# Calabi-Yau theorem

darknmt

June 8, 2017

## Contents



## 1 Calabi conjecture

We start with the following fact (which is an exercise in Daniel Huybrechts, *Complex geometry - an introduction*)

**Proposition 0.1** (Ricci and first Chern class). *Let $(X, g)$ be a compact Kahler manifold, then $iRic(X, g)$ is the curvature of the Chern connection on the canonical bundle $K_X$. In other words, $Ric(X, g)$ is in $Ric(X, g) \in -2\pi c_1(K_X)$ where $c_1(K_X)$ is the first Chern class of $K_X$.*

The Calabi conjecture asked whether there exists for each form $R \in c_1(K_X)$ a metric $g'$ with $Ric(X, g') = R$. We prefer to work with the fundamental form instead of the metric $g$ as the former is antisymmetric.

**Definition 1.** *The quadruple $(h, g, \omega, J)$ is said to be compatible if $g \circ J = g$ and $\omega(a, b) = g(Ja, b)$ and $h = g - i\omega$.*

**Remark 1.** 1. *When $J$ is fixed, one of $h, g, \omega$ that is invariant by $J$ determines the two others.*

2. *For a compatible quadruple, the condition $\nabla J = 0$ is equivalent to $d\omega = 0$. The fundamental form $\omega$ that satisfies $d\omega = 0$ is called a Kahler form.*

## 2 Reduction to local charts, Yau theorem

**$h, g, \omega$ in local coordinates.** We note by $h_{i\bar{j}} = h(\partial_{x_i}, \partial_{x_j}) = 2g_{\mathbb{C}}(\partial_{z_i}, \partial_{z_j})$. By straightforward calculation one has

$$\begin{aligned}\omega &= -\frac{1}{2} Imh_{i\bar{j}}(dx^i \wedge dx^j + dy^i \wedge dy^j) + Reh_{i\bar{j}} dx^i \wedge dy^j \\ &= \frac{i}{2} h_{i\bar{j}} dz^i \wedge d\bar{z^j}\end{aligned}$$

and the condition $d\omega = 0$ is equivalent to

$$\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial h_{k\bar{j}}}{\partial z_i}$$

We also note by $h^{i\bar{j}}$ the inverse transposed of $h_{i\bar{j}}$, i.e. $h^{i\bar{j}} h_{k\bar{j}} = \delta^k_j$

**Definition 2.** *Let $X$ be an almost complex manifold (manifold with an almost complex structure). Then $d : \wedge^n T^*X \longrightarrow \wedge^{n+1} T^*X$ sends $\wedge^{p,q} T^*M$ to $\wedge^{p+1,q} T^*M \oplus \wedge^{p,q+1} T^*M$. We denote by $\partial$ and $\bar{\partial}$ the component of $d$ in $\wedge^{p+1,q} T^*M$ and $\wedge^{p,q+1} T^*M$ respectively.*

It would be convenient to define $d^c = i(\bar{\partial} - \partial)$ then obviously $dd^c = 2i\partial\bar{\partial}$.

**The Ricci curvature.** The Ricci curvature is given by

$$Ric_\omega = -\frac{1}{2} dd^c \log\det(h_{i\bar{j}})$$

**$dd^c$ lemma .** We then can state the $dd^c$ lemma

**Lemma 1.** *Let $\alpha$ be a real, (1,1)-form on a compact Kahler manifold $M$. Then $\alpha$ is $d$ -exact if and only if there exists $\eta \in C^\infty(M)$ globally defined such that $\alpha = dd^c\eta$.*

**Yau theorem.** The $dd^c$ lemma tells us every form $R \in c_1(K_X)$ is of form $Ric_\omega + dd^c\eta$. If one varies the Hermitian product $h_{i\bar{j}}$ to $h_{i\bar{j}} + \phi_{i\bar{j}}$ then the new Ricci curvature is $dd^c \log\det(h_{i\bar{j}} + \phi_{i\bar{j}})$. The Calabi conjecture can be restated as the existence of $\phi$ such that $h_{i\bar{j}} + \phi_{i\bar{j}}$ is definite positive and

$$dd^c \left(\log\det(h_{i\bar{j}} + \phi_{i\bar{j}}) - \log\det(h_{i\bar{j}}) - eta\right) = 0 \tag{1}$$

The functions $f$ that satisfies $dd^c f = 0$ are called *pluriharmonic*. They also satisfy the maximum principle. By compactness of $X$, these functions on $X$ are exactly constant functions. Therefore 1 is equivalent to

$$\det(h_{i\bar{j}} + \phi_{i\bar{j}}) = e^{c+\eta} \det(h_{i\bar{j}})$$

or

$$(\omega + dd^c\phi)^n = e^{c+\eta}\omega^n$$

where $\omega^n$ denotes the repeated wedge product. Note that $(\omega + dd^c\phi)^n - \omega^n = 0$, one has $\int_M (\omega + dd^c\phi)^n = V$, the conjecture of Calabi is therefore a consequence of the following theorem.

**Theorem 2** (Yau)**.** *Given a function $f \in C^\infty(M), f > 0$ such that $\int_M f\omega^n = V$. There exists, and unique up to constant, $\phi \in C^\infty(M)$ such that $\omega + dd^c\phi > 0$ and*

$$(\omega + dd^c\phi)^n = f\omega^n$$

# 3 A sketch of proof

The uniqueness is straightforward. We will prove the existence of $\phi$ under the constraint $\int_M \phi\omega^n = 0$ (which will be useful to prove that $(N)$ is locally diffeomorphism later). We will prove that the set $S$ of $t \in [0, 1]$ such that there exists $\phi_t \in C^{k+2,\alpha}(M)$ with $\int_M \phi_t\omega^n = 0$ that satisfies

$$(\omega + dd^c\phi_t)^n = (tf + 1 - t)\omega^n \tag{2}$$

is both open and close in $[0, 1]$, therefore is the entire interval as $0 \in S$ is non empty.

To see that $S$ is open, one only has to prove that the function $\mathcal{N}$ defined by

$$\phi \mapsto \mathcal{N}(\phi) = \frac{\det(h_{i\bar{j}} + \phi_{i\bar{j}})}{\det(h_{i\bar{j}})}$$

or in other words $(\omega + dd^c\phi)^n = \mathcal{N}(\phi)\omega^n$, is a local diffeomorphism. The differential of $\mathcal{N}$ is given by

$$D\mathcal{N}(\phi).\eta = \mathcal{N}\tilde{\Delta}\eta$$

with $\eta$ varies in $\{\eta \in C^{k,\alpha}(M) : \int_M \eta\omega^n = 0\}$. But it is known that $\Delta$ is bijective between

$$\left\{\eta \in C^{k+2,\alpha}(M) : \int_M \eta = 0\right\} \longrightarrow \left\{f \in C^{k,\alpha}(M) : \int_M f = 0\right\}$$

Therefore $\mathcal{N}$ is a local diffeomorphism and $S$ is open.

The proof that $S$ is closed is more technical. The idea in general is to do 3 things:

1. Using Arzela-Ascoli, it suffices to show that $\{\phi_t : t \in S\}$ is bounded in $C^{k+2,\alpha}$. Therefore up to a subsequence, one has the uniform convergence of $\phi_{t_n}$ to all partial derivatives of order $\leq k+1$. The \$k+2\$-th order follows from (2).

2. Using Schauder theory, prove that the above bound follows from a priori estimate: There exists $\alpha \in (0,1)$ and $C(X, \|f\|_{1,1}, 1/\inf_M f) > 0$ such that every $\phi \in C^4(X)$ satisfying $(\omega + dd^c\phi)^n = f\omega^n$ and $\int_M \phi\omega^n = 0$ has
$$\|\phi\|_{2,\alpha} \leq C$$
.

3. Establish the priori estimate.