Minimal immersions of $\mathbb{S}^2$

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On Sep 22, 2019 I gave a talk at I2M, Marseille about Sacks–Uhlenbeck's work, here is note of my talk.

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1 Brief view of Sacks and Uhlenbeck's strategy.

Let M and N be compact Riemannian manifolds (without boundary), M is a surface and N is isometrically embedded in $\mathbb{R}^k$. It was showed by Eells and Sampson [**?**] that if N is negatively curved than any map from M to N is homotopic to a harmonic map. The idea of Sacks and Uhlenbeck in [**?**] consists of (1) approximating the energy functional E by a family E_α satisfying Palais-Smale condition, whose *nontrivial* critical values can be more easily proved to exist and (2) trying to prove that the critical maps s_α of E_α converge in C^1-topology.

We will first review the general machinery of Morse-Palais-Smale theory and prove the existence of s_α. The convergence of s_α in the case of surface is due to the facts that energy functional E is a conformal invariant of M, in particular E is invariant by homotheties (i.e. E remains unchanged when we zoom in and out), which allows us to justify the C^1-convergence (under conditions of N) except at finitely many points using a local estimate and a suitable covering of M.

Sacks and Uhlenbeck used an extension result for harmonic map, in an elegant argument to prove that if the above sequence $\{s_\alpha\}$ fails to converge at a point, for a certain surface M, then one has a nontrivial harmonic map from $\mathbb{S}^2$ to N. Therefore if such sequence $\{s_\alpha\}$ from $\mathbb{S}^2$ to N exists, for example when $\pi_k(N)$ is nontrivial for a certain $k \geq 2$ then, whether s_α converges or not, there exists a nontrivial harmonic map from $\mathbb{S}^2$ to N.

Finally, the theory of branched immersion of surfaces by Gulliver-Osserman-Royden [**?**] can be applied to show that the harmonic map obtained this way is a conformal, branched, minimal immersion of $\mathbb{S}^2$ to N.

2 General machinery by Morse-Palais-Smale.

2.1 Perturbed functionals E_α.

Let $s : M \longrightarrow N \hookrightarrow \mathbb{R}^k$ be a map from a compact surface M to a compact Riemannian manifold N isometrically embedded into $\mathbb{R}^k$. Recall that the energy functional of s is given by $E(s) := \frac{1}{2}\int_M |ds|^2 dV_M = \frac{1}{2}\int_M \langle s^* g_N, g_M\rangle dV_M$. The perturbed energy functionals are

$$E_\alpha(s) := \int_M \left(1 + |ds|^2\right)^\alpha dV, \quad \alpha \geq 1$$

We will suppose, by rescaling the metric g_M of M that the volume of M is 1, so when $\alpha = 1$, $E_1 = 1 + 2E(s)$ is just the previously defined energy. Using $(a+b)^\alpha \geq a^\alpha + b^\alpha$ and Jensen's inequality, one has $E_\alpha(s) \geq 1 + (2E(s))^\alpha$ for all $\alpha \geq 1$. Also, since we only interest in the case α close to 1, let us also suppose that α from now on is smaller than 2.

By Sobolev embedding, one has $W^{1,2\alpha}(M, \mathbb{R}^k) \subset C^0(M, \mathbb{R}^k)$ compactly for all $\alpha > 1$. It then makes sense to talk about $W^{1,2\alpha}(M, N) \subset C^0(M, N)$ which consist of elements of $W^{1,2\alpha}(M, \mathbb{R}^k) \subset C^0(M, \mathbb{R}^k)$ whose image lies in N.

Theorem 1 (Palais)**.** *The spaces $C^\infty(M, N) \subset W^{1,2\alpha}(M, N) \subset C^0(M, N)$, where $\alpha > 1$, are of the same homotopy type and the inclusions are homotopy equivalences. In particular, their connected components are naturally in bijection.*

We will also need a version of Morse theory for Banach manifolds, also developed by R. Palais in [**?**]. For the terminologies, in the same way that a manifold is modeled by$\mathbb{R}^n$, a *Banach manifold* is modeled by Banach spaces. A *Finsler manifold* is a Banach manifold with a norm on its tangent space that is comparable with the norm of Banach charts.

Theorem 2 (Morse theory for Banach manifolds)**.** *1. If F is a C^2 functional on a complete C^2 Finsler manifold L, F is bounded below and F satisfies Palais-Smale condition (C) then*

(a) The functional F admits minimum on each connected component of L.

(b) *If F has no critical value in $[a, b]$ then the sublevel $\{F \leq b\}$ retracts by deformation to the sublevel $\{F \leq a\}$.*

2. *The pair $(L, F) = (W^{1,2\alpha}(M, N), E_\alpha)$ with $\alpha > 1$ satisfies the condition of the first part.*

The *Palais-Smale condition* is as follows:

(C): Let $S \subset L$ be a subset on which $|F|$ is bounded, but $|dF|$ is not bounded away from 0. Then there exists a critical point of F in $\bar{S}$.

The strategy to prove Theorem 2 is, as in finite dimensional case, to use a pseudo-gradient flow of F whose existence is due to a partition of unity of L (instead of a Riemannian metric on L). The role of Palais-Smale condition in the proof is as follows: Suppose that $\{x_n\}$ is a sequence in a connected component L_1 of L such that $F(x_n)$ tends to $\inf_{L_1} F$, then using the pseudo-gradient flow of F, we can suppose that $|dF(x_n)|$ is arbitrarily small, in particular, we can suppose that $|dF(x_n)| \to 0$. Choose a sequence $\{y_n\}$ of regular points near x_n such that $F(y_n) \to \inf_{L_1} F$ and $|dF(y_n)| \to 0$ and use (C) for $S = \{y_n\}$, one obtains a limit point y_∞ of $\{y_n\}$, hence also of $\{x_n\}$, which minimises F.

As a consequence of Theorem 2, one has:

Corollary 3 (Component-wise minimum of E_α)**.** *The minimum of E_α in each connected component C of $W^{1,2\alpha}(M, N)$, $\alpha > 1$ is taken by some $s_\alpha \in C^\infty(M, N)$ and there exists $B > 0$ depending on the component C such that*

$$\min_C E_\alpha \leq (1 + B^2)^\alpha$$

Proof. By Theorem 2, E_α admits minimum at s_α on each component C of $W^{1,2\alpha}(M, N)$. By writing down the Euler-Lagrange equation of E_α and apply regularity estimates, one can prove that s_α is actually smooth. By Theorem 1, the preimage of C by inclusion $C^\infty(M, N) \subset W^{1,2\alpha}(M, N)$ is a connected component C' of $C^\infty(M, N)$ over which s_α is the minimum of E_α. Take $B = \sup_M |du|$ for an arbitrary element $u \in C'$ and the conclusion follows. □

Remark 4. *Corollary 3 is trivialised when $W^{1,2\alpha}(M, N)$ is connected (for one α or equivalently for all α). In this case, s_α is a constant map and $B = 0$.*

To establish a nontrivial analog of Corollary 3 in the case where the spaces of maps from M to N are connected, we will have to look at the submanifold $N_0 \cong N$ formed by constant maps.

2.2 Tubular neighborhood of the submanifold of trivial maps.

Fix $y \in N$, considered as a constant maps in N_0. We will summarise a few facts about the tangent space of $W^{1,2\alpha}(M, N)$ at y in the following Remark.

These facts come from the *differential structure* of the Banach manifold $W^{1,2\alpha}(M, N)$ that so far has not been introduced, since we only consider $W^{1,2\alpha}(M, N)$ as a closed subset of $W^{1,2\alpha}(M, \mathbb{R})^{\oplus k}$ (so only a topological structure was given). We summarise here, and refer to [?], how a differential structure is given to $W^{k,p}(M,N)$ with k,p such that $W^{k,p}(M) \hookrightarrow C^0(M)$:

- Let ξ be a finite dimensional vector bundle over a compact manifold M, then $W^{k,p}(\xi,M)$ can be defined as the Banach space of sections of ξ that are locally $W^{k,p}$. A norm of $W^{k,p}(\xi,M)$ can be given using a metric of ξ and a volume form of M, but by compactness of M, its equivalent class is independent of such choices.
- Let E be a fiber bundle over M, in our case, $E = N \times M$, and $s \in C^0(E)$ be a continuous section. It can be proved that there exists an open subset ξ of E containing s such that $\xi \to M$ has a vector bundle structure. We say that $s \in W^{k,p}(E,M)$ if $s \in W^{k,p}(\xi,M)$ and it turns out that this definition is independent of the choice of ξ. This defines $W^{k,p}(E,M)$ set-theoretically.
- The differential structure of $W^{k,p}(E,M)$ is given by the atlas $W^{k,p}(\xi,M)$.

Remark 5. *1. The tangent $T_yW^{1,2\alpha}(M,N)$ can be identified with $W^{1,2\alpha}(M,T_yN)$. The subspace T_yN_0 contains constant maps from M to T_yN.*

2. The fiber $\mathcal{N}_y$ over y of the normal bundle $\mathcal{N}$ of N_0 can be identified with

$$\mathcal{N}_y = \left\{ v \in W^{1,\alpha}(M,T_yN) : \int_M v dV = 0 \right\}$$

The exponential map on $TW^{1,2\alpha}(M,N)$ can be defined as follows:

$$\begin{aligned} e:\ TW^{1,2\alpha}(M,N) &\longrightarrow W^{1,2\alpha}(M,N) \\ (s,v) &\longmapsto \left(x \mapsto \exp_{s(x)} v(x)\right) \end{aligned}$$

where $s \in W^{1,2\alpha}(M,N)$ and $v \in T_sW^{1,2\alpha}(M,N)$ is a $W^{1,2\alpha}$ vector field along $s(x)$. With the representation of normal bundle $\mathcal{N}$ as Remark 5, the restriction of e on $\mathcal{N}$ is given by

$$\begin{aligned} e\big|_{\mathcal{N}} :\ \mathcal{N} &\longrightarrow W^{1,2\alpha}(M,N) \\ (y,v) &\longmapsto \left(x \mapsto \exp_y(v(x))\right) \end{aligned}$$

where $y \in N_0 \cong N$ and $v \in W^{1,2\alpha}(M,T_yN)$.

Lemma 6. *The restriction $e\big|_{\mathcal{N}}$ of e on $\mathcal{N}$ is a local diffeomorphism mapping a neighborhood of the zero-section of $\mathcal{N}$ onto a neighborhood of N_0 in $W^{1,2\alpha}(M,N)$.*

Proof. It can be calculated that

$$de_{(y,0)}(a,v) = (x \mapsto a + v(x)) \in T_yW^{1,2\alpha}(M,N) = W^{1,2\alpha}(M,T_yN)$$

for $a \in T_yN$ and $v \in \mathcal{N}_y \subset W^{1,2\alpha}(M,T_yN)$. It is invertible since a is tangential to N_0 and $v \in \mathcal{N}_y$ is in the normal component. The Inverse function theorem applies. $\square$

2.3 Critical values of E_α.

The exponential map previously defined on the normal bundle of N_0 in $W^{1,2\alpha}(M,N)$ allows us to retract by deformation a small neighborhood of N_0 to N_0. We will prove that if the energy $E_\alpha(s)$ is sufficiently close to $1 = E_\alpha(N_0)$ then s is sufficiently $W^{1,2\alpha}$-close to N_0 and hence can be retracted to N_0, in other words, $E_\alpha^{-1}[1, 1+\delta]$ retracts by deformation to $N_0 = E_\alpha^{-1}(1)$.

Proposition 7. *Given $\alpha > 1$, there exists $\delta > 0$ depending on α such that $E_\alpha^{-1}[1, 1+\delta]$ retracts by deformation to $E_\alpha^{-1}(1) = N_0$.*

Proof. Let $s \in E_\alpha^{-1}[1, 1+\delta]$, using $(a+b)^\alpha \geq a^\alpha + b^\alpha$, one has

$$1 + \delta > \int_M (1 + |ds|^2)^\alpha dV > 1 + \int_M |ds|^{2\alpha} dV$$

therefore $\|ds\|_{L^{2\alpha}} \leq \delta^{1/2\alpha}$. By Poincaré-Wirtinger inequality, $\|s - \int_M s\|_{W^{1,2\alpha}} \leq C\delta^{1/4}$ where C is the Poincaré-Wirtinger constant.

By Sobolev embedding, $\max_M |s - \int_M s| \leq C_\alpha \|s - \int_M s\|_{W^{1,2\alpha}}$ where the Sobolev constant C_α can no longer be chosen uniformly in $\alpha \to 1$. Fix an $x_0 \in M$, one has

$$d_{W^{1,2\alpha}}(s, N_0) \leq \|s - s(x_0)\|_{W^{1,2\alpha}} \leq \left\| s - \int_M s \right\|_{W^{1,2\alpha}} + \left| \int_M s - s(x_0) \right| \leq C_\alpha \delta^{1/4}$$

Now choose $\delta \ll 1$ depending on α such that s is in the neighborhood of N_0 given by Lemma 6, s can be written as

$$s(x) = e(y, v(x)) = \exp_y v(x)$$

where $y \in N_0$ and $v \in W^{1,2\alpha}(M, T_y N)$ depend continuously on $s \in W^{1,2\alpha}(M,N)$. We can define the deformation retraction by

$$\begin{aligned} \sigma :\ E_\alpha^{-1}[1, 1+\delta] \times [0,1] &\longrightarrow E_\alpha^{-1}[1, 1+\delta] \\ (s,t) &\longmapsto \Big(x \mapsto \exp_y tv(x) \Big) \end{aligned}$$

It is clear that σ is continuous and σ_0 is a retraction. The only thing to check is that the image of σ remains in $E_\alpha^{-1}[1, 1+\delta]$ at all time. This can be checked by showing that $\frac{d}{dt} E_\alpha(\sigma_t) \geq 0$, hence $E_\alpha(\sigma_t) \leq E_\alpha(\sigma_1) \leq 1 + \delta$ for all $0 \leq t \leq 1$. □

We will now prove the existence of nontrivial critical value of E_α in an interval $(1, B)$ for a certain $B > 1$ sufficiently big independently of $\alpha > 1$.

Fix $z_0 \in M$ and consider the map

$$\begin{aligned} p :\ C^0(M,N) &\longrightarrow N \\ s &\longmapsto f(z_0) \end{aligned}$$

then p is a fiber bundle and therefore is a *Serre fibration*. In fact fix $q_0 \in N$ then for all $q \in N$ near q_0, there is a vector field v_q supported in a small ball centered at q_0 such that the flow of v_q from time 0 to 1 turns q_0 to q, i.e. $\Phi_{v_q}{}_0^1(q_0) = q$, and that v_q varies continuously in q. Then any fiber $p^{-1}(q)$ can be identified with $p^{-1}(q_0)$ using the flow of v_q. We will denote by $\Omega(M, N)$ the topological fiber of p.

We will use a few facts from algebraic topology, briefly summarised here.

Fact 1. 1. *(Long exact sequence of homotopy) Let $p : E \longrightarrow B$ be a fiber bundle of fiber $F = p^{-1}(b_0) \ni f_0$, then one has the following long exact sequence*

$$\dots \xrightarrow{\partial} \pi_n(F) \xrightarrow{\iota_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \dots \longrightarrow \pi_0(E) \longrightarrow 0$$

where $\iota : F \longrightarrow E$ is the inclusion.

2. *If p admits a global section s, then one has a retraction s_* of p_*:*

$$\begin{tikzcd} \pi_n(E) \arrow[r, shift left, "p_*"] & \pi_n(B) \arrow[l, shift left, "s_*"] \end{tikzcd}$$

hence p_* is surjective and ∂ factors through 0, which gives us the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_n(F) \arrow[r, "\iota_*"] & \pi_n(E) \arrow[r, shift left, "p_*"] & \pi_n(B) \arrow[l, shift left, "s_*"] \arrow[r] & 0 \end{tikzcd}$$

where p_* admits a retraction s_*, so the short exact sequence splits and we have

$$\pi_n(E) \cong \pi_n(F) \oplus \pi_n(B).$$

Now apply this result to the fiber bundle $p : C^0(M, N) \longrightarrow N$ of fiber $\Omega(M, N)$, which has N_0 as a global section, one obtains

$$\pi_n(C^0(M, N)) \cong \pi_n(N) \oplus \pi_n(\Omega(M, N)).$$

Theorem 8 (Nontrivial critical value of E_α)**.** *If $C^0(M, N)$ is not connected, or if $\Omega(M, N)$ is not contractible, then there exists $B > 0$ such that for all $\alpha > 1$, E_α has critical values in the interval $(1, (1 + B^2)^\alpha)$.*

In particular, if $M = \mathbb{S}^2$ and if the universal covering $\tilde{N}$ of N is not contractible then E_α has critical values in $(1, (1 + B^2)^\alpha)$.

Proof. If $C^0(M, N)$ is not connected, one only needs to apply Corollary 3 to a connected component of $W^{1,2\alpha}(M, N)$ not containing N_0. We now suppose that $C^0(M, N)$ is connected and $\Omega(M, N)$ is not contractible.

In this case, there exists $n > 0$ such that $\pi_n(\Omega(M, N))$ is nontrivial and contains a nonzero element $\gamma : \mathbb{S}^n \longrightarrow \Omega(M, N)$ which is not homotopic to any $\tilde{\gamma} : \mathbb{S}^n \longrightarrow N_0$ in $\pi_n(C^0(M, N))$.

Choose $B := \max_{\theta\in\mathbb{S}^n, x\in M} |d\gamma(\theta)(x)|$ then by definition

$$E_\alpha(\gamma(\theta)) \le (1+B^2)^\alpha \quad \forall\theta\in\mathbb{S}^n, \alpha>1.$$

If E_α has no critical value in $[1+\frac{\delta_\alpha}{2}, (1+B^2)^\alpha]$ where δ_α is given by Proposition 7, then by Theorem 2, $E_\alpha^{-1}[1,(1+B^2)^\alpha]$ retracts by deformation to $E_\alpha^{-1}[1,1+\delta_\alpha]$ which retracts by deformation to $E_\alpha^{-1}(1) = N_0$. But this means that γ is homotopic to a certain $\tilde{\gamma}\in\pi_n(N)$, which is a contradiction.

As an application, if $M=\mathbb{S}^2$ and the universal covering $\tilde{N}$ is not contractible then the long exact sequence of homotopy for the bundle $\tilde{N}\longrightarrow N$ with fiber of dimension 0, gives

$$\pi_n(\tilde{N}) = \pi_n(N), \quad \forall n\ge 2.$$

Since $\tilde{N}$ is simply-connected and not contractible, there exists $n\ge 2$ such that $0\ne\pi_n(\tilde{N}) = \pi_n(N) = \pi_{n-2}(\Omega(\mathbb{S}^2,N))$, where the last equality follows from Exponential law in Pointed category (the role of cartesian product is played by smash product) and the fact that smash product of spheres is a sphere. The general argument applies. $\square$

3 Local results: Estimates and extension.

We will say that the map $s: M\longrightarrow N$ is a critical point of E_α on a small disc $D(R)\subset M$ if s satisfies the Euler-Lagrange equation of E_α (as functional on $W^{1,2\alpha}(M,N)$) on $D(R)$.

Remark 9. *Rescaling $(D(R), g_M)$, where $R\ll 1$ and g_M is ϵ-close to the Euclidean metric, to the unit disc D one obtains a metric $\tilde{g}_M$ that is still ϵ-close to Euclidean metric. The curvature of $\tilde{g}_M$ is R^2 times smaller than that of g_M.*

If $s: D(R)\longrightarrow N$ is a critical map of E_α on $D(R)$, then the composition $\tilde{s}$ of s and the rescaling operator $D\longrightarrow D(R)$ satisfies the Euler-Lagrange equation of $\tilde{E}_\alpha = R^{2(1-\alpha)}\int_D (R^2+|d\tilde{s}|^2)^\alpha d\tilde{V}$ where $d\tilde{V}$ is the volume form of the rescaled metric $\tilde{g}_M$. We will abusively use the same notation for $\tilde{s}$ and s and regard s as a map on the unit disc D.

Lemma 10 (Sacks-Uhlenback's Main estimate). *For all $p\in(1,+\infty)$, there exists $\epsilon>0$ and $\alpha_0>1$ depending on p such that if*

- *$s: (D,\tilde{g})\longrightarrow N$ is a critical map of E_α on $D(R)$*
- *$E(s)<\epsilon$, $1<\alpha<\alpha_0$*

then

$$\|ds\|_{W^{1,p}(D')} < C(p,D')\|ds\|_{L^2(D)}, \quad \textit{for all disc } D'\Subset D$$

Remark 11. *In fact α_0,ϵ and $C(p,D')$ depend on the rescaled metric $\tilde{g}$ on D, but if $R\ll 1$ and $\tilde{g}$ is very close to Euclidean metric, then one can choose these parameters independently of $\tilde{g}$.*

A consequence of (the proof of) Lemma 10 is the following global result:

Theorem 12 (Critical maps of low energy are trivial). *There exists $\epsilon' > 0$ and $\alpha_0 > 1$ such that if*

- *$s : M \longrightarrow N$ is critical map of E_α*
- *$E(s) < \epsilon'$, $1 < \alpha < \alpha_0$*

then $s \in N_0$ and $E(s) = 0$.

We proved in the last section that, under certain algebraic topological condition on N, E_α admits critical value $v_\alpha \in (1, (1 + B^2)^\alpha)$. We now can conclude that, by Theorem 12, the critical values v_α are bounded away from 1, i.e. $\inf_\alpha v_\alpha > 1$.

We will also need the following extension theorem:

Theorem 13 (Extension of harmonic maps). *If $s : D \setminus \{0\} \longrightarrow N$ is a harmonic map with finite energy $E(s) < \infty$, then s extends to a smooth harmonic map $\tilde{s} : D \longrightarrow N$.*

4 Convergence of critical maps of E_α.

We proved in Theorem 8 that if $C^0(M, N)$ is not connected or if $\Omega(M, N)$ is not contractible, then there exists a family $\{s_\alpha\}$ of critical maps of E_α with bounded, nontrivial energy $E_\alpha(s_\alpha) < B$. Since

- $\int_M |ds_\alpha|^2 \leq (E_\alpha(s_\alpha) - 1)^{1/\alpha}$ is bounded uniformly on α
- $\|s_\alpha\|_{L^\infty}$ is bounded by compactness of N.

the $W^{1,2}(M, \mathbb{R}^k)$-norms of $\{s_\alpha\}$ are bounded. By reflexivity of Sobolev spaces, there exists a subsequence $\{s_\beta\}$ weakly converging to s in $W^{1,2}(M, \mathbb{R}^k)$ with

$$\|s\|_{W^{1,2}} \leq \liminf_{\beta \to 1} \|s_\beta\|_{W^{1,2}}$$

We do not know at this moment if the convergence is C^0, or if s is continuous, or even if the image of s remains in N. The following key lemma answer these questions on a small disc of M in the case the energy of s_α is small.

Lemma 14 (Key). *There exists an $\epsilon > 0$, in fact given by the Main estimate Lemma 10 with $p = 4$, such that if*

- *$s_\alpha : D(R) \longrightarrow N \subset \mathbb{R}^k$ are critical maps of E_α in $W^{1,2\alpha}(D(R), N)$,*
- *$E(s_\alpha) < \epsilon$ and s_α converges weakly to s in $W^{1,2}(D(R), \mathbb{R}^k)$,*

then

- *the restriction of s on $\overline{D(R/2)}$ is smooth harmonic map with image in N,*

- $s_\alpha \to s$ *in* $C^1(\overline{D(R/2)}, N))$.

Remark 15. *There are two different ways to define convergence of a sequence s_n to s in $C^1(\Omega)$ on an open set Ω:*

1. *The sequence s_α and s extend to $C^1(\bar{\Omega})$ and have finite norm $\max_\Omega |s| + \max_\Omega |ds|$ and $\max_\Omega |s_\alpha| + \max_\Omega |ds_\alpha|$ and*

$$\max_\Omega |s_\alpha - s| + \max_\Omega |ds - ds_\alpha| \to 0.$$

   In this case, we will say that s_α converges to s in $C^1(\bar{\Omega})$.

2. *$C^1(\Omega)$ is topologised by a family of seminorms $\Gamma_K : s \longmapsto \max_K |s| + \max_K |ds|$ for $K \Subset \Omega$. This makes $C^1(\Omega)$ a Fréchet topological vector space. If the sequence s_α converges to s under this topology then we will say that s_α converges uniformly to s on compacts of Ω.*

Proof. We consider s_α and s as maps from the unit disc D to $\mathbb{R}^k$, then by Main estimate Lemma 10 for $p = 4$, since $E(s_\alpha) < \epsilon$, one has:

$$\|ds_\alpha\|_{W^{1,4}(D(1/2),\mathbb{R}^k)} \leq C(4, D(1/2))\|ds_\alpha\|_{L^2(D)} = C(4, D(1/2))E(s_\alpha)^{1/2}$$

So $\{s_\alpha\}$ is bounded in $W^{2,4}(D(1/2), \mathbb{R}^k)$ which is embedded compactly into $C^1(\overline{D(1/2)}, \mathbb{R}^k)$.

We now can prove that s_α converges strongly to s in $C^1(\overline{D(1/2)}, \mathbb{R}^k)$: If there was a subsequence $\{s_\beta\}$ whose restriction to $\overline{D(1/2)}$ remains C^1-away from s, then by compactness of $W^{1,4}(D(1/2), \mathbb{R}^k) \hookrightarrow C^1(\overline{D(1/2)}, \mathbb{R}^k)$, we can suppose that $\{s_\beta\}$ converges in C^1 to a certain $\bar{s} \neq s$ on $\overline{D(1/2)}$. But as a subsequence of $\{s_\alpha\}$, $\{s_\beta\}$ converges weakly to s on D, hence on $\overline{D(1/2)}$, we than obtain a contradiction using the uniqueness of weak limit.

By considering the Euler-Lagrange equation and letting $\alpha \to 0$, one concludes that s is a harmonic map from $D(1/2)$ to N. □

The global convergence of $\{s_\alpha\}$ can be established by a well-chosen covering of M by small balls or radius R.

Proposition 16. *Let $s_\alpha : M \longrightarrow N \subset \mathbb{R}^k$ be critical maps of E_α on M such that s_α converges weakly to s in $W^{1,2}(M, \mathbb{R}^k)$ and $E(s_\alpha) < B$. Then there exists $l = l(B, N)$ such that given any $m > 0$, one can find a sequence $\{x_{m,1}, \ldots, x_{m,l}\} \subset M$ and a subsequence $\{s_{\alpha(m)}\}$ of $\{s_\alpha\}$ such that*

$$s_{\alpha(m)} \longrightarrow s \text{ in } C^1\left(M \setminus \bigcup_{i=1}^{l} D(x_{m,i}, 2^{-m+1}), N\right)$$

Proof. We cover M by finitely many balls $D(y_i, 2^{-m})$ such that each point is covered at most h times by the bigger balls $D(y_i, 2^{-m+1})$. By Lemma **??**, h can be chosen independently of m as $2^{-m} \to 0$.

Since $\sum_i \int_{D(y_i,2^{-m+1})} |ds_\alpha|^2 < Bh$, choosing $l = \lceil \frac{Bh}{2\epsilon} \rceil$, we see that there are at most l balls $D(y_{\alpha,i}, 2^{-m+1})$ with centers depending on α, on which the energy $E(s_\alpha)$ is more than ϵ. Passing to a subsequence $\{s_{\alpha(m)}\}$ of $\{s_\alpha\}$, we can suppose that $\{y_{\alpha(m),i}\}$ converges to $x_{m,i}$ as $\{\alpha(m)\} \to 1$. But since the points $\{y_i\}$ are of finite number and separated, $y_{\alpha(m),i} \equiv x_{m,i}$ eventually and we can suppose that the bad balls $D(y_{\alpha(m),i})$ where energy of $s_{\alpha(m)}$ surpasses ϵ are the same for every $s_{\alpha(m)}$.

Now apply Lemma 14 to the sequence $\{s_{\alpha(m)}\}$ on all the other 2^{-m+1}-balls, one sees that $\{s_{\alpha(m)}\}$ converges in C^1 to s on all $\overline{D(y_i, 2^{-m})}$ except those centered at $x_{m,i}$. The conclusion follows. □

Using a diagonal argument, we can find a subsequence $\{s_\beta\}$ of $\{s_\alpha\}$ that converges to s uniformly on compacts of $M \setminus \{x_1, \ldots, x_l\}$.

Theorem 17 (Convergence of $\{s_\alpha\}$)**.** *Let $s_\alpha : M \longrightarrow N \subset \mathbb{R}^k$ be critical maps of E_α on M such that s_α converges weakly to s in $W^{1,2}(M, \mathbb{R}^k)$ and $E(s_\alpha) < B$. Then there exist at most l points $x_1, \ldots, x_l$ in M, where l is given by Proposition 16, and a subsequence $\{s_\beta\}$ of $\{s_\alpha\}$ such that*

$$s_\beta \longrightarrow s \text{ in } C^1(M \setminus \{x_1, \ldots, x_l\}, \mathbb{R}^k) \text{ uniformly on compacts.}$$

Proof. By passing to a subsequence $\{m_k\}$ of $\{m\}$, we can suppose that $\{x_{m,i}\}$ converges to x_i in M. Choose the diagonal subsequence $\{s_\beta\}$ from $\{s_{\alpha(m)}\}$ that consists of $s_{\alpha(m)(a_m)}$ where a_m is sufficiently big such that $\alpha(m)(a_m)$ is increasing and $\|s_{\alpha(m)(b)} - s_{\alpha(m)(c)}\|_{C^1(M \setminus \cup_i D(x_{m,i}, 2^{-m+1})} < \frac{1}{m}$ for all $b, c \geq a_m$. Then the sequence $\{s_\beta\}$ converges uniformly on compacts of $M \setminus \{x_1, \ldots, x_l\}$ because $\{\bigcup_i D(x_{m,i}, 2^{-m+1})\}_m$ is an exhaustive family of compacts of $M \setminus \{x_1, \ldots, x_l\}$. □

Remark 18. *With the same notation as Theorem 17,*

1. *The image $s(M \setminus \{x_1, \ldots, x_l\})$ lies in N. Also, using the Euler-Lagrange equation, one sees that s is a (smooth) harmonic map from $M \setminus \{x_1, \ldots, x_l\}$ to N.*

2. *Since $E(s) \leq \|s\|^2_{W^{1,2}} \leq \liminf_{\alpha \to 1} \|s_\alpha\|^2 < +\infty$, $s\big|_{M \setminus \{x_1, \ldots, x_l\}}$ extends to a harmonic map $\tilde{s} : M \longrightarrow N$. We can therefore suppose that the limit s of Theorem 17 is smooth harmonic map on M and of image in N.*

5 Nontrivial harmonic maps from $\mathbb{S}^2$.

We will now prove the existence of nontrivial harmonic maps from $\mathbb{S}^2$ to a compact Riemannian manifold N satisfying the conditions of Theorem 8.

The following theorem does not suppose any condition on N.

Theorem 19. *Let M be a compact surface and s_α be critical maps of E_α. Suppose that*

- *s_α converges in C^1 to s uniformly on compacts of $M \setminus \{x_1, \ldots, x_l\}$ but not on $M \setminus \{x_2, \ldots, x_l\}$.*

- $E(s_\alpha) < B$

Then there exists a nontrivial harmonic map $s_* : \mathbb{S}^2 \longrightarrow N$.

Before proving the theorem, let us state its corollary.

Corollary 20 (Nontrivial harmonic map from $\mathbb{S}^2$)**.** *If the universal covering* $\tilde{N}$ *of* N *is not contractible then there exists a nontrivial harmonic map* $s : \mathbb{S}^2 \longrightarrow N$.

Proof. By Theorem 8 and Theorem 12, there exist critical maps $s_\alpha : \mathbb{S}^2 \longrightarrow N$ of E_α corresponding to critical values $E_\alpha(s_\alpha)$ in $(1+\delta, B)$. We claim that $\{s_\alpha\}$ cannot converge in $C^1(M)$ to a trivial harmonic map $s \in N_0$. In fact, if it did,

$$1+\delta \leq \lim_{\alpha\to 1} \int_M (1+|ds_\alpha|^2)^\alpha dV = \int_M (1+|ds|^2) dV = 1$$

which is contradictory.

Therefore, we only have two possibilities:

- $\{s_\alpha\}$ does not converge in $C^1(M)$ to s, then by Theorem 19, there exists a nontrivial harmonic map $s_* : \mathbb{S}^2 \longrightarrow N$.
- If $\{s_\alpha\}$ converges in $C^1(M)$ to a certain $\tilde{s}$, then as argued above, $\tilde{s}$ is nontrivial.

In both cases, nontrivial harmonic map from $\mathbb{S}^2$ to N exists. $\square$

Let us now prove Theorem 19.

Proof of Theorem 19. If there is no C^1 convergence near x_1, we claim that:

Assertion 1. *For all* $C > 0$ *and* $\delta > 0$, *there exists* $\alpha > 1$ *arbitrarily close to 1 such that*

$$\max_{\overline{D}(x_1, 2\delta)} |ds_\alpha| > C.$$

Moreover, we can suppose that $\max_{\overline{D}(x_1,2\delta)} |ds_\alpha| = \max_{D(x_1,\delta)} |ds_\alpha|$.

Suppose that was not the case, then there exist $C, \delta > 0$ such that $\max_{D(x_1,2\delta)} |ds_\alpha| \leq C$ for all $\alpha > 1$ sufficiently close to 1. Choose a radius $R \ll \delta$ such that

$$\int_{D(x_1,R)} |ds_\alpha|^2 \leq \pi R^2 C^2 < \epsilon$$

It suffices to apply Key lemma 14 to see that $s_\alpha \to s$ in $C^1(D(x_1, R/2))$, hence s_α converges to s in $C^1(M \setminus \{x_2, \ldots, x_l\})$ uniformly on compacts. Moreover, since $\{ds_\alpha\}$ converges uniformly to ds on $\overline{D}(x_1, 2\delta) \setminus D(x_1, \delta)$, we can suppose, with α sufficiently close to 1, that the maximum is actually attained in $D(x_1, \delta)$.

Therefore, we can choose a sequence $\{C_n\}$ increasing to $+\infty$ and $\{\delta_n\}$ decreasing to 0, such that $C_n\delta_n$ diverges to $+\infty$ and there exists a sequence $\{\alpha_n\}$ decreasing to 1 such that

$$|ds_{\alpha_n}(y_n)| := \max_{D(x_1,\delta_n)} |ds_{\alpha_n}| = \max_{D(x_1,2\delta_n)} |ds_{\alpha_n}| = C_n$$

We define

$$\begin{aligned} \tilde{s}_{\alpha_n} :\ D(\delta_n C_n) &\longrightarrow N \\ x &\longmapsto s_{\alpha_n}(y_n + C_n^{-1}x) \end{aligned}$$

then $|d\tilde{s}_{\alpha_n}(0)| = \max_{D(C_n\delta_n)} |d\tilde{s}_{\alpha_n}| = 1$.

Fix any large $R < +\infty$, since $C_n\delta_n \to +\infty$, $\tilde{s}_{\alpha_n}$ is eventually defined on $D(R)$ and is a critical point of E_{α_n} with respect to a metric $\tilde{g}_n$ on $D(R)$ converging to the Euclidean metric. The energy $E(\tilde{s}_{\alpha_n}|_{D(C_n\delta_n)}, \tilde{g}_n) = E(\tilde{s}_{\alpha_n}|_{D(y_n,\delta_n)}, g_M) \leq B$.

We claim that Proposition 16 and Theorem 17 remain correct when $M = D(R)$ and s_α are critical maps of E_α with respect to metrics $\tilde{g}_\alpha$ converging to the Euclidean metric. To be precise:

Assertion 2. *Let $\tilde{s}_\alpha :\ (D(R), \tilde{g}_\alpha) \longrightarrow N \subset \mathbb{R}^k$ be critical maps of E_α such that*

- *s_α converges weakly to s_* in $W^{1,2}(D(R), Euclid)$,*
- *$E(s_\alpha) < B$*

then there exists at most l points $\{x_1, \ldots, x_l\}$ in $\overline{D}(R)$ and a subsequence $\{s_\beta\}$ such that s_β converges to s_* in $C^1(\overline{D}(R/2) \setminus \{x_1, \ldots, x_l\}, \mathbb{R}^k)$ uniformly on compacts, and s_* is harmonic in $D(R/2)$.

The two ingredients of the proof of Proposition 16 and Theorem 17 to be investigated are the covering and the estimate from Lemma 10. For the estimates, we already remarked that the parameters $\alpha_0, \epsilon, C(p, D')$ of Lemma 10 can be chosen independent of the metric $\tilde{g}_\alpha$ if they are close to Euclidean. For the covering, the investigation is not on the constant h, which can be chosen to be $3^{\dim M}$, but on how small the radius of the covering balls must be, but Lemma **??** states that their size is dictated by the Ricci curvature and sectional curvature of $\tilde{g}_\alpha$, which are also uniformly bounded.

Using Assertion 2, passing to a subsequence of $\{\tilde{s}_{\alpha_n}\}$ if necessary, we can suppose that $\tilde{s}_{\alpha_n} \to s_*$ in $C^1(D(R), \mathbb{R}^k)$. Note that there is no singular point where $\{\tilde{s}_{\alpha_n}\}$ fails to converge because $|d\tilde{s}_{\alpha_n}|$ is bounded uniformly on $D(R)$ (hence cannot explode as in Assertion 1). We can also choose, by a diagonal argument, a subsequence of $\{\tilde{s}_{\alpha_n}\}$ that converges to s_* in $C^1(\mathbb{R}^2)$ uniformly on compacts.

It is clear that $s_* : \mathbb{R}^2 \longrightarrow N$ is harmonic and nontrivial because

$$|ds_*(0)|_{\text{Euclid}} = \lim_{\alpha_n \to 1} |d\tilde{s}_{\alpha_n}(0)|_{\tilde{g}_{\alpha_n}} = 1.$$

Also,
$$\int_{D(R)} |ds_*|^2 dE = \lim_{\alpha_n \to 1} \int_{D(R)} |d\tilde{s}_{\alpha_n}|^2 dV_{\tilde{g}_\alpha} \leq \limsup_{\alpha \to 1} 2E(s_\alpha|_{D(x_1, 2\delta_n)}) < 2B$$
which means the energy of s_* on $\mathbb{R}^2$ is bounded above by $2B$.

Now since $(\mathbb{R}^2, \mathrm{Euclid})$ is conformal to $\mathbb{S}^2 \setminus \{p\}$, s_* can be seen as a harmonic map on $\mathbb{S}^2 \setminus \{p\}$ with the same (finite) energy. By Extension theorem 13, s_* extends to a nontrivial harmonic map from $\mathbb{S}^2$ to N. □

Remark 21. *1. We can have a better estimate of $E(s_*)$. For any $R > 0$, one has*

$$E(s_*|_{D(R)}) + E(s|_{M \setminus \cup_{i=1}^l D(x_i, \delta_n)}) \leq \limsup_{\alpha_n \to 1} \left[E(s_{\alpha_n}|_{D(x_1, \delta_n)}) + E(s_{\alpha_n}|_{M \setminus \cup_{i=1}^l D(x_i, \delta_n)}) \right]$$

Let $\delta \to 0$ then $R \to +\infty$, one has

$$E(s_*) + E(s) \leq \limsup_{\alpha \to 1} E(s_\alpha).$$

2. The proof of Theorem 19 also gives a constraint on the image of s_*: since $s_*(D(R)) \subset \overline{\cup_{1<\beta<\alpha} s_\beta(D(x_1, 2\delta))}$ for all α arbitrarily close to 1 and δ arbitrarily small, one has

$$s_*(\mathbb{S}^2) \subset \bigcap_{\delta \to 0} \bigcap_{\alpha \to 1} \overline{\bigcup_{1<\beta<\alpha} s_\beta(D(x_1, \delta))}$$

6 Minimal immersions of $\mathbb{S}^2$.

We use the following result:

Theorem 22 ([**?**], [**?**], [**?**]). *If $s : \mathbb{S}^2 \longrightarrow N$ is a nontrivial harmonic map and $\dim N \geq 3$, then s is a C^∞ conformal, branched, minimal immersion.*

The "minimal" part follows from [**?**], the "branched" part follows from [**?**] and the "conformal" part follows from [**?**] and the fact that there is no nontrivial holomorphic quadratic differential on $\mathbb{S}^2$. Theorem 19 gives:

Theorem 23. *If the universal covering $\tilde{N}$ of N is not contractible then there exists a C^∞ conformal, branched, minimal immersion $s : \mathbb{S}^2 \longrightarrow N$.*