

# Symmetric spaces and Lie groups

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## 1 Symmetric space

By de Rham decomposition, we now focus more on the building blocks: Riemannian manifolds with irreducible holonomy. The theory of Lie groups allows us to understand a block if it is *symmetric*.

**Definition 1.** *A Riemannian manifold  $M$  is called symmetric if for every  $x \in M$ , there exists an isometry  $s_x$  of  $M$  such that  $x$  is an isolated fixed point and  $s_x^2 = Id$ .*

Let  $x \in M$  and  $v \in T_x M$ , we note by  $\exp_x(v)$  the point of distance  $|v|$  in the geodesic starting in  $x$  with velocity  $v/|v|$ . We remark that any isometry  $s_x$  with  $s_x^2 = Id$  and  $x$  as isolated fixed point satisfies

$$s_x(\exp_x(v)) = \exp_x(-v) \tag{1}$$

In fact the eigenvalues of  $T_x s_x$  have to be 1 or  $-1$ , but as  $x$  is an isolated fixed point one has  $T_x s_x = -Id$ . Then  $s_x$  as an isometry sends the geodesic starting at  $x$  with velocity  $v$  to one starting at  $s_x(x) = x$  with velocity  $(s_x)_* v = -v$  and we have (1).

Equation (1) tells us that  $s_x$  is a reflection of center  $x$  on every geodesic passing by  $x$ . We can compose two reflections  $s_x, s_y$  to form a translation on the geodesic connecting  $x$  and  $y$ . This shows that a symmetric space is

complete and the group of isometries of the form  $s_x \circ s_y$  acts transitively on  $M$ .

**Theorem 1** (Symmetric space). *Let  $M$  be a symmetric Riemannian manifold then*

1.  $M$  is complete.
2. Fix  $x_0 \in M$ , let  $G$  be the group generated by the isometries of form  $s_x \circ s_y$ ,  $x, y \in M$  and  $H$  is the subgroup containing elements of  $G$  that fix  $x_0$ , then  $G$  is Lie subgroup of  $\text{Isom}(M)$  connected by arc,  $H$  is a closed Lie subgroup of  $G$  and  $M$  is isometric to  $G/H$ . Moreover the holonomy group of  $M$  is  $H$ .

**Remark 1.** In general, for a Lie group  $G$  and a closed Lie subgroup  $H$ , if  $G$  has a metric left-invariant by  $G$  and right-invariant by  $H$  (i.e. the metric on  $\mathfrak{g}$  is invariant by action of  $H$  by adjoint) then

$$\mathfrak{g} = \mathfrak{h} \oplus^\perp \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

But if  $G/H$  is symmetric then one has the following extra information

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

It turns out that this condition is quite strong and allowed E. Cartan to classify all such pairs  $(\mathfrak{g}, \mathfrak{h})$ .

## 2 Locally symmetric space

The previous results can be extended to locally symmetric spaces.

**Proposition-Definition 2.** *Let  $M$  is a Riemannian manifold, the followings are equivalent*

1. For every  $x \in M$ , there exists a neighborhood  $U$  of  $x$  and an isometry  $s_x : U \rightarrow U$  such that  $s_x^2 = \text{Id}$  and  $x$  is the unique fixed point of  $s_x$ .
2. The curvature tensor  $R$  satisfies

$$\nabla R = 0$$

If they are satisfied,  $M$  is called locally symmetric.

**Theorem 2** (Locally symmetric space). *Let  $M$  be a locally symmetric Riemannian manifold, then there exists a unique symmetric simply connected Riemannian manifold  $N$  such that  $M$  and  $N$  are locally isometric, i.e. for every  $x \in M$  and  $y \in N$ , there exists neighborhoods  $U$  of  $x$  and  $V$  of  $y$  that are isometric.*

As a result, the reduced holonomy of  $M$  is the same as the holonomy of  $N$ .

### 3 Annex: Group of isometries as Lie group

We explain in this annex some subtle details: how can a group of isometries be a manifold. We state, with Montgomery-Zippin, *Transformation groups* as reference, the following general result:

**Theorem 3** (faithful + locally compact  $\implies$  Lie). *Let  $G$  be a group acting faithfully on a connected manifold  $M$  of class  $C^k$  such that each action is  $C^1$  and  $G$  is locally compact. Then  $G$  is a Lie group and the map  $G \times M \longrightarrow M$  is  $C^1$ .*

Note that we equip a group of isometries with the **compact-open topology**, as  $M$  is locally compact and therefore second-countable (i.e. the topology admits a countable base), we see that a group of isometries is also second-countable. It suffices to prove the local compactness for the group of (all) isometries as this property is inherited by its closed subgroup. The detail can be found in Kobayashi-Nomizu's *Foundations of differential geometry* (Volume I, Theorem 4.7).

**Theorem 4.** *Let  $M$  be a connected, locally-compact metric space and  $G$  be the group of isometries of  $M$ , then*

1.  $G$  is locally compact.
2.  $G_a$  the subset of isometries fixing a point  $a \in M$  is compact.
3. If, in addition,  $M$  is compact then  $G$  is also compact.