Topological characterisation of affine $\mathbb{C}$-schemes (Stein manifolds)

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This is the note of my presentation as final exam of the course *Introduction au h-principe de Gromov* given by Patrick Massot at Departement de Mathématiques d'Orsay. I presented how Cieliebak and Eliashberg [**?**] applied h-principe to characterise the topology of Stein manifold. Further discussion can be found in the book [**?**].

1 The statement of Cieliebak-Eliashberg about topology of Stein manifold.

Stein manifolds are affine schemes over $\mathbb{C}$. There are many equivalent definitions of Stein manifolds, but we will use the following two:

- They are complex manifolds that can be properly embedded into $\mathbb{C}^N$ for some $N \in \mathbb{Z}_{>0}$.

- They are complex manifolds that admit an *exhaustive*, *strictly plurisubharmonic* (PSH) function ϕ. The exhaustive part means that ϕ is a proper, real-value function bounded below, and the strictly plurisubharmonic part means that $-dd^c\phi(v, Jv) > 0$ for all tangent vector v, where $d^c\phi = d\phi \circ J$.

Strictly PSH condition locally reads: $\frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\xi^i\bar{\xi}^j > 0$, this means that the restriction of ϕ on each complex line is subharmonic, i.e. it satisfies the sub-mean property.

Since we will frequently change the complex structure on manifold, it is better to make it appear explicitly in the notation of plurisubharmonicity. So instead of saying a function ϕ is (strictly) PSH with respect to the complex structure J, we will say that ϕ is J-convex.

Example 1 (Stein manifolds)**.** *Here are a few example of Stein manifolds:*

1. *The complex affine spaces $\mathbb{C}^n$, a sub complex manifold of $\mathbb{C}^N$.*
2. *Let X be a closed sub complex manifold of $\mathbb{P}^N_{\mathbb{C}}$ and $H \subset \mathbb{P}^N_{\mathbb{C}}$ be a complex hyperplan. Then $X \setminus H \hookrightarrow \mathbb{P}^N_{\mathbb{C}} \setminus H = \mathbb{C}^N$ is a Stein manifold.*

Our goal will be to answer the following question:

Question. Topologically, what are Stein manifolds.

In other words, we want to find a necessary and sufficient condition of a differential manifold V (without boundary) such that we can equip V with a complex structure J that makes it a Stein manifold. It is easy to see that the following two conditions are necessary:

1. V admits an almost complex structure.
2. V is an open manifold, i.e. there is no compact connected component of V. This is an immediate consequence of Maximum modulus principle for holomorphic functions.

There is another less obvious necessary condition:

Remark 1. *Let (V, J, ϕ) be a Stein manifold. By a generic perturbation we can suppose that ϕ is a Morse function (the Hessian of ϕ is non-degenerate at the critical points of ϕ). Then the index of any critical point p of ϕ is less than $n = \dim_{\mathbb{C}} V$.*

The remark follows from a kind of Pigeonhole principle. Suppose that the index of p is strictly bigger than n, then there exists a complex dimension in T_pV where the Hessian of ϕ is definite negative, this means that there is a complex curve C passing by p on which the function ϕ admits a local maximum at p. This contradicts the pluriharmonicity.

It was proved by Cieliebak and Eliashberg that these conditions are sufficient.

Theorem 1 (Cieliebak-Eliashberg)**.** *Let V be a smooth manifold of real dimension $2n > 4$ and J be an almost complex structure on V and ϕ be an exhaustive Morse function without any critical points of index $k > n$. Then there exists an integrable complex structure $\tilde{J}$ such that $(V, \tilde{J})$ is a Stein manifold.*

In fact, the authors proved that one can obtain $\tilde{J}$ by a homotopy of J and the function ϕ is $\tilde{J}$-convex. In the next part, we will not only modify the complex structure J, but also modify the function ϕ to a PSH function.

2 Morse theory and strategy of the proof.

3 J-convex functions

4 Extension of complex structure.

5 Extension of J-convex function.