Contents

1 First variation of renormalised energy. *<2018-10-26 Fri>* **1**
- 1.1 The results. 1
- 1.2 Proofs. 3

2 Log term in energy of $f : \overline{\mathbb{H}^2} \longrightarrow \overline{\mathbb{H}^{n+1}}$. **4**

3 Commutative diagram revisited **4**

4 LaTeX in Inkscape: Incompatibility between `ghostscript` and `pstoedit` *<2018-03-30 Fri>* **4**

5 LaTeX indentation in org-mode *<2018-02-20 Tue>* **4**

6 A (decent) map of mathematics *<2017-10-17 Tue>* **4**

7 My 2016-2017 internship *<2017-07-31 Mon>* **4**

1 First variation of renormalised energy. *<2018-10-26 Fri>*

1.1 The results.

I will start by writing down the result before giving the proofs. Let (Σ, i) be a Riemann surface and (M, g) be an asymptotically hyperbolic Riemannian four-manifold. We are interested in critical points of renormalised energy $\mathcal{E}_{\text{norm}}$ because these maps form a special class among harmonic maps whose energy is critical to all pertubation (and not just compact pertubation).

Even though we obtained an explicit formula for renormalised energy where a density appears

$$\mathcal{E}_{\text{norm}} = -4\pi\chi + \int_\Sigma \left[2K(f^*g)\sqrt{\det_h f^*g} + \operatorname{Tr}_h f^*g \right] dA_h,$$

it is not easy to compute its variation directly from it. The following approach gives us a formula that although is not explicit, allows us to see the following phenomenon.

Theorem 1 (Critical points of $\mathcal{E}_{\text{norm}}$ = Harmonic + good germ on the boundary). *Let $f : (\Sigma, i) \longrightarrow (M, g)$ be a harmonic map from a Riemann surface Σ to a AH Riemannian four-manifold M such that $q(f)_{\bar{g}} = 0$ on $\partial\Sigma$.*

There is a boundary quantity $\mathcal{B}_2(f)$ that only concerns the germ of f on $\partial\Sigma$ such that:

$$f \text{ is critical to } \mathcal{E}_{\text{norm}} \text{ if and only if } \mathcal{B}_2(f) = 0.$$

We know that the ϵ-energy must be writen as

$$E_\epsilon(f) = \frac{1}{\epsilon}\text{length}(\gamma) + \mathcal{E}_{\text{norm}} + O(\epsilon)$$

where γ is the boundary curve of f and the length is taken under the metric $\bar{g}$. Now given a pertubation $\{f_t\}$ of f_0, we can recover the variation of $\mathcal{E}_{\text{norm}}$ by looking at the $O(1)$ term of $\frac{dE_\epsilon}{dt}(f_t)$. It turns out that in order to compute the variation of E_ϵ, we only need to do the following 2 tasks:

- Extend the computation made by Eells-Sampson [?] to manifold with boundary.
- Take care of the variation of the domain of integration $\Sigma_\epsilon(f_t) := f_t^{-1}(r \geq \epsilon)$ which also depends on t where r is the boundary defining function (bdf).

Proposition 1.1 (Variation of E_ϵ)**.** *For any pertubation $\{f_t\}$ of $f : (\Sigma, \partial\Sigma, i) \longrightarrow (M, \partial M, g)$, one has*

$$\frac{d}{dt}E_\epsilon(f_t) = \int_{\Sigma_\epsilon(f_0)} \left\langle \tau(f_0), \frac{df_t}{dt} \right\rangle_g dV_h + \frac{1}{\epsilon^2}\int_{\partial\Sigma_\epsilon(f_0)} \left\langle 2\frac{\partial f_0}{\partial n} + \frac{\nabla^g r(f_0)}{|d(r \circ f_0|_h}, \frac{df_t}{dt} \right\rangle_{\bar{g}} ds_h \tag{1}$$

where

- *h is a metric in the conformal class i of Σ,*
- *$\frac{\partial}{\partial n}$ is the h-unit normal vector of $\partial\Sigma_\epsilon(f_0)$ in $\Sigma_\epsilon(f_0)$,*
- *$\tau(f_0)$ is the tension field of f_0 with respect to the metric h on Σ and g on M.*

Denote $\mathcal{B} := \int_{\partial\Sigma_\epsilon(f_0)} \left\langle 2\frac{\partial f_0}{\partial n} + \frac{\nabla^g r(f_0)}{|d(r\circ f_0|_h}, \frac{df_t}{dt} \right\rangle_{\bar{g}} ds_h =: \mathcal{B}_0 + \mathcal{B}_1\epsilon + \mathcal{B}_2\epsilon^2 + O(\epsilon^3)$ then

- *$\mathcal{B}_i$ only depends on the germ of f_0 on $\partial\Sigma$*
- *$\mathcal{B}_0 = 2\int_{\partial\Sigma} \left\langle \frac{\partial f_0}{\partial n} + \frac{\nabla^{\bar{g}} r(f_0)}{|\nabla^{\bar{g}} r(f_0)|_{\bar{g}}}, \frac{df_t}{dt} \right\rangle_{\bar{g}} ds_h = 0$ where n is the $f^*\bar{g}$-normal vector of Σ on the boundary.*

As a straightforward consequence, one has

Proposition 1.2. *Given a map $f : (\Sigma, \partial\Sigma, i) \longrightarrow (M, \partial M, g)$, then:*

1. *f_0 is harmonic and $\mathcal{B}_2(f_0) = 0$ if and only if f_0 is a critical point of $\mathcal{E}_{\text{norm}}$.*
2. *The first variation of $\mathcal{E}_{\text{norm}}$ can be writen as:*

$$\frac{d\,\mathcal{E}_{\text{norm}}}{dt}(f_t) = \int_\Sigma \left\langle \tau(f_0), \frac{df_t}{dt} \right\rangle_g dV_h$$
$$+ \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} \left[\int_{\partial\Sigma_\epsilon(f_0)} \left\langle 2\frac{\partial f_0}{\partial n} + \frac{\nabla^g r(f_0)}{|d(r\circ f_0|_h}, \frac{df_t}{dt} \right\rangle_{\bar g} ds_h + \int_{\Sigma_\epsilon(f_0)} \left\langle \nabla^{\bar g}_{\dot\gamma} \frac{\dot\gamma}{|\dot\gamma|_h}, \frac{df_t}{dt} \right\rangle_{\bar g} \right]$$

where γ is the boundary curve of f, $\frac{\partial}{\partial n}$ is the h-unit normal vector of $\partial\Sigma_\epsilon$ in Σ_ϵ.

1.2 Proofs.

We **??** knew that the variation of energy functional can be represented by the tension field, in case of manifolds with boundary, we obtain an additional term from Stokes formula:

$$\frac{d}{dt}E_\epsilon(f_t) = \frac{d}{dt}\int_{\Sigma_\epsilon(f_0)} \mathrm{Tr}_h(f_t^* g) + \frac{d}{dt}\int_{\Sigma_\epsilon(f_t)} \mathrm{Tr}_h(f_0^* g) \tag{2}$$

$$= \int_{\Sigma_\epsilon(f_0)} \left\langle \tau(f_0), \frac{df_t}{dt} \right\rangle_g dV_h + 2\int_{\partial\Sigma_\epsilon(f_0)} \left\langle \frac{\partial f_0}{\partial n}, \frac{df_t}{dt} \right\rangle_g + \frac{d}{dt}\int_{\Sigma_\epsilon(f_t)} \mathrm{Tr}_h(f_0^* g) dV_h \tag{3}$$

Equation (1) is a straightforward application of the following lemma for $\Omega_t = \Sigma_\epsilon(f_t)$, $F = \mathrm{Tr}_h(f_0^* g)$ and $r_t = r \circ f_t$.

Lemma 2 (Riemannian Cavalieri). *Let Ω be a domain in Σ and $\{r_t\}_{t=\overline{0,1}}$ be a familly of functions on Ω where dr_t are non-zero and $\Omega_t \subset \Omega$ be subdomains of Ω defined by $\Omega_t = \{r_t \geq \epsilon\}$. Then for any function F on Ω, one has*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\Omega_t} F dV_h = \int_{\partial\Omega_0} \frac{r_1}{|\nabla^h r_0|_h} F ds_h$$

where $r_1 = \left.\frac{dr_t}{dt}\right|_{t=0}$.

Proof. Let us prove the lemma in case Ω_t only has one connected component with non-empty interior, since the number of such components does not change for t near 0 (this is because r_0 has no critical point in Ω). Let us also suppose that $r_1 \geq 0$ meaning that the domain Ω_t becomes bigger as t increases from 0. This is because one can always partition the curve $\gamma = \partial\Omega_0$ into pieces where $r_1 > 0, r_1 < 0$ or $r_1 = 0$ and the area difference of pieces touching the $r_1 = 0$ parts is of $O(t^2)$.

The difference $\Omega_t \setminus \Omega_0$ is the region where $\epsilon - r_1 t + O(t^2) \leq r_0 \leq \epsilon$, therefore $\Omega_t \setminus \Omega_0$ is of $O(t^2)$ difference from the region $\Phi_t = \{\exp_{\gamma(s)} \frac{\theta r_1 \nabla^h r_0}{|\nabla^h r_0|^2} : \theta \in [0,t], s \in [0,1]\}$. Therefore

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\Omega_t} F dV_h = \left.\frac{d}{dt}\right|_{t=0} \int_0^t \int_\gamma F \frac{r_1}{|\nabla^h r_0|_h} \mathrm{vol}\ ds_h d\theta = \int_\gamma F \frac{r_1}{|\nabla^h r_0|_h} ds_h.$$

where $\frac{r_1}{|\nabla^h r_0|_h} \mathrm{vol}(s,\theta) ds_h d\theta$ is the pullback of the volume form by exponential map $(s,\theta) \mapsto \exp_{\gamma(s)} \frac{\theta r_1 \nabla^h r_0}{|\nabla^h r_0|_h^2}$, so $\mathrm{vol}(s,0) = 1$. □

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