

Divisors, Picard group and Kodaira embedding theorem

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1 Divisors and Picard group

1.1 Holomorphic line bundles and first Chern class

A complex line bundle is a 2 dimensional vector bundle with a complex structure on each fiber, i.e. each change of coordinates $g_{ij} : U_j \cap U_i \times \mathbb{R}^2 \rightarrow U_i \cap U_j \times \mathbb{R}^2$ is i -linear, i.e. g_{ij} can be represented by a function $U_i \cap U_j \rightarrow \mathbb{C}$.

A holomorphic line bundle is a complex line bundle that is also a complex manifold with the projection being holomorphic. In the same notation, the g_{ij} are now holomorphic functions.

A hermitian metric on a line bundle L is a positive sesquilinear form on each fiber. To define the Chern form of L , let U be an open set of X over which L is trivialized and s_x is a holomorphic section of L over U that is non-vanishing, then one defines

$$\omega_{L,h} = \frac{1}{2\pi i} \partial \bar{\partial} \log |s|_h^2$$

which is independant of s since the ratio of two different s is in $\mathcal{O}^*(U)$.

Remark 1. *A Chern form is a real (1,1) form.*

Proposition 0.1. *The set of isomorphic class of holomorphic line bundle is in one-to-one correspondance to $H^1(X, \mathcal{O}_X^*)$*

The proof of this fact is straightforward, but it is worth to remark that this result convinces us that the natural mapping $\check{H}^1(\mathcal{U}, X) \longrightarrow \check{H}^1(\mathcal{V}, X)$ where \mathcal{U} is a finer open covering than \mathcal{V} is injective, since a line bundle is completely defined by *one* set of change of coordinates (g_{ij}) .

Now the Chern class of a holomorphic line bundle is the class of $\omega_{L,h}$ in $H^2(X, \mathbb{Z})$, which turns out to be independent of h and is in fact lies in the image of $H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}) = H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. In fact, the class of $\omega_{L,h}$ can be defined using the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

where the injective arrow is the multiplication by $i2\pi$ and the surjective one is exponential. The Chern map is in fact $H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z})$. To prove this, one uses a double complex whose horizontal is the de Rham resolution and vertical is the Čech resolution and diagram chasing.

1.2 Divisors, line bundles and sheaves

Remark 2. *1. A holomorphic line bundle is the same as a locally free \mathcal{O}_X -module of rank 1.*

2. An isomorphic class of line bundles is the same as a locally free isomorphic sheaf of \mathcal{O}_X -module of rank 1.

1.2.1 From divisors to Picard group

A divisor is a formal sum of irreducible hypersurface, which can also be interpreted as an element of $\check{H}^0(X, K_X^*/\mathcal{O}_X^*)$, which gives a mapping $Div(X) \rightarrow \check{H}^0(X, K_X^*/\mathcal{O}_X^*)$ with principal divisors being exactly sent to elements of K_X^*/\mathcal{O}_X^* coming from K_X^* . multiplicative).

Since the following sequence

$$0 \rightarrow \mathcal{O}_X^* \rightarrow K_X^* \rightarrow K_X^*/\mathcal{O}_X^* \rightarrow 0$$

is exact, one has an application $\mathcal{O} : Div(X) = H^0(X, K_X^*/\mathcal{O}_X^*) \rightarrow H^1(X, \mathcal{O}_X^*) = Pic(X)$. The kernel of \mathcal{O} corresponds to the the space of principal divisors. It is however worth having details of the application \mathcal{O} .

Let $D = (U_i, f_i) \in Div(X)$ where f_i are meromorphic function on U_i with $f_i/f_j \in \mathcal{O}_X^*$, then $\mathcal{O}(D)$ is defined as following:

$$\mathcal{O}(D)(U_i) = f_i^{-1} \mathcal{O}_X(U_i)$$

. Note that if D is effective, i.e. $f_i \in \mathcal{O}_X(U_i)$ then $\mathcal{O}(D)$ is the sheaf of holomorphic functions vanishing on D .

Remark 3. *To resume, here are some basic consequence of the above discussion: If D is effective then*

1. *If D is effective then $H^0(X, \mathcal{O}(D)) \neq 0$.*
2. *If D is effective then $\mathcal{O}(-D)$ is the sheaf of holomorphic function vanishing on D . Therefore $\mathcal{O}(-D)$ can be viewed as a ideal subsheaf of K_X and one has the following exact sequence:*

$$0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

where \mathcal{O}_D is the sheaf of "regular functions" on D .

3. *If L is a holomorphic line bundle and $0 \neq s \in H^0(X, L)$ then $\mathcal{O}(Z(s)) \equiv L$*
4. *If D is effective then $\mathcal{O}(D)$ has a non-zero global section, for example section $1 = (U_i, f_i)$*

1.2.2 The corresponding line bundle of $\mathcal{O}(Y)$

Proposition 0.2. *Let Y be a hypersurface of X , then the line bundle $\mathcal{O}(Y)$ is isomorphic to $\mathcal{N}_{Y,X}$ the normal line bundle of Y in X .*

By consequence, $K_Y = (K_X \otimes \mathcal{O}(Y))|_Y$.

2 Example: Projective space

2.1 $\mathcal{O}(d)$ and its sections

Let's have some examples for the point of view discussed above, starting with the torsion sheaves $\mathcal{O}(d)$.

The sheaf $\mathcal{O}(-1)$, called tautological sheaf, is an invertible sheaf on $\mathbb{P}_{\mathbb{C}}^n$ such that the fiber over $l \in \mathbb{P}_{\mathbb{C}}^n$ of the corresponding line bundle is l itself. Let $l = [x_0 : \dots : x_n]$ in U_i then a point in l is of form $t_i[\frac{x_0}{x_i} : \dots : \frac{x_n}{x_i}]$ with coordinates in U_i being t_i . So the change of coordinates from chart U_i to U_j is, since $\frac{t_j}{x_j} = \frac{t_i}{x_i}$:

$$g_{ji} = \frac{x_i}{x_j}$$

One notes by $\mathcal{O}(1)$ the dual of $\mathcal{O}(-1)$ and $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$ and $\mathcal{O}(-d) = \mathcal{O}(-1)^{\otimes d}$

Now if an invertible sheaf \mathcal{L} with $\mathcal{L}(U_i) = \frac{1}{f_i}\mathcal{O}_X(U_i)$, the change of coordinates of the corresponding line bundle from chart U_i to U_j is $g_{ji} = \frac{f_j}{f_i}$. So for $\mathcal{O}(1)$, one has $\frac{f_j}{f_i} = \frac{x_j}{x_i}$, i.e. there exists a linear combination A of x_0, \dots, x_n such that $f_i = \frac{A}{x_i}$ is a holomorphic function corresponding to the 1-section viewed in chart U_i . As presented in the previous section, **$\mathcal{O}(1)$ is the associated line bundle of a hyperplane defined by the equation $A = 0$.**

Similarly, $\mathcal{O}(d)$ is the associated line bundle of a hypersurface A_d defined by a homogenous equation of degree d , and $\mathcal{O}(d)$ is the line bundle associated to the sheaf of holomorphic functions vanishing on A_d .

2.2 Line bundles and maps to projective space, Veronese embedding

A linearly independent family s_0, \dots, s_N of global sections of a holomorphic line bundle L defines a holomorphic map $X \setminus Bs((s_i)) \rightarrow \mathbb{P}_{\mathbb{C}}^N$ where $Bs((s_i))$ is the set of basepoints of (s_i) where all the sections s_i vanish.

The global sections of $\mathcal{O}(d)$ over $\mathbb{P}_{\mathbb{C}}^n$ are $H^0(\mathbb{P}_{\mathbb{C}}^n, \mathcal{O}(d)) = \mathbb{C}[z_0, \dots, z_n]_d$ the vector space of homogenous polynomial of degree d . The corresponding projective map is in fact a embedding, called Veronese embedding.

2.3 Canonical bundle and Euler sequence

$$K_{\mathbb{P}_{\mathbb{C}}^n} = \mathcal{O}(-n-1)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus n} \longrightarrow \mathcal{T}_{\mathbb{P}_{\mathbb{C}}^n} \longrightarrow 0$$

3 Blowing-up

3.1 Blowing-up

Proposition 0.3 (Adjunction). *Let X be a complex manifold, $x \in X$ and $\pi : \hat{X} \longrightarrow X$ is the blow-up of X at x and E be the corresponding exceptional divisor, then*

$$K_{\hat{X}} = \pi^* K_X \otimes \mathcal{O}((n-1)E).$$

As a consequence, $\mathcal{O}(E)|_E = \mathcal{O}(-1)$ where $\mathcal{O}(-1)$ is the tautologic sheaf over $E = \mathbb{P}_{\mathbb{C}}^{n-1}$.

Proof. The appearance of the number $n-1$ is natural and can be explained as follow. First note that

1. in $\hat{X} \setminus E$, there is no difference between $K_{\hat{X}}$ and K_X .
2. π_* send every tangent vector of E to the tangent vector 0 at $x \in X$,
3. the pull-back $\pi^*\omega$ of an n -form on X always vanishes on E therefore cannot generate $K_{\hat{X}}$.

Our correction of this should be dividing $\pi^*\omega$ by f^k where f is the equation defining E in \hat{X} , i.e. tensoring π^*K_X by an appropriate multiple of $\mathcal{O}(E)$ depending on the order of vanishing of $\pi^*\omega$ at E , which we claim to be $n-1$.

Here is the argument I used to convince myself: this vanishing order is that of the ratio of $\pi^*\omega$ and a non-zero n -form (says the standard in the base formed by $n-1$ tangent vectors e_i^E of E and the normal vector v of E in X), i.e. the vanishing order of $\pi^*\omega(e_i^E, v)$. Each e_i^E plugged into $\pi^*\omega$ adds one order of vanishing resulting in $n-1$.

Here is the argument I would use to convince others: WLOG, suppose that $X = \mathbb{C}^n$ and $x = 0$, then \hat{X} can be seen as a subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$ with the coordinates in each chart $U_i = \{(x_1, \dots, x_n, [p_1 : \dots : p_n]) : p_i \neq 0\}$ being $(\frac{p_1}{p_i}, \dots, \frac{p_n}{p_i}, \zeta_i)$ with $z_k = \frac{p_k}{p_i} \zeta_i$. The map π is given in local coordinates as

$$(\frac{p_1}{p_i}, \dots, \frac{p_n}{p_i}, \zeta_i) \mapsto (\frac{p_1}{p_i} \zeta_i, \dots, \zeta_i, \dots, \frac{p_n}{p_i} \zeta_i)$$

The pull-back of ω is

$$d\left(\frac{p_1}{p_i}\zeta_i\right) \wedge \cdots \wedge d\zeta_i \wedge \cdots \wedge d\left(\frac{p_n}{p_i}\zeta_i\right) = \zeta_i^{n-1} d\left(\frac{p_1}{p_i}\right) \wedge \cdots \wedge d\zeta_i \wedge \cdots \wedge d\left(\frac{p_n}{p_i}\right)$$

which vanishes with order $n - 1$.

For the consequence, note that $\mathcal{K}_E = \mathcal{O}(-n) = (K_{\hat{X}} \otimes \mathcal{O}(E))|_E = (\pi^*K_X \otimes \mathcal{O}(nE))|_E$, but π^*K_X is trivial over E , therefore $\mathcal{O}(E)|_E = \mathcal{O}(-1)$. \square

4 Kodaira vanishing theorem

Theorem 1 (Kodaira vanishing). *Let X be a compact complex manifold of dimension n and L is a positive holomorphic line bundle on X , i.e. there exists a hermitian metric h on L such that the Chern form $\omega_{L,h}$ is positive (i.e. a Kahler form). Then*

$$H^q(X, \Omega_X^p \otimes \mathcal{L}) = 0 \quad \forall p + q > n.$$

In particular,

$$H^i(X, K_X \otimes \mathcal{L}) = 0 \quad \forall i > 0.$$

Proof. Since $\mathcal{H}^{0,q}(E) \simeq H^q(X, \mathcal{E})$ for all hermitian holomorphic vector bundle E with \mathcal{E} the corresponding sheaf of holomorphic sections. One needs to prove that all harmonic form in $\mathcal{A}_X^{p,q}(L)$ vanishes for $p + q > n$. This comes from the following two identities: Let ∇ be the Chern connection on L and $\nabla = \nabla' + \nabla''$ be its decomposition to $(1, 0)$ and $(0, 1)$ operators and Δ'_L be the Laplacian corresponding to ∇' then

1. $\Delta_L = \Delta'_L + 2\pi[L, \Lambda]$ where L and Λ are Lefschetz operators.
2. $[L, \Lambda] = (k - n)Id_{\mathcal{A}_X^k}$

Therefore $0 \leq (\alpha, \Delta'_L \alpha)_{L^2} = 2\pi(n - k)(\alpha, \alpha)_{L^2} \leq 0$ \square

5 The traditional proof of Kodaira embedding theorem

Theorem 2 (Kodaira embedding). *Let X be a compact complex manifold with L a positive holomorphic line bundle on X . Then L is generated by finitely many of its global sections and X can be embedded in a projective space \mathbb{CP}^N with N sufficiently large.*

Proof. The following approach is straight-forward: one shows that at every $x \in X$, there is a global (holomorphic) section s_x of $L^{\otimes m_x}$ such that $s_x(x) \neq 0$ then by compactness one can choose finitely many such sections and m_x which can be guaranteed to generate every germs at x of $L^{\otimes m}$ with $m = \max m_x$. That is one needs to prove that

$$H^0(X, \mathcal{L}^{\otimes m_x}) \twoheadrightarrow H^0(x, \mathcal{L}^{\otimes m_x}|_x)$$

is surjective. Let $\pi : \hat{X} \rightarrow X$ be the blow-up of X at x and $E = \pi^{-1}(x)$ be the corresponding exceptional divisor then one has

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\begin{tikzcd}
H^0(X, \mathcal{L}^{\otimes m_x}) \ar[r, twoheadrightarrow] \ar[d] & H^0(x, \mathcal{L}^{\otimes m_x}|_x) \\
H^0(\hat{X}, \pi^* \mathcal{L}^{\otimes m_x}) \ar[r, twoheadrightarrow] & H^0(E, \pi^* \mathcal{L}^{\otimes m_x}|_E)
\end{tikzcd}
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\label{fig:kodaira-blowup}

$$\begin{array}{ccc} H^0(X, \mathcal{L}^{\otimes m_x}) & \twoheadrightarrow & H^0(x, \mathcal{L}^{\otimes m_x}|_x) \\ \downarrow & & \downarrow \\ H^0(\hat{X}, \pi^* \mathcal{L}^{\otimes m_x}) & \twoheadrightarrow & H^0(E, \pi^* \mathcal{L}^{\otimes m_x}|_E) \end{array}$$

Figure 1: Insert caption [fig:kodaira-blowup]

where the vertical arrows are isomorphic (easy to see, maybe the right one needs Hartog). So one only needs to see that $H^0(\hat{X}, \pi^* \mathcal{L}^{\otimes m_x}) \rightarrow H^0(E, \pi^* \mathcal{L}^{\otimes m_x}|_E)$. Since E is a divisor of \hat{X} , one has $0 \rightarrow \mathcal{O}(E) \rightarrow \mathcal{O}_{\hat{X}} \rightarrow \mathcal{O}_E \rightarrow 0$ hence $0 \rightarrow \mathcal{O}(E) \otimes \pi^* \mathcal{L}^{\otimes m_x} \rightarrow \pi^* \mathcal{L}^{\otimes m_x} \rightarrow \pi^* \mathcal{L}^{\otimes m_x}|_E \rightarrow 0$.

It remains to prove that $H^1(\hat{X}, \mathcal{O}(E) \otimes \pi^* \mathcal{L}^{\otimes m_x})$, or by 1, that $\mathcal{O}(E) \otimes \pi^* \mathcal{L}^{\otimes m_x} \otimes K_{\hat{X}}^{-1} = \mathcal{O}(-nE) \otimes \pi^*(\mathcal{L}^{\otimes m_x} \otimes K_X^{-1})$ is positive, where we used the fact that $K_{\hat{X}} = \pi^* K_X \otimes \mathcal{O}_{\hat{X}}((n-1)E)$.

Note that on $\hat{X} \setminus E$, one can choose m_x large enough such that $\mathcal{L}^{\otimes m_x} \otimes K_X^{-1} \times \mathcal{O}(-nE)$ is positive. It remains to observe $E \subset \hat{X}$ which is in fact \mathbb{CP}^{n-1} . But $\mathcal{O}(-E)|_E \equiv \mathcal{O}(1)$ is positive, which concludes the proof. \square

6 An analytic proof by Donaldson