

Hodge decomposition and Kodaira embedding theorem

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This is my review of lectures 15-19 of Denis Auroux course whose goal is to establish Hodge theory for compact Kähler varieties and present a proof of Donaldson for the Kodaira embedding theorem.

1 Hodge theory

1.1 Operators and their dual

1.1.1 Scalar product on $\Omega^k(M)$

The scalar product on V induces one on $\Omega^k(V)$ by setting $\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle u_i, v_j \rangle)$.

Example 1. $\langle \sum \alpha_I dx^I, \beta_J dx^J \rangle = \sum \alpha_I \beta_I$ if $\{\frac{\partial}{\partial x^i}\}$ form an orthonormal basis.

1.1.2 Hodge star and Hodge dual

Definition 1. The **Hodge star** is defined from $\Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ such that $\alpha \wedge * \beta = \langle \alpha, \beta \rangle \text{vol}$ where vol is the volume form.

Remark 1. 1. An example: $*dx^I = dx^{I^C}$ if $\{\frac{\partial}{\partial x^i}\}$ form an orthonormal basis and the complement I^C is chosen so that $\text{sgn}(I, I^C) = 1$.

2. Note that $** = (-1)^{k(n-k)}$

The **Hodge dual** of an operator P will be defined such that $\langle P\alpha, \beta \rangle_{L^2} = \langle \alpha, P^*\beta \rangle_{L^2}$ where the $\langle \cdot, \cdot \rangle_{L^2}$ is the integral of $\langle \cdot, \cdot \rangle$ over M . For example,

Definition 2. Let d be the coboundary operator then $d^* : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ is defined by $d^* = (-1)^{n(k-1)+1} * d *$

Definition 3. The **de Rham-Laplace** operator is defined by

$$\Delta = dd^* + d^*d = (d + d^*)^2$$

The space of **harmonic forms** is $\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) : \Delta\alpha = 0\}$.

Remark 2. 1. $\Delta^* = \Delta$.

2. $\langle \Delta\alpha, \alpha \rangle = \|d^*\alpha\|^2 + \|d\alpha\|^2$

3. A harmonic form is closed and co-closed.

1.2 Elliptic theory and Hodge theorem for Riemannian manifolds

1.2.1 Symbol of a differential operator

Definition 4. A mapping $L : \Gamma(E) \longrightarrow \Gamma(F)$ where E, F are vector bundles on a manifold M is called a **differential operator** of order k if in local coordinates,

$$L(s) = \sum_{|\alpha| \leq k} A_\alpha(x) \frac{\partial^{|\alpha|} s}{\partial x^\alpha}$$

where $A_\alpha(x)$ is a matrix with C^∞ coefficients.

The **symbol** of L is $\sigma_k(L, \xi) = \sum_\alpha A_\alpha \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} \in \text{Hom}(E_x, F_x)$ where $\xi = \sum \xi_i dx^i \in T^*M$ in the same coordinate as A_α .

Remark 3. 1. $A_\alpha(x)$ depends on the local coordinates and does not transform naturally when one passes from one coordinates to another. In other words, $A_\alpha(x)$ is not in $\text{Hom}(E_x, F_x)$.

2. However, the definition of differential operator does not depend on local coordinates.
3. The symbol transforms naturally (linearly) between coordinates.

From the third remark, one can define:

Definition 5. A differential operator L is called **elliptic** if its symbol $L(x, \xi) : E_x \longrightarrow F_x$ is isomorphic.

1.2.2 Elliptic operators

Theorem 1 (Elliptic operator). Every elliptic operator $L : \Gamma(E) \longrightarrow \Gamma(F)$

1. has a pseudoinverse, i.e. there exists $P : \Gamma(F) \longrightarrow \Gamma(E)$ such that $L \circ P - id_{\Gamma(F)}$ and $P \circ L - id_{\Gamma(E)}$ are smooth operators.
2. is extended to a Fredholm operator $L_s : W^s(E) \longrightarrow W^{s-k}(F)$, i.e. $\ker L = \ker L_s$ and $\text{coker } L_s$ are finite dimensional, $\text{Im } L_s$ is closed.

Moreover, if $L : \Gamma(E) \longrightarrow \Gamma(E)$ is elliptic and self-adjoint then there exists $H_L, G_L : \Gamma(E) \longrightarrow \Gamma(E)$ such that

1. $\text{Im } H_L \subset \ker L$, $id_{\Gamma(E)} = H_L + L \circ G_L = H_L + G_L \circ L$.
2. H_L, G_L extend to $W^s(E) \longrightarrow W^s(E)$.
3. $\Gamma(E) = \ker L \oplus_{\perp L^2} \text{Im } L \circ G_L$.

Theorem 2 (Hodge). Let M be a compact, oriented Riemannian manifold, then

1. $\Omega^k(M) = \mathcal{H}^k(M) \oplus_{\perp L^2} \text{Im } d \oplus_{\perp L^2} \text{Im } d^*$.
2. The projection $\mathcal{H}^k(M) \longrightarrow H_{dR}^k(M, \mathbb{R})$ is isomorphic. In other words, each class is uniquely represented by a harmonic form.

1.3 Hodge decomposition for Kähler manifolds

In case of Kähler manifolds, one has the Hodge decomposition of cohomology which comes from the following two remarks:

1. The Hodge star $* : \Omega^{p,q} \longrightarrow \Omega^{n-q,n-p}$. This is due to the compatible complex structure.

2. The auxiliary operator $L : \alpha \longrightarrow \omega \wedge \alpha$ and its relation with d . This is due to the compatible symplectic structure.

We resume in the following table the definition, domain and Hodge dual of some operators.

Operator	Domain	Definition	Dual
L	$\Omega^{p,q} \longrightarrow \Omega^{p+1,q+1}$	$\alpha \mapsto \omega \wedge \alpha$	$L^* = (-1)^{p+q} * L *$
d_c	$\Omega^k \longrightarrow \Omega^{k+1}$	$J^{-1}dJ$	$d_c^* = (-1)^{k+1} J d^* J$
∂	$\Omega^{p,q} \longrightarrow \Omega^{p+1,q}$		$\partial^* = - * \bar{\partial} *$
$\bar{\partial}$	$\Omega^{p,q} \longrightarrow \Omega^{p,q+1}$		$\bar{\partial}^* = - * \partial *$
\square	$\Omega^{p,q} \longrightarrow \Omega^{p,q}$	$\partial \partial^* + \partial^* \partial$	
$\bar{\square}$	$\Omega^{p,q} \longrightarrow \Omega^{p,q}$	$\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$	

In case of Kähler manifold, one has the following relation between these operators.

Lemma 3. *lem: In a compact Kähler manifold, one has*

1. $[L, d] = [L^*, d^*] = 0$
2. $[L, d^*] = d_c$
3. $[L^*, d] = -d_c^*$
4. $[L^*, d_c] = d^*$

Therefore,

1. $\Delta_c = d_c d_c^* + d_c^* d_c = \Delta$
2. ∂^* is adjoint to ∂ and $\bar{\partial}^*$ to $\bar{\partial}$.
3. $\Delta = 2\square = 2\bar{\square}$

One equip Ω^k with the following Hermitian product

$$\langle \phi, \psi \rangle_{L^2} = \int_M \phi \wedge * \bar{\psi}$$

under which the $\Omega^{p,q}$ are orthogonal.

One now applies the elliptic theory for $\bar{\square} : \Omega^{p,q} \longrightarrow \Omega^{p,q}$ with $\mathcal{H}_{\bar{\square}}^{p,q} = \ker \bar{\square}$ then one sees that

Theorem 4 (Hodge decomposition). *1. Each class in the Dolbeault cohomology $H_{\bar{\partial}}^{p,q}(M)$ contains exactly one harmonic form of $\mathcal{H}_{\bar{\square}}^{p,q} = \ker \bar{\square}$*

2. $H^k(M) = \mathcal{H}_{\Delta} = \bigoplus_{p+q=k} \mathcal{H}_{\bar{\square}}^{p,q} = \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(M)$.

1.4 Hodge symmetries

Let $h^{p,q} = \dim_{\mathbb{R}} H_{\bar{\partial}}^{p,q}(M)$ and $h^k = \dim H_{dR}^k(M, \mathbb{R})$ then one has $h^k = \sum_{p+q=k} h^{p,q}$. The $h^{p,q}$ are usually written down as Hodge's diamond

$$\begin{array}{cccc} h^{n,n} & h^{n,n-1} & \dots & h^{n,0} \\ h^{n-1,n} & h^{n-1,n-1} & \dots & h^{n-1,0} \\ \dots & \dots & \dots & \dots \\ h^{0,n} & h^{0,n-1} & \dots & h^{0,0} \end{array}$$

with the symmetries

1. $h^{p,q} = h^{q,p}$ given by conjugation.
2. $h^{p,q} = h^{n-q,n-p}$ given by the Hodge star.

2 Kodaira embedding theorem