

# Calabi-Yau theorem

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August 1, 2017

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## 1 Calabi conjecture

In complex geometry, one usually defines the *Ricci curvature* to be the real  $(1,1)$ -form  $\rho$  with  $\rho(u, v) = \text{Ric}(Ju, v) = \text{tr}(w \mapsto R(w, v).Ju)$ , as it has the advantage of being an antisymmetric form.

We will call  $\rho$  the Ricci form when it is easy to confuse with the Ricci curvature tensor in Riemannian geometry. We start with the following fact (which is exercise 4.A.3 in Huybrechts, *Complex geometry: an introduction*).

**Remark 1.** *For our convenience when talking about positivity, we would rather use the anticanonical bundle. Then  $K_X^{-1}$  is positive (resp. semi-positive) if and only if Ric is positive definite (resp. positive semi-definite) as a symmetric form.*

We start with the following fact (which is exercise 4.A.3 in Daniel Huybrechts, *Complex geometry: an introduction*)

**Proposition 0.1** (Ricci curvature and first Chern class). *Let  $(X, g)$  be a compact Kähler manifold. Then  $i\rho(X, g)$  is the curvature of the Chern connection on the canonical bundle  $K_X$ . In other words,  $\rho(X, g) \in -2\pi c_1(K_X)$  where  $c_1(K_X)$  is the first Chern class of  $K_X$ .*

**Definition 1.** The quadruple  $(h, g, \omega, J)$  is said to be compatible if  $g \circ J = g$  and  $\omega(a, b) = g(Ja, b)$  and  $h = g - i\omega$ .

**Remark 2.** 1. When  $J$  is fixed, one of  $h, g, \omega$  that is invariant by  $J$  determines the two others.

2. For a compatible quadruple, the condition  $\nabla J = 0$  is equivalent to  $d\omega = 0$ . The fundamental form  $\omega$  that satisfies  $d\omega = 0$  is called a Kähler form.

The Calabi conjecture asked whether for each form  $R \in c_1(K_X)$  one can find a metric  $g'$  whose new fundamental form  $\omega'$  is in the same class of  $\omega$  and  $\text{Ric}(X, g') = R$ . We prefer to work with the fundamental form instead of the metric  $g$  as the former is antisymmetric and its derivative is hence easy to define.

## 2 Reduction to local charts, Yau theorem

**$h, g, \omega$  in local coordinates.** We note by  $h_{i\bar{j}} = h(\partial_{x_i}, \partial_{x_j}) = 2g_{\mathbb{C}}(\partial_{z_i}, \partial_{z_j})$ . By straightforward calculation one has

$$\begin{aligned}\omega &= -\frac{1}{2} \text{Im} h_{i\bar{j}} (dx^i \wedge dx^j + dy^i \wedge dy^j) + \text{Re} h_{i\bar{j}} dx^i \wedge dy^j \\ &= \frac{i}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j\end{aligned}$$

and the condition  $d\omega = 0$  is equivalent to

$$\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial h_{k\bar{j}}}{\partial z_i}$$

We also note by  $h^{i\bar{j}}$  the inverse transposed of  $h_{i\bar{j}}$ , i.e.  $h^{i\bar{j}} h_{k\bar{j}} = \delta_j^k$

**Definition 2.** Let  $X$  be an almost complex manifold (manifold with an almost complex structure). Then  $d : \bigwedge^n T^*X \longrightarrow \bigwedge^{n+1} T^*X$  sends  $\bigwedge^{p,q} T^*M$  to  $\bigwedge^{p+1,q} T^*M \oplus \bigwedge^{p,q+1} T^*M$ . We denote by  $\partial$  and  $\bar{\partial}$  the component of  $d$  in  $\bigwedge^{p+1,q} T^*M$  and  $\bigwedge^{p,q+1} T^*M$  respectively.

It would be convenient to define  $d^c = i(\bar{\partial} - \partial)$  then obviously  $dd^c = 2i\partial\bar{\partial}$ .

**The Ricci curvature.** The Ricci curvature form is given in local coordinates by

$$\text{Ric}_\omega = -\frac{1}{2} dd^c \log \det(h_{i\bar{j}})$$

**$dd^c$  lemma .** We then can state the  $dd^c$  lemma

**Lemma 1.** *Let  $\alpha$  be a real,  $(1,1)$ -form on a compact Kähler manifold  $M$ . Then  $\alpha$  is  $d$ -exact if and only if there exists  $\eta \in C^\infty(M)$  globally defined such that  $\alpha = dd^c\eta$ .*

**Yau's theorem.** The  $dd^c$  lemma tells us that every form  $R \in c_1(K_X)$  is of form  $Ric_\omega + dd^c\eta$ . If one varies the Hermitian product  $h_{i\bar{j}}$  to  $h_{i\bar{j}} + \phi_{i\bar{j}}$  then the new Ricci curvature is  $dd^c \log \det(h_{i\bar{j}} + \phi_{i\bar{j}})$ . The Calabi conjecture can be restated as the existence of  $\phi$  such that  $h_{i\bar{j}} + \phi_{i\bar{j}}$  is definite positive and

$$dd^c (\log \det(h_{i\bar{j}} + \phi_{i\bar{j}}) - \log \det(h_{i\bar{j}}) - \eta) = 0 \quad (1)$$

The functions  $f$  that satisfies  $dd^c f = 0$  are called *pluriharmonic*. They also satisfy the maximum principle. By compactness of  $X$ , these functions on  $X$  are exactly constant functions. Therefore (1) is equivalent to

$$\det(h_{i\bar{j}} + \phi_{i\bar{j}}) = e^{c+\eta} \det(h_{i\bar{j}})$$

or by  $dd^c$  lemma:

$$(\omega + dd^c\phi)^n = e^{c+\eta}\omega^n$$

where  $\omega^n$  denotes the repeated wedge product. Note that  $(\omega + dd^c\phi)^n - \omega^n$  is exact, one has  $\int_M (\omega + dd^c\phi)^n = V$ , the conjecture of Calabi is therefore a consequence of the following theorem.

**Theorem 2 (Yau).** *Given a function  $f \in C^\infty(M)$ ,  $f > 0$  such that  $\int_M f\omega^n = V$ . There exists, uniquely up to constant,  $\phi \in C^\infty(M)$  such that  $\omega + dd^c\phi > 0$  and*

$$(\omega + dd^c\phi)^n = f\omega^n$$

### 3 A sketch of proof

The uniqueness is straightforward. In fact if  $\phi$  and  $\psi$  both satisfy  $\omega + dd^c\phi > 0$ ,  $\omega + dd^c\psi > 0$  and  $(\omega + dd^c\phi)^n = (\omega + dd^c\psi)^n$  then  $D(\phi - \psi) = 0$  as

$$0 = \int_M (\phi - \psi)((\omega + dd^c\phi)^n - (\omega + dd^c\psi)^n) = \int_M d(\phi - \psi) \wedge d^c(\phi - \psi) \wedge T$$

where

$$T = \sum_{j=0}^{n-1} (\omega + dd^c\phi)^j \wedge (\omega + dd^c\psi)^{n-1-j}$$

is a closed (strongly) positive  $(n-1, n-1)$ -form.

We will prove the existence of  $\phi$  under the constraint  $\int_M \phi \omega^n = 0$  (which will be useful to prove that  $(N)$  is locally diffeomorphism later). We will prove that the set  $S$  of  $t \in [0, 1]$  such that there exists  $\phi_t \in C^{k+2, \alpha}(M)$  with  $\int_M \phi_t \omega^n = 0$  that satisfies

$$(\omega + dd^c \phi_t)^n = (tf + 1 - t)\omega^n \quad (2)$$

is both open and close in  $[0, 1]$ , therefore is the entire interval as  $0 \in S$ .

To see that  $S$  is open, one only has to prove that the function  $\mathcal{N}$  defined by

$$\phi \mapsto \mathcal{N}(\phi) = \frac{\det(h_{i\bar{j}} + \phi_{i\bar{j}})}{\det(h_{i\bar{j}})}$$

or in other words  $(\omega + dd^c \phi)^n = \mathcal{N}(\phi)\omega^n$ , is a local diffeomorphism. The differential of  $\mathcal{N}$  is given by

$$D\mathcal{N}(\phi). \eta = \mathcal{N} \Delta \eta$$

with  $\eta$  varies in  $\{\eta \in C^{k, \alpha}(M) : \int_M \eta \omega^n = 0\}$ . and  $\Delta$  is the Laplace-Beltrami operator which is known to be bijective between

$$\left\{ \eta \in C^{k+2, \alpha}(M) : \int_M \eta = 0 \right\} \longrightarrow \left\{ f \in C^{k, \alpha}(M) : \int_M f = 0 \right\}$$

Therefore  $\mathcal{N}$  is a local diffeomorphism and  $S$  is open.

The proof that  $S$  is closed is more technical and is accomplished in 3 steps:

1. Using Arzela-Ascoli theorem, it suffices to show that  $\{\phi_t : t \in S\}$  is bounded in  $C^{k+2, \alpha}$ . Then up to a subsequence, one has the uniform convergence of  $\phi_{t_n}$  and all its partial derivatives of order  $\leq k+1$ . The  $k+2$ -th order follows from (2).
2. Using Schauder theory, prove that the above bound follows from a *a priori estimate*:  
There exists  $\alpha \in (0, 1)$  and  $C(X, \|f\|_{1,1}, 1/\inf_M f) > 0$  such that every  $\phi \in C^4(X)$  satisfying  $(\omega + dd^c \phi)^n = f\omega^n$ ,  $\omega + dd^c \phi > 0$  and  $\int_M \phi \omega^n = 0$  (we will call such  $\phi$  *admissible*) has

$$\|\phi\|_{2, \alpha} \leq C.$$

3. Establish the a priori estimate.

To achieve the a priori estimate, one firstly bounds  $\phi$  in  $C^0$ , then bound  $\|\Delta\phi\|$  and finally establishes the  $C^{2,\alpha}$  estimate. We will give here some detail of the first step. For more detail, see Z. Blocki, *The Calabi-Yau Theorem*.

*Proof of the  $C^0$ -estimate.* Since  $\phi$  is defined up to an additive constant, what we mean by the  $C^0$ -estimate for  $\phi$  is in fact the estimate of

$$\text{osc}_M \phi := \max_M \phi - \min_M \phi$$

by a constant  $C$  that depends only on  $M$  and  $f$ . Without losing of generality, one assumes that  $\int_M \omega^n = 1$  and  $\max_M \phi = -1$ . Therefore  $\|\phi\|_p \leq \|\phi\|_q$  for  $p \leq q < \infty$ .

One has

$$\int_M (-\phi)^p (f-1) \omega^n = \int_M (-\phi)^p dd^c \phi \wedge \left( \sum_{j=0}^{n-1} (\omega + dd^c \phi)^j \wedge \omega^{n-1-j} \right) \quad (3)$$

$$= p \int_M (-\phi)^{p-1} d\phi \wedge d^c \phi \wedge \left( \omega^{n-1} + \sum_{j=1}^{n-1} (\omega + dd^c \phi)^j \wedge \omega^{n-1-j} \right) \quad (4)$$

$$\geq p \int_M (-\phi)^{p-1} d\phi \wedge d^c \phi \wedge \omega^{n-1} \quad (5)$$

$$= \frac{4p}{(p+1)^2} \int_M d(-\phi)^{(p+1)/2} \wedge d^c(-\phi)^{(p+1)/2} \wedge \omega^{n-1} \quad (6)$$

$$= \frac{c_n p}{(p+1)^2} \|D(-\phi)^{(p+1)/2}\|_2^2 \quad (7)$$

where we used the fact that  $\omega + dd^c \phi > 0$  in the inequality, and  $c_n$  is a constant depending only on  $n$ .

Now we use the following Sobolev inequality on  $M$  (i.e. use Sobolev inequality in each chart as a domain of  $\mathbb{R}^m$  then add up the results):

$$\|v\|_{mq/(m-q)} \leq C(M, q) (\|v\|_q + \|Dv\|_q), \quad \forall v \in W^{1,q}(M), q < m$$

with  $v = \phi$ ,  $m = 2n$  the real dimension of  $M$  and  $q = 2$ , then use (7) to bound the  $D\phi$  term:

$$\|(-\phi)^{(p+1)/2}\|_{2n/(n-1)} \leq C_M \left[ \|(-\phi)^{(p+1)/2}\|_2 + \frac{p+1}{\sqrt{p}} \left( \int_M (-\phi)^p (f-1) \omega^n \right)^{1/2} \right]$$

Replace  $p + 1$  by  $p$  and use the fact that  $|\phi| \geq 1$ , one has

$$\|\phi\|_{np/(n-1)} \leq (Cp)^{1/p} \|\phi\|_p, \quad \forall p \geq 2$$

where  $C$  depends only on  $M$  and  $\|f\|_\infty$ .

Repeatedly apply this inequality (this technique is called *Moser's iteration*) one has  $\|\phi\|_{p_{k+1}} \leq (Cp_k)^{1/p_k} \|\phi\|_{p_k}$  where the sequence  $p_k$  is defined by  $p_0 = 2$  and  $p_{k+1} = \frac{n}{n-1} p_k = 2(\frac{n}{n-1})^k$  and

$$\|\phi\|_\infty = \lim_{k \rightarrow \infty} \|\phi\|_{p_k} \leq \|\phi\|_2 \prod_{j=0}^{\infty} (Cp_j)^{1/p_j}$$

with  $\prod_{j=0}^{\infty} (Cp_j)^{1/p_j} = (n/(n-1))^{n(n-1)/2} (2C)^{n/2}$

The fact that  $\|\phi\|_2$  is bounded follows directly from the following lemma.  $\square$

**Lemma 3** ( $L^p$ -boundedness). *For any admissible  $\phi$  with  $\max_M \phi = -1$  one has*

$$\|\phi\|_p \leq C(M, p), \quad \forall 1 \leq p \leq \infty$$

*Proof.* We will prove the lemma with  $p = 1$  first. Let  $g$  be the local potential of the Kähler form  $\omega$ , i.e. a function defined on each chart (not necessarily agrees on zones where charts are glued together) such that  $\omega = dd^c g = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$  where  $g_{i\bar{j}}$  can also be interpreted as  $\frac{\partial^2}{\partial z_i \partial \bar{z}_j} g$ . We also suppose that the function  $g$  is negative on every chart. The fact that  $\omega + dd^c \phi > 0$  is rewritten as  $(g_{i\bar{j}} + \phi_{i\bar{j}}) > 0$  in local coordinates.

Note  $u = g + \phi$  the potential of  $\omega + dd^c \phi$  locally defined on each chart, then  $u$  is negative and plurisubharmonic (psh). For every  $x \in B(y, R)$  one has

$$u(x) \leq \frac{1}{\text{vol}(B(x, 2R))} \int_{B(x, 2R)} u \leq \frac{1}{\text{vol}(B(y, 2R))} \int_{B(y, R)} u$$

where the first inequality is due to plurisubharmonicity and the second is due to  $u \leq 0$ . Therefore

$$\|u\|_{L^1(B(y, R))} \leq \text{vol}(B(y, 2R)) \inf_{B(y, R)} |u|,$$

hence

$$\|\phi\|_{L^1(B(y, R))} \leq \|u\|_{L^1(B(y, R))} \leq \text{vol}(B(y, 2R)) \left( \inf_{B(y, R)} |\phi| + \max_M |g| \right)$$

To see that  $\|\phi\|_1$  is bounded, we apply the following Lemma 4 to the covering of  $M$  by finitely many ball  $B(y_i, R_i)$ ,  $c_i = \text{vol}(B(y_i, 2R_i))$ ,  $d_i = c_i \max_M |g|$  and  $r = 1$ .

The case  $p > 1$  follows analogously using the following estimate: if  $u$  is negative and psh in  $B(y, 2R)$  then

$$\|u\|_{L^p(B(y, R))} \leq C(n, p, R) \|u\|_{L^1(B(y, 2R))}$$

□

**Lemma 4** (Combinatoric). *Let  $M$  be a connected compact manifold covered by finitely many local charts  $\{V_i\}_{i=1}^l$  and  $r, c_i, d_i > 0$ . Then for any continuous function  $\phi$  globally defined on  $M$  such that*

$$\|\phi\|_{L^1(V_i)} \leq c_i \inf_{V_i} |\phi| + d_i, \quad \min_M |\phi| \leq r,$$

one has  $\|\phi\|_1 := \sum_i \|\phi\|_{L^1(V_i)} \leq C(\{V_i\}, \{c_i\}, \{d_i\}, r)$

*Proof.* Let  $p$  be a point in  $M$  where  $|\phi|$  attains its minimum. Since  $M$  is connected, for every  $V_i$ , there exists a sequence  $V_{i_k}, 0 \leq k \leq l$  such that

$$i_0 = i, \quad V_{i_k} \cap V_{i_{k+1}} \neq \emptyset, \quad p \in V_{i_l}$$

One has

$$\begin{aligned} \|\phi\|_{L^1(V_{i_k})} &\leq c_{i_k} \inf_{V_{i_k}} |\phi| + d_{i_k} \leq c_{i_k} \inf_{V_{i_k} \cap V_{i_{k+1}}} |\phi| + d_{i_k} \\ &\leq c_{i_k} \frac{1}{\text{vol}(V_{i_k} \cap V_{i_{k+1}})} \|\phi\|_{L^1(V_{i_{k+1}})} + d_{i_k} \end{aligned}$$

Repeatedly apply this inequality for  $k = 0, \dots, l-1$ , one has

$$\begin{aligned} \|\phi\|_{L^1(V_i)} &\leq A(i, \{V_j\}, \{c_j\}, \{d_j\}) \|\phi\|_{L^1(V_{i_l})} + B(i, \{V_j\}, \{c_j\}, \{d_j\}) \\ &\leq A(i, \{V_j\}, \{c_j\}, \{d_j\}) (c_{i_l} r + d_{i_l}) + B(i, \{V_j\}, \{c_j\}, \{d_j\}) \end{aligned}$$

Take the sum for all  $i = 0, \dots, l$  and the result follows. □

## 4 Calabi-Yau manifold

Recall that we defined a Calabi-Yau manifold to be a compact Riemannian manifold of dimension  $2n$  with holonomy contained in  $SU(n)$ . We also remark, using parallel transport, the existence of a compatible complex structure ( $U(n)$  suffices) and a holomorphic form non-vanishing at every point. We present here some equivalent definitions of compact Calabi-Yau manifolds.

**Theorem 5.** *Let  $X$  be a compact manifold of Kähler type and complex dimension  $n$  then:*

1. *The followings are equivalent*
  - (a) *There exists a Kähler metric such that the global holonomy is in  $SU(n)$ .*
  - (b) *There exists a holomorphic  $(n,0)$  form that vanishes nowhere.*
  - (c) *The canonical bundle  $K_X$  is trivial.*
  - (d) *The structure group of  $X$  can be reduced to  $SU(n)$ .*
2. *The following are equivalent. If  $X$  is simply-connected, they are equivalent with the 4 statements above.*
  - (a) *There exists a Kähler metric such that the local holonomy is in  $SU(n)$ .*
  - (b) *The canonical bundle  $K_X$  is flat.*
  - (c) *There exists a Kähler metric such that the Ricci curvature vanishes.*
  - (d) *The first Chern class vanishes.*

The proof is straightforward (see Manuscript) with the only non-trivial part is when one needs Calabi-Yau theorem to construct Ricci-flat metric.  
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