

# Divisors, Picard group and Kodaira embedding theorem

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## 1 Divisors and Picard group

### 1.1 Holomorphic line bundles and first Chern class

A complex line bundle is a 2 dimensional vector bundle with a complex structure on each fiber, i.e. each change of coordinates  $g_{ij} : U_j \cap U_i \times \mathbb{R}^2 \rightarrow U_i \cap U_j \times \mathbb{R}^2$  is  $i$ -linear, i.e.  $g_{ij}$  can be represented by a function  $U_i \cap U_j \rightarrow \mathbb{C}$ .

A holomorphic line bundle is a complex line bundle that is also a complex manifold with the projection being holomorphic. In the same notation, the  $g_{ij}$  are now holomorphic functions.

A hermitian metric on a line bundle  $L$  is a positive sesquilinear form on each fiber. To define the Chern form of  $L$ , let  $U$  be an open set of  $X$  over which  $L$  is trivialized and  $s_x$  is a holomorphic section of  $L$  over  $U$  that is non-vanishing, then one defines

$$\omega_{L,h} = \frac{1}{2\pi i} \partial \bar{\partial} \log |s|_h^2$$

which is independent of  $s$  since the ratio of two different  $s$  is in  $\mathcal{O}^*(U)$ .

**Remark 1.** *A Chern form is a real (1,1) form.*

**Proposition 0.1.** *The set of isomorphic class of holomorphic line bundle is in one-to-one correspondance to  $H^1(X, \mathcal{O}_X^*)$*

The proof of this fact is straightforward, but it is worth to remark that this result convinces us that the natural mapping  $\check{H}^1(\mathcal{U}, X) \rightarrow \check{H}^1(\mathcal{V}, X)$  where  $\mathcal{U}$  is a finer open covering than  $\mathcal{V}$  is injective, since a line bundle is completely defined by *one* set of change of coordinates  $(g_{ij})$ .

Now the Chern class of a holomorphic line bundle is the class of  $\omega_{L,h}$  in  $H^2(X, \mathbb{Z})$ , which turns out to be independent of  $h$  and is in fact lies in the image of  $H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{R}) = H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ . In fact, the class of  $\omega_{L,h}$  can be defined using the following exact sequence:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0$$

where the injective arrow is the multiplication by  $i2\pi$  and the surjective one is exponential. The Chern map is in fact  $H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z})$ . To prove this, one uses a double complex whose horizontal is the de Rham resolution and vertical is the Čech resolution and diagram chasing.

## 1.2 Divisors, line bundles and sheaves

**Remark 2.** *1. A holomorphic line bundle is the same as a locally free  $\mathcal{O}_X$ -module of rank 1.*

*2. An isomorphic class of line bundles is the same as a locally free isomorphic sheaf of  $\mathcal{O}_X$ -module of rank 1.*

### 1.2.1 From divisors to Picard group

A divisor is a formal sum of irreducible hypersurface, which can also be interpreted as an element of  $\check{H}^0(X, K_X^*/\mathcal{O}_X^*)$ , which gives a mapping  $Div(X) \rightarrow \check{H}^0(X, K_X^*/\mathcal{O}_X^*)$  with principal divisors being exactly sent to elements of  $K_X^*/\mathcal{O}_X^*$  coming from  $K_X^*$ . multiplicative).

Since the following sequence

$$0 \rightarrow \mathcal{O}_X^* \rightarrow K_X^* \rightarrow K_X^*/\mathcal{O}_X^* \rightarrow 0$$

is exact, one has an application  $\mathcal{O} : Div(X) = H^0(X, K_X^*/\mathcal{O}_X^*) \rightarrow H^1(X, \mathcal{O}_X^*) = Pic(X)$ . The kernel of  $\mathcal{O}$  corresponds to the the space of principal divisors. It is however worth having details of the application  $\mathcal{O}$ .

Let  $D = (U_i, f_i) \in Div(X)$  where  $f_i$  are meromorphic function on  $U_i$  with  $f_i/f_j \in \mathcal{O}_X^*$ , then  $\mathcal{O}(D)$  is defined as following:

$$\mathcal{O}(D)(U_i) = f_i^{-1} \mathcal{O}_X(U_i)$$

. Note that if  $D$  is effective, i.e.  $f_i \in \mathcal{O}_X(U_i)$  then  $\mathcal{O}(D)$  is the sheaf of holomorphic functions vanishing on  $D$ .

**Remark 3.** *To resume, here are some basic consequence of the above discussion: If  $D$  is effective then*

1. *If  $D$  is effective then  $H^0(X, \mathcal{O}(D)) \neq 0$ .*
2. *If  $D$  is effective then  $\mathcal{O}(-D)$  is the sheaf of holomorphic function vanishing on  $D$ . Therefore  $\mathcal{O}(-D)$  can be viewed as a ideal subsheaf of  $K_X$  and one has the following exact sequence:*

$$0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

*where  $\mathcal{O}_D$  is the sheaf of "regular functions" on  $D$ .*

3. *If  $L$  is a holomorphic line bundle and  $0 \neq s \in H^0(X, L)$  then  $\mathcal{O}(Z(s)) \equiv L$*
4. *If  $D$  is effective then  $\mathcal{O}(D)$  has a non-zero global section, for example section  $1 = (U_i, f_i)$*

### 1.2.2 The corresponding line bundle of $\mathcal{O}(Y)$

**Proposition 0.2.** *Let  $Y$  be a hypersurface of  $X$ , then the line bundle  $\mathcal{O}(Y)$  is isomorphic to  $\mathcal{N}_{Y,X}$  the normal line bundle of  $Y$  in  $X$ .*

*By consequence,  $K_Y = (K_X \otimes \mathcal{O}(Y))|_Y$ .*

## 2 Example: Projective space

### 2.1 $\mathcal{O}(d)$ and its sections

Let's have some examples for the point of view discussed above, starting with the torsion sheaves  $\mathcal{O}(d)$ .

The sheaf  $\mathcal{O}(-1)$ , called tautological sheaf, is an invertible sheaf on  $\mathbb{P}_{\mathbb{C}}^n$  such that the fiber over  $l \in \mathbb{P}_{\mathbb{C}}^n$  of the corresponding line bundle is  $l$  itself. Let  $l = [x_0 : \dots : x_n]$  in  $U_i$  then a point in  $l$  is of form  $t_i[\frac{x_0}{x_i} : \dots : \frac{x_n}{x_i}]$  with coordinates in  $U_i$  being  $t_i$ . So the change of coordinates from chart  $U_i$  to  $U_j$  is, since  $\frac{t_j}{x_j} = \frac{t_i}{x_i}$ :

$$g_{ji} = \frac{x_i}{x_j}$$

One notes by  $\mathcal{O}(1)$  the dual of  $\mathcal{O}(-1)$  and  $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$  and  $\mathcal{O}(-d) = \mathcal{O}(-1)^{\otimes d}$

Now if an invertible sheaf  $\mathcal{L}$  with  $\mathcal{L}(U_i) = \frac{1}{f_i}\mathcal{O}_X(U_i)$ , the change of coordinates of the corresponding line bundle from chart  $U_i$  to  $U_j$  is  $g_{ji} = \frac{f_j}{f_i}$ . So for  $\mathcal{O}(1)$ , one has  $\frac{f_j}{f_i} = \frac{x_j}{x_i}$ , i.e. there exists a linear combination  $A$  of  $x_0, \dots, x_n$  such that  $f_i = \frac{A}{x_i}$  is a holomorphic function corresponding to the 1-section viewed in chart  $U_i$ . As presented in the previous section,  **$\mathcal{O}(1)$  is the associated line bundle of a hyperplane defined by the equation  $A = 0$ .**

Similarly,  $\mathcal{O}(d)$  is the associated line bundle of a hypersurface  $A_d$  defined by a homogenous equation of degree  $d$ , and  $\mathcal{O}(d)$  is the line bundle associated to the sheaf of holomorphic functions vanishing on  $A_d$ .

### 2.2 Line bundles and maps to projective space, Veronese embedding

A linearly independent family  $s_0, \dots, s_N$  of global sections of a holomorphic line bundle  $L$  defines a holomorphic map  $X \setminus Bs((s_i)) \rightarrow \mathbb{P}_{\mathbb{C}}^N$  where  $Bs((s_i))$  is the set of basepoints of  $(s_i)$  where all the sections  $s_i$  vanish.

The global sections of  $\mathcal{O}(d)$  over  $\mathbb{P}_{\mathbb{C}}^n$  are  $H^0(\mathbb{P}_{\mathbb{C}}^n, \mathcal{O}(d)) = \mathbb{C}[z_0, \dots, z_n]_d$  the vector space of homogenous polynomial of degree  $d$ . The corresponding projective map is in fact a embedding, called Veronese embedding.

### 2.3 Canonical bundle and Euler sequence

$$K_{\mathbb{P}_{\mathbb{C}}^n} = \mathcal{O}(-n-1)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus n} \longrightarrow \mathcal{T}_{\mathbb{P}_{\mathbb{C}}^n} \longrightarrow 0$$

### 3 Blowing-up

#### 3.1 Blowing-up

**Proposition 0.3** (Adjunction). *Let  $X$  be a complex manifold,  $x \in X$  and  $\pi : \hat{X} \longrightarrow X$  is the blow-up of  $X$  at  $x$  and  $E$  be the corresponding exceptional divisor, then*

$$K_{\hat{X}} = \pi^* K_X \otimes \mathcal{O}((n-1)E).$$

*As a consequence,  $\mathcal{O}(E)|_E = \mathcal{O}(-1)$  where  $\mathcal{O}(-1)$  is the tautologic sheaf over  $E = \mathbb{P}_{\mathbb{C}}^{n-1}$ .*

*Proof.* The appearance of the number  $n-1$  is natural and can be explained as follow. First note that

1. in  $\hat{X} \setminus E$ , there is no difference between  $K_{\hat{X}}$  and  $K_X$ .
2.  $\pi_*$  send every tangent vector of  $E$  to the tangent vector 0 at  $x \in X$ ,
3. the pull-back  $\pi^*\omega$  of an  $n$ -form on  $X$  always vanishes on  $E$  therefore cannot generate  $K_{\hat{X}}$ .

Our correction of this should be dividing  $\pi^*\omega$  by  $f^k$  where  $f$  is the equation defining  $E$  in  $\hat{X}$ , i.e. tensoring  $\pi^*K_X$  by an appropriate multiple of  $\mathcal{O}(E)$  depending on the order of vanishing of  $\pi^*\omega$  at  $E$ , which we claim to be  $n-1$ .

Here is the argument I used to convince myself: this vanishing order is that of the ratio of  $\pi^*\omega$  and a non-zero  $n$ -form (says the standard in the base formed by  $n-1$  tangent vectors  $e_i^E$  of  $E$  and the normal vector  $v$  of  $E$  in  $X$ ), i.e. the vanishing order of  $\pi^*\omega(e_i^E, v)$ . Each  $e_i^E$  plugged into  $\pi^*\omega$  adds one order of vanishing resulting in  $n-1$ .

Here is the argument I would use to convince others: WLOG, suppose that  $X = \mathbb{C}^n$  and  $x = 0$ , then  $\hat{X}$  can be seen as a subset of  $\mathbb{C}^n \times \mathbb{P}^{n-1}$  with the coordinates in each chart  $U_i = \{(x_1, \dots, x_n, [p_1 : \dots : p_n]) : p_i \neq 0\}$  being  $(\frac{p_1}{p_i}, \dots, \frac{p_n}{p_i}, \zeta_i)$  with  $z_k = \frac{p_k}{p_i} \zeta_i$ . The map  $\pi$  is given in local coordinates as

$$(\frac{p_1}{p_i}, \dots, \frac{p_n}{p_i}, \zeta_i) \mapsto (\frac{p_1}{p_i} \zeta_i, \dots, \zeta_i, \dots, \frac{p_n}{p_i} \zeta_i)$$

The pull-back of  $\omega$  is

$$d\left(\frac{p_1}{p_i}\zeta_i\right) \wedge \cdots \wedge d\zeta_i \wedge \cdots \wedge d\left(\frac{p_n}{p_i}\zeta_i\right) = \zeta_i^{n-1} d\left(\frac{p_1}{p_i}\right) \wedge \cdots \wedge d\zeta_i \wedge \cdots \wedge d\left(\frac{p_n}{p_i}\right)$$

which vanishes with order  $n - 1$ .

For the consequence, note that  $\mathcal{K}_E = \mathcal{O}(-n) = (K_{\hat{X}} \otimes \mathcal{O}(E))|_E = (\pi^*K_X \otimes \mathcal{O}(nE))|_E$ , but  $\pi^*K_X$  is trivial over  $E$ , therefore  $\mathcal{O}(E)|_E = \mathcal{O}(-1)$ .  $\square$

## 4 Kodaira vanishing theorem

**Theorem 1** (Kodaira vanishing). *Let  $X$  be a compact complex manifold of dimension  $n$  and  $L$  is a positive holomorphic line bundle on  $X$ , i.e. there exists a hermitian metric  $h$  on  $L$  such that the Chern form  $\omega_{L,h}$  is positive (i.e. a Kahler form). Then*

$$H^q(X, \Omega_X^p \otimes \mathcal{L}) = 0 \quad \forall p + q > n.$$

In particular,

$$H^i(X, K_X \otimes \mathcal{L}) = 0 \quad \forall i > 0.$$

*Proof.* Since  $\mathcal{H}^{0,q}(E) \simeq H^q(X, \mathcal{E})$  for all hermitian holomorphic vector bundle  $E$  with  $\mathcal{E}$  the corresponding sheaf of holomorphic sections. One needs to prove that all harmonic form in  $\mathcal{A}_X^{p,q}(L)$  vanishes for  $p + q > n$ . This comes from the following two identities: Let  $\nabla$  be the Chern connection on  $L$  and  $\nabla = \nabla' + \nabla''$  be its decomposition to  $(1, 0)$  and  $(0, 1)$  operators and  $\Delta'_L$  be the Laplacian corresponding to  $\nabla'$  then

1.  $\Delta_L = \Delta'_L + 2\pi[L, \Lambda]$  where  $L$  and  $\Lambda$  are Lefschetz operators.
2.  $[L, \Lambda] = (k - n)Id_{\mathcal{A}_X^k}$

Therefore  $0 \leq (\alpha, \Delta'_L \alpha)_{L^2} = 2\pi(n - k)(\alpha, \alpha)_{L^2} \leq 0$   $\square$

## 5 The traditional proof of Kodaira embedding theorem

**Theorem 2** (Kodaira embedding). *Let  $X$  be a compact complex manifold with  $L$  a positive holomorphic line bundle on  $X$ . Then  $L$  is generated by finitely many of its global sections and  $X$  can be embedded in a projective space  $\mathbb{CP}^N$  with  $N$  sufficiently large.*

*Proof.* The following approach is straight-forward: one shows that at every  $x \in X$ , there is a global (holomorphic) section  $s_x$  of  $L^{\otimes m_x}$  such that  $s_x(x) \neq 0$  then by compactness one can choose finitely many such sections and  $m_x$  which can be guaranteed to generate every germs at  $x$  of  $L^{\otimes m}$  with  $m = \max m_x$ . That is one needs to prove that

$$H^0(X, \mathcal{L}^{\otimes m_x}) \twoheadrightarrow H^0(x, \mathcal{L}^{\otimes m_x}|_x)$$

is surjective. Let  $\pi : \hat{X} \rightarrow X$  be the blow-up of  $X$  at  $x$  and  $E = \pi^{-1}(x)$  be the corresponding exceptional divisor then one has

```
\begin{tikzcd}
H^0(X, \mathcal{L}^{\otimes m_x}) \ar[r, twoheadrightarrow] \ar[d] & H^0(x, \mathcal{L}^{\otimes m_x}|_x) \\
H^0(\hat{X}, \pi^* \mathcal{L}^{\otimes m_x}) \ar[r, twoheadrightarrow] & H^0(E, \pi^* \mathcal{L}^{\otimes m_x}|_E)
\end{tikzcd}
```

\label{fig:kodaira-blowup}

$$\begin{array}{ccc} H^0(X, \mathcal{L}^{\otimes m_x}) & \twoheadrightarrow & H^0(x, \mathcal{L}^{\otimes m_x}|_x) \\ \downarrow & & \downarrow \\ H^0(\hat{X}, \pi^* \mathcal{L}^{\otimes m_x}) & \twoheadrightarrow & H^0(E, \pi^* \mathcal{L}^{\otimes m_x}|_E) \end{array}$$

Figure 1: Insert caption [fig:kodaira-blowup]

where the vertical arrows are isomorphic (easy to see, maybe the right one needs Hartog). So one only needs to see that  $H^0(\hat{X}, \pi^* \mathcal{L}^{\otimes m_x}) \rightarrow H^0(E, \pi^* \mathcal{L}^{\otimes m_x}|_E)$ . Since  $E$  is a divisor of  $\hat{X}$ , one has  $0 \rightarrow \mathcal{O}(E) \rightarrow \mathcal{O}_{\hat{X}} \rightarrow \mathcal{O}_E \rightarrow 0$  hence  $0 \rightarrow \mathcal{O}(E) \otimes \pi^* \mathcal{L}^{\otimes m_x} \rightarrow \pi^* \mathcal{L}^{\otimes m_x} \rightarrow \pi^* \mathcal{L}^{\otimes m_x}|_E \rightarrow 0$ .

It remains to prove that  $H^1(\hat{X}, \mathcal{O}(E) \otimes \pi^* \mathcal{L}^{\otimes m_x})$ , or by 1, that  $\mathcal{O}(E) \otimes \pi^* \mathcal{L}^{\otimes m_x} \otimes K_{\hat{X}}^{-1} = \mathcal{O}(-nE) \otimes \pi^*(\mathcal{L}^{\otimes m_x} \otimes K_X^{-1})$  is positive, where we used the fact that  $K_{\hat{X}} = \pi^* K_X \otimes \mathcal{O}_{\hat{X}}((n-1)E)$ .

Note that on  $\hat{X} \setminus E$ , one can choose  $m_x$  large enough such that  $\mathcal{L}^{\otimes m_x} \otimes K_X^{-1} \times \mathcal{O}(-nE)$  is positive. It remains to observe  $E \subset \hat{X}$  which is in fact  $\mathbb{CP}^{n-1}$ . But  $\mathcal{O}(-E)|_E \equiv \mathcal{O}(1)$  is positive, which concludes the proof.  $\square$

## 6 An analytic proof by Donaldson