Two theorems of Hartogs

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1 Hartogs' theorem for subharmonic function

The first Hartogs theorem concerns the convergence of harmonic functions. It says that under certain conditions, the convergence, apriori pointwise, is actually uniform on every compacts.

Theorem 1 (Hartogs on subharmonic functions)**.** *Let Ω be a domain and $u_k \in SH(\Omega)$ a sequence of subharmonic function such that*

1. *Uniformly bounded on compacts: $u_k\big|_K < M_K$.*
2. *Pointwise limit is continuous: $\limsup_k u_k(x) = C$*

Then for every $K \subset \Omega$ compact and $\varepsilon > 0$, there exists $N(K,\varepsilon) > 0$ such that $u_k < C + \varepsilon$ for all $k > N(K,\varepsilon)$

Proof. By covering Ω with an increasing sequence of compact K_n that $K_n \Subset Int(K_{n+1})$, one can suppose that $u_k < M$ on Ω. One can also suppose that $M = 0$. Note that it suffices to prove that for any $x \in \Omega$, there exists $N(x,\varepsilon)$ such that $u_k < C + \varepsilon$ on a neighborhood U_x of x for all $k > N(x,\varepsilon)$, then the conclusion follows by compactness of K.

One has $\lim_{k\to+\infty}\int_{B(x,R)} u_k = C|B(R)|$ by monotone convergence, so $\int_{B(x,R)} u_k < (C+\varepsilon)|B(R)|$ for $k > N_1(x,\varepsilon)$. For any $r \ll R$ and $y \in B(x,r)$ one has $\int_{B(y,R+r)} u_k \leq \int_{B(x,R)} u_k < (C+\varepsilon)|B(R)|$. Therefore

$$u_k(y) \leq (C+\varepsilon)\frac{|B(R)|}{|B(R+r)|} \leq C + 2\varepsilon \qquad \forall r \ll R, k > N_1(x,\varepsilon)$$

which shows that $u_k < C + 2\varepsilon$ for in a small neighborhood $B(x,r)$ of x. $\square$

Remark 1. *The Theorem 1 above can be easily generalized by replacing the constant C by a continuous function f.*

2 Hartogs theorem of separately holomorphicity

The second result of Hartogs that I want to present here is about the founding notion of holomorphicity for several complex variables.

Theorem 2 (Hartogs for separate holomorphicity)**.** *A function f separately holomorphic on each variable then f is smooth and hence is completely holomorphic*

The strategy is to establish the following steps:

1. f is locally bounded.
2. f is continuous.
3. f is smooth, hence is completely holomorphic.

The second and third steps are not difficult. In fact when one knows that f is locally bounded, one can prove that f is continuous by Schwartz lemma on each variable with appropriate scaling.

$$f(z_1,\dots,z_i,\dots\xi_n) - f(z_1,\dots,\xi_i,\dots,\xi_n) \leq |1 - \overline{f(\dots,z_i,\dots)} f(\dots,\xi_i,\dots)| \left|\frac{z_i - \xi_i}{1 - z_i\bar{\xi_i}}\right| \quad \forall |z_i|, |\xi| < 1, |f| <$$

When f is continuous, one may refind Cauchy integral formulae and differentiability follows by dominated convergence.

So the remaining point is to prove that f is locally bounded, which can be done using Baire theorem and the first Hartogs result, Theorem 1.

2.1 Application of Baire theorem.

We will prove Theorem 2 by induction on the dimension n. We can therefore suppose that with the last variable z_n fixed, the function is completely holomorphic on the $n-1$ first variables. Fix a closed n-polydisc $\mathbb{D}^n \ni x = 0$, denote

$$W_L = \{(z_1, \dots, z_{n-1}) \in \mathbb{D}^{n-1} : \quad |f(z_1, \dots, z_n)| \leq L \quad \forall z_n \in \mathbb{D}^1\}$$

then

1. $\bigcup_{L \in \mathbb{N}} W_L = \mathbb{D}^n$ since for fixed $(z_1, \dots, z_{n-1}) \in \mathbb{D}^{n-1}$, the function f is continuous on z_{n-1}.
2. Each W_L is closed since for fixed z_n, the function f is continuous on $n-1$ first variables.

Therefore by Baire theorem, there exists L large enough such that W_L contains an open set of $\mathbb{D}^{n-1}$. Therefore there exists a strip $U_{n-1} \times \mathbb{D} \subset \mathbb{D}^n$ on which the function f is holomorphic.

We will extend this strip using the following lemma

Lemma 3. *lem:ext-strip Let f be a separately holomorphic function defined on a neighborhood of $\mathbb{D}^n$ such that*

1. *f is continuous on a neighborhood of the strip $\mathbb{D}_\rho \times \cdots \times \mathbb{D}_\rho \times \mathbb{D}$,*
2. *f is completely holomorphic on the first $n-1$ variables when the last one is fixed,*

then f is completely holomorphic on $\mathbb{D}^n$

This lemma can be proved using the series decomposition of f.

2.2 Analysis of series decomposition.

Since f is completely holomorphic on the first $n-1$ variables when z_n is fixed, one has

$$f(z_1, \dots, z_n) = \sum_\alpha a_\alpha(z_n) z^\alpha, \qquad a_\alpha(z_n) = \partial^\alpha f(0, z_n)/\alpha! \text{ is holomorphic in } z_n$$

Then the fact that for fixed z_n, the holomorphic function $f(z', z_n)$ is well-defined on $z' \in \mathbb{D}^{n-1}$ shows that

$$\limsup_{|\alpha| \to \infty} |a_\alpha(z_n)|^{1/|\alpha|} \leq 1. \tag{1}$$

Moreover, Cauchy integral on $\mathbb{D}_\rho \times \cdots \times \mathbb{D}_\rho \times \mathbb{D}$ shows that

$$|a_\alpha(z_n)| = |\partial^\alpha f(0, z_n)|/\alpha! \leq \frac{1}{\rho^{|\alpha|}} M \tag{2}$$

where M is an upper bound of $|f|$ on the strip.

Let $u_\alpha = \frac{1}{|\alpha|} \log |a_\alpha|$ be a subharmonic function of $z_n \in \mathbb{D}$, (1) and (2) show that $\limsup_{|\alpha|\to\infty} u_\alpha \leq 0$ and $u_\alpha \leq \frac{1}{|\alpha|} \log M - \log \rho$ hence uniformly bounded.

By Theorem 1, one has $|a_\alpha(z_n)|^{1/|\alpha|} \leq 1+\varepsilon$ for $|\alpha|$ sufficient large. Letting $\varepsilon \to 0$, one sees that the series converge normally in the interieur of $\mathbb{D}^{n-1}$, hence by Cauchy-Montel the limit f is holomorphic in the interieur of $\mathbb{D}^n$.