Global existence for nonlinear heat equation and harmonic maps between Riemannian manifolds

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Contents

Let M be a compact Riemannian manifold. We want to solve the following nonlinear heat equation where $F : M \longrightarrow M' \subset B \subset V = \mathbb{R}^N$:

$$\frac{dF_t}{dt} = -\Delta F_t + \Gamma(F_t)(\nabla F_t)^2$$

We have proved that the solution exists in short-time and is smooth whenever it exists. We will now establish long-time existence using continuity method, that is we will show that if the solution exists on $[\alpha, \omega_n]$ where ω_n is an increasing sequence to ω, then the solution exists on $[\alpha, \omega]$. We then apply short-time existence to gain a small open interval where solution still exists. We then conclude that the solution exists globally on $[\alpha, +\infty)$ since this interval is connected.

The crucial step to prove that the solution can be extended on $[\alpha, \omega]$ is to uniformly bound all of its derivatives in time of evolution $[\alpha, \omega)$. These estimates will also be useful to justify the convergence of F_t in $C^\infty(M)$ to a smooth function F_∞ which will eventually be a harmonic map from M to M'.

Recall that we proved in Corollary **??** the boundedness of $\|F_t\|_{W^{2,2}(M)}$ by a constant C depending only on curvatures of M, M' and the initial total energies. Since $\frac{dF_t}{dt}$ relates to spatial derivatives of F by the nonlinear heat equation, it is easy to see that $\|F_t\|_{W^{2,2}(M\times[\tau,\tau+\delta])}$ is bounded by a constant independent of τ. Again, we will denote $W^{k,p}(M \times [\beta, \gamma])$ by $W^{k,p}([\beta, \gamma])$.

Theorem 1 ($W^{2,2}$-boundedness). *There exist a constant C depending only on δ, the metrics and initial total energies such that*

$$\|F\|_{W^{2,2}(\tau,\tau+\delta)} \leq C \quad \text{for all } \alpha \leq \tau < \omega - \delta.$$

Proof. Since

$$\|F\|^2_{W^{2,2}([\tau,\tau+\delta])} \leq \int_\tau^{\tau+\delta} \|F_t\|^2_{W^{2,2}(M)} dt + 2\int_\tau^{\tau+\delta} \|\Delta F_t\|^2_{L^2} dt + 2\int_\tau^{\tau+\delta} \left\|\Gamma(F_t)(\nabla F_t)^2\right\|^2_{L^2} dt$$

The first term and the second term are bounded by $C^2\delta$, the third one, since $\Gamma(F_t)$ is bounded, by $C^2\delta$ where C is a constant only depending on the metrics and initial total energies. □

The estimates of higher derivatives of F will be established in the following order: first in $W^{2,p}$ for all p norm then in $W^{k,p}$ for all k, p, then in C^∞.

1 Estimate of higher derivatives.

Lemma 2 ($W^{2,p}$-boundedness). *For all $p \in (1, +\infty)$, there exists a constant $C > 0$ depending only on δ, p, the metrics and initial energies such that for all $\alpha + \delta \leq \tau \leq \omega - \delta$:*

$$\|F\|_{W^{2,p}([\tau,\tau+\delta])} \leq C$$

Proof. Applying Gårding Inequality to the parabolic equation $AF = \Gamma(F)(\nabla F)^2$ where $A := \frac{\partial}{\partial t} + \Delta$ is the heat operator, one has

$$\|F\|_{W^{2,p}([\tau,\tau+\delta])} \leq C\left(\|\Gamma(F)(\nabla F)^2\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} + \|F\|_{W^{2,2}([\tau-\frac{\delta}{3},\tau+\delta])}\right)$$

The second term of RHS is already bounded by applying Theorem 1 to $\frac{4\delta}{3}$. For the first term:

$$\left\|\Gamma(F)(\nabla F)^2\right\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \leq C(M')\||\nabla F|^2\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} = C(M')\|e(F)\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])}.$$

Recall that, by Theorem **??**, the potential density satisfies $\frac{de}{dt} + \Delta e - Ce \leq 0$ for certain constant C depending only on the metric of M, by Maximum principle (Theorem **??**), one has $e \leq \psi_\tau$ where ψ_τ is the solution of $\begin{cases} \frac{d}{dt}\psi_\tau + \Delta\psi_\tau - C\psi_\tau = 0 \\ \psi_\tau\big|_{\tau-\frac{\delta}{2}} = e\big|_{\tau-\frac{\delta}{2}} \end{cases}$ We apply Gårding Inequality again for ψ_τ and obtain

$$\|e(F)\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \leq \|\psi_\tau\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \leq C\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2},\tau+\delta])}. \qquad (1)$$

Now apply L^1-Comparison Theorem **??** to ψ_τ, one has

$$\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2},\tau+\delta])} \leq \int_0^{3\delta/2} \|\psi_\tau\big|_{\tau-\frac{\delta}{2}}\|_{L^1} e^{Bt} dt \leq \int_0^{3\delta/2} e^{Bt} dt . \|e\big|_{\tau-\frac{\delta}{2}}\|_{L^1} \leq C. \tag{2}$$

The lemma follows from (1) and (2). □

We can now estimate higher order derivatives.

Theorem 3 ($W^{k,p}$-boundedness)**.** *For all $p \in (1,+\infty)$ and $k < +\infty$, there exists C depending only on k, p, the metrics and initial energies such that*

$$\|F\|_{W^{k,p}([\tau,\tau+\delta])} \leq C$$

for all $\alpha + \delta \leq \tau \leq \omega - \delta$.

Proof. Applying Gårding Inequality to the equation $\frac{dF}{dt} + \Delta F_t = \Gamma(F)(\nabla F)^2$ then Regularity Theorem **??** for polynomial differential operator, one has for $\epsilon \ll \delta$:

$$\begin{aligned} \|F\|_{W^{k,p}([\tau,\tau+\delta])} &\leq C_\epsilon \left(\|F\|_{W^{2,p}([\tau-\epsilon,\tau+\delta])} + \|\Gamma(F)(\nabla F)^2\|_{W^{k-2,p}([\tau-\epsilon,\tau+\delta])} \right) \\ &\leq C_\epsilon \left(1 + C \left(1 + \|F\|_{W^{s,q}([\tau-\epsilon,\tau+\delta])} \right)^{q/p} \right) \end{aligned}$$

as long as $k-1 < s$ and $\frac{1}{p} > \frac{k}{s}.\frac{1}{q}$. Therefore if $\|F\|_{W^{s,q}([\tau,\tau+\delta])} \leq C(\delta,s,q)$ for all $\beta \leq \tau \leq \omega - \delta$ and $q \in (1,+\infty)$, we just proved that

$$\|F\|_{W^{k,p}([\tau,\tau+\delta])} \leq C(\epsilon,k,p)$$

for all $\begin{cases} \beta + \epsilon \leq \tau \leq \omega - \delta \\ k < s+1, p \in (1,+\infty) \end{cases}$ since $\|F\|_{W^{s,q}([\tau-\epsilon,\tau+\delta])} \leq 2C(\delta,s,q)$.

One can then conclude by induction on k, with step $\frac{1}{2}$, starting with $k = 2$ and $\epsilon = \frac{\delta}{2}$ and each time dividing ϵ by 2. □

2 Global existence for nonlinear heat equation.

Theorem 4 (Global existence)**.** *The solution of nonlinear heat equation*

$$\frac{dF}{dt} = -\Delta F + \Gamma(F)(\nabla F)^2 \tag{3}$$

with smooth initial condition exists globally for all time $t > \alpha$.

Proof. Let F_n be a sequence of solution of (3) on $[\alpha, \omega_n]$ with ω_n increasing to ω then they coincide by uniqueness of solution the equation. As discussed in the beginning of this part, it is sufficient to prove that the solution extends to $[\alpha, \omega]$. Let F be the solution on $[\alpha, \omega)$ such that $F\big|_{[\alpha,\omega_n]} = F_n$, then by Theorem 3, for all $\tau \in [\alpha, \omega - \delta)$:

$$\|D_t^u D_x^v F\|_{L^\infty(M\times[\tau,\tau+\delta])} \leq C_{\text{Sobolev}} \|D_t^u D_x^v F\|_{W^{k,p}(M\times[\tau,\tau+\delta])} \leq C_{\text{Sobolev}}.C(k,p,\delta)$$

where, if we choose k sufficiently large, C_{Sobolev} is the constant of Sobolev imbedding $W^{k,p}(M \times [0,\delta]) \hookrightarrow C(M \times [0,\delta])$ and $C(k,p,\delta)$ is the constant provided by Theorem 3.

So all partial derivatives of F is uniformly bounded on $[\alpha + \delta, \omega)$. This proves that F extends to a solution on $[\alpha, \omega]$. In fact $F\big|_\tau := F\big|_{M\times\{\tau\}}$ converges in $C^\infty(M)$ as $\tau \to \omega$, since $\|D^\alpha F\big|_\tau - D^\alpha F\big|_{\tau'}\|_{L^\infty} \leq \max_{\|\beta\|=\|\alpha\|+1} \|D^\beta F\|_{L^\infty}|\tau - \tau'|$. $\square$

We have just proved the first part of the following theorem. The second part is a reformulation of Theorem **??** of Eells and Sampson.

Theorem 5. 1. *Let M, M' be compact Riemannian manifolds with* $\text{Riem}(M') \leq 0$. *Then for every smooth map $f_0 : M \longrightarrow M' \subset B \subset \mathbb{R}^N$, the non-linear heat equation*

$$\begin{cases} \frac{df_t}{dt} = \tau(f_t), & \textit{for all } t \geq 0 \\ f\big|_{t=0} = f_0, \end{cases}$$

admit a globally defined smooth solution f_t. Moreover, all derivatives $D^\alpha f_t$ remains uniformly bounded as $t \to +\infty$.

2. *For a suitable sequence t_n increasing to $+\infty$ the sequence f_{t_n} converges in $C^\infty(M)$ to a function f_∞ with $\tau(f_\infty) = 0$. Therefore any map $f_0 : M \longrightarrow M'$ is homotopic to a harmonic map.*

Proof. For any sequence t_n, one can extract from $\{f_{t_n}\}$, since their derivatives are uniformly bounded, a subsequence $\{f_{t_{n_i}}\}$ converging in $C^k(M, \mathbb{R}^N)$. By a diagonalisation argument, one can extract from any sequence $\{f_{t_n}\}$ a subsequence converging in $C^\infty(M, \mathbb{R}^N)$ to f_∞. Abusively denote this subsequence by $\{f_{t_n}\}$, by Theorem **??**

$$\lim_{n\to\infty} K(f_{t_n}) = \lim_{n\to\infty} \int_M |\tau(f_{t_n})|^2 = 0$$

Therefore $\tau(f_{t_n}) \to 0$ in $L^2(M)^{\oplus N}$. But also $\tau(f_{t_n}) \to \tau(f_\infty)$ in $C^\infty(M, \mathbb{R}^N)$, one has $\tau(f_\infty) = 0$. The homotopic conlusion follows by rescaling the deformation time between f_{t_n} and $f_{t_{n+1}}$ to $\frac{1}{2^n}$. $\square$