Counter example: Harmonic map that does not satisfy stronger boundary condition

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Let $f : (\bar{\mathbb{H}}^2 = \{(t,s) : t \geq 0\}, \text{Euclidean}) \longrightarrow (\bar{\mathbb{H}}^2 = \{(x,y) : x \geq 0\}, \text{Poincare})$ be a harmonic map. The harmonic equation of f reads

$$\begin{cases} \Delta x + \frac{1}{x}\left[\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 - \left(\frac{\partial x}{\partial t}\right)^2 - \left(\frac{\partial x}{\partial s}\right)^2\right] & = 0 \\ \Delta y - \frac{2}{x}\left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial s} + \frac{\partial x}{\partial t}\frac{\partial y}{\partial t}\right) & = 0 \end{cases}$$

where $\Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}$.

The quadratic differential w.r.t the Euclidean metric on the target space (the metric $\bar{g}$) is

$$\bar{q} := \langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial z} \rangle_E = \left[\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 - \left(\frac{\partial x}{\partial t}\right)^2 - \left(\frac{\partial y}{\partial t}\right)^2\right] - 2i\left(\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial s}\frac{\partial y}{\partial t}\right)$$

Choose $x := x(t)$ depending only on t and $y := y(s)$ depending only on s, the harmonic equations become $\begin{cases} x_{tt} + \frac{1}{x}(y_s^2 - x_t^2) & = 0 \\ y_{ss} & = 0 \end{cases}$ Choose $y := s$ then solve $x'' + \frac{1}{x}(1 - x'^2) = 0$ for x. We also want $x(0) = 0$ and $x'(0) > 0$ so that $x \geq 0$ when $t \geq 0$. The function $x(t) := \sinh(t)$ satisfies all these conditions.

The quadratic differential is

$$\bar{q} = 1 - x'(t)^2 = -\sinh^2 t = -t^2 + O(t^4)$$

So the constant term and the first order term of $\bar{q}$ both vanish, but the second order term does not. This means f only satisfies the intermediate boundary condition, and not the stronger boundary condition.