

Global existence for nonlinear heat equation and harmonic maps between Riemannian manifolds

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Let M be a compact Riemannian manifold. We want to solve the following nonlinear heat equation where $F : M \rightarrow M' \subset B \subset V = \mathbb{R}^N$:

$$\frac{dF_t}{dt} = -\Delta F_t + \Gamma(F_t)(\nabla F_t)^2$$

We have proved that the solution exists in short-time and is smooth whenever it exists. We will now establish long-time existence using continuity method: we will show that if the solution exists on $[\alpha, \omega_n]$ where ω_n is an increasing sequence to ω , then the solution exists on $[\alpha, \omega]$. We then apply short-time existence to gain a small open interval where solution still exists. We then conclude that the solution exists globally on $[\alpha, +\infty)$ since this interval is connected.

The crucial step to prove that the solution can be extended on $[\alpha, \omega]$ is to uniformly bound all of its derivatives in time of evolution $[\alpha, \omega]$. These estimates will also be useful to justify the convergence of F_t in $C^\infty(M)$ to a smooth function F_∞ which will eventually be a harmonic map from M to M' .

Recall that we proved in Corollary ??, under the hypothesis of negative curvature, the boundedness of $\|F_t\|_{W^{2,2}(M)}$ by a constant C depending only on curvatures of M, M' and the initial total energies. Since $\frac{dF_t}{dt}$ relates to spatial derivatives of F by the nonlinear heat equation, it is easy to see that $\|F_t\|_{W^{2,2}(M \times [\tau, \tau + \delta])}$ is bounded by a constant independent of τ . We will denote $W^{k,p}(M \times [\beta, \gamma])$ by $W^{k,p}([\beta, \gamma])$.

Theorem 1 ($W^{2,2}$ -boundedness). *Suppose $\text{Riem}(M') \leq 0$. There exists a constant C depending only on δ , the metrics and initial total energies such that*

$$\|F\|_{W^{2,2}(\tau, \tau+\delta)} \leq C \quad \text{for all } \alpha \leq \tau < \omega - \delta.$$

Proof. Since

$$\|F\|_{W^{2,2}([\tau, \tau+\delta])}^2 \leq \int_{\tau}^{\tau+\delta} \|F_t\|_{W^{2,2}(M)}^2 dt + 2 \int_{\tau}^{\tau+\delta} \|\Delta F_t\|_{L^2}^2 dt + 2 \int_{\tau}^{\tau+\delta} \|\Gamma(F_t)(\nabla F_t)^2\|_{L^2}^2 dt$$

The first term and the second term are bounded by $C^2\delta$, the third one, since $\Gamma(F_t)$ is bounded, by $C^2\delta$ where C is a constant only depending on the metrics and initial total energies. \square

The estimates of higher derivatives of F will be established in the same strategy as the bootstrap: first in $W^{2,p}$ for all p then in $W^{k,p}$ for all k, p , then in C^∞ .

1 Estimate of higher derivatives.

Lemma 2 ($W^{2,p}$ -boundedness). *Suppose $\text{Riem}(M') \leq 0$. For all $p \in (1, +\infty)$, there exists a constant $C > 0$ depending only on δ , p , the metrics and initial energies such that for all $\alpha + \delta \leq \tau \leq \omega - \delta$:*

$$\|F\|_{W^{2,p}([\tau, \tau+\delta])} \leq C$$

Proof. Applying Gårding Inequality to the parabolic equation $AF = \Gamma(F)(\nabla F)^2$ where $A := \frac{\partial}{\partial t} + \Delta$ is the heat operator, one has

$$\|F\|_{W^{2,p}([\tau, \tau+\delta])} \leq C \left(\|\Gamma(F)(\nabla F)^2\|_{L^p([\tau-\frac{\delta}{3}, \tau+\delta])} + \|F\|_{W^{2,2}([\tau-\frac{\delta}{3}, \tau+\delta])} \right)$$

The second term of RHS is already bounded by applying Theorem 1 to $\frac{4\delta}{3}$. For the first term:

$$\|\Gamma(F)(\nabla F)^2\|_{L^p([\tau-\frac{\delta}{3}, \tau+\delta])} \leq C(M') \|\nabla F\|_{L^p([\tau-\frac{\delta}{3}, \tau+\delta])}^2 = C(M') \|e(F)\|_{L^p([\tau-\frac{\delta}{3}, \tau+\delta])}.$$

Recall that by Theorem ??, the potential density satisfies $\frac{de}{dt} + \Delta e - Ce \leq 0$ for certain constant C depending only on the metric of M . By Maximum principle (Theorem ??), one has $e \leq \psi_\tau$ where ψ_τ is the solution of

$$\begin{cases} \frac{d}{dt} \psi_\tau + \Delta \psi_\tau - C \psi_\tau = 0 \\ \psi_\tau|_{\tau-\frac{\delta}{2}} = e|_{\tau-\frac{\delta}{2}} \end{cases}$$

We apply Gårding Inequality again for ψ_τ and obtain

$$\|e(F)\|_{L^p([\tau-\frac{\delta}{3}, \tau+\delta])} \leq \|\psi_\tau\|_{L^p([\tau-\frac{\delta}{3}, \tau+\delta])} \leq C\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2}, \tau+\delta])}. \quad (1)$$

Now apply L^1 -Comparison Theorem ?? to ψ_τ , one has

$$\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2}, \tau+\delta])} \leq \int_0^{3\delta/2} \|\psi_\tau|_{\tau-\frac{\delta}{2}}\|_{L^1} e^{Bt} dt \leq \int_0^{3\delta/2} e^{Bt} dt \cdot \|e|_{\tau-\frac{\delta}{2}}\|_{L^1} \leq C. \quad (2)$$

The lemma follows from (1) and (2). \square

We can now estimate higher order derivatives.

Theorem 3 ($W^{k,p}$ -boundedness). *Suppose $\text{Riem}(M') \leq 0$. For all $p \in (1, +\infty)$ and $k < +\infty$, there exists C depending only on k, p , the metrics and initial energies such that*

$$\|F\|_{W^{k,p}([\tau, \tau+\delta])} \leq C$$

for all $\alpha + \delta \leq \tau \leq \omega - \delta$.

Proof. Applying Gårding Inequality to the equation $\frac{dF}{dt} + \Delta F_t = \Gamma(F)(\nabla F)^2$ then Regularity estimate for the quadratic term (Theorem ??), one has for $\epsilon \ll \delta$:

$$\begin{aligned} \|F\|_{W^{k,p}([\tau, \tau+\delta])} &\leq C_\epsilon \left(\|F\|_{W^{2,p}([\tau-\epsilon, \tau+\delta])} + \|\Gamma(F)(\nabla F)^2\|_{W^{k-2,p}([\tau-\epsilon, \tau+\delta])} \right) \\ &\leq C_\epsilon \left(1 + C \left(1 + \|F\|_{W^{s,q}([\tau-\epsilon, \tau+\delta])} \right)^{q/p} \right) \end{aligned}$$

as long as $k-1 < s$ and $\frac{1}{p} > \frac{k}{s} \cdot \frac{1}{q}$. Therefore if $\|F\|_{W^{s,q}([\tau, \tau+\delta])} \leq C(\delta, s, q)$ for all $\beta \leq \tau \leq \omega - \delta$ and $q \in (1, +\infty)$, we just proved that

$$\|F\|_{W^{k,p}([\tau, \tau+\delta])} \leq C(\epsilon, k, p)$$

for all $\begin{cases} \beta + \epsilon \leq \tau \leq \omega - \delta \\ k < s+1, p \in (1, +\infty) \end{cases}$ since $\|F\|_{W^{s,q}([\tau-\epsilon, \tau+\delta])} \leq 2C(\delta, s, q)$.

One can then conclude by induction on k , with step $\frac{1}{2}$, starting with $k=2$ and $\epsilon = \frac{\delta}{2}$ divided by 2 after each induction step. \square

2 Global existence for nonlinear heat equation.

Theorem 4 (Global existence). *Suppose $\text{Riem}(M') \leq 0$. The solution of nonlinear heat equation*

$$\frac{dF}{dt} = -\Delta F + \Gamma(F)(\nabla F)^2 \quad (3)$$

with smooth initial condition exists globally for all time $t > \alpha$.

Proof. Let F_n be a sequence of solution of (3) on $[\alpha, \omega_n]$ with ω_n increasing to ω then they coincide by uniqueness of the solution. As discussed in the beginning of this part, it is sufficient to prove that the solution extends to $[\alpha, \omega]$. Let F be the solution on $[\alpha, \omega)$ such that $F|_{[\alpha, \omega_n]} = F_n$, then by Theorem 3, for all $\tau \in [\alpha, \omega - \delta]$:

$$\|D_t^u D_x^v F\|_{L^\infty(M \times [\tau, \tau + \delta])} \leq C_{\text{Sobolev}} \|D_t^u D_x^v F\|_{W^{k,p}(M \times [\tau, \tau + \delta])} \leq C_{\text{Sobolev}} \cdot C(k, p, \delta)$$

where, if we choose k sufficiently large, C_{Sobolev} is the constant of Sobolev imbedding $W^{k,p}(M \times [0, \delta]) \hookrightarrow C(M \times [0, \delta])$ and $C(k, p, \delta)$ is the constant provided by Theorem 3.

So all partial derivatives of F are uniformly bounded on $[\alpha, \omega)$. This proves that F extends to a solution on $[\alpha, \omega]$. In fact $F|_\tau := F|_{M \times \{\tau\}}$ converges in $C^\infty(M)$ as $\tau \rightarrow \omega$, since

$$\|D^\alpha F|_\tau - D^\alpha F|_{\tau'}\|_{L^\infty} \leq \max_{\|\beta\|=\|\alpha\|+1} \|D^\beta F\|_{L^\infty} |\tau - \tau'|.$$

□

We have just proved the first part of the following theorem.

Theorem 5 (Eells-Sampson). *1. Let M, M' be compact Riemannian manifolds with $\text{Riem}(M') \leq 0$. Then for every smooth map $f_0 : M \rightarrow M' \subset B \subset \mathbb{R}^N$, the nonlinear heat equation*

$$\begin{cases} \frac{df_t}{dt} = \tau(f_t), & \text{for all } t \geq 0 \\ f|_{t=0} = f_0, \end{cases}$$

admits a globally defined smooth solution f_t . Moreover, all derivatives $D^\alpha f_t$ remain bounded as $t \rightarrow +\infty$.

2. For a suitable sequence $\{t_n\}$ increasing to $+\infty$ the sequence $\{f_{t_n}\}$ converges in $C^\infty(M)$ to a function f_∞ with $\tau(f_\infty) = 0$. Therefore any map $f_0 : M \rightarrow M'$ is homotopic to a harmonic map.

Proof. For any sequence $\{t_n\}$, one can extract from $\{f_{t_n}\}$, since their derivatives are uniformly bounded, a subsequence $\{f_{t_{n_i}}\}$ converging in $C^k(M, \mathbb{R}^N)$. By a diagonal argument, one can extract from any sequence $\{f_{t_n}\}$ a subsequence converging in $C^\infty(M, \mathbb{R}^N)$ to f_∞ . Abusively denote this subsequence by $\{f_{t_n}\}$, by Theorem ??

$$\lim_{n \rightarrow \infty} K(f_{t_n}) = \lim_{n \rightarrow \infty} \int_M |\tau(f_{t_n})|^2 = 0$$

Therefore $\tau(f_{t_n}) \rightarrow 0$ in $L^2(M)^{\oplus N}$. But also $\tau(f_{t_n}) \rightarrow \tau(f_\infty)$ in $C^\infty(M, \mathbb{R}^N)$, one has $\tau(f_\infty) = 0$. \square