Berger classification and remarks on parallel structures

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Our story so far

De Rham decomposition theorem theorem allows us to split a Riemannian manifold under certain conditions (complete and connected) as Riemannian product of complete connected manifold with $irreducible\ holonomy\ representation$. We will now interest in manifolds with irreducible holonomies. If the manifold is $locally\ symmetric$ then one can prove that it is isometric to the homogeneous space G/H with H (the holonomy) a closed Lie subgroup of G. The theory of Lie groups developed by E. Cartan gave a complete list of these spaces.

Berger classification for non-symmetric manifolds

Theorem 1 (Berger classification). For the non-symmetric irreducible manifold, the holonomy representation has to be one of the following

- 1. SO(n)
- 2. $U(m) \subset SO(2m)$
- 3. $SU(m) \subset SO(2m)$
- 4. $Sp(r) \subset SO(4r)$
- 5. $SO(r)Sp(1) \subset SO(4r)$
- 6. $G_2 \subset SO(7)$

7. $Spin(7) \subset SO(8)$.

where n = 2m = 4r is the dimension.

Here are some notations, note always that

$$Sp(m) \subset SU(2m) \subset U(2m) \subset SO(4m)$$

- 1. If $Hol(g) \subset U(m) \subset SO(2m)$, g is called a Kähler metric.
- 2. If $Hol(g) \subset SU(m) \subset SO(2m)$, g is called a Calabi-Yau metric. We will see that a Calabi-Yau metric is a Kähler metric that is also Ricci-flat.
- 3. If $Hol(g) \subset Sp(m) \subset SO(4m)$ then g is called a hyperkahler metric.
- 4. G_2 and Spin(7) are called exceptional holonomies

To sum up: hyperkahler \longrightarrow Calabi-Yau \longrightarrow Kähler.

But what do we mean by $U(n) \subset SO(2n)$? To embed U(n) in SO(2n) one needs to identify \mathbb{C} and \mathbb{R}^{2n} , this can be done using an almost complex structure J of \mathbb{R}^n . We will prove that when we change the almost complex structure, the embedded image of U(n) in SO(2n) always remains in the same conjugacy class, which corresponds to the fact that while holonomy representation is well-defined, the holonomy group in SO(2n) is only defined up to its conjugacy class.

Almost complex structure

Definition 1. A <u>(almost) complex structure</u> J on a vector space V is an automorphism $J: V \longrightarrow V$ with $J^2 = -Id_V$. If V has a scalar product g, we suppose in addition that $g \circ J = J$.

A (almost) complex structure J on manifold M is a vector bundle automorphism $J:TM\longrightarrow TM$ that satisfies $J_x^2=-Id_{T_xM}$ for every $x\in M$. If M is a Riemannian manifold, we assume in addition that $g\circ J=g$.

Let us first have a look at a complex structure J on a fiber (vector space) V. Here are some direct consequences:

Complexification. g and J extend in an unique way over $V_{\mathbb{C}}$ to a hermitian product $g_{\mathbb{C}}$ and a \mathbb{C} -linear automorphism (also noted by J). One also has $g_{\mathbb{C}} \circ J = g_{\mathbb{C}}$.

Eigenspaces. The complexified space $V_{\mathbb{C}}$ is decomposed to $V_{\mathbb{C}} = V^{1,0} \oplus^{\perp} V^{0,1}$ where $V^{1,0}$ and $V^{0,1}$ are eigenspaces (complex vector space) corresponding to eigenvalues i and -i of J on $V_{\mathbb{C}}$. The orthogonality is by $g_{\mathbb{C}}$. The complex conjugate $\sum z_i x_i \mapsto \sum \bar{z}_i x_i$ where $z_i \in \mathbb{C}$ and $x_i \in V$ maps $V^{1,0}$ to $V^{0,1}$. Their dimensions are therefore the same.

Hermitian form. The fundamental form ω of (V, J) is defined by

$$\omega(a,b) = g(Ja,b) = -g(a,Jb)$$
 on V

which is an antisymmetric real 2-form with $\omega \circ J = \omega$. V equipped with the following Hermitian form

$$h(a,b) = g(a,b) - i\omega(a,b)$$
 on V

in the sense that h(.,.) is \mathbb{R} -linear with h(Ja,b)=ih(a,b) and h(a,Jb)=-ih(a,b).

Identification. One usually identifies (V, J) and $(V^{1,0}, i)$ as vector spaces equipped with complex structure, using the following map:

$$\iota_J: x \mapsto \frac{1}{2}(x - iJ(x))$$

which is \mathbb{C} -linear in the sense of complex structure $\iota_J(Jx) = i\iota_J(x)$. Note that one (V, -J) is also isomorphic to $(V^{0,1}, i)$ by the conjugate of $\iota_J : x \mapsto \frac{1}{2}(x + iJ(x))$.

Now note that on we have on (V, J) an hermitian product h(., .) and on $(V^{1,0}, i)$ the restricted hermitian product $g_{\mathbb{C}}$ of $V_{\mathbb{C}}$. The following lemma gives their relation (the proof is straightforward computation, see Manuscript).

Lemma 2. The identification $(V, J) = (V^{1,0}, i)$ by ι_J gives

$$\frac{1}{2}h = g_{\mathbb{C}}|_{V^{1,0}}$$

We can now embed U(n) to SO(2n), in other words $U(V^{1,0})$ to SO(V)

by the map $\phi \mapsto \tilde{\phi}$ as follow: $\begin{array}{ccc} V & \stackrel{\tilde{\phi}}{\longrightarrow} V \\ \downarrow \iota_J & & \downarrow \iota_J \\ V^{1,0} & \stackrel{\phi}{\longrightarrow} V^{1,0} \end{array}$

Note that the correspondance $\phi \leftrightarrow \tilde{\phi}$ is one-to-one between $\{\phi: V^{1,0} \longrightarrow V^{1,0}\mathbb{R}\text{-linear}\}$ and $\{\tilde{\phi}: V \longrightarrow V \mathbb{R}\text{-linear}\}$. Then

- 1. ϕ is \mathbb{C} -linear if and only if $\tilde{\phi}J = J\tilde{\phi}$.
- 2. ϕ preserves $g_{\mathbb{C}}$ if and only if $\tilde{\phi}$ preserves h. Taking the real and imaginary part, the latter is equivalent to the fact that $\tilde{\phi}$ preserves g and ω .
- 3. Every $\mathbb C$ -linear $\tilde{\phi}$ preserves orientation of $V^{1,0}$ as $\mathbb R^{2n}$ (note that the fact that $\tilde{\phi}$ preserves orientation or not is independent of how one identifies $V^{1,0}$ and $\mathbb R^{2n}$

Hence for every J, $\phi \mapsto \tilde{\phi}$ gives a embedding of $U(V^{1,0})$ to SO(V). By each a orthonormal base of $V^{1,0}$ and that of V give a embedding $U(n) \subset SO(2n)$.

Remark 1. The image of U(n) in SO(2n) may depends on J and the orthonormal base of V, but its conjugacy class in SO(2n) is <u>uniquely defined</u>. This is because every complex structure J is, up to a orthonormal conjugation,

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Complexified dual and forms, prelude to Kähler geometry

We state first some linear algebra facts, whose proofs are tedious and can be consulted in the Manuscript.

Lemma 3 (Linear algebra facts). 1. Let $V = W_1 \oplus W_2$ be R-module then the exterior algebra of V splits into

$$\wedge^n V = \bigoplus_{p+q=n} \wedge^p W_1 \otimes \wedge^q W_2$$

We remark that the tensor product here is formal, and not related to the tensor product defining the exterior algebra.

2. If V has a complex structure J then J gives a complex structure on $V^* = Hom_{\mathbb{R}}(V, \mathbb{R})$, naturally by $\phi \mapsto \phi \circ J$.

One has

$$(V^*)_{\mathbb{C}} = Hom_{\mathbb{R}}(V, \mathbb{C}) \equiv Hom_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$$

and

$$(V^*)^{1,0} = Hom_{\mathbb{C}}((V,J),\mathbb{C}), \quad (V^*)^{0,1} = Hom_{\mathbb{C}}((V,-J),\mathbb{C})$$

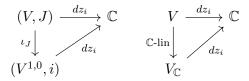


Figure 1: Two natural ways to define dz_i on $V^{1,0}$

where $Hom_{\mathbb{C}}(V,\mathbb{C})$ denotes the set of \mathbb{R} -linear morphisms that preserves complex structures (\mathbb{C} is implicitly with the complex structure $z\mapsto iz$)

Therefore $(V^*)_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$ is rewriten as

$$Hom_{\mathbb{R}}(V,\mathbb{C}) = Hom_{\mathbb{C}}((V,J),\mathbb{C}) \oplus Hom_{\mathbb{C}}((V,-J),\mathbb{C})$$

Using the first point of Lemma 3, one has

$$\wedge^{n}(V^{*})_{\mathbb{C}} = \bigoplus_{p+q=n} \wedge^{p,q} (V^{*})_{\mathbb{C}}$$

where $\wedge^{p,q}T^*_{\mathbb{C}}M$ denotes the \mathbb{C} -vector space of forms that are p times \mathbb{C} -linear and q times \mathbb{C} -antilinear.

Note one can easily find in V an orthonormal basis ∂_{x_i} , ∂_{y_i} with $J(\partial_{x_i}) = \partial_{y_i}$. We clarify here the definition and implicit identifications of basic objects such as dz_i and $d\bar{z}_i$.

Object	Where it belongs/ properties	${\mathbb C}$ -linear extension/ proper
$\partial_{z_i} = \iota_J(\partial_{x_i}) = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$	$V^{1,0}$	$dz_i(\partial_{z_i}) = \delta_{i,j}, dz_i(\partial_{\bar{z_i}}) = 0$
$\partial_{\bar{z}_i} = \iota_{-J}(\partial_{x_i}) = \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$	$V^{0,1}$	$d\bar{z}_i(\partial_{z_i}) = 0, d\bar{z}_i(\partial_{\bar{z}_i}) = \delta_{i,j}$
$dz_i = dx_i + idy_i$	$Hom_{\mathbb{C}}((V,J),\mathbb{C}), \mathbb{C}$ -linear	$Hom_{\mathbb{C}}(V_{\mathbb{C}},\mathbb{C})$, null on $V^{0,1}$,
$d\bar{z}_i = dx_i - idy_i$	$Hom_{\mathbb{C}}((V,-J),\mathbb{C}), \mathbb{C}$ -antilinear	$Hom_{\mathbb{C}}(V_{\mathbb{C}},\mathbb{C})$, null on $V^{1,0}$

Remark 2. One can note a subtlety here: there are two natural ways to extend dz_i to $V^{1,0}$

- 1. by first make a $\mathbb C$ -linear extension on $V_{\mathbb C}$, then make a restriction on $V^{1,0}$
- 2. using the identification $(V, J) \equiv (V^{1,0}, i)$

but these two coincide, as they are all \mathbb{C} -linear and satisfy $dz_i(\partial_{z_j}) = \delta_{i,j}, dz_i(\partial_{\bar{z}_j}) = 0$. Same story with $d\bar{z}_i$. See Figure and Figure

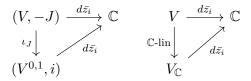


Figure 2: Two natural ways to define $d\bar{z}_i$ on $V^{0,1}$

Definition 2 (Theorem). The following properties are equivalent and X is called a Kähler manifold if one of them is satisfied.

- 1. X is a complex manifold, equipped with a Hermitian structure h(.,.) compatible with the complex structure J, and a fundamental form ω , and $d\omega = 0$.
- 2. X is a Riemannian manifold equipped with a parallel complex structure.
- 3. X is a complex manifold, equipped with a Hermitian structure such that the Chern connection on $T^{1,0}X$ is, up to an identification by ι_J , the Levi-Civita connection
- 4. X is a complex manifold, equipped with a Hermitian structure such that the Chern connection on $T^{1,0}X$ is torsionless

We call a complex manifold X of Kähler type if there exists a Hermitian structure under which X is Kähler.

The proof is straightforward, see Manuscript for details.