Harmonic maps of Riemannian manifolds

Tien NGUYEN MANH

Feb 20, 2018

Contents

This is my reading note for [**?**].

1 Harmonic maps

1.1 Variational approach: energy integral and tension field

Notation. Let M, M', M'' be Riemannian manifolds of dimension n, n' and n', n'' respectively. We will use $i, j, k, \dots, \alpha, \beta, \gamma, \dots, a, b, c$ for local coordinates of M, M', M''. Let $f : M \longrightarrow M', f' : M' \longrightarrow M''$ be a smooth maps, one denotes

$$f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}, \quad f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k f_k^\alpha$$

so that $\nabla g = g_i dx^i$ and $\nabla(\nabla g) = g_{ij} dx^i \otimes dx^j$

Definition 1. *The **energy desity** of f at $p \in m$ is defined by*

$$e(f)(p) = \frac{1}{2}\langle g(p), f^g(p)\rangle_p = \frac{1}{2} g^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$$

*and the **energy functional** of f is*

$$E(f) = \int_M e(f)dV = \frac{1}{2}\int_M g^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta} |\det(g_{ij})|^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$$

We recall that the inner product is between 2 tensors of type (p, q) $S = S^{i_1 \dots i_p}_{j_1 \dots j_q}, T = T^{k_1 \dots k_p}_{l_1 \dots l_q}$ is $\prod_{m,n} g_{i_m k_m} g^{j_n l_m} S^{i_1 \dots i_p}_{j_1 \dots j_q} T^{k_1 \dots k_p}_{l_1 \dots l_q}$

Remark 1. *Under any orthonormal basis of $T_P M$ and $T_{f(P)}N$, one can see that the energy density is non-negative at every point. Hence $E(f) = 0$ if and only if $e(f) = 0$ at all points if and only if f is constant.*

Definition 2. *Let σ be a symmetric function of n variables and α be a symmetric (0,2) tensor field, one can define the $\sigma-$**energy desity** of α at $P \in M$ to be $\sigma(\beta_1, \dots, \beta_n)(P)$ where β_i are eigenvalues of the linear operator $(g^{ik}\alpha_{ij})_{k,j}$. The σ-**energy** of α is $I_\sigma(\alpha) := \int_M \sigma(\alpha)dV$*

Take $\alpha = f^* g'$, one calls $\sigma(\alpha)$ the /σ-energy density of f and $I_\sigma(\alpha)$ the σ-energy of f.

Example 1. *For example, the energy functional $E(f)$ is $I_{\frac{\sigma_1}{2}}(f)$. $V(f) := I_{\sigma_n^{1/2}}(f)$ is called the **volume** of f.*

Lemma 1 (variation of the energy)**.** *Let $f_t : M \longrightarrow M'$ be a smooth family of smooth maps between Riemannian manifolds for $t \in (t_0, t_1)$. Then*

$$\frac{d}{dt}E(f_t) = -\int_M \left(\Delta f_t^\gamma + g^{ij}\Gamma'^\gamma_{\alpha\beta} f^\alpha_{t,i} f^\beta_{t,j}\right) g'_{\gamma\nu} \frac{\partial f_t^\nu}{\partial t} dV, \qquad \forall t \in (t_0, t_1)$$

Proof. Direct computation. □

Definition 3. 1. *A **vector field along** $f : M \longrightarrow M'$ is a smooth application $v : M \longrightarrow TM'$ such that $\pi \circ v = f$ where $\pi : TM' \longrightarrow M'$ is the canonical projection. In other words, it is the association of each point $P \in M$ a tangent vector at $f(P)$*

2. *The **tension field** of f is the following vector field along f defined by*

$$\tau(f)^\gamma := \Delta f^\gamma + g^{ij}\Gamma'^\gamma_{\alpha\beta} f_i^\alpha f_j^\beta$$

By the Lemma 1, $\tau(f)$ is the unique vector field along f such that $\frac{d}{dt}E(f_t) = -\int_M \langle \tau(f), \frac{df_t}{dt}\rangle$. In particular, if f_t is the variation of f along a vector field v along f, i.e. $f_t(P) = \exp_{f(P)}(tv(P))$ then $\nabla_v E(f) = -\langle \tau(f), v\rangle$ along f.

3. *$f : M \longrightarrow M'$ is called **harmonic** if $\tau(f) = 0$, or equivalently f is a critical point of E.*

In normal coordinates of M at P and M' at $f(P)$, the tension field of f is given by

$$\tau^\gamma(f)(P) = \sum_i \frac{\partial^2 f^\gamma}{\partial (x^i)^2}(P)$$

Remark 2. 1. *If M' is flat, i.e. $R'_{\alpha\beta\gamma\delta} = 0$ then $\tau(f)^\gamma = \Delta f^\gamma$ is linear in f.*

2. *Since $\tau(f)$ depends locally on f, isometries and covering maps are harmonic.*

Proposition 1.1 (holomorphicity implies harmonicity)**.** *Holomorphic maps between Kahler manifolds are harmonic.*

Proof. We recall that exponential functions $\exp_P : T_PM \longrightarrow M'$ on a Kahler manifold M are holomorphic for any $P \in M$. In fact, let $v \in T_PM$ and $\delta v \in T_v(T_PM)$ be a tangent vector at v and denote abusively by J the complex structure of the complex vector space T_PM and that of M, one needs to see that

$$D\exp_P(v).J\delta v = J(\exp_P(v))D\exp_P(v).\delta v \tag{1}$$

In fact, let Y_1, Y_2 be Jacobi fields along $U(t) = \exp_P(tv)$ the geodesics of M starting at P in direction v with $Y_1(0) = Y_2(0) = 0, \dot{Y}_1(0) = \delta v, \dot{Y}_2(0) = J\delta v$ then the LHS of (1) is $Y_2(1)$, and the RHS is $J(U(1))Y_1(1)$. Then one can see that $Y_2(t) - J(U(t))Y_1(t) = 0$ for every $t \in [0,1]$ since it is true at $t = 0$ and the derivative with respect to t vanishes as $\nabla_{\dot{U}} J = 0$.

Therefore, at a point P of a Kahler manifold M, there exist holomorphic coordinates $z^j = x^j + iy^j$ of M in a neighborhood of P such that $\{x_j, y_j : j = \overline{1, n/2}\}$ are normal coordinates centered in P. Using such coordinates for $P \in M$ and $f(P) \in M'$, one has $f^\gamma = 0$ since f^γ is holomorphic and $\Gamma'^\gamma_{\alpha\beta}(P) = 0$ by normality, it follows that $\tau(f) = 0$ at every point $P \in M$. $\square$

1.2 Formulation using connection on vector bundle

Setup and notation. Let E be a metric vector bundle over a Riemannian manifold M, i.e. each fiber of E is equiped with an inner product that we denote by $(g'_{\alpha\beta})$. The metric of M is denoted by (g_{ij}). Let n and m be the dimension of M of the fiber.

Covariant derivatives and exterior derivatives. We recall that a **covariant derivative** or a **connection** $\tilde{\nabla}$ of E is uniquely determined in a local coordinates by an $m \times m$ matrix A of 1-form on M, in other words an 1-form on M with value in $Hom_M(E, E)$ which depends on the local frame of E (i.e. A is not a tensor with value in E). A is called the **connection form** of $\tilde{\nabla}$. Locally

$$\tilde{\nabla}_X(s^\alpha \tilde{e}_\alpha) = (\nabla_X s^\alpha)\tilde{e}_\alpha + A^\alpha_\beta(X) s^\beta \tilde{e}_\alpha.$$

When one prefers to work with forms other than tensors with value in E, one uses an **exterior derivative**, a map $D : A^p(M, E) \longrightarrow A^{p+1}(M, E)$ which turns an p-form with value in E to an $p + 1$-form with value in E. Locally

$$D(s^\alpha \tilde{e}_\alpha) = (ds^\alpha)\tilde{e}_\alpha + A^\alpha_\beta \wedge s^\beta \tilde{e}_\alpha.$$

and

$$D^2(s^\alpha \tilde{e}_\alpha = (dA + A \wedge A) \wedge s.$$

One notes $\Theta := dA + A \wedge A$, which is an $m \times m$ matrix of 2-forms of M. Unlike A, Θ, seen as an 2-form with value in $Hom_M(E, E)$ does not depend on the local frame of E, i.e. Θ transforms as a (0,2) tensor with value in E, called the **curvature form**.

The fibrewise metric structure of E and the metric tensor of M give rise to a pointwise inner product of (p, q) tensors of M with value in E, in particular a pointwise inner product $(s, s') \mapsto s \cdot s'$ from $A^p(M, E) \times A^p(M, E)$ to $C^\infty(M)$. Integrated over M, the pointwise inner product gives rise to a global inner product $\langle \cdot, \cdot \rangle$, i.e. a true inner product of $A^p(M, E)$. One denotes by $\delta : A^{p+1}(M, E) \longrightarrow A^p(M, E)$ the adjoint operator of $D : A^p(M, E) \longrightarrow A^{p+1}(M, E)$ with respect to this inner product, i.e. $\langle Ds, s' \rangle_{A^{p+1}(M,E)} = \langle s, \delta s' \rangle_{A^p(M,E)}$ for all $s \in A^p(M, E), s' \in A^{p+1}(M, E)$.

Laplacian operator and harmonic forms. The **Laplacian operator** is defined as a endomorphism of $A^p(M, E)$ given by

$$\tilde{\Delta} = -(D\delta + \delta D)$$

and a form $s \in A^p(M, E)$ is called **harmonic** if $\tilde{\Delta} s = 0$. Since the Laplacian operator represents the *Dirichlet integral*, i.e.

$$\langle Ds, Ds' \rangle + \langle \delta s, \delta s' \rangle = \langle -\tilde{\Delta} s, s' \rangle,$$

one has $\tilde{\Delta} s = 0$ if and only if $Ds = \delta s = 0$.

Riemannian connected bundle. The metric vector bundle E over M is called a **Riemannian-connected bundle** if it has a connection $\tilde{\nabla}$ under which the metric g' of E is parallel, i.e. $\tilde{\nabla}g' = 0$, in other words, the matrix A in a orthonormal frame is anti-symmetric: $A + {}^{\mathrm{t}}A = 0$. Unless explicitly indicated, we always suppose that our metric vector bundle E is Riemannian-connected and the metric g' is parallel to the connection being used.

Example 2. *The case of our interest is when we have a smooth map $f : M \longrightarrow M'$ and $E = f^*TM'$ is a metric vector bundle over M under the metric g' induced from M'. Taking the connection $\tilde{\nabla}$ to be the Levi-Civita connection ∇' on M', meaning*

$$\tilde{\nabla}_X s = \nabla'_{f_* X} s,$$

for any vector field s along f, one can see that E is a Riemannian-connected bundle over M.

Lemma 2. *Let E be a Riemannian-connected bundle and $s \in A^1(M, E)$, one has*

1. *$\delta s = (\delta s)^\alpha \tilde{e}_\alpha \in A^0(M, E)$ where*

$$(\delta s)^\alpha = -g^{ij} \left(\nabla_i s_j^\alpha + A_{\beta i}^\alpha s_j^\beta \right),$$

2. *$\Delta s = (\Delta s)_i dx^i$ where $(\Delta s)_i$ is an $m \times m$ matrix given by*

$$(\Delta s)_i = \tilde{\nabla}^k \tilde{\nabla}_k s_i - {}^{\mathrm{t}}\!\left(\Theta_i^h - \mathrm{Ric}_i^h \right) s_h$$

where:

- *the indices i, h, k correspond to local coordinates of M, not a frame of M',*
- *Θ_i^h is the curvature form of $\tilde{\nabla}$ with its indices raised by the metric g of M,*
- *$\mathrm{Ric}_i^h = \mathrm{Ric}_i^h I_m$ is the Ricci curvature tensor of (M, g) with indices raised by the metric g, multiplied by the identity $m \times m$ matrix,*
- *$\tilde{\nabla}^k = g^{hk} \tilde{\nabla}_h$.*

3. *With $s \cdot s'$ denoting the pointwise inner product of $A^1(M, E)$ and $\langle \cdot, \cdot \rangle$ denoting the metric g' of E, one has*

$$\frac{1}{2}\Delta(s \cdot s) = -s \cdot \Delta s - \langle \tilde{\nabla}_i s_k, \tilde{\nabla}^i s^k \rangle_E - \left\langle {}^{\mathrm{t}}\!\left(\Theta_i^h - \mathrm{Ric}_i^h \right) s_h, s^i \right\rangle_E \quad (2)$$

Proof. Computational in nature. □

Remark 3. *1. We note by $Q(s)$ the last term of (2), then $Q(s)$ is a (2,0) tensor on M with value in $E^* \otimes E^*$ where E^* is the dualised bundle of E. In practice, $Q(s)$ is an $mn \times mn$ matrix with coefficients*

$$Q(s)^{hi}_{\alpha\beta} = g^{hk} h^{ij} \left[\left(g'_{\alpha\gamma} \Theta^{\gamma}_{\beta} \right)_{kj} - g'_{\alpha\beta} \mathrm{Ric}_{kj} \right]$$

.

2. *Since $\int_M \Delta(s \cdot s) dV = 0$, if s is harmonic, one has*

$$\begin{aligned} \int_M Q(s) dV &= - \int_M \langle \tilde{\nabla}_i s_k, \tilde{\nabla}^i s^k \rangle_E dV \\ &= - \int_M \| \nabla_i s^{\alpha}_k dx^i \otimes dx^k \otimes \tilde{e}_{\alpha} \|^2_{A^2(M,E)} dV \leq 0 \end{aligned} \tag{3}$$

1.3 The case of $E = f^* TM'$

1.3.1 Energy functional and tension field

Our interest will be the case of Example 2 where $E = f^* TM'$ for some smooth map $f : M \longrightarrow M'$ of Riemannian manifolds is a Riemmanian-connected bundle over M with the connection $\tilde{\nabla}$ given by the Levi-Civita connection of M'.

The tangent map $Tf : TM \longrightarrow TM'$ can be interpreted as a form f_* in $A^1(M, E)$. The energy functional can be rewritten as

$$E(f) = \frac{1}{2} \int_M f^{\alpha}_i f^{\beta}_j g^{ij} g'_{\alpha\beta} dV = \frac{1}{2} \langle f_*, f_* \rangle_{A^1 M, E}.$$

Proposition 2.1. *Let $f : M \longrightarrow M'$ and $E = f^* TM'$ be the Riemannian-connected bundle over M. Then:*

1. *$A^{\beta}_{\alpha} = \Gamma'^{\beta}_{\gamma\alpha} f^{\gamma}_i dx^i$ where $\Gamma'^{\beta}_{\gamma,\alpha}$ are Christoffel symbols of (M', g').*
2. *$Df_* = 0$ where f_* is considered as an element of $A^1(M, E)$. Hence $\tilde{\Delta} f_* = -D\delta f_*$.*
3. *The tension field of f is $\tau f_* = -\delta f_*$.*

Proof. 1. We will use the fact that $\tilde{\nabla} g' = 0$. Given two section $s = s^{\alpha} \tilde{e}_{\alpha}, t = t^{\beta} \tilde{e}_{\beta}$ of E, expanding $\nabla_i (s \cdot t) = (\tilde{\nabla}_i s) \cdot t + s \cdot \tilde{\nabla}_i t$, one has

$$s^{\alpha} t^{\beta} \frac{\partial g'_{\alpha\beta}}{\partial x^i} = s^{\alpha} t^{\beta} \left(A^{\gamma}_{\alpha i} g'_{\gamma\beta} + A^{\gamma}_{\beta i} g'_{\alpha\gamma} \right)$$

Taking s, t to be of small support, $\alpha = \beta$ and substituing $A^\gamma_{\alpha i} = \Gamma'^\nu_{\gamma\alpha} f^\gamma_i$, one obtains the first statement.

2. By direct computation:

$$Df_* = \left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \Gamma'^\alpha_{\gamma\beta} f^\gamma_i f^\beta_j \right) dx^j \wedge dx^i \otimes \tilde{e}_\alpha,$$

which is the product of a symmetric quantity in (i, j) and an anti-symmetric one, hence 0.

3. Using the first part of Lemma 2 for $s = f_* = f^\alpha_i dx^i \otimes \tilde{e}_\alpha$, one has $\delta f_* = -g^{ij} \left(\nabla_i \nabla_j f^\gamma + \Gamma'^\gamma_{\alpha\beta} f^\alpha_i f^\beta_j \right) \tilde{e}_\gamma = -\tau(f)$

□

It follows immediately that

Corollary 2.1. *$f : M \longrightarrow M'$ is a harmonic map of Riemannian manifolds if and only if f_* is harmonic as form in $A^1(M, f^*TM')$.*

1.3.2 Fundamental form, some results in case of signed curvature

Definition 4. *The **fundamental form** of a map $f : M \longrightarrow M'$ of Riemannian manifolds is the (0,2) symmetric tensor on M with value in $E = f^*TM'$ defined by*

$$\beta(f) := \tilde{\nabla} f_* = \left(f^\gamma_{ij} + \Gamma'^\gamma_{\alpha\beta} f^\alpha_i f^\beta_j \right) dx^i \otimes dx^j \otimes \tilde{e}_\gamma.$$

*The function f is called **totally symmetric** if $\beta(f) = 0$ identically on M.*

Remark 4. 1. *The tension field $\tau(f) = g^{ij}(\beta(f))_{ij}$ is the trace of the fundamental form.*

2. *If f is totally geodesic then it is harmonic.*

When $s = f_*$, Lemma 2 and Remark 3 become, with no more than direct computation. The appearance of the Riemann curvature tensor R' of (M', g') is due to the formula

$$R'^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma'^\rho{}_{\nu\sigma} - \partial_\nu \Gamma'^\rho{}_{\mu\sigma} + \Gamma'^\rho{}_{\mu\lambda} \Gamma'^\lambda{}_{\nu\sigma} - \Gamma'^\rho{}_{\nu\lambda} \Gamma'^\lambda{}_{\mu\sigma}.$$

Lemma 3. *1. $Q(f_*)$ is given by*

$$Q(f_*) = -R'_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta g^{ik} g^{jl} - \mathrm{Ric}^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$$

and

$$Q(f_*)^{ij}_{\alpha\beta} = -R'_{\alpha\beta\gamma\delta} f_k^\gamma f_l^\delta g^{ik} g^{jl} - \mathrm{Ric}^{ij} g'_{\alpha\beta}.$$

2. If f is harmonic then

$$\Delta e(f) = |\beta(f)|^2 + Q(f_*)$$

where $|\beta(f)|$ is the pointwise norm of $\beta(f)$.

The previous computation of $Q(f_*)$ in term of Riemannian curvature of M' and Ricci curvature of M give the following result in the case where the curvature of M and M' are of definite sign.

Notation. Given a Riemannian manifold M, we will use the following notation:

1. $\mathrm{Ric} \geq 0$ (resp. $\mathrm{Ric} > 0$) if the Ricci curvature is positive semi-definite (resp. positive definite) as symmetric bilinear form.
2. $\mathrm{Ric} \leq 0$ (resp. $\mathrm{Ric} < 0$) if $-\mathrm{Ric} \geq 0$ (resp. $-\mathrm{Ric} > 0$).
3. $\mathrm{Riem} \geq 0$ (resp. $\mathrm{Riem} > 0$) if the Riemann curvature tensor satisfies $R_{ijhk} u^{ij} u^{hk} \geq C(P) u_{ij} u^{ij}$ at $P \in M$ for all anti-symmetric (0,2) tensor u on M, where $C(P) \geq 0$ (resp. $C(P) > 0$)
4. $\mathrm{Riem} \leq 0$ (resp. $\mathrm{Riem} < 0$) if $-\mathrm{Riem} \geq 0$ (resp. $-\mathrm{Riem} > 0$).

Remark 5. *By symmetries of the curvature tensor,* $\mathrm{Riem} \geq 0$ *if and only if $R_{ijhk} u^{ij} u^{hk} \geq 0$ at $P \in M$ for all (0,2) tensor u on M, where $C(P) \geq 0$.*

Corollary 3.1. *Let $f : M \longrightarrow M'$ be a map of Riemannian manifolds.*

1. *If f is harmonic and $Q(f_*) \geq 0$ then f is totally geodesic and $e(f)$ is constant.*
2. *If $\mathrm{Ric}(M) \leq 0$ and $\mathrm{Riem}(M') \leq 0$ then f is harmonic if and only if f is totally geodesic.*
3. *Under the same condition as 2),*

- *If* $\mathrm{Ric}(M) < 0$ *at one point of* M *then all harmonic maps are constant.*
- *If* $\mathrm{Riem}(M') < 0$ *everywhere in the image of* f *and* f *is harmonic, then* f *is either constant or maps* M *onto a closed geodesic of* M'*.*

Proof. All the statements are consequence of 2) of Lemma 3 and the fact that $\int_M \Delta e(f) dV = 0$ noticing that

- $-\mathrm{Ric}^{ij} f_i^\alpha f_j^\beta g'_{\alpha\beta}$ is $-\mathrm{Ric} \otimes g'$ applied doubly to $f_i^\alpha dx^i \otimes \tilde{e}_\alpha$.
- $-R'_{\alpha\beta\gamma\delta} f_k^\gamma f_l^\delta g^{ik} g^{jl}$ is $-\tilde{R}' \otimes g \otimes g$ applied to $f_i^\alpha f_j^\beta \tilde{e}_\alpha \otimes \tilde{e}_\beta \otimes dx^i \otimes dx^j$ where $\tilde{R}'$ is the bilinear form $(u, v) \mapsto R'_{\alpha\beta\gamma\delta} u^{\alpha\beta} v^{\gamma\delta}$.

For 3), if $\mathrm{Ric}(M) < 0$ at one point $P \in M$ then at that point $f_i^\alpha dx^i \tilde{e}_\alpha = 0$, meaning $f_* = 0$, hence $e(f)$ vanishes at P. Since $e(f)$ has to be constant, it vanishes identically, which implies that f is constant.

If $\mathrm{Riem}(M') < 0$, one has $f_i^\alpha f_j^\beta \tilde{e}_\alpha \otimes \tilde{e}_\beta$ is symmetric, i.e. $f_i^\alpha f_j^\beta = f_j^\alpha f_i^\beta$ for all i, j, α, β, meaning that the ratio $f_i^\alpha : f_j^\alpha$ is independent of α, i.e. $(\nabla f^\alpha)_{\alpha \in \overline{1,m}}$ are colinear, i.e. the image of Tf is of one dimension, which leads to the conclusion, as we will see later that a totally geodesic map transform geodesic to geodesic. □

1.3.3 Example: Riemannian immersion

1.3.4 Example: Riemannian submersion

1.4 Composition of maps

2 Deformation of maps

Emacs 25.3.1 (Org mode 9.0.5)