

# Berger classification and remarks on parallel structures

June 25, 2021

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## 1 Our story so far

De Rham decomposition theorem allows us to split a Riemannian manifold under certain conditions (complete and connected) as Riemannian product of complete connected manifold with *irreducible holonomy representation*. If an irreducible building block is *locally symmetric*, the theory of Lie groups developed by E. Cartan gave a complete list of holonomy of these spaces. We now shift our focus on non-symmetric irreducible manifolds.

## 2 Berger classification of non-symmetric irreducible manifolds

**Theorem 1** (Berger classification). *For a non-symmetric irreducible manifold, the holonomy representation has to be one of the following*

1.  $SO(n)$ ,
2.  $U(m) \subset SO(2m)$ ,
3.  $SU(m) \subset SO(2m)$ ,

4.  $Sp(r) \subset SO(4r)$ ,
5.  $SO(r)Sp(1) \subset SO(4r)$ ,
6.  $G_2 \subset SO(7)$ ,
7.  $Spin(7) \subset SO(8)$ .

where  $n = 2m = 4r$  is the (real) dimension.

Here are some notations, note always that

$$Sp(m) \subset SU(2m) \subset U(2m) \subset SO(4m)$$

1. If  $Hol(g) \subset U(m) \subset SO(2m)$ ,  $g$  is called a *Kähler metric*.
2. If  $Hol(g) \subset SU(m) \subset SO(2m)$ ,  $g$  is called a *Calabi-Yau metric*. We will see that a Calabi-Yau metric is a Kähler metric that is also Ricci-flat.
3. If  $Hol(g) \subset Sp(m) \subset SO(4m)$  then  $g$  is called a *hyperkähler metric*.
4.  $G_2$  and  $Spin(7)$  are called *exceptional holonomies*

To sum up: hyperkähler  $\longrightarrow$  Calabi-Yau  $\longrightarrow$  Kähler.

**But what do we mean by  $U(n) \subset SO(2n)$ ?** To embed  $U(n)$  in  $SO(2n)$  one needs to identify  $\mathbb{C}$  and  $\mathbb{R}^{2n}$ , this can be done using an almost complex structure  $J$  of  $\mathbb{R}^{2n}$ . We will prove that when we change the almost complex structure, the embedded image of  $U(n)$  in  $SO(2n)$  always remains in the same conjugacy class, which corresponds to the fact that while holonomy representation is well-defined, the holonomy group in  $SO(2n)$  is only defined up to its conjugacy class.

### 3 Almost complex structure

**Definition 2.** *A(n) (almost) complex structure  $J$  on a vector space  $V$  is an automorphism  $J : V \longrightarrow V$  with  $J^2 = -Id_V$ . If  $V$  has a scalar product  $g$ , we suppose in addition that  $g \circ J = g$ .*

*A(n) (almost) complex structure  $J$  on manifold  $M$  is a vector bundle automorphism  $J : TM \longrightarrow TM$  that satisfies  $J_x^2 = -Id_{T_x M}$  for every  $x \in M$ . If  $M$  is a Riemannian manifold, we assume in addition that  $g \circ J = g$ .*

Let us first have a look at a complex structure  $J$  on a fiber (vector space)  $V$ . Here are some direct consequences:

**Complexification.**  $g$  and  $J$  extend in an unique way over  $V_{\mathbb{C}}$ , the complexification of  $V$ , to a Hermitian product  $g_{\mathbb{C}}$  and a  $\mathbb{C}$ -linear automorphism (also noted by  $J$ ) and one still has  $g_{\mathbb{C}} \circ J = g_{\mathbb{C}}$ .

**Eigenspaces.** The complexified space  $V_{\mathbb{C}}$  is decomposed to  $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$  where  $V^{1,0}$  and  $V^{0,1}$  are eigenspaces (complex vector space) corresponding to eigenvalues  $i$  and  $-i$  of  $J$  on  $V_{\mathbb{C}}$ . The orthogonality is by  $g_{\mathbb{C}}$ . The complex conjugate  $\sum z_i x_i \mapsto \sum \bar{z}_i x_i$  where  $z_i \in \mathbb{C}$  and  $x_i \in V$  maps  $V^{1,0}$  to  $V^{0,1}$ . Their dimensions are therefore the same.

**Hermitian form.** The fundamental form  $\omega$  of  $(V, J)$  is defined by

$$\omega(a, b) = g(Ja, b) = -g(a, Jb) \quad \text{on } V$$

which is an antisymmetric real 2-form with  $\omega \circ J = \omega$ .  $V$  equipped with the following Hermitian form

$$h(a, b) = g(a, b) - i\omega(a, b) \quad \text{on } V$$

in the sense that  $h(., .)$  is  $\mathbb{R}$ -bilinear with  $h(Ja, b) = ih(a, b)$  and  $h(a, Jb) = -ih(a, b)$ .

**Identification.** One usually identifies  $(V, J)$  and  $(V^{1,0}, i)$  as vector spaces equipped with complex structure, using the following map:

$$\iota_J : x \mapsto \frac{1}{2}(x - iJ(x))$$

which is  $\mathbb{C}$ -linear in the sense of complex structure:  $\iota_J(Jx) = i\iota_J(x)$ . Note that  $(V, -J)$  is also isomorphic to  $(V^{0,1}, i)$  by the conjugate of  $\iota_J$ :  $x \mapsto \frac{1}{2}(x + iJ(x))$ .

Now note that on we have on  $(V, J)$  an hermitian product  $h(., .)$  and on  $(V^{1,0}, i)$  the restricted Hermitian product  $g_{\mathbb{C}}$  of  $V_{\mathbb{C}}$ . The following lemma gives their relation (the proof is straightforward computation, see Manuscript).

**Lemma 3.** *The identification  $(V, J) = (V^{1,0}, i)$  by  $\iota_J$  gives*

$$\frac{1}{2}h = g_{\mathbb{C}}|_{V^{1,0}}$$

We can now embed  $U(n)$  to  $SO(2n)$ , in other words  $U(V^{1,0})$  to  $SO(V)$  by the map

$$\phi \mapsto \tilde{\phi} \text{ as follow: } \begin{array}{ccc} V & \xrightarrow{\tilde{\phi}} & V \\ \downarrow \iota_J & & \downarrow \iota_J \\ V^{1,0} & \xrightarrow{\phi} & V^{1,0} \end{array}$$

Note that the correspondance  $\phi \leftrightarrow \tilde{\phi}$  is one-to-one between  $\{\phi : V^{1,0} \longrightarrow V^{1,0} \text{ } \mathbb{R}\text{-linear}\}$  and  $\{\tilde{\phi} : V \longrightarrow V \text{ } \mathbb{R}\text{-linear}\}$ . Then

1.  $\phi$  is  $\mathbb{C}$ -linear if and only if  $\tilde{\phi}J = J\tilde{\phi}$ .
2.  $\phi$  preserves  $g_{\mathbb{C}}$  if and only if  $\tilde{\phi}$  preserves  $h$ . Taking the real and imaginary part, the latter is equivalent to the fact that  $\tilde{\phi}$  preserves  $g$  and  $\omega$ .

3. Every  $\mathbb{C}$ -linear  $\tilde{\phi}$  preserves orientation of  $V^{1,0}$  as  $\mathbb{R}^{2n}$  (note that the fact that  $\tilde{\phi}$  preserves orientation or not is independent of how one identifies  $V^{1,0}$  and  $\mathbb{R}^{2n}$ ).

Hence for every  $J$ ,  $\phi \mapsto \tilde{\phi}$  gives a embedding of  $U(V^{1,0})$  to  $SO(V)$ . An orthonormal base of  $V^{1,0}$  and that of  $V$  give a embedding  $U(n) \subset SO(2n)$ .

**Remark 4.** The image of  $U(n)$  in  $SO(2n)$  may depends on  $J$  and the orthonormal base of  $V$ , but its conjugacy class in  $SO(2n)$  is uniquely defined. This is because every complex structure  $J$  is, up to a orthonormal conjugation,

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

## 4 Complexified dual and forms, prelude to Kähler geometry

We state first some linear algebra facts, whose proofs are tedious and can be consulted in the Manuscript.

**Lemma 5** (Linear algebra facts). 1. Let  $V = W_1 \oplus W_2$  be  $\mathbb{R}$ -module then the exterior algebra of  $V$  splits into

$$\bigwedge^n V = \bigoplus_{p+q=n} \bigwedge^p W_1 \otimes \bigwedge^q W_2$$

Note that the tensor product here is formal, and not related to the tensor product defining the exterior algebra.

2. If  $V$  has a complex structure  $J$  then  $J$  gives a complex structure on  $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ , naturally by  $\phi \mapsto \phi \circ J$ .

One has

$$(V^*)_{\mathbb{C}} = \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \equiv \text{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$$

and

$$(V^*)^{1,0} = \text{Hom}_{\mathbb{C}}((V, J), \mathbb{C}), \quad (V^*)^{0,1} = \text{Hom}_{\mathbb{C}}((V, -J), \mathbb{C})$$

where  $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$  denotes the set of  $\mathbb{R}$ -linear morphisms that preserves complex structures ( $\mathbb{C}$  is implicitly with the complex structure  $z \mapsto iz$ )

Therefore  $(V^*)_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$  is rewritten as

$$\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \text{Hom}_{\mathbb{C}}((V, J), \mathbb{C}) \oplus \text{Hom}_{\mathbb{C}}((V, -J), \mathbb{C})$$

Using the first point of Lemma 5, one has

$$\bigwedge^n (V^*)_{\mathbb{C}} = \bigoplus_{p+q=n} \bigwedge^{p,q} (V^*)_{\mathbb{C}}$$

where  $\text{big} \wedge^{p,q} T_{\mathbb{C}}^* M$  denotes the  $\mathbb{C}$ -vector space of forms that are  $p$  times  $\mathbb{C}$ -linear and  $q$  times  $\mathbb{C}$ -antilinear.

Note one can easily find in  $V$  an orthonormal basis  $\partial_{x_i}, \partial_{y_i}$  with  $J(\partial_{x_i}) = \partial_{y_i}$ . We clarify here the definition and implicit identifications of basic objects such as  $dz_i$  and  $d\bar{z}_i$ .

Object	Where it belongs/ properties	Extension/ properties
$\partial_{z_i} = \iota_J(\partial_{x_i})$ $= \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$	$V^{1,0}$ , form a $\mathbb{C}$ -base	$dz_i(\partial_{z_j}) = \delta_{i,j}$ , $dz_i(\partial_{\bar{z}_j}) = 0$
$\partial_{\bar{z}_i} = \iota_{-J}(\partial_{x_i})$ $= \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$	$V^{0,1}$ , form a $\mathbb{C}$ -base	$d\bar{z}_i(\partial_{z_j}) = 0$ , $d\bar{z}_i(\partial_{\bar{z}_j}) = \delta_{i,j}$
$dz_i = dx_i + idy_i$	$\text{Hom}_{\mathbb{C}}((V, J), \mathbb{C}) \equiv \text{Hom}_{\mathbb{C}}(V^{1,0}, \mathbb{C})$ , $\mathbb{C}$ -linear	$\text{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ , null on $V^{0,1}$
$d\bar{z}_i = dx_i - idy_i$	$\text{Hom}_{\mathbb{C}}((V, -J), \mathbb{C}) \equiv \text{Hom}_{\mathbb{C}}(V^{0,1}, \mathbb{C})$ , $\mathbb{C}$ -antilinear	$\text{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ , null on $V^{1,0}$

**Remark 6.** One can note that there are two natural ways to extend  $dz_i$  to  $V^{1,0}$

1. by first make a  $\mathbb{C}$ -linear extension on  $V_{\mathbb{C}}$ , then make a restriction on  $V^{1,0}$
2. using the identification  $(V, J) \equiv (V^{1,0}, i)$

but these two coincide, as there exists a unique form  $\mathbb{C}$ -linear  $dz_i$  that satisfies  $dz_i(\partial_{z_j}) = \delta_{i,j}$ ,  $dz_i(\partial_{\bar{z}_j}) = 0$ . Same story with  $d\bar{z}_i$ . See Figure 1 and Figure 2

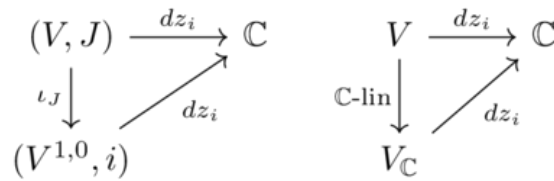


Figure 1: Two natural ways to define  $dz_i$  on  $V^{1,0}$ . They give the same form.

**Definition 7 (Theorem).** The following properties are equivalent and  $X$  is called a Kähler manifold if one of them is satisfied.

$$\begin{array}{ccc}
(V, -J) & \xrightarrow{d\bar{z}_i} & \mathbb{C} \\
\downarrow \iota_J & \nearrow d\bar{z}_i & \\
(V^{0,1}, i) & & 
\end{array}
\qquad
\begin{array}{ccc}
V & \xrightarrow{d\bar{z}_i} & \mathbb{C} \\
\downarrow \mathbb{C}\text{-lin} & \nearrow d\bar{z}_i & \\
V_{\mathbb{C}} & & 
\end{array}$$

Figure 2: Two natural ways to define  $d\bar{z}_i$  on  $V^{0,1}$ . They give the same form.

1.  $X$  is a complex manifold, equipped with a Hermitian structure  $h(\cdot, \cdot)$  compatible with the complex structure  $J$ , and a fundamental form  $\omega$  with  $d\omega = 0$ .
2.  $X$  is a Riemannian manifold with a parallel complex structure.
3.  $X$  is a complex manifold, equipped with a Hermitian structure such that the Chern connection on  $T^{1,0}X$  is, up to an identification by  $\iota_J$ , the Levi-Civita connection.
4.  $X$  is a complex manifold, equipped with an Hermitian structure such that the Chern connection on  $T^{1,0}X$  is torsionless.

We call a complex manifold  $X$  Kähler type if there exists a Hermitian structure under which  $X$  is Kähler.

The proof is straightforward. The only part that is not trivial is that a parallel almost complex structure has to come from a complex atlas, i.e. atlas of  $X$  such that each transition map preserves the complex structure. Such almost complex structures are called *integrable*.

To prove this one uses the following (1,2)-tensor called *Nijenhuis tensor* of a (1,1)-tensor  $A$ , defined by:

$$N_A(X, Y) = -A^2[X, Y] + A[AX, Y] + A[X, AY] - [AX, AY]$$

and the following theorem

**Theorem 8** (Newlander–Nirenberg). *An almost complex structure on  $M$  with vanishing  $N_J$  is integrable.*

The proof that a parallel almost complex structure  $J$  has  $N_J = 0$  is computational in nature and can be found in the Manuscript.

## 5 Symplectic holonomy

One can look at symplectic group  $Sp(r)$  from the following two points of view:

1.  $Sp(r)$  is subgroup of  $Aut_{\mathbb{H}}(\mathbb{H}^r)$  of elements preserving a quaternion Hermitian form  $q$ , where  $\mathbb{H}$  is the algebra of quaternions.

$$2. Sp(r) = SU(2r) \cap Sp(2r, \mathbb{C}).$$

The second point of view explains how  $Sp(r)$  is embedded in  $SO(4r)$ . Let us consider  $Sp(r)$  from the first point. In our context, let  $V$  be a tangent space at one point of the manifold  $M$ , that is a  $4r$  real dimensional vector space, one can regard  $Sp(r)$  as the group of automorphisms of  $V$  preserving the Riemannian metric  $g|_V$  and the complex structures  $I, J$  (hence  $K = IJ$ ) satisfying  $IJ = -JI$ . Hence we have the following remark:

**Remark 9.** *The following properties are equivalent for a Riemannian manifold  $M$ :*

1.  $Hol(M) \subset Sp(r) \subset SO(2r)$ .
2. *There exists on  $M$  three parallel complex structures  $I, J, K$  that satisfy  $K = IJ = -JI$ .*
3. *There exists on  $M$  a parallel complex structure  $I$  and a holomorphic (w.r.t  $I$ ), parallel, 2-form  $\varphi$  that is non-degenerate at a point (hence every point).*

We note that the holomorphic 2-form in the third point is given by

$$\varphi = \omega_I + \sqrt{-1}\omega_K$$

where  $\omega_I$  and  $\omega_K$  are fundamental forms with respect to complex structures  $I$  and  $K$ , and  $M$  is regarded under the complex structure  $I$ .

The implication (2)  $\implies$  (3) is actually exercise 1.2.5 (page 40) of Daniel Huybrechts, *Complex geometry: an introduction* (see Manuscript).

For the implication (3)  $\implies$  (2), note that the real and imaginary part of  $\varphi$  are parallel, they correspond to complex structures  $J$  and  $K$  on  $M$ . Since  $\varphi$  is a (2,0)-form w.r.t  $I$ , one has  $\varphi(Iu, v) = i\varphi(u, v)$ , taking the real part and using the fact that  $g$  is non-degenerate, one has  $K = IJ = -JI$ .