(|z|)$ *where* χ *is a cut-off function that is* 1 *on a neighborhood of* 0
- *The 2-form* $\alpha = -2i\partial\bar{\partial}h$ *is well-defined everywhere except* 0*, and vanishes on a neighborhood of* 0*. Denote by* α^{naiv} *its extension to* M*.*
- *Recall the fact that every smooth 2-form on a compact, connected, Riemann surface with boundary can be writen as* $\alpha^{\text{naiv}} = -2i\partial\bar{\partial}\varphi$*, pose* $g = h - \varphi$*.*

For Riemann surface without boundary, the equation is $-2i\partial\bar{\partial}g = \delta_0 - 2i\int_M \partial\bar{\partial}g$ *and the fact to evoke is that any smooth 2-form* α *with* $\int_M \alpha = 0$ *is of form* $\alpha = -2i\partial\bar{\partial}\varphi$

We will suppose that M^n is a Riemannian manifold with injectivity radius $\delta_0 > 0$, and of bounded curvature. Compact manifolds, for example, fall in this category.

8.1 Parametrix and the Green's formula

Definition 20. *A* ***Green's function*** $G(p,q)$ *of a compact Riemannian manifold is a function defined on* $(M \times M) \setminus \Delta_M$ *such that*

1. *$\Delta_q^{\mathrm{dist}} G(p,q) = \delta_p(q)$ if M has boundary.*

2. *$\Delta_q^{\mathrm{dist}} G(p,q) = \delta_p(q) - V^{-1}$*

where Δ_q^{dist} concerns the distribution derivatives and V is the volume of M.

*Let $p, q \in M$ be distinct points, the **parametrix** H is defined by*

$$H(p,q) = \begin{cases} [(n-2)\omega_{n-1}]^{-1} r^{2-n}\chi(r), & \text{if } n > 2 \\ -(2\pi)^{-1}\chi(r)\log r, & \text{if } n = 2 \end{cases}$$

where $r = d(p,q)$, $\chi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ is smooth, $\chi = 1$ in a neighborhood of 0 and $\chi(t) = 0$ if $t > \delta_0$.

Recall that in the geodesic polar coordinates, i.e. the polar coordinates on the tangent T_pM at $p \in M$, identified with a neighborhood of $p \in M$, the metric g is given by

$$g : ds^2 = dr^2 + r^2 g_{\theta_i\theta_j}(r,\theta) d\theta^i d\theta^j$$

and one denotes $|g_\theta| := \det(g_{\theta_i\theta_j})$, therefore $|g| = \det(g_{ij}) = r^{2(n-1)}|g_\theta|$

Lemma 86. *If a function $\varphi \in C^2$ defined locally around $p \in M$ and φ is radial, i.e. $\varphi = f(r)$ in a small geodesic ball $B(p,\delta)$ then*

$$-\Delta\varphi = f'' + \frac{n-1}{r} f' + f' \partial_r \log\sqrt{|g_\theta|}$$

Proof. One has

$$\begin{aligned} \Delta\varphi &= -\operatorname{Tr}\Big(\nabla_i(g^{kj}\partial_j\varphi\; e_k)\Big)_{i,k} = -\partial_i(g^{ij}\partial_j\varphi) - g^{kj}\partial_j\varphi\Gamma^i_{ik} \\ &= -|g|^{-1/2}\partial_i(g^{ij}|g|^{1/2}\partial_j\varphi) \end{aligned}$$

since $\Gamma^i_{ik} = \partial_k \log\sqrt{|g|} = \frac{\partial_i|g|}{2|g|}$. One concludes by substituting $|g| = r^{2n-2}|g_\theta|$ and noticing that $g^{r\theta_i} = g^{\theta_i\theta_j} = 0$ $(i \neq j)$. $\square$

Remark 42. *1. The Laplacian of the metric g, viewed in polar geodesic coordinates centered at p, i.e. in the tangent space T_pM is not the Euclidean Laplacian of T_pM, however the difference if $O(r)$ since $\partial_r \log\sqrt{|g_\theta|} \leq Ar$ where the bound A is given by Ricci curvature, see the Volume comparison theorem.*

2. *Applied the formula for $q \mapsto H(p,q)$, one has*
$$\Delta_q^{\mathrm{naiv}} H(p,q) = [(n-2)\omega_{n-1}]^{-1} r^{1-n} \left((n-3)\chi' - r\chi'' + ((n-2)\chi - r\chi')\partial_r \log \sqrt{|g_\theta|}\right) \tag{8.2}$$
therefore $\Delta_q^{\mathrm{naiv}} H(p,q) \leq B r^{2-n}$ where B does not depend on p.

3. *Unlike the case of Remark 41 where we know the exact fundamental solution and the form α^{naiv} has no singularity, there is no reason for that this holds true for $\Delta_q^{\mathrm{naiv}} H(p,q)$. However, we proved that the order of singularity at $q = p$ can be controlled.*

Proposition 86.1 (Green's formula)**.** *For any function $\psi \in C^2(M)$, one has*
$$\psi(p) = \int_M H(p,q)\Delta\psi(q)dV(q) - \int_M \Delta_q^{\mathrm{naiv}} H(p,q)\psi(q)dV(q) \tag{8.3}$$
where $\Delta_q^{\mathrm{naiv}} H(p,q)$ denotes the pointwise derivative of $H(p,q)$, not the distribution derivative.

Remark 43. 1. *In other words, the theorem says that $\Delta_q^{\mathrm{dist}} H(p,q) = \Delta_q^{\mathrm{naiv}} H(p,q) + \delta_p(q)$ where Δ_q^{dist} is the distribution derivative. In particular, if there is no concern about regularity of the distance function $d(p,q)$ (as in the Euclidean case), allowing us to take the cut-off function $\chi = 1$ in the definition of parametrix, then $\Delta_q^{\mathrm{naiv}} H(p,q) = 0$ and $\Delta_q^{\mathrm{dist}} H(p,q) = \delta_p(q)$ which is not a surprise since $H(p,q)$ is also the Green's function.*

2. *Taking $\psi = 1$, one has*
$$\int_M \Delta_q^{\mathrm{naiv}} H(p,q) = -1$$

3. *Multiplying (8.3) by $\phi(p)$ and integrate over M, one has*
$$\int_M \phi(q)\psi(q)dV(q) = \int_M \left(\int_M H(p,q)\phi(p)dV(p)\right)\Delta\psi(q)dV(q) - \int_M \left(\int_M \Delta_q^{\mathrm{naiv}} H(p,q)\phi(p)\right)$$
hence in distribution sense
$$\phi(q) = \Delta_q \int_M H(p,q)\phi(p)dV(p) - \int_M \Delta_q^{\mathrm{naiv}} H(p,q)\phi(p) \tag{8.4}$$
The equation (8.4) is called the transposition of equation (8.3) and what we have just done is a rigourous proof of the following heuristic justification of (8.4): "Take the derivative Δ_q inside the integral, then use $\int_M \delta_p(q)\phi(p)dV(p)$ "=" $\phi(q)$ ".

Proof. The intuition is clear:

- since one only modifies the fundamental solution at points q far from p, one only needs to recompense by $\Delta_q^{\text{naiv}} H(p,q)$
- there may be trouble near p caused by the difference between the Euclidean Laplacian and the metric Laplacian, however as explained by Remark 42, this difference is $O(r)$ as $r \to 0$.

For a rigorous proof, one calculates $\int_M H(p,q)\Delta\psi(q)dV(q)$ by decomposing M to $B(p,\epsilon)$ and $M \setminus B(p,\epsilon)$ with $0 < \epsilon < \delta_0$ tending to 0 eventually, then

$$\begin{aligned}\int_{M\setminus B(p,\epsilon)} H(p,q)\Delta\psi(q)dV(q) &= \int_{M\setminus B(p,\epsilon)} \left(\Delta_q^{\text{naiv}} H(p,q)\psi(q) + d(\psi \wedge *dH - H \wedge *d\psi)\right) dV(q) \\ &= \int_{M\setminus B(p,\epsilon)} \Delta_q^{\text{naiv}} H(p,q)\psi(q)dV(q) + \int_{\partial B(p,\epsilon)} (\psi \wedge *dH - H \wedge *d\psi)dV(q)\end{aligned}$$

by Stokes' theorem, where $*$ denotes the Hodge star. Therefore

$$\int_M H(p,q)\Delta\psi(q)dV(q) = \int_M \Delta_q^{\text{naiv}} H(p,q)\psi(q)dV(q) + I_1 + I_2$$

where $I_1 = \lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}(\psi\wedge*dH - H\wedge*d\psi)$ and $I_2 = \lim_{\epsilon\to 0}\int_{B(p,\epsilon)} H(p,q)\Delta\psi(q)dV(q)$.

Now $I_2 = \psi(p)$ since $(\frac{\sin(b\epsilon)}{b\epsilon})^{n-1} \le dV/dE \le (\frac{\sin(\alpha\epsilon)}{\alpha\epsilon})^{n-1}$ in $B(p,\epsilon)$ by Volume comparison theorem where b^2 is an upper bound of sectional curvature and $(n-1)\alpha^2$ is a lower bound of Ricci curvature ($\alpha \in \mathbb{C}$), and since $\Delta\psi(q) - \Delta_E\psi(q) = O(\varepsilon)$ in $B(p,\epsilon)$ where Δ_E is the Euclidean Laplacian.

For I_1, with ϵ small enough such that $\chi = 1$, one has $|H \wedge *d\psi| \le \text{const}\,\epsilon^{2-n}(*d\psi)$. By straightforward computation:

$$\begin{aligned} dH &= -\omega_{n-1}^{-1} r^{1-n} dr, \quad dV = r^{n-1}\sqrt{|g_\theta|}dr \wedge d\theta^1 \wedge \cdots \wedge d\theta^{n-1} \\ *dH &= -\omega_{n-1} r^{2n-2}\sqrt{|g_\theta|}d\theta^1 \wedge \cdots \wedge d\theta^{n-1}\end{aligned}$$

hence $\int_{\partial B(p,\epsilon)} H \wedge *d\psi = O(\epsilon)$ and $\int_{\partial B(p,\epsilon)} \psi \wedge *dH = O(\epsilon^{2n-2})$. Therefore $I_1 = 0$ and the conclusion follows. □

8.2 Existence of Green's function on compact Riemannian manifolds

Our goal is to prove the following theorem

Theorem 87 (Existence of Green's function)**.** *Let M^n be a compact Riemannian manifold without boundary, there exists a Green's function $G(p,q)$ of the Laplacian such that*

1. Green's function. *For all $\varphi \in C^2(M)$,*
$$\varphi(p) = V^{-1}\int_M \varphi(q)dV(q) + \int_M G(p,q)\Delta\varphi(q)dV(q) \tag{8.5}$$
2. Smooth. *$G \in C^\infty((M \times M) \setminus \Delta_M)$.*
3. Radial estimates. *There exists a constant k such that*
$$|G(p,q)| \leq \begin{cases} k(1+|\log r|), & \text{if } n = 2 \\ kr^{2-n}, & \text{if } n > 2 \end{cases} \tag{8.6}$$
for $r = d(p,q)$. Moreover, one has the derivative estimates:
$$|\nabla_q G(p,q)| \leq kr^{1-n}, \qquad \left|\nabla_q^2 G(p,q)\right| \leq kr^{-n}, \tag{8.7}$$
4. *G is bounded below. Since G is defined upto a constant, one can choose the constant so that $G > 0$.*
5. Constant integral. *The integral $\int_M G(p,q)dV(p)$ is constant in q. Since G is defined upto a constant, one can choose the constant so that $\int_M G(p,q)dV(p) = 0$.*
6. Symmetric. *$G(p,q) = G(q,p)$ for $p \neq q$ in M.*

For a better notation, let us replace $\Delta_p U(p,q)$ by $\Delta_2 U(p,q)$. Recall that we already know how to solve the equation $\Delta u = f$ for $f \in L^2(M)$, this means we can solve $\Delta_2 U(p,q) = f_p(q)$ for double-integrable functions f_p, or briefly we can *solve* L^2 functions. Now, define
$$(X * Y)(p,q) := \int_M X(p,r)Y(r,q)dV(r)$$
if the integration is possible and if it commutes with derivation, one has
$$\Delta_2(F_1 * H) = F_1 * \Delta_2^{\text{dist}} H = F_1 + F_1 * \Delta_2^{\text{naiv}} H$$
So if one can *solve* $F_1 * \Delta^{\text{naiv}} H$, then one can *solve* F_1, i.e. if $\Delta_2 E_2 = F_1 * \Delta_2^{\text{naiv}} H$ then take $E_1 := F_1 * H - E_2$, one has $\Delta_2 E_1 = F_1$.

Now in order to prove that $\delta_\Delta - V^{-1}$ can be solved, it remains to check that
$$\delta_\Delta * (\Delta_2^{\text{naiv}} H)^{*k} \in L^2(M) \quad \text{for } k \gg 1. \tag{8.8}$$
This is the content of the following lemma.

Lemma 88. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $X, Y : (\Omega \times \Omega) \setminus \Delta_\Omega \longrightarrow \mathbb{R}$ be continuous functions such that*

$$|X(p,q)| \leq \operatorname{const} d(p,q)^{\alpha-n}, \qquad |Y(p,q)| \leq \operatorname{const} d(p,q)^{\beta-n}, \qquad \alpha, \beta \in (0,n)$$

then

$$Z(p,q) := \int_\Omega X(p,r)Y(r,q)dV(r)$$

is continuous in $(\Omega \times \Omega) \setminus \Delta_\Omega$ and

$$|Z(p,q)| \leq \begin{cases} \operatorname{const} d(p,q)^{\alpha+\beta-n}, & \text{if } \alpha+\beta < n \\ \operatorname{const}(1 + |\log d(p,q)|), & \text{if } \alpha+\beta = n \\ \operatorname{const}, & \text{if } \alpha+\beta > n \end{cases}$$

In the case $\alpha+\beta > n$, Z admits a continuous extension to $\Omega \times \Omega$. The result also holds for compact Riemannian manifolds.

Proof. It suffices to consider p, q closed to each other. Let $d(p,q) = 2\rho$. Decompose $\Omega = (\Omega \cap B(p,\rho)) \cup (\omega \setminus B(q,3\rho)) \cup \Omega \cap (B(q,3\rho) \setminus B(p,\rho))$, then

$$\left| \int_{\Omega \cap B(p,\rho)} X(p,r)Y(r,q)dV(r) \right| \leq C\rho^{\alpha+\beta-n}$$

$$\left| \int_{\Omega \cap B(q,3\rho) \setminus B(p,\rho)} X(p,r)Y(r,q)dV(r) \right| \leq C\rho^{\alpha+\beta-n}$$

$$\left| \int_{\Omega \setminus B(q,3\rho)} X(p,r)Y(r,q)dV(r) \right| \leq C \int_\rho^D \frac{dr}{r^{n-\alpha-\beta-1}}$$

where D is the diameter of Ω. For compact Riemannian manifold, take $\rho \ll \delta_0$, the injectivity radius and use Comparison theorem, one has the same estimates. $\square$

Back to the proof of Theorem 87, one can see that it suffices to choose $k > \frac{n}{2}$ in (8.8). The rigorous proof is given below.

Proof of Theorem ref:thm:existence-green. Carefully do the algebraic part of the above argument, one poses

$$G(p,q) = H(p,q) + \sum_{i=1}^{k-1} (-\Delta_2^{\text{naiv}} H)^{*i} * H + F_k(p,q)$$

where $F_k(p,q)$ satisfies

$$\Delta_2 F_k(p,q) = (-\Delta_2^{\text{naiv}} H)^{*k} - V^{-1}$$

This is possible if one chooses $k > n/2$ since by repeated application of Lemma 88, $(-\Delta_2^{\text{naiv}} H)^{*k}$ is continuous. By regularity result of equation $\Delta u = f$, the function $q \mapsto F_k(p,q)$ is in $C^2(M \setminus \{p\})$. Each function $F_k(p,\cdot)$ is uniquely defined up to a constant, choose the constant such that $\int_M G(p,q)dV(q) = 0$, then the function $p \mapsto \int_M F_k(p,q)dV(q)$ is continuous. The condition 1) of the Theorem can be verified without difficulty. Moreover, since $\Delta_2 G(p,q) = 0$ if $q \neq p$, the function $q \mapsto G(p,q)$ is C^∞.

We will prove such $G(p,q)$ satisfies the statements 2-6, starting from a weaker form 2-) of 2), that is we will prove that $p \mapsto G(p,q)$ is continuous, then using this, we will prove 3-6, and eventually come back to prove 2 completely.

For 2-) we will use the following fact:

Fact. If $\Delta u = f$ and $f \in C^0(M)$ (hence $u \in C^2(M)$ and $\int_M u = 0$, then one has $\sup |u| \leq C \sup |f|$ where $C > 0$ is a constant.

Denote $\Gamma_i := (-\Delta_2^{\text{naiv}} H)^{*i}$ and apply the result for $u = F(p,\cdot) - V^{-1}\int_M F(p,q)dV(q)$ and $f = \Gamma_k(p,\cdot)$, one has

$$\sup \left| F(p,\cdot) - F(r,\cdot) - V^{-1}\int_M (F(p,\cdot) - F(r,\cdot)) \right| \leq C \sup_q |\Gamma_k(p,q) - \Gamma_k(r,q)|$$

Then the continuity of $p \mapsto F(p,\cdot)$ under C^0 topology is given by

- $p \mapsto \int_M F(p,\cdot)$ is continuous by the previous choice of constant.
- The uniform continuity of Γ_k on $M \times M$, which is the result of its continuity and the compactness of $M \times M$.

Hence $p \mapsto G(p,q)$ is continuous on $M \setminus \{q\}$ for all $q \in M$.

For 3), fix $p \in M$ and let $r = d(p,q)$ be small, then $H(p,q) = O(r^{2-n}$, $(\Gamma_i * H)(p,q) = O(r^{2i+2-n})$ by Lemma 88 and $F(p,q) = O(1)$ if $n > 2$. Hence $G(p,q) = O(r^{2-n})$, where here the constant in $O(r^{2-n})$, if checked carefully, does not depend on p. The case $n = 2$ can be treated similarly. For the derivative estimates, note that $\nabla_q G(p,q) = \nabla_q H(p,q) + \sum_{i=1}^{k-1}(\Gamma_i * \nabla_2 H)(p,q) + \nabla_q F(p,q)$ and $\nabla_q^2 G(p,q) = \nabla_q^2 H(p,q) + \sum_{i=1}^{k-1}(\Gamma_i * \nabla_2^2 H)(p,q) + \nabla_q^2 F(p,q)$ where the commutative of derivation and integration can be justified by Lebesgue's Dominated convergence. In both case, the dominant terms as $q \to p$ are $\nabla_q H(p,q)$ and $\nabla_q^2 H(p,q)$ respectively, which is $O(r^{1-n})$ and $O(r^{-n})$ where the constants in big-O do not depend on p.

For 4), note that $H(p,q)$ is the dominant term of $G(p,q)$ as $q \to p$ and $H(p,q) > 0$, one see that $G(p,q) > 0$ in a neighborhood of Δ_M. By the compactness of M and the continuity of G outside of Δ_M, one sees that G is bounded below.

To prove 5), take to transposition of (8.5), i.e. multiply by $\psi(p)$ and integrate, as in Remark 43, one obtains

$$\Delta_q \int_M G(p,q)\psi(p)dV(p) = \psi(q) - V^{-1}\int_M \psi(p)dV(p) \tag{8.9}$$

Substitute $\psi = 1$, one sees that $q \mapsto \int_M G(p,q)dV(p)$ is harmonic on M, hence is constant by compactness of M.

We will now prove 6). It follows from (8.5) that

$$\Delta_q \int_M G(p,q)\psi(q)dV(q) = \Delta_q \psi(q) \tag{8.10}$$

Also, from the transposition (8.9), replace ψ by $\Delta\psi$, one has

$$\Delta_q \int_M G(p,q)\Delta\psi(p)dV(p) = \Delta_q \psi(q)$$

Swap p and q and subtract to (8.10), one has

$$\Delta_p \int_M \left(G(p,q) - G(q,p)\right)\Delta\psi(q)dV(q) = 0$$

Hence $\int_M \left(G(p,q) - G(q,p)\right)\Delta\psi(q) = C$ const. Integrate by $p \in M$ and use the fact that we chose $\int_M G(q,p)dV(p) = 0$, one has $C = 0$, meaning that $\Delta_q \left(G(p,q) - G(q,p)\right) = C(p)$, being independent of q. By swapping p, q, one has $C(p) = -C(q)$ for all $p \neq q$. Since M contains more than 3 points, these constants are 0. Hence $G(p,q) = G(q,p)$.

Now coming back to 2), since $G(p,q) = G(q,p)$, we see that $p \mapsto G(p,q)$ is C^∞ for all $q \in M$. It remains to prove that $p \mapsto \nabla_q^h G(p,q)$ is continuous on $M \setminus \{q\}$, then Schwarz's lemma applies. For that, one may try the following argument:

$$\Delta_p \nabla_q^h G(p,q) = \nabla_q^h \Delta_p G(p,q) = 0, \quad p \in M \setminus [q\}$$

hence $p \mapsto \nabla_q^h G(p,q)$ is C^∞. It is however difficult to justify the commutativity of derivations, which is equivalent to

$$\int_M \nabla_q^h G(p,q)\Delta\varphi(p)dV(p) = \nabla_q^h \int_M G(p,q)\Delta\varphi(p)dV(p), \tag{8.11}$$

that is the ability to derive in the integral sign. A justification for this can be done in the case $h \leq 2$ using estimates of 3).

A simpler way is to note that it suffices to prove the continuity of $p \mapsto \nabla_q^h G(p,q)$ for p in a small open set V with $\bar{V}$ not containing q. Then claim that $\Delta_p \nabla_q^h G(p,q) = \nabla_q \Delta_p G(p,q) = 0$ as distributions on V, which is equivalent to (8.11) for all test functions φ with $\operatorname{supp} \varphi \in V$. Then Dominated convergence applies since $|\nabla_q^{h+1} G(p,q)| \leq C d(q, \bar{V})^{1-n-h}$ hence is bounded. $\square$

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