Interpolation theory and Sobolev spaces on compact manifolds

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1 Motivation

We will define a more general notion of Sobolev spaces on compact manifold than those in [?] and [?], where Sobolev spaces on a (Riemannian) manifold $W^{k,p}(M)$ of dimension n are defined for $k \in \mathbb{Z}_{\geq 0}$ and for *uniform weight*, i.e. a function $f \in W^{k,p}(M)$ is supposed to be weakly differentiable up to order k in every variables $x_1, \ldots, x_n$ in each smooth coordinates. The space $W^{k,p}(M)$ in this case can be defined by density with respect to a certain norm involving derivatives $\frac{\partial f}{\partial x^\alpha}$.

Meanwhile, the suitable function spaces to solve parabolic equations are those whose regularity in time is half of that in space, i.e. we will solve parabolic equations on the Sobolev spaces $W^{k,p}(M \times T)$ of functions k times regular in M and $k/2$ times regular in T. We cannot always, (for example when k is odd) find a simple norm involving derivatives of f in order to define $W^{k,p}$ by density. This generalisation will be done using Stein's multipliers.

Another generalisation will be made is to allow the manifold to have boundary. Even when we only want to solve parabolic equation on manifold M without boundary, the underlying space is $M \times [0, T]$ which has boundary. Moreover, we will have to discuss the notion of trace in order to use the initial condition at $t = 0$.

All manifolds will be compact, with no given metric. This is not really a generalisation since on compact manifolds, Sobolev spaces $W^{k,p}(M)$, as defined in [**?**] and [**?**] set theoretically do not depend on the metric and (the equivalent class of) their norms also independent of the metric.

We will mainly follow the discussion in [**?**], where the author also works on manifold with *corner*, i.e. irregular boundary. The corners, modeled by $\mathbb{R}^{n-k} \times \mathbb{R}^k_{\geq 0}$, appear naturally, for example at the boundary ∂M in $t = 0$. The extra effort to cover the case of corners is not much (see [**?**, page 50]) and essentially algebraic.

The advantage of this approach (Stein's multipliers and interpolation theory) over the definition by density is that it is based on an algebraic framework compatible with compact operators and capable of formulating analytic result economically. For example, 3 classical results of elliptic equations (existence and approximate solution, regularity, Garding's inequality) can be nicely encoded in a commutative diagram and can be proved at the same time (and the proof is essentially algebraic), see this post.

2 Preparatory material

We will recall here basic elements of Fourier transform on the space of tempered distributions and then we will have a quick review of interpolation theory.

2.1 Stein's multiplier

Let $X = \mathbb{R}^n$ be the Euclidean space, coordinated by $x_1, \ldots, x_n$ and $\mathcal{E} = \mathbb{R}^n$, coordinated by $\xi_1, \ldots, \xi_n$ be the frequency domain of X. Recall that Fourier transform is an isomorphism in the following three levels

1. The Schwartz space of rapidly decreasing smooth functions $\mathcal{S}(X)$ whose elements are smooth and decrease more rapidly then any rational func-

tion. The Schwartz space are topologized by the family of semi-norms $|f|_{\alpha,\beta} = \sup_X |x^\alpha D_x^b f(x)|$.

2. The space $L^2(X)$ of doubly-integrable functions.

3. The space of tempered distributions, i.e. the dual space $\mathcal{S}^*(X)$ of $\mathcal{S}(X)$ under the weak-* topology given by $\mathcal{S}(X)$.

To simplify the notation, we use $D_x^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ and $P(D) = \sum_\alpha c_\alpha D^\alpha$ for any polynomial P.

Recall that for any $u \in \mathcal{S}(X)$ and for any polynomial P, one has $\widehat{P(D)u} = P(\xi)\hat{u}(\xi)$. This can be extended to non-polynomial function of $M(D)$ of D by

$$\widehat{M(D)u} := M(\xi)\hat{u}(\xi)$$

where M is a slowly growing function, i.e. $D^\alpha M(\xi)$ grows slower than certain polynomial as $|\xi| \to \infty$.

The following theorem give a criteria of the function M such that $M(D) : \mathcal{S}(X) \longrightarrow \mathcal{S}(X)$ extend to $L^p(X) \longrightarrow L^p(X)$.

Theorem 1 (Stein)**.** *If for any* primitive *index $\alpha = (\alpha_1, \dots, \alpha_n)$, i.e. each α_i being* 0 *or* 1 *(there are exactly 2^n primitive indices), one has*

$$|\xi^\alpha D^\alpha M(\xi)| \le C_\alpha$$

then $M(D)$ extend to a bounded linear operator on $L^p(X)$.

Definition 1. *1. A slowly growing function W on $\mathcal{E}$ with $W(\xi) > 0$ is called a* ***weight*** *if for all primitive index α, one has*

$$|\xi^\alpha D^\alpha W(\xi)| \le C_\alpha W(\xi).$$

2. The ***Sobolev space*** *$W^{k,p}(X, W)$ with respect to weight W, $k \in \mathbb{R}, 1 < p < \infty$ is the vector space*

$$W^{k,p}(X, W) = \Big\{ u \in \mathcal{S}^*(X) : \ W(D)^k u \in L^p(X) \Big\}$$

normed by $\|u\|_{W^{k,p}} = \|W(D)^k u\|_{L^p}$.

Example 1 (Weight given by $\Sigma = (\sigma_1, \dots, \sigma_n)$)**.** *Note by $\sigma := \mathrm{lcm}(\sigma_1, \dots, \sigma_n)$ then $W_\Sigma(\xi) = \left(1 + \xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n}\right)^{1/2\sigma}$ is a weight. We will only use weights of this type in our discussion. The index $\Sigma = (\sigma_1, \dots, \sigma_n)$ is chosen according to the differential operator in the elliptic/parabolic equation. In particular, for Laplace equation, one choose $\Sigma = (1, \dots, 1)$ and for heat equation $\Sigma = (1, 2, \dots, 2)$ where* 1 *is in the time component.*

Remark 1. *1. If W_1, W_2 are weights then $W_1 + sW_2, W_1W_2, W_1^s (s > 0)$ are also weights.*

2. *The operator $W(D): W^{k+r}, p(X, W) \longrightarrow W^{k,p}(X)$ is bounded.*

3. *Given another weight $V(\xi) \leq CW(\xi)$, by Stein's criteria (Theorem 1) one has a bounded embedding $W^{k,p}(X, W) \hookrightarrow W^{k,p}(X, V)$.*

The Sobolev space $W^{k,p}(X, W_\Sigma$ has a simple definition by density when $\sigma \mid k$. Given an index $\alpha = (\alpha_1, \ldots, \alpha_n)$, note by $\|\alpha\| := \sum_{i=1}^n \alpha_i \frac{\sigma}{\sigma_i}$.

Theorem 2 (Equivalent norm when $\sigma \mid k$)**.** *If $k > 0$ and $\sigma \mid k$ and $1 < p < \infty$, then given $u \in \mathcal{S}^*(X)$, one has*

1. *$u \in W^{k,p}(X)$ if and only if $D^\alpha u \in L^p(X)$ for all $\|\alpha\| \leq k$ and the norm $\sum_{\|\alpha \leq k\|} \|D^\alpha u\|_{L^p}$ is equivalent to $\|u\|_{W^{k,p}}$.*

2. *$u \in W^{-k,p}$ if and only if there exists $g_\alpha \in L^p$ such that $u = \sum_{\|\alpha\| \leq k} D^\alpha g_\alpha$ and $\|u\|_{W^{-k,p}}$ is equivalent to*

$$\inf \left\{ \sum_{\|\alpha\| \leq k} \|g_\alpha\|_{L^p} : \ u = \sum_{\|\alpha\| \leq k} D^\alpha g_\alpha \right\}$$

Example 2. *1. When $\sigma_1 = \cdots = \sigma_n = 1$, one has the familiar Sobolev spaces.*

2. *For (the weight of) heat equation, $W^{2,p}$ cam be defined by density using the norm*

$$\|u(t,x)\| = \left\| \frac{\partial u}{\partial t} \right\|_{L^p} + \|Du\|_{L^p} + \|Du\|_{L^p}$$

where L^p stands for $L^p(X \times [0,T])$.

2.2 Holomorphic interpolation of Banach spaces

The Interpolation theory is based on the following Three-lines theorem whose proof follows from the classic Hadamard's three-lines theorem (the case $A = \mathbb{C}$) and how we define complex Banach spaces and holomorphic functions taking value there.

Theorem 3 (Three-lines)**.** *Let A be a complex Banach space and $h: S = \{0 \leq \operatorname{Re} z \leq 1\} \subset \mathbb{C} \longrightarrow A$ be a holomorphic function, i.e. continuous and holomorphic in the interior such that h is bounded at infinity, i.e. $h(x+iy) \to 0$ as $y \to \infty$. Let $M(x) := \sup_y \|h(x+iy)\|$ then one has*

$$M(x) \leq M(1)^x M(0)^{1-x}$$

Let A_0, A_1 be complex Banach spaces such that

1. A_0, A_1 can be continuously embedded into a Hausdorff topological complex vector space E such that the complex structures are compatible with each others, i.e. the linear embeddings preserve complex structures.

2. The intersection $A_0 \cap A_1$ in E is dense in $(A_i, \|\|_{A_i})$ for $i = 0, 1$.

such (A_0, A_1) is called an **interpolatable** pair.

The norms of $A_0 \cap A_1$ and $A_0 + A_1$ are defined such that the these spaces are Banach and the diagram

$$0 \longrightarrow A \cap B \longrightarrow A \oplus B \longrightarrow A + B \longrightarrow 0 \tag{1}$$

commutes and the arrows are continuous. By Open mapping theorem, this means that the norm on $A_0 \cap A_1$ is equivalent to $\|x\|_{A_0 \cap A_1} = \|x\|_{A_0} + \|x\|_{A_1}$ and the norm on $A_0 + A_1$ is equivalent to $\|x\|_{A_0+A_1} = \inf_{x=x_0+x_1, x_i \in A_i} \{\|x_0\|_{A_0} + \|x\|_{A_1}\}$.

Remark 2. *A pair (A_0, A_1) of Banach spaces may give different interpolatable pairs depending how they are embedded into a common space E. It is not difficult to see that the data of* interpolatable pair *is uniquely determined by 2 complex Banach spaces U, V (which are eventually $A \cap B$ and $A + B$) and the diagram*

$$\begin{tikzcd}
& & 0 \arrow[d, bend right] & & \\
& & A_0 \arrow[u, bend right] \arrow[d, bend right] \arrow[dr] & & \\
0 \arrow[r] & U \arrow[ur] \arrow[r] \arrow[dr] & A_0 \oplus A_1 \arrow[u, bend right] \arrow[r] \arrow[d, bend right] & V \arrow[r] & 0 \\
& & A_1 \arrow[u, bend right] \arrow[d, bend right] \arrow[ur] & & \\
& & 0 \arrow[u, bend right] & &
\end{tikzcd} \tag{2}$$

in which

1. *All arrows are continuous and compatible with complex structures. The vertical sequence is exact, the horizontal sequence is exact and canonical.*

2. *The diagonal arrows from U to A_0, A_1 are injective and of dense image in A_0, A_1.*

3. *The maps composed by the diagonal arrows $U \to A_i \to V$ are injective for $i = 0, 1$. Since the two maps are additive inverse, it suffices to have injectivity for one of them.*

In the language that we will use to solve linear equation, the diagram (2) *is equivalent to the square*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & V \end{tikzcd}$$

being exact.

The following construction will give a family of complex subspace A_θ of $A_0 + A_1$ containing $A_0 \cap A_1$ for $0 \le \theta \le 1$ that interpolates A_0 and A_1 that satisfies the following properties, called interpolation inequalities

Theorem 4 (Interpolation inequality for common elements)**.** *Let $a \in A_0 \cap A_1$ then $a \in A_\theta$ and*

$$\|a\|_{A_\theta} \le 2\|a\|_{A_1}^{\theta}\|a\|_{A_0}^{1-\theta}$$

and

Theorem 5 (Interpolation inequality for operators)**.** *Given interpolatable pairs (A_0, A_1) and $B_0, B_1)$, and T a bounded linear operator $T : A_0 \longrightarrow B_0$ and $T : A_1 \longrightarrow B_1$ such that T is well-defined on $A_0 \cap A_1$. Then T extends linearly and continuously to $T : \ A_0 + A_1 \longrightarrow B_0 + B_1$, that is*

$$\begin{tikzcd} 0 \arrow[r] & A_0 \cap A_1 \arrow[r] \arrow[d, "T"] & A_0 \oplus A_1 \arrow[r] \arrow[d, "T \oplus T"] & A_0 + A_1 \arrow[r] \arrow[d, "T"] & 0 \\ 0 \arrow[r] & B_0 \cap B_1 \arrow[r] & B_0 \oplus B_1 \arrow[r] & B_0 + B_1 \arrow[r] & 0 \end{tikzcd} \qquad (3)$$

Also, T defines a bounded operator $T : \ A_\theta \longrightarrow B_\theta$ and

$$\|T\|_{L(A_\theta, B_\theta)} \le 2\|E\|_{L(A_1,B_1)}^{\theta}\|E\|_{L(A_0,B_0)}^{1-\theta}$$

To define A_θ, let

$$\mathcal{H}(A_0, A_1) := \{h : S \longrightarrow A_0 + A_1 : \ h \text{ is holomorphic and } h(z) \to 0 \text{ as } |y| \to \infty, h(iy) \in A_0, \ h(1+iy) \in$$

where, as above, S denotes the strip $0 \le \operatorname{Re} z \le 1$. Then $\mathcal{H}(A_0, A_1)$ is a Banach space with the norm

$$\|h\|_{\mathcal{H}(A_0,A_1)} := \sup_y \|h(iy)\|_{A_0} + \sup_y \|h(1+iy)\|_{A_1}$$

The space A_θ is defined set-theoretically as the space of all value in $A_0 + A_1$ that a function $h \in \mathcal{H}(A_0, A_1)$ can take at $\theta \in [0,1] \in S$. Therefore, set-theoretically A_θ coincides with A_0 and A_1 when $\theta = 0$ and $\theta = 1$. To define the norm on A_θ, let

$$\mathcal{K}_\theta(A_0, A_1) := \{h \in \mathcal{H}(A_0, A_1) : \ h(\theta) = 0\}$$

then $\mathcal{K}_\theta(A_0, A_1)$ is a closed complex subspace of the Banach space $\mathcal{H}(A_0, A_1)$. Then $A_\theta := \mathcal{H}(A_0, A_1)/\mathcal{K}_\theta(A_0, A_1)$ has the natural quotient norm inherited from $\mathcal{H}(A_0, A_1)$ and is still a Banach space.

It is not difficult to see that the norm on A_θ coincides with the norm $\|\cdot\|_{A_0}, \|\cdot\|_{A_1}$ when $\theta = 0$ or $\theta = 1$

Theorem 4 follows from the this lemma when one takes h to be a constant, and is in $A_0 \cap A_1$.

Lemma 6. *If $h \in \mathcal{H}(A_0, A_1)$ then $\|h(\theta)\|_{A_\theta} \le 2M_1^\theta M_0^{1-\theta}$ where*

$$M_0 := \sup_y \|h(iy)\|_{A_0}, \quad M_1 := \sup_y \|h(1+iy)\|_{A_1}$$

Proof. The A_θ-norm of $h(\theta)$ only depends on the value of h at θ, one can therefore replace h by a function of form $h_{c,\epsilon}(z) = \exp(c(z-\theta) + \epsilon z^2)h(z)$, then let $\epsilon \to 0$ and choose the optimal c, which is $e^c = M_0/M_1$. □

Theorem 5 follows from Theorem 4 and the very definition of quotient norm.

Remark 3. *The optimal constant, as given by the proofs, is $\theta^{-\theta}(1-\theta)^{\theta-1} < 2$*

The interest of holomorphic interpolation theory comes from the fact that interpolation of Sobolev spaces are still Sobolev spaces, which, together with Theorem 5 and Theorem 4, gives a class of useful inequalities generally called interpolation inequalities.

Theorem 7 (Interpolation of Sobolev spaces)**.** *Let $p, q \in (1, +\infty)$ and $k, l \in \mathbb{R}$ and $X = \mathbb{R}^n$. Take*

$$A_0 := W^{k,p}(X), \quad A_1 := W^{l,q}(X)$$

then $A_\theta = W^{s,r}(X)$ where

$$\theta l + (1-\theta)k = s, \qquad \theta\frac{1}{q} + (1-\theta)\frac{1}{p} = \frac{1}{r}$$

The holomorphic interpolation behaves predictably with direct sum and compact operators

Theorem 8. *Let* $(A_0, A_1), (B_0, B_1)$ *be interpolatable pairs and denotes by* $(A \oplus B)_\theta$ *be the interpolation of* $A_0 \oplus B_0$ *and* $A_1 \oplus B_1$ *then one has* $(A \oplus B)_\theta \cong A_\theta \oplus B_\theta$ *by a canonical isomorphism.*

Proof. The set-theoretical bijection is easy to see: note that there is a natural inclusion $(A \oplus B)_\theta \hookrightarrow A_\theta \oplus B_\theta$, which is also a bijection because $\mathcal{H}(A_0 \oplus B_0, A_1 \oplus B_1) = \mathcal{H}(A_0, A_1) \oplus \mathcal{H}(B_0, B_1)$.

The most difficult part is to know what we mean by *isomorphism*. In fact the two norms (the interpolation norm and the direct-sum norm) do not coincide, but they are equivalent. One can prove, with basic sup-inf analysis that

$$\frac{1}{2} \| \cdot \|_{A_\theta \oplus B_\theta} \leq \| \cdot \|_{(A \oplus B)_\theta} \leq \| \cdot \|_{A_\theta \oplus B_\theta}$$

□

Theorem 8 can be generalised to the following result.

Theorem 9 (*)**.** *Let* (X_0, X_1) *and* (Y_0, Y_1) *be interpolatable pairs. Suppose that there are inclusion* $X_0 \hookrightarrow Y_0$ *and* $X_1 \hookrightarrow Y_1$ *with closed images in* Y_0 *and* Y_1 *respectively and the inclusions agree on* $X_0 \cap X_1$ *as mappings from* $X_0 \cap X_1$ *to* $Y_0 + Y_1$*. Moreover, suppose that the image of* $X_0 + X_1$ *in* $Y_0 + Y_1$ *is closed. Then there is a natural inclusion* $X_\theta \hookrightarrow Y_\theta$ *with closed image in* Y_θ

Remark 4.

1. *The condition* $X_0 + X_1 \hookrightarrow Y_0 + Y_1$ *being of closed image is redundant if* $X_1 \hookrightarrow X_0$ *and* $Y_1 \hookrightarrow Y_0$*, as in the case of interpolation of* certain *Sobolev spaces on manifolds. In general context, one can also check that this condition holds for the maps* $\iota_{k,p}$ *and* $\iota_{l,q}$ *in Definition 2 of Sobolev spaces using the fact that the* ι *admit left-inverse given by* $\{\tilde{\psi}_i\}$*. See Remark 7.*

2. *If one has two exact sequences*

$$0 \longrightarrow X_i \longrightarrow Y_i \longrightarrow Z_i \longrightarrow 0, \quad i = 0, 1 \tag{4}$$

whose arrows commute with ones from the intersection and ambient spaces of interpolatable pairs $(X_0, X_1), (Y_0, Y_1), (Z_0, Z_1)$ *then, since the images of* $X_i \longrightarrow Y_i$ *being kernel of* $Y_i \longrightarrow Z_i$ *are closed, one has the inclusion for interpolation spaces, also of closed image:*

$$0 \longrightarrow X_\theta \longrightarrow Y_\theta, \quad 0 \leq \theta \leq 1$$

I am not sure if this exact sequence can be extended to Z_θ *in general.*

3. *In particular, if the sequences in* (4) *split, meaning that one can find a retraction* $0 \longrightarrow Z_i \longrightarrow Y_i$*, then by applying the theorem for the retractions, one sees that the interpolation sequence extend to* Z_θ*, i.e.*

$$0 \longrightarrow X_\theta \longrightarrow Y_\theta \longrightarrow Z_\theta \longrightarrow 0$$

and also splits, meaning $Y_\theta \cong X_\theta \oplus Z_\theta$*. Applying this results to the split-exact sequences*

$$0 \longrightarrow A_i \longrightarrow A_i \oplus B_i \longrightarrow B_i \longrightarrow 0$$

one then obtains Theorem 8.

Proof. The inclusion $X_\theta \hookrightarrow Y_\theta$ is natural and due to the fact that $\mathcal{H}(X_0, X_1) \subset \mathcal{H}(Y_0, Y_1)$. The equivalence of the interpolation norm X_θ and the norm inherited from Y_θ on X_θ requires more than a simple sup-inf analysis as in the proof of Theorem 8 since $\mathcal{H}(X_0, X_1)$ is strictly included in $\mathcal{H}(Y_0, Y_1)$. What we can say is that the interpolation norm X_θ dominates the interpolation norm of Y_θ, since it involves the infimum on the smaller set. In other words, it means that the inclusion $X_\theta \hookrightarrow Y_\theta$ is continuous. Therefore, to establish the equivalence of norms, it remains, by Open mapping theorem, to check that the image of $X_\theta \hookrightarrow Y_\theta$ is closed.

Since

$$\begin{tikzcd}
X_\theta \arrow[rr] \arrow[d, equal] & & Y_\theta \arrow[d, equal] \\
\mathcal{H}(X_0, X_1)/\mathcal{K}_\theta(X_0, X_1) & & \mathcal{H}(Y_0, Y_1)/\mathcal{K}_\theta(Y_0, Y_1) \\
\mathcal{H}(X_0, X_1) \arrow[rr] \arrow[u, two heads] & & \mathcal{H}(Y_0, Y_1) \arrow[u, two heads]
\end{tikzcd}$$

it suffices to show that the image $\mathcal{H}(X_0, X_1) \hookrightarrow \mathcal{H}(Y_0, Y_1)$ is closed, meaning if $\mathcal{H}(X_0, X_1) \ni h_n \to h$ in $\mathcal{H}(Y_0, Y_1)$, then h must take value in $X_0 + X_1$. By the equivalence of the norm on X_i and the restricted norm from Y_i, $i = 0, 1$, one sees that $h(iy) \in A_0$ and $h(1 + iy) \in A_1$.

Since $X_0 + X_1$ is closed in $Y_0 + Y_1$, any holomorphic map $\mathcal{H}(Y_0, Y_1) \ni f : S \longrightarrow Y_0 + Y_1$ passes holomorphically to the quotient $S \longrightarrow (Y_0 + Y_1)/(X_0 + X_1)$. The fact that h takes value in $X_0 + X_1$ follows from Maximum modulus principle for holomorphic functions. $\square$

Theorem 10 (Interpolation of compact embedding)**.** *If* $A_1 \hookrightarrow A_0$ *is a compact embedding, then* $A_1 \cong A_0 \cap A_1 \hookrightarrow A_0$ *is a compact embedding where the first* $\cong$ *denotes the same space with equivalent norms.*

Proof. It follows from Theorem 4:

$$\|x_m - x_n\|_{A_\theta} \leq 2\|x_m - x_n\|_{A_0}^{1-\theta}\|x_m - x_n\|_{A_1}^{\theta}$$

Hence if $\{x_n\}$ is a bounded sequence in A_1, it converges in A_0 and therefore A_θ. $\square$

The previous Theorem 4, together with Theorem 7 also gives a proof of Kondrachov's Theorem, that is the embedding $W^{k,p}(X) \hookrightarrow W^{l,p}(X)$ is compact if $k > h >\geq 0$. This follows from the following 2 remarks

1. The case $l = 0$ and $k \gg 1$ follows from the embedding $W^{k,p} \hookrightarrow C^1$ and Ascoli's theorem. Hence by Theorem 4, one has the compactness embedding if $k \gg 1$ and $l < k$.

2. For the case of small k, note that

$$W^{k+r,p}(X) \twoheadrightarrow W^{k,p}(X) : \ v \mapsto W(D)^r u$$

   is surjective and any $u \in W^{k,p}(X)$ can be lifted to an element $\tilde{u} \in W^{k+r,p}(X)$ of the same norm. In fact, if $W(\xi)^k \hat{u} \in L^p$ then choose $\tilde{u}$ such that $\widehat{\tilde{u}} = W(\xi)^{-r}\hat{u}$. Kondrachov's theorem follows from the diagram:

$$\begin{tikzcd} W^{k+r,p}(X) \arrow[r, two heads] \arrow[d, hook, "\text{compact}"'] & W^{k,p}(X) \arrow[d, hook] \\ W^{h+r,p}(X) \arrow[r, two heads] & W^{h,p}(X) \end{tikzcd}$$

Remark 5. *The advantage of this proof is that it is valid for weighted Sobolev spaces over manifolds.*

3 Sobolev spaces on compact manifold without boundary

Let M be a compact manifold without boundary. We fix a finite atlas of M by chart $\varphi_i : \ M \supset U_i \longrightarrow V_i \subset \mathbb{R}^n$ such that the transitions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : V_j \longrightarrow V_i$ are of strictly positive and bounded derivatives, i.e. $C(\alpha)^{-1} \geq D^\alpha \varphi_{ij} \leq C(\alpha)$ for all indices α. We will called such atlas a good atlas. One can always obtain such atlas by shrinking a bit each chart of a given atlas of M. Let ψ_i be a partition of unity subordinated to $\{U_i\}$

Definition 2. *1. The **Sobolev spaces** $W^{k,p}(M)$ is defined as*

$$W^{k,p}(M) := \left\{ f \in \mathcal{S}(M)^* : \ (\psi_i f) \circ \varphi_i^{-1} \in W^{k,p}(\mathbb{R}^n) \right\}$$

with the norm

$$\|f\|_{W^{k,p}} = \sum_i \|(\psi_i f) \circ \varphi_i^{-1}\|_{W^{k,p}(\mathbb{R}^n)}$$

2. *Weighted Sobolev spaces can be defined when M has a foliation structure, i.e. M is locally modeled by $0 \subsetneq F_1 \subsetneq \cdots \subset F_k \subsetneq \mathbb{R}^n$ where F_i are vector subspace of $\mathbb{R}^\ltimes$ of dimension $0 < n_1 < \cdots < n_k < n$ respectively and F_k are preserved by the transition maps φ_{ij}, for example when M is a product of manifolds of lower dimension. Then the above definition extends to weighted Sobolev spaces with weight $\sigma_1 = \cdots = \sigma_{n_1}$, $\sigma_{n_1+1} = \cdots = \sigma_{n_2}, \ldots \sigma_{n_k+1} = \cdots = \sigma_n$.*

Remark 6. *1. One can define $\mathcal{S}(M)^*$ as the dual space of $\mathcal{S}(M) = C^\infty(M)$ under Schwartz topology with respect to any metric, since by compactness any two metrics on M are comparable. The distributions $\psi_i f$ are tempered because they are compactly supported.*

2. *One can identify $C^\infty(M)$ with a subspace of $\mathcal{S}^*(M)$ that is contained in any Sobolev space $W^{k,p}(M)$ by fixing a Riemannian metric g on M. The map $C^\infty(M) \hookrightarrow \mathcal{S}^*(M)$ may depend on the g, but its image does not. Similarly, one can also identify an element of $\mathbb{W}^{k,p}(\mathbb{R}^n)$ supported in V_i with an element in $W^{k,p}(M)$.*

3. *If one uses another good atlas U_i' or a different partition of unity, one obtains the same set $W^{k,p}(M)$ and an equivalent norm. To see this, let us call two good atlas* compatible *if their union is also a good atlas, then the statement holds for two compatible atlas by comparing their union. Moreover, for any two arbitrary good atlas $\{U_i\}, \{U_j'\}$, one can find a good atlas compatible to both of them by shrinking their union.*

By definition, one has an inclusion $\iota : \ W^{k,p}(M) \hookrightarrow \bigoplus_i W^{k,p}(\mathbb{R}^n)$. Also ι is of closed image because one can find a projection $\pi : \ \bigoplus W^{k,p}(\mathbb{R}^n) \longrightarrow W^{k,p}(M)$ with $\pi \circ \iota = \mathrm{Id}$. In fact, let $\tilde{\psi}_i$ be functions supported in U_i that equal 1 in the support of ψ_i, then

$$\pi : \ g \mapsto \sum \tilde{\psi}_i.(g \circ \varphi_i)$$

works. The continuity of π follows from straight-forward calculations.

The closedness of image of ι is equivalent to the fact that $W^{k,p}(M)$ is complete.

Remark 7. *Although ι preserves the norm of $W^{k,p}(M)$ and has a right-inverse, it is far from being an isomorphism (it is not surjective). Each component of an element in the image of ι tends to 0 on the boundary of V_i (take $k \gg 1$ then everyone is continuous by Sobolev embedding, there is no subtlety in what we mean by "tends to 0"). [**?**, page 54] seems to claim that ι is an isomorphism and apply Theorem 8 repeatedly to deduce Theorem 7 for Sobolev spaces on manifold, then the Sobolev embedding $W^{k,p} \hookrightarrow C^l(M)$ and Kondrachov's theorem.*

The above results are true and the correction is not difficult (use Theorem 9).

From the remark, one has

Theorem 11 (Interpolation of Sobolev spaces on manifold)**.** *Theorem 7 holds for Sobolev spaces $W^{k,p}(M)$ on compact manifold M.*

4 Sobolev spaces on compact manifold with boundary

In this part, we will define the Sobolev spaces $W^{k,p}(M/\mathcal{A})$ where $k \in \mathbb{R}, p \in (1,\infty)$ and M is manifolds with boundary and $\mathcal{A}$ is union of connected components of ∂M the boundary of M. These spaces contain $W^{k,p}(M)$ "functions" who vanish on A. The appearance of the parameter $\mathcal{A}$ is because we will later take $M = M' \times [0,T]$ where M' is a manifold without boundary where we want to solve heat equation, and the natural $\mathcal{A}$ would be $\partial M \times \{0\}$. We also want that when the new definition coincides with the case of no boundary when $\mathcal{A} = \emptyset$

Suppose that we already define the Sobolev spaces on $X \times Y^+$ where $X = \mathbb{R}^n$ and $Y^+ = \mathbb{R}_{\geq 0}$, that is the space $W^{k,p}(X \times Y^+) = W^{k,p}(X \times Y^+/\emptyset)$ and $W^{k,p}(X \times Y^+, X \times \{0\})$. Then then we define the space $W^{k,p}(M/\mathcal{A})$ in analog of Definition 2 as follows

Definition 3. *1. The **Sobolev spaces** $W^{k,p}(M/\mathcal{A})$ where A is a connected component of ∂M is defined as*

$$W^{k,p}(M/\mathcal{A}) := \left\{ f \in \mathcal{S}(M)^* : \ (\psi_i f) \circ \varphi_i^{-1} \in W^{k,p}(R_i/\mathcal{A}_i) \right\}$$

where $\mathcal{A}_i = \varphi_i(U_i \cap \mathcal{A})$ and R_i is the Euclidean space containing U_i, that is either $\mathbb{R}^{n+1}$ when $\mathcal{A}_i = \emptyset$ or $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ when $\mathcal{A}_i \subset \mathbb{R}^n \times \{0\}$. The norm is given by

$$\|f\|_{W^{k,p}} = \sum_i \|(\psi_i f) \circ \varphi_i^{-1}\|_{W^{k,p}(R_i/\mathcal{A}_i)}$$

2. As before, weighted Sobolev spaces can be defined when M has a foliation structure compatible with its boundary.

The fact that different good atlas and different partition of unity defines the same space $W^{k,p}(M/\mathcal{A})$ (as a subset of $\mathcal{S}^*(M)$) with equivalents norm comes from the following lemma, which is just a formulation of arguments in the case of no boundary. Its proof follows from 2 remarks: On reduces the lemma, by interpolation inequality, to the case k is a multiple of σ and use the criteria in Theorem 2. Then in this case, use the boundedness of derivative of the transition map.

Lemma 12. *Let $(U, \mathcal{A}_U)$ and $(V, \mathcal{A}_V)$ be subsets of $(X \times Y^+, X \times \{0\})$ and $\varphi_{VU} : (U, \mathcal{A}_U) \longrightarrow (V, \mathcal{A}_V)$ being a diffeomorphism mapping $\mathcal{A}_U \subset \partial U$ to $\mathcal{A}_V \subset \partial V$ bijectively and of bounded derivatives. Let $0 \leq \psi \leq 1$ be a smooth function compactly supported in U. Then the linear mapping T : $\mathcal{S}^*(X \times Y^+/X \times \{0\}) \longrightarrow \mathcal{S}^*(X \times Y^+/X \times \{0\})$: $f \longrightarrow \psi.(f \circ \varphi_{VU}^{-1})$ extends to a bounded operator from $W^{k,p}(U/\mathcal{A}_U) \longrightarrow W^{k,p}(V, \mathcal{A}_V)$.*

The Sobolev spaces on half-plan $X \times Y^+$ therefore has to have the following properties: they have to be Banach spaces, satisfying Interpolation theorem 7 and having $\mathcal{S}(M)$ as a dense subspace. We will sketch rapidly the (well known) ideas to define Sobolev spaces on half-plan and the trace operator in the next sections.

4.1 Sobolev spaces on half-plan

In this section, the Sobolev spaces on $X \times Y$ or $X \times Y^+$ are defined with weight $(\sigma_1, \ldots, \sigma_n, \rho)$ and $\sigma := \mathrm{lcm}(\sigma_1, \ldots, \sigma_n, \rho)$.

4.1.1 Smooth extensions

Let $\mathcal{S}(X \times Y^+)$ denote the space of smooth, rapidly decreasing functions (and all of their derivatives) on $X \times Y^+$ and $\mathcal{S}(X \times Y^+/0)$ denotes the subspace of functions who vanish, together with all their derivatives, at $X \times \{0\}$. Idem for $\mathcal{S}(X \times Y^-)$ and $\mathcal{S}(X \times Y^-/0)$. The following exact sequence is obvious and the arrows are continuous under Schwartz topology.

$$0 \longrightarrow \mathcal{S}(X \times Y^-/0) \xrightarrow{Z_-} \mathcal{S}(X \times Y) \xrightarrow{C_+} \mathcal{S}(X \times Y^+) \longrightarrow 0 \tag{5}$$

where Z_- be the extension by 0 and C_+ be the cut-off operator.

It is however not obvious that the sequence in (5) splits. Algebraically this is equivalent to the fact that C_+ admits a retraction, noted E_+ since it is in fact an extension to the negative half-plan, which is continuous under Schwartz topology. The construction of E_+ is as follows

$$
\begin{aligned}
E_+ : \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y) \\
f &\longmapsto \left((x,y) \longmapsto \begin{cases} f(x,y), & \text{if } y \geq 0 \\ \int_0^\infty \varphi(\lambda) f(x, -\lambda y) d\lambda, & \text{if } y < 0 \end{cases} \right)
\end{aligned}
$$

This pushes the difficult part to the construction of φ, which is resolved by the following lemma.

Lemma 13. *There exists a smooth function $\varphi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ such that $\int_0^{+\infty} x^n |\varphi(x)| dx < \infty \quad \forall n \in \mathbb{Z}$ and*

$$\int_0^{+\infty} x^n \varphi(x) dx = (-1)^n \quad \forall n \in \mathbb{Z} \setminus \{0\}$$

Moreover, $\varphi(\frac{1}{x}) = -x\varphi(x)$ for all $x > 0$.

In fact, the function

$$\varphi(x) = \frac{e^4}{\pi} . \frac{e^{-(x^{1/4}+x^{-1/4})} \sin(x^{1/4} - x^{-1/4})}{1+x}$$

works. The continuity of operator E_+ comes from these properties of φ and basic justification of Lebesgue's Dominated convergence. The projection R_- of Z_- in the sequence (5) is constructed algebraically:

$$
\begin{aligned}
R_- \ \mathcal{S}(X \times Y) &\longrightarrow \mathcal{S}(X \times Y^-/0) \\
f &\longmapsto f - E_+ C_+ f
\end{aligned}
$$

which is also continuous in Schwartz topology. To resume, one has the split exact sequence

$$0 \longrightarrow \mathcal{S}(X \times Y^-/0) \underset{R_-}{\overset{Z_-}{\rightleftarrows}} \mathcal{S}(X \times Y) \underset{E_+}{\overset{C_+}{\rightleftarrows}} \mathcal{S}(X \times Y^+) \longrightarrow 0 \tag{6}$$

idem for $\mathcal{S}(X \times Y^+/0$ and $\mathcal{S}(X \times Y^-)$ and the operators Z_+, C_-, E_- and R_+.

Also, note that

$$\langle E_+ f, g \rangle = \langle f, R_+ g \rangle \tag{7}$$

where the first coupling is on $\mathcal{S}(X \times Y) \times \mathcal{S}(X \times Y)$ and the second is on $\mathcal{S}(X \times Y^+) \times \mathcal{S}(X \times Y^+/0)$.

Remark 8. *1. The two couplings satisfy* $\langle D^\alpha u, v\rangle = (-1)^{|\alpha|}\langle u, D^\alpha v\rangle$.

2. *The second coupling give two natural identifications*

$$\mathcal{S}(X\times Y^+/0) \hookrightarrow \mathcal{S}^*(X\times Y^+), \quad \mathcal{S}(X\times Y^+) \hookrightarrow \mathcal{S}^*(X\times Y^+/0)$$

while the first one gives $\mathcal{S}(X\times Y) \hookrightarrow \mathcal{S}^*(X\times Y)$.

3. (7) *shows that* E_+ *and* R_+ *are adjoint, strictly speaking* E_+ *is the restriction of* R_+^*, *that is*

$$\begin{tikzcd}
\mathcal{S}(X\times Y^+) \arrow[r, "E_+"] \arrow[d, hook] & \mathcal{S}(X\times Y) \arrow[d, hook] \\
\mathcal{S}^*(X\times Y^+/0) \arrow[r, "R_+^*"] & \mathcal{S}^*(X\times Y)
\end{tikzcd}$$

Also, one has $\langle C_- f, g\rangle = \langle f, Z_- g\rangle$, hence similarly, one has

$$\begin{tikzcd}
\mathcal{S}(X\times Y^-/0) \arrow[r, "Z_-"] \arrow[d, hook] & \mathcal{S}(X\times Y) \arrow[d, hook] \\
\mathcal{S}^*(X\times Y^-) \arrow[r, "C_-^*"] & \mathcal{S}^*(X\times Y)
\end{tikzcd}$$

To resume, one can extend the sequence in (5) to the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}(X\times Y^-/0) \arrow[r, shift left, "Z_-"] \arrow[d, hook] & \mathcal{S}(X\times Y) \arrow[l, shift left, "R_-"] \arrow[r, shift left, "C_+"] \arrow[d, hook] & \mathcal{S}(X\times Y^+) \arrow[l, shift left, "E_+"] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{S}^*(X\times Y^-) \arrow[r, shift left, "C_-^*"] & \mathcal{S}^*(X\times Y) \arrow[l, shift left, "E_-^*"] \arrow[r, shift left, "Z_+^*"] & \mathcal{S}^*(X\times Y^+/0) \arrow[l, shift left, "R_+^*"] \arrow[r] & 0
\end{tikzcd} \tag{8}$$

We will define Sobolev spaces $W^{k,p}(X\times Y^-/0)$ and $W^{k,p}(X\times Y^+)$ so that the Sobolev spaces form an intermediate row in diagram, since the center cell $\mathcal{S}(X\times Y) \subset W^{k,p}(X\times Y) \subset \mathcal{S}^*(X\times Y)$ is already defined, there is only one natural way to do this.

Definition 4. *1. The **Sobolev space upper on half-plan** is*

$$W^{k,p}(X\times Y^+) := \left\{ f \in \mathcal{S}^*(X\times Y^+/0) :\ \exists g \in W^{k,p}(X\times Y), f = Z_+^* g \right\}$$

with norm $\|f\|_{W^{k,p}(X\times Y^+)} = \inf_g \|g\|_{W^{k,p}(X\times Y)}$.

2. *The* ***Sobolev space on lower half-plan with vanishing trace***

$$W^{k,p}(X\times Y^-/0) := \Big\{ f\in\mathcal{S}^*(X\times Y^-):\ C_-^* f\in W^{k,p}(X\times Y)\Big\}$$

with the induced norm $\|f\|_{W^{k,p}(X\times Y^-/0)} := \|C_-^* f\|_{W^{k,p}(X\times Y)}$.

Remark 9. *1. In other words,* $W^{k,p}(X\times Y^-/0) = {C_-^*}^{-1}(W^{k,p}(X\times Y))$ *and* $W^{k,p}(X\times Y^+) = Z_+^*(W^{k,p}(X\times Y))$ *and they are given by the induced norm and the quotient norm of* $W^{k,p}(X\times Y)$ *respectively. The operator* C_-^* *and* Z_+^* *are by definition bounded under Sobolev norm.*

2. *The topology of* $W^{k,p}(X\times Y)$ *being finer than the induced of weak-* topology from* $\mathcal{S}^*(X\times Y)$*, the restricted operator* $Z_+^*\big|_{W^{k,p}} : W^{k,p}(X\times Y)\longrightarrow \mathcal{S}^*(X\times Y^+/0)$ *is continuous, hence* $\ker Z_+^*\big|_{W^{k,p}}\subset W^{k,p}(X\times Y)$*, which is also the image by* C_-^* *of* $W^{k,p}(X\times Y^-/0)$*, is a closed subspace of the Banach space* $W^{k,p}(X\times Y)$*. Therefore* $W^{k,p}(X\times Y^-/0)$ *and* $W^{k,p}(X\times Y^+)$ *are Banach spaces.*

3. *Idem for* $W^{k,p}(X\times Y^+/0)$ *and* $W^{k,p}(X\times Y^-)$.

Theorem 14. *1. For all* $k\in\mathbb{R}$ *and* $p\in(1,\infty)$*, the three lines of the following diagram are split-exact and the arrows of the second lines are bounded operators under Sobolev norms.*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}(X\times Y^-/0) \arrow[r, shift left, "Z_-"] \arrow[d, hook] & \mathcal{S}(X\times Y) \arrow[l, shift left, "R_-"] \arrow[r, shift left, "C_+"] \arrow[d, hook] & \mathcal{S}(X\times Y^+) \arrow[l, shift left, "E_+"] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & W^{k,p}(X\times Y^-/0) \arrow[r, shift left, "C_-^*"] \arrow[d, hook] & W^{k,p}(X\times Y) \arrow[l, shift left, "E_-^*"] \arrow[r, shift left, "Z_+^*"] \arrow[d, hook] & W^{k,p}(X\times Y^+) \arrow[l, shift left, "R_+^*"] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{S}^*(X\times Y^-) \arrow[r, shift left, "C_-^*"] & \mathcal{S}^*(X\times Y) \arrow[l, shift left, "E_-^*"] \arrow[r, shift left, "Z_+^*"] & \mathcal{S}^*(X\times Y^+/0) \arrow[l, shift left, "R_+^*"] \arrow[r] & 0
\end{tikzcd} \tag{9}$$

2. *The subspaces* $\mathcal{S}(X\times Y^-/0)$ *and* $\mathcal{S}(X\times Y^+)$ *are dense in* $W^{k,p}(X\times Y^0/0)$ *and* $W^{k,p}(X\times Y^+)$ *respectively.*

3. *Interpolation theorem 7 holds for* $W^{k,p}(X\times Y^-/0)$ *and* $W^{k,p}(X\times Y^+)$.

Proof. The commutativity of the diagram is purely algebraic. The continuity of C_-^* and Z_+^* in the $W^{k,p}$-row follows from the definition of norms in this row.

The only non-trivial part is the continuity of E_-^* and R_+^* in the $W^{k,p}$-row, and it suffices to prove that $C_-^* E_-^*$ and $R_+^* Z_+^*$ are bounded as automorphism of $W^{k,p}(X \times Y)$. This follows from direct computation of these norm in the case $\sigma \mid k \in \mathbb{R}$ and interpolation inequality (Theorem 5) for intermediate k.

Once the continuity of E_-^* and R_+^* is established, the density of $\mathcal{S}(X \times Y^-/0)$ follows straight-forwardly and we see that $W^{k,p}(X \times Y^-/0)$ and $W^{l,p}(X \times Y^-/0)$ are interpolatable (the two spaces share a dense subspace). Theorem 9 applies and shows that Theorem 7 holds for $W^{k,p}(X \times Y^-/0)$.

Idem for the side of $\mathcal{S}(X \times Y^+) \subset W^{k,p}(X \times Y^+)$. □

Remark 10. *By dualizing the diagram* (9) *and using the fact that the dual space of $W^{k,p}(X \times Y)$ is $W^{-k,p'}(X \times Y)$, one can prove that the dual space of $W^{k,p}(X \times Y^+)$ is $W^{-k,p'}(X \times Y^+/0)$.*

4.1.2 Functoriality of D_x and equivalent definitions

The following discussion appeared as 4 lemmas in [**?**, page 38-42] in the proof of Vanishing trace theorem 16. The reason of this paragraph is that I do not want to consider these results as technical details, and I think these ideas can be presented without much computation.

Note that the weight $W(\xi,\eta) = \left(1 + \xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n} + \eta^{2\rho}\right)^{1/2\sigma}$ is comparable to $W(\xi) + W(\eta)$ where

$$W(\xi) = \left(1 + \xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n}\right)^{1/2\sigma}, \quad W(\eta) = \left(1 + \eta^{2\rho}\right)^{1/2\sigma}$$

and also $W^k(\xi,\eta)$ is comparable to $W(\xi)^k + W(\eta)^k$. Hence $W(D_x)^l : W^{k,p}(X \times Y) \longrightarrow W^{k-l,p}(X \times Y)$ is a bounded operator.

The vertical arrows in the following diagram are the vertical arrows of (9). The dashed horizontal arrow indicates that it is established only in the center cells $W^{k,p}(X \times Y) \longrightarrow W^{k-l,p}(X \times Y)$.

$$\begin{tikzcd} & \mathcal{S}-\text{row} \arrow[dl] \arrow[dr] & \\ W^{k,p}-\text{row} \arrow[rr, dashed, "W(D_x)^l"] \arrow[dr] & & W^{k-l,p}-\text{row} \arrow[dl] \\ & \mathcal{S}^*-\text{row} & \end{tikzcd} \tag{10}$$

We will see that the dashed arrow can be extended to a full arrow, that is 3 arrows between the $W^{k,p}$-row and $W^{k-l,p}$-row that are compatible with the diagram (9).

One can construct $W(D_x)^l$ arrows from $W^{k,p}(X\times Y^-/0) \longrightarrow W^{k-l,p}(X\times Y^-/0)$ and $W^{k,p}(X\times Y^+) \longrightarrow W^{k-l,p}(X\times Y^+)$ as adjoint of $W(D_x)^l$ on S(X× Y^+) and $\mathcal{S}(X\times Y^+/0)$. They are by definition continuous on the weak-* topology. It is easy to see that if we can prove that these two $W(D_x)^l$ arrows commute with C^*_-, E^*_- and Z^*_+, R^*_+ on $W^{k,p}$-row and $W^{k-l,p}$-row, then by the continuity of the $W(D_x)^l$ arrow from $W^{k,p}(X\times Y) \longrightarrow W^{k-l,p}(X\times Y)$, these $W(D_x)^l$ arrows are bounded in $W^{k,p}$ norm.

The two new $W(D_x)^l$ arrows commute with all "$\longrightarrow$" arrows in the $W^{k,p}$-row of (9), i.e. C^*_- and Z^*_+, since for smooth functions, D_x commutes with Z_- (extension by 0) and C_+ (cut-off).

The fact that $W(D_x)^l$ commutes with the "$\longleftarrow$" arrows, i.e. E^*_- and R^*_+ because the way we define extension operators E_+ and R_- for smooth functions that satisfy

$$\begin{array}{ccc} \mathcal{S}(X\times Y^+) & \xrightarrow{E_+} & \mathcal{S}(X\times Y) \\ \downarrow{\scriptstyle W(D_x)^l} & & \downarrow{\scriptstyle W(D_x)^l} \\ \mathcal{S}(X\times Y^+) & \xrightarrow{E_+} & \mathcal{S}(X\times Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{S}(X\times Y) & \xrightarrow{R_-} & \mathcal{S}(X\times Y^-/0) \\ \downarrow{\scriptstyle W(D_x)^l} & & \downarrow{\scriptstyle W(D_x)^l} \\ \mathcal{S}(X\times Y) & \xrightarrow{R_-} & \mathcal{S}(X\times Y^-/0) \end{array}$$

Remark 11. *There is no functoriality of D_y since for $y<0$*

$$D^l_y E_+ f(x,y) = \int_0^\infty (-\lambda)^l \varphi(\lambda) D^l_y f(x,-\lambda y)d\lambda \neq E_+ D^l_y f(x,y)$$

meaning that the D_y does not commute with E_+.

However $D^l_y E_+ f \in L^p(X\times Y)$ if and only if $E_+ D^l_y f \in L^p(X\times Y)$ if and only if $D^l_y f \in L^p(X\times Y^+)$. Moreover the 3 L^p norms are equivalent.

The density of $\mathcal{S}(X\times Y^-/0)$ and $\mathcal{S}(X\times Y^+)$ in the corresponding $W^{k,p}$ shows that the new $W^{k,p}$ spaces can also be defined by density using the $W^{k,p}$-norm of the extension (Z_- and E_+ respectively) from half-plan to the whole plan. By the continuity of R^*_+ in the second row of (9) when $k=0$, one see that the L^p-norms of the extensions by Z_- and E_+ are equivalent to the L^p norm on the half-plan. Therefore, one has the following analog of Theorem 2.

Theorem 15. *Given $k>0$ and $\sigma \mid k$,*

1. *If $f \in \mathcal{S}^*(X\times Y^+/0)$ then*
   (a) *$f \in W^{k,p}(X\times Y^+)$ if and only if $D^\alpha_x D^\beta_y f \in L^p(X\times Y^+)$ for $\|(\alpha,\beta)\| \leq k$.*

(b) $f \in W^{-k,p}(X \times Y^+)$ *if and only if there exists* $g_{\alpha\beta} \in L^p(X \times Y^+)$ *such that* $f = \sum_{\|(\alpha,\beta)\| \leq k} D_x^\alpha D_y^\beta g_{\alpha\beta}$.

2. *If* $f \in \mathcal{S}^*(X \times Y^+)$ *then*

   (a) $f \in W^{k,p}(X \times Y^+/0)$ *if and only if* $D_x^\alpha D_y^\beta f \in L^p(X \times Y^+)$ *for* $\|(\alpha, \beta)\| \leq k$.

   (b) $f \in W^{-k,p}(X \times Y^+/0)$ *if and only if there exists* $g_{\alpha\beta} \in L^p(X \times Y^+)$ *such that* $f = \sum_{\|(\alpha,\beta)\| \leq k} D_x^\alpha D_y^\beta g_{\alpha\beta}$.

4.2 Trace theorems

To make the notation more intuitive, we abusively denote the horizontal arrows in the $W^{k,p}$-row and the $\mathcal{S}^*$-row by their corresponding arrows in the $\mathcal{S}$-row (i.e. their restriction on the space of smooth functions), that is we will use Z_-, C_+, R_-, E_+ instead of $C_-^*, Z_+^*, E_-^*, R_+^*$.

The goal of this section is to define the restriction of a function $f \in W^{k,p}(X \times Y^+)$ on $X \times \{0\}$. The pointwise restriction of f does not make sense because f is only defined up to a negligible set (i.e. of Lebesgue measure 0). The strategy is to take a sequence $f_n \in \mathcal{S}(X \times Y^+)$ $W^{k,p}$-converging to f and to see if $\{f_n\big|_{X \times \{0\}}\}$ converges in $L^p(X \times \{0\})$. If it does one calls the limit trace of f on $X \times \{0\}$. Theorem 16, Example 12 and Theorem 18 show that one should expect

- high regularity of f, i.e. k large enough, so that the limit exists,
- a drop of regularity of the restriction.

From diagram (9) and its opposite version (with all + sign replaced by − and vice versa), there is a natural inclusion $\iota :\ W^{k,p}(X \times Y^+/0)$ to $W^{k,p}(X \times Y^+)$, by, for smooth functions, first extending by zero, then cut-off

$$\begin{tikzcd}
W^{k,p}(X \times Y^+/0) \arrow[rr, hook, "\iota"] \arrow[dr, "Z_+"'] & & W^{k,p}(X \times Y^+) \\
& W^{k,p}(X \times Y) \arrow[ur, "C_+"'] &
\end{tikzcd}$$

Theorem 16 (Vanishing trace)**.** *If* $p \in (1, +\infty)$ *and* $-1 + \frac{1}{p} < \rho\frac{k}{\sigma} < \frac{1}{p}$ *then* ι *is an isomorphic*

Proof. Define

$$\begin{aligned} M_+(\lambda):\ \mathcal{S}(X\times Y^+) &\longrightarrow \mathcal{S}(X\times Y^+)\\ f(x,y) &\longmapsto f(x,\lambda y)\end{aligned}$$

Since $\langle M_+(\lambda)f,g\rangle = \langle f, N_+(\lambda)g\rangle$ for all $f\in\mathcal{S}(X\times Y^+), g\in\mathcal{S}(X\times Y^+/0)$ and $\lambda>0$ where $N_+(\lambda)g(x,y) := \lambda^{-1}g(x,\lambda^{-1}y)$, one sees that $M_+(\lambda)$ extends to $\mathcal{S}^*(X\times Y^+/0)\longrightarrow\mathcal{S}^*(X\times Y^+/0)$ and that one extension of it is $N_+^*(\lambda)$ the adjoint of $N_+(\lambda)$:

$$\begin{tikzcd} \mathcal{S}(X\times Y^+) \arrow[r, "M_+(\lambda)"] \arrow[d, hook] & \mathcal{S}(X\times Y^+) \arrow[d, hook] \\ \mathcal{S}^*(X\times Y^+/0) \arrow[r, "N_+^*(\lambda)"] & \mathcal{S}^*(X\times Y^+/0) \end{tikzcd}$$

We abusively call $N_+^*(\lambda)$ by $M_+(\lambda)$. We will let $\lambda\to+\infty$, the operator $M_+(\lambda)$ intuitively "shrinks" to the boundary $X\times\{0\}$.

Lemma 17. *For $k\geq 0,\lambda\geq 1$, $M_+(\lambda): W^{k,p}(X\times Y^+)\longrightarrow W^{k,p}(X\times Y^+)$ is bounded and*

$$\|M_+(\lambda)f\|_{W^{k,p}(X\times Y^+)} \leq C\lambda^{\frac{\rho k}{\sigma}-\frac{1}{p}}\|f\|_{W^{k,p}(X\times Y^+)}$$

where C does not depend on λ.

The proof of the Lemma 17 is straightforward: it suffices to prove the boundedness in the case $\sigma\mid k$ an use interpolation inequality 5, also one can suppose that $f\in\mathcal{S}(X\times Y^+)$. Note that $(\frac{\partial}{\partial y})^l M_+(\lambda) = \lambda^l M_+(\lambda)(\frac{\partial}{\partial y})^l$ while $\frac{\partial}{\partial x}$ commutes with $M_+(\lambda)$, hence in general $|D^\alpha_{(x,y)}M_+(\lambda)f|\leq\lambda^{k\rho/\sigma}|D^\alpha_{(x,y)}f|$ for all $\|\alpha\|\leq k,\lambda\geq 1$. The $-\frac{1}{p}$ in the exponent of λ is due to: $\|M_+(\lambda)g\|_{L^p} = \lambda^{-1/p}\|g\|_{L^p}$.

Back to Theorem 16, let $f\in\mathcal{S}(X\times Y^+)$ and define $\tilde{M}(\lambda)f$ to be f on $X\times Y^+$ and $M_-(\lambda)C_-E_+f$ on $X\times Y^-$, then $\tilde{M}(\lambda)f\in W^{\sigma/\rho,p}(X\times Y)$. Note that $D_y\tilde{M}(\lambda)f$ is not continuous at $X\times\{0\}$ but is still in $L^p(X\times Y)$ because f and $M_-(\lambda)C_-E_+f$ agrees on $X\times\{0\}$. Suppose we can prove that as $\lambda\to+\infty$ the sequence $\tilde{M}(\lambda)f$ converges to $\tilde{M}f$ in $W^{k,p}(X\times Y)$ then $C_-\tilde{M}f = \lim_{\lambda\to+\infty}M_-(\lambda)C_-E_+f = 0$. One obtains, by exactness of the second row of diagram (9), existence of a $g\in W^{k,p}(X\times Y^+/0)$ such that $\tilde{M}f = Z_+g$. Moreover, since $C_+\tilde{M}(\lambda)f = f$ for all $\lambda>0$, one has $C_+\tilde{M}f = f$, hence $\iota g = C_+Z_+g = C_+\tilde{M}f = f$.

It remains to prove the existence of such $\tilde{M}f$. By Lemma 17 and the fact that all $\tilde{M}(\lambda)f$ are the same on $X \times Y^+$, one has

$$\|\tilde{M}(\lambda)f - \tilde{M}(2\lambda)f\|_{W^{k,p}(X\times Y)} \leq 2C\lambda^{\frac{\rho k}{\sigma}-\frac{1}{p}}\|f\|_{W^{k,p}(X\times Y^+)}$$

Therefore if $\frac{\rho k}{\sigma} < \frac{1}{p}$, the sequence $\tilde{M}(2^n)f$ converge in $W^{k,p}(X \times Y)$ to $\tilde{M}f$. $\square$

Remark 12. *If $\rho = \sigma_i$ then $\sigma = 1$, take $k = 0$ then the Theorem 16 claims that $\mathcal{S}(X \times Y^+/0)$ is dense in $L^P(X \times Y^+) \supset \mathcal{S}(X \times Y^+)$, or equivalently any smooth function $f \in \mathcal{S}(X \times Y^+)$ not necessarily vanishes on $X \times \{0\}$ can be L^p-approximated by smooth functions with all derivative vanishes on $X \times \{0\}$. This means that one cannot define any notion of trace on $X \times \{0\}$ that varies continuously under the L^p norm.*

In case of high regularity $\frac{\rho k}{\sigma} > \frac{1}{p}$, one can define a meaningful notion of trace.

Theorem 18 (Well-defined trace)**.** *If $\frac{\rho k}{\sigma} > \frac{1}{p}$ then the restriction map*

$$\begin{aligned} B : \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X) \\ f(x,y) &\longmapsto f(x,0) \end{aligned}$$

extends to a bounded operator, abusively noted by $B: \ W^{k,p}(X \times Y^+) \longrightarrow L^p(X)$.

Definition 5. *We call $\partial W^{k,p}(X \times Y^+) := W^{k,p}(X \times Y^+)/\ker B$ the **space of boundary value** of function in $W^{k,p}(X \times Y^+)$.*

Theorem 18 can be strengthen by remarking that if $\sigma := \operatorname{lcm}(\sigma_1, \ldots, \sigma_n, \rho) = \operatorname{lcm}(\sigma_1, \ldots, \sigma_n)$ and if $W(\xi)$ denotes the weight $(1 + \xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n})^{1/2\sigma}$ then B and $W(D_x)$ commute, i.e.

$$\begin{tikzcd} W^{k,p}(X \times Y^+) \arrow[r, "B"] \arrow[d, "W(D_x)^l"'] & L^p(X) \subset \mathcal{S}^*(X) \arrow[d, "W(D_x)^l"] \\ W^{k-l,p}(X \times Y^+) \arrow[r, "B"] & L^p(X) \end{tikzcd}$$

as long as $\frac{\rho(k-l)}{\sigma} > \frac{1}{p}$. Therefore, one has

Theorem 19 (Regularity of trace)**.** *If $0 \le l < k - \frac{\sigma}{\rho p}$ then the trace operator B in Theorem 18 actually of image in $W^{l,p}(X)$ and the operator*

$$B: \ W^{k,p}(X \times Y^+) \longrightarrow W^{l,p}(X)$$

is bounded.

Proof of Theorem 18. It suffices to prove that $\|Bf\|_{L^p(X)} \le C\|f\|_{W^{k,p}(X\times Y^+)}$ for all $f \in \mathcal{S}(X \times Y^+)$ and $1 \ge \frac{\rho k}{\sigma} > \frac{1}{p}$ (for higher k, embed in the $W^{k,p}$ smaller k). Define

$$\begin{aligned} T_v: \ \mathcal{S}(X \times Y^+) &\longrightarrow \mathcal{S}(X \times Y^+) \\ f &\longmapsto \left((x,y) \longmapsto \frac{1}{v}\int_0^v f(x, y+w)dw\right) \end{aligned}$$

for $v > 0$. One can check that T_v extends to a bounded operator $T_v : W^{k,p}(X\times Y^+) \longrightarrow W^{k,p}(X\times Y^+)$ for all $k \ge 0$ and that \[$\begin{cases} \|D_yT_vf\|_{L^p(X\times Y^+)} \le Cv^{-1}\|f\|_{L^p(X\times Y^+)}, \\ \|D_yT_vf\|_{L^p(X\times Y^+)} \le C\|f\|_{W^{\sigma/\rho,p}(X\times Y^+)} \end{cases}$ \] hence by Interpolation inequality 5, one obtains for all $0 \le k \le \sigma/\rho$: $\|D_yT_vf\|_{L^p(X\times Y^+)} \le Cv^{\rho k/\sigma - 1}\|f\|_{W^{k,p}(X\times Y^+)}$ hence

$$\|D_y(T_{v/2} - T_vf)\|_{L^p(X\times Y^+)} \le Cv^{\rho k/\sigma - 1}\|f\|_{W^{k,p}(X\times Y^+)} \tag{11}$$

Similarly, one can prove that for all $0 \le k \le \sigma/\rho$: $\|(\mathrm{Id} - T_v)f\|_{L^p(X\times Y^+)} \le Cv^{\rho k/\sigma}\|f\|_{W^{k,p}(X\times Y^+)}$ therefore

$$\|(T_{v/2} - T_v)f\|_{L^p(X\times Y^+)} \le Cv^{\rho k/\sigma}\|f\|_{W^{k,p}(X\times Y^+)} \tag{12}$$

Moreover, using Hölder inequality and Fundamental theorem of calculus, one has: if $g \in \mathcal{S}(X \times Y^+)$ then

$$\|Bg\|_{L^p(X)} \le C\|g\|^{1/p'}_{L^p(X\times Y^+)}\|D_yg\|^{1/p}_{L^p(X\times Y^+)} \tag{13}$$

Substitute g by $(T_{v/2} - T_v)f$ in (13) then use apply (11) and (12), one has

$$\|B(T_{v/2} - T_v)\|_{L^p(X)} \le Cv^{\frac{\rho k}{\sigma} - \frac{1}{p}}\|f\|_{W^{k,p}(X\times Y^+)}$$

Therefore if $\frac{1}{p} < \frac{\rho k}{\sigma} \le 1$, the sequence $BT_{2^-n}f$ converges in $L^p(X)$ and the limit is of L^p-norm less than $C\|f\|_{W^{k,p}(X\times Y^+)}$. Since f is continuous, the limit is $f\big|_{X\times\{0\}}$. The theorem follows. $\square$

Remark 13. *The fact that the condition on l in Theorem 19 is an open condition explains why we topologize the space of boundary value $\partial W^{k,p}(X \times Y^+)$ by the quotient $W^{k,p}$-norm instead of any $W^{l,p}$-norm. On one hand, we have completeness for free. On the other hand, since $B:\ W^{k,p}(X \times Y^+) \longrightarrow L^p(X)$ factorizes to any $W^{l,p}(X)$, the quotient $W^{k,p}$-norm is stronger than any $W^{l,p}$-norm.*

In the proof of Theorem 16, we glue a function $f_+ \in \mathcal{S}(X \times Y^+)$ with $f_- \in \mathcal{S}(X \times Y^-)$ of the same value on $X \times \{0\}$ and the result is a function in $W^{\sigma/\rho,p}(X \times Y)$. This can be generalised as follow

Theorem 20 (Patching theorem)**.** *If $p \in (1, +\infty)$ and $\frac{1}{p} < \rho\frac{k}{\sigma} < 1 + \frac{1}{p}$, then given $f_+ \in W^{k,p}(X \times Y^+)$ and $f_- \in W^{k,p}(X \times Y^-)$ such that $Bf_+ = Bf_-$ in $L^p(X)$, one define $f \in L^p(X \times Y)$ such that $f = f_+$ on $X \times Y^+$ and $f = f_-$ on $X \times Y^-$. Then actually $f \in W^{k,p}(X \times Y)$.*

4.3 Trace operator on manifold

The following paragraph does not appear in [?] because of Remark 7.

To resume, we have defined Sobolev spaces on manifold with boundary as the space of currents whose cut-off restrictions on each chart are in $W^{k,p}$. Also we have defined trace operator of Sobolev spaces on half-plan in a vision to extend the notion to manifold.

Let $f \in W^{k,p}(M/\mathcal{A})$ and $\mathcal{B}$ be a connected component of ∂M. With the same notation as Definition 3, f gives the data of $f_i = (\psi_i f) \circ \varphi_i^{-1} \in W^{k,p}(R_i/\mathcal{A}_i)$ the cut-off restriction of f on each chart using a partition of unity $\{\psi_i\}_i$ subordinated to a good atlas $(U_i)_i$ of M, where R_i is an Euclidean space of the same dimension as M ($\mathcal{A}_i = \emptyset$), or a half-plan ($\mathcal{A}_i \subset \partial R_i$). Note that $U_i \cap \mathcal{B}$ is a good atlas of $\mathcal{B}$ and ψ_i is still a partition of unity subordinated to this atlas, therefore take $g_i \in W^{l,p}(\partial R_i)$ to be trace of f_i on the image of $\mathcal{B}$ of each chart. It remains to check that the data g_i correspond to a unique element $g \in W^{l,p}(\mathcal{B})$. Recall that we have the following diagram:

$$0 \longrightarrow W^{l,p}(\mathcal{B}) \underset{\pi}{\overset{\iota}{\rightleftarrows}} \bigoplus_i W^{l,p}(\partial R_i)$$

where ι admits a projection π given by the cut-off functions $\tilde{\psi}_i$ that we choose to be the same ones used for M. Hence to see that $(g_i)_i$ is in the image of ι, it suffices to check that $\iota \circ \pi((g_i)_i) = (g_i)_i$ which should be straightforward, since $\sum_i \tilde{\psi}_i \psi_i = 1$.

Now that we defined a trace operator $B: W^{k,p}(M) \longrightarrow L^p(\partial M)$ that factor through $W^{k,p}(M) \longrightarrow W^{l,p}(\partial M)$ for all $0 \leq l < k - \frac{\sigma}{\rho p}$, we can define the space of boundary value of function in $W^{k,p}(M)$ by

$$\partial W^{k,p}(M) := W^{k,p}(M)/\ker B$$

which has a finer topology than its image in any $W^{l,p}(\partial M)$ for $0 \leq l < k - \frac{\sigma}{\rho p}$.