Berger classification and remarks on parallel structures

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1 Our story so far

De Rham decomposition theorem allows us to split a Riemannian manifold under certain conditions (complete and connected) as Riemannian product of complete connected manifold with *irreducible holonomy representation.* If an irreducible building block is *locally symmetric*, the theory of Lie groups developed by E. Cartan gave a complete list of holonomy of these spaces. We now shift our focus on non-symmetric irreducible manifolds.

2 Berger classification of non-symmetric irreducible manifolds

Theorem 1 (Berger classification)**.** *For a non-symmetric irreducible manifold, the holonomy representation has to be one of the following*

1. $SO(n)$,

2. $U(m) \subset SO(2m)$*,*

3. $SU(m) \subset SO(2m)$*,*

4. $Sp(r) \subset SO(4r)$*,*

5. $SO(r)Sp(1) \subset SO(4r)$*,*

6. $G_2 \subset SO(7)$*,*

7. $Spin(7) \subset SO(8)$*.*

where $n = 2m = 4r$ *is the (real) dimension.*

Here are some notations, note always that

$$Sp(m) \subset SU(2m) \subset U(2m) \subset SO(4m)$$

1. If $Hol(g) \subset U(m) \subset SO(2m)$, g is called a *Kähler metric.*

2. If $Hol(g) \subset SU(m) \subset SO(2m)$, g is called a *Calabi-Yau* metric. We will see that a Calabi-Yau metric is a Kähler metric that is also Ricci-flat.

3. If $Hol(g) \subset Sp(m) \subset SO(4m)$ then g is called a *hyperkähler* metric.

4. G_2 and Spin(7) are called *exceptional holonomies*

To sum up: hyperkähler $\longrightarrow$ Calabi-Yau $\longrightarrow$ Kähler.

But what do we mean by $U(n) \subset SO(2n)$**?** To embed $U(n)$ in $SO(2n)$ one needs to identify $\mathbb{C}$ and $\mathbb{R}^{2n}$, this can be done using an almost complex structure J of $\mathbb{R}^{2n}$. We will prove that when we change the almost complex structure, the embedded image of $U(n)$ in $SO(2n)$ always remains in the same conjugacy class, which corresponds to the fact that while holonomy representation is well-defined, the holonomy group in $SO(2n)$ is only defined up to its conjugacy class.

3 Almost complex structure

Definition 1. *A(n) (almost) complex structure* J *on a vector space* V *is an automorphism* $J: V \longrightarrow V$ *with* $J^2 = -Id_V$*. If* V *has a scalar product* g*, we suppose in addition that* $g \circ J = J$*.*

A(n) (almost) complex structure J *on manifold* M *is a vector bundle automorphism* $J : TM \longrightarrow TM$ *that satisfies* $J_x^2 = -Id_{T_xM}$ *for every* $x \in M$*. If* M *is a Riemannian manifold, we assume in addition that* $g \circ J = g$*.*

Let us first have a look at a complex structure J *on a fiber* (vector space) V. Here are some direct consequences:

Complexification. g and J extend in an unique way over $V_{\mathbb{C}}$, the complexification of V, to a Hermitian product $g_{\mathbb{C}}$ and a $\mathbb{C}$ -linear automorphism (also noted by J) and one still has $g_{\mathbb{C}} \circ J = g_{\mathbb{C}}$.

Eigenspaces. The complexified space $V_{\mathbb{C}}$ is decomposed to $V_{\mathbb{C}} = V^{1,0} \oplus^{\perp} V^{0,1}$ where $V^{1,0}$ and $V^{0,1}$ are eigenspaces (complex vector space) corresponding to eigenvalues i and $-i$ of J on $V_{\mathbb{C}}$. The orthogonality is by $g_{\mathbb{C}}$. The complex conjugate $\sum z_i x_i \mapsto \sum \bar{z}_i x_i$ where $z_i \in \mathbb{C}$ and $x_i \in V$ maps $V^{1,0}$ to $V^{0,1}$. Their dimensions are therefore the same.

Hermitian form. The fundamental form ω of (V, J) is defined by

$$\omega(a, b) = g(Ja, b) = -g(a, Jb) \quad \text{on } V$$

which is an antisymmetric real 2-form with $\omega \circ J = \omega$. V equipped with the following Hermitian form

$$h(a, b) = g(a, b) - i\omega(a, b) \quad \text{on } V$$

in the sense that $h(.,.)$ is $\mathbb{R}$ -bilinear with $h(Ja, b) = ih(a, b)$ and $h(a, Jb) = -ih(a, b)$.

Identification. One usually identifies (V, J) and $(V^{1,0}, i)$ as vector spaces equipped with complex structure, using the following map:

$$\iota_J : x \mapsto \frac{1}{2}(x - iJ(x))$$

which is $\mathbb{C}$ -linear in the sense of complex structure: $\iota_J(Jx) = i\iota_J(x)$. Note that $(V, -J)$ is also isomorphic to $(V^{0,1}, i)$ by the conjugate of ι_J: $x \mapsto \frac{1}{2}(x + iJ(x))$.

Now note that on we have on (V, J) an hermitian product $h(.,.)$ and on $(V^{1,0}, i)$ the restricted Hermitian product $g_{\mathbb{C}}$ of $V_{\mathbb{C}}$. The following lemma gives their relation (the proof is straightforward computation, see Manuscript).

Lemma 2. *The identification $(V, J) = (V^{1,0}, i)$ by ι_J gives*

$$\frac{1}{2}h = g_{\mathbb{C}}|_{V^{1,0}}$$

We can now embed $U(n)$ to $SO(2n)$, in other words $U(V^{1,0})$ to $SO(V)$ by the map $\phi \mapsto \tilde{\phi}$ as follow:

$$\begin{tikzcd} V \arrow[r, "\tilde{\phi}"] \arrow[d, "\iota_J"] & V \arrow[d, "\iota_J"] \\ V^{1,0} \arrow[r, "\phi"] & V^{1,0} \end{tikzcd}$$

Note that the correspondance $\phi \leftrightarrow \tilde{\phi}$ is one-to-one between $\{\phi:\ V^{1,0} \longrightarrow V^{1,0}\ \mathbb{R}\text{-linear}\}$ and $\{\tilde{\phi}:\ V \longrightarrow V\ \mathbb{R}\text{-linear}\}$. Then

1. ϕ is $\mathbb{C}$ -linear if and only if $\tilde{\phi}J = J\tilde{\phi}$.
2. ϕ preserves $g_{\mathbb{C}}$ if and only if $\tilde{\phi}$ preserves h. Taking the real and imaginary part, the latter is equivalent to the fact that $\tilde{\phi}$ preserves g and ω.
3. Every $\mathbb{C}$ -linear $\tilde{\phi}$ preserves orientation of $V^{1,0}$ as $\mathbb{R}^{2n}$ (note that the fact that $\tilde{\phi}$ preserves orientation or not is independent of how one identifies $V^{1,0}$ and $\mathbb{R}^{2n}$).

Hence for every J, $\phi \mapsto \tilde{\phi}$ gives a embedding of $U(V^{1,0})$ to $SO(V)$. An orthonormal base of $V^{1,0}$ and that of V give a embedding $U(n) \subset SO(2n)$.

Remark 1. *The image of $U(n)$ in $SO(2n)$ may depends on J and the orthonormal base of V, but its conjugacy class in $SO(2n)$ is uniquely defined. This is because every complex structure J is, up to a orthonormal conjugation,*

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

4 Complexified dual and forms, prelude to Kähler geometry

We state first some linear algebra facts, whose proofs are tedious and can be consulted in the Manuscript.

Lemma 3 (Linear algebra facts)**.** *1. Let $V = W_1 \oplus W_2$ be R -module then the exterior algebra of V splits into*

$$\bigwedge^n V = \bigoplus_{p+q=n} \bigwedge^p W_1 \otimes \bigwedge^q W_2$$

Note that the tensor product here is formal, and not related to the tensor product defining the exterior algebra.

2. If V has a complex structure J then J gives a complex structure on $V^* = Hom_{\mathbb{R}}(V, \mathbb{R})$, naturally by $\phi \mapsto \phi \circ J$.

One has

$$(V^*)_{\mathbb{C}} = Hom_{\mathbb{R}}(V, \mathbb{C}) \equiv Hom_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$$

and

$$(V^*)^{1,0} = Hom_{\mathbb{C}}((V, J), \mathbb{C}), \quad (V^*)^{0,1} = Hom_{\mathbb{C}}((V, -J), \mathbb{C})$$

where $Hom_{\mathbb{C}}(V, \mathbb{C})$ denotes the set of $\mathbb{R}$ -linear morphisms that preserves complex structures ($\mathbb{C}$ is implicitly with the complex structure $z \mapsto iz$)

Therefore $(V^*)_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$ is rewriten as

$$Hom_{\mathbb{R}}(V, \mathbb{C}) = Hom_{\mathbb{C}}((V, J), \mathbb{C}) \oplus Hom_{\mathbb{C}}((V, -J), \mathbb{C})$$

Using the first point of Lemma 3, one has

$$\bigwedge^{n} (V^*)_{\mathbb{C}} = \bigoplus_{p+q=n} \bigwedge^{p,q} (V^*)_{\mathbb{C}}$$

where $big \wedge^{p,q} T^*_{\mathbb{C}} M$ denotes the $\mathbb{C}$ -vector space of forms that are p times $\mathbb{C}$ -linear and q times $\mathbb{C}$ -antilinear.

Note one can easily find in V an orthonormal basis $\partial_{x_i}, \partial_{y_i}$ with $J(\partial_{x_i}) = \partial_{y_i}$. We clarify here the definition and implicit identifications of basic objects such as dz_i and $d\bar{z}_i$.

Object	**Where it belongs/ properties**	**Extension/ properties**
$\partial_{z_i} = \iota_J(\partial_{x_i})$ $= \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$	$V^{1,0}$, form a $\mathbb{C}$ -base	$dz_i(\partial_{z_j}) = \delta_{i,j}$, $dz_i(\partial_{\bar{z}_j}) = 0$
$\partial_{\bar{z}_i} = \iota_{-J}(\partial_{x_i})$ $= \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$	$V^{0,1}$, form a $\mathbb{C}$ -base	$d\bar{z}_i(\partial_{z_j}) = 0$, $d\bar{z}_i(\partial_{\bar{z}_j}) = \delta_{i,j}$
$dz_i = dx_i + idy_i$	$Hom_{\mathbb{C}}((V, J), \mathbb{C}) \equiv Hom_{\mathbb{C}}(V^{1,0}, \mathbb{C})$, $\mathbb{C}$ -linear	$Hom_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$, null on $V^{0,1}$
$d\bar{z}_i = dx_i - idy_i$	$Hom_{\mathbb{C}}((V, -J), \mathbb{C}) \equiv Hom_{\mathbb{C}}(V^{0,1}, \mathbb{C})$, $\mathbb{C}$ -antilinear	$Hom_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$, null on $V^{1,0}$

Remark 2. *One can note that there are two natural ways to extend dz_i to $V^{1,0}$*

1. *by first make a $\mathbb{C}$ -linear extension on $V_{\mathbb{C}}$, then make a restriction on $V^{1,0}$*

2. *using the identification $(V, J) \equiv (V^{1,0}, i)$*

but these two coincide, as there exists a unique form $\mathbb{C}$ -linear dz_i that satisfies $dz_i(\partial_{z_j}) = \delta_{i,j}, dz_i(\partial_{\bar{z}_j}) = 0$. Same story with $d\bar{z}_i$. See Figure 1 and Figure 2

$$\begin{tikzcd}
(V, J) \arrow[r, "dz_i"] \arrow[d, "\iota_J"'] & \mathbb{C} \\
(V^{1,0}, i) \arrow[ur, "dz_i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "dz_i"] \arrow[d, "\mathbb{C}\text{-lin}"'] & \mathbb{C} \\
V_{\mathbb{C}} \arrow[ur, "dz_i"'] &
\end{tikzcd}$$

Figure 1: Two natural ways to define dz_i on $V^{1,0}$. They give the same form.

$$\begin{tikzcd}
(V, -J) \arrow[r, "d\bar{z}_i"] \arrow[d, "\iota_J"'] & \mathbb{C} \\
(V^{0,1}, i) \arrow[ur, "d\bar{z}_i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "d\bar{z}_i"] \arrow[d, "\mathbb{C}\text{-lin}"'] & \mathbb{C} \\
V_{\mathbb{C}} \arrow[ur, "d\bar{z}_i"'] &
\end{tikzcd}$$

Figure 2: Two natural ways to define $d\bar{z}_i$ on $V^{0,1}$. They gives the same form.

Definition 2 (Theorem)**.** *The following properties are equivalent and X is called a Kähler manifold if one of them is satisfied.*

1. *X is a complex manifold, equipped with a Hermitian structure $h(.,.)$ compatible with the complex structure J, and a fundamental form ω with $d\omega = 0$.*

2. *X is a Riemannian manifold with a parallel complex structure.*

3. *X is a complex manifold, equipped with a Hermitian structure such that the Chern connection on $T^{1,0}X$ is, up to an identification by ι_J, the Levi-Civita connection.*

4. *X is a complex manifold, equipped with an Hermitian structure such that the Chern connection on $T^{1,0}X$ is torsionless.*

We call a complex manifold X <u>of Kähler type</u> if there exists a Hermitian structure under which X is Kähler.

The proof is straightforward. The only part that is not trivial is that a parallel almost complex structure has to come from a complex atlas, i.e. atlas of X such that each transition map preserves the complex structure. Such almost complex structures are called *integrable.*

To prove this one uses the following (1,2)-tensor called *Nijenhuis tensor* of a (1,1)-tensor A, defined by:

$$N_A(X,Y) = -A^2[X,Y] + A[AX,Y] + A[X,AY] - [AX,AY]$$

and the following theorem

Theorem 4 (Newlander–Nirenberg)**.** *An almost complex structure on M with vanishing N_J is integrable.*

The proof that a parallel almost complex structure J has $N_J = 0$ is computational in nature and can be found in the Manuscript.

5 Symplectic holonomy

One can look at symplectic group $Sp(r)$ from the following two points of view:

1. $Sp(r)$ is subgroup of $Aut_{\mathbb{H}}(\mathbb{H}^r)$ of elements preserving a quaternion Hermitian form q, where $\mathbb{H}$ is the algebra of quaternions.

2. $Sp(r) = SU(2r) \cap Sp(2r, \mathbb{C})$.

The second point of view explains how $Sp(r)$ is embedded in $SO(4r)$. Let us consider $Sp(r)$ from the first point. In our context, let V be a tangent space at one point of the manifold M, that is a $4r$ real dimensional vector space, one can regard $Sp(r)$ as the group of automorphisms of V preserving the Riemannian metric $g|_V$ and the complex structures I, J (hence $K = IJ$) satisfying $IJ = -IJ$. Hence we have the following remark:

Remark 3. *The following properties are equivalent for a Riemannian manifold M:*

1. *$Hol(M) \subset Sp(r) \subset SO(2r)$.*

2. *There exists on M three parallel complex structures I, J, K that satisfy $K = IJ = -JI$.*

3. *There exists on M a parallel complex structure I and a holomorphic (w.r.t I), parallel, 2-form φ that is non-degenerate at a point (hence every point).*

We note that the holomorphic 2-form in the third point is given by

$$\varphi = \omega_J + \sqrt{-1}\omega_K$$

where ω_I and ω_K are fundamental forms with respect to complex structures I and K, and M is regarded under the complex structure I.

The implication (2) $\implies$ (3) is actually exercise 1.2.5 (page 40) of Daniel Huybrechts, *Complex geometry: an introduction* (see Manuscript).

For the implication (3) $\implies$ (2), note that the real and imaginary part of φ are parallel, they correspond to complex structures J and K on M. Since φ is a (2,0)-form w.r.t I, one has $\varphi(Iu, v) = i\varphi(u, v)$, taking the real part and using the fact that g is non-degenerate, one has $K = IJ = -JI$. Emacs 25.3.1 (Org mode 9.0.5)