From Busemann function to Cheeger-Gromoll splitting

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May 29, 2017

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We will prove the following result by Cheeger and Gromoll by a slightly modified approach of A. Besse.

Theorem 1 (Cheeger-Gromoll)**.** *Let M be a complete, connected Riemannian manifold with non negative Ricci curvature. Suppose that M contains a line then M is isometric to $M' \times \mathbb{R}$ with M' a complete, connected Riemannian manifold with non negative Ricci curvature.*

1 Busemann function

Let γ be a geodesic ray. We construct the Busemann function b associated to the ray as

$$b(x) = \lim_{t \to +\infty} f_t(x) = t - d(x, \gamma(t))$$

where the limit exists because the sequence f_t is non-decreasing and bounded by $d(x, \gamma(0))$. The convergence is also uniform in every compact.

In Euclidean space for example, the Busemann function is the orthogonal projection on γ. We will see that in a Riemannian manifold with non negative curvature, the Busemann function will serve as a projection.

Now with a fixed $x_0 \in M$, the tangent vector at x_0 of the geodesics connecting x_0 and $\gamma(t)$ is in the unit sphere of T_xM, which is compact. Let

X be a limit point of these tangents vectors and pose

$$b_{X,t}(x) = b(x_0) + t - d(x, C_X(t))$$

where $C_X(t)$ is the geodesic flow starting at x_0 with velocity X.

Remark 1. *1. From the construction of X, one has $b(x_0)+t = b(C_X(t))$, therefore $b_{X,t} \leq b$ withe quality in x_0. We say that b is supported by $b_{X,t}$ at x_0. In general a function f is* supported *by g at x_0 if $f(x_0) = g(x_0)$ and $f \geq g$ in a neighborhood of x_0.*

2. $b_{X,t}$ is smooth and a computation in local coordinate gives $\Delta b_{X,t} \geq -\frac{\dim M - 1}{t}$

3. $\|\nabla b_{X,t}\| = 1$

The estimation given on the second point is established using Jacobi fields:

Lemma 2. *The function $f(x) = d(x, x_0)$ satisfies at a point x out of the cut-locus of x_0:*

$$\nabla f(x) \leq \frac{n-1}{l}$$

where $n = \dim M, l = d(x, x_0) = f(x)$ in Riemannian manifold M with non-negative Ricci curvature.

Proof. Let $N(t), t \in \overline{0, l}$ be the velocity of the geodesic γ from x_0 to x, and $E_1, \ldots, E_{n-1}, N$ be a parallel frame along γ. Let J_i be the unique Jacobi fields along γ with $J_i(l) = E_i(l)$ and $J_i(0) = 0$ (existence and uniqueness of J_i is due to the fact that x is not in the cut-locus).

Then basic manipulation of Jacobi fields gives (without the fact that curvature is non-negative):

$$\Delta f(x) = \int_0^l \sum_{i=1}^{n-1} < \Delta_N J_i, \Delta_N J_i > - < R(N, J_i)J_i, N > = \sum_{i=1}^{n-1} I_\gamma(J_i, J_i)$$

where I_γ is the index form of γ. Note that the Jacobi fields J_i coincide with the fields $\frac{t}{l}E(t)$ at 0 and l, therefore by the Fundamental inequality of index form

$$I_\gamma(J_i, J_i) \leq I_\gamma(\frac{t}{l}E_i, \frac{t}{l}E_i)$$

hence

$$\Delta f(x) \leq \int_0^l \sum_{i=1}^{n-1} < \Delta_N \frac{t}{l}E_i, \Delta_N \frac{t}{l}E_i > - < R(N, \frac{t}{l}E_i)\frac{t}{l}E_i, N >$$

The curvature term being $\frac{t^2}{l^2}Ric(N,N)$ hence non-negative, one has

$$\Delta f(x) \leq \int_0^l \sum_{i=1}^{n-1} < \Delta_N \frac{t}{l} E_i, \Delta_N \frac{t}{l} E_i >= \frac{n-1}{l}$$

$\square$

We also note that it suffices to show that b is harmonic. In fact, from the smoothness one has $\nabla b(x_0) = \nabla b_{X,t}(x_0)$, which means $\|\nabla b\| = 1$ at every point in M. For each point $y \in M$, there exists a unique x with $b(x) = 0$ and time t when the flow of ∇b arrive at x. M is therefore homeomorphic to $\bar{M} \times \mathbb{R}$ by the map $F : y \mapsto (x,t)$ map. To see that this map is isometric, it remains to prove that the gradient field ∇b is parallel. In fact, $\bar{M}$ being equiped with the restriction of the metric on M, the isometry of F is equivalent to the isometry of Φ^t the flow of ∇b in every time t, which means $\frac{d}{dt} < \Phi^t_* u, \Phi^t_* u >$ vanishes at $t = 0$

$$\frac{d}{dt} < \Phi^t_* u, \Phi^t_* u >= 2 < \nabla_{\partial t} \Phi^t_* u, u > |_{t=0} = 2 < \nabla_u \nabla f, u >$$

where for the second equality we used Schwarz lemma for $\Phi(t,x) = \Phi^t(x)$. The vanishing of $< \nabla_u \nabla f(x), u >$ for every vector u is, by bilinearity, equivalent to that of $\nabla_u \nabla f$ for every u, meaning that ∇f is parallel. The last one is a simple computation:

$$Ric(N,N) = -N(\Delta b) - \|\nabla N\|^2$$

where $\|\nabla N\|^2 = \sum_{i=1}^{n-1} < \nabla_{E_i} N, E_j >^2$. We see that $N = \nabla b$ is parallel if $\Delta b = 0$.

Remark 2. 1. *One can show (see A. Besse) that every gradient field ∇b of norm 1 at every point is actually harmonic.*

2. *Using de Rham decomposition, one has directly the splitting of M since N is parallel and M is complete.*

2 Harmonicity

The Busemann function associated to a geodesic ray is subharmonic, it is a consequence of the following lemma.

Lemma 3. *In a connected Riemannian manifold, if a continuous function f is supported at any point x by a family f_ϵ (depending on x) with $\Delta(f_\epsilon) \leq \epsilon$, then f can not attain maximum (unless when f is constant).*

Proof. Given a small geodesic ball B, suppose that we have a function h on B with $\Delta h < 0$ in B and $f + h$ attains maximum at x in the interior of B. Then $f_\epsilon + h$ also attains maximum at x, which means $\Delta f_\epsilon + \Delta h \geq 0$, which is contradictory.

For the construction of the function h, one suppose that B is small enough such that $f|_{\partial B} \leq max_B f =: f(x_0)$ and equality is not attained at every points in ∂B. Then choose

$$h = \eta(e^{\alpha\phi} - 1)$$

with and $\phi(x) = -1$ if $x \in \partial B$ and $f(x) = f(x_0)$, $\phi(x_0) = 0$, $\nabla\phi \neq 0$ and a large α such that

$$\Delta h = \eta(-\alpha^2\|\nabla\phi\| + \alpha\Delta\phi)e^{\alpha\phi}.$$

is negative. □

Now for subharmonicity of b, given a harmonic function h that coincides with b in the boundary ∂B of a geodesic ball B, then $b - h$ is supported by $b_{X,t} - h$ with $\Delta(b_{X,t} - h) \to 0$ as t tends to $+\infty$, therefore $b - h \leq (b - h)|_{\partial B} = 0$ in B. hence b is subharmonic.

Corollary 3.1. *The Busemann function of a geodesic ray in a Riemannian manifold M with non-negative Ricci curvature is subharmonic.*

Now let b_+ be the function previously constructed for the ray $\gamma|_{[0,+\infty[}$ and b_- the Busemann function for the ray $\tilde{\gamma}|_{[0,+\infty[}$ where $\tilde{\gamma}(t) = \gamma(-t)$. Note that $b_+ + b_- \leq 0$ with equality on the line γ, but the sum is subharmonic therefore by maximum principle $b_+ = -b_-$ and b is harmonic hence smooth. The splitting theorem of Cheeger-Gromoll follows.

3 Application

Theorem 1 gives the following result from A. Beauville, *Variétés kahleriennes dont la première classe de Chern est nulle* (Theorem 1)

Theorem 4. *Let M be a compact Riemannian manifold with non-negative Ricci curvature, then the universal covering of M is of form $\mathcal{M} = \mathbb{R}^n \times \bar{M}$ where $\bar{M}$ does not contain any lines. Then*

1. *$\bar{M}$ is compact.*
2. *If M is Ricci-flat then $\bar{M} = \prod V_i \times X_j$ where V_i and X_j are simply-connected Kahler manifold with $Hol(V_i) = SU(m_i/2)$ and $Hol(X_j) = Sp(r_j/4)$ where $m_i = \dim V_i$ and $r_j = \dim X_j$.*

3. *M is, upto a finite covering, the product $T \times \prod V_i \times \prod X_j$ where T is a complex torus.*

Note that the second point follow directly from Berger classification. The third point is a consequence of Bieberbach theorem. We will prove here the first point, which is in fact Theorem 2 in J.Cheeger- D.Gromoll, *The splitting theorem for manifold of nonnegative ricci curvature.*

Proof. It suffices to prove that if $\bar{M}$ is not compact, then it contains a line. In fact, it is easy to see that such $\bar{M}$ must contains a minimal ray. In fact it is obvious that with a fixed $p \in M$ the function

$$F : v \mapsto \inf\{t > 0 : d(p, exp_p(tv)) < t\}$$

defined on the unit ball U_p of $T_p\bar{M}$ is upper semi-continuous. Therefore if $F(v) < \infty$ for all v unit tangent vector at p then F is bounded in U_p by a constant c. Therefore $\bar{M} \subset exp_p(cU_p)$ which is compact (contradiction). Therefore there exists a minimal ray at every point $p \in \bar{M}$.

The existence of a line in general might not be true, the only extra property of $\bar{M}$ that we will need is that it has a (fundamental) domain K compact and a family σ_i of isometries such that $\bar{M} = \cup_i g_i K$.

Let us first prove that such domain K exists. Remarque that every isometry of $\mathcal{M}$ acts separately on $\mathcal{M}$, i.e. of form $\sigma(u) = (\sigma_1(x), \sigma_2(y))$ for $u = (x, y) \in \mathcal{M}$ with g_1, g_2 isometries of $\mathbb{R}^n$ and $\bar{M}$. In fact such σ maps the $ Rn$-component of $T_u\mathcal{M}$ to the corresponding component of $T_\sigma(u)\mathcal{M}$ since a vector in this component is characterized by the fact that its geodesics is a line. As $\mathbb{R}^n \times \{y\}$ is a totally geodesic submanifold of $\mathcal{M}$, this means that $\sigma(\mathbb{R}^n \times \{y\}) = \mathbb{R}^n \times \{y\}$, and hence $\sigma(x \times \bar{M}) = \sigma(x) \times \bar{M}$ by isometry.

Let G_1 (resp. G_2) be the group formed by σ_1 (resp. σ_2) the $\mathbb{R}^n$ (resp. $\bar{M}$ of $\sigma = (\sigma_1, \sigma_2) \in \pi_1(M)$ that acts on $\mathcal{M}$ we have a surjection $M = \mathcal{M}/\pi_1 \longrightarrow \mathbb{R}^n/G_1 \times \bar{M}/G_2$, meaning that M/G_2 is compact. Such domain K can be chosen, for example, as a ball in $\bar{M}$ large enough that its image in the quotient M/G_2 contains every equivalent classes.

Now let γ be a minimal ray starting from $p \in M$, for each $x \in \gamma$ there exists an isometry σ of $\bar{M}$ such that $\sigma(x) \in K$. By compactness of K, there exists a sequence $t_n \to +\infty$ with $x_n = \gamma(t_n)$, $v_n = \dot{\gamma}(t_n)$ that satisfies $y_n = \sigma_n(x_n) \to y \in K$ and $(\sigma_n)_* v_n \to v \in T_y\bar{M}$ in the tangent bundle $T\bar{M}$. Then the geodesic of $\bar{M}$ starting at y with vector v is a line. In fact it suffices to prove that $d(exp_y(tv), exp_y(-sv)) = s + t$ for $s, t > 0$: for n large enough that $t_n > s$ then

$$d(exp_{y_n}(tv_n), exp_{y_n}(-sv_n)) = s + t$$

then let $n \to +\infty$ and the result follows. $\square$