Moser's Isotopy method and Darboux theorem

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1 Symplectic geometry does not exist

We will prove a symplectic manifold, i.e. a smooth manifold equiped with a closed everywhere non-degenerate 2-form, does not have local invariant. This is a significant difference between symplectic manifold and riemannian manifold, whose local invariance is the curvature.

To do this, one uses a trick of Moser which in the compact case show that isotopic symplectic structures ω_0 and ω_1 are strongly isotopic, i.e. $\psi_t^*\omega_t = \omega_0$

2 Isotopy method

2.1 Moser's trick

Let M be a closed manifold (compact, without boundary) and ω_t is a family of symplectic structures on M such that

$$\frac{d}{dt}\omega_t = d\sigma_t$$

then there exists diffeomorphism ψ_t of M such that $\psi_t^*\omega_t = \omega_0$

Construction of ψ_t. One constructs ψ_t by its flow $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ such that

$$0 = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) = \psi_t^*\left(d\sigma_t + X_t \neg d\omega_t + d(X_t \neg \omega_t)\right)$$

Since ω_t are closed and non-degenerate, it suffits to choose X_t, which exists uniquely, such that $X_t \neg \omega = \sigma_t$.

2.2 Application: Darboux theorem and Moser Stability

Using this trick, we can prove the following theorem of Darboux.

Lemma 1. *Let M be a closed manifold with symplectic structures ω_0 and ω_1 such that they coincide on a fiber T_qM. Then there exists neighborhoods $\mathcal{N}_0, \mathcal{N}_1$ of q and a diffeomorphism $\psi : \mathcal{N}_0 \longrightarrow \mathcal{N}_1$ such that $\psi^*\omega_1 = \omega_0$.*

Proof. We remark that it is enough to prove that there exists σ locally defined near q with $\omega_1 - \omega_0 = d\sigma$ where $\sigma = 0$ on T_qM. In fact, let $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ one then has a neighborhood $\mathcal{N}_0$ of q such that ω_t are non-degenerate and the field X_t constructed by Moser's technique ($X_t = 0$ at q) has its flow well-defined at time $t = 1$ when starting at $\mathcal{N}_0$. Then ψ_1 and $\mathcal{N}_1$ is what we want.

One then uses another trick to construct σ: Take any Riemannian metric on M and let ϕ_t be constructed using the geodesic flow and retricting $\mathcal{N}_0$ to a geodesic ball such that $\phi_0|_{\mathcal{N}_0} = q$, $\phi_1 = Id$ and $d\phi_t(q) = Id_{T_qM}$. Then

$$\omega_1 - \omega_0 = \int_0^1 \frac{d}{dt}\phi_t^*(\omega_1 - \omega_0)dt = \int_0^1 \phi_t^* d(Y_t \neg (\omega_1 - \omega_0)) = d\left(\int_0^1 dt \phi_t^*(Y_t \neg (\omega_1 - \omega_0)\right).$$

It is straight-forward to see that the σ constructed this way works. □

The theorem of Darboux follows easily from Lemma 1.

Theorem 2 (Darboux). *Every two symplectic form ω_0, ω_1 on a closed manifold M are locally isomorphic.*