Some results in one complex variable

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Contents

Theorem 1 (Runge). Let Ω be a domain of \mathbb{C} and K a compact in Ω . Then the followings are equivalent

- 1. K has no hole in Ω , i.e. there is no connected component C of $\Omega \setminus K$ with $\bar{C} \subset \Omega$.
- 2. Every holomorphic function in K can be $\|.\|_{K,\infty}$ -approached by holomorphic functions in Ω .
- 3. For every $x \in \Omega \setminus K$, there exists a holomorphic function $f_x \in \mathcal{O}_{\Omega}$ such that $|f(x)| > \sup_K |f|$.

Theorem 2 (Mittag-Leffler). Let (a_i) be a discrete sequence of points in Ω and f_i be meromorphic functions with pole only at a_i . Then there exists a meromorphic function f with poles only at (a_i) such that $f - f_i$ has removable singularity at a_i .

Proof. Suppose that (f_i) are globally defined in Ω . Choose an exhaustive sequence $(\hat{K}_j = K_j)_j$ that increases slower than (a_i) , i.e. $a_i \notin K_i$. By 1 for $K_i \subset \Omega$, there exist $g_i \in \mathcal{O}(\Omega)$ with $||g_i - f_i||_{K_i,\infty} < 2^{-i}$. Pose

$$f = \sum_{i} (f_i - g_i).$$