Parametrix and Green's function of Laplacian operator on Riemannian manifolds

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Mar 22, 2018

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This post is the second part of my reading note for [**?**]. The first part is here.

Recall that in the Euclidean space $\mathbb{R}^n$, one obtains a representation of the solution u of equation $\Delta u = f$ by

- first solving for an explicit radial solution of $\Delta G = \delta_0$. In particular, $G = [(n-2)\omega_{n-1}]^{-1} r^{2-n}$ if $n > 2$ and $G = -(2\pi)^{-1}\log(r)$ if $n = 2$
- then tensoring G by f, one has the solution $u = G * f$ of $\Delta u = f$

To generalise this argument for Riemannian manifolds, there are a few points that have to be modified:

1. Since it does not make sense to add/substract points of a manifold, one will need to find different fundamental solutions for different points, so instead of fundamental solution, we will find the Green's function $G = G(p,q)(p,q \in M)$. The convolution will be replace by the following operation on functions X, Y defined on $(M \times M) \setminus \Delta_M$ where Δ_M denotes the diagonal:

$$(X * Y)(p,q) = \int_M X(p,r)Y(r,q)dV(r)$$

.

2. The distance function $q \mapsto d(p,q)$ is only smooth near p, outside of the cut-locus, the best one can says is that the function is Lipschitz. Since cut-loci are almost impossible to calculate or visualise (the cut-locus of an ellipsoid is still a conjecture, according to [**?**]), one will cut-off the Euclidean solution, try to solve the equation near p and later add a correcting term. This inspires the definition of parametrix.

3. Another reason that we have to approximate the exact solution by parametrix, that also explain the iteration in Theorem 2, is that the expression of Laplacian, even in the geodesic polar coordinate and even near the origin, involves the metric, hence the Euclidean fundamental solution is not yet a solution even near the origin.

Remark 1. *To give a simplified analogy of what we will be doing, let us prove the existence of "Green's function" on Riemann surfaces (with boundary, so that we do not have to deal with the volume). The "Laplace equation" is*

$$-2i\partial\bar{\partial}g = \delta_0 \tag{1}$$

where the LHS is a 2-form and the RHS is a generalised 2-form in the sense of current. Contrary to the previous point 3, one knows the exact local solution of (1)*, namely* $z \mapsto -(2\pi)^{-1}\log(|z|)$*. Therefore, the argument will be simplified as:*

- *Given a holomorphic chart of a point* $0 \in M$*, pose* $h(z) := -(2\pi)^{-1}\log(|z|)\chi(|z|)$ *where* χ *is a cut-off function that is* 1 *on a neighborhood of* 0
- *The 2-form* $\alpha = -2i\partial\bar{\partial}h$ *is well-defined everywhere except* 0*, and vanishes on a neighborhood of* 0*. Denote by* α^{naiv} *its extension to* M*.*
- *Recall the fact that every smooth 2-form on a compact, connected, Riemann surface with boundary can be writen as* $\alpha^{\text{naiv}} = -2i\partial\bar{\partial}\varphi$*, pose* $g = h - \varphi$*.*

For Riemann surface without boundary, the equation is $-2i\partial\bar{\partial}g = \delta_0 - 2i\int_M \partial\bar{\partial}g$ *and the fact to evoke is that any smooth 2-form* α *with* $\int_M \alpha = 0$ *is of form* $\alpha = -2i\partial\bar{\partial}\varphi$

We will suppose that M^n is a Riemannian manifold with injectivity radius $\delta_0 > 0$, and of bounded curvature. Compact manifolds, for example, fall in this category.

1 Parametrix and the Green's formula

Definition 1. *A **Green's function** $G(p,q)$ of a compact Riemannian manifold is a function defined on $(M \times M) \setminus \Delta_M$ such that*

1. $\Delta_q^{\mathrm{dist}} G(p,q) = \delta_p(q)$ *if M has boundary.*
2. $\Delta_q^{\mathrm{dist}} G(p,q) = \delta_p(q) - V^{-1}$

where Δ_q^{dist} concerns the distribution derivatives and V is the volume of M.

*Let $p, q \in M$ be distinct points, the **parametrix** H is defined by*

$$H(p,q) = \begin{cases} [(n-2)\omega_{n-1}]^{-1} r^{2-n} \chi(r), & \text{if } n > 2 \\ -(2\pi)^{-1} \chi(r) \log r, & \text{if } n = 2 \end{cases}$$

where $r = d(p,q)$, $\chi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ is smooth, $\chi = 1$ in a neighborhood of 0 and $\chi(t) = 0$ if $t > \delta_0$.

Recall that in the geodesic polar coordinates, i.e. the polar coordinates on the tangent T_pM at $p \in M$, identified with a neighborhood of $p \in M$, the metric g is given by

$$g : ds^2 = dr^2 + r^2 g_{\theta_i \theta_j}(r,\theta) d\theta^i d\theta^j$$

and one denotes $|g_\theta| := \det(g_{\theta_i \theta_j})$, therefore $|g| = \det(g_{ij}) = r^{2(n-1)} |g_\theta|$

Lemma 1. *If a function $\varphi \in C^2$ defined locally around $p \in M$ and φ is radial, i.e. $\varphi = f(r)$ in a small geodesic ball $B(p,\delta)$ then*

$$-\Delta\varphi = f'' + \frac{n-1}{r} f' + f' \partial_r \log \sqrt{|g_\theta|}$$

Proof. One has

$$\begin{aligned} \Delta\varphi &= -\operatorname{Tr}\Big(\nabla_i (g^{kj} \partial_j \varphi \; e_k)\Big)_{i,k} = -\partial_i (g^{ij} \partial_j \varphi) - g^{kj} \partial_j \varphi \Gamma^i_{ik} \\ &= -|g|^{-1/2} \partial_i (g^{ij} |g|^{1/2} \partial_j \varphi) \end{aligned}$$

since $\Gamma^i_{ik} = \partial_k \log \sqrt{|g|} = \frac{\partial_i |g|}{2|g|}$. One concludes by substituting $|g| = r^{2n-2} |g_\theta|$ and noticing that $g^{r\theta_i} = g^{\theta_i \theta_j} = 0$ $(i \neq j)$. □

Remark 2. 1. *The Laplacian of the metric g, viewed in polar geodesic coordinates centered at p, i.e. in the tangent space T_pM is not the Euclidean Laplacian of T_pM, however the difference if $O(r)$ since $\partial_r \log \sqrt{|g_\theta|} \leq Ar$ where the bound A is given by Ricci curvature, see the Volume comparison theorem.*

2. *Applied the formula for $q \mapsto H(p,q)$, one has*

$$\Delta_q^{\text{naiv}} H(p,q) = [(n-2)\omega_{n-1}]^{-1} r^{1-n} \left((n-3)\chi' - r\chi'' + ((n-2)\chi - r\chi')\partial_r \log\sqrt{|g_\theta|} \right) \tag{2}$$

therefore $\Delta_q^{\text{naiv}} H(p,q) \leq Br^{2-n}$ where B does not depend on p.

3. *Unlike the case of Remark 1 where we know the exact fundamental solution and the form α^{naiv} has no singularity, there is no reason for that this holds true for $\Delta_q^{\text{naiv}} H(p,q)$. However, we proved that the order of singularity at $q = p$ can be controlled.*

Proposition 1.1 (Green's formula)**.** *For any function $\psi \in C^2(M)$, one has*

$$\psi(p) = \int_M H(p,q)\Delta\psi(q)dV(q) - \int_M \Delta_q^{\text{naiv}} H(p,q)\psi(q)dV(q) \tag{3}$$

where $\Delta_q^{\text{naiv}} H(p,q)$ denotes the pointwise derivative of $H(p,q)$, <u>not</u> the distribution derivative.

Remark 3. 1. *In other words, the theorem says that $\Delta_q^{\text{dist}} H(p,q) = \Delta_q^{\text{naiv}} H(p,q) + \delta_p(q)$ where Δ_q^{dist} is the distribution derivative. In particular, if there is no concern about regularity of the distance function $d(p,q)$ (as in the Euclidean case), allowing us to take the cut-off function $\chi = 1$ in the definition of parametrix, then $\Delta_q^{\text{naiv}} H(p,q) = 0$ and $\Delta_q^{\text{dist}} H(p,q) = \delta_p(q)$ which is not a surprise since $H(p,q)$ is also the Green's function.*

2. *Taking $\psi = 1$, one has*

$$\int_M \Delta_q^{\text{naiv}} H(p,q) = -1$$

3. *Multiplying (3) by $\phi(p)$ and integrate over M, one has*

$$\int_M \phi(q)\psi(q)dV(q) = \int_M \left(\int_M H(p,q)\phi(p)dV(p) \right) \Delta\psi(q)dV(q) - \int_M \left(\int_M \Delta_q^{\text{naiv}} H(p,q)\phi(p) \right) \psi(q)$$

hence in distribution sense

$$\phi(q) = \Delta_q \int_M H(p,q)\phi(p)dV(p) - \int_M \Delta_q^{\text{naiv}} H(p,q)\phi(p) \tag{4}$$

The equation (4) is called the <u>transposition</u> of equation (3) and what we have just done is a rigourous proof of the following heuristic justification of (4): "Take the derivative Δ_q inside the integral, then use $\int_M \delta_p(q)\phi(p)dV(p)$ "=" $\phi(q)$".

Proof. The intuition is clear:

- since one only modifies the fundamental solution at points q far from p, one only needs to recompense by $\Delta_q^{\text{naiv}} H(p,q)$
- there may be trouble near p caused by the difference between the Euclidean Laplacian and the metric Laplacian, however as explained by Remark 2, this difference is $O(r)$ as $r \to 0$.

For a rigorous proof, one calculates $\int_M H(p,q)\Delta\psi(q)dV(q)$ by decomposing M to $B(p,\epsilon)$ and $M \setminus B(p,\epsilon)$ with $0 < \epsilon < \delta_0$ tending to 0 eventually, then

$$\begin{aligned}\int_{M\setminus B(p,\epsilon)} H(p,q)\Delta\psi(q)dV(q) &= \int_{M\setminus B(p,\epsilon)} \left(\Delta_q^{\text{naiv}} H(p,q)\psi(q) + d(\psi \wedge *dH - H \wedge *d\psi)\right) dV(q) \\ &= \int_{M\setminus B(p,\epsilon)} \Delta_q^{\text{naiv}} H(p,q)\psi(q)dV(q) + \int_{\partial B(p,\epsilon)} (\psi \wedge *dH - H \wedge *d\psi)dV(q)\end{aligned}$$

by Stokes' theorem, where $*$ denotes the Hodge star. Therefore

$$\int_M H(p,q)\Delta\psi(q)dV(q) = \int_M \Delta_q^{\text{naiv}} H(p,q)\psi(q)dV(q) + I_1 + I_2$$

where $I_1 = \lim_{\epsilon\to 0} \int_{\partial B(p,\epsilon)} (\psi \wedge *dH - H \wedge *d\psi)$ and $I_2 = \lim_{\epsilon\to 0} \int_{B(p,\epsilon)} H(p,q)\Delta\psi(q)dV(q)$.

Now $I_2 = \psi(p)$ since $(\frac{\sin(b\epsilon)}{b\epsilon})^{n-1} \leq dV/dE \leq (\frac{\sin(\alpha\epsilon)}{\alpha\epsilon})^{n-1}$ in $B(p,\epsilon)$ by Volume comparison theorem where b^2 is an upper bound of sectional curvature and $(n-1)\alpha^2$ is a lower bound of Ricci curvature ($\alpha \in \mathbb{C}$), and since $\Delta\psi(q) - \Delta_E\psi(q) = O(\varepsilon)$ in $B(p,\epsilon)$ where Δ_E is the Euclidean Laplacian.

For I_1, with ϵ small enough such that $\chi = 1$, one has $|H \wedge *d\psi| \leq \text{const}\, \epsilon^{2-n}(*d\psi)$. By straightforward computation:

$$\begin{aligned} dH &= -\omega_{n-1}^{-1} r^{1-n} dr, \quad dV = r^{n-1}\sqrt{|g_\theta|}dr \wedge d\theta^1 \wedge \cdots \wedge d\theta^{n-1} \\ *dH &= -\omega_{n-1} r^{2n-2}\sqrt{|g_\theta|}d\theta^1 \wedge \cdots \wedge d\theta^{n-1}\end{aligned}$$

hence $\int_{\partial B(p,\epsilon)} H \wedge *d\psi = O(\epsilon)$ and $\int_{\partial B(p,\epsilon)} \psi \wedge *dH = O(\epsilon^{2n-2})$. Therefore $I_1 = 0$ and the conclusion follows. □

2 Existence of Green's function on compact Riemannian manifolds

Our goal is to prove the following theorem

Theorem 2 (Existence of Green's function)**.** *Let M^n be a compact Riemannian manifold without boundary, there exists a Green's function $G(p,q)$ of the Laplacian such that*

1. Green's function. *For all $\varphi \in C^2(M)$,*

$$\varphi(p) = V^{-1}\int_M \varphi(q)dV(q) + \int_M G(p,q)\Delta\varphi(q)dV(q) \tag{5}$$

2. Smooth. $G \in C^\infty((M \times M) \setminus \Delta_M)$.

3. Radial estimates. *There exists a constant k such that*

$$|G(p,q)| \leq \begin{cases} k(1+|\log r|), & \text{if } n=2 \\ kr^{2-n}, & \text{if } n>2 \end{cases} \tag{6}$$

for $r = d(p,q)$. Moreover, one has the derivative estimates:

$$|\nabla_q G(p,q)| \leq kr^{1-n}, \qquad \left|\nabla_q^2 G(p,q)\right| \leq kr^{-n}, \tag{7}$$

4. *G is bounded below. Since G is defined upto a constant, one can choose the constant so that $G > 0$.*

5. Constant integral. *The integral $\int_M G(p,q)dV(p)$ is constant in q. Since G is defined upto a constant, one can choose the constant so that $\int_M G(p,q)dV(p) = 0$.*

6. Symmetric. *$G(p,q) = G(q,p)$ for $p \neq q$ in M.*

For a better notation, let us replace $\Delta_p U(p,q)$ by $\Delta_2 U(p,q)$. Recall that we already know how to solve the equation $\Delta u = f$ for $f \in L^2(M)$, this means we can solve $\Delta_2 U(p,q) = f_p(q)$ for double-integrable functions f_p, or briefly we can *solve* L^2 functions. Now, define

$$(X * Y)(p,q) := \int_M X(p,r)Y(r,q)dV(r)$$

if the integration is possible and if it commutes with derivation, one has

$$\Delta_2(F_1 * H) = F_1 * \Delta_2^{\text{dist}} H = F_1 + F_1 * \Delta_2^{\text{naiv}} H$$

So if one can *solve* $F_1 * \Delta^{\text{naiv}} H$, then one can *solve* F_1, i.e. if $\Delta_2 E_2 = F_1 * \Delta_2^{\text{naiv}} H$ then take $E_1 := F_1 * H - E_2$, one has $\Delta_2 E_1 = F_1$.

Now in order to prove that $\delta_\Delta - V^{-1}$ can be solved, it remains to check that

$$\delta_\Delta * (\Delta_2^{\text{naiv}} H)^{*k} \in L^2(M) \quad \text{for } k \gg 1. \tag{8}$$

This is the content of the following lemma.

Lemma 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $X, Y : (\Omega \times \Omega) \setminus \Delta_\Omega \longrightarrow \mathbb{R}$ be continuous functions such that*

$$|X(p,q)| \leq \text{const}\, d(p,q)^{\alpha-n}, \qquad |Y(p,q)| \leq \text{const}\, d(p,q)^{\beta-n}, \qquad \alpha, \beta \in (0,n)$$

then

$$Z(p,q) := \int_\Omega X(p,r)Y(r,q)dV(r)$$

is continuous in $(\Omega \times \Omega) \setminus \Delta_\Omega$ and

$$|Z(p,q)| \leq \begin{cases} \text{const}\, d(p,q)^{\alpha+\beta-n}, & \text{if } \alpha+\beta < n \\ \text{const}(1 + |\log d(p,q)|), & \text{if } \alpha+\beta = n \\ \text{const}, & \text{if } \alpha+\beta > n \end{cases}$$

In the case $\alpha+\beta > n$, Z admits a continuous extension to $\Omega \times \Omega$. The result also holds for compact Riemannian manifolds.

Proof. It suffices to consider p, q closed to each other. Let $d(p,q) = 2\rho$. Decompose $\Omega = (\Omega \cap B(p,\rho)) \cup (\omega \setminus B(q,3\rho)) \cup \Omega \cap (B(q,3\rho) \setminus B(p,\rho))$, then

$$\left| \int_{\Omega \cap B(p,\rho)} X(p,r)Y(r,q)dV(r) \right| \leq C\rho^{\alpha+\beta-n}$$

$$\left| \int_{\Omega \cap B(q,3\rho) \setminus B(p,\rho)} X(p,r)Y(r,q)dV(r) \right| \leq C\rho^{\alpha+\beta-n}$$

$$\left| \int_{\Omega \setminus B(q,3\rho)} X(p,r)Y(r,q)dV(r) \right| \leq C \int_\rho^D \frac{dr}{r^{n-\alpha-\beta-1}}$$

where D is the diameter of Ω. For compact Riemannian manifold, take $\rho \ll \delta_0$, the injectivity radius and use Comparison theorem, one has the same estimates. □

Back to the proof of Theorem 2, one can see that it suffices to choose $k > \frac{n}{2}$ in (8). The rigorous proof is given below.

Proof of Theorem ref:thm:existence-green. Carefully do the algebraic part of the above argument, one poses

$$G(p,q) = H(p,q) + \sum_{i=1}^{k-1} (-\Delta_2^{\text{naiv}} H)^{*i} * H + F_k(p,q)$$

where $F_k(p,q)$ satisfies

$$\Delta_2 F_k(p,q) = (-\Delta_2^{\text{naiv}} H)^{*k} - V^{-1}$$

This is possible if one chooses $k > n/2$ since by repeated application of Lemma 3, $(-\Delta_2^{\text{naiv}} H)^{*k}$ is continuous. By regularity result of equation $\Delta u = f$, the function $q \mapsto F_k(p,q)$ is in $C^2(M \setminus \{p\})$. Each function $F_k(p,\cdot)$ is uniquely defined up to a constant, choose the constant such that $\int_M G(p,q)dV(q) = 0$, then the function $p \mapsto \int_M F_k(p,q)dV(q)$ is continuous. The condition 1) of the Theorem can be verified without difficulty. Moreover, since $\Delta_2 G(p,q) = 0$ if $q \neq p$, the function $q \mapsto G(p,q)$ is C^∞.

We will prove such $G(p,q)$ satisfies the statements 2-6, starting from a weaker form 2-) of 2), that is we will prove that $p \mapsto G(p,q)$ is continuous, then using this, we will prove 3-6, and eventually come back to prove 2 completely.

For 2-) we will use the following fact:

Fact. If $\Delta u = f$ and $f \in C^0(M)$ (hence $u \in C^2(M)$ and $\int_M u = 0$, then one has $\sup |u| \leq C \sup |f|$ where $C > 0$ is a constant.

Denote $\Gamma_i := (-\Delta_2^{\text{naiv}} H)^{*i}$ and apply the result for $u = F(p,\cdot) - V^{-1}\int_M F(p,q)dV(q)$ and $f = \Gamma_k(p,\cdot)$, one has

$$\sup \left| F(p,\cdot) - F(r,\cdot) - V^{-1}\int_M (F(p,\cdot) - F(r,\cdot)) \right| \leq C \sup_q |\Gamma_k(p,q) - \Gamma_k(r,q)|$$

Then the continuity of $p \mapsto F(p,\cdot)$ under C^0 topology is given by

- $p \mapsto \int_M F(p,\cdot)$ is continuous by the previous choice of constant.
- The uniform continuity of Γ_k on $M \times M$, which is the result of its continuity and the compactness of $M \times M$.

Hence $p \mapsto G(p,q)$ is continuous on $M \setminus \{q\}$ for all $q \in M$.

For 3), fix $p \in M$ and let $r = d(p,q)$ be small, then $H(p,q) = O(r^{2-n}$, $(\Gamma_i * H)(p,q) = O(r^{2i+2-n})$ by Lemma 3 and $F(p,q) = O(1)$ if $n > 2$. Hence $G(p,q) = O(r^{2-n})$, where here the constant in $O(r^{2-n})$, if checked carefully, does not depend on p. The case $n = 2$ can be treated similarly. For the derivative estimates, note that $\nabla_q G(p,q) = \nabla_q H(p,q) + \sum_{i=1}^{k-1}(\Gamma_i * \nabla_2 H)(p,q) + \nabla_q F(p,q)$ and $\nabla_q^2 G(p,q) = \nabla_q^2 H(p,q) + \sum_{i=1}^{k-1}(\Gamma_i * \nabla_2^2 H)(p,q) + \nabla_q^2 F(p,q)$ where the commutative of derivation and integration can be justified by Lebesgue's Dominated convergence. In both case, the dominant terms as $q \to p$ are $\nabla_q H(p,q)$ and $\nabla_q^2 H(p,q)$ respectively, which is $O(r^{1-n})$ and $O(r^{-n})$ where the constants in big-O do not depend on p.

For 4), note that $H(p,q)$ is the dominant term of $G(p,q)$ as $q \to p$ and $H(p,q) > 0$, one see that $G(p,q) > 0$ in a neighborhood of Δ_M. By the compactness of M and the continuity of G outside of Δ_M, one sees that G is bounded below.

To prove 5), take to transposition of (5), i.e. multiply by $\psi(p)$ and integrate, as in Remark 3, one obtains

$$\Delta_q \int_M G(p,q)\psi(p)dV(p) = \psi(q) - V^{-1}\int_M \psi(p)dV(p) \tag{9}$$

Substitute $\psi = 1$, one sees that $q \mapsto \int_M G(p,q)dV(p)$ is harmonic on M, hence is constant by compactness of M.

We will now prove 6). It follows from (5) that

$$\Delta_q \int_M G(p,q)\psi(q)dV(q) = \Delta_q\psi(q) \tag{10}$$

Also, from the transposition (9), replace ψ by $\Delta\psi$, one has

$$\Delta_q \int_M G(p,q)\Delta\psi(p)dV(p) = \Delta_q\psi(q)$$

Swap p and q and subtract to (10), one has

$$\Delta_p \int_M \left(G(p,q) - G(q,p)\right)\Delta\psi(q)dV(q) = 0$$

Hence $\int_M \left(G(p,q) - G(q,p)\right)\Delta\psi(q) = C\,\text{const}$. Integrate by $p \in M$ and use the fact that we chose $\int_M G(q,p)dV(p) = 0$, one has $C = 0$, meaning that $\Delta_q\left(G(p,q) - G(q,p)\right) = C(p)$, being independent of q. By swapping p, q, one has $C(p) = -C(q)$ for all $p \neq q$. Since M contains more than 3 points, these constants are 0. Hence $G(p,q) = G(q,p)$.

Now coming back to 2), since $G(p,q) = G(q,p)$, we see that $p \mapsto G(p,q)$ is C^∞ for all $q \in M$. It remains to prove that $p \mapsto \nabla_q^h G(p,q)$ is continuous on $M \setminus \{q\}$, then Schwarz's lemma applies. For that, one may try the following argument:

$$\Delta_p \nabla_q^h G(p,q) = \nabla_q^h \Delta_p G(p,q) = 0, \quad p \in M \setminus [q\}$$

hence $p \mapsto \nabla_q^h G(p,q)$ is C^∞. It is however difficult to justify the commutativity of derivations, which is equivalent to

$$\int_M \nabla_q^h G(p,q)\Delta\varphi(p)dV(p) = \nabla_q^h \int_M G(p,q)\Delta\varphi(p)dV(p), \tag{11}$$

that is the ability to derive in the integral sign. A justification for this can be done in the case $h \leq 2$ using estimates of 3).

A simpler way is to note that it suffices to prove the continuity of $p \mapsto \nabla_q^h G(p,q)$ for p in a small open set V with $\bar{V}$ not containing q. Then claim that $\Delta_p \nabla_q^h G(p,q) = \nabla_q \Delta_p G(p,q) = 0$ as distributions on V, which is equivalent to (11) for all test functions φ with $\operatorname{supp} \varphi \in V$. Then Dominated convergence applies since $|\nabla_q^{h+1} G(p,q)| \leq C d(q, \bar{V})^{1-n-h}$ hence is bounded. $\square$