

# Bogomolov-Beauville classification

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## 1 From the Riemannian results of de Rham and Berger

We will first prove a (conceptually) straightforward result of de Rham decomposition and Berger classification. The following theorem is taken from Beauville's article

**Theorem 1** (Beauville). *Let  $X$  be a compact Kähler manifold with flat Ricci curvature, then*

1. *The universal covering space  $\tilde{X}$  of  $X$  decomposes isometrically and holomorphically as*

$$\tilde{X} = E \times \prod_i V_i \times \prod_j X_j$$

*where  $E = \mathbb{C}^k$ ,  $V_i$  and  $X_j$  are simply-connected compact manifolds of real dimension  $2m_i$  and  $4r_j$  with irreducible homonomy  $SU(m_i)$  for  $V_i$  and  $Sp(r_j)$  for  $X_j$ . One also has uniqueness in the strong sense as in de Rham decomposition.*

2. *There exists a finite covering space  $X'$  of  $X$  such that*

$$X' = T \times \prod_i V_i \times \prod_j X_j$$

*where  $T$  is a complex torus.*

*Proof.* Note that the first point is obtained directly from Cheeger-Gromoll splitting and de Rham decomposition: The one-dimensional parallel subspaces (of trivial holonomy) are regrouped to  $E$ . By Cheeger-Gromoll splitting,  $\tilde{X} = E \times M$  where  $M$  contains no line and is compact (note that we use compactness of  $X$  here). The irreducible factors in  $M$  are not symmetric spaces as Ricci curvature of symmetric spaces is non-degenerate. Holonomy of these factors are  $SU(m_i)$  and  $Sp(r_j)$  according to Berger list since they are Kähler manifolds and Ricci-flat. It remains to prove the second point.

We will regard each element of  $\pi_1(X)$  by its isometric, free, proper action on  $\tilde{X}$ . As pointed out the arguments in our discussion of uniqueness of de Rham decomposition, every isometry of  $\tilde{X}$  to itself preserves the components  $T_{x_0}E$ ,  $T_{x_i}V_i$  and  $T_{x_j}X_j$  of  $T_x\tilde{X}$ , each isometry  $\phi$  of  $\tilde{X}$  is of form  $(\phi_1, \phi_2)$  where  $\phi_1 \in Isom(E)$  and  $\phi_2 \in Isom(M)$ .

We will use here the fact that if  $M$  is a Kähler manifold, compact and Ricci-flat then  $Isom(M)$  equipped with compact-open topology is discrete, therefore finite, which will be proved later (Lemma 3). We note  $\Gamma := \{\phi = (\phi_1, \phi_2) \in \pi_1(X), \phi_2 = Id_M\}$  and sometime abusively regard  $\Gamma$  as a subgroup of  $Isom(E)$ . Note that  $\Gamma$  is a normal subgroup of  $\pi_1(X)$  with finite index since the quotient is isomorphic to a subgroup of  $Isom(M)$ . Therefore  $\tilde{X}/\Gamma = E/\Gamma \times M$  is compact as a finite cover of  $X$ .

We apply the following theorem of Bieberbach.

**Theorem 2** (Bieberbach). *Let  $E = \mathbb{R}^n$  be an Euclidean space and  $\Gamma$  be a subgroup of  $Isom(E)$  that satisfies*

1.  $\Gamma$  *is discrete under compact-open topology.*
2.  $E/\Gamma$  *is compact.*

*Then the subgroup  $\Gamma'$  of translations in  $\Gamma$  is of finite index.*

Suppose that the two conditions are satisfied then the theorem gives:  $\tilde{X}/\Gamma' = E/\Gamma' \times M = T \times \prod_i V_i \times \prod_j X_j$  is a finite cover of  $\tilde{X}/\Gamma$  as  $\Gamma'$  is a normal subgroup of  $\Gamma$ :

**Fact.** The subgroup of translations in  $Isom(E)$ , where  $E = \mathbb{R}^n$  is an Euclidean space, is normal.

Therefore  $X' = \tilde{X}/\Gamma'$  is a finite cover of  $X$  that we want to find.

It remains to prove that  $\Gamma$  is discrete, which is a consequence of

1.  $\pi_1(X)$  is discrete, without limit point in  $Isom(E) \times Isom(M)$  (obvious).
2.  $Isom(M)$  is compact.

In fact given any  $\phi = (\phi_1, \phi_2) \in Isom(E) \times Isom(M)$ , there exists by (1.) a neighborhood  $\mathcal{U}_1(\phi_1, \phi_2) \times \mathcal{U}_2(\phi_1, \phi_2)$  of  $\phi$  in  $Isom(E) \times Isom(M)$  such that all points of  $\pi_1(X)$  lying in this region project to  $\phi_1$ . By (2.) we can find a neighborhood  $\mathcal{U}_1$  of  $\phi_1$  in  $Isom(E)$  small enough that  $\mathcal{U}_1(\phi_1) \times Isom(M) \subset \cup_{\phi_2 \in Isom(M)} \mathcal{U}_1(\phi_1, \phi_2) \times \mathcal{U}_2(\phi_1, \phi_2)$ . Therefore the projection of  $\pi_1(X)$  to  $Isom(E)$  is discrete, by consequence  $\Gamma$  is discrete.  $\square$

**Lemma 3.** *Let  $M$  be is a compact, simply-connected, Ricci-flat, Kähler manifold, then the group  $Aut(M)$  of automorphism of  $M$  equipped with compact-open topology is discrete, therefore  $Isom(M)$  is discrete, hence finite.*

*Proof.* The idea is that since  $Aut(M)$  is a Lie group, it suffices to prove that its Lie algebra is of dimension 0. This is done using these facts.

1. The Lie algebra of  $Aut(M)$  can be identified with the vector space of holomorphic vector fields on  $M$ .
2. *Bochner's principle:* All holomorphic tensor fields on a compact, Ricci-flat Kähler manifold are parallel.
3. The only invariant vector of the holonomy representation of  $M$  is 0 (obvious).

$\square$

Bochner principle for holomorphic vector fields comes from the following identity (called *Weitzenböck formula*):

$$\Delta(\frac{1}{2}\|X\|^2) = \|\Delta X\|^2 + g(X, \nabla \text{div} X) + Ric(X, X)$$

for every vector field  $X$ . If  $X$  is holomorphic then it is harmonic and has  $\text{div} X = 0$ . The fact that  $M$  is Ricci-flat gives  $\Delta(\frac{1}{2}\|X\|^2) = \|\nabla X\|^2$  and the function  $\|X\|^2$  is subharmonic, therefore constant since  $M$  is compact. We then have  $\nabla X = 0$ , i.e.  $X$  is parallel. The method of Bochner also works for tensor fields of any type in a Ricci-flat Kähler manifold and one also has  $\Delta(\|\tau\|^2) = \|\nabla \tau\|^2$  and that every holomorphic tensor field is parallel. See P. Petersen, *Riemannian geometry* and A. Besse, *Einstein Manifolds* for more detail.

## 2 Towards a classification for complex manifold

To obtain a translation of Theorem 1 in a context of complex manifolds (without any preferred metric a priori), we study the 2 building blocks: manifolds with holonomy  $SU(m)$  and  $Sp(r)$ . To be clear, recall that a complex manifold  $X$  is called of Kähler type if one can equip  $X$  with an Hermitian structure whose fundamental form  $\omega$  satisfies  $d\omega = 0$ . When we say  $X$  is of Kähler type, we refer to  $X$  as a complex manifold without fixing a metric on  $X$ .

### 2.1 Special unitary manifolds (proper Calabi-Yau manifolds)

**Remark 1.** *Let  $X$  be a compact Kähler manifold with holonomy  $SU(m)$  and complex dimension  $m \geq 3$  then:*

1.  $H^0(X, \Omega_X^p) = 0$  for all  $0 < p < m$ , by consequence  $\chi(\mathcal{O}_X) = 1 + (-1)^m$ .
2.  $X$  is projective, that is  $X$  can be embedded into  $\mathbb{P}^N$  as zero-locus of some (finitely) homogeneous polynomials.
3.  $\pi_1(X)$  is finite and if  $m$  is even,  $X$  is simply connected.

The first point is in fact algebraic in nature: it comes from the fact that the representation of  $SU(m)$  over  $\bigwedge^p T_x^* M$  is irreducible for all  $p$  et non-trivial for  $0 < p < m$ , therefore the action of  $SU(m)$  on  $\bigwedge^p T_x^* M$  for  $0 < p < m$  has no invariant element, hence  $H^0(X, \Omega_X^p) = 0$ .

The second point follows the following facts:

1. (Kodaira's theorem) A compact Kähler manifold with  $H^{2,0} = 0$  can be embedded in  $\mathbb{P}^N$ .
2. (Chow's theorem) A compact complex manifold embedded in  $\mathbb{P}^N$  is algebraic, i.e. defined by a finite number of homogeneous polynomials.

The third point is a direct consequence of Riemann-Hurwitz formula. In fact, the universal cover  $\tilde{X}$  of  $X$  is of holonomy  $SU(m)$ . This is due to the following remarks:  $Hol(X) \supset Hol(X') \supset Hol_0(X') = Hol_0(X)$  and  $Hol_0(X) = Hol(X) = SU(m)$  as  $SU(n)$  is connected.

By Theorem 1,  $\tilde{X}$  is compact by Lemma 3 a finite covering of  $X$  as  $\pi_1(X)$  is finite. As  $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\tilde{X}}) = 2$ , one has  $X = \tilde{X}$ , hence  $X$  is simply-connected.

**Theorem 4.** *Given a compact manifold  $X$  of Kähler type and complex dimension  $m$ , the following properties are equivalent*

1. *There exists a compatible metric  $g$  over  $X$  such that  $Hol(X, g) = SU(m)$ .*
2.  *$K_X$  is trivial and  $H^0(X', \Omega_{X'}^p) = 0$  for every  $0 < p < m$  and  $X'$  a finite covering of  $X$ .*

*Proof.* (1) implies (2) as a finite covering space  $X'$  of a special unitary manifold  $X$  is still a special unitary.

For the implication (2)  $\implies$  (1): by Yau's theorem we equip  $X$  with a Ricci-flat metric, by Theorem 1, there exists a finite cover  $X' = T \times \prod_i V_i \times \prod_j X_j$  where  $T$  is a complex torus,  $Hol(V_i) = SU(m_i)$ ,  $Hol(X_j) = Sp(r_j)$ . But  $H^0(X', \Omega_{X'}^p) = 0$  for  $0 < p < m$ ,  $X'$  has to be one of the  $V_i$  as  $H^0(X_j, \Omega_{X_j}^2)$  and  $H^0(V_i, \Omega_{V_i}^{m_i})$  do not vanish. Therefore  $Hol(X') = SU(m)$ , hence  $Hol(X) = SU(m)$ .  $\square$

Theorem 4 allows us to check if a manifold  $X$  is special unitary by looking at the  $h^{0,p}(0 < p < m)$  coefficients of the Hodge diamond of  $X$  and its finite covers. We can see, by this criteria that the following examples are special unitary manifolds. All of them are algebraically constructed, since a construction by glueing local charts is difficult (or impossible).

**Example 1** (Special unitary manifold). 1. *Elliptic curves over  $\mathbb{C}$  are special unitary, as any statement starting with "for every  $0 < p < 1$ " is formally true.*

2. *A K3 surface (simply-connected surface with trivial canonical bundle) is special unitary, its Hodge diamond is given below.*
3. *A quintic threefold (hypersurface of degree 5 in 4-dimensional projective space) is a special unitary manifold, the Hodge diamond of which is given is given below. In particular, the Fermat quintic defined by*

$$\{(z_0 : z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^4 : \sum z_i^5 = 0\}$$

4. *In general, any smooth hypersurface  $X$  of  $\mathbb{CP}^{m+1}$  of degree  $m+2$  satisfies  $h^{0,p} = 0$  for all  $0 < p < m$ . If  $X$  is simply-connected then it is a special unitary manifold.*

Table 1: Hodge diamond of a K3 surface.

$$\begin{array}{cccc}
 & & 1 & \\
 & 0 & & 0 \\
 1 & & 20 & & 1 \\
 & 0 & & 0 \\
 & & 1 & 
 \end{array}$$

Table 2: Hodge diamond of a quintic threefold.

$$\begin{array}{cccccc}
 & & & 1 & & \\
 & & 0 & & 0 & \\
 & 0 & & 1 & & 0 \\
 1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 \\
 & & 0 & & 0 & \\
 & & & 1 & & 
 \end{array}$$

## 2.2 Irreducible symplectic and hyperkähler manifolds

**Remark 2.** *Let  $X$  be a compact Kähler manifold with holonomy  $Sp(r)$  and complex dimension  $2r$  then:*

1. *There exists a holomorphic 2-form  $\varphi$  non-degenerate at every points.*
2.  *$H^0(X, \Omega_X^{2l+1}) = 0, H^0(X, \Omega_X^{2l}) = \mathbb{C}\varphi^l$  for all  $0 \leq l \leq r$ . By consequence  $\chi(\mathcal{O}_X) = r + 1$ .*
3.  *$X$  is simply-connected.*

The first point of the remark follows directly from our discussion of Berger classification.

The second point is algebraic in nature: The representation of  $Sp(r)$  on  $\bigwedge^p T_x^* M$  splits into

$$\bigwedge^p T_x^* M = P_p \oplus P_{p-2}\varphi(x) \oplus P_{p-4}\varphi^2(x) \oplus \dots \quad (1)$$

where  $P_k, 0 \leq k \leq r$  are irreducible, non-trivial for  $k > 0$  and  $\varphi(x) \in \bigwedge^2 T_x^* M$  uniquely defined up to a constant. Therefore the only invariant elements are  $c\varphi^{p/2}$  where  $c$  is a scalar.

For the last point, one uses the same arguments as Remark 1.

**Theorem 5.** *Given a compact manifold  $X$  of Kähler type and complex dimension  $2r$ , then:*

1. *The following properties are equivalent.  $X$  is called hyperkähler if it satisfies one of them.*
  - (a) *There exists a compatible metric  $g$  such that  $\text{Hol}(X, g) \subset \text{Sp}(r)$ .*
  - (b) *There exists a compatible symplectic structure: a 2-form that is closed, holomorphic and non-degenerate at every point.*
2. *The following properties are equivalent.  $X$  is called irreducible symplectic if it satisfies one of them.*
  - (a) *There exists a compatible metric  $g$  such that  $\text{Hol}(X, g) = \text{Sp}(r)$*
  - (b)  *$X$  is simply-connected and there exists (uniquely up to a constant) a compatible symplectic structure on  $X$ .*

By "compatible", we mean "compatible with the complex structure".

*Proof.* 1. The fact that (a) implies (b) is obvious. For the other way: since  $K_X$  is trivial (existence of global non-null section) by Yau's theorem we equip  $X$  with a Ricci-flat metric, then the symplectic structure  $\varphi$  of  $X$  is parallel by Bochner's principle. Hence the holonomy is in  $\text{Sp}(r)$ .

2. For the implication (a)  $\implies$  (b), it suffices to notice that the invariant elements  $\varphi$  in the decomposition (1) is unique. For the direction (b)  $\implies$  (a), note that  $X$  can be equipped with a Calabi-Yau metric by the (b)  $\implies$  (a) part of (1.), by Theorem 1,  $X = \prod_{j=1}^m X_j$  where  $X_j$  are irreducible compact Kähler manifolds. The symplectic structure  $\varphi$  on  $X$ , restricted on each  $X_j$ , gives a symplectic structure  $\varphi_j$  of  $X_j$ . But any form  $\sum_j \lambda_j \varphi_j$  is another symplectic structure of  $X$ , one must have  $m = 1$  by uniqueness of  $\varphi$ . □

**Example 2.** 1. *One can notice a trivial example: Every special unitary manifold of 2 complex dimensions is irreducible symplectic because  $\text{SU}(2)$  is isomorphic to  $\text{Sp}(1)$ .*

2. *Let  $X$  be a smooth cubic hypersurface in  $\mathbb{CP}^{n+1}$  and  $F(X) = \{L \in \text{Gr}(1, \mathbb{CP}^{n+1}), L \subset X\} \subset \text{Gr}(1, \mathbb{CP}^{n+1})$  the manifold formed by lines in  $X$ .  $F(X)$  is non-empty when  $n > 1$ , smooth if  $X$  is smooth and of dimension  $2n - 4$ . Beauville and Donagi proved that for  $n = 4$ ,  $F(X)$  is irreducible symplectic, therefore hyperkähler.*

### 2.3 Decomposition for complex manifold with vanishing Chern class

Theorem 1 can be translated to a decomposition for complex manifold in the following way:

**Theorem 6** (Bogomolov-Beauville classification). *Let  $X$  be a compact manifold of Kähler type of vanishing first Chern class.*

1. *The universal covering space  $\tilde{X}$  of  $X$  is isomorphic to a product  $E \times \prod_i V_i \times \prod_j X_j$  where  $E = \mathbb{C}^k$  and*
  - (a) *Each  $V_i$  is a projective simply-connected manifold of complex dimension  $m_i \geq 3$ , with trivial  $K_{V_i}$  and  $H^0(V_i, \Omega_{V_i}^p) = 0$  for  $0 < p < m_i$*
  - (b) *Each  $X_j$  is an hyperkähler manifold.*

*This decomposition is unique up to an order of  $i$  and  $j$ .*

2. *There exists a finite cover  $X'$  of  $X$  isomorphic to the product  $T \times \prod_i V_i \times \prod_j X_j$ .*

The theorem follows directly from Theorem 1, the only point that needs proof is the uniqueness, which will be achieved in two steps:

1. Prove the uniqueness in the case that  $X$  is simply-connected.
2. Prove that every isomorphism  $\phi : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z$  is splitted as  $\phi = (\phi_1, \phi_2)$  where  $\phi_1 : \mathbb{C}^k \longrightarrow \mathbb{C}^h$  and  $\phi_2 : Y \longrightarrow Z$  are isomorphisms (by consequence  $h = k$ ).

These two steps will be accomplished in the following two lemmas

**Lemma 7.** *Let  $Y = \prod_j Y_j$  be a finite product of compact, simply-connected manifold of Kähler type with vanishing Chern class. The Calabi-Yau metrics of  $Y$  are then  $g = \sum_i pr_j^* g_j$  where  $g_j$  are Calabi-Yau metrics of  $Y_j$ .*

*Proof.* Let  $g$  be a Calabi-Yau metric of  $Y$  and  $[\omega]$  its class in  $H^{1,1}(Y)$ . Since  $Y_j$  are simply-connected,  $[\omega] = \sum_j pr_j^* [\omega_j]$ . By Yau's theorem, there exist unique Calabi-Yau metrics  $g_j$  of  $Y_j$  in each class  $[\omega_j]$ . The metric  $g' = \sum_j pr_j^* g_j$  is in the same class  $\omega$  of  $g$  and is also a Calabi-Yau metric, hence  $g = g' = \sum_j pr_j^* g_j$ .  $\square$



This lemma asserts that when our manifolds  $Y, Y_j$  are equipped with appropriate Calabi-Yau metrics, the decomposition map is also a (Riemannian) isometric, we therefore obtain uniqueness of  $V_i, X_j$  from uniqueness of Theorem 1.

**Lemma 8.** *Let  $Y, Z$  be compact, simply-connected manifold of Kähler type, then any isomorphism  $u : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z$  is splitted as  $\phi = (\phi_1, \phi_2)$  where  $\phi_1 : \mathbb{C}^k \longrightarrow \mathbb{C}^h$  and  $\phi_2 : Y \longrightarrow Z$  are isomorphisms.*

*Proof.* It is clear that the composed function  $u_1 : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z \longrightarrow \mathbb{C}^h$  is constant in  $Y$ , i.e.  $u_1(t, y) = u_1(t)$  as holomorphic functions on  $Y$  are constant, therefore  $u(t, y) = (u_1(t), u_2(t, y))$ . As  $u$  is isomorphic, one has  $h \leq k$  then by the same argument for  $u^{-1}$ , one has  $h = k$ ,  $u_1$  is an isomorphism and  $u_2(t, \cdot)$  is an isomorphism from  $Y$  to  $Z$ .  $u_2(0, \cdot)^{-1} \circ u_2(t, \cdot)$  is then a curve in  $Aut(Y)$ , which is discrete by Lemma 3. Therefore  $u_2(t, \cdot) = u_2(0, \cdot)$  independent of  $t$ .  $\square$

‘ Emacs 25.3.1 (Org mode 9.0.5)