Short-time existence and regularity for nonlinear heat equation: Polynomial differential operators and Besov spaces

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May 27, 2018

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We will establish in this part a regularity estimate for differential operator with coefficient depending nonlinearly in x and $f(x)$. Although the result can be stated using only Sobolev spaces, it is natural for the proof to make a detour to Besov space where we can use theorem 5.

We will then apply the regularity estimate for the nonlinear part of the heat operator in order to setup a bootstrap scheme that eventually will prove that any $W^{2,p}$ solution of nonlinear heat equation that is initially C^∞ will be always C^∞.

We will also prove short-time existence using well-known method of Implicit function theorem for Banach spaces. Since the solution is smooth, we can apply Theorem **??** to conclude that the it remains in $M' \subset \mathbb{R}^N$.

1 Polynomial differential operator.

Definition 1. *We say that P is a **polynomial differential operator of type** (n,k) if P is of the form*

$$P(F) = \sum c_{\alpha_1,\dots,\alpha_\nu}(x, F(x)) D^{\alpha_1} F^{a_1} \dots D^{\alpha_n u} F^{a_\nu}$$

where the coefficients $c_{\alpha_1,\dots,\alpha_n u}$ depend smoothly and nonlinearly on x and F and $\alpha_i \in \mathbb{R}^N$ are indices with the weighted norm $\|\alpha_i\| \le k$ and $\sum \|\alpha_i\| \le n$.

Example 1. *On $M \times [\alpha, \omega]$ the nonlinear heat operator $PF := \frac{dF}{dt} - \tau(F_t)$ is a polynomial differential operator of type (2,2). The tension field alone is of type (2,1).*

1.1 A regularity estimate for polynomial differential operator.

Our goal in this part is to prove the following estimate for polynomial differential operator, in which X will be $M \times [\alpha, \omega]$.

Theorem 1 (Regularity of polynomial differential operator)**.** *Let X be a compact Riemannian manifold, $B \subset \mathbb{R}^N$ is a large Euclidean ball and P be a polynomial differential operator of type (n,k) on X. Suppose that*

$$r \ge 0, \quad p, q \in (1, \infty), \quad r + k < s, \quad \frac{1}{p} > \frac{r+n}{s} \frac{1}{q}. \tag{1}$$

Then for all $F \in C(X, B) \cap W^{s,q}(X)$, one has $Pf \in W^{r,p}(X)$ and

$$\|PF\|_{W^{r,p}} \le C \left(1 + \|F\|_{W^{s,q}}\right)^{q/p}.$$

where C is a constant independent of F.

We will prove that the result is *local*, in a sense to be defined. Then we will prove the local statement using Besov spaces.

Proof (reduction of Theorem 1 to a local statement). Let $\{\varphi_i : U_i \longrightarrow V_i\}$ be an atlas of M. We denote a point in U_i by x and its coordinates in V_i by ξ. Let $\sum \psi_i = 1$ be a partition of unity subordinated to $\{U_i\}$ and $\tilde{\psi}_i$ be smooth functions supported in U_i with $0 \le \tilde{\psi}_i \le 1$ and $\tilde{\psi}_i = 1$ in the support of ψ_i, as in the definition of Sobolev spaces on manifold. We suppose the following local statement is true:

Lemma 2 (Local statement). *Let P be a polynomial differential operator of type (n,k) and coefficients $c_{\alpha_1,\dots,\alpha_\nu}(x,F)$ are smooth and vanish when $x\in\mathbb{R}^{\dim X}$ is outside of a compact. Let $B\subset\mathbb{R}^N$ be a large Euclidean ball and r,p,q,s as in (1). Then for all compactly supported $F\in C(\mathbb{R}^{\dim X},B)\cap W^{s,q}(\mathbb{R}^{\dim X})$, one has*

$$\|PF\|_{W^{r,p}}\leq C\left(1+\|F\|_{W^{s,q}}\right)^{q/p}$$

where the constant C depends only on B and the support of F, and not on F.

One has

$$\|PF\|_{W^{r,p}}:=\sum_i\|\psi_i PF\|_{W^{r,p}}$$

where viewed in the chart U_i, each $\psi_i(x)PF(x)$ is $\sum_\alpha \psi_i(\xi).c_\alpha(\xi,g_i).D^\alpha g_i$ where $g_i=f_i\circ\varphi_i^{-1}$ is f_i viewed in the chart. Since $\tilde{\psi}_i=1$ in the support of ψ_i, one has

$$\psi_i(\xi).c_\alpha(\xi,g_i).D^\alpha g_i=\psi_i(\xi).c_\alpha(\xi,\tilde{\psi}_i g_i)D^\alpha(\tilde{\psi}_i g_i)$$

hence

$$\|\psi_i(\xi).c_\alpha(\xi,g_i).D^\alpha g_i\|_{W^{r,p}}\leq C\left(1+\|\tilde{\psi}_i g_i\|_{W^{s,q}}\right)^{q/p}\leq C\left(1+\|F\|_{W^{s,q}}\right)^{q/p}.$$

Therefore $\|PF\|_{W^{r,p}}\leq mC\left(1+\|F\|_{W^{s,q}}\right)^{q/p}$ where m is the number of charts we used to cover M. □

Remark 1. *The use of partition of unity in the last proof is to decompose $PF=\sum\psi_i PF$ and not $F=\psi_i F$ since we no longer have linearity of the operator P in F.*

1.2 Review of Besov spaces $B^{s,p}$.

In this part, $X=\mathbb{R}^n$ coordinated by $(x_1,\dots,x_n)$ with weight $(\sigma_1,\dots,\sigma_n)$. We define

$$T_j^v f(x_1,\dots,x_n):=f(x_1,\dots,x_j+v,\dots,x_n),\quad \Delta_j^v:=T_j^v-\mathrm{Id}$$

for $f\in\mathcal{S}(X)$.

For the notation, we will denote the Besov spaces by $B^{s,p}$ with $s\in\mathbb{R}_{>0}\backslash\mathbb{Z}$ and $p\in(1,\infty)$ so that they look similar to Sobolev space $W^{s,p}$. In a more standard notation, our spaces $B^{s,p}$ are denoted by $B^s_{p,p}$

Definition 2. *We define $B^{s,p}$ as the completion of $\mathcal{S}(X)$ under the norm*

$$\|f\|_{B^{s,p}} := \sum_{\|\gamma\|<s} \|D^\gamma\|_{L^p} + \sum_{s-\frac{\sigma}{\sigma_j}<\|\gamma\|<s} \sup_v \frac{\|\Delta_j^v D^\gamma f\|_{L^p}}{|v|^{(s-\|\gamma\|)\sigma_j/\sigma}}$$

We cite here some well-known facts

1. While Sobolev spaces with non-integral regularity are complex interpolation of integral ones, Besov spaces are their real interpolation.

2. Besov spaces $B^{s,p}(X)$ are reflexive Banach space with their dual spaces being $B^{-s,p}(X)$.

Theorem 3. *If $r < s$ then*

$$W^{s,p}(X) \subset B^{s,p}(X) \subset W^{r,p}(X).$$

Theorem 4 (Multiplication)**.** *For $f, g \in \mathcal{S}(X)$ and* $\begin{cases} 0 < \alpha < 1, \tilde{p} \leq p, \tilde{q} \leq q, \tilde{r} \leq r \\ \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \frac{1}{\tilde{p}} + \frac{1}{q} = \frac{1}{p} + \frac{1}{\tilde{q}} = \frac{1}{\tilde{r}} \end{cases}$,
one has

$$\|fg\|_{B^{\alpha,\tilde{r}}} \leq C\left(\|f\|_{B^{\alpha,\tilde{p}}}\|g\|_{L^q} + \|f\|_{L^p}\|g\|_{B^{\alpha,\tilde{q}}}\right) \tag{2}$$

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q} \tag{3}$$

Therefore by density (2) *is true for all $f \in L^p \cap B^{\alpha,\tilde{p}}, g \in L^q \cap B^{\alpha,\tilde{q}}$ and* (3) *is true for all $f \in L^p, g \in L^q$.*

The reason for which we use the Besov norm is the following estimate:

Theorem 5 (Composition)**.** *Let $\Gamma(x, y)$ be a continuous, nonlinear function of variables $x \in \mathbb{R}^n, y \in \mathbb{R}^N$. Suppose that Γ vanishes for all x outside of a compact in $\mathbb{R}^n$ and Γ is C-Lipschitz in y, and define*

$$\Gamma f := (x \mapsto \Gamma(x, f(x))) .$$

Then

$$\|\Gamma f\| \leq C\left(1 + \|f\|_{B^{\alpha,p}}\right)$$

1.3 Proof of the local estimate.

Since $B^{r+\epsilon,p}(X) \subset W^{r,p}(X)$, by increasing r a bit, we can suppose that $r \notin \mathbb{Z}$ and replace the $W^{r,p}$ norm in the statement by the $B^{r,p}$ norm, that is to estimate:

$$\|PF\|_{B^{r,p}} = \sum_{\|\gamma\|<r} \|D^\gamma(PF)\|_{L^p} + \sum_{r-\sigma/\sigma_j<\|\gamma\|<r} \frac{\|\Delta_j^v D^\gamma(PF)\|_{L^p}}{|v|^{(r-\|\gamma\|)\sigma_j/\sigma}}$$

where

$$D^\gamma(PF) = \sum c_{\beta_1,\dots,\beta_\mu}(x,F) D^{\beta_1} f^{b_1} \dots D^{\beta_\mu} f^{b_\mu} \tag{4}$$

with $\max \|\beta_i\| \le k + \|\gamma\|$ and $\sum \|\beta_i\| \le n + \|\gamma\|$.

Using $\Delta_j^v(fg) = \Delta_j^v f\ T_j^v g + f\Delta_j^v g$, one can see that $\Delta_j^v D^\gamma(PF)$ is a sum of terms of 2 types:

$$\Delta_j^v c_{\beta_1,\dots,\beta_\mu}\ T_j^v(D^{\beta_1} f^{b_1}) \dots T_j^v(D^{\beta_\mu} f^{b_\mu}) \tag{5}$$

and

$$c_{\beta_1,\dots,\beta_\mu}\ D^{\beta_1} f^{b_1}\ \dots\ D^{\beta_{i-1}} f^{b_{i-1}}\ \Delta_j^v(D^{\beta_i} f^{b_i})\ T_j^v(D^{\beta_{i+1}} f^{b_{i+1}})\ \dots\ T_j^v(D^{\beta_\mu} f^{b_\mu}) \tag{6}$$

Our strategy is to use Theorem 4 to estimate the terms (4), (5) and (6) as follows, where we denote $\|g\|_p := \|g\|_{L^p}$

$$\left\| c_{\beta_1,\dots,\beta_\mu}(x,F)\ D^{\beta_1} f^{b_1} \dots D^{\beta_\mu} f^{b_\mu} \right\|_p \le \|c_{\beta_1,\dots,\beta_\mu}\|_\infty . \|D^{\beta_1} f^{b_1}\|_{p_1} \dots \|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \tag{7}$$

$$\left\| \Delta_j^v c_{\beta_1,\dots,\beta_\mu}\ T_j^v(D^{\beta_1} f^{b_1}) \dots T_j^v(D^{\beta_\mu} f^{b_\mu}) \right\|_p \le \|\Delta_j^v c_{\beta_1,\dots,\beta_\mu}\|_{\tilde{p}_0} . \|D^{\beta_1} f^{b_1}\|_{p_1} \dots \|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \tag{8}$$

$$\begin{aligned} &\left\| c_{\beta_1,\dots,\beta_\mu}\ D^{\beta_1} f^{b_1}\ \dots\ D^{\beta_{i-1}} f^{b_{i-1}}\ \Delta_j^v(D^{\beta_i} f^{b_i})\ T_j^v(D^{\beta_{i+1}} f^{b_{i+1}})\ \dots\ T_j^v(D^{\beta_\mu} f^{b_\mu}) \right\|_p \le \\ &\|c_{\beta_1,\dots,\beta_\mu}\|_\infty . \|D^{\beta_1} f^{b_1}\|_{p_1} \dots \|D^{\beta_{i-1}} f^{b_{i-1}}\|_{p_{i-1}} . \|\Delta_j^v(D^{\beta_i} f^{b_i})\|_{\tilde{p}_i} . \|D^{\beta_{i+1}} f^{b_{i+1}}\|_{p_{i+1}} \dots \|D^{\beta_\mu} f^{b_\mu}\|_{p_\mu} \end{aligned} \tag{9}$$

Then continue by bounding the Δ_j^v terms:

$$\|\Delta_j^v c_{\beta_1,\dots,\beta_\mu}\|_{\tilde{p}_0} \le |v|^{\theta\sigma_j/\sigma} C(1 + \|F\|_{B^{\theta,\tilde{p}_0}}) \le |v|^{\theta\sigma_j/\sigma} C(1 + \|F\|_{W^{\theta,\tilde{p}_0}}) \tag{10}$$

using Theorem 5, where C is the Lipschitz constant of $c_{\beta_1,\dots,\beta_\mu}(x,F)$ in F, which exists because $c_{\beta_1,\dots,\beta_\mu}$ is smooth and F always remains in a large Euclidean ball B. The next Δ_j^v term to bound is, using Theorem 3:

$$\|\Delta_j^v(D^{\beta_i}f^{b_i})\|_{\tilde{p}_i} \leq |v|^{\theta\sigma_j/\sigma}\|f^{b_i}\|_{B^{\|\beta_i\|+\theta,\tilde{p}_i}} \leq |v|^{\theta\sigma_j/\sigma}\|f^{b_i}\|_{W^{\|\beta_i\|+\theta,\tilde{p}_i}} \tag{11}$$

And finally plugging (10) and (11) in (8) and (9), and noting that $\|c_{\beta_1,\dots,\beta_\mu}\|_\infty\|$ in (7) is bounded by a constant, it remains to estimate $\|f^{b_i}\|_{W^{\|\beta_i\|,p_i}}$, $\|f^{b_i}\|_{W^{\|\beta_i\|+\theta,\tilde{p}_i}}$ and $\|F\|_{W^{\theta,\tilde{p}_0}}$ in term of $\|F\|_{W^{s,q}}$, for which we will use the following consequence of Interpolation inequality.

Lemma 6. *Let $0 \leq r \leq s$ and $p,q \in (1,\infty)$ such that $0 < \frac{1}{p} - \frac{r}{s}\frac{1}{q} < 1 - \frac{r}{s}$. Then for all compactly supported $F \in C(X,B) \cap W^{s,q}$ where $B \subset \mathbb{R}^N$ is a large Euclidean ball, one has*

$$\|F\|_{W^{r,p}} \leq C\|F\|_\infty^{1-r/s}\|F\|_{W^{s,q}}^{r/s} \leq C'\|F\|_{W^{s,q}}^{r/s}$$

where C depends only on B and the support of F, but not F.

Proof. Since F is bounded, $f^\alpha \in W^{s,q} \cap W^{0,v}$ for all $v > 1$. By Interpolation inequality

$$\|f^\alpha\|_{W^{r,p}} \leq 2\|f^\alpha\|_{W^{s,q}}^{r/s}\|f^\alpha\|_{W^{0,v}}^{1-r/s}$$

then choose v with $(1-\frac{r}{s})\frac{1}{v} = \frac{1}{p} - \frac{r}{s}\frac{1}{q}$. ∎

To apply Lemma 6, we have to choose $p_i, \tilde{p}_i, \tilde{p}_0, \theta$ such that $\begin{cases} 0 < \frac{1}{p_i} - \frac{\|\beta_i\|}{s}\frac{1}{q} < 1 - \frac{\|\beta_i\|}{s}, \\ 0 < \frac{1}{\tilde{p}_i} - \frac{\|\beta_i+\theta\|}{s}\frac{1}{q} < 1 - \frac{\|\beta_i+\theta\|}{s} \\ 0 < \frac{1}{\tilde{p}_0} - \frac{\theta}{s}\frac{1}{q} < 1 - \frac{\theta}{s} \end{cases}$

We choose $\frac{1}{p_i}$ just a bit bigger than $\frac{\|\beta_i\|}{s}\frac{1}{q}$, $\frac{1}{\tilde{p}_i}$ just a bit bigger than $\frac{\|\beta_i+\theta\|}{s}\frac{1}{q}$ and $\frac{1}{\tilde{p}_0}$ just a bit bigger than $\frac{\theta}{s}\frac{1}{q}$. We will now come back to justify the estimates (7), (8), (9). Since F is bounded in B and compactly supported in V, we see that $\|f^\alpha\|_p \leq C(B,V)\|f^\alpha\|_q$ if $p \leq q$. Therefore,

1. For (7), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{p_1} + \cdots + \frac{1}{p_\mu}$$

which is true because the RHS is is a bit bigger than $\frac{1}{qs}\sum\|\beta_i\| \leq \frac{n+\|\gamma\|}{qs} < \frac{n+r}{qs}$.

2. For (8), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{\tilde{p}_0} + \frac{1}{p_1} + \cdots + \frac{1}{p_\mu}$$

where the RHS is is a bit bigger than $\frac{\theta}{s}\frac{1}{q} + \frac{1}{qs}\sum \|\beta_i\| \leq \frac{n+\|\gamma\|+\theta}{qs}$.

3. For (9), it is sufficient to have

$$\frac{1}{p} \geq \frac{1}{p_1} + \cdots + \frac{1}{\tilde{p}_i} + \cdots + \frac{1}{p_\mu}$$

where the RHS is is a bit bigger than $\frac{\theta}{s}\frac{1}{q} + \frac{1}{qs}\sum \|\beta_i\| \leq \frac{n+\|\gamma\|+\theta}{qs}$.

It is sufficient then to take $\theta = r - \|\gamma\|$. Now the estimates (7), (8), (9) can be continued as

$$RHS(7) \leq \prod_i \|f^{b_i}\|_{W^{s,q}}^{\|\beta_i\|/s} \leq \|F\|_{W^{s,q}}^{\frac{n+\|\gamma\|}{s}} \leq \|F\|_{W^{s,q}}^{q/p} \tag{12}$$

$$RHS(8) \leq |v|^{\theta\sigma_j/\sigma}\left(1 + \|F\|_{W^{s,q}}^{\theta/s}\right)\prod_i \|f^{b_i}\|_{W^{s,q}}^{\|\beta_i\|/s} \leq |v|^{\theta\sigma_j/\sigma}\left(1 + \|F\|_{W^{s,q}}^{\theta/s}\right)\|F\|_{W^{s,q}}^{q/p} \tag{13}$$

$$RHS(9) \leq |v|^{\theta\sigma_j/\sigma}\left(1 + \|f^{b_i}\|_{W^{s,q}}^{\frac{\|\beta_i\|+\theta}{s}}\right)\prod_{u\neq i} \|f^{b_u}\|_{W^{s,q}}^{\|\beta_u\|/s} \leq |v|^{\theta\sigma_j/\sigma}\left(1 + \|F\|_{W^{s,q}}^{\frac{\|\beta_i\|+\theta}{s}}\right)\|F\|_{W^{s,q}}^{q/p} \tag{14}$$

While (12) gives $\|D^\gamma(PF)\|_p \leq C\|F\|_{W^{s,q}}^{q/p}$, the last two (13) and (14) give

$$\sum_{s-\frac{\sigma}{\sigma_j}<\|\gamma\|<s} \sup_v \frac{\|\Delta_j^v D^\gamma(PF)\|_p}{|v|^{(r-\|\gamma\|)\sigma_j/\sigma}} \leq C\left(1 + \|F\|_{W^{q,s}}^{(n+r)/s}\right)$$

2 Regularity for nonlinear heat equation.

Let $p > \dim M + 2$, using the regularity estimate for polynomial differential operator, we now can prove

Theorem 7 (Bootstrap for nonlinear heat equation)**.** *Let $F :\ M\times[\alpha,\omega] \longrightarrow B$ such that $F \in W^{2,p}(M\times[\alpha,\omega])$ and $\frac{dF_t}{dt} = \tau(F_t)$, i.e.*

$$\frac{dF^\alpha}{dt} = -\Delta F^\alpha + g^{ij}\Gamma'^\alpha_{\beta\gamma}(F)F_i^\beta F_j^\gamma$$

and $F\big|_{M\times\{\alpha\}}$ is smooth. Then F is smooth on $M\times[\alpha,\omega]$.

Remark 2. *Note that since $p > \dim M + 2 = \dim(M \times [\alpha, \omega]) + 1$, if $F \in W^{2,p}(M \times [\alpha, \alpha+\epsilon])$ then F and $\frac{\partial F}{\partial x^i}$ are in $C(M \times [\alpha, \alpha+\epsilon])$ by Sobolev imbeddings. It makes sense then to talk about:*

1. *the restriction and boundary condition at time $t = \alpha$ (in fact, by Trace theorem, $p > 1$ is enough).*

2. *the pointwise condition $F : M \times [\alpha, \alpha + \epsilon] \longrightarrow B \subset V$.*

Proof. We define the operators $PF := g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F^{\beta}_i F^{\gamma}_j$ of type (2,1) and $AF := \frac{dF}{dt} + \Delta F$ of type (2,2). As in Theorem **??**, we will abusively denote $W^{k,p}(M \times [\beta, \gamma])$ by $W^{k,p}([\beta, \gamma])$. Our bootstrap scheme consists of 3 steps:

1. Prove that $F \in W^{2,\tilde{p}}([\pi, \omega])$ for every $\pi > \alpha$ and $\tilde{p} \in (1, \infty)$. By compactness of M, it is sufficient to prove this for a sequence $\tilde{p} \to +\infty$.

2. Prove that F is C^∞ for all time $t > \alpha$.

3. Prove that F is C^∞ on $M \times [\alpha, \omega]$.

Step 1. By Theorem 1, $AF = PF \in W^{r,q}([\alpha, \omega])$ whenever $r < 1$ and $\frac{1}{q} > (\frac{r}{2} + 1)\frac{1}{p}$. Apply Theorem **??**, for all $\pi > \alpha$, $F \in W^{r+2,q}([\pi, \omega]) \subset W^{2,\tilde{p}}([\pi, \omega])$ for $\frac{1}{\tilde{p}} = \frac{1}{q} - \frac{r}{\dim M + 1}$. Choose $\frac{1}{q}$ very close to $(\frac{r}{2} + 1)\frac{1}{p}$, one sees that the condition on $\tilde{p}$ is $\frac{1}{\tilde{p}} > (\frac{r}{2} + 1)\frac{1}{p} - \frac{r}{p-1}$, which will be satisfied if $\frac{1}{\tilde{p}} > (1 - \frac{r}{2})\frac{1}{p}$, i.e. for all $\tilde{p} < \frac{p}{1-r/2}$. It remains to repeat this result to finish the first step. We will say $F \in W^{2,*}([\pi, \omega])$ for $F \in W^{2,p}([\pi, \omega])$ for all $p \in (1, \infty)$.

Step 2. By Theorem 1, for all $r < 1$, one has $AF = PF \in W^{r,*}([\pi, \omega])$, therefore by Theorem **??**, $F \in W^{r+2,*}([\pi, \omega])$. Iterate this result and one has $F \in W^{k,*}([\pi, \omega])$ for all $k \in [2, \infty)$ and $\pi > \alpha$. So F is smooth for $t > \alpha$.

Step 3. We apply regularity result (Theorem **??**) for elliptic operator A and boundary operators $B^0 : F \mapsto F\big|_{M \times \{\alpha\}}$ and $B^1 : F \mapsto F\big|_{M \times \{\omega\}}$, both are of weight 0: Since for q, r in Step 1, one has $AF = PF \in W^{r,q}([\alpha, \omega])$ and $B^j F \in \partial W^{r,q}$, therefore $F \in W^{r+2,q}([\alpha, \omega]) \subset W^{2,\tilde{p}}([\alpha, \omega])$ for the same $\tilde{p}$ as Step 1. This proves that $F \in W^{2,*}([\alpha, \omega])$, which also means that one has $F \in W^{r+2,q}([\alpha, \omega])$ with no additional condition on q except $q \in (1, \infty)$. Iterate and one obtains the regularity of F on $[\alpha, \omega]$. □

Remark 3. *The first 2 steps were to prove the regularity of $F\big|_{M \times \{\omega\}}$, which was then used as a boundary condition in order to apply regularity result for elliptic operator on manifold with boundary.*

3 Short-time existence for nonlinear heat equation.

We will choose as always $p > \dim M + 2$. As before, M is a compact Riemannian manifold and $f \ : M \longrightarrow B \subset V = \mathbb{R}^N$ where B is a large Euclidean ball ball.

Theorem 8 (Short-time existence)**.** *Let $F_\alpha : M \longrightarrow B$ be a smooth map, then there exists $\epsilon > 0$ depending on F_α and $F : M \times [\alpha, \alpha+\epsilon] \longrightarrow B$ such that $F \in W^{2,p}(M \times [\alpha, \alpha+\epsilon])$ with $F\big|_{M\times\{\alpha\}} = F_\alpha$ and*

$$\frac{dF_t}{dt} = \tau(F_t) \quad \text{on } M \times [\alpha, \alpha+\epsilon]$$

Proof. We find F as a sum $F = F_b + F_\#$ where $F_b \in C^\infty(M \times [\alpha, \omega])$ satisfies the initial condition and $F_\# \in W^{2,p}(M \times [\alpha, \alpha+\epsilon]/\alpha)$.

The nonlinear heat operator is

$$\begin{aligned} T : W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N} &\longrightarrow L^p(M \times [\alpha, \omega])^{\oplus N} \\ F_\# &\longmapsto \tau(F_b + F_\#) \end{aligned}$$

where $\tau(F)^\alpha = \Delta F^\alpha + g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^\beta F_j^\gamma$, which can be rewritten as $\tau(F) = -\Delta F + \Gamma'(F)(\nabla F)^2$. The derivative of T at $F_\#$ in direction $k \in W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N}$ is

$$DT(F_\#)k = \Delta k + D\Gamma'(F) \cdot k.(\nabla F)^2 + 2\Gamma'(F)\nabla F.\nabla k,$$

or in local coordinates:

$$DT(F_\#)^\alpha = g^{ij}\left(\frac{\partial^2 k^\alpha}{\partial x^i \partial x^j} - k_l^\alpha \Gamma_{ij}^l\right) + g^{ij}\frac{\partial \Gamma'^{\alpha}_{\beta\gamma}}{\partial y^\delta} k^\delta F_i^\beta F_j^\gamma + 2g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(F)F_i^\beta F_j^\gamma$$

which is of form $DT(F_\#)k = -\Delta k - a(x, F)\nabla k - b(x, F)k$ where a, b are smooth.

Therefore if we note by

$$\begin{aligned} H : W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N} &\longrightarrow L^p(M \times [\alpha, \omega])^{\oplus N} \\ F_\# &\longmapsto (\frac{d}{dt} - \tau)(F_b + F_\#) \end{aligned}$$

then the derivative of H at $F_\# = 0$ is

$$DH(0) \cdot k = \frac{dk}{dt} + \Delta k + a(x, F_b)\nabla k + b(x, F_b)k$$

which by Theorem **??** is an isomorphism from $W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N}$ to $W^{0,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N} = L^p(M \times [\alpha, \omega])^{\oplus N}$. This shows that H is a local isomorphism mapping a neighborhood of 0 to a neighborhood of $(\frac{d}{dt} - \tau)F_b$.

Define $g_\epsilon \in L^p(M \times [\alpha, \omega])^{\oplus N}$ by

$$g_\epsilon := \begin{cases} 0, & \text{if } t \in [\alpha, \alpha + \epsilon] \\ (\frac{d}{dt} - \tau)F_b, & \text{if } t > \alpha + \epsilon \end{cases}$$

which is arbitrarily $L^p(M \times [\alpha, \omega])$-close to $(\frac{d}{dt} - \tau)F_b$ for $0 < \epsilon \ll 1$. There exists therefore $F_\# \in W^{2,p}(M \times [\alpha, \omega]/\alpha)^{\oplus N}$ such that $H(F_\#) = g_\epsilon$, meaning that the function $F = F_b + F_\# : M \longrightarrow V$ satisfies $F\big|_{M \times \{\alpha\}} = F_\alpha$ and $\frac{dF}{dt} - \tau(F_t) = 0$ for $t \in [\alpha, \alpha + \epsilon]$.

By Regularity Theorem 7, F is C^∞ for $t \in [\alpha, \alpha + \epsilon]$. Theorem **??** assures that the image of F is in B, hence in M' for $t \in [\alpha, \alpha + \epsilon]$. $\square$