Hodge decomposition and Kodaira embedding theorem

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Contents

This is my review of lectures 15-19 of Denis Auroux course whose goal is to estabish Hodge theory for compact Kähler varieties and present a proof of Donaldson for the Kodaira embedding theorem.

1 Hodge theory

1.1 Operators and their dual

1.1.1 Scalar product on $\Omega^k(M)$

The scalar product on V induces one on $\Omega^k(V)$ by setting $\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k\rangle = \det(\langle u_i, v_j\rangle)$.

Example 1. *$\langle \sum \alpha_I dx^I, \beta_J dx^J\rangle = \sum \alpha_I \beta_I$ if $\{\frac{\partial}{\partial x^i}\}$ form an orthonormal basis.*

1.1.2 Hodge star and Hodge dual

Definition 1. *The* ***Hodge star*** *is defined from* $\Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ *such that* $\alpha \wedge *\beta = \langle \alpha, \beta \rangle vol$ *where vol is the volume form.*

Remark 1. 1. *An example:* $*dx^I = dx^{I^C}$ *if* $\{\frac{\partial}{\partial x^i}\}$ *form an orthonormal basis and the complement* I^C *is chosen so that* $sgn(I, I^C) = 1$.

2. *Note that* $** = (-1)^{k(n-k)}$

The **Hodge dual** of an operator P will be defined such that $\langle P\alpha, \beta \rangle_{L^2} = \langle \alpha, P^*\beta \rangle_{L^2}$ where the $\langle \cdot, \cdot \rangle_{L^2}$ is the integral of $\langle \cdot, \cdot \rangle$ over M. For example,

Definition 2. *Let* d *be the coboundary operator then* $d^* : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ *is defined by* $d^* = (-1)^{n(k-1)+1} * d*$

Definition 3. *The* ***de Rham-Laplace*** *operator is defined by*

$$\Delta = dd^* + d^*d = (d + d^*)^2$$

The space of ***harmonic forms*** *is* $\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) : \Delta\alpha = 0\}$.

Remark 2. 1. $\Delta^* = \Delta$.

2. $\langle \Delta\alpha, \alpha \rangle = \|d^*\alpha\|^2 + \|d\alpha\|^2$

3. *A harmonic form is closed and co-closed.*

1.2 Elliptic theory and Hodge theorem for Riemannian manifolds

1.2.1 Symbol of a differential operator

Definition 4. *A mapping* $L : \Gamma(E) \longrightarrow \Gamma(F)$ *where* E, F *are vector bundles on a manifold* M *is called a* ***differential operator*** *of order* k *if in local coordinates,*

$$L(s) = \sum_{|\alpha| \leq k} A_\alpha(x) \frac{\partial^| \alpha| s}{\partial x^\alpha}$$

where $A_\alpha(x)$ *is a matrix with* C^∞ *coefficients.*

The ***symbol*** *of* L *is* $\sigma_k(L, \xi) = \sum_\alpha A_\alpha \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n} \in Hom(E_x, F_x)$ *where* $\xi = \sum \xi_i dx^i \in T^*M$ *in the same coordinate as* A_α.

Remark 3. 1. $A_\alpha(x)$ *depends on the local coordinates and does not transform naturally when one passes from one coordinates to another. In other words,* $A_\alpha(x)$ *is not in* $Hom(E_x, F_x)$.

2. *However, the definition of differential operator does not depend on local coordinates.*

3. *The symbol transforms naturally (linearly) between coordinates.*

From the third remark, one can define:

Definition 5. *A differential operator L is called **elliptic** if its symbol $L(x,\xi) : E_x \longrightarrow F_x$ is isomorphic.*

1.2.2 Elliptic operators

Theorem 1 (Elliptic operator)**.** *Every elliptic operator $L : \Gamma(E) \longrightarrow \Gamma(F)$*

1. *has a pseudoinverse, i.e. there exists $P : \Gamma(F) \longrightarrow \Gamma(E)$ such that $L \circ P - id_{\Gamma(F)}$ and $P \circ L - id_{\Gamma(E)}$ are smooth operators.*

2. *is extended to a Fredhom operator $L_s : W^s(E) \longrightarrow W^{s-k}(F)$, i.e. $\ker L = \ker L_s$ and cokerL_s are finite dimensional, $\operatorname{Im} L_s$ is closed.*

Moreover, if $L : \Gamma(E) \longrightarrow \Gamma(E)$ is elliptic and self-adjoint then there exists $H_L, G_L : \Gamma(E) \longrightarrow \Gamma(E)$ such that

1. $\operatorname{Im} H_L \subset \ker L$, $id_{\Gamma(E)} = H_L + L \circ G_L = H_L + G_L \circ L$.

2. H_L, G_L *extend to* $W^s(E) \longrightarrow W^s(E)$.

3. $\Gamma(E) = \ker L \oplus_{\perp L^2} \operatorname{Im} L \circ G_L$.

Theorem 2 (Hodge)**.** *Let M be a compact, oriented Riemannian manifold, then*

1. $\Omega^k(M) = \mathcal{H}^k(M) \oplus_{\perp L^2} \operatorname{Im} d \oplus_{\perp L^2} \operatorname{Im} d^*$.

2. *The projection $\mathcal{H}^k(M) \longrightarrow H^k_{dR}(M,\mathbb{R})$ is isomorphic. In other words, each class is uniquely represented by a harmonic form.*

1.3 Hodge decomposition for Kähler manifolds

In case of Kähler manifolds, one has the Hodge decomposition of cohomology which comes from the following two remarks:

1. The Hodge star $* : \Omega^{p,q} \longrightarrow \Omega^{n-q,n-p}$. This is due to the compatible complex structure.

2. The auxilary operator $L : \alpha \longrightarrow \omega \wedge \alpha$ and its relation with d. This is due to the compatible symplectic structure.

We resume in the following table the definition, domain and Hodge dual of some operators.

Operator	Domain	Definition	Dual
L	$\Omega^{p,q} \longrightarrow \Omega^{p+1.q+1}$	$\alpha \mapsto \omega \wedge \alpha$	$L^* = (-1)^{p+q} * L*$
d_c	$\Omega^k \longrightarrow \Omega^{k+1}$	$J^{-1} dJ$	$d_c^* = (-1)^{k+1} J d^* J$
∂	$\Omega^{p,q} \longrightarrow \Omega^{p+1,q}$		$\partial^* = - * \bar{\partial} *$
$\bar{\partial}$	$\Omega^{p,q} \longrightarrow \Omega^{p,q+1}$		$\bar{\partial}^* = - * \partial *$
$\square$	$\Omega^{p,q} \longrightarrow \Omega^{p,q}$	$\partial\partial^* + \partial^*\partial$	
$\bar{\square}$	$\Omega^{p,q} \longrightarrow \Omega^{p,q}$	$\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$	

In case of Kähler manifold, one has the following relation between these operators.

Lemma 3. *lem: In a compact Kähler manifold, one has*

1. $[L, d] = [L^*, d^*] = 0$
2. $[L, d^*] = d_c$
3. $[L^*, d] = -d_c^*$
4. $[L^*, d_c] = d^*$

Therefore,

1. $\Delta_c = d_c d_c^* + d_c^* d_c = \Delta$
2. ∂^* *is adjoint to* ∂ *and* $\bar{\partial}^*$ *to* $\bar{\partial}$.
3. $\Delta = 2\square = 2\bar{\square}$

One equip Ω^k with the following Hermitian product

$$\langle \phi, \psi \rangle_{L^2} = \int_M \phi \wedge * \bar{\psi}$$

under which the $\Omega^{p,q}$ are orthogonal.

One now applies the elliptic theory for $\bar{\square} : \Omega^{p,q} \longrightarrow \Omega^{p,q}$ with $\mathcal{H}^{p,q}_{\bar{\square}} = \ker \square$ then one sees that

Theorem 4 (Hodge decomposition)**.** 1. *Each class in the Dolbeault cohomology* $H^{p,q}_{\bar{\partial}}(M)$ *contains exactly one harmonic form of* $\mathcal{H}^{p,q}_{\bar{\square}} = \ker \bar{\square}$

2. $H^k(M) = \mathcal{H}_\Delta = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\square}} = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(M)$.

1.4 Hodge symmetries

Let $h^{p,q} = \dim_{\mathbb{R}} H^{p,q}_{\bar{\partial}}(M)$ and $h^k = \dim H^k_{dR}(M, \mathbb{R})$ then one has $h^k = \sum_{p+q=k} h^{p,q}$. The $h^{p,q}$ are usually written down as Hodge's diamond

$$\begin{array}{llll} h^{n,n} & h^{n,n-1} & \dots & h^{n,0} \\ h^{n-1,n} & h^{n-1,n-1} & \dots & h^{n-1,0} \\ \dots & \dots & \dots & \dots \\ h^{0,n} & h^{0,n-1} & \dots & h^{0,0} \end{array}$$

with the symmetries

1. $h^{p,q} = h^{q,p}$ given by conjugation.
2. $h^{p,q} = h^{n-q,n-p}$ given by the Hodge star.

2 Kodaira embedding theorem