Classification theorem in Riemannian geometry

darknmt

August 19, 2017

Contents

The PDF version of this page is automatically generated (without any guarantee of readability or accuracy) and can be downloaded by replacing `html` in the page address by `pdf`. For example `/html/sheaf-cohomology.html` should become `/pdf/sheaf-cohomology.pdf`.

This is part of my reading in the summer of 2017. I will try not to mention the details here as they will be covered in future posts. The big question is to

1. first determine which groups can be a holonomy group (results of E. Cartan then Berger).
2. then study Bogomolov-Beauville result: hyperkahler manifolds and Calabi-Yau manifolds are building blocks of Kahler manifold with vanishing Chern class.

1 What is holonomy and why it is important

Holonomy is the action of closed curves starting from a point x in a Riemannian manifold M to the tangent space T_xM. It is important because

1. Knowing the holonomy means knowing the parallel tensors, in fact each parallel tensor is uniquely determined by its value on a fiber T_xM (for other fibers just transport), and this value has to be invariant by the action of holonomy.

2. The smaller the holonomy is, the more structure the manifold can have. For example, if somehow one knows that the holonomy group is in $SU(n)$ for a manifold of dimension $2n$, then as $SU(n)$ is exactly those that preserve a almost complex structure J, by transporting J, we see that M has a almost complex structure.

3. It serves as a geometric invariant (unlike the cohomology, which are topological invariants)

Note that we defined the holonomy group by its representation. In most case, we only consider Levi-Civita connection and the holonomy representation is then orthogonal.

2 De Rham decomposition

De Rham theorem allows us to decompose a Riemannian manifold under certain conditions (complete and connected) as Riemannian product of complete connected manifold with *irreducible holonomy representation*.

The idea behind this results is to prove that if a fiber T_xM decompose to subspaces stable by holonomy, then by parallel transport TM also decompose. But by direct computation each component of TM is involutive and by Frobenius theorem locally tangent to a submanifold of M. The completeness allows us to join the local pieces.

Now it remains to deal with irreducible holonomies. If the manifold is *locally symmetric* then one can prove that it is isometric to the homogeneous space G/H with H (the holonomy) a closed Lie subgroup of G. The theory of Lie groups developed by E. Cartan gave a complete list of these spaces.

3 Berger classification for non-symmetric manifolds

For the non-symmetric manifold, Berger proved that a irreducible holonomy has to be one of the following

1. $SO(n)$

2. $U(m) \subset SO(2m)$

3. $SU(m) \subset SO(2m)$

4. $Sp(m) \subset SO(4m)$

5. $SO(m)Sp(1) \subset SO(4m)$

6. $G_2 \subset SO(7)$

7. $\mathrm{Spin}(7) \subset SO(8)$

where n is the dimension. Here are some notations, note always that

$$Sp(m) \subset SU(2m) \subset U(2m) \subset SO(4m)$$

1. If $Hol(g) \subset U(m) \subset SO(2m)$, g is called a *Kahler metric.*

2. If $Hol(g) \subset SU(m) \subset SO(2m)$, g is called a *Calabi-Yau* metric. We will see that a Calabi-Yau metric is a Kahler metric that is also Ricci-flat.

3. If $Hol(g) \subset Sp(m) \subset SO(4m)$ then g is called a *hyperkahler* metric.

4. G_2 and $\mathrm{Spin}(7)$ are called *exceptional holonomies*

Therefore hyperkahler $\longrightarrow$ Calabi-Yau $\longrightarrow$ Kahler

4 Chern class and Calabi conjecture (Yau theorem)

The Chern class $c_1(M)$ of a Riemannian manifold is the class in $H^{1,1}(M)$ of the Ricci curvature. It is easy to see that two different kahler metric give the same Chern class. The conjecture posed by Calabi asks **whether we can modify the kahler metric to attain, as Ricci curvature, every** $(1,1)$ **form in the original Chern class**. The response, which is positive, was given by S.T. Yau.

Theorem 1 (Calabi-Yau)**.** *Let M be a compact manifold with a Kahler form ω and the corresponding Ricci curvature R_ω. Let R be a $(1,1)$ form cohomologous to R_ω, then there exists a Kahler form ω' cohomologous to ω such that R' is the Ricci curvature of ω'.*

5 Bogomolov-Beauville classification

Bogomolov result, later refined by Beauville showed that the building blocks of Kahler manifolds with vanishing Chern class are the complex torus, simply-connected Hyperkahler manifolds or simply-connected Calabi-Yau manifolds.

Theorem 2 (Bogomolov-Beauville)**.** *Let M be a compact manifold of Kahler type with $c_1(M) = 0$ then there exists a finite etale covering space $\tilde{M}$ of M of form*

$$\tilde{X} = T \times \prod_i V_i \prod_j X_j$$

where V_i is a simply-connected, projective manifold with $H^0(V_i, \Omega^*_{V_i}) = \mathbb{C} \oplus \mathbb{C}\omega$, $\omega \in \Omega^{n_i}$ and X_j irreducible symplectic manifold.

Emacs 25.2.1 (Org mode 9.0.5)