# From Busemann function to Cheeger-Gromoll splitting

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We will prove the following result by Cheeger and Gromoll by a slightly modified approach of A. Besse.

**Theorem 1** (Cheeger-Gromoll). *Let $M$ be a complete, connected Riemannian manifold with non-negative Ricci curvature. Suppose that $M$ contains a line then $M$ is isometric to $M' \times \mathbb{R}$ with $M'$ a complete, connected Riemannian manifold with non-negative Ricci curvature.*

Note that the notion of geodesic ray or geodesic line used here is rather strict: A geodesic line $\gamma$ is a geodesic parameterized by $\mathbb{R}$ such that the distance between two point is exactly the distance *on the geodesic*, for example, geodesic line, if it passes by $p \in M$ with velocity $v$ of norm 1, satisfies

$$d(exp_p(tv), exp_p(-sv)) = s + t, \quad \forall s, t > 0$$

# 1 Busemann function

Let $\gamma$ be a geodesic ray. We construct the Busemann function $b$ associated to the ray as

$$b(x) = \lim_{t\to+\infty} (t - d(x, \gamma(t)))$$

where the limit exists because the sequence $f_t : x \mapsto t - d(x, \gamma(t))$ is non-decreasing and bounded above by $d(x, \gamma(0))$. The convergence is also uniform in every compact of $M$.

In Euclidean space for example, the Busemann function is the orthogonal projection on $\gamma$. We will see that in a Riemannian manifold with non negative curvature, the Busemann function will serve as a projection.

Now with a fixed $x_0 \in M$, the tangent vectors at $x_0$ of the geodesics connecting $x_0$ and $\gamma(t)$ is in the unit sphere of $T_xM$, which is compact. Let $X$ be a limit point of these tangents vectors, we defined

$$b_{X,t}(x) = b(x_0) + t - d(x, exp_{x_0}(tX))$$

where $exp_{x_0}(tX)$ is the geodesic starting at $x_0$ with velocity $X$.

**Remark 1.** 1. *From the construction of* $X$*, one has* $b(x_0)+t = b(exp_{x_0}(tX))$*, therefore* $b_{X,t} \leq b$ *with equality in* $x_0$*. We say that* $b$ *is supported by* $b_{X,t}$ *at* $x_0$*. In general a function* $f$ *is supported by* $g$ *at* $x_0$ *if* $f(x_0) = g(x_0)$ *and* $f \geq g$ *in a neighborhood of* $x_0$*.*

2. $b_{X,t}$ *is smooth and a computation in local coordinate gives* $\Delta b_{X,t} \geq -\frac{\dim M - 1}{t}$

3. $\|\nabla b_{X,t}\| = 1$

The estimation given on the second point of Remark 1 is established using Jacobi fields:

**Lemma 2.** *The function* $f(x) = d(x, x_0)$ *satisfies at a point* $x$ *out of the cut-locus of* $x_0$*:*

$$\nabla f(x) \leq \frac{n-1}{l}$$

*where* $n = \dim M, l = d(x, x_0) = f(x)$ *in Riemannian manifold* $M$ *with non-negative Ricci curvature.*

*Proof.* Let $N(t), 0 \leq t \leq l$ be the velocity of the geodesic $\gamma$ from $x_0$ to $x$, and $E_1, \ldots, E_{n-1}, N$ be a parallel frame along $\gamma$. Let $J_i$ be the unique Jacobi fields along $\gamma$ with $J_i(l) = E_i(l)$ and $J_i(0) = 0$ (existence and uniqueness of $J_i$ is due to the fact that $x$ is not in the cut-locus).

Then basic manipulation of Jacobi fields gives (without the fact that curvature is non-negative):

$$\Delta f(x) = \int_0^l dt \sum_{i=1}^{n-1} (< \nabla_N J_i, \nabla_N J_i > - < R(N, J_i)J_i, N >) = \sum_{i=1}^{n-1} I_\gamma(J_i, J_i)$$

where $I_\gamma$ is the index form of $\gamma$. Note that the Jacobi fields $J_i$ coincide with the fields $\frac{t}{l}E(t)$ at 0 and $l$, therefore by the *fundamental inequality* of index form (see Sakai Takashi, *Riemannian geometry* for details about Jacobi fields and Fundamental inequality of index form):

$$I_\gamma(J_i, J_i) \leq I_\gamma(\frac{t}{l}E_i, \frac{t}{l}E_i)$$

hence

$$\Delta f(x) \leq \int_0^l \sum_{i=1}^{n-1} < \nabla_N \frac{t}{l}E_i, \nabla_N \frac{t}{l}E_i > - < R(N, \frac{t}{l}E_i)\frac{t}{l}E_i, N >$$

The curvature term being $\frac{t^2}{l^2}Ric(N, N)$ therefore non-negative, one has

$$\Delta f(x) \leq \int_0^l dt \sum_{i=1}^{n-1} < \nabla_N \frac{t}{l}E_i, \nabla_N \frac{t}{l}E_i > = \frac{n-1}{l}.$$

$\square$

We also note that for Theorem 1 it suffices to show that $b$ is harmonic. In fact, from the smoothness one has $\nabla b(x_0) = \nabla b_{X,t}(x_0)$, which means $\|\nabla b\| = 1$ at every point in $M$. For each point $y \in M$, there exists a unique $x$ with $b(x) = 0$ and time $t$ when the flow of $\nabla b$ starting at $x$ arrives at $y$. $M$ is therefore homeomorphic to $\bar{M} \times \mathbb{R}$ by the map $F : y \mapsto (x, t)$. In order that $F$ is an isometry, it suffices to prove that the gradient field $\nabla b$ is parallel. In fact, $\bar{M}$ being equiped with the restriction of the metric on $M$, the fact that $F$ is isometric is equivalent to the the fact that the flow $\Phi^t$ of $\nabla b$ is isometric for every time $t$, which means $\frac{d}{dt} < \Phi^t_* u, \Phi^t_* u >$ vanishes at $t = 0$. But

$$\frac{d}{dt} < \Phi^t_* u, \Phi^t_* u > |_{t=0} = 2 < \nabla_{\partial t} \Phi^t_* u, u > |_{t=0} = 2 < \nabla_u \nabla b, u >$$

where in the second equality we used Schwarz lemma for commuting derivatives of $\Phi(t,x) = \Phi^t(x)$. The vanishing of $< \nabla_u \nabla b(x), u >$ for every vector $u$ is, by bilinearity, equivalent to that of $\nabla_u \nabla b$ for every $u$, meaning that $\nabla b$ is parallel.

The fact that $\nabla b$ is parallel is due to a simple computation:

$$Ric(N,N) = -N(\Delta b) - \|\nabla N\|^2$$

where $\|\nabla N\|^2 = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} < \nabla_{E_i} N, E_j >^2$. We see that $N = \nabla b$ is parallel if $\Delta b = 0$.

**Remark 2.** 1. *One can show (see A. Besse) that every gradient field $\nabla b$ of norm 1 at every point is actually harmonic.*

2. *Using de Rham decomposition, one has directly the splitting of $M$ if it is simply-connected since $N$ is parallel and $M$ is complete.*

## 2 Harmonicity

The Busemann function associated to a geodesic ray is subharmonic, it is a consequence of the following lemma.

**Lemma 3.** *In a connected Riemannian manifold, if a continuous function $f$ is supported at any point $x$ by a family $f_\epsilon$ (depending on $x$) with $\Delta(f_\epsilon) \leq \epsilon$, then $f$ can not attains its maximum (unless when $f$ is constant).*

*Proof.* Given a small geodesic ball $B$, suppose that we have a function $h$ on $B$ with $\Delta h < 0$ in $B$ and $f + h$ attains maximum at $x$ in the interior of $B$. Then $f_\epsilon + h$ also attains maximum at $x$, which means $\Delta f_\epsilon + \Delta h \geq 0$, which is contradictory.

For the construction of the function $h$, one suppose that $B$ is small enough such that $f|_{\partial B} \leq max_B f =: f(x_0)$ and equality is not attained at every points in $\partial B$. Then choose

$$h = \eta(e^{\alpha\phi} - 1)$$

with and $\phi(x) = -1$ if $x \in \partial B$ and $f(x) = f(x_0)$, $\phi(x_0) = 0$, $\nabla\phi \neq 0$ and a large $\alpha$ such that

$$\Delta h = \eta(-\alpha^2 \|\nabla\phi\| + \alpha\Delta\phi)e^{\alpha\phi}.$$

is negative. □

Now for subharmonicity of $b$, given a harmonic function $h$ that coincides with $b$ in the boundary $\partial B$ of a geodesic ball $B$, then $b-h$ is supported by $b_{X,t}-h$ with $\Delta(b_{X,t}-h) \to 0$ as $t$ tends to $+\infty$, therefore $b-h \leq (b-h)|_{\partial B} = 0$ in $B$. We have just proved the following lemma:

**Corollary 3.1.** *The Busemann function of a geodesic ray in a Riemannian manifold $M$ with non-negative Ricci curvature is subharmonic.*

Now let $b_+$ be the function previously constructed for the ray $\gamma|_{[0,+\infty[}$ and $b_-$ the Busemann function for the ray $\tilde{\gamma}|_{[0,+\infty[}$ where $\tilde{\gamma}(t) = \gamma(-t)$. Note that $b_+ + b_- \leq 0$ with equality on the line $\gamma$, but the sum is subharmonic therefore by maximum principle $b_+ + -b_- = 0$ and $b$ is harmonic therefore smooth. The splitting theorem of Cheeger-Gromoll follows.

# 3 Application

A consequence of Theorem 1 is the following result from J.Cheeger- D.Gromoll, *The splitting theorem for manifold of nonnegative Ricci curvature* (Theorem 2)

**Theorem 4.** *Let $M$ be a compact Riemannian manifold with non-negative Ricci curvature, then the universal covering space of $M$ is of form $\tilde{M} = \mathbb{R}^n \times \bar{M}$ where $\bar{M}$ does not contain any lines. Then $\bar{M}$ is compact.*

*Proof.* It suffices to prove that if $\bar{M}$ is not compact, then it contains a line. In fact, it is easy to see that such $\bar{M}$ must contains a (strict) geodesic ray. In fact it is obvious that with a fixed $p \in M$ the function

$$F : v \mapsto \inf\{t > 0 : d(p, exp_p(tv)) < t\}$$

defined on the unit ball $U_p$ of $T_p\bar{M}$ is upper semi-continuous. Therefore if $F(v) < \infty$ for all unit tangent vector $v$ at $p$ then $F$ is bounded above in $U_p$ by a constant $c$. Therefore $\bar{M} \subset exp_p(cU_p)$ which is compact (contradiction). Therefore there exists a minimal ray at every point $p \in \bar{M}$.

The existence of a line in general might not be true, the only extra property of $\bar{M}$ that we will need is that it has a (fundamental) domain $K$ compact and a family $\sigma_i$ of isometries such that $\bar{M} = \cup_i \sigma_i K$.

Let us first prove that such domain $K$ exists. Remark that every isometry of $\mathcal{M}$ acts separately on $\mathcal{M}$, i.e. of form $\sigma(u) = (\sigma_1(x), \sigma_2(y))$ for $u = (x, y) \in \mathcal{M}$ with $g_1, g_2$ isometries of $\mathbb{R}^n$ and $\bar{M}$. This can be seen by the uniqueness part of de Rham decomposition or simply by noticing that a tangent vector in the $T_x\mathbb{R}^n$ component is characterized by the fact that its geodesics is a line. As the action of $\pi_1(M)$ on $\tilde{M}$ is free and proper, it has a fundamental domain $H$. We then can choose $K$ to be the projection of $H$ on $\bar{M}$ and $\{\sigma_i\}$ to be the projection of $\pi_1(M)$ on $Isom(\bar{M})$.

Now let $\gamma$ be a minimal ray starting from $p \in M$, for each $x \in \gamma$ there exists an isometry $\sigma$ of $\bar{M}$ such that $\sigma(x) \in K$. By compactness of $K$, there exists a sequence $t_n \to +\infty$ with $x_n = \gamma(t_n)$, $v_n = \dot{\gamma}(t_n)$ that satisfies $y_n = \sigma_n(x_n) \to y \in K$ and even more, $(\sigma_n)_* v_n \to v \in T_y\bar{M}$ in the tangent bundle $T\bar{M}$. Then the geodesic of $\bar{M}$ starting at $y$ with vector $v$ is a line. In fact it suffices to prove that $d(exp_y(tv), exp_y(-sv)) = s + t$ for $s, t > 0$, but for $n$ large enough that $t_n > s$ one has

$$d(exp_{y_n}(tv_n), exp_{y_n}(-sv_n)) = s + t$$

then let $n \to +\infty$, one sees that $\bar{M}$ contains a geodesic line, which is contradictory. $\square$

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