Bogomolov-Beauville classification

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June 14, 2017

Contents

1 From the Riemannian results of de Rham and Berger

We will first prove a (conceptually) straightforward result of de Rham decomposition and Berger classification. The following theorem is taken from Beauville's article

Theorem 1 (Beauville)**.** *Let X be a compact Kähler manifold with flat Ricci curvature, then*

1. *The universal covering space $\tilde{X}$ of X decomposes isometrically as*

$$\tilde{X} = E \times \prod_i V_i \times \prod_j X_j$$

where $E = \mathbb{C}^k$, V_i and X_j are simply-connected compact manifolds of dimension $2m_i$ and $4r_j$ with irreducible homonomy $SU(m_i)$ for V_i and $Sp(r_j)$ for X_j. One also has uniqueness in the strong sense as in de Rham decomposition.

2. *There exists a finite etale covering space X' of X such that*

$$X' = T \times \prod_i V_i \times \prod_j X_j$$

where T is a complex torus.

Proof. Note that the first point is obtained directly from de Rham decomposition: The one-dimensional parallel subspaces (of trivial holonomy) are regrouped to E. By Cheeger-Gromoll splitting, $\tilde{X} = E \times M$ where M contains no line and is compact (note that we use compactness of X here). The irreducible factors in M are not symmetric spaces as Ricci curvature of symmetric spaces is non-degenerate. Holonomy of these factors are $SU(m_i)$ and $Sp(r_j)$ according to Berger list since they are Kähler manifolds and Ricci-flat. It remains to prove the second point.

We will regard each element of $\pi_1(X)$ by its isometric, free, proper action on $\tilde{X}$. As pointed out the arguments in our discussion of uniqueness of de Rham decomposition, every isometry of $\tilde{X}$ to itself preserves the components $T_{x_0}E$, $T_{x_i}V_i$ and $T_{x_j}X_j$ of $T_x\tilde{X}$, each isometry ϕ of $\tilde{X}$ is of form (ϕ_1, ϕ_2) where $\phi_1 \in Isom(E)$ and $\phi_2 \in Isom(M)$.

We will use here the fact that if M is a Kähler manifold, compact and Ricci-flat then $Isom(M)$ equiped with compact-open topology is discrete, therefore finite, which will be proved later (Lemma 3). We note $\Gamma := \{\phi = (\phi_1, \phi_2) \in \pi_1(X),\ \phi_2 = Id_M\}$ and sometime abusively regard Γ as a subgroup of $Isom(E)$. Note that Γ is a normal subgroup of $\pi_1(X)$ with finite index since the quotient is isomorphic to $Isom(M)$. Therefore $\tilde{X}/\Gamma = E/\Gamma \times M$ is compact as a finite covering of X.

We apply the following theorem of Bieberbach.

Theorem 2 (Bieberbach)**.** *Let $E = \mathbb{R}^n$ an Euclidean space and Γ be a subgroup of $Isom(E)$ satisfies*

1. *Γ is discrete under compact-open topology.*

2. *E/Γ is compact.*

Then the subgroup Γ' of translations in Γ is of finite index.

Suppose that the two conditions are satisfied and the theorem gives $\tilde{X}/\Gamma' = E/\Gamma' \times M = T \times \prod_i V_i \times \prod_j X_j$ is a finite covering of $\tilde{X}/\Gamma$ as Γ' is a normal subgroup of Γ since

Fact. The subgroup of translations in $Isom(E)$, where $E = \mathbb{R}^{\ltimes}$ is an Euclidean space, is normal.

Therefore $X' = \tilde{X}/\Gamma'$ is a finite covering of X that we want to find.

It remains now to prove that Γ is discrete, which is a consequence of

1. $\pi_1(X)$ is discrete, without limit point (obvious).
2. $Isom(M)$ is finite (see lemma 3)

In fact given any $\phi = (\phi_1, \phi_2) \in Isom(E) \times Isom(M)$, there exists by (1.) a neighborhood $\mathcal{U}_1(\phi_1, \phi_2) \times \mathcal{U}_2(\phi_1, \phi_2)$ of ϕ in $Isom(E) \times Isom(M)$ such that all points of $\pi_1(X)$ lying in this region project to ϕ_1. By (2.) we can find a neighborhood $\mathcal{U}_1$ of ϕ_1 in $Isom(E)$ small enough that $\mathcal{U}_1(\phi_1) \times Isom(M) \subset \cup_{\phi_2 \in Isom(M)} \mathcal{U}_1(\phi_1, \phi_2) \times \mathcal{U}_2(\phi_1, \phi_2)$. Therefore the projection of $\pi_1(X)$ to $Isom(E)$ is discrete, by consequence Γ is discrete. □

Lemma 3. *Let M be is a compact, simply-connected, Ricci-flat, Kähler manifold, then the group $Aut(M)$ of automorphism of M equiped with compact-open topology is discrete, therefore $Isom(M)$ is discrete, hence finite.*

Proof. The idea is that since $Aut(M)$ is a Lie group, it suffices to prove that its Lie algebra is of dimension 0. This is done using these facts.

1. The Lie algebra of $Aut(M)$ can be identified with the vector space of holomorphic vector fields on M.
2. *Bochner's principle*: All holomorphic tensor fields on a compact, Ricci-flat Kähler manifold are parallel. This can be seen by the identity $\Delta(\|\tau\|^2) = \|D\tau\|^2$
3. The only invariant vector of the holonomy representation of M is 0 (obvious).

□

2 Towards a classification for complex manifold

To obtain a translation of Theorem 1 in a context of complex manifolds (without any prefered metric a priori), we study the 2 building blocks. To be clear, recall that a complex manifold X is called of Kähler type if one can equipe X with an Hermitian structure whose fundamental form ω satisfies $d\omega = 0$. When we say X is of Kähler type, we refer to X as a complex manifold without fixing a metric on X. We resume here some results, see the manuscript for their proofs.

2.1 Special unitary manifold (proper Calabi-Yau manifold)

Remark 1. *Let X be a compact Kähler manifold with holonomy $SU(m)$ and complex dimension m then:*

1. *$H^0(X, \Omega_X^p) = 0$ for all $0 < p < m$, by consequence $\chi(\mathcal{O}_X) = 1+(-1)^m$.*
2. *X is* projective*, that is X can be embedded into $\mathbb{P}^N$ as zero-locus of some (finitely) homogeneous polynomials.*

The first point is in fact algebraic in nature: it comes from the fact that the representation of $SU(m)$ over $\wedge^p T_x^* M$ is irreducible for all p et non-trivial for $0 < p < m$, therefore the action of $SU(m)$ on $\wedge^p T_x^* M$ for $0 < p < m$ has no invariant element, hence $H^0(X, \Omega_X^p) = 0$.

The second point follows the following facts:

1. A compact Kähler manifold with $H^{2,0}$ can be embedded in $\mathbb{P}^N$.
2. (Chow's theorem) A compact complex manifold embedded in $\mathbb{P}^N$ is algebraic, i.e. defined by a finite number of homogeneous polynomials.

Theorem 4. *Given a compact manifold X of Kähler type and complex dimension m, the following properties are equivalent*

1. *There exists a compatible metric g over X such that $Hol(X, g) = SU(m)$.*
2. *K_X is trivial and $H^0(X', \Omega_{X'}^p) = 0$ for every $0 < p < m$ and X' a finite covering of X.*

Proof. (1) implies (2) as a finite covering space X' of a special unitary manifold X is still a special unitary. This is due to the following remarks: $Hol(X) \supset Hol(X') \supset Hol_0(X') = Hol_0(X)$ and $Hol_0(X) = Hol(X) = SU(m)$ as $SU(n)$ is connected.

For the second point, by Yau's theorem we equip X with a Ricci-flat metric, by Theorem 1, there exists a finite covering $X' = T \times \prod_i V_i \times \prod_j X_j$ where T is a complex torus, $Hol(V_i) = SU(m_i), Hol(X_j) = Sp(r_j)$. But $H^0(X', \Omega_{X'}^p) = 0$ for $0 < p < m$, X' has to be one of the V_i as $H^0(X_j, \Omega_{X_j}^{2r_j})$ and $H^0(V_i, \Omega_{V_i}^{m_i})$ do not vanish. Therefore $Hol(X) \supset Hol(X') = SU(m)$, hence $Hol(X) = SU(m)$. $\square$

2.2 Symplectic manifold

Remark 2. *Let X be a compact Kähler manifold with holonomy $Sp(r)$ and complex dimension $2r$ then:*

1. *There exists a holomorphic 2-form φ non-degenerate at every point.*
2. $H^0(X, \Omega_X^{2l+1}) = 0, H^0(X, \Omega_X^{2l}) = \mathbb{C}\varphi^l$ *for all* $0 \leq l \leq r$. *By consequence* $\chi(\mathcal{O}_X) = r + 1$.

Theorem 5. *Given a compact manifold X of Kähler type and complex dimension $2r$, then:*

- *The followings are equivalent:*
  1. *There exists a compatible metric g such that $Hol(X, g) \subset Sp(r)$.*
  2. *There exists a symplectic structure: a 2-form that is closed, holomorphic and non-degenerate at every point*
- *The followings are equivalent, if X is called irreducible symplectic or hyperkahler if it satisfies one of them.*
  1. *There exists a compatible metric g such that $Hol(X, g) = Sp(r)$*
  2. *X is simply-connected and there exists (uniquely up to a constant) a symplectic structure on X.*

2.3 Decomposition for complex manifold with vanishing Chern class

Theorem 1 can be translated to a decomposition for complex manifold in the following way:

Theorem 6 (Bogomolov- Beauville classification)**.** *Let X be a compact manifold of Kähler type with vanishing Chern class.*

1. *The universal covering $\tilde{X}$ of X is isomorphic to a product $\mathbb{E} \times \prod_i V_i \times \prod_j X_j$ where $E = \mathbb{C}^k$ and*
   (a) *Each V_i is a projective simply-connected manifold of complex dimension $m_i \geq 3$, with trivial K_{V_i} and $H^0(V_i, \Omega_{V_i}^p) = 0$ for $0 < p < m_i$*
   (b) *Each X_j is an irreducible compact symplectic manifold of Kähler type.*

This decomposition is unique up to an order of i and j.

2. *There exists a finite covering X' of X isomorphic to the product $T \times \prod_i V_i \times \prod_j X_j$.*

The theorem follows directly from Theorem 1, the only point that needs proof is the uniqueness, which will be achieved in two steps:

1. Prove the uniqueness in the case that X is simply-connected.

2. Prove that every isomorphism $\phi : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z$ is splitted as $\phi = (\phi_1, \phi_2)$ where $\phi_1 : \mathbb{C}^k \longrightarrow \mathbb{C}^h$ and $\phi_2 : Y \longrightarrow Z$ are isomorphisms (by consequence $h = k$).

These two steps will be accomplished in the following two lemmas

Lemma 7. *Let $Y = \prod_j Y_j$ be a compact, simply-connected manifold of Kähler type with vanishing Chern class. The Calabi-Yau metrics of Y are then $g = \sum_l pr_j^* g_l$ where g_l are Calabi-Yau metrics of Y_l.*

Proof. Let g be a Calabi-Yau metric of Y and $[\omega]$ its class in $H^{1,1}(Y)$. Since Y_j are simply-connected, $[\omega] = \sum_j pr_j^*[\omega_j]$. By Yau's theorem, there exist unique Calabi-Yau metrics g_j of Y_j in each class $[\omega_j]$. The metric $g' = \sum_j pr_j^* g_j$ is in the same class ω of g and is also a Calabi-Yau metric, hence $g = g' = \sum_j pr_j^* g_j$. □

This lemma asserts that when our manifolds Y, Y_j are equiped with appropriate Calabi-Yau metrics, the decomposition map is also a (Riemannian) isometric, we therefore obtain uniqueness of V_i, X_j from uniqueness of Theorem 1.

Lemma 8. *Let Y, Z be compact, simply-connected manifold of Kähler type, then any isomorphism $u : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z$ is splitted as $\phi = (\phi_1, \phi_2)$ where $\phi_1 : \mathbb{C}^k \longrightarrow \mathbb{C}^h$ and $\phi_2 : Y \longrightarrow Z$ are isomorphisms.*

Proof. It is clear that the function $u_1 : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z \longrightarrow \mathbb{C}^h$ is constant in Y, i.e. $u_1(t, y) = u_1(t)$ as holomorphic functions on Y are constant. Therefore $u(t, y) = (u_1(t), u_2(t, y))$, as u is isomorphic, one has $h \leq k$ then by the same argument for u^{-1}, one has $h = k$, u_1 is an isomorphism and $u_t(\cdot) := u_2(t, \cdot)$ is an isomorphism from Y to Z. $u_0^{-1} \circ u_t$ is then a curve in $Aut(Y)$, which is discrete by Lemma 3. Hence $u_t = u_0$ independent de t. □