Polynomial differential operators and Besov spaces

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Definition 1. *We say that P is a **polynomial differential operator of type** (n,k) if P is of the form*

$$P(F) = \sum c_{\alpha_1,\dots,\alpha_\nu}(x, F(x)) D^{\alpha_1} F^{a_1} \dots D^{\alpha_\nu} F^{a_\nu}$$

where the coefficients $c_{\alpha_1,\dots,\alpha_n u}$ depend smoothly and nonlinearly on x and F and $\alpha_i \in \mathbb{R}^N$ are indices with the weighted norm $\|\alpha_i\| \le k$ and $\sum \|\alpha_i\| \le n$.

Example 2. *On $M \times [\alpha, \omega]$ the tension field $\tau(F) := -\Delta F^\alpha + g^{ij}\Gamma'^\alpha_{\beta\gamma}(F) F^\beta_i F^\gamma_j$ is a polynomial differential operator of type (2,2). The quadratic term alone is of type (2,1).*

1 A regularity estimate for polynomial differential operator.

Our goal in this part is to prove the following estimate for polynomial differential operator, in which X will be $M \times [\alpha, \omega]$.

Theorem 3 (Regularity of polynomial differential operator)**.** *Let X be a compact Riemannian manifold, $B \subset \mathbb{R}^N$ is a large Euclidean ball and P be a polynomial differential operator of type (n,k) on X. Suppose that*

$$r \ge 0, \quad p, q \in (1, \infty), \quad r + k < s, \quad \frac{1}{p} > \frac{r+n}{s}\frac{1}{q}. \tag{1}$$

Then for all $F \in C(X, B) \cap W^{s,q}(X)$, one has $PF \in W^{r,p}(X)$ and

$$\|PF\|_{W^{r,p}} \le C\left(1 + \|F\|_{W^{s,q}}\right)^{q/p}.$$

where C is a constant independent of F.

We will prove that the result is *local*, in a sense to be defined. Then we will prove the local statement using Besov spaces.

Proof (reduction of Theorem 3 to a local statement). Let $\{\varphi_i : U_i \longrightarrow V_i\}$ be an atlas of M. We denote a point in U_i by x and its coordinates in V_i by ξ. Let $\sum \psi_i = 1$ be a partition of unity subordinated to $\{U_i\}$ and $\tilde{\psi}_i$ be smooth functions supported in U_i with $0 \le \tilde{\psi}_i \le 1$ and $\tilde{\psi}_i = 1$ in the support of ψ_i, as in the definition of Sobolev spaces on manifold. We suppose the following local statement is true:

Lemma 4 (Local statement)**.** *Let P be a polynomial differential operator of type (n,k) and coefficients $c_{\alpha_1,\ldots,\alpha_\nu}(x,F)$ are smooth and vanish when $x \in \mathbb{R}^{\dim X}$ is outside of a compact. Let $B \subset \mathbb{R}^N$ be a large Euclidean ball and r,p,q,s as in* (1)*. Then for all compactly supported $F \in C(\mathbb{R}^{\dim X}, B) \cap W^{s,q}(\mathbb{R}^{\dim X})$, one has*

$$\|PF\|_{W^{r,p}} \le C\,(1 + \|F\|_{W^{s,q}})^{q/p}$$

where the constant C depends only on B and the support of F, and not on F.

One has

$$\|PF\|_{W^{r,p}} := \sum_i \|\psi_i PF\|_{W^{r,p}}$$

where viewed in the chart U_i, each $\psi_i(x)PF(x)$ is $\sum_\alpha \psi_i(\xi).c_\alpha(\xi, g_i).D^\alpha g_i$ where $g_i = f_i \circ \varphi_i^{-1}$ is f_i viewed in the chart. Since $\tilde{\psi}_i = 1$ in the support of ψ_i, one has

$$\psi_i(\xi).c_\alpha(\xi, g_i).D^\alpha g_i = \psi_i(\xi).c_\alpha(\xi, \tilde{\psi}_i g_i) D^\alpha(\tilde{\psi}_i g_i)$$

hence by the local statement:

$$\|\psi_i(\xi).c_\alpha(\xi, g_i).D^\alpha g_i\|_{W^{r,p}} \le C\,(1 + \|\tilde{\psi}_i g_i\|_{W^{s,q}})^{q/p} \le C\,(1 + \|F\|_{W^{s,q}})^{q/p}.$$

Therefore $\|PF\|_{W^{r,p}} \le mC\,(1 + \|F\|_{W^{s,q}})^{q/p}$ where m is the number of charts we used to cover M. □

Remark 5. *The use of partition of unity in the last proof is to decompose $PF = \sum \psi_i PF$ and not $F = \psi_i F$ since we no longer have linearity of the operator P in F.*

2 Review of Besov spaces $B^{s,p}$.

In this part, $X = \mathbb{R}^n$ coordinated by $(x_1, \ldots, x_n)$ with weight $(\sigma_1, \ldots, \sigma_n)$. We define

$$T_j^v f(x_1, \ldots, x_n) := f(x_1, \ldots, x_j + v, \ldots, x_n), \quad \Delta_j^v := T_j^v - \mathrm{Id}$$

for $f \in \mathcal{S}(X)$.

For the notation, we will denote the Besov spaces by $B^{s,p}$ with $s \in \mathbb{R}_{>0} \setminus \mathbb{Z}$ and $p \in (1,\infty)$ so that they look similar to Sobolev space $W^{s,p}$. In a more standard notation, our spaces $B^{s,p}$ are denoted by $B^s_{p,p}$

Definition 6. *We define $B^{s,p}$ as the completion of $\mathcal{S}(X)$ under the norm*

$$\|f\|_{B^{s,p}} := \sum_{\|\gamma\|<s} \|D^\gamma f\|_{L^p} + \sum_{s-\frac{\sigma}{\sigma_j}<\|\gamma\|<s} \sup_v \frac{\|\Delta_j^v D^\gamma f\|_{L^p}}{|v|^{(s-\|\gamma\|)\sigma_j/\sigma}}$$

We cite here some well-known facts

1. While Sobolev spaces with non-integral regularity are complex interpolation of integral ones, Besov spaces are their real interpolation.

2. Besov spaces $B^{s,p}(X)$ are reflexive Banach spaces with their dual spaces being $B^{-s,p'}(X)$ where $\frac{1}{p}+\frac{1}{p'}=1$.

Theorem 7. *If $r<s$ then*

$$W^{s,p}(X) \subset B^{s,p}(X) \subset W^{r,p}(X).$$

Theorem 8 (Multiplication)**.** *For $f,g \in \mathcal{S}(X)$ and $\begin{cases} 0<\alpha<1, \tilde{p}\le p, \tilde{q}\le q, \tilde{r}\le r \\ \frac{1}{p}+\frac{1}{q}=\frac{1}{r}, \frac{1}{\tilde{p}}+\frac{1}{q}=\frac{1}{p}+\frac{1}{\tilde{q}}=\frac{1}{\tilde{r}} \end{cases}$, one has*

$$\|fg\|_{B^{\alpha,\tilde{r}}} \le C\left(\|f\|_{B^{\alpha,\tilde{p}}}\|g\|_{L^q} + \|f\|_{L^p}\|g\|_{B^{\alpha,\tilde{q}}}\right) \tag{2}$$

$$\|fg\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q} \tag{3}$$

Therefore by density (2) *is true for all $f \in L^p \cap B^{\alpha,\tilde{p}}, g \in L^q \cap B^{\alpha,\tilde{q}}$ and* (3) *is true for all $f \in L^p, g \in L^q$.*

The reason for which we use the Besov norm is the following estimate:

Theorem 9 (Composition)**.** *Let $\Gamma(x,y)$ be a continuous, nonlinear function of variables $x \in \mathbb{R}^n, y \in \mathbb{R}^N$. Suppose that Γ vanishes for all x outside of a compact in $\mathbb{R}^n$ and Γ is C-Lipschitz in y, and define*

$$\Gamma f := (x \longmapsto \Gamma(x, f(x))).$$

Then

$$\|\Gamma f\| \le C\left(1+\|f\|_{B^{\alpha,p}}\right)$$

3 Proof of the local estimate.

Since $B^{r+\epsilon,p}(X) \subset W^{r,p}(X)$, by increasing r a bit, we can suppose that $r \notin \mathbb{Z}$ and replace the $W^{r,p}$ norm in the statement by the $B^{r,p}$ norm, that is to estimate:

$$\|PF\|_{B^{r,p}} = \sum_{\|\gamma\|<r} \|D^\gamma(PF)\|_{L^p} + \sum_{r-\sigma/\sigma_j<\|\gamma\|<r} \frac{\|\Delta_j^v D^\gamma(PF)\|_{L^p}}{|v|^{(r-\|\gamma\|)\sigma_j/\sigma}}$$

where

$$D^\gamma(PF) = \sum c_{\beta_1,\dots,\beta_\mu}(x,F) D^{\beta_1} f^{b_1} \dots D^{\beta_\mu} f^{b_\mu} \tag{4}$$

with $\max \|\beta_i\| \leq k + \|\gamma\|$ and $\sum \|\beta_i\| \leq n + \|\gamma\|$.

Using $\Delta_j^v(fg) = \Delta_j^v f \; T_j^v g + f \Delta_j^v g$, one can see that $\Delta_j^v D^\gamma(PF)$ is a sum of terms of 2 types:

$$\Delta_j^v c_{\beta_1,\dots,\beta_\mu} \; T_j^v(D^{\beta_1} f^{b_1}) \dots T_j^v(D^{\beta_\mu} f^{b_\mu}) \tag{5}$$

and

$$c_{\beta_1,\dots,\beta_\mu} \; D^{\beta_1} f^{b_1} \; \dots \; D^{\beta_{i-1}} f^{b_{i-1}} \; \Delta_j^v(D^{\beta_i} f^{b_i}) \; T_j^v(D^{\beta_{i+1}} f^{b_{i+1}}) \; \dots \; T_j^v(D^{\beta_\mu} f^{b_\mu}) \tag{6}$$

Our strategy is to use Theorem 8 to estimate the terms (4), (5) and (6) as follows, where we denote $\|g\|_p := \|g\|_{L^p}$

$$\left\| c_{\beta_1,\dots,\beta_\mu}(x,F) \; D^{\beta_1} f^{b_1} \dots D^{\beta_\mu} f^{b_\mu} \right\|_p \leq \|c_{\beta_1,\dots,\beta_\mu}\|_\infty . \|D^{\beta_1} f^{b_1}\|_{p_1} \dots \|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \tag{7}$$

$$\left\| \Delta_j^v c_{\beta_1,\dots,\beta_\mu} \; T_j^v(D^{\beta_1} f^{b_1}) \dots T_j^v(D^{\beta_\mu} f^{b_\mu}) \right\|_p \leq \|\Delta_j^v c_{\beta_1,\dots,\beta_\mu}\|_{\tilde{p}_0} . \|D^{\beta_1} f^{b_1}\|_{p_1} \dots \|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \tag{8}$$

$$\begin{aligned} &\left\| c_{\beta_1,\dots,\beta_\mu} \; D^{\beta_1} f^{b_1} \; \dots \; D^{\beta_{i-1}} f^{b_{i-1}} \; \Delta_j^v(D^{\beta_i} f^{b_i}) \; T_j^v(D^{\beta_{i+1}} f^{b_{i+1}}) \; \dots \; T_j^v(D^{\beta_\mu} f^{b_\mu}) \right\|_p \leq \\ &\|c_{\beta_1,\dots,\beta_\mu}\|_\infty . \|D^{\beta_1} f^{b_1}\|_{p_1} \dots \|D^{\beta_{i-1}} f^{b_{i-1}}\|_{p_{i-1}} . \|\Delta_j^v(D^{\beta_i} f^{b_i})\|_{\tilde{p}_i} . \|D^{\beta_{i+1}} f^{b_{i+1}}\|_{p_{i+1}} \dots \|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \end{aligned} \tag{9}$$

Then continue by bounding the Δ_j^v terms:

$$\|\Delta_j^v c_{\beta_1,\dots,\beta_\mu}\|_{\tilde{p}_0} \leq |v|^{\theta\sigma_j/\sigma} C(1 + \|F\|_{B^{\theta,\tilde{p}_0}}) \leq |v|^{\theta\sigma_j/\sigma} C(1 + \|F\|_{W^{\theta,\tilde{p}_0}}) \tag{10}$$

using Theorem 9, where C is the Lipschitz constant of $c_{\beta_1,\dots,\beta_\mu}(x,F)$ in F, which exists because $c_{\beta_1,\dots,\beta_\mu}$ is smooth and F always remains in a large Euclidean ball B. The next Δ_j^v term to bound is, using Theorem 7:

$$\|\Delta_j^v(D^{\beta_i} f^{b_i})\|_{\tilde{p}_i} \leq |v|^{\theta\sigma_j/\sigma} \|f^{b_i}\|_{B^{\|\beta_i\|+\theta,\tilde{p}_i}} \leq |v|^{\theta\sigma_j/\sigma} \|f^{b_i}\|_{W^{\|\beta_i\|+\theta,\tilde{p}_i}} \tag{11}$$

And finally plugging (10) and (11) in (8) and (9), and noting that $\|c_{\beta_1,\dots,\beta_\mu}\|_\infty$ in (7) is bounded by a constant, it remains to estimate $\|f^{b_i}\|_{W^{\|\beta_i\|,p_i}}$, $\|f^{b_i}\|_{W^{\|\beta_i\|+\theta,\tilde{p}_i}}$ and $\|F\|_{W^{\theta,\tilde{p}_0}}$ in term of $\|F\|_{W^{s,q}}$, for which we will use the following consequence of Interpolation inequality.

Lemma 10. *Let $0 \leq r \leq s$ and $p, q \in (1,\infty)$ such that $0 < \frac{1}{p} - \frac{r}{s}\frac{1}{q} < 1 - \frac{r}{s}$. Then for all compactly supported $F \in C(X,B) \cap W^{s,q}$ where $B \subset \mathbb{R}^N$ is a large Euclidean ball, one has*

$$\|F\|_{W^{r,p}} \leq C\|F\|_\infty^{1-r/s} \|F\|_{W^{s,q}}^{r/s} \leq C' \|F\|_{W^{s,q}}^{r/s}$$

where C, C' depend only on B and the support of F, but not F.

Proof. Since F is bounded, $f^\alpha \in W^{s,q} \cap W^{0,v}$ for all $v > 1$. By Interpolation inequality

$$\|f^\alpha\|_{W^{r,p}} \leq 2\|f^\alpha\|_{W^{s,q}}^{r/s}\|f^\alpha\|_{W^{0,v}}^{1-r/s}$$

then choose v with $(1-\frac{r}{s})\frac{1}{v} = \frac{1}{p} - \frac{r}{s}\frac{1}{q}$. □

To apply Lemma 10, we have to choose $p_i, \tilde{p}_i, \tilde{p}_0, \theta$ such that $\begin{cases} 0 < \frac{1}{p_i} - \frac{\|\beta_i\|}{s}\frac{1}{q} < 1 - \frac{\|\beta_i\|}{s}, \\ 0 < \frac{1}{\tilde{p}_i} - \frac{\|\beta_i+\theta\|}{s}\frac{1}{q} < 1 - \frac{\|\beta_i+\theta\|}{s} \\ 0 < \frac{1}{\tilde{p}_0} - \frac{\theta}{s}\frac{1}{q} < 1 - \frac{\theta}{s} \end{cases}$

We choose $\frac{1}{p_i}$ just a bit bigger than $\frac{\|\beta_i\|}{s}\frac{1}{q}$, $\frac{1}{\tilde{p}_i}$ just a bit bigger than $\frac{\|\beta_i+\theta\|}{s}\frac{1}{q}$ and $\frac{1}{\tilde{p}_0}$ just a bit bigger than $\frac{\theta}{s}\frac{1}{q}$. We will now come back to justify the estimates (7), (8), (9). Since F is bounded in B and compactly supported in an open set V, we see that $\|f^\alpha\|_p \leq C(B,V)\|f^\alpha\|_q$ if $p \leq q$. Therefore,

1. For (7), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{p_1} + \cdots + \frac{1}{p_\mu}$$

   which is true because the RHS is is a bit bigger than $\frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|}{qs} < \frac{n+r}{qs} < \frac{1}{p}$.

2. For (8), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{\tilde{p}_0} + \frac{1}{p_1} + \cdots + \frac{1}{p_\mu}$$

   where the RHS is is a bit bigger than $\frac{\theta}{s}\frac{1}{q} + \frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|+\theta}{qs}$.

3. For (9), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{p_1} + \cdots + \frac{1}{\tilde{p}_i} + \cdots + \frac{1}{p_\mu}$$

   where the RHS is is a bit bigger than $\frac{\theta}{s}\frac{1}{q} + \frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|+\theta}{qs}$.

It is sufficient then to take $\theta = r - \|\gamma\|$. Now the estimates (7), (8), (9) can be continued as

$$RHS(7) \leq \prod_i \|f^{b_i}\|_{W^{s,q}}^{\|\beta_i\|/s} \leq \|F\|_{W^{s,q}}^{\frac{n+\|\gamma\|}{s}} \leq \|F\|_{W^{s,q}}^{q/p} \tag{12}$$

$$RHS(8) \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|F\|_{W^{s,q}}^{\theta/s}\right)\prod_i \|f^{b_i}\|_{W^{s,q}}^{\|\beta_i\|/s} \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|F\|_{W^{s,q}}^{\theta/s}\right)\|F\|_{W^{s,q}}^{q/p} \tag{13}$$

$$RHS(9) \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|f^{b_i}\|_{W^{s,q}}^{\frac{\|\beta_i\|+\theta}{s}}\right)\prod_{u\neq i}\|f^{b_u}\|_{W^{s,q}}^{\|\beta_u\|/s} \leq |v|^{\theta\sigma_j/\sigma}\left(1+\|F\|_{W^{s,q}}^{\frac{\|\beta_i\|+\theta}{s}}\right)\|F\|_{W^{s,q}}^{q/p} \tag{14}$$

While (12) gives $\|D^\gamma(PF)\|_p \leq C\|F\|_{W^{s,q}}^{q/p}$, the last two (13) and (14) give

$$\sum_{s-\frac{\sigma}{\sigma_j}<\|\gamma\|<s} \sup_v \frac{\|\Delta_j^v D^\gamma(PF)\|_p}{|v|^{(r-\|\gamma\|)\sigma_j/\sigma}} \leq C\left(1+\|F\|_{W^{q,s}}^{(n+r)/s}\right)$$

We proved the local statement Lemma 4.