From Busemann function to Cheeger-Gromoll splitting

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We will prove the following result by Cheeger and Gromoll by a slightly modified approach of A. Besse.

Theorem 1 (Cheeger-Gromoll)**.** *Let M be a complete, connected Riemannian manifold with non-negative Ricci curvature. Suppose that M contains a line then M is isometric to $M' \times \mathbb{R}$ with M' a complete, connected Riemannian manifold with non-negative Ricci curvature.*

Note that the notion of geodesic ray or geodesic line used here is rather strict: A geodesic line γ is a geodesic parameterized by $\mathbb{R}$ such that the distance between two point is exactly the distance *on the geodesic*, for example, geodesic line, if it passes by $p \in M$ with velocity v of norm 1, satisfies

$$d(exp_p(tv), exp_p(-sv)) = s + t, \quad \forall s, t > 0$$

1 Busemann function

Let γ be a geodesic ray. We construct the Busemann function b associated to the ray as

$$b(x) = \lim_{t \to +\infty} (t - d(x, \gamma(t)))$$

where the limit exists because the sequence $f_t: x \mapsto t - d(x, \gamma(t))$ is non-decreasing and bounded above by $d(x, \gamma(0))$. The convergence is also uniform in every compact of M.

In Euclidean space for example, the Busemann function is the orthogonal projection on γ. We will see that in a Riemannian manifold with non negative curvature, the Busemann function will serve as a projection.

Now with a fixed $x_0 \in M$, the tangent vectors at x_0 of the geodesics connecting x_0 and $\gamma(t)$ is in the unit sphere of $T_x M$, which is compact. Let X be a limit point of these tangents vectors, we defined

$$b_{X,t}(x) = b(x_0) + t - d(x, exp_{x_0}(tX))$$

where $exp_{x_0}(tX)$ is the geodesic starting at x_0 with velocity X.

Remark 1. 1. *From the construction of X, one has $b(x_0)+t = b(exp_{x_0}(tX))$, therefore $b_{X,t} \leq b$ with equality in x_0. We say that b is supported by $b_{X,t}$ at x_0. In general a function f is* <u>*supported*</u> *by g at x_0 if $f(x_0) = g(x_0)$ and $f \geq g$ in a neighborhood of x_0.*

2. *$b_{X,t}$ is smooth and a computation in local coordinate gives $\Delta b_{X,t} \geq -\frac{\dim M - 1}{t}$*

3. *$\|\nabla b_{X,t}\| = 1$*

The estimation given on the second point of Remark 1 is established using Jacobi fields:

Lemma 2. *The function $f(x) = d(x, x_0)$ satisfies at a point x out of the cut-locus of x_0:*

$$\nabla f(x) \leq \frac{n-1}{l}$$

where $n = \dim M, l = d(x, x_0) = f(x)$ in Riemannian manifold M with non-negative Ricci curvature.

Proof. Let $N(t), 0 \leq t \leq l$ be the velocity of the geodesic γ from x_0 to x, and $E_1, \ldots, E_{n-1}, N$ be a parallel frame along γ. Let J_i be the unique Jacobi fields along γ with $J_i(l) = E_i(l)$ and $J_i(0) = 0$ (existence and uniqueness of J_i is due to the fact that x is not in the cut-locus).

Then basic manipulation of Jacobi fields gives (without the fact that curvature is non-negative):

$$\Delta f(x) = \int_0^l dt \sum_{i=1}^{n-1} (< \nabla_N J_i, \nabla_N J_i > - < R(N, J_i)J_i, N >) = \sum_{i=1}^{n-1} I_\gamma(J_i, J_i)$$

where I_γ is the index form of γ. Note that the Jacobi fields J_i coincide with the fields $\frac{t}{l}E(t)$ at 0 and l, therefore by the *fundamental inequality* of index form (see Sakai Takashi, *Riemannian geometry* for details about Jacobi fields and Fundamental inequality of index form):

$$I_\gamma(J_i, J_i) \leq I_\gamma(\frac{t}{l}E_i, \frac{t}{l}E_i)$$

hence

$$\Delta f(x) \leq \int_0^l \sum_{i=1}^{n-1} < \nabla_N \frac{t}{l}E_i, \nabla_N \frac{t}{l}E_i > - < R(N, \frac{t}{l}E_i)\frac{t}{l}E_i, N >$$

The curvature term being $\frac{t^2}{l^2}Ric(N,N)$ therefore non-negative, one has

$$\Delta f(x) \leq \int_0^l dt \sum_{i=1}^{n-1} < \nabla_N \frac{t}{l}E_i, \nabla_N \frac{t}{l}E_i >= \frac{n-1}{l}.$$

$\square$

We also note that for Theorem 1 it suffices to show that b is harmonic. In fact, from the smoothness one has $\nabla b(x_0) = \nabla b_{X,t}(x_0)$, which means $\|\nabla b\| = 1$ at every point in M. For each point $y \in M$, there exists a unique x with $b(x) = 0$ and time t when the flow of ∇b starting at x arrives at y. M is therefore homeomorphic to $\bar{M} \times \mathbb{R}$ by the map $F : y \mapsto (x,t)$. In order that F is an isometry, it suffices to prove that the gradient field ∇b is parallel. In fact, $\bar{M}$ being equiped with the restriction of the metric on M, the fact that F is isometric is equivalent to the the fact that the flow Φ^t of ∇b is isometric for every time t, which means $\frac{d}{dt} < \Phi^t_* u, \Phi^t_* u >$ vanishes at $t = 0$. But

$$\frac{d}{dt} < \Phi^t_* u, \Phi^t_* u > |_{t=0} = 2 < \nabla_{\partial t}\Phi^t_* u, u > |_{t=0} = 2 < \nabla_u \nabla b, u >$$

where in the second equality we used Schwarz lemma for commuting derivatives of $\Phi(t,x) = \Phi^t(x)$. The vanishing of $< \nabla_u \nabla b(x), u >$ for every vector u is, by bilinearity, equivalent to that of $\nabla_u \nabla b$ for every u, meaning that ∇b is parallel.

The fact that ∇b is parallel is due to a simple computation:

$$Ric(N,N) = -N(\Delta b) - \|\nabla N\|^2$$

where $\|\nabla N\|^2 = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} < \nabla_{E_i} N, E_j >^2$. We see that $N = \nabla b$ is parallel if $\Delta b = 0$.

Remark 2. 1. *One can show (see A. Besse) that every gradient field ∇b of norm 1 at every point is actually harmonic.*

2. *Using de Rham decomposition, one has directly the splitting of M if it is simply-connected since N is parallel and M is complete.*

2 Harmonicity

The Busemann function associated to a geodesic ray is subharmonic, it is a consequence of the following lemma.

Lemma 3. *In a connected Riemannian manifold, if a continuous function f is supported at any point x by a family f_ϵ (depending on x) with $\Delta(f_\epsilon) \leq \epsilon$, then f can not attains its maximum (unless when f is constant).*

Proof. Given a small geodesic ball B, suppose that we have a function h on B with $\Delta h < 0$ in B and $f + h$ attains maximum at x in the interior of B. Then $f_\epsilon + h$ also attains maximum at x, which means $\Delta f_\epsilon + \Delta h \geq 0$, which is contradictory.

For the construction of the function h, one suppose that B is small enough such that $f|_{\partial B} \leq max_B f =: f(x_0)$ and equality is not attained at every points in ∂B. Then choose

$$h = \eta(e^{\alpha\phi} - 1)$$

with and $\phi(x) = -1$ if $x \in \partial B$ and $f(x) = f(x_0)$, $\phi(x_0) = 0$, $\nabla\phi \neq 0$ and a large α such that

$$\Delta h = \eta(-\alpha^2\|\nabla\phi\| + \alpha\Delta\phi)e^{\alpha\phi}.$$

is negative. □

Now for subharmonicity of b, given a harmonic function h that coincides with b in the boundary ∂B of a geodesic ball B, then $b - h$ is supported by $b_{X,t} - h$ with $\Delta(b_{X,t} - h) \to 0$ as t tends to $+\infty$, therefore $b - h \leq (b - h)|_{\partial B} = 0$ in B. We have just proved the following lemma:

Corollary 3.1. *The Busemann function of a geodesic ray in a Riemannian manifold M with non-negative Ricci curvature is subharmonic.*

Now let b_+ be the function previously constructed for the ray $\gamma|_{[0,+\infty[}$ and b_- the Busemann function for the ray $\tilde{\gamma}|_{[0,+\infty[}$ where $\tilde{\gamma}(t) = \gamma(-t)$. Note that $b_+ + b_- \leq 0$ with equality on the line γ, but the sum is subharmonic therefore by maximum principle $b_+ + -b_- = 0$ and b is harmonic therefore smooth. The splitting theorem of Cheeger-Gromoll follows.

3 Application

A consequence of Theorem 1 is the following result from J.Cheeger- D.Gromoll, *The splitting theorem for manifold of nonnegative Ricci curvature* (Theorem 2)

Theorem 4. *Let M be a compact Riemannian manifold with non-negative Ricci curvature, then the universal covering space of M is of form $\tilde{M} = \mathbb{R}^n \times \bar{M}$ where $\bar{M}$ does not contain any lines. Then $\bar{M}$ is compact.*

Proof. It suffices to prove that if $\bar{M}$ is not compact, then it contains a line. In fact, it is easy to see that such $\bar{M}$ must contains a (strict) geodesic ray. In fact it is obvious that with a fixed $p \in M$ the function

$$F : v \mapsto \inf\{t > 0 : d(p, exp_p(tv)) < t\}$$

defined on the unit ball U_p of $T_p\bar{M}$ is upper semi-continuous. Therefore if $F(v) < \infty$ for all unit tangent vector v at p then F is bounded above in U_p by a constant c. Therefore $\bar{M} \subset exp_p(cU_p)$ which is compact (contradiction). Therefore there exists a minimal ray at every point $p \in \bar{M}$.

The existence of a line in general might not be true, the only extra property of $\bar{M}$ that we will need is that it has a (fundamental) domain K compact and a family σ_i of isometries such that $\bar{M} = \cup_i \sigma_i K$.

Let us first prove that such domain K exists. Remark that every isometry of $\mathcal{M}$ acts separately on $\mathcal{M}$, i.e. of form $\sigma(u) = (\sigma_1(x), \sigma_2(y))$ for $u = (x, y) \in \mathcal{M}$ with g_1, g_2 isometries of $\mathbb{R}^n$ and $\bar{M}$. This can be seen by the uniqueness part of de Rham decomposition or simply by noticing that a tangent vector in the $T_x\mathbb{R}^n$ component is characterized by the fact that its geodesics is a line. As the action of $\pi_1(M)$ on $\tilde{M}$ is free and proper, it has a fundamental domain H. We then can choose K to be the projection of H on $\bar{M}$ and $\{\sigma_i\}$ to be the projection of $\pi_1(M)$ on $Isom(\bar{M})$.

Now let γ be a minimal ray starting from $p \in M$, for each $x \in \gamma$ there exists an isometry σ of $\bar{M}$ such that $\sigma(x) \in K$. By compactness of K, there exists a sequence $t_n \to +\infty$ with $x_n = \gamma(t_n)$, $v_n = \dot{\gamma}(t_n)$ that satisfies $y_n = \sigma_n(x_n) \to y \in K$ and even more, $(\sigma_n)_* v_n \to v \in T_y\bar{M}$ in the tangent bundle $T\bar{M}$. Then the geodesic of $\bar{M}$ starting at y with vector v is a line. In fact it suffices to prove that $d(exp_y(tv), exp_y(-sv)) = s + t$ for $s, t > 0$, but for n large enough that $t_n > s$ one has

$$d(exp_{y_n}(tv_n), exp_{y_n}(-sv_n)) = s + t$$

then let $n \to +\infty$, one sees that $\bar{M}$ contains a geodesic line, which is contradictory. $\square$