

# Two theorems of Hartogs

darknmt

April 4, 2018

## Contents

<b>1</b>	<b>Hartogs' theorem for subharmonic function</b>	<b>1</b>
<b>2</b>	<b>Hartogs theorem of separately holomorphicity</b>	<b>2</b>
2.1	Application of Baire theorem. . . . .	3
2.2	Analysis of series decomposition. . . . .	3

## 1 Hartogs' theorem for subharmonic function

The first Hartogs theorem concerns the convergence of harmonic functions. It says that under certain conditions, the convergence, apriori pointwise, is actually uniform on every compacts.

**Theorem 1** (Hartogs on subharmonic functions). *Let  $\Omega$  be a domain and  $u_k \in SH(\Omega)$  a sequence of subharmonic function such that*

1. *Uniformly bounded on compacts:  $u_k|_K < M_K$ .*
2. *Pointwise limit is continuous:  $\limsup_k u_k(x) = C$*

*Then for every  $K \subset \Omega$  compact and  $\varepsilon > 0$ , there exists  $N(K, \varepsilon) > 0$  such that  $u_k < C + \varepsilon$  for all  $k > N(K, \varepsilon)$*

*Proof.* By covering  $\Omega$  with an increasing sequence of compact  $K_n$  that  $K_n \Subset \text{Int}(K_{n+1})$ , one can suppose that  $u_k < M$  on  $\Omega$ . One can also suppose that  $M = 0$ . Note that it suffices to prove that for any  $x \in \Omega$ , there exists  $N(x, \varepsilon)$  such that  $u_k < C + \varepsilon$  on a neighborhood  $U_x$  of  $x$  for all  $k > N(x, \varepsilon)$ , then the conclusion follows by compactness of  $K$ .

One has  $\lim_{k \rightarrow +\infty} \int_{B(x,R)} u_k = C|B(R)|$  by monotone convergence, so  $\int_{B(x,R)} u_k < (C + \varepsilon)|B(R)|$  for  $k > N_1(x, \varepsilon)$ . For any  $r \ll R$  and  $y \in B(x, r)$  one has  $\int_{B(y, R+r)} u_k \leq \int_{B(x, R)} u_k < (C + \varepsilon)|B(R)|$ . Therefore

$$u_k(y) \leq (C + \varepsilon) \frac{|B(R)|}{|B(R+r)|} \leq C + 2\varepsilon \quad \forall r \ll R, k > N_1(x, \varepsilon)$$

which shows that  $u_k < C + 2\varepsilon$  for in a small neighborhood  $B(x, r)$  of  $x$ .  $\square$

**Remark 1.** *The Theorem 1 above can be easily generalized by replacing the constant  $C$  by a continuous function  $f$ .*

## 2 Hartogs theorem of separately holomorphicity

The second result of Hartogs that I want to present here is about the founding notion of holomorphicity for several complex variables.

**Theorem 2** (Hartogs for separate holomorphicity). *A function  $f$  separately holomorphic on each variable then  $f$  is smooth and hence is completely holomorphic*

The strategy is to establish the following steps:

1.  $f$  is locally bounded.
2.  $f$  is continuous.
3.  $f$  is smooth, hence is completely holomorphic.

The second and third steps are not difficult. In fact when one knows that  $f$  is locally bounded, one can prove that  $f$  is continuous by Schwartz lemma on each variable with appropriate scaling.

$$f(z_1, \dots, z_i, \dots, \xi_n) - f(z_1, \dots, \xi_i, \dots, \xi_n) \leq |1 - \overline{f(\dots, z_i, \dots)} f(\dots, \xi_i, \dots)| \left| \frac{z_i - \xi_i}{1 - \overline{z_i} \xi_i} \right| \quad \forall |z_i|, |\xi_i| < 1, |f| < 1$$

When  $f$  is continuous, one may refine Cauchy integral formulae and differentiability follows by dominated convergence.

So the remaining point is to prove that  $f$  is locally bounded, which can be done using Baire theorem and the first Hartogs result, Theorem 1.

## 2.1 Application of Baire theorem.

We will prove Theorem 2 by induction on the dimension  $n$ . We can therefore suppose that with the last variable  $z_n$  fixed, the function is completely holomorphic on the  $n - 1$  first variables. Fix a closed  $n$ -polydisc  $\mathbb{D}^n \ni x = 0$ , denote

$$W_L = \{(z_1, \dots, z_{n-1}) \in \mathbb{D}^{n-1} : |f(z_1, \dots, z_n)| \leq L \quad \forall z_n \in \mathbb{D}^1\}$$

then

1.  $\bigcup_{L \in \mathbb{N}} W_L = \mathbb{D}^n$  since for fixed  $(z_1, \dots, z_{n-1}) \in \mathbb{D}^{n-1}$ , the function  $f$  is continuous on  $z_n$ .
2. Each  $W_L$  is closed since for fixed  $z_n$ , the function  $f$  is continuous on  $n - 1$  first variables.

Therefore by Baire theorem, there exists  $L$  large enough such that  $W_L$  contains an open set of  $\mathbb{D}^{n-1}$ . Therefore there exists a strip  $U_{n-1} \times \mathbb{D} \subset \mathbb{D}^n$  on which the function  $f$  is holomorphic.

We will extend this strip using the following lemma

**Lemma 3.** *lem:ext-strip Let  $f$  be a separately holomorphic function defined on a neighborhood of  $\mathbb{D}^n$  such that*

1.  *$f$  is continuous on a neighborhood of the strip  $\mathbb{D}_\rho \times \dots \times \mathbb{D}_\rho \times \mathbb{D}$ ,*
2.  *$f$  is completely holomorphic on the first  $n - 1$  variables when the last one is fixed,*

*then  $f$  is completely holomorphic on  $\mathbb{D}^n$*

This lemma can be proved using the series decomposition of  $f$ .

## 2.2 Analysis of series decomposition.

Since  $f$  is completely holomorphic on the first  $n - 1$  variables when  $z_n$  is fixed, one has

$$f(z_1, \dots, z_n) = \sum_{\alpha} a_{\alpha}(z_n) z^{\alpha}, \quad a_{\alpha}(z_n) = \partial^{\alpha} f(0, z_n) / \alpha! \text{ is holomorphic in } z_n$$

Then the fact that for fixed  $z_n$ , the holomorphic function  $f(z', z_n)$  is well-defined on  $z' \in \mathbb{D}^{n-1}$  shows that

$$\limsup_{|\alpha| \rightarrow \infty} |a_{\alpha}(z_n)|^{1/|\alpha|} \leq 1. \tag{1}$$

Moreover, Cauchy integral on  $\mathbb{D}_\rho \times \cdots \times \mathbb{D}_\rho \times \mathbb{D}$  shows that

$$|a_\alpha(z_n)| = |\partial^\alpha f(0, z_n)|/\alpha! \leq \frac{1}{\rho^{|\alpha|}} M \quad (2)$$

where  $M$  is an upper bound of  $|f|$  on the strip.

Let  $u_\alpha = \frac{1}{|\alpha|} \log |a_\alpha|$  be a subharmonic function of  $z_n \in \mathbb{D}$ , (1) and (2) show that  $\limsup_{|\alpha| \rightarrow \infty} u_\alpha \leq 0$  and  $u_\alpha \leq \frac{1}{|\alpha|} \log M - \log \rho$  hence uniformly bounded.

By Theorem 1, one has  $|a_\alpha(z_n)|^{1/|\alpha|} \leq 1 + \varepsilon$  for  $|\alpha|$  sufficient large. Letting  $\varepsilon \rightarrow 0$ , one sees that the series converge normally in the interior of  $\mathbb{D}^{n-1}$ , hence by Cauchy-Montel the limit  $f$  is holomorphic in the interior of  $\mathbb{D}^n$ .  
Emacs 25.3.1 (Org mode 9.0.5)