Weighted monotonicity theorems and applications to minimal surfaces in hyperbolic space

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Abstract

We show that there is a weighted version of monotonicity theorem corresponding to each function on a Riemannian manifold whose Hessian is a multiple of the metric tensor. Such function appears in the Euclidean space, the hyperbolic space $\mathbb{H}^n$ and the round sphere $\mathbb{S}^n$ as the distance function, the Minkowskian coordinates of $\mathbb{R}^{n,1}$ and the Euclidean coordinates of $\mathbb{R}^{n+1}$.

In $\mathbb{H}^n$, we show that the time-weighted monotonicity theorem implies the unweighted version of Anderson [And82]. Applications include upper bounds for Graham–Witten renormalised area of minimal surfaces in term of the length of boundary curve and a complete computation of Alexakis–Mazzeo degrees defined in [AM10].

An argument on area-minimising cones suggests the existence of a minimal surface in $\mathbb{H}^4$ bounded by the Hopf link $\{zw = \epsilon > 0, |z|^2 + |w|^2 = 1\}$ other than the pair of disks. We give an explicit construction of a minimal annulus in $\mathbb{H}^4$ with this property and obtain by the same method its sister in $\mathbb{S}^4$.

A weighted monotonicity theorem is also proved in Riemannian manifolds whose sectional curvature is bounded from above.

1 Introduction

Let h be a function on a Riemannian manifold (M, g) with $\operatorname{Hess} h = Ug$. We will prove a monotonicity theorem for minimal surfaces (k-dimensional orientable submanifolds with vanishing mean curvature) in M where the area functional is weighted by U. The theorem also holds for exterior extension of a minimal surface by the gradient flow of h in the same fashion as the extended monotonicity theorem of Ekholm, White and Wienholtz [EWW02].

The existence of such function h was proved by Cheeger and Colding [CC96] to be equivalent to the metric locally being a warped product. The manifold M can therefore be specialised to be the hyperbolic space $\mathbb{H}^n$ or the round sphere $\mathbb{S}^n$, on which the Minkowskian coordinates ξ_α and the Euclidean coordinates x_i satisfy $\operatorname{Hess} \xi_\alpha = \xi_\alpha g_{\mathbb{H}^n}$ and $\operatorname{Hess} x_i = -x_i g_{\mathbb{S}^n}$ respectively. On $\mathbb{R}^n$, there are only 2 ways to write the Euclidean metric locally as a warped product, either by the coordinate functions x_i, in which case the weighted monotonicity theorem trivialises, or by the distance function $\rho = \sum_{i=1}^n x_i^2$ whose corresponding monotonicity theorem is the classical one.

In hyperbolic space, an unweighted monotonicity theorem was proved by Anderson in [And82]. We will show by a Comparison Lemma (Lemma 25) that the monotonicity theorem corresponding to the time coordinate implies the unweighted version. For any surface not necessarily minimal, the Comparison Lemma says that if its density with respect to a weight function is increasing then so is the density with respect to any *weaker* weight. In particular, for the unit ball $\mathbb{B}^n$ equipped with the Poincaré, Euclidean and round-sphere metrics, one has the following chain of monotonicity

$$\text{time-weighted } g_{\mathbb{H}^n} \gg \text{unweighted } g_{\mathbb{H}^n} \gg g_E \gg \text{unweighted } g_{\mathbb{S}^n} \gg \text{weighted } g_{\mathbb{S}^n}.$$

A surface having increasing density with respect to an area functional in the chain automatically has increasing density with respect to any area functional on the right of it. This is partly the reason why our statements concerning minimal surfaces in $\mathbb{H}^n$, which are time-monotone, outnumber what we can say about their counterparts in $\mathbb{S}^n$ which are at the opposite end of the chain.

While minimal surfaces in $\mathbb{S}^n$ are weighted-monotone, the Clifford torus of $\mathbb{S}^3$ shows that unweighted monotonicity does not hold even in a hemisphere. Despite this, there is a way to use Lemma 25 to obtain meaningful statements about the unweighted area. We illustrate this by giving a geometric proof of some results in [CLY84] and [HS74].

One application of the monotonicity chain is the following statement about minimising cones from [And82].

Theorem 1 (cf. Theorem 9 of [And82]). *If a k-dimensional radial cone C_γ, constructed over a submanifold γ on the sphere at infinity of the Poincaré model is Euclidean area-minimising, then it is the only complete minimal surface asymptotic to γ.*

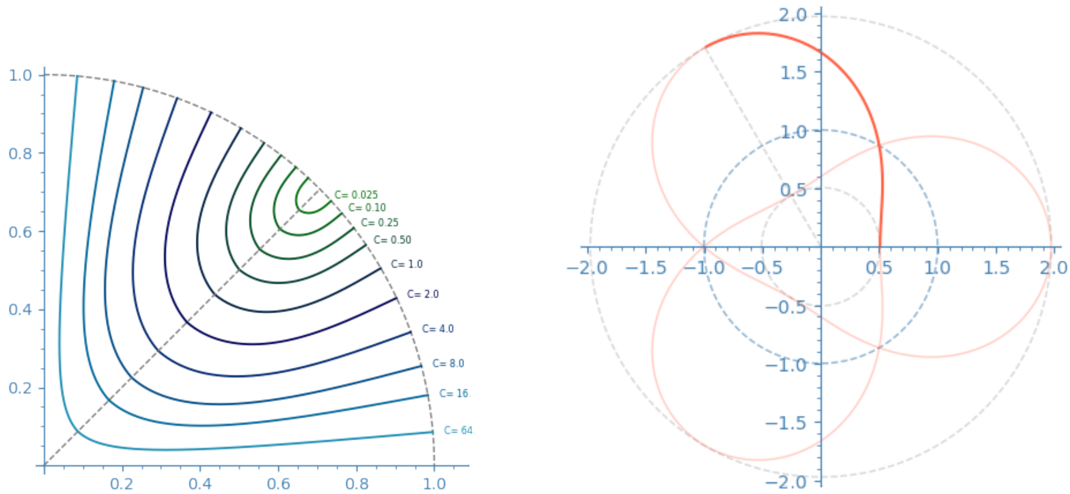
In particular, area minimising cones in hyperbolic space are exactly those which are minimal in Euclidean space.

The first half of Theorem 1 is slightly stronger than the second half: Knowing that the pair of 2-planes $zw = 0$ in $\mathbb{C}^2$ is hyperbolic area-minimising only allows us to rule out minimal surfaces of $\mathbb{H}^4$ that agree with the planes on a neighborhood of infinity.

The pair of 2-planes in $\mathbb{R}^4 \cong \mathbb{C}^2$ that contains the Hopf link $\{zw = \epsilon, |z|^2 + |w|^2 = 1\}$ is not Euclidean area-minimising when $\epsilon \in (0, \frac{1}{2})$. Theorem 1 therefore suggests another minimal surface in the Poincaré ball filling the link. We will point out an explicit family of minimal annuli with this property as the orbit of curves in the real plane $\operatorname{Im} z = \operatorname{Im} w = 0$ by the quaternionic rotation $(z, w) \mapsto (ze^{i\theta}, we^{-i\theta})$. The same construction also yields minimal annuli fibred by Hopf links under any metric $g = e^{2\varphi(\rho)} g_E$ of $\mathbb{R}^4$ that is conformal to the Euclidean metric by a factor φ depending only on $\rho = |z|^2 + |w|^2$.

Proposition 2. *Let M_C be the surface in $\mathbb{R}^4$ given by rotating the following real plane curve:*

$$\sin^2 \psi = \frac{e^{-4\varphi}}{C^2 \rho^2}, \quad C > 0 \tag{1}$$



(a) in $\mathbb{H}^4$ with C in $\{.025, .1, .25, .5, 1, 2, 4, 8, 16, 64\}$.

(b) in $\mathbb{S}^4$ with $C = 6.15$. The angle $\theta_C \approx \frac{2\pi}{3}$. The unit circle represents the Clifford torus.

Figure 1: The profile curve of M_C

where ψ is the angle formed by the tangent of the curve at a point p and the radial direction $\overrightarrow{Op}$. Then M_C is minimal under the metric $g = e^{2\varphi} g_E$. Up to $SO(4)$, the annuli M_C and the 2-planes are the only minimal surfaces obtained as orbit of a real plane curve by quaternionic rotation.

In the round four-sphere, the family $\{M_C\}$ starts with a Clifford torus and end with a totally geodesic $\mathbb{S}^2$. There is a countable number of parameter C in between for which the annulus M_C can be rearranged periodically to obtain a minimally immersed torus with one-dimensional self-intersection. The profile curves of M_C in $\mathbb{H}^4$ and $\mathbb{S}^4$ are drawn in Figure 1 for different values of C.

Denote by ω_k the Euclidean k-volume of $\mathbb{S}^k$. Another application of monotonicity theorem is:

Corollary 3. *The boundary at infinity of a complete k-dimensional minimal surface containing the center of Poincaré ball has Euclidean $(k-1)$-volume at least ω_{k-1}.*

Definition 4. *Let γ^{k-1} be a submanifold of the sphere at infinity and p be an interior point of the ball. We say that p is in the* center set *of γ if the Euclidean volume of γ in the Poincaré model centered at p is at least ω_{k-1}.*

Corollary 3 can be rephrased as: A complete minimal surface of $\mathbb{H}^n$ is contained in the center set of its boundary. Together with the convex hull introduced in [And82], the center set poses sufficient restriction on minimal surface filling a given curve to prove:

Theorem 5. *Let $L^{k-1} := L_1 \sqcup L_2$ be a separated union of two links L_1, L_2 in $\mathbb{S}^{n-1}$. There is a way to rearrange L in its isotopy class such that any minimal surface Σ^k in $\mathbb{H}^n$ filling L is a disjoint union of minimal surfaces filling each L_i.*

Proof. We isotope L so that L_1 (respectively L_2) is contained in a small ball centered at the North (respectively South) pole of the Poincaré ball and so that the Euclidean volume of L is less than $\frac{1}{2}\omega_{k-1}$. It suffices to prove that Σ has no intersection with the equatorial hyperplane. By convexity, such intersection is contained in a small ball centered at the origin O. If it was non-empty, by a small Möbius transform we could suppose that Σ contains O while keeping the Euclidean length of L less than ω_{k-1}. This contradicts Corollary 3. $\square$

It was proved in [AM10] that:

Theorem 6 (Alexakis–Mazzeo)**.** *Let $\mathcal{M}_{g,b}$ be the space of properly embedded, connected $C^{3,\alpha}$ two-dimensional minimal surfaces in $\mathbb{H}^3$ of genus g and b boundary components, $\mathcal{C}_b$ be the space of $C^{3,\alpha}$ closed, embedded curves of b connected components in $\mathbb{S}^2$ and $\Pi_{g,b} : \mathcal{M}_{g,b} \longrightarrow \mathcal{C}_b$ be the map which sends a surface to its boundary. Then*

1. *$\mathcal{M}_{g,b}$ is a Banach manifold.*
2. *$\Pi_{g,b}$ is proper, Fredholm, of index 0 and it has a well-defined degree $d(g,b)$.*

Since the only minimal surface filling a round circle of $\mathbb{S}^2$ is a totally geodesic disk, one has $d(0,1) = 1$ and $d(g,1) = 0$ for any $g \geq 1$. It follows from Theorem 5 that:

Theorem 7. *All Alexakis–Mazzeo degrees are zero except $d(0,1)$.*

It is of great interest to try to replace $\mathbb{H}^3$ in the statement of Theorem 6 by $\mathbb{H}^4$. The connected components of $\mathcal{C}_b$ would be by definition knots ($b = 1$) or links ($b > 1$) and Alexakis–Mazzeo degrees would be knot/link invariants. This direction was studied by Alves and Fine and their work will appear later. The results above suggest that one can distinguish the Hopf link from two unlinked circles by this method. It follows from Theorem 5 that the degree among surfaces of Euler characteristic zero of the latter vanishes while the existence of M_C suggests that it is non-zero for the former.

The total area of a complete two-dimensional minimal surface in $\mathbb{H}^n$ is necessarily infinite. By looking at its expansion as the surface runs to infinity, Graham and Witten [GW99] were able to renormalise the area to a finite number $\mathcal{A}_R$. Both the time monotonicity and the space monotonicity theorems can be used to obtain upper bounds of this number. Note that the zero set of a space coordinate is a totally geodesic hyperplane which separates the sphere at infinity $\mathbb{S}_\infty$ into two halves. The *doubled hyperbolic metric* is obtained by putting the $(n-1)$-dimensional hyperbolic metric on each of them.

Theorem 8. *Let $\Sigma^2 \subset \mathbb{H}^n$ be a minimal surface with boundary $\gamma^1 \in \mathbb{S}_\infty$. Then*

$$\mathcal{A}_R(\Sigma) + \sup_{\textit{round } \tilde{g}} |\gamma|_{\tilde{g}} \leq 0 \tag{2}$$

where the supremum is taken among metrics of curvature +1 in the standard conformal class of $\mathbb{S}_\infty$.

By choosing any point of Σ as center of the Poincaré ball and applying Corollary 3, one has:

Corollary 9. *$\mathcal{A}_R(\Sigma) \leq -2\pi$ for any minimal surface Σ^2 in $\mathbb{H}^n$.*

Corollary 9 can also be proved from the space version of Theorem 8.

Theorem 10. *If the space coordinate $\xi_1 \geq \alpha > 0$ on a minimal surface $\Sigma^2 \subset \mathbb{H}^n$ then*

$$\mathcal{A}_R(\Sigma) + \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)|\gamma_\infty|_{\tilde{g}} \leq 0$$

where $\tilde{g}$ is the doubled hyperbolic metric associated to ξ_1.

During the preparation of this paper, Theorem 8 and Corollary 9 were independently proved by Bernstein in [Ber21] using the work of Choe and Gulliver [CG92b]. It was via [Ber21] that the author learned about [CG92b] and its companion [CG92a]. Theorem 20 and 22 are indeed Theorem 3 of [CG92a], the time-weighted area in $\mathbb{H}^n$ and the weighted area in $\mathbb{S}^n$ (Section 3) were called *modified volume* there. The fact that a minimal surface in $\mathbb{H}^n$ has less area than the cone built upon its boundary (Proposition 2 of [CG92b], Corollary 31 in this text) was crucial to the proof of sharp isoperimetric inequality $4\pi A + A^2 \leq L^2$, where A is the area of a two-dimensional minimal surface and L is the length of its boundary. By this fact, since $4\pi A + A^2$ is increasing in A, it suffices to check the inequality only for cones. The Comparison Lemma 25 allows us to arrive at Corollary 31 in a simpler way than [CG92b].

One can hope, since a monotonicity theorem is an inequality, that the results above still hold when the Hessian of the function h is comparable to the metric as symmetric 2-tensors. Such function arises naturally as the distance function in a Riemannian manifold whose sectional curvature is bounded from above. We explore this idea in Section 5. We note that although the unweighted monotonicity theorem does not hold in case of positive curvature, an unweighted monotonicity *inequality* was obtained by Scharrer [Sch21].

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2 The hyperbolic space and the sphere as warped spaces.

A metric on a Riemannian manifold $M = N \times [a, b]$ is a *warped product* if it has the form

$$g = dr^2 + f^2(r) g_N \tag{3}$$

where $r \in [a, b]$ and g_N is a Riemannian metric on N. It can be checked that an anti-derivative h of the warping function f satisifies $\mathrm{Hess}(h) = f'(r)g$. On the other hand, if such function h exists, the space is locally warped by its level sets [CC96].

Proposition 11 (cf. [CC96]). *Suppose that there exists a function h on (M, g) with no critical point, whose level set are connected and*

$$\operatorname{Hess} h = U.g \tag{4}$$

for a function $U \in C^0(M)$. Then:

1. *$U = U(h)$ is a function of h, i.e. a composition of h and a function $U : \mathbb{R} \longrightarrow \mathbb{R}$. The function $V := |dh|^2 \in C^1(M)$ is also a function of h and one has $U = \frac{1}{2}V'$.*
2. *The metrics g_a, g_b induced from g on level sets $h^{-1}(a)$ and $h^{-1}(b)$ are related by $\frac{g_a}{V(a)} = \frac{g_b}{V(b)}$ via the inverse gradient flow of h. This defines a metric $\tilde{g}$ on level sets under which the flow is isometric. The metric g on M pulls back via the flow map $h^{-1}(a) \times \operatorname{Range}(h) \longrightarrow M$ to*

$$g = \frac{V(h)}{V(a)} g_a + \frac{dh^2}{V(h)} = V(h)\tilde{g} + \frac{dh^2}{V(h)},$$

which is a warped product after a change of variable $dr = \frac{dh}{V(h)^{1/2}}$.

Proof. For any vector field v, one has

$$v(V) = 2g(\nabla_v \nabla h, \nabla h) = 2\operatorname{Hess}(h)(v, \nabla h) = 2Ug(v, \nabla h)$$

It follows, by first taking v to be any vector field tangent to level sets of h, then to be the inverse gradient $u := \frac{\nabla h}{|\nabla h|^2}$, that V is constant on the level sets, and as a function of h, $V' = 2U$.

For the second part, let v_t be a vector field of M tangent to level sets $h^{-1}(t)$ given by pushing forward via the flow of u a vector field v_a tangent to $h^{-1}(a)$. The Lie bracket $[v_t, u]$ vanishes by definition and for all time t,

$$\frac{d}{dt}|v_t|^2 = 2g(\nabla_u v_t, v_t) = 2g(\nabla_{v_t} u, v_t) = \frac{2}{|\nabla h|^2}\operatorname{Hess}(h)(v_t, v_t) = \frac{V'}{V}|v_t|^2.$$

This means that $\frac{|v_t|^2}{V(t)}$ is constant along the flow and so $\frac{g_a}{V(a)} = \frac{g_b}{V(b)}$ for all $a, b \in \operatorname{Range}(h)$. $\square$

We note that $\tilde{g}$ is the metric g_N in (3). We will use $\tilde{g}$ to denote the metric $V(h)^{-1}g$ on M.

In applications, we will only assume that the function h satisfies (4) on M and it can have critical points, as in the following examples.

Example 12. *In Euclidean space $\mathbb{R}^n$, the only functions satisfying* (4) *are the coordinates $x_i, i = 1, \ldots, n$ with $U = 0, V = 1$ and the square of distance $\rho := \frac{1}{2}\sum_{i=1}^n x_i^2$ with $U = 1, V = 2\rho$.*

Example 13. *In the unit sphere $\mathbb{S}^n = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$, the Euclidean coordinates x_i satisfy* (4) *with $U = -x_i, V = 1 - x_i^2$.*

Example 14. *In the hyperbolic space* $\mathbb{H}^n = \{(\xi_0, \ldots, \xi_n) \in \mathbb{R}^{n,1} : \xi_0^2 - \sum_{i=1}^n \xi_i^2 = 1, \xi_0 > 0\}$*, the Minkowskian coordinates* ξ_α *satisfies* (4) *with* $U = \xi_\alpha, V = \xi_\alpha^2 + |\partial_{\xi_\alpha}|^2$*, where* $|\partial_{\xi_\alpha}|^2$ *is the Minkowskian norm, which is -1 for time-like unit vectors and +1 for space-like ones.*

Unlike the round sphere, the hyperbolic space can be written as a warped product in two different ways up to isometry. Each interior point corresponds to a unique time coordinate ξ_0 *that it minimises and each oriented totally geodesic hyperplane corresponds to a unique space coordinate* ξ_1 *that vanishes on it. Note that no other level set of* ξ_1 *is totally geodesic.*

We remark that metric $\tilde{g}$ in the case $(\mathbb{R}^n, \rho)$, $(\mathbb{S}^n, x_i)$ and $(\mathbb{H}^n, \xi_0)$ is the round metric on $\mathbb{S}^{n-1}$. For $(\mathbb{H}^n, \xi_1)$, $\tilde{g}$ is the doubled hyperbolic metric on $\mathbb{S}^{n-1}$.

3 Monotonicity Theorems and Comparison Lemma

The following lemma often appears as $\operatorname{div}_\Sigma X^\Sigma = \operatorname{div}_\Sigma X + k(\Sigma)X$ where Σ is a submanifold of M, X is a vector field along Σ and X^Σ is its tangent component to Σ. Similarly, we will denote the gradient vector in M of a function h by ∇h and its projection to Σ by $\nabla^\Sigma h$.

Lemma 15 (Leibniz rule)**.** *Let* $f : (\Sigma^k, g_\Sigma) \longrightarrow (M^n, g)$ *be a map between Riemannian manifolds and* $\tau(f)$ *be its tension field, then for any* C^2 *function h on M, one has*

$$\Delta_\Sigma(h \circ f) = \operatorname{Tr}_\Sigma f^* \operatorname{Hess} h + dh.\tau(f) \tag{5}$$

In particular, the Laplacian of h on a submanifold Σ *is given by*

$$\Delta_\Sigma h = \operatorname{Tr}_\Sigma \operatorname{Hess} h \tag{6}$$

if Σ *is either minimal or tangent to the gradient of h at the point in question.*

3.1 Weighted monotonicity

Condition (4) forces all non-degenerate critical points of h to be either local maxima or local minima. The functions in Examples 12, 13, 14 fall into two types:

1. h has no other critical value than its minimum $h_{\min}$ and the sublevel sets $\{h \leq t\}$ are compact. This is the case of $(\mathbb{R}^n, \rho)$, $(\mathbb{S}^n \setminus \{\mathrm{pt}\}, \mathrm{x_i})$ and $(\mathbb{H}^n, \xi_0)$.

2. h has no critical point and the sublevel sets are no longer compact, as in the case of $(\mathbb{H}^n, \xi_1)$.

Given a subset γ of a level set of h, the *h-tube* $T_\gamma(t_1, t_2)$ is obtained by flowing γ along the gradient field of h from level $h = t_1$ to level $h = t_2$. When h is of the first type and $t_1 = h_{\min}$, this is visually a cone.

For a function h of the first type, we define the *weighted area* and the *weighted density* of a surface Σ^k to be

$$A_h(\Sigma)(t) := \int_{\Sigma, h \leq t} U, \qquad \Theta_h^A(t) := \frac{A_h(t)}{\frac{\omega_{k-1}}{k} V^{k/2}(t)}. \tag{7}$$

Note that the denominator of Θ_h^A is up to constant the weighted area of a tube $T_\gamma(h_{\min}, t)$:

$$A_h(T_\gamma) = \frac{|\gamma|}{k}\left(V(t)^{k/2} - V(h_{\min})^{k/2}\right) = \frac{|\gamma|}{k}V(t)^{k/2} \tag{8}$$

where $|\gamma|$ is the volume of γ under the metric $\tilde{g}$.

For the second type, it is possible that the integral in (7) does not converge. One can remedy this by only counting area in the region $h \geq h_0$. We will assume that Σ intersects the level set $h^{-1}(h_0)$ at a smooth $(k-1)$-dimensional submanifold γ_0 and define the weighted area and the weighted density by

$$B_h(\Sigma)(t) := \int_{\Sigma, h_0 \leq h \leq t} U(h) + \frac{1}{k}\int_{\gamma_0} |\nabla^\Sigma h|, \qquad \Theta_h^B := \frac{B_h(t)}{\frac{\omega_{k-1}}{k}V^{k/2}(t)}. \tag{9}$$

which are finite for a large class of surfaces. The denominator of Θ_h^B is again the weighted area of a tube $T_\gamma(h_0, t)$:

$$B_h(T_\gamma) = \frac{|\gamma|}{k}(V(t)^{k/2} - V(h_0)^{k/2}) + \frac{|\gamma|}{k}V(h_0)^{k/2} = \frac{|\gamma|}{k}V(t)^{k/2}. \tag{10}$$

Theorem 16 (Weighted Monotonicity)**.** *Suppose that h is a C^2 function on a Riemannian manifold M that satisfies* (4) *with U and $V = |\nabla h|^2$ being functions of h such that $U = \frac{1}{2}V'$. Let Σ^k be a minimal surface in M. Assume that the integral in* (7) *(respectively* (9)*) is finite, then $\frac{d}{dt}\Theta_h^A$ (respectively $\frac{d}{dt}\Theta_h^B$) has the same sign as U.*

Moreover, the conlusion still holds for an extension $\tilde{\Sigma}$ of a minimal surface $\Sigma^k \subset \{h \leq t_0\}$ whose boundary γ^{k-1} is piecewise smooth and contained in $h^{-1}(t_0)$ by an exterior h-tube $T_\gamma(t_0, t)$ built upon γ.

In both case, $\frac{d}{dt}\Theta_h^A$ (respectively $\frac{d}{dt}\Theta_h^B$) vanishes if and only if the gradient of h is tangent to Σ.

Proof. It follows from Lemma 15 that $\Delta_\Sigma h = kU$ on Σ. By Stokes' theorem,

$$A_h(t) = \int_{\Sigma, h \leq t} U(h) = \frac{1}{k}\int_{\Sigma, h\leq t} \Delta_\Sigma h = \frac{1}{k}\int_{\Sigma, h=t} \nabla^\Sigma h \cdot n = \frac{1}{k}\int_{\Sigma, h=t} |\nabla^\Sigma h| \tag{11}$$

because the outer normal of $\{h \leq t\}$ in Σ is $n = \frac{\nabla^\Sigma h}{|\nabla^\Sigma h|}$. By the coarea formula,

$$\frac{dA_h}{dt} = U(t)\int_{\Sigma, h=t} \frac{1}{|\nabla^\Sigma h|}.$$

Combining this with (11) and $|\nabla^\Sigma h|^2 \leq V(h)$, one has $\frac{1}{U}\frac{dA_h}{dt} \geq \frac{kA}{V}$, or $\frac{1}{U}\frac{d}{dt}(\frac{A_h}{V^{k/2}}) \geq 0$.

Similarly, one has $B_h(t) = \frac{1}{k}\int_{\Sigma, h=t}|\nabla^\Sigma h|$, and $\frac{dB_h}{dt} = U(t)\int_{\Sigma, h=t}\frac{1}{|\nabla^\Sigma h|}$ and the same conclusion is drawn for the second type function.

For cone extension, it suffices to rewrite equation (11) when $t > t_0$ as

$$kA_h(t) = \left(\int_{\Sigma, h\leq t_0} + \int_{T_\gamma(t_0,t)}\right)\Delta h = \int_{\tilde{\Sigma}, h=t} |\nabla^{\tilde{\Sigma}} h| + \int_\gamma \left(|\nabla^\Sigma h| - |\nabla^M h|\right) \leq \int_{\tilde{\Sigma}, h=t} |\nabla^{\tilde{\Sigma}} h|.$$

$\square$

Remark 17. *1. If Σ contains a multiple of γ, the area of the tube should be counted with multiplicity.*

2. We only need the "$\leq$" sign in (11) *and hence it suffices that $\Delta_\Sigma h \geq kU(h)$. Theorem 16 still holds if* $\operatorname{Hess} h \geq U.g$*, provided that U and $V = |dh|^2$ are still functions of h and that $U = \frac{1}{2}V'$.*

Remark 18. *When Σ is a stationary rectifiable k-current, the proof of Theorem 16 can be adapted in the same fashion as [And82] and [EWW02]. We replace the integration by part* (11) *by the first variation formula of current, which reads $\int_\Sigma \operatorname{div}_\Sigma X \, d\|\Sigma\| = 0$ where X is any smooth vector field and $d\|\Sigma\|$ is the mass measure. We recover $kA_h - \frac{V}{U}A'_h \leq 0$ by choosing $X := \chi(h)\nabla h$ where χ is a decreasing function that approximates the characteristic function of $[-\infty, t]$, and by noting that* $\operatorname{div}_\Sigma X = \chi'|\nabla^\Sigma h|^2 + \chi\Delta_\Sigma h \geq \chi' V + k\chi U$.

For the tube extension and the formulation of B_h when h is of the second type, we replace the intersection $\gamma = \Sigma \cap h^{-1}(t_0)$ (or $\gamma_0 = \Sigma \cap h^{-1}(h_0)$ respectively) by an $\mathcal{H}^{k-1}$-rectifiable set such that the pair (Σ, γ) is strongly stationary. *This means that $\int_\Sigma \operatorname{div}_\Sigma X \leq \int_\gamma |X^\perp|$ for any smooth vector field X whose normal component to γ is $X^\perp$, or equivalently that there exists an $\mathcal{H}^{k-1}$-measurable normal vector field ν on γ with* $\sup|\nu| \leq 1$ *such that $\int_\Sigma \operatorname{div}_\Sigma X = \int_\gamma g(X, \nu)$. The definition* (9) *should be rewritten for strongly stationary pair (Σ, γ_0) as*

$$B_h(\Sigma)(t) := \int_\Sigma U(h) - \frac{1}{k}\int_{\gamma_0} g(\nabla h, \nu_0).$$

Theorem 16 can also be extended for harmonic maps. Given a map $f : \Sigma \longrightarrow M$, we define its *dimension at a point* $p \in \Sigma$ to be the ratio $\frac{|df_p|^2}{|df_p|_o^2}$ of the tensor norm of the derivative at p (called energy density) and its operator norm, or $+\infty$ if the latter vanishes. Note that when df_p is non-zero and conformal, this is the dimension of Σ. The *dimension* of f, defined as the smallest dimension among all points, will play the role of k in our argument.

The *weighted Dirichlet energy* of f in the region $h \leq t$ is defined as $E_h(t) := \int_{\Sigma, h\circ f \leq t} U|df|^2$ or $E_h(t) := \int_{\Sigma, h_0 \leq h\circ f \leq t} U|df|^2 + \int_{\Sigma, h\circ f = h_0} |d(h \circ f)|$ depending on the type of h. The *weighted density* is $\Theta_h(t) := \frac{E_h(t)}{V(t)^{k/2}}$.

Theorem 19. *Let h, U, V be as in Theorem 16 and $f : \Sigma \longrightarrow M$ be a harmonic map. Then $\frac{d}{dt}\Theta_h$ has the same sign as U.*

Proof. By Lemma 15 one has $\Delta(h \circ f) = U|df|^2$ and by integration by part, $E_h(t) = \int_{\Sigma, h\circ f = t} |d(h \circ f)|$. One then compares E_h with its derivative obtained from coarea formula $\frac{dE_h}{dt} = U(t) \int_{\Sigma, h\circ f = t} \frac{|df|^2}{|d(h\circ f)|}$. The definition of k guarantees $\frac{|df|^2}{|d(h\circ f)|} \geq k\frac{|d(h\circ f)|}{|dh|^2}$ and therefore $U^{-1}\frac{dE_h(t)}{dt} \geq \frac{k}{V}E_h$. □

We will restate Theorem 16 when M is the hyperbolic space and the sphere.

Given a time coordinate ξ_0 of $\mathbb{H}^n$, the *time-weighted area* functional is defined as

$$A_{\xi_0}(\Sigma)(t) := \int_{\Sigma, 1 \leq \xi_0 \leq t} \xi_0,$$

For a totally geodesic copy of $\mathbb{H}^k$ in $\mathbb{H}^n$ passing by $\xi_0^{-1}(1)$, it is given by $A_{\xi_0}(\mathbb{H}^k)(t) = \frac{\omega_{k-1}}{k}(t^2-1)^{k/2}$. Define the *time-weighted density* by $\Theta_{\xi_0}(\Sigma)(t) := \frac{A_{\xi_0}(\Sigma)(t)}{A_{\xi_0}(\mathbb{H}^k)(t)}$ and substitute $h = \xi_0$ into Theorem 16, one has:

Theorem 20 (Time Monotonicity)**.** *The time-weighted density of an extension by exterior ξ_0-tube of a minimal surface is increasing on $(1, +\infty)$.*

To state the weighted monotonicity theorem corresponding to a space coordinate ξ_1, we will assume that the surface Σ^k is contained in the region $\{\xi_1 \geq 0\}$ and that its boundary consists of a (possibly empty) part γ_0 in $\xi_1^{-1}(0)$ and a part γ_∞ in $\mathbb{S}_\infty$, both of them are disjoint from the equator $\xi_1^{-1}(0) \cap \mathbb{S}_\infty$. The *space-weighted area* functional is defined by

$$B_{\xi_1}(\Sigma)(t) := \int_{\Sigma, 0 \leq \xi_1 \leq t} \xi_1 + \frac{1}{k} \int_{\Sigma, \xi_1 = 0} |\nabla^\Sigma \xi_1|$$

and is finite for such surface Σ. In particular, if γ^{k-1} is a submanifold in the interior of $\xi_1^{-1}(0)$ with hyperbolic volume $|\gamma|$, the ξ_1-tube $T_\gamma(0, t)$ built upon γ has space-weighted area $\frac{|\gamma|}{k}(t^2+1)^{k/2}$. The *space-weighted density* is defined as $\Theta_{\xi_1}(t) := \frac{B_{\xi_1}(\Sigma)(t)}{\frac{\omega_{k-1}}{k}(t^2+1)^{k/2}}$ and is constant for such tube.

Theorem 21 (Space Monotonicity)**.** *The space-weighted density of an extension by exterior ξ_1-tube of a minimal surface is increasing on $(0, +\infty)$.*

When M is the round sphere $\mathbb{S}^n$ seen as a warped space using an Euclidean coordinate $x = x_i$, we define the *weighted area* and *weighted density* by

$$A_x(\Sigma)(t) := \int_{\Sigma, x \geq t} x, \qquad \Theta_x(\Sigma)(t) := \frac{A_x(\Sigma)(t)}{A_x(\mathbb{S}^k)(t)}.$$

A totally geodesic $\mathbb{S}^k \subset \mathbb{S}^n$ passing by $x^{-1}(1)$ has $A_x(\mathbb{S}^k)(t) = \frac{\omega_{k-1}}{k}(1-t^2)^{k/2}$. It follows from Theorem 16 for $h = 1 - x$ that:

Theorem 22 (Weighted monotonicity in $\mathbb{S}^n$)**.** *The weighted density of an extension by exterior x-tube of a minimal surface in $\mathbb{S}^n$, is decreasing on $(0, 1)$ and increasing on $(-1, 0)$.*

3.2 Comparison lemma

It is useful to weight the area functional of (7) by a function $P(h)$ other than $U(h)$. The *P-area* is defined by

$$A_P(\Sigma)(t) := \int_{\Sigma, h \leq t} P(h), \quad B_P(\Sigma)(t) := \int_{\Sigma, h_0 \leq h \leq t} P(h) + \frac{1}{k} \int_{h = h_0} |\nabla^\Sigma h|.$$

We normalise the P-area of a minimal surface by that of an h-tube, which is up to a factor

$$Q(t) := \begin{cases} \omega_{k-1} \int_{h \le t} P(h) V^{\frac{k}{2}-1}(h) dh, & \text{if } h \text{ is of the first type,} \\ \omega_{k-1} \left(\int_{h=h_0}^{t} P(h) V^{\frac{k}{2}-1}(h) dh + \frac{1}{k} V(h_0)^{k/2} \right) & \text{if } h \text{ is of the second type} \end{cases} \tag{12}$$

and define the *P-density* as $\Theta_P(\Sigma)(t) := \frac{A_P(t)}{Q(t)}$ or $\frac{B_P(t)}{Q(t)}$ respectively. When $P = U$, these are the weighted area/density defined in (7) and (9). We will always assume that the weight function is positive in the relevant region of M.

Example 23. *1. On the round sphere $\mathbb{S}^n$, modeled as the compactification of $\mathbb{R}^n$ with $g_{\mathbb{S}^n} = \frac{4}{(1+r^2)^2} g_E$, the function $x := \frac{1-r^2}{1+r^2}$ is the Euclidean coordinate of Example 13 that is maximised at the origin and minimised at infinity. The Euclidean area of $\mathbb{R}^n$ is a P-area with $P = (1+x)^{-k}$.*

2. In the Poincaré ball $\mathbb{B}^n$ with metric $g_{\mathbb{H}^n} = \frac{4}{(1-r^2)^2} g_E$, the function $\xi_0 := \frac{1+r^2}{1-r^2}$ is the time coordinate minimised at the center. The Euclidean area corresponds to $P = (1+\xi_0)^{-k}$. The unweighted area of the round metric $g_{\mathbb{S}^n}$ corresponds to $P = \xi_0^{-k}$ and the x-weighted area above corresponds to $P = \xi_0^{-k-1}$.

Definition 24. *Given 2 weight functions P_1, P_2 whose tube area Q_1, Q_2 are defined by* (12)*, we say that P_1 is* weaker *than P_2 if $\frac{P_1}{Q_1} \le \frac{P_2}{Q_2}$, in other words if $\frac{d}{dt}\frac{Q_1}{Q_2} \le 0$, i.e. the P_2-area of a k-dimensional tube increases faster than its P_1-area. This is obviously a transitive relation.*

Lemma 25 (Comparison). *Let $\Sigma^k \subset M^n$ be any surface (not necessarily minimal), P_1, P_2 be two non-negative continuous weights and Θ_1, Θ_2 be the corresponding densities.*

1. *Suppose that P_1 is weaker than P_2 and that $\frac{d\Theta_2}{dt} \ge 0$, then one has $\Theta_1 \le \Theta_2$ and*

$$\frac{d\Theta_1}{dt} \ge \frac{Q_2}{P_2}\frac{P_1}{Q_1}\frac{d\Theta_2}{dt}.$$

   In particular, the density Θ_1 is also increasing.

2. *On the other hand, if P_2 is weaker than P_1 and $\frac{d\Theta_2}{dt} \ge 0$, then one has $\Theta_1 \ge \Theta_2$.*

Proof. One has $P_1^{-1}\frac{dA_{P_1}}{dt} = P_2^{-1}\frac{dA_{P_2}}{dt}$ from coarea formula, therefore

$$\frac{Q_1}{P_1}\frac{d\Theta_1}{dt} + \omega_{k-1} V^{\frac{k}{2}-1}\Theta_1 = \frac{Q_2}{P_2}\frac{d\Theta_2}{dt} + \omega_{k-1} V^{\frac{k}{2}-1}\Theta_2 \tag{13}$$

which can be rearranged into

$$P_1^{-1}\frac{d}{dt}\left(Q_1(\Theta_1 - \Theta_2)\right) = \left(\frac{Q_2}{P_2} - \frac{Q_1}{P_1}\right)\frac{d\Theta_2}{dt} \tag{14}$$

For the second part of the Lemma, it follows from the hypothesis that the RHS of (14) is positive, and therefore $Q_1(\Theta_1 - \Theta_2)$ is an increasing function. The latter vanishing at $t = 0$ for both types of function h means that $\Theta_1 \geq \Theta_2$ for all time.

For the first part, the RHS of (14) is negative by hypothesis and therefore $\Theta_2 \geq \Theta_1$. The rest of the conclusion follows by substituing this into (13). $\square$

Remark 26. *In the Poincaré ball $\mathbb{B}^n$ (see Example 23), let $P_1 = \xi_0^{-k-1}$, $P_2 = \xi_0^{-k}$, $P_3 = (1+\xi_0)^{-k}$, $P_4 = 1$ and $P_5 = \xi_0$. It can be checked that P_i is weaker than P_{i+1}. Lemma 25 says that there is the following chain of monotonicity:*

$$\text{time-weighted } g_{\mathbb{H}^n} \gg \text{unweighted } g_{\mathbb{H}^n} \gg g_E \gg \text{unweighted } g_{\mathbb{S}^n} \gg \text{weighted } g_{\mathbb{S}^n} \tag{15}$$

where any surface $\Sigma^k \subset \mathbb{B}^n$ having increasing density of one area functional in the chain will automatically have increasing density of any area functional following it.

Let $\tilde{\Sigma}^k$ be a tube extension of a minimal surface as in Theorem 16. It turns out that the additional information gained in the tube region $h \geq t_0$ from the monotonicity of the P-density can be rewriten as a comparison of the P-area of Σ and that of the tube.

Lemma 27. *The density $\Theta_P(t)$ of $\tilde{\Sigma}$ is increasing on $t \geq t_0$ if and only if*

$$A_P(\Sigma)(t_0) \leq A_P(T_\gamma(h_{\min}, t_0)) \quad (\textit{respectively } B_P(\Sigma)(t_0) \leq B_P(T_\gamma(h_0, t_0)))$$

when h is of first (respectively second) type.

Proof. It follows directly from $\frac{Q(t)}{Q(t_0)} = \frac{A_P(T_\gamma(h_{\min},t))}{A_P(T_\gamma(h_{\min},t_0))}$ (or $\frac{B_P(T_\gamma(h_0,t))}{B_P(T_\gamma(h_0,t_0))}$ for second type). $\square$

Corollary 28. *Let P be a weaker weight than U. Then in the region $U \geq 0$, the P-density of a minimal surface is an increasing function and its P-area in $\{h \leq t\}$ (respectively $\{h_0 \leq h \leq t\}$) is less than that of the h-cone (respectively tube) with the same boundary.*

4 Applications to minimal surfaces in $\mathbb{S}^n$ and $\mathbb{H}^n$

It is well known that there is no closed minimal surfaces in any hemisphere of $\mathbb{S}^n$. To see this, let x be the Euclidean coordinate that is positive on the hemisphere then any closed minimal surface has $\int x = 0$. Quantitatively, one has:

Corollary 29. *Let $\Sigma^k \subset \mathbb{S}^n$ be a minimal surface (or its extension by x-tube), γ_t be its intersection with $x^{-1}(t)$, $t \in [0,1]$. Let $|\gamma_t|_{\tilde{g}}$ be the volume of γ_t under the metric $\tilde{g}$ of Proposition 11.* [1]

1. *One has*

$$A_x(\Sigma)(t) := \int_{\Sigma, x \geq t} x \leq \frac{|\gamma_t|_{\tilde{g}}}{k}(1-t^2)^{k/2} \tag{16}$$

[1] This is the volume of its image by the radial projection centered at $x^{-1}(1)$ onto the equator $x^{-1}(0)$.

2. *Let m be the density of Σ at the North pole, then $\Theta_x(t) \geq m$, i.e.*

$$A_x(\Sigma)(t) \geq m\frac{\omega_{k-1}}{k}(1-t^2)^{k/2} \tag{17}$$

In particular, if Σ contains the North pole $x^{-1}(1)$ then the $g_{\mathbb{S}^n}$-volume of γ_t is bigger than that of a great $(k-1)$-sphere in $x^{-1}(t)$, which is $\omega_{k-1}(1-t^2)^{k/2}$.

Proof. The first part follows from Corollary 28, the second part from $\lim_{t\to 1}\Theta_x(t) = m$. □

Although the weight x is weaker than the uniform weight, we can still obtain a statement about the unweighted area from the second half of Lemma 25. We illustrate the technique by giving a geometric proof of the following volume bound from [CLY84]. Our technique can be used in spaces with curvature bounded from above (see Proposition 46).

Corollary 30 (cf. Corollary 2 of [CLY84]). *1. Let Σ^k be a minimal surface (or its extension by x-tube) in $\mathbb{S}^n$ that contains the North pole $O = x^{-1}(1)$ with multiplicity m. Suppose that Σ has no boundary in the interior of the geodesic ball $B(O,s)$ for a certain $s < \frac{\pi}{2}$, then the volume of $\Sigma \cap B(O,s)$ is at least m times that of the ball of radius s in $\mathbb{S}^k$, i.e.*

$$A(\Sigma \cap B(O,s)) \geq m\omega_{k-1}\int_{r=0}^{s}\frac{\sin^{k-1}(br)}{b^{k-1}}dr$$

2. *Any closed k-dimensional minimal surface of $\mathbb{S}^n$ has volume at least ω_k. Equality happens only for totally geodesic spheres.*

Proof. The first part follows from Lemma 25 and Corollary 29: $\Theta_1(t) \geq \Theta_x(t) \geq m$ for all $0 \leq t \leq 1$, where Θ_1 is the unweighted density.

For the second part, Σ may not contain the South pole $-O$, but we still have the same lower bound for the $(-x)$-weighted area of Σ in the southern hemisphere because

$$A_x(\Sigma \cap B(O,\frac{\pi}{2})) - A_{-x}(\Sigma \cap B(-O,\frac{\pi}{2})) = \int_\Sigma x = -\int_\Sigma \Delta x = 0.$$

□

4.1 Time monotonicity

The non-existence of closed minimal surfaces in the hyperbolic space, or Minkowskian hemisphere, can be seen via the time coordinate ξ_0 which would have vanishing integral over such surface. Each choice of interior point $p \in \mathbb{H}^n$ as center of the Poincaré ball defines a time coordinate $\xi_0 = \frac{1+r^2}{1-r^2}$ and a metric $\tilde{g}_p = \frac{g_{\mathbb{H}^n}}{\xi_0^2-1}$ on $\mathbb{H}^n \setminus \{p\}$ which extends to the Euclidean metric on the boundary of the ball. Conversely, all round metrics in the standard conformal class of $\mathbb{S}^{n-1}$ are obtained this way.

Corollary 31 below is the hyperbolic version of Corollary 29 and it follows from Corollary 28 for the time-weighted area and the unweighted area.

Corollary 31. *Let $\Sigma^k \subset \mathbb{H}^n$ be a minimal surface (or its extension by ξ_0-tube), γ_t be its intersection with $\xi_0^{-1}(t)$, $t \in (1, +\infty)$. Let $|\gamma_t|_{\tilde{g}}$ be the $\tilde{g}$-volume of γ_t.* [2]

1. *The time-weighted area and the unweighted area of Σ in the region $\{\xi_0 \leq t\}$ are less than those of the radial cone of section γ_t, i.e.*

$$A_{\xi_0}(\Sigma)(t) := \int_{\Sigma, \xi_0 \leq t} \xi_0 \leq \frac{|\gamma_t|_{\tilde{g}}}{k}(t^2-1)^{k/2}, \quad A(\Sigma)(t) \leq |\gamma_t|_{\tilde{g}} \int_1^t (s^2-1)^{\frac{k}{2}-1} ds \tag{18}$$

2. *Let m be the density of Σ at the origin, then*

$$A_{\xi_0} \geq m \frac{\omega_{k-1}}{k}(t^2-1)^{k/2} \tag{19}$$

In particular, if Σ contains the center p then γ_t has $g_{\mathbb{H}^n}$-volume at least $\omega_{k-1}(t^2-1)^{k/2}$ and the boundary of Σ on the sphere at infinity has Euclidean volume at least ω_{k-1}.

Remark 32. *The $\tilde{g}$-volume $|\gamma|_{\tilde{g}}$ was called the* angle *of γ in [CG92b] and the area comparison with cone* (18) *was proved there without assuming that the boundary lies on one level set of ξ_0. However, because Theorem 16 holds for extension of minimal surfaces by ξ_0-tube, we can always reduce to this case.*

4.2 Area-minimising cones and the annuli M_C

Area-minimising cones in Euclidean space appear naturally as *oriented tangent cones* of an area-minimising surface (see [Mor16]). It follows by Corollary 28 that in $\mathbb{R}^n$ and $\mathbb{H}^n$, minimising cones are the only minimal surfaces bounded by their section. It is clear that a hyperbolic minimising cone is also Euclidean minimising. The converse is also true, as pointed out in [And82]. We provide here a proof using the Comparison Lemma 25.

Proof of Theorem 1. A $\mathbb{H}^n$-minimal surface Σ^k asymptotic to γ would satisfies Euclidean monotonicity and thus by Lemma 27 would have Euclidean area smaller than that of C_γ. This contradicts the hypothesis on C_γ. □

Theorem 1 can be illustrated in the Poincaré four-ball $\mathbb{B}^4 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 \leq 1\}$ as follows. We denote by L_ϵ the Hopf link cut out in $\mathbb{S}^3$ by the complex curve $C_\epsilon : zw = \epsilon$, $\epsilon \in [0, \frac{1}{2})$, of $\mathbb{C}^2$. Each L_ϵ bounds a pair of 2-planes which, because of the complex curves, is not Euclidean-minimising unless $\epsilon = 0$. Therefore the pair of planes, although totally geodesic, is not hyperbolic-minimising.

It is possible to construct explicitly the analog of C_ϵ in any radially conformally flat metric $g = e^{2\varphi(\rho)} g_E$. We assume that φ is a function of $\rho := |z|^2 + |w|^2$ and look for g-minimal surfaces that are invariant by the $\mathbb{S}^1$ action $(z,w) \mapsto (ze^{i\theta}, we^{-i\theta})$. If $\mathbb{C}^2$ is identified with the space of quaternions by $(z,w) \mapsto z + jw$ then this action corresponds to multiplication on the left by $e^{i\theta}$.

[2] This is also the Euclidean volume of its image under the radial projection centered at p onto $\mathbb{S}_\infty$.

We will obtain such surfaces by rotating a curve in the real plane $\operatorname{Im} z = \operatorname{Im} w = 0$. Such curve is given by $zw = F(\rho)$ where F is a real function on ρ. The minimal surface equation is equivalent to the following second order ODE of F

$$\frac{x'}{x} - \frac{y'}{y} + \frac{1}{\rho} + \frac{\varphi'}{2}\left[8 + \rho\left(\frac{x^2}{y^2} - 4\right)\frac{F'}{F}\right] = 0, \quad \text{where } x = F - F'\rho, \quad y = \frac{F'}{2}\sqrt{\rho^2 - 4F^2}$$

which should reduce to a first order ODE. This is because we can obtain more solution curves by rotating a given one in the real plane. This second rotation corresponds to multiplying on the right of $z + jw$ by $e^{j\alpha}$ and it commutes with the left multiplication by $e^{i\theta}$.

Concretely, by a change of variable $F = \frac{\rho}{2}\sin\theta(\rho)$ the previous ODE reduces either to the first order Bernoulli equation $\theta'^2 = -\rho^2 + C^2\rho^4 e^{4\varphi}$ for a parameter $C > 0$, or to $\theta' = 0$ which corresponds to pairs of 2-planes. This is the same parameter C of Proposition 2. In Figure 1, the solution curves in $\mathbb{H}^4$ and $\mathbb{S}^4$ are drawn with different values of C.

Remark 33. 1. *When g is the Euclidean metric,* (1) *is the hyperbola equation.*

2. *For the hyperbolic space, $e^\varphi = \frac{2}{1-\rho}$. The total angle θ_C wiped by the profile curve converges to 0 as $C \to 0$ and to $\frac{\pi}{2}$ as $C \to \infty$. The profile curves expectedly meet the unit circle at right angle.*

3. *For the round sphere, $e^\varphi = \frac{2}{1+\rho}$ and the parameter C can only be chosen in $(4, \infty)$. The angle θ_C can take any value between $\frac{\pi}{2}$ ($C = \infty$, M_C being a totally geodesic $\mathbb{S}^2$) and $\frac{\pi}{\sqrt{2}}$ ($C = 4$, M_C being a Clifford torus). In particular, if θ_C is a rational multiple of π in this interval, we can close the surface by repeating the profile curve. This produces a countable family of immersed tori in $\mathbb{S}^4$ that are invariant by the quaternionic rotation.*

4.3 Convex hull and renormalised area of minimal surfaces in $\mathbb{H}^n$

We are interested in minimal surfaces Σ^k that are asymptotic to a properly embedded submanifold γ^{k-1} in $\mathbb{S}_\infty$. The Poincaré ball model $\mathbb{B}^n$ provides a C^∞ compactification $\overline{\mathbb{H}^n}$ of $\mathbb{H}^n$, we will assume that all surfaces considered in this section are C^2 near $\mathbb{S}_\infty$ and that its interior lies strictly inside the interior of $\mathbb{B}^n$.

It was observed in [And82] that if the boundary of a minimal surface lies on one half of the hyperbolic space cut out by a totally geodesic hyperplane H then the entire surface also lies on that side. One can see this by applying maximum principle[3] to the space coordinate ξ_1 whose zero set is H, which satisfies $\Delta\xi_1 = k\xi_1$ on any minimal surface of dimension k. It follows that a minimal surface is contained in the *convex hull* of its boundary, defined the the intersection of all half spaces containing it. An immediate application of the convex hull is that such Σ meets the sphere at infinity at a right angle, i.e. the normal vector of $\mathbb{S}_\infty$ is contained in the tangent of Σ.

[3] When Σ is merely an integral current whose boundary is supported in $\xi_1 < 0$, replace maximum principle by the first variation formula with the perturbative vector field $X = \chi(\xi_1)\nabla\xi_1$ where χ is an increasing function supported in $[0, \infty]$.

Using the convex hull and the unweighted monotonicity theorem, Anderson solved the asymptotic Plateau problem. In [And82], an area-minimising locally integral k-current asymptotic to a given immersed closed $(k-1)$-submanifold of $\mathbb{S}_\infty$ was constructed.

We note that there is also a notion of convex hull for minimal surfaces in $\mathbb{S}^n$ that could be contained in one hemisphere due to Lawson [Law70].

A *boundary defining function* of $\mathbb{H}^n$ is a non-negative function ρ on the compactification $\overline{\mathbb{H}^n}$ that vanishes exactly on $\mathbb{S}_\infty$ and exactly to first order. Such function is called *special* if $|d\ln\rho|_{g_{\mathbb{H}^n}} = 1$ on a neighborhood of the boundary. It was proved by Graham and Witten in [GW99] that for any C^2 minimal surface Σ^2 and any special boundary defining function ρ, the area functional has the expansion

$$A(\Sigma \cap \{\rho \geq \epsilon\}) = \frac{|\gamma|_{\bar{g}}}{\epsilon} + \mathcal{A}_R + O(\epsilon) \tag{20}$$

where $\bar{g} = \rho^2 g$ and the coefficient $\mathcal{A}_R$, called *renormalised area* of Σ, is independent of the choice of ρ.

The function $\rho = \xi_0^{-1}$ can be used as a boundary defining function. Although it is not special, it is third order close to one: The function $\bar{\rho} = 2\frac{1-r}{1+r}$ is special and $\rho = \frac{\bar{\rho}}{1+\frac{\bar{\rho}^2}{4}} = \bar{\rho} + O(\bar{\rho}^3)$. Because the metrics $\bar{g}$ and $\tilde{g}$ coincide on $\mathbb{S}_\infty$, Graham–Witten expansion (20) can be rewritten as

$$A(\Sigma)(t) = |\gamma|_{\tilde{g}} t + \mathcal{A}_R + O(t^{-1}) \tag{21}$$

where $A(\Sigma)(t)$ is the unweighted area of Σ in the region $\xi_0 \leq t$.

4.4 An upper bound of renormalised area

Proof of Theorem 8. It suffices to prove that

$$\mathcal{A}_R(\Sigma) + |\gamma|_{\tilde{g}} \leq 0 \tag{22}$$

for any minimal surface Σ (not necessarily containing the origin). Let $\gamma_t := \Sigma \cap \xi_0^{-1}(t)$ and T be the radial cone with section γ_t. By Corollary 31, one has

$$A(\Sigma)(t) \leq A(T) = |\gamma_t|_{\tilde{g}}(t-1) \quad \forall t \geq 1.$$

Because Σ meets $\mathbb{S}_\infty$ at right angle, one has $|\gamma_t|_{\tilde{g}} = |\gamma|_{\tilde{g}} + O(t^{-2})$, which together with the Graham–Witten expansion 21 implies (22) □

Remark 34. 1. *The renormalised area of a totally geodesic copy of $\mathbb{H}^2$ in $\mathbb{H}^n$ is -2π. The estimate* (2) *is verified in this case because round circles of $\mathbb{S}^{n-1}$ have length at most 2π.*

2. *The upper bound $\mathcal{A}_R \leq -2\pi$ for minimal disks was known in [AM10], which proved that $\mathcal{A}_R(\Sigma) \leq -2\pi\chi$ where χ is the Euler characteristic of Σ.*

3. *The estimate* (2) *can also be seen as an upper bound for the length of a curve γ under round metrics. Note that the space of round metrics in the standard conformal sphere is not compact and we used the fact that γ bounds a C^2 surface. The spiral curve*

$$\theta(r) = \left(r^{1/4} - r^{-1/4}\right)^2 \tag{23}$$

which turns around the origin of $\mathbb{R}^2 \cup \{\infty\} \cong \mathbb{S}^2$ infinitely many times has finite length under any round metric, but its length can be arbitrarily large. On the other hand, one can prove that round metric lengths have an upper bound when the curve has positive normal injectivity radius.

4.5 Space monotonicity

We can also estimate renormalised area using the space coordinate ξ_1 of Example 14. Recall that the weights P_i in the Comparison Lemma 25 only need to be C^0.

Lemma 35. *For any $k \geq 2$ and $\alpha > 0$, the following weights are weaker than $U(\xi_1) = \xi_1$ in the region $\xi_1 \geq 0$:*

$$P_1(t) = \begin{cases} t, & \textit{when } t \leq \alpha \\ \alpha, & \textit{when } t > \alpha \end{cases} \qquad \textit{and} \qquad P_2(t) = \alpha \tanh \frac{t}{\alpha}$$

The function $\rho = |\xi_1|^{-1}$ is a boundary defining function except on the equator and it is special up to third order. This is because the function $l = |\operatorname{arsinh} \xi_1|$ which computes the distance to the totally geodesic hyperplane $\xi_1^{-1}(0)$ satisfies $|dl| = 1$, hence $\bar{\rho} = 2\exp(-l)$ is a special boundary defining function. The latter is related to ρ by $\rho = \frac{\bar{\rho}}{1-(\bar{\rho}/2)^2} = \bar{\rho} + O(\bar{\rho}^3)$.

The Graham–Witten expansion (21) is still applicable to ρ and can be rewriten as

$$A(\Sigma \cap \{\xi_1 \leq t\}) = |\gamma_\infty|_{\tilde{g}} t + \mathcal{A}_R + O(t^{-1}) \tag{24}$$

where $\tilde{g}$ is the doubled hyperbolic metric described in Example 14, which coincides with $\rho^2 g_{\mathbb{H}^n}$ on $\mathbb{S}_\infty$. The space versions of Corollary 31 and Theorem 8 are as follow.

Corollary 36. *Let Σ be a minimal surface with boundary as in the paragraph preceding Theorem 21. Then for all $t > 0$,*

$$\frac{(t^2+1)^{k/2}}{k} \int_{\Sigma, \xi_1 = 0} |\nabla^\Sigma \xi_1| \leq B_{\xi_1} \leq \frac{|\gamma_t|_{\tilde{g}}}{k} (t^2+1)^{k/2}$$

In particular, $|\gamma_\infty|_{\tilde{g}} \geq \int_{\Sigma, \xi_1 = 0} |\nabla^\Sigma \xi_1|$.

Theorem 37. *Assume in addition to the hypothesis of Corollary 36, that $k = 2$. Then for all $\alpha > 0$, one has*

$$\alpha \mathcal{A}_R(\Sigma) - \int_{\Sigma, 0 \leq \xi_1 \leq \alpha} (\alpha - \xi_1) + \frac{1}{2} \int_{\xi_1 = 0} |\nabla^\Sigma \xi_1| \leq \frac{1-\alpha^2}{2} |\gamma_\infty|_{\tilde{g}} \tag{25}$$

In particular, if $\xi_1 \geq \alpha > 0$ on Σ, one obtains the estimate of Theorem 10:

$$\mathcal{A}_R(\Sigma) + \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)|\gamma_\infty|_{\bar{g}} \leq 0 \tag{26}$$

Proof of Theorem 37. By Lemma 35, the monotonicity theorem holds for weight P_1, which by Corollary 28 implies

$$B_{P_1}(\Sigma)(t) \leq B_{P_1}(T_{\gamma_t})(t) = |\gamma_t|_{\bar{g}}\left(\alpha t + \frac{1-\alpha^2}{2}\right) \quad \forall t > 0 \tag{27}$$

where $T_{\gamma_t}(0,t)$ is the ξ_1-tube built upon $\gamma_t := \Sigma \cap \xi_1^{-1}(t)$. By definition

$$B_{P_1}(\Sigma)(t) = \frac{1}{2}\int_{\Sigma,\xi_1=0} |\nabla^\Sigma \xi_1| - \int_{\Sigma,0\leq\xi_1\leq\alpha}(\alpha - \xi_1) + \alpha A(\Sigma)(t) \tag{28}$$

One obtains (25) by replacing $A(\Sigma)(t)$ in (28) with the expansion (24) and by using $|\gamma_t|_{\bar{g}} = |\gamma_\infty|_{\bar{g}} + O(t^{-2})$ which is because Σ meets $\mathbb{S}_\infty$ at a right angle. $\square$

Remark 38. *It is possible to prove Corollary 9 from the estimate* (26)*. Since the function $y := \frac{\xi_1}{\xi_0}$ foliates $\mathbb{H}^n$ into totally geodesic codimension 1, by a Mobius transform, one can put the boundary curve of Σ, and by convexity the entire minimal surface, between level sets $y = \beta$, and $y = \beta + \epsilon, \beta > 0$. This guarantees that $\xi_1 \geq \alpha := \frac{\beta}{\sqrt{1-\beta^2}}$ on the surface. The Mobius transform can be chosen so that the $\bar{g}$-length of the boundary curve is ϵ-near to that of a great circle on $\{y = \beta\} \cap \mathbb{S}_\infty$, which is $2\pi\frac{\sqrt{1-\beta^2}}{\beta}$. Combine all these with* (26) *and send ϵ to 0, one has $\mathcal{A}_R + 2\pi(1 - \frac{1}{2\beta^2}) \leq 0$ for all $\beta > 0$, which means $\mathcal{A}_R \leq -2\pi$. Figure 2 illustrates the level sets of $y = \frac{\xi_1}{\xi_0}$ and those of ξ_1 in the Poincaré model.*

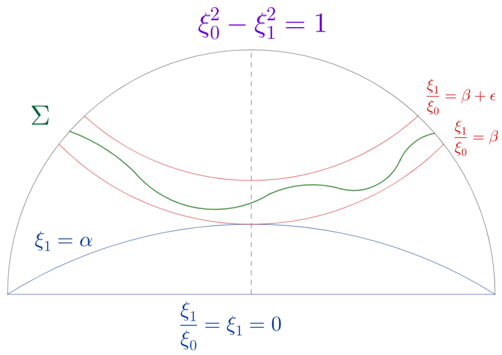

Figure 2: Level sets of y (totally geodesic, in red) and those of ξ_1 (in blue). The $(n-1)$-dimensional disks $y = \beta$ and $\xi_1 = \alpha = \frac{\beta}{\sqrt{1-\beta^2}}$ touch each other at the center.

5 Weighted monotonicity in spaces with curvature bounded from above

Fix a point O in a Riemannian manifold (M^n, g), and let r_{inj} be the injectivity radius at O. The Hessian of the distance function r to O is given by:

$$\operatorname{Hess}_p r(\partial_r, \cdot) = 0, \qquad \operatorname{Hess}_p r(v, v) =: I(v), \quad \forall p \in B(O, r_{\text{inj}}),\ \forall v \perp \partial_r \tag{29}$$

where $I(v) = \int_\Gamma \left(|\dot{V}|^2 - K_M(\dot{\gamma}, V)|V|^2\right)$ is the index form of the Jacobi field V along the geodesic Γ between O and p that interpolates 0 at O and v at p.

When the sectional curvature satisfies $K_M \leq -a^2$ (respectively b^2), one can check that $I(v) \geq a\coth(ar)|v|^2$ (respectively $b\cot(br)|v|^2$). This gives an estimate of the $\operatorname{Hess} r$ on the directions orthogonal to ∂_r. By a change of variable, we can estimate the Hessian in a more isotropic way.

Proposition 39. *Inside $B(O, r_{\text{inj}})$, one has*

1. $\operatorname{Hess}(a^{-2}\cosh ar) \geq \cosh ar.\, g$ *if* $K_M \leq -a^2$.
2. $\operatorname{Hess}(-b^{-2}\cos br) \geq \cos br.\, g$ *if* $K \leq b^2$ *and* $r \leq \frac{\pi}{b}$.

This means that the functions $h = a^{-2}\cosh ar$ and $h = -b^{-2}\cos br$ satisfy $\operatorname{Hess} h \geq U.g$. We note that the functions U, V defined as in Proposition 11 are still functions of h: $U = a^2 h$ (respectively $-b^2 h$) and $V = |\nabla h|^2 = a^2(h^2 - 1)$ (respectively $-b^2(h^2 - 1)$) and one still has $U = \frac{1}{2}V'$.

The *eligible interval* $[0, r_{\max})$ is defined to be $[0, r_{\text{inj}})$ when $K_M \leq -a^2$ and $[0, \min(r_{\text{inj}}, \frac{\pi}{2b}))$ when $K_M \leq b^2$.

Remark 40.
1. *When M is $\mathbb{H}^n$ or $\mathbb{S}^n$, the function h is the time-coordinate ξ_0 and the Euclidean coordinate x in Example 14 and Example 13.*
2. *It follows from maximum principle and Proposition 39 that there exists no closed minimal surface in $B(O, r_{\max})$. If M is Cartan–Hadamard ($r_{\max} = +\infty$) and if the boundary of a minimal surface is contained in a geodesic ball, the entire surface stays inside that ball.*

5.1 Weighted monotonicity theorem

As explained in Remark 17, we can still have weighted monotonicity theorem for the function h. The *weighted area* and *weighted density* of a surface Σ^k are defined to be

$$\bar{A}(\Sigma)(t) := \int_{\Sigma, r \leq t} U, \quad \bar{\Theta}(t) := \frac{\bar{A}(\Sigma)(t)}{Q(t)}, \quad Q(t) := \begin{cases} \frac{\omega_{k-1}}{k}\frac{\sinh^k at}{a^k}, & \text{when } K_M \leq -a^2 \\ \frac{\omega_{k-1}}{k}\frac{\sin^k bt}{b^k}, & \text{when } K_M \leq b^2 \end{cases}$$

Note that Q is the weighted area of a ball of radius t in the k-dimensional space-forms of curvature $-a^2$ or b^2 and that the density converges to 1 as t decreases to 0.

Theorem 41. *Let M be a Riemannian manifold with sectional curvature $K_M \leq -a^2$ or $K_M \leq b^2$ and $\Sigma^k \subset M$ be an extension of a minimal surface by geodesic cone, then the density $\bar{\Theta}(\Sigma)(t)$ is an increasing function in the eligible interval.*

By a different argument than that of Corollary 29 and Corollary 31, one can prove that the intersection curve of a minimal surface with a geodesic sphere of M is longer than the great circle of the sphere of same radius in space-form.

Corollary 42. *Let $\Sigma^k \subset M$ be a minimal surface containing the point O with multiplicity m and $l_t := \mathrm{vol}_{k-1}\left(\Sigma \cap r^{-1}(t)\right)$. For all t in the eligible interval, one has*

$$Q(t) \leq \bar{A}(\Sigma)(t) \leq \frac{l_t}{k} V(t)^{1/2}$$

In particular, one has

$$l_t \geq \begin{cases} m\omega_{k-1}\left(\frac{\sinh at}{a}\right)^{k-1}, & \text{if } K_M \leq -a^2 \\ m\omega_{k-1}\left(\frac{\sin bt}{b}\right)^{k-1}, & \text{if } K_M \leq b^2 \end{cases}$$

Proof. Instead of Lemma 27 (see Remark 47), the upper estimate of $\bar{A}$ follows from (11):

$$\bar{A}(\Sigma)(t) \leq \frac{1}{k}\int_{\Sigma, h=t} |\nabla^\Sigma h| \leq \frac{V(t)^{1/2}}{k} l_t.$$

□

5.2 Comparison lemma

It is more convenient see a weight P as a non-negative continuous function of r instead of h. The *P-area* is defined as $A_P(\Sigma)(t) := \int_{\Sigma, r\leq t} P(r)$ and the *P-density* is $\Theta_P := \frac{A_P}{Q}$ where $Q = Q_P$ is P-area of a ball of radius r in space-form:

$$Q(t) := \begin{cases} \omega_{k-1}\int_{r\leq t} P(r)\frac{\sinh^{k-1} ar}{a^{k-1}}dr, & \text{when } K_M \leq -a^2 \\ \omega_{k-1}\int_{r\leq t} P(r)\frac{\sin^{k-1} br}{b^{k-1}}dr, & \text{when } K_M \leq b^2 \end{cases} \tag{30}$$

Lemma 43 (Comparison)**.** *Let $\Sigma^k \subset M$ be any surface not necessarily minimal and P_1, P_2 be two non-negative, continuous weight functions. Define Q_1, Q_2 from P_1, P_2 as in* (30)*.*

1. *If P_1 is weaker than P_2, i.e. $\frac{P_1}{Q_1} \leq \frac{P_2}{Q_2}$, and $\frac{d}{dt}\Theta_2 \geq 0$ in the eligible interval, then one has $\Theta_1 \leq \Theta_2$ and*
$$\frac{d\Theta_1}{dt} \geq \frac{Q_2}{P_2}\frac{P_1}{Q_1}\frac{d\Theta_2}{dt} \geq 0$$

2. *If P_2 is weaker than P_1 and $\frac{d}{dt}\Theta_2 \geq 0$ in the eligible interval, then one has $\Theta_1 \geq \Theta_2$*

We note that it is necessary to mention a or b in order to compare two weights. However, it can be checked that

Lemma 44. *For any $a, b \geq 0$ and $u \geq v \geq 0$,*

1. *$P_1 = \cosh vr$ is weaker than $P_2 = \cosh ur$ when $K_M \leq -a^2$,*
2. *$P_1 = \cos ur$ is weaker than $P_2 = \cos vr$ in the interval $t \leq \frac{\pi}{2u}$ when $K_M \leq b^2$.*

Remark 45. *1. It follows from Lemma 44 and Theorem 41 that for negatively curved space $K_M \leq -a^2$, the monotonicity theorem holds for any weight $P_u = \cosh ur$ with $u \in [0, a)$ and in particular the uniform weight $P_0 = 1$. One recovers the Theorem 1 of [And82].*

2. When $K_M \leq b^2$, the monotonicity theorem holds for any weight $P_u = \cos ur$ with $u \in [b, \infty)$ and the first part of Comparison Lemma could not be used on the uniform weight. However, one can still use the second part to obtain $\Theta(t) \geq \bar{\Theta}(t) \geq$ multiplicity at O.

Proposition 46. *Suppose that $K \leq b^2$, the minimal surface Σ^k contains a point O with multiplicity m and it has no boundary in the interior of $B(O, t)$ for certain $t < r_{\max}$. Then*

$$A(\Sigma \cap B(O, t)) \geq m\omega_{k-1} \int_{r=0}^{t} \frac{\sin^{k-1}(br)}{b^{k-1}} dr$$

In particular, if M is simply connected, with curvature pinched between $\frac{b^2}{4}$ and b^2 and $\Sigma \subset M$ is a closed minimal surface, then

$$A(\Sigma) \geq \frac{1}{2}\omega_k b^{-k}. \tag{31}$$

A weaker version of inequality (31), with $\frac{1}{2}\omega_k$ replaced by the volume of the *unit k-ball*, was proved in [HS74].

Remark 47. *Lemma 27 does not generalise because the P-area of a geodesic cone in M is no longer proportional to Q.*

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