From Busemann function to Cheeger-Gromoll splitting

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We will prove the following result by Cheeger and Gromoll by a slightly modified approach of A. Besse.

Theorem 1 (Cheeger-Gromoll)**.** *Let M be a complete, connected Riemannian manifold with non negative Ricci curvature. Suppose that M contains a line then M is isometric to $M' \times \mathbb{R}$ with M' a complete, connected Riemannian manifold with non negative Ricci curvature.*

1 Busemann function

Let γ be a geodesic ray. We construct the Busemann function b associated to the ray as

$$b(x) = \lim_{t \to +\infty} f_t(x) = t - d(x, \gamma(t))$$

where the limit exists because the sequence f_t is non-decreasing and bounded by $d(x, \gamma(0))$. The convergence is also uniform in every compact.

In Euclidean space for example, the Busemann function is the orthogonal projection on γ. We will see that in a Riemannian manifold with non negative curvature, the Busemann function will serve as a projection.

Now with a fixed $x_0 \in M$, the tangent vector at x_0 of the geodesics connecting x_0 and $\gamma(t)$ is in the unit sphere of T_xM, which is compact. Let X be a limit point of these tangents vectors and pose

$$b_{X,t}(x) = b(x_0) + t - d(x, C_X(t))$$

where $C_X(t)$ is the geodesic flow starting at x_0 with velocity X.

Remark 1. 1. *From the construction of X, one has $b(x_0)+t = b(C_X(t))$, therefore $b_{X,t} \leq b$ withe quality in x_0. We say that b is supported by $b_{X,t}$ at x_0. In general a function f is* supported *by g at x_0 if $f(x_0) = g(x_0)$ and $f \geq g$ in a neighborhood of x_0.*

2. *$b_{X,t}$ is smooth and a computation in local coordinate gives $\Delta b_{X,t} \leq \frac{\dim M - 1}{t}$*

3. $\|\nabla b_{X,t}\| = 1$

We also note that it suffices to show that b is harmonic. In fact, from the smoothness one has $\nabla b(x_0) = \nabla b_{X,t}(x_0)$, which means $\|\nabla b\| = 1$ at every point in M. For each point $y \in M$, there exists a unique x with $b(x) = 0$ and time t when the flow of ∇b arrive at x. M is therefore homeomorphic to $\bar{M} \times \mathbb{R}$ by the above $y \mapsto (x,t)$ map. To see that this map is isometric, it remains to prove that ∇b is parallel, which follows from harmonicity of b

$$Ric(\nabla b, \nabla b) = -\|\nabla(\nabla b)\| - (\nabla b).(\Delta b)$$

we see that ∇b is parallel if $\Delta b = 0$.

Remark 2. *One can show (see A. Besse) that every gradient field ∇b of norm 1 at every point is actually harmonic.*

2 Harmonicity

The Busemann function associated to a geodesic ray is subharmonic, it is a consequence of the following lemma.

Lemma 2. *In a connected Riemannian manifold, if a continuous function f is supported at any point x by a family f_ϵ (depending on x) with $\Delta(f_\epsilon) \leq \epsilon$, then f can not attain maximum (unless when f is constant).*

Proof. Given a small geodesic ball B, suppose that we have a function h on B with $\Delta h < 0$ in B and $f + h$ attains maximum at x in the interior of B. Then $f_\epsilon + h$ also attains maximum at x, which means $\Delta f_\epsilon + \Delta h \geq 0$, which is contradictory.

For the construction of the function h, one suppose that B is small enough such that $f|_{\partial B} \leq max_B f =: f(x_0)$ and equality is not attained at every points in ∂B. Then choose

$$h = \eta(e^{\alpha\phi} - 1)$$

with and $\phi(x) = -1$ if $x \in \partial B$ and $f(x) = f(x_0)$, $\phi(x_0) = 0$, $\nabla\phi \neq 0$ and a large α such that

$$\Delta h = \eta(-\alpha^2 \|\nabla\phi\| + \alpha\Delta\phi)e^{\alpha\phi}.$$

is negative. □

Now for subharmonicity of b, given a harmonic function h that coincides with b in the boundary ∂B of a geodesic ball B, then $b - h$ is supported by $b_{X,t} - h$ with $\Delta(b_{X,t} - h) \to 0$ as t tends to $+\infty$, therefore $b - h \leq (b-h)|_{\partial B} = 0$ in B. hence b is subharmonic.

Corollary 2.1. *The Busemann function of a geodesic ray in a Riemannian manifold M with non-negative Ricci curvature is subharmonic.*

Now let b_+ be the function previously constructed for the ray $\gamma|_{[0,+\infty[}$ and b_- the Busemann function for the ray $\tilde{\gamma}|_{[0,+\infty[}$ where $\tilde{\gamma}(t) = \gamma(-t)$. Note that $b_+ + b_- \leq 0$ with equality on the line γ, but the sum is subharmonic therefore by maximum principle $b_+ = -b_-$ and b is harmonic hence smooth. The splitting theorem of Cheeger-Gromoll follows.