Calabi-Yau theorem

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Contents

1 Calabi conjecture

In complex geometry, one usually defines the *Ricci curvature* to be the real (1,1)-form ρ with $\rho(u,v) = Ric(Ju,v) = \mathrm{tr}(w \mapsto R(w,v).Ju)$, as it has the advantage of being an antisymmetric form.

We will call ρ the Ricci form when it is easy to confuse with the Ricci curvature tensor in Riemannian geometry. We start with the following fact (which is exercise 4.A.3 in Huybrechts, *Complex geometry: an introduction*).

Remark 1. *For our convenience when talking about positivity, we would rather use the anticanonical bundle. Then K_X^{-1} is positive (resp. semi-positive) if and only if Ric is positive definite (resp. positive semi-definite) as a symmetric form.*

We start with the following fact (which is exercise 4.A.3 in Daniel Huybrechts, *Complex geometry: an introduction*)

Proposition 0.1 (Ricci curvature and first Chern class)**.** *Let (X,g) be a compact Kähler manifold. Then $i\rho(X,g)$ is the curvature of the Chern connection on the canonical bundle K_X. In other words, $\rho(X,g) \in -2\pi c_1(K_X)$ where $c_1(K_X)$ is the first Chern class of K_X.*

Definition 1. *The quadruple* (h, g, ω, J) *is said to be* <u>*compatible*</u> *if* $g \circ J = g$ *and* $\omega(a, b) = g(Ja, b)$ *and* $h = g - i\omega$.

Remark 2. 1. *When* J *is fixed, one of* h, g, ω *that is invariant by* J *determines the two others.*

2. *For a compatible quadruple, the condition* $\nabla J = 0$ *is equivalent to* $d\omega = 0$. *The fundamental form* ω *that satisfies* $d\omega = 0$ *is called a* <u>*Kähler form*</u>.

The Calabi conjecture asked whether for each form $R \in c_1(K_X)$ one can find a metric g' whose new fundamental form ω' is in the same class of ω and $Ric(X, g') = R$. We prefer to work with the fundamental form instead of the metric g as the former is antisymmetric and its derivative is hence easy to define.

2 Reduction to local charts, Yau theorem

h, g, ω in local coordinates. We note by $h_{i\bar{j}} = h(\partial_{x_i}, \partial_{x_j}) = 2g_{\mathbb{C}}(\partial_{z_i}, \partial_{z_j})$. By straightforward calculation one has

$$\begin{aligned}\omega &= -\frac{1}{2} Imh_{i\bar{j}}(dx^i \wedge dx^j + dy^i \wedge dy^j) + Reh_{i\bar{j}} dx^i \wedge dy^j \\ &= \frac{i}{2} h_{i\bar{j}} dz^i \wedge d\bar{z^j}\end{aligned}$$

and the condition $d\omega = 0$ is equivalent to

$$\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial h_{k\bar{j}}}{\partial z_i}$$

We also note by $h^{i\bar{j}}$ the inverse transposed of $h_{i\bar{j}}$, i.e. $h^{i\bar{j}} h_{k\bar{j}} = \delta_j^k$

Definition 2. *Let* X *be an almost complex manifold (manifold with an almost complex structure). Then* $d : \bigwedge^n T^*X \longrightarrow \bigwedge^{n+1} T^*X$ *sends* $\bigwedge^{p,q} T^*M$ *to* $\bigwedge^{p+1,q} T^*M \oplus \bigwedge^{p,q+1} T^*M$. *We denote by* ∂ *and* $\bar{\partial}$ *the component of* d *in* $\bigwedge^{p+1,q} T^*M$ *and* $\bigwedge^{p,q+1} T^*M$ *respectively.*

It would be convenient to define $d^c = i(\bar{\partial} - \partial)$ then obviously $dd^c = 2i\partial\bar{\partial}$.

The Ricci curvature. The Ricci curvature form is given in local coordinates by

$$Ric_\omega = -\frac{1}{2} dd^c \log \det(h_{i\bar{j}})$$

dd^c lemma . We then can state the dd^c lemma

Lemma 1. *Let α be a real, (1,1)-form on a compact Kähler manifold M. Then α is d -exact if and only if there exists $\eta \in C^\infty(M)$ globally defined such that $\alpha = dd^c\eta$.*

Yau's theorem. The dd^c lemma tells us that every form $R \in c_1(K_X)$ is of form $Ric_\omega + dd^c\eta$. If one varies the Hermitian product $h_{i\bar{j}}$ to $h_{i\bar{j}} + \phi_{i\bar{j}}$ then the new Ricci curvature is $dd^c \log\det(h_{i\bar{j}} + \phi_{i\bar{j}})$. The Calabi conjecture can be restated as the existence of ϕ such that $h_{i\bar{j}} + \phi_{i\bar{j}}$ is definite positive and

$$dd^c \left(\log\det(h_{i\bar{j}} + \phi_{i\bar{j}}) - \log\det(h_{i\bar{j}}) - \eta\right) = 0 \tag{1}$$

The functions f that satisfies $dd^c f = 0$ are called *pluriharmonic*. They also satisfy the maximum principle. By compactness of X, these functions on X are exactly constant functions. Therefore (1) is equivalent to

$$\det(h_{i\bar{j}} + \phi_{i\bar{j}}) = e^{c+\eta}\det(h_{i\bar{j}})$$

or by dd^c lemma:

$$(\omega + dd^c\phi)^n = e^{c+\eta}\omega^n$$

where ω^n denotes the repeated wedge product. Note that $(\omega + dd^c\phi)^n - \omega^n$ is exact, one has $\int_M (\omega + dd^c\phi)^n = V$, the conjecture of Calabi is therefore a consequence of the following theorem.

Theorem 2 (Yau)**.** *Given a function $f \in C^\infty(M), f > 0$ such that $\int_M f\omega^n = V$. There exists, uniquely up to constant, $\phi \in C^\infty(M)$ such that $\omega + dd^c\phi > 0$ and*

$$(\omega + dd^c\phi)^n = f\omega^n$$

3 A sketch of proof

The uniqueness is straightforward. In fact if ϕ and ψ both satisfy $\omega + dd^c\phi > 0$, $\omega + dd^c\psi > 0$ and $(\omega + dd^c\phi)^n = (\omega + dd^c\psi)^n$ then $D(\phi - \psi) = 0$ as

$$0 = \int_M (\phi - \psi)((\omega + dd^c\phi)^n - (\omega + dd^c\psi)^n) = \int_M d(\phi - \psi) \wedge d^c(\phi - \psi) \wedge T$$

where

$$T = \sum_{j=0}^{n-1} (\omega + dd^c\phi)^j \wedge (\omega + dd^c\psi)^{n-1-j}$$

is a closed (strongly) positive $(n-1, n-1)$ -form.

We will prove the existence of ϕ under the constraint $\int_M \phi\omega^n = 0$ (which will be useful to prove that (N) is locally diffeomorphism later). We will prove that the set S of $t \in [0,1]$ such that there exists $\phi_t \in C^{k+2,\alpha}(M)$ with $\int_M \phi_t\omega^n = 0$ that satisfies

$$(\omega + dd^c\phi_t)^n = (tf + 1 - t)\omega^n \tag{2}$$

is both open and close in $[0,1]$, therefore is the entire interval as $0 \in S$.

To see that S is open, one only has to prove that the function $\mathcal{N}$ defined by

$$\phi \mapsto \mathcal{N}(\phi) = \frac{\det(h_{i\bar{j}} + \phi_{i\bar{j}})}{\det(h_{i\bar{j}})}$$

or in other words $(\omega + dd^c\phi)^n = \mathcal{N}(\phi)\omega^n$, is a local diffeomorphism. The differential of $\mathcal{N}$ is given by

$$D\mathcal{N}(\phi).\eta = \mathcal{N}\Delta\eta$$

with η varies in $\{\eta \in C^{k,\alpha}(M) : \int_M \eta\omega^n = 0\}$. and Δ is the Laplace-Beltrami operator which is known to be bijective between

$$\left\{\eta \in C^{k+2,\alpha}(M) : \int_M \eta = 0\right\} \longrightarrow \left\{f \in C^{k,\alpha}(M) : \int_M f = 0\right\}$$

Therefore $\mathcal{N}$ is a local diffeomorphism and S is open.

The proof that S is closed is more technical and is accomplished in 3 steps:

1. Using Arzela-Ascoli theorem, it suffices to show that $\{\phi_t : t \in S\}$ is bounded in $C^{k+2,\alpha}$. Then up to a subsequence, one has the uniform convergence of ϕ_{t_n} and all its partial derivatives of order $\leq k+1$. The $k+2$ -th order follows from (2).

2. Using Schauder theory, prove that the above bound follows from *a priori estimate*:
   There exists $\alpha \in (0,1)$ and $C(X, \|f\|_{1,1}, 1/\inf_M f) > 0$ such that every $\phi \in C^4(X)$ satisfying $(\omega + dd^c\phi)^n = f\omega^n$, $\omega + dd^c\phi > 0$ and $\int_M \phi\omega^n = 0$ (we will call such ϕ *admissible*) has

$$\|\phi\|_{2,\alpha} \leq C.$$

3. Establish the a priori estimate.

To achieve the a priori estimate, one firstly bounds ϕ in C^0, then bound $\|\Delta\phi\|$ and finally establishs the $C^{2,\alpha}$ estimate. We will give here some detail of the first step. For more detail, see Z. Blocki, *The Calabi-Yau Theorem.*

Proof of the C^0-estimate. Since ϕ is defined up to an additive constant, what we mean by the C^0 -estimate for ϕ is in fact the estimate of

$$\mathrm{osc}_M\phi := \max_M \phi - \min_M \phi$$

by a constant C that depends only on M and f. Without losing of generality, one assumes that $\int_M \omega^n = 1$ and $\max_M \phi = -1$. Therefore $\|\phi\|_p \leq \|\phi\|_q$ for $p \leq q < \infty$.

One has

$$\int_M (-\phi)^p (f-1)\omega^n = \int_M (-\phi)^p dd^c\phi \wedge \left(\sum_{j=0}^{n-1} (\omega + dd^c\phi)^j \wedge \omega^{n-1-j} \right) \tag{3}$$

$$= p\int_M (-\phi)^{p-1} d\phi \wedge d^c\phi \wedge \left(\omega^{n-1} + \sum_{j=1}^{n-1} (\omega + dd^c\phi)^j \wedge \omega^{n-1-j} \right) \tag{4}$$

$$\geq p\int_M (-\phi)^{p-1} d\phi \wedge d^c\phi \wedge \omega^{n-1} \tag{5}$$

$$= \frac{4p}{(p+1)^2} \int_M d(-\phi)^{(p+1)/2} \wedge d^c(-\phi)^{(p+1)/2} \wedge \omega^{n-1} \tag{6}$$

$$= \frac{c_n p}{(p+1)^2} \|D(-\phi)^{(p+1)/2}\|_2^2 \tag{7}$$

where we used the fact that $\omega + dd^c\phi > 0$ in the inequality, and c_n is a constant depending only on n.

Now we use the following Sobolev inequality on M (i.e. use Sobolev inequality in each chart as a domain of $\mathbb{R}^m$ then add up the results):

$$\|v\|_{mq/(m-q)} \leq C(M,q)(\|v\|_q + \|Dv\|_q), \quad \forall v \in W^{1,q}(M), q < m$$

with $v = \phi$, $m = 2n$ the real dimension of M and $q = 2$, then use (7) to bound the $D\phi$ term:

$$\|(-\phi)^{(p+1)/2}\|_{2n/(n-1)} \leq C_M \left[\|(-\phi)^{(p+1)/2}\|_2 + \frac{p+1}{\sqrt{p}} \left(\int_M (-\phi)^p (f-1)\omega^n \right)^{1/2} \right]$$

Replace $p+1$ by p and use the fact that $|\phi| \geq 1$, one has

$$\|\phi\|_{np/(n-1)} \leq (Cp)^{1/p} \|\phi\|_p, \quad \forall p \geq 2$$

where C depends only on M and $\|f\|_\infty$.

Repeatedly apply this inequality (this technique is called *Moser's iteration*) one has $\|\phi\|_{p_{k+1}} \leq (Cp_k)^{1/p_k} \|\phi\|_{p_k}$ where the sequence p_k is defined by $p_0 = 2$ and $p_{k+1} = \frac{n}{n-1} p_{k-1} = 2(\frac{n}{n-1})^k$ and

$$\|\phi\|_\infty = \lim_{k\to\infty} \|\phi\|_{p_k} \leq \|\phi\|_2 \prod_{j=0}^{\infty} (Cp_j)^{1/p_j}$$

with $\prod_{j=0}^{\infty} (Cp_j)^{1/p_j} = (n/(n-1))^{n(n-1)/2} (2C)^{n/2}$

The fact that $\|\phi\|_2$ is bounded follows directly from the following lemma.

$\square$

Lemma 3 (L^p-boundedness)**.** *For any admissible* ϕ *with* $\max_M \phi = -1$ *one has*

$$\|\phi\|_p \leq C(M,p), \quad \forall 1 \leq p \leq \infty$$

Proof. We will prove the lemma with $p = 1$ first. Let g be the local potential of the Kähler form ω, i.e. a function defined on each chart (not necessarily agrees on zones where charts are glued together) such that $\omega = dd^c g = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ where $g_{i\bar{j}}$ can also be intepreted as $\frac{\partial^2}{\partial z_i \partial \bar{z}_j} g$. We also suppose that the function g is negative on every chart. The fact that $\omega + dd^c\phi > 0$ is rewritten as $(g_{i\bar{j}} + \phi_{i\bar{j}}) > 0$ in local coordinates.

Note $u = g + \phi$ the potential of $\omega + dd^c\phi$ locally defined on each chart, then u is negative and plurisubharmonic (psh). For every $x \in B(y,R)$ one has

$$u(x) \leq \frac{1}{\mathrm{vol}(B(x,2R)} \int_{B(x,2R)} u \leq \frac{1}{\mathrm{vol}(B(y,2R))} \int_{B(y,R)} u$$

where the first inequality is due to plurisubharmonicity and the second is due to $u \leq 0$. Therefore

$$\|u\|_{L^1(B(y,R))} \leq \mathrm{vol}(B(y,2R)) \inf_{B(y,R)} |u|,$$

hence

$$\|\phi\|_{L^1(B(y,R))} \leq \|u\|_{L^1(B(y,R))} \leq \mathrm{vol}(B(y,2R)) (\inf_{B(y,R)} |\phi| + \max_M |g|)$$

To see that $\|\phi\|_1$ is bounded, we apply the following Lemma 4 to the covering of M by finitely many ball $B(y_i, R_i)$, $c_i = \mathrm{vol}(B(y_i, 2R_i))$, $d_i = c_i \max_M |g|$ and $r = 1$.

The case $p > 1$ follows analoguously using the following estimate: if u is negative and psh in $B(y, 2R)$ then

$$\|u\|_{L^p(B(y,R)} \leq C(n, p, R)\|u\|_{L^1(B(y,2R))}$$

$\square$

Lemma 4 (Combinatoric)**.** *Let M be a connected compact manifold covered by finitely many local charts $\{V_i\}_{i=1}^l$ and $r, c_i, d_i > 0$. Then for any continuous function ϕ globally defined on M such that*

$$\|\phi\|_{L^1(V_i)} \leq c_i \inf_{V_i} |\phi| + d_i, \quad \min_M |\phi| \leq r,$$

one has $\|\phi\|_1 := \sum_i \|\phi\|_{L^1(V_i)} \leq C(\{V_i\}, \{c_i\}, \{d_i\}, r)$

Proof. Let p be a point in M where $|\phi|$ attains its minimum. Since M is connected, for every V_i, there exists a sequence $V_{i_k}, 0 \leq k \leq l$ such that

$$i_0 = i, \quad V_{i_k} \cap V_{i_{k+1}} \neq \emptyset, \quad p \in V_{i_l}$$

One has

$$\begin{aligned}
\|\phi\|_{L^1(V_{i_k})} &\leq c_{i_k} \inf_{V_{i_k}} |\phi| + d_{i_k} \leq c_{i_k} \inf_{V_{i_k} \cap V_{i_{k+1}}} |\phi| + d_{i_k} \\
&\leq c_{i_k} \frac{1}{\mathrm{vol}(V_{i_k} \cap V_{i_{k+1}})} \|\phi\|_{L^1(V_{i_{k+1}})} + d_{i_k}
\end{aligned}$$

Repeatedly apply this inequality for $k = 0, \ldots, l-1$, one has

$$\begin{aligned}
\|\phi\|_{L^1(V_i)} &\leq A(i, \{V_j\}, \{c_j\}, \{d_j\})\|\phi\|_{L^1(V_{i_l})} + B(i, \{V_j\}, \{c_j\}, \{d_j\}) \\
&\leq A(i, \{V_j\}, \{c_j\}, \{d_j\})(c_{i_l} r + d_{i_l}) + B(i, \{V_j\}, \{c_j\}, \{d_j\})
\end{aligned}$$

Take the sum for all $i = 0, \ldots, l$ and the result follows. $\square$

4 Calabi-Yau manifold

Recall that we defined a Calabi-Yau manifold to be a compact Riemannian manifold of dimension $2n$ with holonomy contained in $SU(n)$. We also remark, using parallel transport, the existence of a compatible complex structure ($U(n)$ suffices) and a holomorphic form non-vanishing at every point. We present here some equivalent definitions of compact Calabi-Yau manifolds.

Theorem 5. *Let X be a compact manifold of Kähler type and complex dimension n then:*

1. *The followings are equivalent*

   (a) *There exists a Kähler metric such that the global holonomy is in $SU(n)$.*

   (b) *There exists a holomorphic $(n, 0)$ form that vanishes nowhere.*

   (c) *The canonical bundle K_X is trivial.*

   (d) *The structure group of X can be reduced to $SU(n)$.*

2. *The following are equivalent. If X is simply-connected, they are equivalent with the 4 statements above.*

   (a) *There exists a Kähler metric such that the local holonomy is in $SU(n)$.*

   (b) *The canonical bundle K_X is flat.*

   (c) *There exists a Kähler metric such that the Ricci curvature vanishes.*

   (d) *The first Chern class vanishes.*

The proof is straightforward (see Manuscript) with the only non-trivial part is when one needs Calabi-Yau theorem to construct Ricci-flat metric.