Berger classification and remarks on parallel structures

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Our story so far

De Rham decomposition theorem theorem allows us to split a Riemannian manifold under certain conditions (complete and connected) as Riemannian product of complete connected manifold with *irreducible holonomy representation*. We will now interest in manifolds with irreducible holonomies. If the manifold is *locally symmetric* then one can prove that it is isometric to the homogeneous space G/H with H (the holonomy) a closed Lie subgroup of G. The theory of Lie groups developed by E. Cartan gave a complete list of these spaces.

Berger classification for non-symmetric manifolds

Theorem 1 (Berger classification)**.** *For the non-symmetric irreducible manifold, the holonomy representation has to be one of the following*

1. $SO(n)$
2. $U(m) \subset SO(2m)$
3. $SU(m) \subset SO(2m)$
4. $Sp(r) \subset SO(4r)$
5. $SO(r)Sp(1) \subset SO(4r)$
6. $G_2 \subset SO(7)$

7. $Spin(7) \subset SO(8)$

where $n = 2m = 4r$ is the dimension.

Here are some notations, note always that

$$Sp(m) \subset SU(2m) \subset U(2m) \subset SO(4m)$$

1. If $Hol(g) \subset U(m) \subset SO(2m)$, g is called a *Kahler metric.*
2. If $Hol(g) \subset SU(m) \subset SO(2m)$, g is called a *Calabi-Yau* metric. We will see that a Calabi-Yau metric is a Kahler metric that is also Ricci-flat.
3. If $Hol(g) \subset Sp(m) \subset SO(4m)$ then g is called a *hyperkahler* metric.
4. G_2 and Spin(7) are called *exceptional holonomies*

To resume: hyperkahler $\longrightarrow$ Calabi-Yau $\longrightarrow$ Kahler

But wait.. you said $U(n) \subset SO(2n)$? To embed $U(n)$ in $SO(2n)$ one need to identify $\mathbb{C}$ and $\mathbb{R}^{2n}$, this can be done using an almost complex structure J of $\mathbb{R}^n$. We will prove that when we change the almost complex structure, the embeded image of $U(n)$ in $SO(2n)$ always remains in the same conjugacy class, which works out perfectly with the fact that while holonomy representation is well-defined, the holonomy group in $SO(2n)$ is only defined up to its conjugacy class as one has to fix a basis

Almost complex structure

Definition 1. *A (almost) complex structure J on a vector space V is an automorphism $J : V \longrightarrow V$ with $J^2 = -Id_V$. If V has a scalar product g, we suppose in addition that $g \circ J = J$.*

A (almost) complex structure J on manifold M is a vector bundle automorphism $J : TM \longrightarrow TM$ that satisfies $J_x^2 = -Id_{T_xM}$ for every \$x$\in M$. If M is a Riemannian manifold, we supose in addition that $g \circ J = g$.

Let us first have a look at a complex structure J *on a fiber* (vector space) V. Here are some direct consequences:

The complexifieds. g and J extends in an unique way over $V_{\mathbb{C}}$ to a hermitian product $g_{\mathbb{C}}$ and a $\mathbb{C}$ -linear automorphism (also noted by J). One also has $g_{\mathbb{C}} \circ J = g_{\mathbb{C}}$.

Eigenspaces. The complexified space $V_{\mathbb{C}}$ is decomposed to $V_{\mathbb{C}} = V^{1,0} \oplus^{\perp} V^{0,1}$ where $V^{1,0}$ and $V^{0,1}$ are eigenspaces (complex vector space) corresponding to eigenvalues i and $-i$ of J on $V_{\mathbb{C}}$. The orthogonality is by $g_{\mathbb{C}}$. The complex conjugate $\sum z_i x_i \mapsto \sum \bar{z}_i x_i$ where $z_i \in \mathbb{C}$ and $x_i \in V$ maps $V^{1,0}$ to $V^{0,1}$. Their dimensions are therefore the same.

Hermitian form. The <u>fundamental form</u> ω of (V, J) is defined by

$$\omega(a, b) = g(Ja, b) = -g(a, Jb) \quad \text{on } V$$

which is an antisymmetric real 2-form with $\omega \circ J = \omega$. V equiped with the following <u>Hermitian form</u>

$$(a, b) = g(a, b) - i\omega(a, b) \quad \text{on } V$$

in the sense that $(.,.)$ is $\mathbb{R}$ -linear with $(Ja, b) = i(a, b)$ and $(a, Jb) = -i(a, b)$.

Identification. One usually identifies (V, J) and $(V^{1,0}, i)$ as vector spaces equiped with complex structure, using the following map:

$$\iota_J : x \mapsto \frac{1}{2}(x - iJ(x))$$

which is $\mathbb{C}$ -linear in the sense of complex structure $\iota_J(Jx) = i\iota_J(x)$. Note that one (V, J) is also isomorphic to $(V^{0,1}, -i)$ by the conjugate of ι_J: $x \mapsto \frac{1}{2}(x + iJ(x))$.

Now note that on we have on (V, J) an hermitian product $(.,.)$ and on $(V^{1,0}, i)$ the restricted hermitian product $g_{\mathbb{C}}$ of $V_{\mathbb{C}}$. The following lemma gives their relation (the proof is straightforward computation, see Manuscript).

Lemma 2. *The identification* $(V, J) = (V^{1,0}, i)$ *by* ι_J *give*

$$\frac{1}{2}(.,.) = g_{\mathbb{C}}|_{V^{1,0}}(.,.)$$

We can now embed $U(n)$ to $SO(2n)$, in other words $U(V^{1,0})$ to $SO(V)$ by the map $\phi \mapsto \tilde{\phi}$ as follow:

$$\begin{array}{ccc} V & \xrightarrow{\tilde{\phi}} & V \\ \downarrow{\scriptstyle \iota_J} & & \downarrow{\scriptstyle \iota_J} \\ V^{1,0} & \xrightarrow{\phi} & V^{1,0} \end{array}$$

Note that the correspondance $\phi \leftrightarrow \tilde{\phi}$ is one-to-one between $\{\phi : V^{1,0} \longrightarrow V^{1,0} \mathbb{C}\text{-linear}\}$ and $\{\tilde{\phi} : V \longrightarrow V \ \mathbb{R}\text{-linear}\}$. Then

1. ϕ is $\mathbb{C}$ -linear if and only if $\tilde{\phi}J = J\tilde{\phi}$.

2. ϕ preserves $g_{\mathbb{C}}$ if and only if $\tilde{\phi}$ preserves $(.,.)$. Taking the real and imaginary part, the latter is equivalent to the fact that $\tilde{\phi}$ preserves g and ω.

3. Every $\mathbb{C}$ -linear $\tilde{\phi}$ preserves orientation of $V^{1,0}$ as $\mathbb{R}^2 n$ (note that the fact that $\tilde{\phi}$ preserves orientation or not is independent of how one identifies $V^{1,0}$ and $\mathbb{R}^{2n}$

Hence for every J, $\phi \mapsto \tilde{\phi}$ gives a embedding of $U(V^{1,0})$ to $SO(V)$. By each a orthonormal base of $V^{1,0}$ and that of V give a embedding $U(n) \subset SO(2n)$.

Remark 1. *The image of $U(n)$ in $SO(2n)$ may depends on J and the orthonormal base of V, but its conjugacy class in $SO(2n)$ is uniquely defined. This is because every complex structure J is, up to a orthonormal conjugation,*

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Complexified dual and forms, prelude to Kahler geometry

We state first some linear algebra facts, whose proofs are unremarkable.

Lemma 3 (Linear algebra facts)**.** *1. Let $V = W_1 \oplus W_2$ be R -module then the exterior algebra of V splits into*

$$\Lambda^n V = \oplus_{p+q=n} \Lambda^p W_1 \otimes \Lambda^q W_2$$

. We remark that the tensor product here is formal, and not related to the tensor product defining the exterior algebra.

2. If V has a complex structure J then J gives a complex structure on $V^* = Hom_{\mathbb{R}}(V, \mathbb{R})$, naturally by $\phi \mapsto \phi \circ J$.

One has

$$(V^*)_{\mathbb{C}} = Hom_{\mathbb{R}}(V, \mathbb{C}) \equiv Hom_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$$

and

$$(V^*)^{1,0} = Hom_{\mathbb{C}}((V, J), \mathbb{C}), \quad (V^*)^{0,1} = Hom_{\mathbb{C}}((V, -J), \mathbb{C})$$

where $Hom_C(V, \mathbb{C})$ denotes the set of $\mathbb{R}$ -linear morphisms that preserves complex structures ($\mathbb{C}$ is implicitly with the complex structure $z \mapsto iz$)

Therefore $V_{\mathbb{C}}^* = (V^*)^{1,0} \oplus (V^*)^{0,1}$ is rewriten as

$$Hom_{\mathbb{R}}(V, \mathbb{C}) = Hom_{\mathbb{C}}((V, J), \mathbb{C}) \oplus Hom_{\mathbb{C}}((V, -J), \mathbb{C})$$

Using the first point of Lemma 3, one has

$$\Lambda^n V_{\mathbb{C}}^* = \oplus_{p+q=n} \Lambda^{p,q} V_{\mathbb{C}}^*$$

where $\Lambda^{p,q} T_{\mathbb{C}}^* M$ denotes the $\mathbb{C}$ -vector space of forms p times $\mathbb{C}$ -linear and q times $\mathbb{C}$ -antilinear.

Note one can easily find in V an orthonormal basis $\partial_{x_i}, \partial_{y_i}$ with $J(\partial_{x_i}) = \partial_{y_i}$. We clarify here the definition and implicit identifications of basic objects such as dz_i and $d\bar{z}_i$.

Object	**Where it belongs/ properties**	**$\mathbb{C}$ -linear extension/ properti**
$\partial_{z_i} = \iota_J(\partial_{x_i}) = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$	$V^{1,0}$	$dz_i(\partial_{z_j}) = \delta_{i,j}, dz_i(\partial_{\bar{z}_j)=0}$
$\partial_{\bar{z}_i} = \iota_{-J}(\partial_{x_i}) = \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$	$V^{0,1}$	$d\bar{z}_i(\partial_{z_j}) = 0, d\bar{z}_i(\partial_{\bar{z}_j}) = \delta_{i,j}$
$dz_i = dx_i + idy_i$	$Hom_{\mathbb{C}}((V, J), \mathbb{C})$, $\mathbb{C}$ -linear	$Hom_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$, null on $V^{0,1}$,
$d\bar{z}_i = dx_i - idy_i$	$Hom_{\mathbb{C}}((V, -J), \mathbb{C})$, $\mathbb{C}$ -antilinear	$Hom_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$, null on $V^{1,0}$

Remark 2. *One can note a subtlety here: there are two natural ways to extend dz_i to $V^{1,0}$*

1. *by first make a $\mathbb{C}$ -linear extension on $V_{\mathbb{C}}$, then make a restriction on $V^{1,0}$*

2. *using the identification $(V, J) \equiv (V^{1,0}, i)$*

but these two coincide, as they are all $\mathbb{C}$ -linear and satisfies $dz_i(\partial_{z_j}) = \delta_{i,j}, dz_i(\partial_{\bar{z}_j}) = 0$. Same story with $d\bar{z}_i$.

$$\begin{tikzcd} (V,J) \arrow[r, "dz_i"] \arrow[d, "\iota_J"'] & \mathbb{C} \\ (V^{1,0}, i) \arrow[ur, "dz_i"'] & \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "dz_i"] \arrow[d, "\mathbb{C}\text{-lin}"'] & \mathbb{C} \\ V_{\mathbb{C}} \arrow[ur, "dz_i"'] & \end{tikzcd}$$