Calabi-Yau theorem

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1 Calabi conjecture

We start with the following fact (which is an exercise in Daniel Huybrechts, *Complex geometry - an introduction*)

Proposition 0.1 (Ricci curvature and first Chern class)**.** *Let* (X, g) *be a compact Kähler manifold, then* $iRic(X, g)$ *is the curvature of the Chern connection on the canonical bundle* K_X*. In other words,* $Ric(X, g) \in -2\pi c_1(K_X)$ *where* $c_1(K_X)$ *is the first Chern class of* K_X*.*

The Calabi conjecture asked whether there exists for each form $R \in c_1(K_X)$ a metric g' with $Ric(X, g') = R$. We prefer to work with the fundamental form instead of the metric g as the former is antisymmetric and its derivative is hence easy to define.

Definition 1. *The quadruple* (h, g, ω, J) *is said to be* <u>*compatible*</u> *if* $g \circ J = g$ *and* $\omega(a, b) = g(Ja, b)$ *and* $h = g - i\omega$*.*

Remark 1. *1. When* J *is fixed, one of* h, g, ω *that is invariant by* J *determines the two others.*

2. *For a compatible quadruple, the condition $\nabla J = 0$ is equivalent to $d\omega = 0$. The fundamental form ω that satisfies $d\omega = 0$ is called a* <u>*Kähler form*</u>.

2 Reduction to local charts, Yau theorem

h, g, ω in local coordinates. We note by $h_{i\bar{j}} = h(\partial_{x_i}, \partial_{x_j}) = 2g_{\mathbb{C}}(\partial_{z_i}, \partial_{z_j})$. By straightforward calculation one has

$$\begin{aligned}\omega &= -\frac{1}{2} Im h_{i\bar{j}}(dx^i \wedge dx^j + dy^i \wedge dy^j) + Re h_{i\bar{j}} dx^i \wedge dy^j \\ &= \frac{i}{2} h_{i\bar{j}} dz^i \wedge d\bar{z^j}\end{aligned}$$

and the condition $d\omega = 0$ is equivalent to

$$\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial h_{k\bar{j}}}{\partial z_i}$$

We also note by $h^{i\bar{j}}$ the inverse transposed of $h_{i\bar{j}}$, i.e. $h^{i\bar{j}} h_{k\bar{j}} = \delta^k_j$

Definition 2. *Let X be an almost complex manifold (manifold with an almost complex structure). Then $d : \wedge^n T^*X \longrightarrow \wedge^{n+1} T^*X$ sends $\wedge^{p,q} T^*M$ to $\wedge^{p+1,q} T^*M \oplus \wedge^{p,q+1} T^*M$. We denote by ∂ and $\bar{\partial}$ the component of d in $\wedge^{p+1,q} T^*M$ and $\wedge^{p,q+1} T^*M$ respectively.*

It would be convenient to define $d^c = i(\bar{\partial} - \partial)$ then obviously $dd^c = 2i\partial\bar{\partial}$.

The Ricci curvature. The Ricci curvature is given by

$$Ric_\omega = -\frac{1}{2} dd^c \log\det(h_{i\bar{j}})$$

dd^c lemma . We then can state the dd^c lemma

Lemma 1. *Let α be a real, (1,1)-form on a compact Kähler manifold M. Then α is d -exact if and only if there exists $\eta \in C^\infty(M)$ globally defined such that $\alpha = dd^c\eta$.*

Yau's theorem. The dd^c lemma tells us every form $R \in c_1(K_X)$ is of form $Ric_\omega + dd^c\eta$. If one varies the Hermitian product $h_{i\bar{j}}$ to $h_{i\bar{j}} + \phi_{i\bar{j}}$ then the new Ricci curvature is $dd^c \log\det(h_{i\bar{j}} + \phi_{i\bar{j}})$. The Calabi conjecture can be restated as the existence of ϕ such that $h_{i\bar{j}} + \phi_{i\bar{j}}$ is definite positive and

$$dd^c \left(\log\det(h_{i\bar{j}} + \phi_{i\bar{j}}) - \log\det(h_{i\bar{j}}) - \eta\right) = 0 \tag{1}$$

The functions f that satisfies $dd^c f = 0$ are called *pluriharmonic*. They also satisfy the maximum principle. By compactness of X, these functions on X are exactly constant functions. Therefore (1) is equivalent to

$$\det(h_{i\bar{j}} + \phi_{i\bar{j}}) = e^{c+\eta} \det(h_{i\bar{j}})$$

or

$$(\omega + dd^c\phi)^n = e^{c+\eta}\omega^n$$

where ω^n denotes the repeated wedge product. Note that $(\omega + dd^c\phi)^n - \omega^n$ is exact, one has $\int_M (\omega + dd^c\phi)^n = V$, the conjecture of Calabi is therefore a consequence of the following theorem.

Theorem 2 (Yau)**.** *Given a function $f \in C^\infty(M), f > 0$ such that $\int_M f\omega^n = V$. There exists, and unique up to constant, $\phi \in C^\infty(M)$ such that $\omega + dd^c\phi > 0$ and*

$$(\omega + dd^c\phi)^n = f\omega^n$$

3 A sketch of proof

The uniqueness is straightforward. We will prove the existence of ϕ under the constraint $\int_M \phi\omega^n = 0$ (which will be useful to prove that (N) is locally diffeomorphism later). We will prove that the set S of $t \in [0, 1]$ such that there exists $\phi_t \in C^{k+2,\alpha}(M)$ with $\int_M \phi_t\omega^n = 0$ that satisfies

$$(\omega + dd^c\phi_t)^n = (tf + 1 - t)\omega^n \tag{2}$$

is both open and close in $[0, 1]$, therefore is the entire interval as $0 \in S$ is non empty.

To see that S is open, one only has to prove that the function $\mathcal{N}$ defined by

$$\phi \mapsto \mathcal{N}(\phi) = \frac{\det(h_{i\bar{j}} + \phi_{i\bar{j}})}{\det(h_{i\bar{j}})}$$

or in other words $(\omega + dd^c\phi)^n = \mathcal{N}(\phi)\omega^n$, is a local diffeomorphism. The differential of $\mathcal{N}$ is given by

$$D\mathcal{N}(\phi).\eta = \mathcal{N}\Delta\eta$$

with η varies in $\{\eta \in C^{k,\alpha}(M) : \int_M \eta\omega^n = 0\}$. and Δ is the Laplace-Beltrami operator which is known to be bijective between

$$\left\{\eta \in C^{k+2,\alpha}(M) : \int_M \eta = 0\right\} \longrightarrow \left\{f \in C^{k,\alpha}(M) : \int_M f = 0\right\}$$

Therefore $\mathcal{N}$ is a local diffeomorphism and S is open.

The proof that S is closed is more technical and is accomplished in 3 steps:

1. Using Arzela-Ascoli theorem, it suffices to show that $\{\phi_t : t \in S\}$ is bounded in $C^{k+2,\alpha}$. Therefore up to a subsequence, one has the uniform convergence of ϕ_{t_n} and all its partial derivatives of order $\leq k+1$. The $k+2$ -th order follows from (2).

2. Using Schauder theory, prove that the above bound follows from a *priori estimate*: There exists $\alpha \in (0,1)$ and $C(X, \|f\|_{1,1}, 1/\inf_M f) > 0$ such that every $\phi \in C^4(X)$ satisfying $(\omega + dd^c\phi)^n = f\omega^n$ and $\int_M \phi\omega^n = 0$ has
$$\|\phi\|_{2,\alpha} \leq C$$

3. Establish the priori estimate.

4 Calabi-Yau manifold

Recall that we defined a Calabi-Yau manifold to be a compact Riemannian manifold of dimension $2n$ with holonomy contained in $SU(n)$. We also remark, using parallel transport, the existence of a compatible complex structure ($U(n)$ suffices) and a holomorphic form non-vanishing at every point. We present here some equivalent definitions of compact Calabi-Yau manifolds.

Theorem 3. *Let X be a compact manifold of Kähler type and complex dimension n then:*

1. *The followings are equivalent*

(a) *There exists a Kähler metric such that the global holonomy is in $SU(n)$.*

(b) *There exists a holomorphic $(n, 0)$ form that vanishes nowhere.*

(c) *The canonical bundle K_X is trivial.*

(d) *The structure group of X can be reduced to $SU(n)$.*

2. *The following are equivalent. If X is simply-connected, they are equivalent with the 4 statements above.*

   (a) *There exists a Kähler metric such that the local holonomy is in $SU(n)$.*

   (b) *The canonical bundle K_X is flat.*

   (c) *There exists a Kähler metric such that the Ricci curvature vanishes.*

   (d) *The first Chern class vanishes.*

The proof is straightforward (see Manuscript) with the only exception is that one needs Calabi-Yau theorem to construct Ricci-flat metric.