Global existence for nonlinear heat equation and harmonic maps between Riemannian manifolds

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Contents

Let M be a compact Riemannian manifold. We want to solve the following nonlinear heat equation where $F : M \longrightarrow M' \subset B \subset V = \mathbb{R}^N$:

$$\frac{dF_t}{dt} = -\Delta F_t + \Gamma(F_t)(\nabla F_t)^2$$

We have proved that the solution exists in short-time and is smooth whenever it exists. We will now establish long-time existence using continuity method: we will show that if the solution exists on $[\alpha, \omega_n]$ where ω_n is an increasing sequence to ω, then the solution exists on $[\alpha, \omega]$. We then apply short-time existence to gain a small open interval where solution still exists. We then conclude that the solution exists globally on $[\alpha, +\infty)$ since this interval is connected.

The crucial step to prove that the solution can be extended on $[\alpha, \omega]$ is to uniformly bound all of its derivatives in time of evolution $[\alpha, \omega)$. These estimates will also be useful to justify the convergence of F_t in $C^\infty(M)$ to a smooth function F_∞ which will eventually be a harmonic map from M to M'.

Recall that we proved in Corollary **??**, under the hypothesis of negative curvature, the boundedness of $\|F_t\|_{W^{2,2}(M)}$ by a constant C depending only on curvatures of M, M' and the initial total energies. Since $\frac{dF_t}{dt}$ relates to spatial derivatives of F by the nonlinear heat equation, it is easy to see that $\|F_t\|_{W^{2,2}(M\times[\tau,\tau+\delta])}$ is bounded by a constant independent of τ. We will denote $W^{k,p}(M \times [\beta, \gamma])$ by $W^{k,p}([\beta, \gamma])$.

Theorem 1 ($W^{2,2}$-boundedness)**.** *Suppose* $\mathrm{Riem}(M') \leq 0$*. There exists a constant C depending only on δ, the metrics and initial total energies such that*

$$\|F\|_{W^{2,2}(\tau,\tau+\delta)} \leq C \quad \textit{for all } \alpha \leq \tau < \omega - \delta.$$

Proof. Since

$$\|F\|^2_{W^{2,2}([\tau,\tau+\delta])} \le \int_\tau^{\tau+\delta} \|F_t\|^2_{W^{2,2}(M)}dt + 2\int_\tau^{\tau+\delta} \|\Delta F_t\|^2_{L^2}\, dt + 2\int_\tau^{\tau+\delta} \left\|\Gamma(F_t)(\nabla F_t)^2\right\|^2_{L^2} dt$$

The first term and the second term are bounded by $C^2\delta$, the third one, since $\Gamma(F_t)$ is bounded, by $C^2\delta$ where C is a constant only depending on the metrics and initial total energies. □

The estimates of higher derivatives of F will be established in the same strategy as the bootstrap: first in $W^{2,p}$ for all p then in $W^{k,p}$ for all k, p, then in C^∞.

1 Estimate of higher derivatives.

Lemma 2 ($W^{2,p}$-boundedness)**.** *Suppose* $\mathrm{Riem}(M') \le 0$. *For all* $p \in (1, +\infty)$, *there exists a constant* $C > 0$ *depending only on* δ, p, *the metrics and initial energies such that for all* $\alpha + \delta \le \tau \le \omega - \delta$:

$$\|F\|_{W^{2,p}([\tau,\tau+\delta])} \le C$$

Proof. Applying Gårding Inequality to the parabolic equation $AF = \Gamma(F)(\nabla F)^2$ where $A := \frac{\partial}{\partial t} + \Delta$ is the heat operator, one has

$$\|F\|_{W^{2,p}([\tau,\tau+\delta])} \le C\left(\|\Gamma(F)(\nabla F)^2\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} + \|F\|_{W^{2,2}([\tau-\frac{\delta}{3},\tau+\delta])}\right)$$

The second term of RHS is already bounded by applying Theorem 1 to $\frac{4\delta}{3}$. For the first term:

$$\left\|\Gamma(F)(\nabla F)^2\right\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \le C(M')\||\nabla F|^2\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} = C(M')\|e(F)\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])}.$$

Recall that by Theorem **??**, the potential density satisfies $\frac{de}{dt} + \Delta e - Ce \le 0$ for certain constant C depending only on the metric of M. By Maximum principle (Theorem **??**), one has $e \le \psi_\tau$ where ψ_τ is the solution of

$$\begin{cases} \frac{d}{dt}\psi_\tau + \Delta\psi_\tau - C\psi_\tau = 0 \\ \psi_\tau\big|_{\tau-\frac{\delta}{2}} = e\big|_{\tau-\frac{\delta}{2}} \end{cases}$$

We apply Gårding Inequality again for ψ_τ and obtain

$$\|e(F)\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \le \|\psi_\tau\|_{L^p([\tau-\frac{\delta}{3},\tau+\delta])} \le C\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2},\tau+\delta])}. \tag{1}$$

Now apply L^1-Comparison Theorem **??** to ψ_τ, one has

$$\|\psi_\tau\|_{L^1([\tau-\frac{\delta}{2},\tau+\delta])} \le \int_0^{3\delta/2} \|\psi_\tau\big|_{\tau-\frac{\delta}{2}}\|_{L^1} e^{Bt}dt \le \int_0^{3\delta/2} e^{Bt}dt.\|e\big|_{\tau-\frac{\delta}{2}}\|_{L^1} \le C. \tag{2}$$

The lemma follows from (1) and (2). □

We can now estimate higher order derivatives.

Theorem 3 ($W^{k,p}$-boundedness)**.** *Suppose* $\mathrm{Riem}(M') \leq 0$*. For all* $p \in (1, +\infty)$ *and* $k < +\infty$*, there exists C depending only on k, p, the metrics and initial energies such that*

$$\|F\|_{W^{k,p}([\tau,\tau+\delta])} \leq C$$

for all $\alpha + \delta \leq \tau \leq \omega - \delta$*.*

Proof. Applying Gårding Inequality to the equation $\frac{dF}{dt} + \Delta F_t = \Gamma(F)(\nabla F)^2$ then Regularity estimate for the quadratic term (Theorem **??**), one has for $\epsilon \ll \delta$:

$$\begin{aligned}\|F\|_{W^{k,p}([\tau,\tau+\delta])} &\leq C_\epsilon \left(\|F\|_{W^{2,p}([\tau-\epsilon,\tau+\delta])} + \|\Gamma(F)(\nabla F)^2\|_{W^{k-2,p}([\tau-\epsilon,\tau+\delta])} \right) \\ &\leq C_\epsilon \left(1 + C \left(1 + \|F\|_{W^{s,q}([\tau-\epsilon,\tau+\delta])} \right)^{q/p} \right)\end{aligned}$$

as long as $k - 1 < s$ and $\frac{1}{p} > \frac{k}{s}.\frac{1}{q}$. Therefore if $\|F\|_{W^{s,q}([\tau,\tau+\delta])} \leq C(\delta, s, q)$ for all $\beta \leq \tau \leq \omega - \delta$ and $q \in (1, +\infty)$, we just proved that

$$\|F\|_{W^{k,p}([\tau,\tau+\delta])} \leq C(\epsilon, k, p)$$

for all $\begin{cases} \beta + \epsilon \leq \tau \leq \omega - \delta \\ k < s + 1, p \in (1, +\infty) \end{cases}$ since $\|F\|_{W^{s,q}([\tau-\epsilon,\tau+\delta])} \leq 2C(\delta, s, q)$.

One can then conclude by induction on k, with step $\frac{1}{2}$, starting with $k = 2$ and $\epsilon = \frac{\delta}{2}$ divided by 2 after each induction step. □

2 Global existence for nonlinear heat equation.

Theorem 4 (Global existence)**.** *Suppose* $\mathrm{Riem}(M') \leq 0$*. The solution of nonlinear heat equation*

$$\frac{dF}{dt} = -\Delta F + \Gamma(F)(\nabla F)^2 \tag{3}$$

with smooth initial condition exists globally for all time $t > \alpha$*.*

Proof. Let F_n be a sequence of solution of (3) on $[\alpha, \omega_n]$ with ω_n increasing to ω then they coincide by uniqueness of the solution. As discussed in the beginning of this part, it is sufficient to prove that the solution extends to $[\alpha, \omega]$. Let F be the solution on $[\alpha, \omega)$ such that $F\big|_{[\alpha,\omega_n]} = F_n$, then by Theorem 3, for all $\tau \in [\alpha, \omega - \delta)$:

$$\|D_t^u D_x^v F\|_{L^\infty(M\times[\tau,\tau+\delta])} \leq C_{\text{Sobolev}} \|D_t^u D_x^v F\|_{W^{k,p}(M\times[\tau,\tau+\delta])} \leq C_{\text{Sobolev}}.C(k, p, \delta)$$

where, if we choose k sufficiently large, C_{Sobolev} is the constant of Sobolev imbedding $W^{k,p}(M \times [0, \delta]) \hookrightarrow C(M \times [0, \delta])$ and $C(k, p, \delta)$ is the constant provided by Theorem 3.

So all partial derivatives of F are uniformly bounded on $[\alpha, \omega)$. This proves that F extends to a solution on $[\alpha, \omega]$. In fact $F\big|_\tau := F\big|_{M\times\{\tau\}}$ converges in $C^\infty(M)$ as $\tau \to \omega$, since

$$\|D^\alpha F\big|_\tau - D^\alpha F\big|_{\tau'}\|_{L^\infty} \leq \max_{\|\beta\|=\|\alpha\|+1} \|D^\beta F\|_{L^\infty}|\tau - \tau'|.$$

□

We have just proved the first part of the following theorem.

Theorem 5 (Eells-Sampson)**.**

1. *Let M, M' be compact Riemannian manifolds with* $\mathrm{Riem}(M') \leq 0$. *Then for every smooth map $f_0 : M \longrightarrow M' \subset B \subset \mathbb{R}^N$, the nonlinear heat equation*

$$\begin{cases} \frac{df_t}{dt} = \tau(f_t), & \textit{for all } t \geq 0 \\ f\big|_{t=0} = f_0, \end{cases}$$

   admits a globally defined smooth solution f_t. Moreover, all derivatives $D^\alpha f_t$ remain bounded as $t \to +\infty$.

2. *For a suitable sequence $\{t_n\}$ increasing to $+\infty$ the sequence $\{f_{t_n}\}$ converges in $C^\infty(M)$ to a function f_∞ with $\tau(f_\infty) = 0$. Therefore any map $f_0 : M \longrightarrow M'$ is homotopic to a harmonic map.*

Proof. For any sequence $\{t_n\}$, one can extract from $\{f_{t_n}\}$, since their derivatives are uniformly bounded, a subsequence $\{f_{t_{n_i}}\}$ converging in $C^k(M, \mathbb{R}^N)$. By a diagonal argument, one can extract from any sequence $\{f_{t_n}\}$ a subsequence converging in $C^\infty(M, \mathbb{R}^N)$ to f_∞. Abusively denote this subsequence by $\{f_{t_n}\}$, by Theorem **??**

$$\lim_{n\to\infty} K(f_{t_n}) = \lim_{n\to\infty} \int_M |\tau(f_{t_n})|^2 = 0$$

Therefore $\tau(f_{t_n}) \to 0$ in $L^2(M)^{\oplus N}$. But also $\tau(f_{t_n}) \to \tau(f_\infty)$ in $C^\infty(M, \mathbb{R}^N)$, one has $\tau(f_\infty) = 0$. □