Calabi–Yau metric on K3 surface

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Contents

1 Kähler geometry and Calabi–Yau theorem.

A *hermitian manifold* is a manifold M^{2n} with an *almost complex structure* J such and a *hermitian metric* h, i.e. $J^2 = -1$, h is i-linear, $h(v,u) = \overline{h(u,v)}$, $h(u,u)$ is positive and $h(Ju,v) = ih(u,v) = h(u,-Jv)$ for all $u, v \in C \otimes TM$. Writing $h = g + i\omega$, one sees that g is a Riemannian metric and ω is a nondegenerate 2-form of type (1,1), g and ω are J-invariant, $J^*g = g$, $J^*\omega = \omega$ and one can compute the triplet (g, ω, h) knowing one of the three.

The following 3 conditions are equivalent and imply integrability of J: (i) ω is closed, (ii) ω is parallel w.r.t g, (iii) J is parallel w.r.t g. In this case, one calls M a *Kähler manifold*, ω the *Kähler form* (or metric) and its cohomology class in $H^2(M, \mathbb{R})$ the *Kähler class*. [1] It follows from the following lemma that Kähler metrics of the same cohomology class are parameterised by an open cone of $\Omega^0(M)/\mathbb{R}$.

Lemma 1 ($\partial\bar{\partial}$ Lemma)**.** *A real (1,1) form on a compact Kähler manifold is exact if and only if it can be writen as $i\partial\bar{\partial}\varphi$ where φ is a real function on M, unique up to constant.*

[1] One can also define Kähler manifold as riemannian manifold with *holonomy group* reducing from $SO(2n)$ to $U(n)$. This means that there is a *parallel, compatible* complex structure J, i.e. $\nabla J = 0$ and $g \circ J = J$. These two conditions implies integrability of J, and when J is compatible, $\nabla J = 0$ is equivalent to $d\omega = 0$ where $\omega(u,v) = g(Ju,v)$.

If we choose a basis $\{\partial_{x_i}, \partial_{y_i}\}_{i=\overline{1,n}}$ such that $J\partial_{x_i} = \partial_{y_i}$ then $dz_i = dx_i + idy_i$ are i-eigenvectors of J, i.e. $J(dz_i) = J^*(dz_i) = idz_i$ and $J(d\bar{z}_i) = -id\bar{z}_i$. The hermitian metric can be writen as $h = \sum h_{ij} dz_i \otimes d\bar{z}_j$ where $h_{ij} = h(\partial_{x_i}, \partial_{x_j}) = \overline{h_{ji}}$, and Kähler form is $\omega = \frac{i}{2} h_{ij} dz_i \wedge d\bar{z}_j$. Because J is parallel, the Ricci curvature is J-invariant, and because it is symmetric, we can turn it into a 2-form called *Ricci form* $\rho(u,v) = \mathrm{Ric}(Ju, v)$, which we will abusively denote by Ric and can be computed in local coordinates as

$$\mathrm{Ric} = -i\partial\bar{\partial} \log \det(h_{ij}) \tag{1}$$

Remark 2. 1. *The RHS of* (1) *is globally well-defined: if one writes h in another chart* $\{d\zeta_i = \alpha_{ij} dz_j\}$, *then* $h^z = {}^{\mathrm{t}}\alpha h^{\zeta} \bar{\alpha}$ *and* $\log\det h^z = \log\det h^{\zeta} + \log|\det\alpha|^2$ *and the last term solves* $\partial\bar{\partial} = 0$ *because* $\det\alpha$ *is holomorphic.*

2. *The Ricci curvature of* ω *is completely determined by its volume form* $\frac{\omega^n}{n!} = \det(h_{ij}) \bigwedge_i \frac{i}{2} dz_i \wedge d\bar{z}_i$. *Given 2 Kähler metrics* ω_1, ω_2, *the difference of 2 Ricci forms is* $\mathrm{Ric}_1 - \mathrm{Ric}_2 = -i\partial\bar{\partial}\log\frac{\omega_1^n}{\omega_2^n}$.

3. *Since* $\partial\bar{\partial}\log\det(h_{ij})$ *is the curvature of the canonical bundle* $K_M = \Lambda^{n,0}T^*M$, *the cohomology class of* ρ *only depends on the complex structure of M and not its Kähler metric:* $[\mathrm{Ric}] = -2\pi c_1(K_M)$. *It follows that Ricci-flat Kähler manifolds have real Chern class* $c_1(K_M) = 0$ *which implies, if M is simply connected, that* K_M *is holomorphically trivial.*[2]

Let (M, J, ω_0) be a compact Kähler manifold, by $\partial\bar{\partial}$-Lemma one can write any (1,1)-form $\rho \in -2\pi c_1(K_M)$ as $\rho = \mathrm{Ric}_0 - i\partial\bar{\partial}\eta$ and any Kähler metric cohomologous to ω_0 as $\omega = \omega_0 + i\partial\bar{\partial}\phi$. By the previous remark, ρ is the Ricci form of ω if and only if

$$(\omega_0 + i\partial\bar{\partial}\phi)^n = e^{c+\eta}\omega_0^n =: f\omega_0^n \tag{2}$$

for a constant c. Because $[\omega] = [\omega_0]$ the two metrics have equal volume, i.e. $\int_M (f-1)\omega_0^n = 0$. Yau resolved *Calabi conjecture* by showing that for any f with average volume 1, there exists a potential $\phi \in \Omega^0(M)$ unique up to constant with $\omega_0 + i\partial\bar{\partial}\phi > 0$ satisfying (2).

Theorem 3 (Calabi–Yau)**.** *Let* (M, ω_0, J) *be a compact Kähler manifold and* ρ *be any (1,1) form in* $-2\pi c_1(K_M)$. *There exists a unique Kähler metric* ω *cohomologous to* ω_0 *in* $H^2(M, \mathbb{R})$ *such that its Ricci form is* ρ.

Since we are interested in Ricci-flat metric, we will suppose that the canonical bundle is holomorphically trivial, i.e. there is a global nowhere-vanishing holomorphic (n,0)-form χ uniquely up to constant (due to Maximum Principle), and the Ricci form can be writen as $\mathrm{Ric} = -i\partial\bar{\partial}\log\frac{\omega^n}{\chi\wedge\bar{\chi}}$ by the first point of Remark 2. Because M has no non constant (pluri-) harmonic function, the Ricci flat condition is

$$\omega^n = \lambda\, \chi \wedge \bar{\chi} \tag{3}$$

[2] When M is simply connected, vanishing real Chern class (flatness) implies vanishing integral Chern class (topological triviality). Topologically trivial line bundles that are not holomorphically trivial are the kernel of $c_1 : H^1(M, \mathcal{O}^*) \longrightarrow H^2(M, \mathbb{Z})$, which by the exponential sequence is the Dolbeault cohomology $H^1(M, \mathcal{O})$ which vanishes due to Hodge theory.

for a constant λ depending only $[\omega]$.

One way to generate Kähler metric *locally* is by using a Kähler potential, i.e. we set $\omega_0 = 0$ and look for $\omega =: i\partial\bar{\partial}\phi$. [3] Now if we restrict to rotational potential $\phi = \phi(\rho)$ where $\rho = r^2$ on $\mathbb{C}^2$, the Ricci-flat equation becomes $(\phi')^2 + r^2\phi'\phi'' = \lambda$. Up to multiplicative constant, one has 2 solutions: the Euclidean potential $\phi_E = \rho$ and the Eguchi–Hanson potential $\phi_{EH}(\rho) = -\tanh^{-1}(1+\rho^2)^{1/2}$ that explodes at 0. Both of them give a positive metric. We will not need the close formula for ϕ_{EH}, but it is useful to keep in mind its development at ρ near 0 and $+\infty$.

$$\phi_{EH} = \log\rho + \frac{1}{4}\rho^2 + O(\rho^4), \quad \rho \approx 0 \tag{4}$$

which will show that the corresponding Kähler metric extends to the blowup and

$$\phi_{EH} = \rho - \frac{1}{2}\rho^{-1} + O(\rho^{-3}), \quad \rho \approx +\infty \tag{5}$$

which shows that when ρ is big, the metric looks Euclidean.

2 Kummer surfaces as desingularisation of $T^2/\pm$. Eguchi–Hanson metric on the Blowup of $\mathbb{C}^2/\pm$. Glueing Kähler potentials.

A complex torus $T^2 = \mathbb{C}^2/\Lambda$ has a natural involution $x \mapsto -x$ which admits $2^4 = 16$ fixed points (there 2 choices, integer or half-integer, for each generator of Λ) and they become 16 singular points in the quotient space $T^2/\pm$, whose regular set X is equiped with the induced metric and complex structure of $\mathbb{C}^2$. The neighborhood of each singular is modeled by $\mathbb{C}^2/\pm$ at 0, in term of metric and complex structure. Let Z be the blowup of $T^2/\pm$ at its singular points, we will see soon that Z is Kähler and its canonical bundle is trivial. Our goal is to rediscover Yau's theorem 3 for Z as a glueing problem.

To see how blowing up change a singular point, we isomorph $Y := \mathbb{C}^2/\pm$ with the surface $\mathcal{C} := \{(w_1, w_2, w_3) : w_3^2 = w_1 w_2\} \subset \mathbb{C}^3$ via $(z_1, z_2) \mapsto (z_1^2, z_2^2, z_1 z_2)$. The blowup $\hat{\mathbb{C}}^3$ of $\mathbb{C}^3$ at 0 is obtained by replacing the origin by complex lines passing by it, i.e. a $\mathbb{C}P^2$: $\hat{\mathbb{C}}^3 := \{((w_1, w_2, w_3), [u_1 : u_2 : u_3]) : u_i w_j = u_j w_i\} \subset \mathbb{C}^3 \times \mathbb{C}P^2$. The blowup of $\mathcal{C}$ at 0 is obtained by taking preimage (the so-called *proper transform* of $\mathcal{C}$)

$$\hat{\mathcal{C}} := \{((w_1, w_2, w_3), [u_1 : u_2 : u_3]) \in \mathbb{C}^3 \times \mathbb{C}P^2 : w_3^2 = w_1 w_2,\ u_i w_j = u_j w_i,\ u_3^2 = u_1 u_2\}$$

Proposition 4. *1. The holomorphic (2,0)-form $dz_1 \wedge dz_2$ which is well defined on $\mathcal{C} \setminus \{0\}$ extends to a nowhere vanishing holomorphic (2,0)-form on $\hat{\mathcal{C}}$. This means that Z has trivial canonical bundle.*

2. The Eguchi–Hanson metric $\omega_{EH} = i\partial\bar{\partial}\phi_{EH}$ on $\hat{\mathcal{C}} \setminus \{0\}$ extends to a Ricci-flat metric on $\hat{\mathcal{C}}$.

[3] In fact the $\bar{\partial}$-Poincare lemma means any Kähler metric can be writen locally this way.

Proof. Since the u's are coordinates of $\mathbb{C}P^2$, at least u_1 or u_2 is non-zero. Rewrite dz's in term of dw's and use $\frac{dw_1}{w_1} = 2\frac{dw_3}{w_3} - \frac{dw_2}{w_2}$, one has

$$\begin{aligned}
4dz_1 \wedge dz_2 &= \frac{1}{w_3} dw_1 \wedge dw_2 = \frac{w_1}{w_3}\left(2\frac{dw_3}{w_3} - \frac{dw_2}{w_2}\right) \wedge dw_2 \\
&= \frac{w_3}{w_2}\left(2\frac{dw_3}{w_3} - 2\frac{dw_2}{w_2}\right) \wedge dw_2 = \frac{u_3}{u_2}\left(2\frac{du_3}{u_3} - 2\frac{du_2}{u_2}\right) \wedge dw_2 \\
&= 2\frac{u_3}{u_2} d\log\frac{u_3}{u_2} \wedge dw_2 = 2d\left(\frac{u_3}{u_2}\right) \wedge dw_2
\end{aligned}$$

So the form extends to $\hat{\mathcal{C}}$, the extension is nowhere vanishing because the $\frac{u_3}{u_2}$-direction is tangent to the divisor and the w_2-direction is normal to it.

To prove that the Kähler metric ω_{EH} extends, we make use of the development (4). The similar argument as the above shows that any metric of the form $i\partial\bar{\partial}(\log\rho + \frac{1}{4}\rho^2 + O(\rho^4))$ extends[4]. An even easier way to see that ω_{EH} extends is to rewrite the Kähler with the log term replaced by

$$\begin{aligned}
i\partial\bar{\partial}\log\rho &= i\partial\bar{\partial}\log(|w_1| + |w_2|) = i\partial\bar{\partial}\log\frac{|w_1| + |w_2|}{|w_2|} \\
&= i\partial\bar{\partial}\log\frac{|u_1| + |u_2|}{|u_2|} = i\partial\bar{\partial}\log(|u_1| + |u_2|)
\end{aligned}$$

where the function $\log(|u_1| + |u_2|)$ is well-defined on the entire $\hat{\mathcal{C}}$. □

Remark 5. *The space $Y := \widehat{\mathbb{C}^2/\pm 1}$ equiped with ω_{EH}, often called* Eguchi–Hanson Einstein ALE manifolds, *has* asymptomtically locally Euclidean *structure at infinity (blowing up at the origin obviously does not change anything at infinity). Because of the expansion* (5), *the ALE order is exactly 4.*

Now we will use the expansions 5 to glue Euguchi–Hanson potential ϕ_{EH} to the Euclidean potential outside of a big ball, i.e. let $\phi_{Y,R}$ be an intepolation of ϕ_{EH} on $\rho \leq \frac{R}{4}$ and $\phi_E = \rho$ on $\rho > R$:

$$\phi_{Y,R}(\rho) = \phi_{EH}(\rho) + \beta(\frac{\rho}{R}).(\phi_E - \phi_{EH})(\rho)$$

where $\beta \in C^\infty(\mathbb{R})$ taking value in $[0,1]$, $\beta = 0$ on $(-\infty, \frac{1}{4} + \epsilon)$ and $\beta = 1$ on $[1-\epsilon, +\infty)$. Recall that $\rho = r_Y^2$ where r_Y is the Euclidean distance to the origin on $Y = \widehat{\mathbb{C}^2/\pm 1}$, descended from that of $\mathbb{C}^2$, and that $\phi_E - \phi_{EH} = \frac{1}{2}\rho^{-1} + O(\rho^{-3}) = \frac{1}{2}r_Y^{-2} + O(r_Y^{-6})$, so $|\nabla^k(\phi_{Y,R} - \phi_{EH})| = O(R^{-1-\frac{k}{2}})$ and if we denote $\omega_{Y,R} := i\partial\bar{\partial}\phi_{Y,R}$ to be the Kähler metric coming from the potential $\phi_{Y,R}$ then:

$$\omega_{Y,R} = \omega_{EH} + O(R^{-2}), \qquad \omega_{Y,R}^2 = (1+\eta)\omega_{EH}^2$$

[4]One only needs the first-order term in ρ to vanish.

where η is supported in $\frac{1}{2}R^{1/2} \leq r_X \leq R^{1/2}$ and $|\nabla^k \eta| = O(R^{-2-\frac{k}{2}})$. Therefore $\mathrm{Ric}_{\omega_{Y,R}} = \mathrm{Ric}_{\omega_{EH}} - i\partial\bar{\partial}\log(1+\eta) = O(R^{-3})$ in the region of interpolation, and is 0 elsewhere.

Now we will equip the Kummer surface Z with a family of metrics ω_R given by scaling down Y by R (in length) and glue to X. Concretely, we build a function r_X on X which is globally bounded and equal to the Euclidean distance to singular point in $T^2/\pm 1$, then near each singular point p_i, we glue the truncated $X \setminus \{r_X \leq R^{-1/2}\}$ to the region $\{r_Y \geq R^{1/2}\}$ in Y and construct the (1,1)-form ω_R on Z that is the flat ω_X on the X-side, i.e. $X \setminus \{r_X \leq R^{-1/2}\}$, and is $R^2\omega_{Y,R}$ on Y-side, i.e. $Y \cap \{r_Y \leq R^{1/2}\}$.

Remark 6. 1. *It can be seen from* (1) *that the Ricci curvature is scale-invariant, so as R gets bigger, the 16 Y-parts get smaller in size, and bigger in curvature, but their Ricci curvature stays the same.*

2. *The Kähler class $[\omega_R] \in H^2(Z, \mathbb{R})$ depends on R. This makes sense because in the next section, we are going to find Ricci-flat metric on Z very close to ω_R and in its cohomology class. What we will obtain is a family of Ricci-flat metrics $\tilde{\omega}_R$ (unique in $[\omega_R]$) which are shrinking in size and exploding in curvature near the 16 singular points. This is the reversed process of Einstein metric bubbling as in [**?**, **?**, **?**]. The Gromov–Hausdorff limit of $(Z, \tilde{\omega}_R)$ will be the orbifold X. If in addition, we know that $\tilde{\omega}_{Y,R}$ is C^2-close to $\omega_{Y,R}$.*

   - *The scale-up process at the bubble $(\{r_X \leq R^{1/2}\}, R^2\omega_{Y,R})$ turns $R^2\omega_{Y,R}$ to ω_{EH}. The 16 Einstein ALEs obtained are Eguchi–Hanson ALEs, which by 5 are of* ALE order *4 as expected by [**?**].*
   - *There is no "bubble on bubble". One can check that the energy $\int_Z |Rm_{\tilde{\omega}_R}|^2 \mathrm{vol}_{\omega_R} \approx \int_Z |Rm_{\omega_R}|^2 \mathrm{vol}_{\omega_R} \approx 16 \int_Y |Rm_{EH}|^2 \mathrm{vol}_{EH}$.*

   However, we are only able to find $\tilde{\omega}_R$ via a small L^2_5 pertubation of the Kähler potential of ω_R, which translates to $L^2_3 \hookrightarrow C^{0,\alpha}$ pertubation of the metric.

3 Tube metric and Monge–Ampère equation

A clearer way to to understand the procedure of "scaling down Y and glueing it to X" is via tube metric. We first remark that the quantity $r^-2\omega$ is scale-invariant:

$$r_X^{-2}\omega_R = \begin{cases} r_X^{-2}\omega_E, & \text{in } X \\ R^2 r_Y^{-2} R^{-2}\omega_{R,Y} = r_Y^{-2}\omega_{R,Y} \approx r_Y^{-2}\omega_{EH}, & \text{in } Y \end{cases} \tag{6}$$

Lemma 7 (Tube metric)**.** *Let g_E be the Euclidean metric on $\mathbb{C}^2$ and r be the distance function to the origin, then $r^{-2}g_E$ is the product metric on $S^3 \times \mathbb{R} \cong \mathbb{C}^2 \setminus \{0\}$. Similarly $r^{-2}g_E$ on $\mathbb{C}^2/\pm 1 \setminus \{0\}$ is the product metric on $S^3/\pm 1 \times \mathbb{R}$.*

This can be seen via a change of variable $t = \log r$, so $g_E = dr^2 + r^2 g_{S^3} = r^2 dt^2 + r^2 g_{S^3}$. This means $r^{-2}g_E$ is the product metric $dt^2 + g_{S^3}$ of $S^3 \times \mathbb{R}$.

Now we come back to the two pictures of X and Y and think of X as a manifold with 16 cylindrical ends at the singular points and think of Y as having 1 cylindrical end at infinity of $\mathbb{C}^2/\pm 1$. We mark the $S^3/\pm 1$ sections in X, Y that correspond to $r_X = 1$ and $r_Y = 1$, then as R increase, we move further to to ends by tube-distance $\frac{1}{2}\log R$ and glue the two sections $\{r_X = R^{-1/2}\}$ and $\{r_Y = R^{1/2}\}$ together. If we choose one, in our case X, and turn the tube metric into the finite one by multiplying it by r^2, then it becomes the Gromov–Hausdorff limit in Anderson-Bando-Kasue-Nakajima theorem, while the other becomes the "bubble".

Not only does the tube metric provide a clear geometric view of the glueing procedure, but also an analysis toolbox for solving the Ricci-flat equation.

We want to find a Kähler potential ϕ so that $\tilde{\omega}_R := \omega_R + i\partial\bar{\partial}\phi$ is Ricci-flat, i.e. $\tilde{\omega}_R^2 = (\omega_R + i\partial\bar{\partial}\phi)^2 = \lambda\chi \wedge \bar{\chi}$ where χ is the non-vanishing holomorphic $(2,0)$-form. But since $\omega_R^2 = \frac{1}{1+\eta}\chi \wedge \bar{\chi}$ where η is as before (in the glueing procedure, we also recaled the holomorphic (2,0)-form on Y), i.e. it is supported in $\frac{1}{2}R^{-1/2} \leq r_X \leq R^{1/2}$ and $|\eta| = O(R^{-2})$, one has $(\omega_R + i\partial\bar{\partial}\phi)^2 = (1+\eta)\omega_R^2$.

We then substitute ω_R by the tube metric $\Theta_R := r_X^{-2}\omega_R$ and rearrange, and divide by Θ_R^2 to have $2r_X^{-2} i\partial\bar{\partial}\phi \wedge \Theta_R/\Theta_R^2 + (r_X^{-2} i\partial\bar{\partial}\phi)^2/\Theta_R^2 = \lambda(1+\eta) - 1$. This is a PDE, with nonlinearity appears in highest order, its linearisation at $\phi = 0$ is the first term of LHS. We will do a change the variable $f = r_X\phi$ and multiply the two sides by r_X^3 to have

$$2r_X i\partial\bar{\partial}(r_X^{-1}f) \wedge \Theta_R/\Theta_R^2 + r_X^{-3}(r_X i\partial\bar{\partial}(r_X^{-1}f))^2/\Theta_R^2 = (\lambda(1+\eta) - 1)r_X^3 \tag{7}$$

The justification for this manoeuvre is that it turns the linear term into the Laplacian with respect to the tube metric.

Lemma 8. 1. *If $\omega = r_X^2\Theta$ is Kähler then $r_X i\partial\bar{\partial}(r_X^1 f) = \Delta_\Theta f + V$ where $V = r_X^3\Delta_\omega r_X^{-1}$. In particular, if ω is the Euclidean metric on $\mathbb{C}^2$, then $V \equiv 1$.*

2. *If M is a manifold with cylindrical end, equiped with a tube metric and V is a function on M that is approximately 1 in its end, then $\Delta + V : L^2_{k+2} \longrightarrow L^2_k$ is Fredholm. Elements of its kernel are smooth, have exponential decay in all derivatives.*

3. *Let X_1, X_2 be manifolds, each with a compact part and one cylindrical end of the same section and V_i be potentials on V_i which equal to 1 in the end. Denote by $X_T := X_1 \#_T X_2$ the manifold given by glueing the two ends at distance T away from the compact part and V the potential glued from V_i. If $\Delta_i + V_i$ are invertible on X_i then the operator $\Delta_T + V$ has right inverse when T is sufficiently big, and its norm is bounded independently of T.*

The linearisation of (7) is not invertible, since the function $f = r_X$ is obviously in the kernel. It turns out that this is the only direction in the kernel. So we should modify the RHS of (7) by adding the term τr_X where $\tau \in \mathbb{R}$ and look for pair of solution (f, τ). The linearised operator, which is self-adjoint, is obviously invertible. And if we can solve

$$2r_X i\partial\bar{\partial}(r_X^{-1}f) \wedge \Theta_R/\Theta_R^2 + r_X^{-3}(r_X i\partial\bar{\partial}(r_X^{-1}f))^2/\Theta_R^2 = (\lambda(1+\eta) - 1)r_X^3 + \tau r_X$$

Integrate and use the fact that the ω_R volume is the same as the $\tilde{\omega}_R$ volume, one can see that $\tau = 0$.

Now suppose that we can solve the linearised equation, it remains to solve the nonlinear one (7). To do this, one uses the quantitative Inverse Function theorem.

Theorem 9 (Quantitative IFT). *Let $F : X \longrightarrow Y$ be a C^2 map between Banach space such that $F(a) = b$ and the derivative $F'(a) : X \longrightarrow Y$ of F at a is invertible. Suppose that F'' is bounded on an open set $U \supset B(a, r_U) \ni a$. Then the image $F(U)$ contains a ball centered at b, of radius $r = \min\{\frac{1}{4|F''||F'(a)^{-1}|^2}, \frac{r_U}{2|F'(a)^{-1}|}\}$ where $|F''|$ is the constant such that $|F'(x_1)v - F'(x_2)v| \leq |F''||x_1 - x_2||v|$*

It is often proved by contraction mapping, but before going to the detail let us explain how the quantity $\frac{1}{4|F''|F'(a)^{-1}|^2}$ appears using the Taylor expansion of F: $F(a+\delta) = F(a) + F'(a)\delta + \frac{1}{2}F''(a)\delta^2 + o(\delta^2)$. Infinitesimally to move F in a direction v at $F(a)$, x should be move in the direction $F'(a)^{-1}v$, but to go from "infinitesimally" to "locally" we get interference from higher order term. To guarantee that it is negligible, one needs

$$\left|F''(a)\right| |\delta|^2 \leq \left|F'(a)\delta\right|$$

Since $|F'(a)^{-1}|^{-1}|\delta| \leq |F'(a)\delta|$, one only needs $|F''||\delta|^2 \leq |F'(a)^{-1}|^{-1}|\delta|$, or $|\delta| \leq \frac{1}{|F'(a)^{-1}||F''|}$. To garuantee this, F should move less than $\frac{1}{|F'(a)^{-1}|^2|F''|}$.

Proof. Let $y \in Y$ be a point close to b. Define $x_0 = a$ and $x_{n+1} = G(x_n)$ where $G(x) = x + F_a^{-1}(y - F(x))$. We will prove that if y is sufficiently close to b then G is contraction and so $\{x_n\}$ converges. One has

$$\left|G'\right| = |1 - F_a^{-1}F_x| = |F_a^{-1}(F_a - F_x)| \leq |F_a^{-1}||F''||x-a|$$

So G is $\frac{1}{2}$-contraction, as long as $x \in U$ and $|x-a| \leq \frac{1}{2|F_a^{-1}||F''|}$. One has $|x_{n+1} - x_n| \leq 2^{-n}|x_1 - x_0| \leq 2^{-n}|F_a^{-1}||y-b|$ and so $\{x_n\}$ converges to a limit x_∞ in the ball of radius $2|F_a^{-1}||y-b|$ around a, which we can suppose to lie in $U \cap B(a, \frac{1}{2|F_a^{-1}|F''|})$ if $|y-b| \leq \frac{1}{4|F_a^{-1}|^2|F''|}$. The limit x_∞ satisfies $G(x_\infty) = x_\infty$, i.e. $F(x_\infty) = y$. □