

# From Busemann function to Cheeger-Gromoll splitting

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We will prove the following result by Cheeger and Gromoll by a slightly modified approach of A. Besse.

**Theorem 1** (Cheeger-Gromoll). *Let  $M$  be a complete, connected Riemannian manifold with non-negative Ricci curvature. Suppose that  $M$  contains a line then  $M$  is isometric to  $M' \times \mathbb{R}$  with  $M'$  a complete, connected Riemannian manifold with non-negative Ricci curvature.*

Note that the notion of geodesic ray or geodesic line used here is rather strict: A geodesic line  $\gamma$  is a geodesic parameterized by  $\mathbb{R}$  such that the distance between two point is exactly the distance *on the geodesic*, for example, geodesic line, if it passes by  $p \in M$  with velocity  $v$  of norm 1, satisfies

$$d(\exp_p(tv), \exp_p(-sv)) = s + t, \quad \forall s, t > 0$$

## 1 Busemann function

Let  $\gamma$  be a geodesic ray. We construct the Busemann function  $b$  associated to the ray as

$$b(x) = \lim_{t \rightarrow +\infty} (t - d(x, \gamma(t)))$$

where the limit exists because the sequence  $f_t : x \mapsto t - d(x, \gamma(t))$  is non-decreasing and bounded above by  $d(x, \gamma(0))$ . The convergence is also uniform in every compact of  $M$ .

In Euclidean space for example, the Busemann function is the orthogonal projection on  $\gamma$ . We will see that in a Riemannian manifold with non negative curvature, the Busemann function will serve as a projection.

Now with a fixed  $x_0 \in M$ , the tangent vectors at  $x_0$  of the geodesics connecting  $x_0$  and  $\gamma(t)$  is in the unit sphere of  $T_{x_0}M$ , which is compact. Let  $X$  be a limit point of these tangents vectors, we defined

$$b_{X,t}(x) = b(x_0) + t - d(x, \exp_{x_0}(tX))$$

where  $\exp_{x_0}(tX)$  is the geodesic starting at  $x_0$  with velocity  $X$ .

**Remark 1.** 1. From the construction of  $X$ , one has  $b(x_0) + t = b(\exp_{x_0}(tX))$ , therefore  $b_{X,t} \leq b$  with equality in  $x_0$ . We say that  $b$  is supported by  $b_{X,t}$  at  $x_0$ . In general a function  $f$  is supported by  $g$  at  $x_0$  if  $f(x_0) = g(x_0)$  and  $f \geq g$  in a neighborhood of  $x_0$ .

2.  $b_{X,t}$  is smooth and a computation in local coordinate gives  $\Delta b_{X,t} \geq -\frac{\dim M - 1}{t}$

3.  $\|\nabla b_{X,t}\| = 1$

The estimation given on the second point of Remark 1 is established using Jacobi fields:

**Lemma 2.** The function  $f(x) = d(x, x_0)$  satisfies at a point  $x$  out of the cut-locus of  $x_0$ :

$$\nabla f(x) \leq \frac{n-1}{l}$$

where  $n = \dim M, l = d(x, x_0) = f(x)$  in Riemannian manifold  $M$  with non-negative Ricci curvature.

*Proof.* Let  $N(t), 0 \leq t \leq l$  be the velocity of the geodesic  $\gamma$  from  $x_0$  to  $x$ , and  $E_1, \dots, E_{n-1}, N$  be a parallel frame along  $\gamma$ . Let  $J_i$  be the unique Jacobi fields along  $\gamma$  with  $J_i(l) = E_i(l)$  and  $J_i(0) = 0$  (existence and uniqueness of  $J_i$  is due to the fact that  $x$  is not in the cut-locus).

Then basic manipulation of Jacobi fields gives (without the fact that curvature is non-negative):

$$\Delta f(x) = \int_0^l dt \sum_{i=1}^{n-1} (\langle \nabla_N J_i, \nabla_N J_i \rangle - \langle R(N, J_i)J_i, N \rangle) = \sum_{i=1}^{n-1} I_\gamma(J_i, J_i)$$

where  $I_\gamma$  is the index form of  $\gamma$ . Note that the Jacobi fields  $J_i$  coincide with the fields  $\frac{t}{l}E(t)$  at 0 and  $l$ , therefore by the *fundamental inequality* of index form (see Sakai Takashi, *Riemannian geometry* for details about Jacobi fields and Fundamental inequality of index form):

$$I_\gamma(J_i, J_i) \leq I_\gamma\left(\frac{t}{l}E_i, \frac{t}{l}E_i\right)$$

hence

$$\Delta f(x) \leq \int_0^l \sum_{i=1}^{n-1} \langle \nabla_N \frac{t}{l}E_i, \nabla_N \frac{t}{l}E_i \rangle - \langle R(N, \frac{t}{l}E_i) \frac{t}{l}E_i, N \rangle$$

The curvature term being  $\frac{t^2}{l^2} Ric(N, N)$  therefore non-negative, one has

$$\Delta f(x) \leq \int_0^l dt \sum_{i=1}^{n-1} \langle \nabla_N \frac{t}{l}E_i, \nabla_N \frac{t}{l}E_i \rangle = \frac{n-1}{l}.$$

□

We also note that for Theorem 1 it suffices to show that  $b$  is harmonic. In fact, from the smoothness one has  $\nabla b(x_0) = \nabla b_{X,t}(x_0)$ , which means  $\|\nabla b\| = 1$  at every point in  $M$ . For each point  $y \in M$ , there exists a unique  $x$  with  $b(x) = 0$  and time  $t$  when the flow of  $\nabla b$  starting at  $x$  arrives at  $y$ .  $M$  is therefore homeomorphic to  $\bar{M} \times \mathbb{R}$  by the map  $F : y \mapsto (x, t)$ . In order that  $F$  is an isometry, it suffices to prove that the gradient field  $\nabla b$  is parallel. In fact,  $\bar{M}$  being equipped with the restriction of the metric on  $M$ , the fact that  $F$  is isometric is equivalent to the fact that the flow  $\Phi^t$  of  $\nabla b$  is isometric for every time  $t$ , which means  $\frac{d}{dt} \langle \Phi_*^t u, \Phi_*^t u \rangle$  vanishes at  $t = 0$ . But

$$\frac{d}{dt} \langle \Phi_*^t u, \Phi_*^t u \rangle|_{t=0} = 2 \langle \nabla_{\partial t} \Phi_*^t u, u \rangle|_{t=0} = 2 \langle \nabla_u \nabla b, u \rangle$$

where in the second equality we used Schwarz lemma for commuting derivatives of  $\Phi(t, x) = \Phi^t(x)$ . The vanishing of  $\langle \nabla_u \nabla b(x), u \rangle$  for every vector  $u$  is, by bilinearity, equivalent to that of  $\nabla_u \nabla b$  for every  $u$ , meaning that  $\nabla b$  is parallel.

The fact that  $\nabla b$  is parallel is due to a simple computation:

$$Ric(N, N) = -N(\Delta b) - \|\nabla N\|^2$$

where  $\|\nabla N\|^2 = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \langle \nabla_{E_i} N, E_j \rangle^2$ . We see that  $N = \nabla b$  is parallel if  $\Delta b = 0$ .

- Remark 2.** 1. One can show (see A. Besse) that every gradient field  $\nabla b$  of norm 1 at every point is actually harmonic.
2. Using de Rham decomposition, one has directly the splitting of  $M$  if it is simply-connected since  $N$  is parallel and  $M$  is complete.

## 2 Harmonicity

The Busemann function associated to a geodesic ray is subharmonic, it is a consequence of the following lemma.

**Lemma 3.** *In a connected Riemannian manifold, if a continuous function  $f$  is supported at any point  $x$  by a family  $f_\epsilon$  (depending on  $x$ ) with  $\Delta(f_\epsilon) \leq \epsilon$ , then  $f$  can not attain its maximum (unless when  $f$  is constant).*

*Proof.* Given a small geodesic ball  $B$ , suppose that we have a function  $h$  on  $B$  with  $\Delta h < 0$  in  $B$  and  $f + h$  attains maximum at  $x$  in the interior of  $B$ . Then  $f_\epsilon + h$  also attains maximum at  $x$ , which means  $\Delta f_\epsilon + \Delta h \geq 0$ , which is contradictory.

For the construction of the function  $h$ , one suppose that  $B$  is small enough such that  $f|_{\partial B} \leq \max_B f =: f(x_0)$  and equality is not attained at every points in  $\partial B$ . Then choose

$$h = \eta(e^{\alpha\phi} - 1)$$

with  $\phi(x) = -1$  if  $x \in \partial B$  and  $f(x) = f(x_0)$ ,  $\phi(x_0) = 0$ ,  $\nabla\phi \neq 0$  and a large  $\alpha$  such that

$$\Delta h = \eta(-\alpha^2 \|\nabla\phi\| + \alpha\Delta\phi)e^{\alpha\phi}.$$

is negative. □

Now for subharmonicity of  $b$ , given a harmonic function  $h$  that coincides with  $b$  in the boundary  $\partial B$  of a geodesic ball  $B$ , then  $b - h$  is supported by  $b_{X,t} - h$  with  $\Delta(b_{X,t} - h) \rightarrow 0$  as  $t$  tends to  $+\infty$ , therefore  $b - h \leq (b - h)|_{\partial B} = 0$  in  $B$ . We have just proved the following lemma:

**Corollary 3.1.** *The Busemann function of a geodesic ray in a Riemannian manifold  $M$  with non-negative Ricci curvature is subharmonic.*

Now let  $b_+$  be the function previously constructed for the ray  $\gamma|_{[0,+\infty[}$  and  $b_-$  the Busemann function for the ray  $\tilde{\gamma}|_{[0,+\infty[}$  where  $\tilde{\gamma}(t) = \gamma(-t)$ . Note that  $b_+ + b_- \leq 0$  with equality on the line  $\gamma$ , but the sum is subharmonic therefore by maximum principle  $b_+ + b_- = 0$  and  $b$  is harmonic therefore smooth. The splitting theorem of Cheeger-Gromoll follows.

### 3 Application

A consequence of Theorem 1 is the following result from J.Cheeger- D.Gromoll, *The splitting theorem for manifold of nonnegative Ricci curvature* (Theorem 2)

**Theorem 4.** *Let  $M$  be a compact Riemannian manifold with non-negative Ricci curvature, then the universal covering space of  $M$  is of form  $\tilde{M} = \mathbb{R}^n \times \bar{M}$  where  $\bar{M}$  does not contain any lines. Then  $\bar{M}$  is compact.*

*Proof.* It suffices to prove that if  $\bar{M}$  is not compact, then it contains a line. In fact, it is easy to see that such  $\bar{M}$  must contain a (strict) geodesic ray. In fact it is obvious that with a fixed  $p \in M$  the function

$$F : v \mapsto \inf\{t > 0 : d(p, \exp_p(tv)) < t\}$$

defined on the unit ball  $U_p$  of  $T_p\bar{M}$  is upper semi-continuous. Therefore if  $F(v) < \infty$  for all unit tangent vector  $v$  at  $p$  then  $F$  is bounded above in  $U_p$  by a constant  $c$ . Therefore  $\bar{M} \subset \exp_p(cU_p)$  which is compact (contradiction). Therefore there exists a minimal ray at every point  $p \in \bar{M}$ .

The existence of a line in general might not be true, the only extra property of  $\bar{M}$  that we will need is that it has a (fundamental) domain  $K$  compact and a family  $\sigma_i$  of isometries such that  $\bar{M} = \cup_i \sigma_i K$ .

Let us first prove that such domain  $K$  exists. Remark that every isometry of  $\mathcal{M}$  acts separately on  $\mathcal{M}$ , i.e. of form  $\sigma(u) = (\sigma_1(x), \sigma_2(y))$  for  $u = (x, y) \in \mathcal{M}$  with  $g_1, g_2$  isometries of  $\mathbb{R}^n$  and  $\bar{M}$ . This can be seen by the uniqueness part of de Rham decomposition or simply by noticing that a tangent vector in the  $T_x\mathbb{R}^n$  component is characterized by the fact that its geodesics is a line. As the action of  $\pi_1(M)$  on  $\tilde{M}$  is free and proper, it has a fundamental domain  $H$ . We then can choose  $K$  to be the projection of  $H$  on  $\bar{M}$  and  $\{\sigma_i\}$  to be the projection of  $\pi_1(M)$  on  $Isom(\bar{M})$ .

Now let  $\gamma$  be a minimal ray starting from  $p \in M$ , for each  $x \in \gamma$  there exists an isometry  $\sigma$  of  $\bar{M}$  such that  $\sigma(x) \in K$ . By compactness of  $K$ , there exists a sequence  $t_n \rightarrow +\infty$  with  $x_n = \gamma(t_n)$ ,  $v_n = \dot{\gamma}(t_n)$  that satisfies  $y_n = \sigma_n(x_n) \rightarrow y \in K$  and even more,  $(\sigma_n)_*v_n \rightarrow v \in T_y\bar{M}$  in the tangent bundle  $T\bar{M}$ . Then the geodesic of  $\bar{M}$  starting at  $y$  with vector  $v$  is a line. In fact it suffices to prove that  $d(\exp_y(tv), \exp_y(-sv)) = s + t$  for  $s, t > 0$ , but for  $n$  large enough that  $t_n > s$  one has

$$d(\exp_{y_n}(tv_n), \exp_{y_n}(-sv_n)) = s + t$$

then let  $n \rightarrow +\infty$ , one sees that  $\bar{M}$  contains a geodesic line, which is contradictory.  $\square$

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