

# Moser's Isotopy method and Darboux theorem

Tien

September 15, 2018

## Contents

<b>1</b>	<b>Symplectic geometry does not exist</b>	<b>1</b>
<b>2</b>	<b>Isotopy method</b>	<b>1</b>
2.1	Moser's trick . . . . .	1
2.2	Application: Darboux theorem and Moser Stability . . . . .	2

## 1 Symplectic geometry does not exist

We will prove a symplectic manifold, i.e. a smooth manifold equipped with a closed everywhere non-degenerate 2-form, does not have local invariant. This is a significant difference between symplectic manifold and riemannian manifold, whose local invariance is the curvature.

To do this, one uses a trick of Moser which in the compact case show that isotopic symplectic structures  $\omega_0$  and  $\omega_1$  are strongly isotopic, i.e.  $\psi_t^* \omega_t = \omega_0$

## 2 Isotopy method

### 2.1 Moser's trick

Let  $M$  be a closed manifold (compact, without boundary) and  $\omega_t$  is a family of symplectic structures on  $M$  such that

$$\frac{d}{dt} \omega_t = d\sigma_t$$

then there exists diffeomorphism  $\psi_t$  of  $M$  such that  $\psi_t^* \omega_t = \omega_0$

**Construction of  $\psi_t$ .** One constructs  $\psi_t$  by its flow  $\frac{d}{dt}\psi_t = X_t \circ \psi_t$  such that

$$0 = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) = \psi_t^*(d\sigma_t + X_t\lrcorner d\omega_t + d(X_t\lrcorner\omega_t))$$

Since  $\omega_t$  are closed and non-degenerate, it suffices to choose  $X_t$ , which exists uniquely, such that  $X_t\lrcorner\omega = \sigma_t$ .

## 2.2 Application: Darboux theorem and Moser Stability

Using this trick, we can prove the following theorem of Darboux.

**Lemma 1.** *Let  $M$  be a closed manifold with symplectic structures  $\omega_0$  and  $\omega_1$  such that they coincide on a fiber  $T_qM$ . Then there exists neighborhoods  $\mathcal{N}_0, \mathcal{N}_1$  of  $q$  and a diffeomorphism  $\psi : \mathcal{N}_0 \rightarrow \mathcal{N}_1$  such that  $\psi^*\omega_1 = \omega_0$ .*

*Proof.* We remark that it is enough to prove that there exists  $\sigma$  locally defined near  $q$  with  $\omega_1 - \omega_0 = d\sigma$  where  $\sigma = 0$  on  $T_qM$ . In fact, let  $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$  one then has a neighborhood  $\mathcal{N}_0$  of  $q$  such that  $\omega_t$  are non-degenerate and the field  $X_t$  constructed by Moser's technique ( $X_t = 0$  at  $q$ ) has its flow well-defined at time  $t = 1$  when starting at  $\mathcal{N}_0$ . Then  $\psi_1$  and  $\mathcal{N}_1$  is what we want.

One then uses another trick to construct  $\sigma$ : Take any Riemannian metric on  $M$  and let  $\phi_t$  be constructed using the geodesic flow and restricting  $\mathcal{N}_0$  to a geodesic ball such that  $\phi_0|_{\mathcal{N}_0} = q$ ,  $\phi_1 = Id$  and  $d\phi_t(q) = Id_{T_qM}$ . Then

$$\omega_1 - \omega_0 = \int_0^1 \frac{d}{dt}\phi_t^*(\omega_1 - \omega_0)dt = \int_0^1 \phi_t^*d(Y_t\lrcorner(\omega_1 - \omega_0)) = d\left(\int_0^1 dt\phi_t^*(Y_t\lrcorner(\omega_1 - \omega_0))\right).$$

It is straight-forward to see that the  $\sigma$  constructed this way works.  $\square$

The theorem of Darboux follows easily from Lemma 1.

**Theorem 2 (Darboux).** *Every two symplectic form  $\omega_0, \omega_1$  on a closed manifold  $M$  are locally isomorphic.*