

# Some results in one complex variable

darknmt

April 8, 2018

## Contents

**Theorem 1** (Runge). *Let  $\Omega$  be a domain of  $\mathbb{C}$  and  $K$  a compact in  $\Omega$ . Then the followings are equivalent*

1.  *$K$  has no hole in  $\Omega$ , i.e. there is no connected component  $\mathcal{C}$  of  $\Omega \setminus K$  with  $\bar{\mathcal{C}} \subset \Omega$ .*
2. *Every holomorphic function in  $K$  can be  $\|\cdot\|_{K,\infty}$ -approached by holomorphic functions in  $\Omega$ .*
3. *For every  $x \in \Omega \setminus K$ , there exists a holomorphic function  $f_x \in \mathcal{O}_\Omega$  such that  $|f(x)| > \sup_K |f|$ .*

**Theorem 2** (Mittag-Leffler). *Let  $(a_i)$  be a discrete sequence of points in  $\Omega$  and  $f_i$  be meromorphic functions with pole only at  $a_i$ . Then there exists a meromorphic function  $f$  with poles only at  $(a_i)$  such that  $f - f_i$  has removable singularity at  $a_i$ .*

*Proof.* Suppose that  $(f_i)$  are globally defined in  $\Omega$ . Choose an exhaustive sequence  $(\hat{K}_j = K_j)_j$  that increases slower than  $(a_i)$ , i.e.  $a_i \notin K_i$ . By 1 for  $K_i \subset \Omega$ , there exist  $g_i \in \mathcal{O}(\Omega)$  with  $\|g_i - f_i\|_{K_i,\infty} < 2^{-i}$ . Pose

$$f = \sum_i (f_i - g_i).$$

□