Minimal surfaces and holomorphic curves

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Contents

1 Minimal surfaces in $\mathbb{R}^3$ and Gauss map.

We will start with the following result of S. S. Chern [**?**].

Theorem 1 (Chern)**.** *Let $f : \Sigma^2 \hookrightarrow \mathbb{R}^3$ be a compact oriented surface. The association to each point $p \in \Sigma$ its normal vector at p gives a map*

$$\tilde{f} : \Sigma \longrightarrow \mathbb{S}^2$$

Then Σ is minimal surface if and only if $\tilde{f} : (\Sigma, i) \longrightarrow (\mathbb{S}^2, J)$ is antiholomorphic, where

- *i is the complex structure given by the conformal class of the induced metric of Euclidean metric from $\mathbb{R}^3$,*

- *J is the complex structure on $\mathbb{S}^2$ given by the diffeomorphism*

$$\begin{aligned} \mathbb{S}^2 &\longrightarrow Q_1 = \{z_0^2 + z_1^2 + z_2^2 = 0\} \subset \mathbb{CP}^2 \\ r &\longmapsto [(u_1 + iv_1, u_2 + iv_2, u_3 + iv_3)] \end{aligned}$$

where (u, v, r) form an oriented orthonormal basis of $\mathbb{R}^3$.

This result can also be generalised for surface $\Sigma^2 \subset \mathbb{R}^n$ by associating to each point $p \in \Sigma$ its tangent plane $T_p\Sigma \in \widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$ the space of oriented 2-pane in Rⁿ \). We can then equip $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$ with a complex structure given by the diffeomorphism

$$\begin{aligned} \widetilde{\mathrm{Gr}}_2(\mathbb{R}^n) &\longrightarrow Q_{n-2} = \{z_0^2 + \cdots + z_{n-1}^2\} \subset \mathbb{CP}^{n-1} \\ u \wedge v &\longmapsto [u + iv] = [(u_1 + iv_1, \dots, u_n + iv_n)] \end{aligned}$$

where (u, v) forms an oriented orthonormal basis of $T_p\Sigma$.

The above association $\tilde{f}$ is called the **Gauss map** of the surface Σ.

Remark 1. *The above definition of Q_{n-2} suggests that we should complexify an inner product $\langle \cdot, \cdot \rangle$ on a real vector space V to an inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $V \otimes \mathbb{C}$ so that the latter is symetric, i.e.*

$$\langle iu, v \rangle_{\mathbb{C}} = \langle u, iv \rangle_{\mathbb{C}} = i\langle u, v \rangle_{\mathbb{C}}$$

and not i-antilinear in the second parameter. This way, one has

$$|u + iv|^2 = \sum_i (u_i + iv_i)^2 = |u|^2 - |v|^2 + 2i\langle u, v \rangle = 0 \text{ if } u, v \text{ form an orthogonal basis.}$$

The correspondence that we will define between minimal surface in a Riemannian four-manifold and a holomorphic curve in an almost complex six-manifold is a generalised version of the Gauss map, called *Gauss lift.*

2 Twistor spaces and Gauss lift.

2.1 2-forms in 4 dimensional geometry

Let M be an oriented four-manifold and vol is a volume form on M. Then there is a symmetric bilinear form on $\Lambda^2 T^*M$ given by the wedge product

$$\begin{aligned} \Lambda^2 T^*M &\longrightarrow \mathbb{R} \\ (u, v) &\longmapsto \frac{u \wedge v}{\mathrm{vol}} \end{aligned}$$

which is of signature (3,3), i.e. the (non-unique) maximal vector subspace on which $\wedge$ is positive (resp. negative) are of dimension 3.

Fixing a metric g on M, one can dualise $\wedge$, i.e. define the Hodge star operator

$$\begin{aligned} * : \Lambda^2 T^*M &\longrightarrow \Lambda^2 T^*M \\ v &\longmapsto *v \text{ such that } u \wedge *v = \langle u, v \rangle \,\mathrm{vol}_g \end{aligned}$$

Remark 2. *1. The inner product $\langle \cdot, \cdot \rangle$ here is rescaled from the induced metric on $\Lambda^2 T^*M$ such that $dx^i \wedge dx^j$ is of norm 1 (and not 2) for all orthonormal dx^i, dx^j.*

2. The Hodge star (on 2-forms in 4D) is conformal invariant. In fact, if $g_\theta = e^{2\theta} g$ then $e^{-4\theta}\langle u, *_\theta v \rangle = \langle u, *_\theta v \rangle_\theta = e^{-4\theta} \frac{u \wedge v}{\mathrm{vol}_g} = e^{-4\theta} \langle u, *v \rangle$. On the other hand, knowing $*$, one can recover the conformal class of g.

Since $*$ is auto-adjoint, i.e. $\langle u, *v \rangle = \frac{u \wedge v}{\mathrm{vol}} = \langle *u, v \rangle$, and $(*)^2 = 1$, its Eigenvalues are ± 1 with Eigenspaces $\Lambda_\pm$ of dimension 3.

Similarly, one can define the bilinear form $\wedge$ and the Hodge star $*$ for 2-vectors in $\Lambda^2 TM$.

The above constructions are point-wise, so let us consider a 4 dimensional vector space V that will be in our case the tangent space $T_x M$ at a point $x \in M$.

Remark 3. *The null-cone of 2-vectors (resp. 2-vectors) u such that $u \wedge u = 0$ is exactly the set of simple 2-vectors (resp. 2-forms). This is because of the standard form of 2-vectors (resp. 2-forms), which in 4D can only be $e_1 \wedge f_1$ or $e_1 \wedge f_1 + e_2 \wedge f_2$ where $\{e_i, f_i, i = 1, 2\}$ form a basis of V. These simple 2-forms represents an **oriented 2 dimensional subspace** of V.*

Now let g be a metric on V, as before we can talk about (anti)-self-dual 2-vectors (resp. 2-forms) on V.

Remark 4. *The set of complex structures J compatible with g is in one-to-one correspondence with the set of self-dual 2-vectors (if J preserves orientation) and anti-self-dual 2-vectors (if J reverses orientation). In other words, the musical $J^\sharp$ is of form*

$$J^\sharp = e_1 \wedge e_2 + e_3 \wedge e_4$$

for an orthonormal basis $\{e_i\}$ of g. The orientation of J is given by the orientation of the basis (e_1, e_2, e_3, e_4).

The surjectivity part of the following result may be useful when one needs a good basis of V.

Theorem 2. *The group $SO(V)$ acts separately and isometrically on Λ_+ and Λ_- and the covering*

$$SO(V) \longrightarrow SO(\Lambda_+) \oplus SO(\Lambda_-)$$

is two-to-one.

Proof.

1. The separate action is because $*$ (and hence $\Lambda_\pm$) only depends on g.
2. Surjectivity follows from 2-to-1, connectedness, compactness of the SO groups and dimension: The image of $SO(V)$ has to be a connected, compact subgroup of $SO(\Lambda_+) \oplus SO(\Lambda_-)$ having the same dimension.
3. 2-to-1: This is the only computational part of the proof. Let α be an element of $SO(V)$ sending an orthonormal base e_i to another orthonormal base f_i such that α acts trivially on $\Lambda_\pm$, then clearly α is trivial on Λ^2. One has $e_i \wedge e_j = \alpha(e_i) \wedge \alpha(e_j) = \alpha_i^h e_h \wedge \alpha_j^k e_k$, hence
$$(\delta_i^h \delta_j^k - \delta_i^k \delta_j^h) e_h \wedge e_k = (\alpha_i^h \alpha_j^k - \alpha_i^k \alpha_j^h) e_h \wedge e_k \quad \forall h, k, i, j$$
Hence $(\alpha_i^h)^2 = (\delta_i^h)^2$, meaning that $\alpha = \pm \mathrm{Id}$.

$\square$

An application of Theorem 2 is the immediate (without computation) proof of the following result. Denote by $S_\pm$ the unit sphere of $\Lambda_\pm$.

Corollary 2.1. *For any $\omega_+ \in S_+$ and $\omega_- \in S_-$, the sum $\omega_+ + \omega_-$ is in the null-cone*

Proof. By surjectivity in Theorem 2, there exists $\varphi \in SO(V)$ that maps $\omega_\pm$ to $e_1 \wedge e_2 \pm e_3 \wedge e_4$, hence it maps $\omega_+ + \omega_-$ to $2e_1 \wedge e_2$ which is in the null-cone. It remains to evoke that the null-cone is preserved by $SO(V)$. $\square$

2.2 Twistor spaces and their natural complex structures

Given a Riemannian four-manifold (M, g), then $\Lambda^2 TM$ splits as $\Lambda^2 TM = \Lambda_+ \oplus \Lambda_-$ and one obtains 2 $\mathbb{S}^2$-bundle over M $S_\pm = S(\Lambda_\pm)$ whose fibres are unit spheres in $\Lambda_\pm$. Again, then metric on $\Lambda_\pm$ is renormalised so that $e_1 \wedge e_2 \pm e_3 \wedge e_4$ are of norm 1. In particular, if (e_1, e_2, e_3, e_4) is an oriented basis then an orthonormal basis of S_- is $\{e_1 \wedge e_2 - e_3 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3\}$. The six-dimensional total spaces of these bundles are called **Twistor spaces** over M.

Remark 5. *The distinction between S_+ and S_- only depends on the orientation of M, i.e. if one reverses the orientation, one becomes the other.*

We will show that there are two natural (almost) complex structure on twistor spaces $S_\pm$. By Remark 5, let us define these complex structure on S_- since the same construction applies for S_+.

Take $\omega \in S_-$ an anti-self-dual 2-vector over a point $p \in M$, then by Remark 4 the musical $\omega_\flat$ gives rise to an orientation-reversing complex structure J on T_pM. Under J, $\Lambda^2 TM$ splits as

$$\mathbb{C} \otimes_{\mathbb{R}} \Lambda^2 T_pM = \Lambda_p^{2,0} \oplus \Lambda_p^{1,1} \oplus \Lambda_p^{0,2}$$

where the factors are of complex dimension 1, 4 and 1 respectively and $\omega \in \Lambda_p^{1,1}$. In particular, if ω is given by $\omega = e_1 \wedge e_2 - e_3 \wedge e_4$, which we denote briefly as $12 - 34$ then

1. $\Lambda_p^{1,1}$ is $\mathbb{C}$-generated by $12, 34, 14 + 23.13 - 24$, i.e. $\{\omega, 12 + 34, 14 + 23, 13 - 24\}$ is an orthonormal basis of $\Lambda_p^{1,1}$.
2. $\Lambda^{2,0}$ is $\mathbb{C}$-generated by $(14 - 23) - i(13 + 24)$ and $\Lambda^{0,2}$ is $\mathbb{C}$-generated by $(14 - 23) + i(13 + 24)$.

Writing

$$\mathbb{C} \otimes \Lambda^2 T_pM = (\mathbb{C} \otimes \Lambda_{+.p}) \oplus (\mathbb{C} \otimes \Lambda_{-,p}) = \Lambda_p^{2,0} \oplus \Lambda_p^{1,1} \oplus \Lambda_p^{0,2},$$

one see

- from the first point shows that $\Lambda_p^{1,1} = \mathbb{C}\omega \oplus^\perp (\mathbb{C} \otimes \Lambda_{+,p})$
- from the second point shows that $\mathbb{C} \otimes \Lambda_{-,p} = \left(\Lambda_p^{2,0} \oplus \Lambda_p^{0,2}\right) \oplus^\perp \mathbb{C}\omega$

Therefore the complexified vertical tangent of S_- at ω is $\Lambda_p^{2,0} \oplus \Lambda_p^{0,2}$ which is the complexification of $\mathbb{R}(14 - 23) \oplus \mathbb{R}(13 + 24)$.

The following result allows us to define Koszul-Malgrange almost complex structure on $S_\pm$.

Proposition 2.1. *Let (M^4, g) be a Riemannian four-manifold and $S_\pm$ be its twistor spaces. Then the Levi–Civita connection ∇ reduces to $S_\pm$, meaning that the horizontal 4-planes at $\omega \in S_+ \subset \Lambda^2 T^*M$ (resp. $\omega \in S_- \subset \Lambda^2 T^*M$) are in $T_\omega S_+ \subset T_\omega \Lambda^2 T^*M$ (resp. $T_\omega S_- \subset T_\omega \Lambda^2 T^*M$).*

Definition 1. *The two natural complex structure J_1, J_2 of S_- at ω is given by setting*

$$T^{1,0}(\omega) = \begin{cases} (T^{1,0})^h \oplus \Lambda^{2,0} \text{ for } J_1 \\ (T^{1,0})^h \oplus \Lambda^{0,2} \text{ for } J_2 \end{cases}$$

where $T^{1,0}$ is the holomorphic tangent at the point p, the base point of $\omega \in S_-$ on M, under the complex structure given by ω, parallelly lifted to the horizontal in plane at ω using the Levi–Civita connection of (M, g).

Another way to define these 2 complex structures is to say that

- One has a natural complex structure on the fibers of S_-, under which, if one supposes $\omega = 12 - 34$, $J_1^v(14 - 23) = 13 + 24$ and $J_2^v = -J_1^v$.
- The complex structures on $T_\omega S_- = H \oplus V$ is given by the sum of J_1^v (resp. J_2^v) on the vertical component V and the complex structure (given by ω through horizontal lift) on the horizontal component H.

We will know prove that the almost complex structure J_2 is never integrable. The idea is to say that if this was true than there are very few holomorphic sections, which contradicts the fact that under Koszul–Malgrange [?] complex structure (component-wise complex structure) a section is holomorphic if and only if its image is a complex sub manifold and there are plenty complex sub manifold in an integrable complex manifold.

We will start by explaining two ways to complexify a connection.

2.3 Two different complexifications of a linear map.

The majority of computational details in the proof of holomorphic curve-minimal surface correspondence will be very intuitive if one is able to go back and forth between the following two ways to complexify a linear map $f : (V, J_V) \longrightarrow (W, J_W)$ between complex vector spaces.

	First way (A)	Second way (B)
Definition		

F_2: V⊗ C &$\longrightarrow W\otimes$ C\ iX &$\longmapsto if(X)$

f is holomorphic iff	$F_1(X + iJ_V X) = 0$, i.e. $F_1(\frac{\partial}{\partial \bar{z}}) = 0$	$F_2(V^{1,0}) \subset W^{1,0}$
Advantage	Do not need to complexify W	F_2 preserves type

While the advantage of (B) is clear, the fact that in (A), we do not have to complexify W is quite convenient in certain cases, for example when f is a connection on a complex vector bundle F (with complex structure on each fibre) over an almost complex manifold M. Imagine we have to complexify the fibre and then the defining horizontal planes of certain connection ∇ in order to talk about $\nabla_{\frac{\partial}{\partial z}}$. A section s is then called holomorphic if and only if $\nabla_{\frac{\partial}{\partial \bar{z}}} s = 0$. This means that s is holomorphic as a map from M to the total space F equipped with the Koszul-Malgrange complex structure.

Now let us first use the complexification (B) for the Levi–Civita connection ∇ on S^-. First note that if $s : U \subset M \longrightarrow S_-$ is a section then U can be equipped with a natural complex structure J which is $s(p)$ at every point $p \in U \subset M$. The following result is straight-forward.

Proposition 2.2 (holomorphic section of S_-)**.** *Given a section* $s : (U, J) \longrightarrow (S_-, J_\alpha), \alpha = 1, 2,$ *then*

1. *One only has to check the vertical component to prove s is holomorphic.*
2. *To prove that s is holomorphic, take any* $\xi \in T^{1,0}_p M$ *and check if* $v = \nabla_\xi s$ *is in* $\Lambda^{2,0}$ *(for* J_1*) or in* $\Lambda^{0,2}$ *(for* J_2*).*
3. *s is holomorphic if and only if* $s_* T_p M$ *is a complex subspace of* $T_{s(p)} S_-$*.*

Proof. For the 'if' part of the last statement, note that $s_*(J\xi)$ and $J s_* \xi$ live in the same four-plane $s_* T_p M$ with the same horizontal projection, therefore they coincide. □

Using complexification (B), we can see ∇ as a section of $(\mathbb{C} \otimes T^* M) \otimes (\Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2})$. Denote by D_1, D_2 the projection of ∇ on

$$\mathfrak{S}_1 := ((T^{0,1})^* \otimes \Lambda^{2,0}) \oplus ((T^{1,0})^* \otimes \Lambda^{0,2}),$$

and

$$\mathfrak{S}_2 := ((T^{0,1})^* \otimes \Lambda^{0,2}) \oplus ((T^{1,0})^* \otimes \Lambda^{2,0}).$$

then one has

Proposition 2.3. *For any section s over U and J the complex structure coming from s, there is no* $\Lambda^{1,1}$ *component in* $\nabla_X s$ *for any vector* $X \in T_p M$*. In other words,*

$$\nabla s = D_1 s + D_2 s.$$

Proof. Let (e_1, e_2, e_3, e_4) be an orthonormal frame such that $Je_1 = e_2, Je_3 = e_4$ and $s = e_1 \wedge e_2 + e_3 \wedge e_4$ and denote $\alpha_1 = \frac{1}{2}(e_1 - ie_2)$ and $\alpha_2 = \frac{1}{2}(e_3 - ie_4)$. One has $\alpha_i \in T^{1,0}$ and

$$2is = \alpha_1 \wedge \bar{\alpha}_1 + \alpha_2 \wedge \bar{\alpha}_2$$

hence the $\Lambda^{1,1}$ component of $2i\nabla_X s = \nabla_x \alpha_i \wedge \bar{\alpha}_i + \alpha_i \wedge \nabla_X \bar{\alpha}_i$ is $(2i\nabla_X s)^{1,1} = \lambda_i^j \alpha_j \wedge \bar{\alpha}_i + \tilde{\lambda}_i^j \alpha_i \wedge \bar{\alpha}_j$ where

$$\begin{aligned} \nabla_X \alpha_i &= \lambda_i^j \alpha_j + \ldots \text{ terms in } \bar{\alpha}_j \\ \nabla_X \bar{\alpha}_i &= \tilde{\lambda}_i^j \bar{\alpha}_j + \ldots \text{ terms in } \alpha_j \end{aligned}$$

since $\lambda_i^j = \langle \nabla_X \alpha_i, \bar{\alpha}_j \rangle = -\langle \alpha_i, \nabla_X \bar{\alpha}_j \rangle = -\tilde{\lambda}_j^i$, one has

$$(2i\nabla_X s)^{1,1} = \lambda_i^j \alpha_j \wedge \bar{\alpha}_i - \lambda_j^i \alpha_i \wedge \bar{\alpha}_j = 0.$$

□

2.4 Non-integrability of (S_-, J_2).

It follows from Proposition 2.2 and Proposition 2.3 that

Proposition 2.4. *A section $s : (U, J) \longrightarrow (S_-, J_\alpha), \alpha = 1, 2$ is holomorphic if and only if $D_\alpha s = 0$.*

The equation $D_1 s = 0$ is equivalent to the Nijenhuis tensor of J being 0, therefore one has:

Theorem 3. *Let s be a section of S_- on U and J be the corresponding complex structure on U. Then $D_1 s = 0$ if and only if J is integrable.*

Now let us derive non-integrability of (S_-, J_2) using the previous results. If (S_-, J_2) was integrable, then take any 2 (complex) dimensional sub manifold of S_- that is graph of a non-parallel section s. Then by Proposition 2.2, s would be holomorphic meaning that $D_2 s = 0$ by Proposition 2.4. Also, integrability of J_2 and holomorphicity of s would imply integrability of the complex structure J coming from s on U, which means, by Theorem 3, $D_1 s = 0$. By Proposition 2.3, $\nabla s = D_1 s + D_2 s = 0$ and s would be therefore parallel, which is a contradiction.

Theorem 4 (Non-integrability of J_-)**.** *(S_-, J_-) is never integrable.*

2.5 Gauss lift.

We finalise this section by defining the generalised Gauss map (i.e. Gauss lift) and giving the exact statement of Twistorial correspondence.

Let $f : (\Sigma, i) \longrightarrow (M^4, g)$ be an immersion of a Riemann surface Σ to a Riemannian four-manifold M. Define $\tilde{f}_-$ to be the anti-self-dual projection of the 2-vector associated to $f_* T_p \Sigma$ (Remark 3), i.e.

$$\begin{aligned} \tilde{f}_- : \Sigma &\longrightarrow S_- \\ p &\longmapsto (1 - *)(f_* T_p M) \end{aligned}$$

Remark 6. *Another way to define $\tilde{f}_-$ is to say that it is the unique anti-self-dual 2-vector such that f is holomorphic under the corresponding complex structure (Remark 4).*

It is clear that the Gauss lift can be continuously defined in case where f is a branched immersion [**?**]. We can now state the correspondence in [**?**].

Theorem 5 (Eells-Salamon)**.** *There is a one-to-one correspondence between non-vertical J_2-holomorphic curves in S_- and conformal harmonic map from (Σ, i) to (M, g) given by the Gauss lift.*

3 Twistor correspondence.

The correspondence is between conformal harmonic branched immersions in M and J_2-holomorphic curves in S_- via the Gauss lift. Therefore a branch point of $f : \Sigma \longrightarrow M$ comes from

- either a critical point of the holomorphic curve $\tilde{f}_- : \Sigma \longrightarrow S_-$
- or a point $p \in \Sigma$ where $\tilde{f}_-$ is tangent to the twistor lines $(S_-)_{f(p)}$.

We will show that in both cases, these points are isolated, hence of finite number by compactness of Σ.

3.1 Review: A few local properties of (pseudo)holomorphic curve.

Theorem 6 (Dependence on ∞-jet)**.** *Let $u : \Sigma \longrightarrow S$ be a holomorphic curve between a compact connected Riemann surface Σ and an almost complex manifold S. Then u is uniquely determined by its ∞-jet at one point $p \in \Sigma$, i.e. if there is another holomorphic curve $v : \Sigma \longrightarrow M$ such that the ∞-jets of v and u coincide then the two maps u, v coincide.*

For a proof of this Theorem 6, see [**?**]. The idea is to write down the generalised Cauchy-Riemann equation for u and v then apply a PDE estimate. We are interested in how this theorem can be applied to prove that critical point of u is isolated.

Proposition 6.1 (Isolated critical points)**.** *Given a non-constant holomorphic curve $u;(\Sigma, i) \longrightarrow S$ from a Riemann surface Σ to an almost complex manifold S. Then critical points $p \in \Sigma$ where $du(p) = 0$ are isolated in Σ.*

This result is immediate when S has a holomorphic function ξ_1 locally defined at p, for example when the complex structure on S is integrable. In that case, it suffices to notice that p has to be a critical point of $\xi_1 \circ u$ which, in a local chart of Σ, is a zero of the holomorphic function $\frac{(\partial \xi_1 \circ u)}{\partial z}$.

Proof. We can suppose that $u(0) = 0 \in \mathbb{C}^n, du(0) = 0$ and the complex structure on M is $J(\xi) \in GL(\mathbb{R}^{2n})$ with $J^2 = -Id$ and $J(0) = J_0$ the standard complex structure of $\mathbb{C}^n$. Let s, t be the isothermal coordinates of Σ, the generalised Cauchy–Riemann equation reads

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0 \tag{1}$$

Since u is non-constant, using Theorem 6, one can find an $l \geq 1$ such that

$$|u(z)| = O(|z|^l) \neq O(|z|^{l+1})$$

therefore $J(u(z)) = J_0 + O(|z|^l)$. Take the $|z|^{l-1}$ term in (1), one has

$$\frac{\partial T_l(u)}{\partial s} + J_0 \frac{\partial T_l(u)}{\partial t} = 0$$

where $T_l(u)$ is the expansion of u at 0 upto order l. This means T_l is holomorphic w.r.t J_0, i.e. $T_l(u) = (a_1 z^l, \dots, a_n z^l)$ for a non-zero vector $a \in \mathbb{C}^n$. Now that

$$u(z) = az^l + O(|z|^{l+1}), \quad du(z) = alz^{l-1} + O(|z|^l),$$

one sees that for $|z| < \epsilon$ sufficiently small $du(z) \neq 0$. □

In case where $S \longrightarrow M$ is a twistor space with the twistor lines S_q over $q \in M$ being a complex submanifold, using the same technique, one can prove that

Proposition 6.2 (Vertically tangent points)**.** *Let $u : \Sigma \longrightarrow S$ be a non-vertical holomorphic curve from a compact Riemann surface Σ to a twistor space S, then the points $p \in \Sigma$ where the tangent $u_* T_p\Sigma$ is vertical at $u(p) \in S$ are isolated and therefore of finite number.*

Proof. The statement being local, we can suppose $p = 0 \in \mathcal{U} \subset \Sigma = \mathbb{C}$. Let $v : \mathcal{V} = \mathcal{O}p(0) \longrightarrow S = \mathbb{C}^n$ be the vertical twistor line $z \longmapsto z \times \{0\}$ touching u at $u(0) \equiv v(0) = 0$. This means we also suppose that the restriction of complex structure on $S = \mathbb{C}^n$ onto $\mathbb{C} \times \{0\}$ is the standard one (we can always do this!). Choose isothermal coordinates $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ of $\mathcal{U}, \mathcal{V}$ such that $u_* \frac{\partial}{\partial s} = v_* \frac{\partial}{\partial_s}$ and $u_* \frac{\partial}{\partial t} = v_* \frac{\partial}{\partial_t}$ at 0. In other words, we suppose that $u(0) = v(0) = 0 \in \mathbb{C}^n$ and $du(0) = dv(0)$.

Subtracting the generalised Cauchy–Riemann equations for u and v, one has

$$\partial_s(u - v) + (J(u) - J(v))\partial_t u + J(v)\partial_t(u - v) = 0 \tag{2}$$

Since u is not vertical, by Theorem 6, there exists $l \geq 2$ such that $u - v = O(|z|^l) \neq O(|z|^{l+1})$. Therefore $J(u) - J(v) = O(|z|^l)$. Take the $O(|z|^{l-1})$ part in (2), one sees that $T_l(u - v)$ is holomorphic in the usual sense, hence there exists $0 \neq a \in \mathbb{C}^n$ such that

$$(u - v)(z) = (a_1 z^l, \ldots, a_n z^l)$$

The proof is finished if there exists an $a_i \neq 0, i \in \overline{2, n}$. If not, we can replace v by $\tilde{v} : z \longmapsto (z + a_1 z^l) \times \{0\}$ which is still a holomorphic parametrisation of the twistor line $\mathbb{C} \times \{0\}$, but $|u(z) - v(z)| = O(|z|^{l+1})$. By repeating this argument finitely many times, one reaches a moment when there exists a non-zero a_i for $i \geq 2$. The conclusion then follows. $\square$

3.2 Review: Harmonic maps from a Riemann surface.

First of all this term makes sense because energy of a map $f : \Sigma \longrightarrow M$ is a conformal invariant when $\dim \Sigma = 2$. We can therefore try to write the equation of harmonicity $\tau(f) = 0$ using the complex structure of Σ.

Using a complex coordinate z, one has $df = \nabla_{\partial_z} f dz + \nabla_{\partial_{\bar{z}}} f \; d\bar{z}$ where $\nabla_{\partial_z} f := f_* \partial_z$, $\nabla_{\partial_{\bar{z}}} f := f_* \partial_{\bar{z}}$. The vector bundle $f^* M$ over Σ is equiped with a fiberwise metric and a metric connection inherited from M. Combining this with the Levi–Civita connection on (Σ, h) where $[h] = i$ the complex structure on Σ, one obtains a connection on all $T^* M^{\otimes k} \otimes TM^{\otimes l} \otimes f^* TM$ that is compatible with tensor product and contraction. We now use complexification (B) when we talk about covariant derivative $\nabla_{\partial_z} s$ of a section s.

Remark 7. *∇_{∂_z} and $\nabla_{\partial_{\bar z}}$ commute on f. In fact*

$$\nabla_{\partial_z}\nabla_{\partial_{\bar z}}f-\nabla_{\partial_{\bar z}}\nabla_{\partial_z}f=\nabla^M_{f_*\partial_z}f_*\partial_{\bar z}-\nabla^M_{f_*\partial_{\bar z}}f_*\partial_z=[f_*\partial_z,f_*\partial_{\bar z}]=f_*[\partial_z,\partial_{\bar z}]=0$$

This means that $\nabla_{\partial_z}\nabla_{\partial_{\bar z}}f$ is always a real vector field along f.

We will write down the equation of harmonicity $\mathrm{Tr}_h\,\nabla df=0$ explicitely using $\partial_z,\partial_{\bar z}$ and the fact that $\mathrm{Tr}_h(dz\otimes dz)=\mathrm{Tr}_h(d\bar z\otimes d\bar z)=0$:

$$\begin{aligned}\nabla df &= \nabla_{\partial_z}(\nabla_{\partial_z}f dz+\nabla_{\partial_{\bar z}}f\ d\bar z)\otimes dz+\nabla_{\partial_{\bar z}}(\nabla_{\partial_z}f dz+\nabla_{\partial_{\bar z}}f\ d\bar z)\otimes d\bar z\\ &= \nabla_{\partial_z}\nabla_{\partial_{\bar z}}f(dz\otimes d\bar z+d\bar z\otimes dz)+\text{ terms in } dz\otimes dz \text{ and } d\bar z\otimes d\bar z\end{aligned}$$

in which we claim that the sum

$$\mathcal{A}=(\nabla_{\partial_z}f\nabla_{\partial_z}dz+\nabla_{\partial_{\bar z}}f\nabla_{\partial_z}d\bar z)\otimes dz+(\nabla_{\partial_z}f\nabla_{\partial_{\bar z}}dz+\nabla_{\partial_{\bar z}}f\nabla_{\partial_{\bar z}}d\bar z)\otimes dz$$

is in fact a "term in $dz\otimes dz$ and $d\bar z\otimes d\bar z$". This can be seen by rewriting the first sum in $\mathcal{A}$ as

$$-\Gamma^z_{z\bar z}f_*\partial_z-\Gamma^{\bar z}_{z\bar z}f_*\partial_{\bar z}+\text{ terms in } dz\otimes dz \text{ and } d\bar z\otimes d\bar z$$

where Γ is the (B-complexified) Christoffel symbols of (Σ,h), and noticing that $\Gamma^z_{z\bar z}f_*\partial_z+\Gamma^{\bar z}_{z\bar z}f_*\partial_{\bar z}=f_*\nabla^{\Sigma}_{\partial_z}\partial_{\bar z}$. But

$$\nabla^{\Sigma}_{\partial_z}\partial_{\bar z}=\nabla_{\partial_x}\partial_x+\nabla_{J\partial_x}J\partial_x=0$$

since Riemann surfaces are Kahler manifolds ($SO(2)=U(1)$).

3.3 Proof of Theorem 5.

We have now acquired enough technology to prove Theorem 5 of Eells-Salamon. Keeping the same notation as before, with $f:(\Sigma,i)\longrightarrow(M,g)$ the immersion of the surface Σ and $\tilde f_-$ its Gauss lift, we can suppose that f has no branch point and its Gauss lift does not touch the vertical twistor lines. In other words, we only need to prove a "smooth version" of Theorem 5. This is because the Gauss lift can be continuously defined at branch points, which are of finite number, so there is no problem of associating a holomorphic curve on S_- to a harmonic conformal $f:\Sigma\longrightarrow M$. On the other hand, given a holomorphic curve $u:(\Sigma,i)\longrightarrow(S_-,J_2)$ and denote $f:\Sigma\longrightarrow M$ its projection to M, by Proposition 6.1 and Proposition 6.2, the singular points of f (where f fails to be an immersion) are of finite number. Holomorphicity of u implies that, at regular points of f, f is holomorphic with respect to the complex structure given by u, but this is the defining property of Gauss lift $\tilde f_-$, we have $u\equiv\tilde f_-$ at non-singular points of f. The continuity argument finishes the proof.

It remains now to prove the smooth version of Theorem 5.

Theorem 7 (Smooth version of Eells-Salamon)**.** *Given an immersion $f : (\Sigma^2, J) \longrightarrow (M^4, g)$ of a Riemann surface to a Riemannian four-manifold. Then f is conformal and harmonic if and only if $\tilde{f}_- : (\Sigma, J) \longrightarrow (S_-, J_2)$ is holomorphic.*

Proof. We will denote by σ the 2-form (section of f^*S_-) corresponding to $\tilde{f}_-$, i.e. $(\tilde{f}_-)_*X = \nabla_X\sigma \oplus f_*X$ where the former is vertical and the latter is the horizontal lift of f_*X. Then $\tilde{f}_-$ is holomorphic if and only if, using complexification (A), $(\tilde{f}_-)_*(X + iJX) = 0$, which means, for horizontal and vertical components:

$$f_*(X + iJX) = 0, \text{ which means } f_*X + (\tilde{f}_-)f_*JX = 0 \text{ where } (\tilde{f}_-) \text{ is the corresponding complex struct} \tag{3}$$

$$\nabla_{X+iJX}\sigma = 0, \text{ which means } \nabla_X\sigma + J_2\nabla_{JX}\sigma = 0 \tag{4}$$

This is because the induced complex structure of $f^*S_- \subset S_-$ coincides with the Koszul–Malgrange complex structure of f^*S_- as a vector bundle on Σ (the last statement is because $f : (\Sigma, i) \longrightarrow (f(\Sigma), \tilde{f}_-)$ is holomorphic).

One sees that (3) is equivalent to f being harmonic because f_*X and f_*JX have the same length and are orthogonal in M.

We will prove that under (3), (4) is equivalent to f being harmonic. To see that, we will translate (4) to an equivalent condition using complexification (B).

$$\nabla_X\sigma + J_2\nabla_{JX}\sigma = 0 \forall X \in T\Sigma \iff \nabla_{\frac{\partial}{\partial z}}\sigma = \nabla_{X-iJX}\sigma := \nabla_X\sigma - i\nabla_{JX}\sigma = \nabla_X\sigma - iJ_2\nabla_X\sigma \in T^{1,0}S_-$$

with respect to the complex structure J_2. But $\nabla_{\frac{\partial}{\partial z}}\sigma$ is vertical and $T^{1,0}S_- = (T^{1,0})^h \oplus (\Gamma^{0,2})^v$, one has (4) is equivalent to $\nabla_{\frac{\partial}{\partial z}}\sigma \in \Gamma^{0,2}$. Since we knew from Proposition 2.3 that $\nabla_{\frac{\partial}{\partial z}}\sigma$ only has a $\Lambda^{2,0}$ component and a $\Lambda^{0,2}$ component (under $\tilde{f}_-$), (4) means the former vanishes.

We will denote by δ and $\bar{\delta}$ the covariant derivatives $\nabla_{\frac{\partial}{\partial z}}$ and $\nabla_{\frac{\partial}{\partial \bar{z}}}$ (complexification (B)) respectively. Since f is conformal its Gauss lift σ is a multiple of $i(1-*)(\delta f \wedge \bar{\delta} f)$, let suppose that $\sigma = -ic(1-*)(\delta f \wedge \bar{\delta} f)$ where the scalar c may varies from point to point. One has

$$\delta\sigma = (c^{-1}\delta c)\sigma - ic(1-*)(\delta^2 f \wedge \bar{\delta} f + \delta f \wedge \delta\bar{\delta} f).$$

At the point of M in question and under the complex structure $\tilde{f}_-$, one has

- $\mathbb{C} \otimes \Lambda^2_- = \Lambda^{0,2} \oplus \Lambda^{2,0} \oplus \mathbb{C}\sigma$,

- $\bar{\delta} f$ is of type (0,1) (f is holomorphic under $\tilde{f}_{-}$)
- δf is of type (1,0).

Therefore the $\Lambda^{2,0}$ component of $\delta\sigma$ is

$$-ic\delta f \wedge (\delta\bar{\delta} f)^{1,0} =: A$$

The only way for A to be 0 without $(\delta\bar{\delta} f)^{1,0}$ vanishing is that $(\delta\bar{\delta} f)^{1,0}$ is a non-zero multiple of δf, which is impossible because one always has

$$g(\bar{\delta} f, (\delta\bar{\delta} f)^{1,0}) = g(\bar{\delta} f, \delta\bar{\delta} f) = -g(\delta\bar{\delta} f, \bar{\delta} f) = 0$$

So $A = 0 \iff (\delta\bar{\delta} f)^{1,0} = 0 \iff \delta\bar{\delta} f = 0$ because $\delta\bar{\delta} f$ is real according to Remark 7. $\square$