

# Local results of several complex variables

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## 1 de Rham currents

Let  $M$  be a differential  $m$ -dimensional manifold and  $\mathcal{E}^p(M)$  be the vector space of smooth  $p$ -forms on  $M$  and  $\mathcal{D}^p(M)$  be the space of those with compact support. Then  $\mathcal{E}^p(M), \mathcal{D}^p(M)$  is a topological vector space with the pseudonorms  $p_{K,\Omega}^s(\omega) = \max_{K, |\alpha| \leq s} |D^\alpha u_I|$  where  $K \Subset \Omega$  an coordinated open set. The space of de Rham current with dimension  $p$  / degree  $m - p$  is defined as the dual space of  $\mathcal{D}^p(M)$ , denoted by  $\mathcal{D}'^{m-p}(M)$  or  $D'_p(M)$

**Remark 1.** 1. We are still in  $\mathbb{R}$ , but the definition expands to the complex case, denoted by  $\mathcal{D}'^{m-p,m-q}(M) = \mathcal{D}'_{p,q}(M)$  where  $m$  is the complex dimension of  $M$ .

2. The degree is defined such that the current  $T_\omega : \eta \mapsto \int_M \omega \wedge \eta$  is of the same degree as  $\omega$ . The dimension is defined so that the current  $T_{[Z]} : \eta \mapsto \int_Z \eta$  is of the same dimension as  $Z$ .

**Definition 1.** One has the following operation on  $\mathcal{D}'^{m-p}(M)$ :

1. **Derivative:**  $\langle dT, \omega \rangle = (-1)^{\deg T} \langle T, d\omega \rangle$
2. **Wedge product with a form:**  $\langle T \wedge \eta, \omega \rangle = \langle T, \eta \wedge \omega \rangle$
3. **Pushforward:** If  $F : X \rightarrow Y$  proper on  $\text{supp } T$  then  $\langle F_* T, \omega \rangle = \langle T, F^* \omega \rangle = \langle T, \chi F_* \omega \rangle$  where  $\chi \in C^\infty(M)$  identically 1 on  $\text{supp } T$ . The proper condition is such that the pullback of  $\omega$  is compactly support in  $\text{supp } T$
4. **Pullback:** Let  $F : X \rightarrow Y$  submersion then the pushforward of a form on  $X$  is well-defined by Fubini. One has  $\langle F^* T, \omega \rangle = \langle T, F_* \omega \rangle$

**Remark 2.** 1. The sign of derivative is chosen so that  $dT_\omega = T_{d\omega}$ .

2. Pushforward keeps the dimension, as the arguments are of the same degree.
3. Pullback keeps the codimension, meaning the degree (think  $F^* T_{[Z]} = T_{[F^{-1}(Z)]}$ ).
4. Locally a current is of form  $T = \sum u_I dx^I$  where  $u_I$  are distribution. **Note:** Here distribution are indentified as a current of maximal degree and not zero degree as they naturally are. To be exact, the notation of  $u_I$  is contravariant and its action is  $\varphi dx^1 \wedge \dots \wedge dx^N \mapsto \langle u_I, \varphi \rangle dx^1 \wedge \dots \wedge dx^n / \text{vol}$  where  $\text{vol}$  is a canonical volume form.

The last two remarks explain the sign in the following proposition.

**Proposition 0.1** (Pushforward and Pullback). Let  $F : M_1 \rightarrow M_2$ , submersion if needed, then

1.  $\text{supp } F_* T \subset F(\text{supp } T)$
2.  $d(F_* T) = F_* dT$  (pushforward of a form is still that form)
3.  $F_*(T \wedge F^* g) = (F_* T) \wedge g$

and

1.  $F^*(dT) = (-1)^{m_1-m_2} d(F^* T)$
2.  $F^*(T \wedge g) = (-1)^{m_1-m_2-\deg g} (F^* T) \wedge F^* g$

## 2 Subharmonic and Plurisubharmonic functions

Some properties of holomorphic functions that remain in several variables.

- Cauchy formula
- Analyticity: series development. Therefore its zeroes never form an open set (except for constant)
- Maximum modulus
- Cauchy inequality and Montel's theorem

### 2.1 Subharmonic functions

We are now in the context of  $\mathbb{R}^n$ .

**Theorem 1** (Green kernel). *Let  $\Omega \Subset \mathbb{R}^n$  be a smoothly bounded domain, then there exists uniquely a function  $G_\Omega : \bar{\Omega} \times \bar{\Omega} \rightarrow [-\infty, 0]$ , called the Green kernel of  $\Omega$ , with the following properties:*

1. *Regular:  $G_\Omega$  is  $C^\infty$  on  $\bar{\Omega} \times \bar{\Omega} \setminus \Delta_\Omega$  where  $\Delta_\Omega$  denotes the diagonal,*
2. *Symetric:  $G_\Omega(x, y) = G_\Omega(y, x)$ ,*
3. *Negative:  $G_\Omega(x, y) < 0$  on  $\Omega \times \Omega$  and  $G_\Omega(x, y) = 0$  on  $\partial\Omega \times \Omega$ ,*
4.  *$\Delta_x G_\Omega(x, y) = \delta_y$  on  $\Omega$  for every  $y \in \Omega$ .*

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**Example 1** (case  $\Omega = B(0, r)$ ). *One can take  $G_r = N(x - y) - N(\frac{|y|}{r}(x - \frac{r^2}{|y|^2}y))$  where  $N$  is the Newton kernel (or Newtonian potential, the gravitational potential). Explicitly, one has*

$$G_r(x, y) = \frac{1}{4\pi} \log \frac{|x - y|^2}{r^2 - 2\langle x, y \rangle + \frac{1}{r^2}|x|^2|y|^2} \quad \text{if } n = 2 \quad (1)$$

$$G_r(x, y) = \frac{-1}{(m-2)\text{vol}(S^{m-1})} (|x - y|^{2-m} - (r^2 - 2\langle x, y \rangle + \frac{1}{r^2}|x|^2|y|^2)^{1-m/2}) \quad \text{if } n \geq 3 \quad (2)$$

**Proposition 1.1** (Green-Riesz representation). *For  $u \in C^2(\bar{\Omega}, \mathbb{R})$  one has*

$$u(x) = \int_{\Omega} G_{\Omega}(x, y) \Delta u(y) d\lambda(y) + \int_{\partial\Omega} u(y) \frac{\partial G_{\Omega}}{\partial \nu_y} d\sigma(y)$$

*In particular, for  $\Omega = B(0, r)$ , one has*

$$P_r(x, y) := \frac{\partial G}{\partial \nu_y} = \frac{1}{\text{vol}(S^{m-1})r} \frac{r^2 - |x|^2}{|x - y|^m}$$

*called the Poisson kernel.*

*Proof.* Use the Green-Riesz formula:  $\int_{\Omega} u \Delta v - v \Delta u = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}$ .  $\square$

**Definition 2.** *Let  $\Omega \subset \mathbb{R}^n$  be an open subset and  $u : \Omega \rightarrow [-\infty, \infty)$  a upper semi-continuous function:*

$$\limsup_{x \rightarrow x_0} u(x) \leq u(x_0)$$

*One notes by  $\mu_S(u, a, r)$  and  $\mu_B(u, a, r)$  the average of  $u$  in the sphere and the disk centered in  $a$  of radius  $r$ . Then the following properties are equivalent and a function is called subharmonic if they are verified.*

- 1)  $u(x) \leq P_{a,r}[u](x) \quad \forall a, r, x \in B(a, r) \subset \Omega,$
- 2)  $u(a) \leq \mu_S(u, a, r) \quad \forall B(a, r) \subset \Omega,$
- 2')  $u(a) \leq \mu_S(u, a, r) \quad \text{for } B(a, r_n) \subset \Omega, r_n \rightarrow 0,$
- 3)  $u(a) \leq \mu_B(u, a, r) \quad \forall B(a, r) \subset \Omega,$
- 3')  $u(a) \leq \mu_B(u, a, r) \quad \text{for } B(a, r_n) \subset \Omega, r_n \rightarrow 0,$
- 4) *If  $u \in C^2$ , then  $\Delta u \geq 0$ .*

*The convex cone of subharmonic functions on a domain  $\Omega$  is denoted by  $Sh(\Omega)$ .*

*Proof.* It is obvious that  $(1) \rightarrow (2) \rightarrow (3) \rightarrow (3') \rightarrow (2')$ . To prove  $(2') \rightarrow (1)$  one needs the following 2 facts:

**Lemma 2** (u.s.c function as limit of continuous functions). *Let  $u$  be a u.s.c. function on a compact metric space  $X$ , then there exists a sequence  $u_n$  continuous function on  $X$  that decreases to  $u$  pointwise.*

**Proof.** Let  $\tilde{u}_k(x) = \max\{u(x), -k\}$  to exclude the  $-\infty$  points. Then  $v_k(x) = \sup_{y \in X} (u(y) - kd(x, y))$  works.

**Lemma 3.** (2') implies strict maximum principle (see 3.1).

**Proof.** By restriction to smaller neighborhood, one can suppose that  $u$  attains global maximum at  $x_0$  in  $\Omega$ . Then  $W = \{x \in \Omega : u(x) < u(x_0)\}$  is an open set, and has a point  $y$  in its boundary if  $W$  nonempty. Then (2') is not satisfied at  $y$  since the measure of open arc is nonzero.

Note that if  $u$  is continue than (2')  $\rightarrow$  (1): Let  $h = P_{a,r}[u]$  harmonic then  $u - h$  satisfies (2'), therefore the maximum principle, hence  $u - h \leq (u - h)|_{S(a,r)} = 0$ .

If  $u$  is u.s.c, take a sequence  $v_k$  continuous that decreases to  $u$  and let  $h_k = P_{a,r}[v_k]$  then  $h_k \geq v_k \geq u$  and  $h_k \rightarrow P_{a,r}[u]$  by monotone convergence.  $\square$

**Proposition 3.1.** Let  $u \in Sh(\Omega)$  then

**(Strict) maximum principle.**  $u$  cannot attain local maximum unless it is constant in the corresponding connected component,

**Locally integrable.**  $u$  is  $L^1_{loc}$  on each connected component where  $u \neq -\infty$ ,

**Pointwise decreasing limit** The pointwise limit  $u$  of a decreasing sequence  $u_k$  of subharmonic functions is also subharmonic.

**Regularisation.**  $\mu_S(u, a, \varepsilon), \mu_B(u, a, \varepsilon), \rho_\varepsilon * u$  increase in  $\varepsilon$ . Moreover,  $\rho_\varepsilon * u \in Sh(\Omega)$  and decreases to  $u$  pointwise as  $\varepsilon \rightarrow 0$ .

Moreover, for  $u \in \mathcal{D}'(\Omega)$

**Positive measure.**  $u \in Sh(\Omega)$  iff  $\Delta u \geq 0$  is a positive measure.

*Proof.* **Locally integrable.** To see that  $u \in L^1_{loc}(\Omega)$  if  $\Omega$  is connected and  $u \neq -\infty$ , let  $x$  be a point in the boundary of  $W = \{y \in \Omega : u \text{ integrable in neighborhood of } y\}$ , then apply mean value property in  $a \in W$  such that  $x \in B(a, r)$ .

**Pointwise decreasing limit.** Infimum of a family of u.s.c functions is still u.s.c. The mean value property comes from monotone convergence.

**Regularisation.** Check first for  $C^2$  functions, then regularise. One uses the following Gauss formula:

$$\mu_S(u, a, r) = u(a) + \frac{1}{n} \int_0^r \mu_B(\Delta u, a, t) t dt$$

to see that  $\mu_S$  is increasing in  $r$  and

$$\mu_B(u, a, r) = m \int_0^1 t^{m-1} \mu_S(u, a, rt) dt$$

to see that  $\mu_B$  is increasing. For the convolution, use

$$u * \rho_\varepsilon = \text{vol}(S^{n-1}) \int_0^1 \mu_S(u, a, \varepsilon t) \rho(t) t^{m-1} dt.$$

**Positive measure.**  $\Delta u * \rho_\varepsilon \geq 0$  as function, therefore the limit  $\geq 0$  as measure (dominated convergence). □

**Proposition 3.2** (new harmonic functions from old ones). *Let  $u_k \in Sh(\Omega)$  then*

1. *If  $\{u_k\}$  decrease to  $u$  then  $u \in Sh(\Omega)$ .*
2. *Let  $\chi$  be a convex function, non-decreasing in each variable then  $\chi(u_1, \dots, u_p) \in Sh(\Omega)$ . Therefore,  $\sum u_i$  and  $\max\{u_i\}$  are subharmonic.*

**Proposition 3.3** (Upper regularization). 1. *Let  $u$  be a real function on  $\Omega$  then  $u^*(x) = \lim_{\varepsilon \rightarrow 0} \sup_{x+\varepsilon B} u$ , called the upper envelope of  $u$  is u.s.c and is in fact the smallest u.s.c function greater than  $u$ .*

2. **Choquet lemma.** *Let  $\{u_\alpha\}$  be a family of real function, one defines the upper regularization of  $\{u_\alpha\}$  by  $u^*$  where  $u = \sup_\alpha u_\alpha$ . Then from every such family, on can always find a countable subfamily  $\{v_i\}$  such that  $u^* = v^*$ .*
3. *If  $\{u_\alpha\} \subset Sh(\Omega)$  then  $u^* = u$  a.e. and  $u^* \in Sh(\Omega)$ .*

*Proof.* 1. Obvious.

2. Let  $B_i$  be a countable base of the topology and  $x_{i,j}$  be a sequence such that  $u(x_{i,j}) \rightarrow \sup_{B_i} u$ . Let  $\{u_{i,j,k}\}$  be a countable subfamily such that  $u_{i,j,k}(x_i) \rightarrow u(x_i)$  then it is a suitable subfamily.
3. WLOG, suppose that  $\{u_\alpha\} = \{u_i\}$  countable then  $u$  satisfies the submean value property:  $u(z) \leq \mu_B(u, z, r)$ . By the continuity of  $\mu_B(u, z, r)$  one has  $u^*(z) \leq \mu_B(u, z, r) \leq \mu(u^*, z, r)$  therefore  $u^* \in Sh(\Omega)$  and  $u^*(z) = \lim_{r \rightarrow 0} \mu_B(u^*, z, r) = \lim_{r \rightarrow 0} \mu_B(u, z, r)$ , from which  $u = u^*$  □

## 2.2 Plurisubharmonic functions

The analog of harmonic functions over  $\mathbb{C}$  in multidimensional case  $\Omega \subset \mathbb{C}^n$  is in fact *pluriharmonic functions* which is defined through the notion of plurisubharmonic functions

**Definition 3.** 1. A real function  $u$  is said to be plurisubharmonic if and only if its restriction to any complex line is subharmonic. One denotes by  $Psh(\Omega)$  the space of plurisubharmonic function on  $\Omega$ .

2. In case  $u \in C^2$  on  $\Omega \subset \mathbb{C}^n$ , this is equivalent to

$$H(u)(\zeta) = \sum \frac{\partial^2 u}{\partial z^j \partial \bar{z}^k} \zeta^j \bar{\zeta}^k \geq 0 \quad \forall \zeta \in \mathbb{C}^n$$

where the notation  $H(u)(\zeta)$  is invariant, i.e. if  $f : M_1 \rightarrow M_2$  is holomorphic then  $H(u \circ f)(\zeta) = H(u)df(\zeta)$ .

3. In the general case, this is equivalent to  $H(u)(\zeta) \geq 0 \quad \forall \zeta \in \mathbb{C}^n$  as a measure.

**Remark 3.** 1. The invariance can be noticed using  $\zeta^j = \zeta^j d\zeta^j + \bar{\zeta}^j d\bar{\zeta}^j$  where LHS is interpreted as a vector in  $T\mathbb{C}$ . This allows us to extend the notion of  $Psh(M)$  to any complex manifold  $M$ .

2. By consequence,  $f^*u \in Psh(M_1)$  for all  $u \in Psh(M_2)$  and  $f : M_1 \rightarrow M_2$  holomorphic.

**Proposition 3.4** (new Psh functions from old ones). *The construction of new plurisubharmonic function is the same as that of subharmonic function. Let  $u_k \in Psh(\Omega)$  then*

1. If  $\{u_k\}$  decrease to  $u$  then  $u \in Psh(\Omega)$ .
2. Let  $\chi$  be a convex function, non-decreasing in each variable then  $\chi(u_1, \dots, u_p) \in Psh(\Omega)$ . Therefore,  $\sum u_i$  and  $\max\{u_i\}$  are plurisubharmonic.
3. The upper regularization  $u^*$  where  $u = \sup_{\alpha} u_{\alpha}$  is also plurisubharmonic and  $u = u^*$  almost everywhere.

*Proof.* The only nontrivial proof is the third one where upper envelop in  $\mathbb{C}^{\times}$  and in a line can be different. To fix this, use Choquet lemma 3.3 and dominated convergence,  $u * \rho_{\varepsilon}$  satisfies the submean property on every complex line and decrease to  $u$  a.e.  $\square$

### 2.3 Pluriharmonic functions

**Definition 4.** A function  $u$  is said to be pluriharmonic on  $\Omega$ , denoted  $u \in Ph(\Omega)$  if  $u \in Psh(\Omega)$  and  $-u \in Psh(\Omega) \setminus \setminus$  where  $(\Omega)$ .

This is obviously equivalent to  $H(u) = 0$ , i.e.  $\frac{\partial^2 u}{\partial z^j \partial \bar{z}^k} = 0 \quad \forall j, k$ , i.e.  $\partial \bar{\partial} u = 0$ .

**Remark 4.** 1. By mean value property,  $Ph(\Omega) \subset Harm(\Omega)$ .

2. If  $f \in \mathcal{O}(M)$  then  $\Re f, \Im f \in Ph(M)$

**Theorem 4** (analog of harmonic function). If  $M$  is a complex manifold such that  $H_{dR}^1(X, \mathbb{R}) = 0$  then every pluriharmonic function  $u$  is a real part of a holomorphic function  $f \in \mathcal{O}(M)$

*Proof.* Since  $d(\bar{\partial}u) = 0$ , and  $H_{dR}^1 = 0$ , one has  $\bar{\partial}u = dg$ . Therefore  $d(u - 2\Re g) = (\bar{\partial}u - dg) + (\partial u - d\bar{g}) = 0$ , hence one chooses  $f = 2g + C$  on each connected component.  $\square$

### 3 Resolution of $\bar{\partial}$ , Dolbeault-Grothendieck lemma

The generalized Cauchy formula for several variables is the following (the formula in wikipedia is  $K_{BM}^{0,0}$ )

**Theorem 5** (Bochner–Martinelli–Koppelman formula). The Bochner-Martinelli kernel is the following  $(n, n-1)$ -form on  $\mathbb{C}^n$

$$k_{BM} = (-1)^{n(n-1)/2} \frac{(n-1)!}{(2\pi i)^n} \sum_{1 \leq j \leq n} (-1)^j \frac{\bar{z}_j dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n}{|z|^{2n}}$$

then  $\bar{\partial}k_{BM} = \delta_0$  on  $\mathbb{C}^n$ .

Let  $K_{BM} = \pi^* k_{BM}$  where  $\pi : (z, \zeta) \mapsto z - \zeta$  so that  $\bar{\partial}K_{BM} = [\Delta]$ , then: For any domain  $\Omega \subset \mathbb{C}^n$  bounded with piecewise  $C^1$  boundary and  $v$  a  $(p, q)$ -form of class  $C^1$  on  $\bar{\Omega}$  then

$$v(z) = \int_{\partial\Omega} K_{BM}^{p,q}(z, \zeta) \wedge v(\zeta) + \bar{\partial} \int_{\Omega} K_{BM}^{p,q-1}(z, \zeta) \wedge v(\zeta) + \int_{\Omega} K_{BM}^{p,q}(z, \zeta) \wedge \bar{\partial}v(\zeta)$$

where  $K_{BM}^{p,q}$  denotes the component of  $K_{BM}$  type  $(p, q)$  in  $z$  and type  $(n-p, n-q-1)$  in  $\zeta$

Another consequence of 5 is the *global* resolution of  $\bar{\partial}$  in case of compact support.



**Corollary 5.1.** *If  $v$  is a  $(p, q)$ -form with  $q \geq 1$  on  $\mathbb{C}^n$ , compactly supported, with regularity of class  $C^s$  such that  $\bar{\partial}v = 0$  then there exists an  $(p, q-1)$ -form  $u$  on  $\mathbb{C}^n$  with the same regularity as  $u$  such that  $\bar{\partial}u = v$ . In fact one can take*

$$u(z) = \int_{\mathbb{C}^n} K_{BM}^{p,q-1}(z, \zeta) \wedge v(\zeta)$$

*In case  $(p, q) = (0, 1)$  then  $u$  is compactly support. This means that the compact support  $(0, 1)$ -Dolbeault cohomology  $H_c^{0,1}(\mathbb{C}^n) = 0$ .*

Since  $K_{BM} = O(|z|^{1-2n})$ , one has  $|u(z)| = O(|z|^{1-2n})$  at infinity. Therefore the compact support of  $u$  in case  $(p, q) = (0, 1)$  is explained by Liouville theorem.

The Dolbeault-Grothendieck lemma solves the equation  $\bar{\partial}u = v$  in a local scale if the compact support condition is dropped and gives regular result if  $v$  is a  $(p, 0)$ -form.

**Theorem 6** (Dolbeault-Grothendieck lemma). *Let  $v \in \mathcal{D}'(p, q)(\Omega)$  such that  $\bar{\partial}v = 0$ .*

1. *If  $q = 0$  then  $v = \sum v_I dz^I$  where  $v_I \in \mathcal{O}(\Omega)$ .*
2. *If  $q \geq 1$  then there exists  $\omega \subset \Omega$  and  $u \in \mathcal{D}'(p, q-1)(\Omega)$  such that  $\bar{\partial}u = v$ . Moreover, if  $v \in \mathcal{E}^{p,q}(\Omega)$  then  $u \in \mathcal{E}^{p,q-1}(\Omega)$*

**Corollary 6.1** (Hypoellipticity in bidegree  $(p, 0)$ ).  *$\bar{\partial}$  is hypoellipticity in bidegree  $(p, 0)$ , i.e. if  $\bar{\partial}u = v$ ,  $v$  of bidegree  $(p, 1)$  and  $v$  is  $C^\infty$  then  $u$  is also  $C^\infty$  on the entire domain  $\Omega$ .*

## 4 Extension theorems, Domain of holomorphy

**Theorem 7** (Hartog extension). *Let  $\Omega \subset \mathbb{C}^n$  be a domain and  $K \Subset \Omega$  such that  $\Omega \setminus K$  is connected. Then  $\mathcal{O}(\omega)|_{\Omega \setminus K} = \mathcal{O}(\Omega \setminus K)$  every holomorphic function on  $\Omega \setminus K$  extends to  $\Omega$*

*Proof.* Let  $f \in \mathcal{O}(\Omega \setminus K)$  be the function we want to extend. Let  $\varphi$  be a function with support in a neighborhood of  $K$  and is identically 1 on  $K$  and  $g = (1 - \varphi)f$  which coincides with  $f$  outside of  $\text{supp } \varphi$ . Then  $v = \bar{\partial}g \in \mathcal{D}^{0,1}$  satisfies  $\bar{\partial}v = 0$ , therefore there exists  $u \in C_c^\infty(\mathbb{C}^n)$  with  $\text{supp } u \subset \text{supp } \varphi$  such that  $\bar{\partial}u = v = \bar{\partial}g$ , the holomorphic function  $g - u$  is well-defined on  $\Omega$  and coincides with  $f$  (and  $g$ ) on  $\Omega \setminus \text{supp } \varphi$ , therefore coincides with  $f$  on  $\Omega \setminus K$ .  $\square$

Note that although we do not need  $\Omega$  to be small, this theorem counts as a local result due to the hypothesis that we are in  $\mathbb{C}^n$ .

A global result can be obtained using the Hartog figure, that is the union of an annulus  $\{(z_1, z') : r < |z_1| < R\}$  and an open set in other dimension  $\{(z_1, z') : z' \in \omega \text{ open}\}$ . and use the interpolation  $(z_1, z') \mapsto \int_{C_R} \frac{f(\zeta_1, z')}{\zeta_1 - z_1} d\zeta_1$  to extend  $f$ . The open set in  $z'$ -dimension is to show that the interpolation and  $f$  coincide on it. With one dimension  $z_1$  to form the annulus and another dimension (says  $z_2$  to form the open set, one can extend any holomorphic function to a submanifold of (complex) codimension at least 2.

**Theorem 8** (Riemann extension). *Let  $M$  be a complex manifold and  $N$  a sub complex manifold of codimension  $\geq 2$  then any holomorphic function on  $M \setminus N$  extends uniquely to  $M$ .*

## 4.1 Generalities

An approach to the extension problem on complex manifolds is through the notion of holomorphic hull and holomorphic convexity.

**Definition 5.** 1. *Let  $K$  be a compact in a complex manifold  $M$ . Then the holomorphic hull  $\hat{K}_{\mathcal{O}(M)}$  is the set  $\{z \in M : f(z) \leq \sup_K |f| \quad \forall f \in \mathcal{O}(M)\}$ .*

2. *A complex manifold  $M$  is said to be holomorphically convex if  $\hat{K}_{\mathcal{O}(X)}$  is compact for all compact  $K \subset M$ .*

**Proposition 8.1** (holomorphic hull). *The following statements are obvious*

1.  $\hat{K}$  is a closed subset containing  $K$  and  $\hat{\hat{K}} = \hat{K}$ .
2. If  $f : M_1 \rightarrow M_2$  is holomorphic then  $f(\hat{K}) \subset \widehat{f(K)}$ . (Think inclusion)
3. **Hole filling.** In particular, if  $f : \bar{B} \rightarrow X$  and  $f(\partial B) \subset K$  then  $f(\bar{B}) \subset \hat{K}$ .

**Proposition 8.2** (holomorphically convex). *Let  $M$  be a holomorphically convex complex manifold then*

1.  $M$  admits a exhaustive sequence of compact  $K_\nu$ , i.e.  $K_\nu \Subset K_{\nu+1}$  and  $\widehat{\widehat{K}_\nu} = K_\nu$ .
2.  $M$  is weakly pseudoconvex, i.e. there exists  $\psi \in Psh(M) \cap C^\infty(M)$  such that  $\{\psi < c\}$  are relatively compact, i.e.  $\lim_{K \rightarrow M} \psi|_{M \setminus K} = +\infty$

## 4.2 Case $\Omega \subset \mathbb{C}^n$

**Definition 6.** *Domain of holomorphy*

**Proposition 8.3.** *Let  $\Omega \subset \mathbb{C}^n$  be a domain then:*

1. *If  $\Omega$  is a domain of holomorphy then  $\hat{K}_{\mathcal{O}(\Omega)}$  is compact and  $d(K, \partial\Omega) = d(\hat{K}, \partial\Omega)$ .*
2. *The followings are equivalent:*
  - (a)  *$\Omega$  is a domain of holomorphy.*
  - (b)  *$\Omega$  is holomorphically convex.*
  - (c) *Let  $\{z_k\}$  be a sequence in  $\Omega$  without accumulation in  $\Omega$  and  $c_k \in \mathbb{C}$ . There exists a function  $f \in \mathcal{O}(\Omega)$  such that  $f(z_k) = c_k$ .*
  - (d) *There exists a function  $F \in \mathcal{O}(\Omega)$  that is unbounded locally in any point on  $\partial\Omega$ .*

#+ENDtheorem

### 4.2.1 Different notion of pseudoconvexity

### 4.2.2 Richberg approximation theorem