Elliptic and parabolic equation on compact manifolds

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Contents

1 Commutative diagram and linear PDE. Example: Semi-elliptic equation on $\mathbb{R}^n$

Fix a weight $(\sigma_1, \ldots, \sigma_n)$ on $X = \mathbb{R}^n$ and recall that for an index α, we note $\|\alpha\| := \sum_i \frac{\sigma}{\sigma_i} \alpha_i$.

We will consider in this section a partial differential operator A that is <u>heterogeneous</u>, of <u>constant coefficient</u> and of weight r, i.e.

$$A(D) = \sum_{\|\alpha\|=r} a_\alpha D^\alpha, \quad D^\alpha = \left(\frac{1}{i} \frac{\partial}{\partial x} \right)^\alpha$$

The **symbol** of A is $A(\xi) := \sum_{\|\alpha\|=r} a_\alpha \xi^\alpha$ and A is called **semi-elliptic** if $A(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus 0$

Remark 1. *If A is semi-elliptic then $\sigma \mid r$. In fact choose all $\xi_j = 0$ except $\xi_i \neq 0$, one sees that there must be a non-zero coefficient $a_{(0,\ldots,\frac{r\sigma_i}{\sigma},\ldots,0)}$, i.e. $\frac{r\sigma_i}{\sigma} \in \mathbb{Z}$ for all $i = \overline{1,n}$. Hence $\sigma \mid r$ ($\sigma = \mathrm{lcm}(\sigma_i)$ being a combination of σ_i, look at the same combination of $\frac{r\sigma_i}{\sigma}$).*

It is clear that the operator $A : W^{n,p}(X) \longrightarrow W^{n-r,p}(X)$ is bounded for all $n \in \mathbb{R}$ and the following diagram commutes for every real numbers $k < n$.

$$
\begin{array}{ccc}
W^{n,p}(X) & \xrightarrow{A(D)} & W^{n-r,p}(X) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
W^{k,p}(X) & \xrightarrow{A(D)} & W^{k-r,p}(X)
\end{array}
\tag{1}
$$

Definition 2. *Let E, F, G, H be Banach spaces and l, m, p, q are bounded operator such that the following diagram* (diag:D) *commutes*

$$
\begin{array}{ccc}
E & \xrightarrow{l} & F \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle p} \\
G & \xrightarrow{q} & H
\end{array}
\tag{diag:D}
$$

Then (diag:D) *is said to be an **exact square** if the following associated sequence is exact*

$$
0 \longrightarrow E \xrightarrow{l \oplus m} F \oplus G \xrightarrow{p \ominus q} H \longrightarrow 0
$$

Example 3. *If (A, B) is an interpolatable pair of Banach spaces then*

$$
\begin{array}{ccc}
A \cap B & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & A + B
\end{array}
$$

is exact, where arrows are natural inclusions.

The notion of exact square allows us to reformulate classical results of elliptic equation as

Theorem 4 (Elliptic equation with constant coefficients). *The square* (1) *is exact for all $k < n$ in $\mathbb{R}$. This encodes the following 3 results:*

1. $W^{n,p}(X) \xhookrightarrow{A \oplus i} W^{n-r,p}(X) \oplus W^{k,p}(X)$ *is of closed image, i.e. there exists $C > 0$ such that*

$$
\|f\|_{W^{n,p}(X)} \leq C \left(\|Af\|_{W^{n-r,p}(X)} + \|f\|_{W^{k,p}(X)} \right)
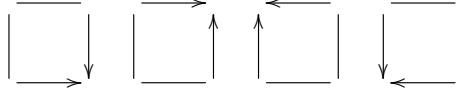$$

which is Gårding's inequality.

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2. $\ker A \ominus i = \operatorname{Im} A \oplus i$, i.e. if $f \in W^{k,p}(X)$ and $Af \in W^{n-r}(X)$ then actually $f \in W^{n,p}(X)$, which is regularity theorem.

3. $\operatorname{Im} A \ominus i = W^{k-r,p}(X)$, i.e. for all $g \in W^{k-r,p}(X)$, there exists $f \in W^{k,p}(X)$ such that $Af - g \in W^{n-r,p}(X)$, which is the existence of approximate solution (the idea behind parametrix).

A way to prove that a square is exact is to show that it splits

Definition 5. *The square* (diag:D) *is called **split** if there exists* l', m', p', q' *such that*

$$
\begin{array}{ccc}
E & \underset{l}{\overset{l'}{\rightleftarrows}} & F \\
{\scriptstyle m'}\big\Updownarrow{\scriptstyle m} & & {\scriptstyle p}\big\Updownarrow{\scriptstyle p'} \\
G & \underset{q'}{\overset{q}{\rightleftarrows}} & H
\end{array}
\tag{2}
$$

commutes in 4 ways:

and splits in 4 ways

i.e. the sum of two circle in each diagram is the identities.

Theorem 6. *1. A split square is exact. In fact, if a square splits, then the associated short sequence splits.*

2. If E, F, G, H are Hilbert spaces then any exact square splits.

Proof of Theorem 4 . Since A is semi-elliptic, there exists $\epsilon > 0$ such that $|A(\xi)| \geq \epsilon \|\xi\|^r$ for $\|\xi\| := (\xi_1^{2\sigma_1} + \cdots + \xi_n^{2\sigma_n})^{1/2\sigma}$. Let $\psi(\xi)$ be a radial function in ξ that is identically 1 for $\|\xi\| \leq 1$ and 0 for $\|\xi\| \geq 2$ and define

$$
G(\xi) := \begin{cases} \frac{1 - \psi(\xi)}{A(\xi)}, & \text{if } \|\xi\| \geq 1 \\ 0, & \text{if } \|\xi\| \leq 1 \end{cases}
$$

Then by Stein's multiplier theorem,

$$
G(D): W^{k-r,p}(X) \longrightarrow W^{k,p}(X) \quad \forall k \in \mathbb{R},
$$
$$
\psi(D): W^{l,p}(X) \longrightarrow W^{k,p}(X) \quad \forall k, l \in \mathbb{R}
$$

3

are bounded operators. We say that $G(D)$ is an *approximate inverse* of $A(D)$ because $G(D)A(D) = A(D)G(D) = 1 - \psi(D)$. It is easy to check that (1) splits:

$$
\begin{array}{ccc}
W^{k,p}(X) & \underset{A(D)}{\overset{G(D)}{\leftrightarrows}} & W^{k-r,p}(X) \\
\psi(D) \Big(\Big\downarrow i & & i \Big\downarrow \Big) \psi(D) \\
W^{l,p}(X) & \underset{G(D)}{\overset{A(D)}{\leftrightarrows}} & W^{l-r,p}(X)
\end{array}
$$

$\square$

The following abstract result shows that solutions of homogeneous equation $Af = 0$ are smooth (also proved in the second point of Theorem 4) and the solution space is of finite dimension.

Theorem 7. *Suppose that the square*

$$
\begin{array}{ccc}
E & \overset{l}{\longrightarrow} & F \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle p} \\
G & \underset{q}{\longrightarrow} & H
\end{array}
$$

is exact and m, p are compact operators. Then l and q have closed image, and their kernels and cokernels are isomorphic through m and p, and are of finite dimensional.

Proof. By basic diagram chasing, one can see that the restriction of m is an isomorphism $\ker l \longrightarrow \ker q$. But m is compact, $\ker l \cong \ker q$ are locally compact, hence of finite dimension.

It is easy to check (with sequential limit) that $\mathrm{Im}\, l$ is closed in F, since $\mathrm{Im}(l \oplus m) = \ker p \ominus q$ is closed and m is compact. So $\mathrm{coker}\, l$ is a Banach space.

Let $p'' : \mathrm{coker}\, l = F/l(E) \longrightarrow H/\overline{q(G)}$ be the map induced by p to the quotients, note that we have to take the closure of $q(G)$ to ensure that the quotient is Banach. Then p'' is obviously continuous and compact. Also p'' is surjective because $F \oplus G \overset{p \ominus q}{\longrightarrow\!\!\!\!\rightarrow} H$.

We will prove that p'' is injective. If $f \in F \setminus l(E)$ then by Hahn-Banach theorem, there exists a linear functional $\lambda \in F^*$ such that $\lambda(f) = 1$ and $\lambda(l(E)) = 0$. One has

$$
0 \longrightarrow H^* \overset{(p \ominus q)^*}{\longrightarrow} F^* \oplus G^* \overset{(l \oplus m)^*}{\longrightarrow\!\!\!\!\rightarrow} E^* \longrightarrow 0
$$

and that $(l \oplus m)^*(\lambda \oplus 0) = 0$, hence there exists $\lambda' \in H^*$ such that $\lambda \oplus 0 = (p \ominus q)^* \lambda'$, i.e. $\lambda' \circ q = 0$ and $\lambda' \circ p = \lambda$, which means λ' vanishes on $q(G)$, hence $\overline{q(G)}$, and that $\lambda'(p(f)) = \lambda(f) = 1$. Hence $p(f) \notin \overline{q(G)}$ and p'' is injective.

The injectivity of p'' has 2 consequences. First, it means that $\mathrm{coker}\, l \cong H/\overline{q(G)}$ by a compact operator, hence the two are locally compact and of finite dimension.

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Second, it proves that $q(G)$ is closed in H. In fact, given $h \in \overline{q(G)}$, by surjectivity of $p \ominus q$, one has $h = pf + qg$ for $f \in F$ and $g \in G$, this means $p''(\bar{f}) = \bar{0} \in H/\overline{q(G)}$, hence $\bar{f} = \bar{0} \in \operatorname{coker} l$, i.e. $f = l(e)$ for some $e \in E$. Therefore

$$h = p \circ l(e) + q(g) = q(m(e) + g) \in q(G)$$

and $p(G) = \overline{p(G)}$ is closed in H. $\qquad\square$

Remark 8. *The proof of Theorem 7 is much simpler for split squares. We presented the version for exact squares because we will use it later. The advantage of using exact squares instead of split square is, as we will see, that among commutative squares, exact squares form a relatively open set, allowing us to "pertube" an exact square and extend the theory to cover the case A of variable coefficients.*

2 Elliptic equation on half-plan $X \times Y^+$. Boundary conditions.

We will quickly review in this part the ideas to solve elliptic equations with constant coefficients on half-plan. This does not require any more abstract (i.e. with diagram) results. The main tasks will be using suitable cut-off function on the frequent space (1) to define the approximate inverse of an elliptic operator on half-plan that is adapted to the boundary structure and (2) to approximately inverse the boundary operators.

We will solve elliptic equation on $X \times Y^+$ where the variables are $x_1, \dots, x_n$ and y, under weight $\Sigma = (\sigma_1, \dots, \sigma_n, \rho)$. Recall that $A(D) = \sum_{\|(\alpha,\beta)\|=r} a_{\alpha\beta} D_x^\alpha D_y^\beta$ with symbol $A(\xi, \eta) = \sum_{\|(\alpha,\beta)\|=r} \xi^\alpha \eta^\beta$.

If $A(D)$ is semi-elliptic then for all $\xi \neq 0$ the polynomial $\eta \mapsto A(\xi, \eta)$ has no real zeros, hence can be factorized to

$$A(\xi, \eta) = A^+(\xi, \eta) A^-(\xi, \eta)$$

where $A^+(\xi, \eta)$ (resp. $A^-(\xi, \eta)$) only has zeros η with $\operatorname{Im} \eta > 0$ (resp. $\operatorname{Im} \eta < 0$).

Remark 9. *1. By semi-ellipticity, the monomial $a_{\alpha\beta} \xi^\alpha \eta^\beta$ with biggest β has index $\alpha = 0$. Hence we can suppose that the leading coefficients, as polynomials in η of A, A^+, A^- are 1.*

2. As polynomial in η, $A^+(\xi, \eta) = \sum_{\beta=0}^m a_\beta^+(\xi) \eta^\beta$ where $m = r\rho/\sigma$ and $a_\beta^+(\xi)$ are Σ-heterogeneous of weight $(m - \beta)\rho$, i.e.

$$a_\beta^+(t^{\sigma/\sigma_1} \xi_1, \dots, t^{\sigma/\sigma_n} \xi_n) = t^{(m-\beta)\rho} a_\beta^+(\xi)$$

Also, the coefficients a_β^+ are smooth in ξ.

We will solve the elliptic equation under some *suitable* boundary conditions. Let B^j, $1 \le j \le m$ be m Σ-heterogeneous *boundary operators* of weights r_j, i.e.

$$B^j(D) = \sum_{\|(\alpha,\beta)\|=r_j} b_{\alpha\beta}^j D_x^\alpha D_y^\beta$$

of symbol

$$B^j(\xi, \eta) = \sum_{\|(\alpha,\beta)\|=r_j} b^j_{\alpha\beta} \xi^\alpha \eta^\beta = \sum_{\|(\alpha,\beta)\|=r_j} b^j_\beta(\xi) \eta^\beta$$

where b^j_β are heterogeneous in ξ (actually polynomials) and of weight $r_j - \beta$.

As our discussion on trace operator, if $k > r_j + \frac{\sigma}{\rho p}$ then B^j extends to a bounded operator

$$W^{k,p}(X \times Y^+) \xrightarrow{\quad B_j \quad} \partial W^{k-r_j,p}(X) \hookrightarrow W^{l,p}(X)$$

$$W^{k-r_j,p}(X \times Y^+)$$

for all $0 \leq l < n - r_j - \frac{\sigma}{\rho p}$.

Definition 10. *We will say that the operators $Bf = (B^1 f, \ldots, B^m f)$ satisfy the **complementary boundary condition (CBC)** if the*

$$\det \left(c^j_\beta(\xi) \right)_{j,\beta} \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

where $c^j_\beta(\xi)$ are the cofficients of the remanders $C^j(\xi, \eta)$ when one divides $B^j(\xi, \eta)$ by $A^+(\xi, \eta)$ as polynomials in η, i.e.

$$B^j(\xi, \eta) \equiv C^j(\xi, \eta) = \sum_{\beta=0}^{m-1} c^j_\beta(\xi) \eta^\beta \qquad \mod A^+(\xi, \eta)$$

Approximate inverse of boundary operator B. The CBC condition allows us to approximately inverse boundary operator B.

Theorem 11 (Approximate inverse of B). *Let $B : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X)^{\oplus m}$ be a boundary operator that satisfies CBC condition, then there exists an operator*

$$H : \mathcal{S}(X)^{\oplus m} \longrightarrow \mathcal{S}(X \times Y^+)$$
$$(h_1, \ldots, h_m) \longmapsto H_1 h_1 + \cdots + H_m h_m$$

such that

1. $(\mathrm{Id} - BH)h = \psi(D_x)h$ *for all $h \in \mathcal{S}(X)^{\oplus m}$.*

2. $(\mathrm{Id} - HB)f = \psi(D_x)f$ *for all $f \in \ker A(D) : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$.*

where $\psi(\xi)$ is the radial smooth cut-off function in ξ that equals 1 when $\|\xi\| \leq 1$ and 0 when $\|\xi\| \geq 2$.

Moreover, if $k > r_j + \frac{\sigma}{\rho p}$ then the operators $H_j : \mathcal{S}(X) \longrightarrow \mathcal{S}(X \times Y^+)$ extends to a bounded operator

$$H_j : \partial W^{n-r_j,p}(X) \longrightarrow W^{k,p}(X \times Y^+)$$

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Sketch of proof. We define $H_j : \mathcal{S}(X) \longrightarrow \mathcal{S}(X \times Y^+)$ by its action on the frequent space of X, in particular, set

$$\widetilde{H_j h}(\xi, \eta) := H_j(\xi, y)\tilde{h}(\xi)$$

where $\tilde{f}$ is the partial (in x) Fourier transform of f and $H_j(\xi, y)$ is given by

$$H_j(\xi, y) := (1 - \psi(\xi)) \int_\Gamma \sum_{\alpha=0}^{m-1} e_j^\alpha(\xi) \frac{A_\alpha^+(\xi, \eta)}{A^+(\xi, \eta)} e^{i\eta y} d\eta$$

where $\Gamma \subset \mathbb{C}$ is a curve enclosing all zeros of $A(\xi, \eta)$ with $\operatorname{Im} \eta > 0$, $(e_j^\alpha(\xi))_{\alpha,j}$ is the inverse matrix of $(c_\beta^j(\xi))_{j,\beta}$ and $A_\alpha^+(\xi, \eta) := \sum_{\beta=0}^{m-\alpha-1} a_{\alpha+\beta+1}^+(\xi)\eta^\beta$. $\qquad\square$

Some auxilary functions. We cannot use the operator G as in the case of whole plan as an inverse of A on the half-plan $X \times Y^+$, since we only have access to the frequent space of X. However we can modify the cut-off function to create an approximate inverse of A on the half-plan.

Let $\varphi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ be the function that we used in the definition of E_+, i.e.

$$\varphi(x) := \frac{e^4}{\pi} \cdot \frac{e^{-(x^{1/4}+x^{-1/4})} \sin(x^{1/4} - x^{-1/4})}{1+x}, \quad x \geq 0$$

with the properties $\int_0^\infty x^n \varphi(x)dx = (-1)^n$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $\int_0^\infty \varphi(x)dx = 0$. Extending φ by 0 for $x < 0$, one still has a smooth function. Define $\chi(y) := -\varphi(-y - 1)$, then $\chi \in \mathcal{S}(Y)$, with support in $(-\infty, -1]$ and

$$\int_{\mathbb{R}} y^n \chi(y)dy = \begin{cases} 0, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \end{cases}$$

In the frequent space of Y, this means $\hat{\chi}(0) = 1$ and $D_\eta^k \hat{\chi}(0) = 0$, i.e. $1 - \hat{\chi}(\eta)$ has a zero of infinite order at $\eta = 0$.

Also, since $\chi = 0$ when $y > -1$, the convolution

$$f \longmapsto \hat{\chi}(D_y)f = \chi * f$$

maps $\mathcal{S}(Y^-/0)$ to itself, hence induces a mapping from $\mathcal{S}(Y^+)$ to itself, since

$$0 \longrightarrow \mathcal{S}(Y^-/0) \longrightarrow \mathcal{S}(Y) \longrightarrow \mathcal{S}(Y^+) \longrightarrow 0$$

(given any $f \in \mathcal{S}(Y^+)$, any extension $\tilde{f}$ of f to $\mathcal{S}(Y)$ has the same restriction of $\hat{\chi}(D_y)\tilde{f}$ on Y^+).

Let $w(\xi, \eta) := \psi(\xi)\hat{\chi}(\eta)$ then w defines an operator

$$w(D) : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$$

In fact, for all $k, l \in \mathbb{R}$, there exists $C > 0$ such that

$$\|w(D)f\|_{W^{k,p}(X \times Y^+)} \leq C\|f\|_{W^{l,p}(X \times Y^+)}.$$

Approximate inverse of elliptic operator A **on half-plan.** The auxilary function w will play the role of ψ in the whole plan case.

Theorem 12 (Approximate inverse of A on $X \times Y^+$). *There exists an operator $G : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$ such that:*

1. $(\mathrm{Id} - AG) = w(D)$

2. *For all $k, l \in \mathbb{R}$, there exists $C > 0$ such that for all $f \in \mathcal{S}(X \times Y^+)$:*

$$\|(\mathrm{Id} - GA)\psi(D_x)\|_{W^{k,p}(X \times Y^+)} \leq C\|f\|_{W^{l,p}(X \times Y^+)}$$

Also G extends to a bounded operator $G : W^{k-r,p}(X \times Y^+) \longrightarrow W^{k,p}(X \times Y^+)$ for all $k \in \mathbb{R}$.

Sketch of proof. In fact G is defined as follows:

$$G_0(\xi, \eta) := \frac{1 - w(\xi, \eta)}{A(\xi, \eta)}$$

which is smooth at $(0,0)$, where $1 - w$ has a zero of infinite order. Then $G_0(D) : \mathcal{S}(X \times Y) \longrightarrow \mathcal{S}(X \times Y)$ extends to $W^{k-r,p}(X \times Y) \longrightarrow W^{k,p}(X \times Y)$. Finally, take $G = C_+ G E_+$, which maps $\mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+)$ by first extending a function to the whole plan, applying G_0 and finally cutting-off. $\qquad\square$

Approximate inverse of the combined operator. Let $\mathcal{C}$ be the combined operator:

$$\mathcal{C} : \mathcal{S}(X \times Y^+) \longrightarrow \mathcal{S}(X \times Y^+) \oplus \mathcal{S}(X)^{\oplus m}$$
$$f \longmapsto (Af, Bf)$$

and define the operator $\mathcal{J}$ as

$$\mathcal{J} : \mathcal{S}(X \times Y^+) \oplus \mathcal{S}(X)^{\oplus m} \longrightarrow \mathcal{S}(X \times Y^+)$$
$$(g, h) \longmapsto Gg + H(h - BGg)$$

then one can prove with straightforward computation that $\mathcal{J}$ is an approximate inverse of $\mathcal{C}$.

Theorem 13 (Approximate inverse of C). *For smooth functions $f \in \mathcal{S}(X \times Y^+)$ and $(g, h) \in \mathcal{S}(X \times Y^+) \oplus \mathcal{S}(X)^{\oplus m}$, one has*

1. $(\mathrm{Id} - \mathcal{C}\mathcal{J})(g, h) = (w(D)g, \psi(D_x)(h - BGg)) =: \lambda(g, h)$

2. $(\mathrm{Id} - \mathcal{J}\mathcal{C})f = \psi(D_x)(\mathrm{Id} - GA)f + (\mathrm{Id} - HB - \psi(D_x))w(D)f =: \mu(f)$

Since G, H extend to Sobolev spaces, one also has

$$\mathcal{J} : W^{k-r,p}(X \times Y^+) \bigoplus_{j=1}^{m} \partial W^{k-r_j,p}(X) \longrightarrow W^{k,p}(X \times Y^+)$$

whenever $k > \frac{\sigma}{\rho p} + \max_j r_j$.

Theorem 14. *In analogue of Theorem 4, one has the following exact-split square*

$$W^{k,p}(X \times Y^+) \underset{\mathcal{C}}{\overset{\mathcal{J}}{\rightleftarrows}} W^{k-r,p}(X \times Y^+) \oplus_{j=1}^{m} \partial W^{k-r_j,p}(X) \tag{3}$$

$$\mu \Big\Uparrow \iota \qquad\qquad\qquad\qquad \iota \Big\Uparrow \lambda$$

$$W^{l,p}(X \times Y^+) \underset{\mathcal{J}}{\overset{\mathcal{C}}{\rightleftarrows}} W^{l-r,p}(X \times Y^+) \oplus_{j=1}^{m} \partial W^{l-r_j,p}(X)$$

for all $k > l > \frac{\sigma}{\rho p} + \max_j r_j$.

3   From local to global.

3.1   Pertubation of exact squares and consequences.

We will extend the result of Theorem 4 (exactness of heterogeneous elliptic operator with constant coefficient on Euclidean plan) in 2 levels: (1) for general elliptic operators (non-heterogeneous and with variable coefficients) and (2) for such operators on compact manifold (with boundary if needed). These 2 generalizations will be done using the same technique: "cube by cube" approximating an exact square.

We topologize the space of commutative squares $\quad E \overset{l}{\longrightarrow} F \quad$ as a closed subspace

$$\begin{array}{ccc} E & \overset{l}{\longrightarrow} & F \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle p} \\ G & \overset{q}{\longrightarrow} & H \end{array}$$

$SQ(E, F, G, H)$ of $L(E, F) \times L(F, H) \times L(E, G) \times L(G, H)$ defined by the equation $q \circ m = p \circ l$.

Theorem 15. *In $SQ(E, F, G, H)$, the exact squares form an open set.*

Instead of giving a proof (see [?, page 75-77]), let us explain why Theorem 15 is true. The commutativity already tells us that the composition of any two consecutive arrows in

$$0 \longrightarrow E \overset{l \oplus m}{\longrightarrow} F \oplus G \overset{p \ominus q}{\longrightarrow} H \longrightarrow 0$$

is 0, and exactness is an extra condition of type "maximal rank", which is an open condition (For matrices, this means the derterminant does not vanish. The analogous phenomenon

for Banach spaces is that a linear map sufficiently close to an invertible one is also invertible).

We will distinguish the following 2 types of cubes that we will use to cover a manifold. We will call the following set an *interior cube*

$$B_\epsilon := \left\{ (x_1, \ldots, x_n) : |x_1| \le \epsilon^{\sigma/\sigma_1}, \ldots |x_n| \le \epsilon^{\sigma/\sigma_n} \right\}$$

and the following an *boundary cube*

$$B_\epsilon^+ := \left\{ (x_1, \ldots, x_n, y) : |x_1| \le \epsilon^{\sigma/\sigma_1}, \ldots |x_n| \le \epsilon^{\sigma/\sigma_n}, 0 \le y \le \epsilon^{\sigma/\rho} \right\}$$

For the second type, we note by ∂_0 the part $y = 0$ of the boundary of B_ϵ^+, and by ∂_e the remaining part.

We will say that the $A := \sum_{\|\alpha\| \le r} a_\alpha(x) D^\alpha$ is **semi-elliptic at** 0 if $A_0 := \sum_{\|\alpha\| = r} a_\alpha(0) D^\alpha$ is a semi-elliptic operator.

Proposition 16 (Approximate operator on interior cube). *Suppose that $A := \sum_{\|\alpha\| \le r} a_\alpha(x) D^\alpha$ is defined in B_{ϵ_0} and A is semi-elliptic at $x = 0$. Fix $-\infty < l < k < +\infty$. Then there exists an $\epsilon > 0$ sufficiently small and an operator $A^\# = \sum_{\|\alpha\| \le r} a_\alpha^\#(x) D^\alpha$ with smooth coefficients defined on X such that*

$$A^\# = \begin{cases} A, & \text{inside } B_\epsilon \\ A_0, & \text{outside } B_{2\epsilon} \end{cases}$$

and the "k,l" square corresponding to $A^\#$, i.e.

$$\begin{array}{ccc} W^{k,p}(X) & \xrightarrow{A^\#} & W^{k-r,p}(X) \\ \downarrow{\iota} & & \downarrow{\iota} \\ W^{l,p}(X) & \xrightarrow{A^\#} & W^{l-r,p}(X) \end{array}$$

is exact.

An analoguous result holds for boundary problem. The setup for boundary problem on half-plan $X \times Y^+$ is as follows.

$$A := \sum_{\|(\alpha,\beta)\| \le r} a_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta$$

$$B^j := \sum_{\|(\alpha,\beta)\| \le r_j} b_{\alpha,\beta}^j(x,y) D_x^\alpha D_y^\beta, \quad j = \overline{1, m}$$

are operators with smooth coefficients on $B_{\epsilon_0}^+$ and

$$A_0 := \sum_{\|(\alpha,\beta)\| = r} a_{\alpha,\beta}(0,0) D_x^\alpha D_y^\beta$$

$$B_0^j := \sum_{\|(\alpha,\beta)\| = r_j} b_{\alpha,\beta}^j(0,0) D_x^\alpha D_y^\beta, \quad j = \overline{1, m}$$

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If A is semi-elliptic at 0 then we say that $\{B^j\}$ satisfy the CBC condition at 0 uf $\{B_0^j\}$ are CBC with respect to A_0. Note that this is an "open condition", i.e. if the condition is satisfied at $(0,0)$ then it is also satisfied in a neighborhood of $(0,0)$ in $X \times \{0\}$. The analoguous result for boundary problem can then be stated.

Proposition 17 (Approximate opearator on boundary cube). *Under the previous setup and with $\frac{\sigma}{\rho p} + \max_j r_j < l < k < +\infty$, for $\epsilon > 0$ sufficiently small, there exists operators $C^{\#} = (A^{\#}, B^{\#})$ with smooth coefficient in $X \times Y^{+}$ agreeing with (A, B) in B_{ϵ}^{+} and with (A_0, B_0) outside of $B_{2\epsilon}^{+}$ such that the square*

$$
\begin{array}{ccc}
W^{k,p}(X \times Y^{+}) & \xrightarrow{\ C^{\#}\ } & W^{k-r,p}(X \times Y^{+}) \oplus_{j=1}^{m} \partial W^{k-r_j,p}(X) \\
\iota \Big\downarrow & & \Big\downarrow \iota \\
W^{l,p}(X \times Y^{+}) & \xrightarrow[\ C^{\#}\ ]{} & W^{l-r,p}(X \times Y^{+}) \oplus_{j=1}^{m} \partial W^{l-r_j,p}(X)
\end{array}
$$

is exact.

We will prove Proposition 16 here to demonstrate how Theorem 15 is employed. Another reason is that the corresponding proof in [?] is not very readable due to a notation/printing issue.

Proof of Proposition 16. We will use the change of coordinates $\tilde{x}_i = \lambda^{-\sigma/\sigma_i} x_i$, which gives a diffeomorphism h_λ from B_{ϵ_0} to $B_{\epsilon_0/\lambda}$, in which the derivative operators are

$$
\left(\frac{\partial}{\partial \tilde{x}_i}\right)_i^{\alpha} = \lambda^{\alpha_i \sigma/\sigma_i} \left(\frac{\partial}{\partial x_i}\right)_i^{\alpha}, \quad \tilde{D}^{\alpha} = \lambda^{\|\alpha\|} D^{\alpha}
$$

The operator A, viewed in h_λ, i.e. the operator $f \mapsto A(f \circ h_\lambda)$, is $\sum_{\|\alpha\| \le r} a_\alpha(\lambda^{\sigma/\sigma_i} \tilde{x}_i) \lambda^{-\|\alpha\|} \tilde{D}^{\alpha}$. We pose

$$
\begin{aligned}
\tilde{A}_\lambda &:= \sum_{\|\alpha\| \le r} \lambda^{r-\|\alpha\|} a_\alpha(\lambda^{\sigma/\sigma_i} \tilde{x}_i) \tilde{D}^{\alpha} \\
\tilde{A}_0 &:= \sum_{\|\alpha\| = r} a_\alpha(0) \tilde{D}^{\alpha} \\
\tilde{A}_\lambda^{*} &:= \varphi(\tilde{x}) \tilde{A}_\lambda + (1 - \varphi(\tilde{x})) \tilde{A}_0
\end{aligned}
$$

where φ is radial in $\tilde{x}$, equals 1 for $\|\tilde{x}\| \le 1$ and 0 for $\|\tilde{x}\| \ge 2$.

The coefficient before $\tilde{D}^{\alpha}$ of $\tilde{A}_\lambda^{*}$ is $\lambda^{r-\|\alpha\|} \left[\varphi(\tilde{x}) a_\alpha(\lambda^{\sigma/\sigma_i} \tilde{x}_i) + (1 - \varphi(\tilde{x})) a_\alpha(0) \delta_{\|\alpha\|=r} \right]$ is the same as that of $\tilde{A}_0$ for $\tilde{x}$ outside of B_2 and C^0-converges to that of $\tilde{A}_0$ inside B_1. Hence for λ sufficiently small the corresponding "k, l" diagram of $\tilde{A}_\lambda^{*}$ is exact, hence so is the diagram of $\lambda^{-r} \tilde{A}_\lambda^{*}$. Choose $\epsilon = \lambda$ and $A^{\#}$ to be $\lambda^{-r} \tilde{A}_\lambda^{*}$ viewed in X through $h\lambda$. $\square$

Remark 18. *To avoid making infinite intersection of open sets, we have to fix k and l first in Proposition 16 and Proposition 17. The approximate operators $A^\#, B^\#$ and the size ϵ of the cube therefore depend on k, l, but this dependence will not be a trouble when we pass from local to global situation.*

The exactness of semi-elliptic operator with variable coefficients on manifold will be establish analytically, meaning through the 3 statements similar to those of Theorem 4. Proposition 16 and 17 can be applied to prove the the local version of these statements.

Lemma 19. *With the same ϵ and k, l as Proposition 16 and the extra condition that $l \geq k - 1$, one has for all $0 < \delta < \epsilon$*

1. $\|f\|_{W^{k,p}(B_\delta)} \leq C \left(\|Af\|_{W^{k-r,p}(B_\epsilon)} + \|f\|_{W^{l,p}(B_\epsilon)} \right)$ *for all $f \in W^{k,p}(B_\epsilon)$.*

2. *If $f \in W^{l,p}(B_\epsilon)$ and $Af \in W^{k-r,p}(B_\epsilon)$ then $f \in W^{k,p}(B_\delta)$.*

3. *If $g \in W^{l-r,p}(B_\delta/\partial)$ then there exists $f \in W^{l,p}(B_\epsilon, \partial)$ such that*

$$g - Af \in W^{k-r,p}(B_\epsilon, \partial B_\epsilon).$$

Proof. Let ψ be a cut-off function that equals 1 on B_δ and 0 outside of B_ϵ and $A^\#$ be the differential operator on X with exact "k, l" diagram given by Proposition 16 which equals A on B_ϵ. The idea of the remaining computation is to use the exactness of $A^\#$ on ψf and the reason for which the local-global passage is not trivial is that the operator $A^\#$ and the multiplication by ψ do not commute. The commutator $[A^\#, \psi]$, however, is of weight at least 1 less than A and with the choice $l \geq k - 1$ the norm $\|[\psi, A^\#]f\|_{W^{k-r,p}(X)}$ is dominated by $\|f\|_{W^{l,p}}$.

1. If $f \in W^{k,p}(B_\epsilon)$ then $\psi f \in W^{k,p}(B_\epsilon, \partial)$ and

$$\begin{aligned}
\|f\|_{W^{k,p}(B_\delta)} \leq \|\psi f\|_{W^{k,p}(X)} &\leq C \left(\|A^\# \psi f\|_{W^{k-r,p}(X)} + \|\psi f\|_{W^{l,p}(X)} \right) \\
&\leq C \left(\|\psi A^\# f\|_{W^{k-r,p}(X)} + \|[\psi, A^\#]f\|_{W^{k-r,p}(X)} + \|\psi f\|_{W^{l,p}(X)} \right) \\
&\leq C' \left(\|Af\|_{W^{k-r,p}(B_\epsilon)} + \|f\|_{W^{l,p}(B_\epsilon)} \right)
\end{aligned}$$

2. Given $f \in W^{l,p}(B_\epsilon)$ and $Af \in W^{k-r,p}(B_\epsilon)$, one has $\psi f \in W^{l,p}(X)$. Also, $[A^\#, \psi]f \in W^{l-r+1,p}(X) \subset W^{k-r,p}(X)$ and $\psi A^\# f = \psi Af \in W^{k-r,p}(X)$, therefore $A^\#(\psi f) \in W^{k-r,p}(X)$. By exactness of $A^\#$, one has $\psi f \in W^{k,r}(X)$, so $f \in W^{k,r}(B_\delta)$.

3. If $g \in W^{l-r,p}(B_\delta/\partial) \subset W^{l-r,p}(X)$, by exactness of $A^\#$ we can find $\tilde{f} \in W^{l,p}(X)$ such that $g - A^\# \tilde{f} \in W^{k-r,p}(X)$. Choose $f = \psi \tilde{f} \in W^{l,p}(B_\epsilon/\partial)$ then

$$g - Af = g - A^\#(\psi \tilde{f}) = \psi(g - A^\# \tilde{f}) + [\psi, A^\#]\tilde{f} \in W^{k-r,p}(B_\epsilon)$$

since $\psi(g - A^\# \tilde{f}) \in W^{k-r,p}(B_\epsilon)$ and $[\psi, A^\#]\tilde{f} \in W^{l-r+1,p}(B_\epsilon) \subset W^{k-r,p}(B_\epsilon)$.

$\square$

Lemma 20. *With (A, B) and ϵ, k, l as in Proposition 17 with the extra condition $l \geq k - 1$, then for all $\delta < \epsilon$, one has*

1. $\|f\|_{W^{k,p}(B_\delta^+)} \leq C \left(\|Af\|_{W^{k-r,p}(B_\epsilon^+)} + \sum_{j=1}^m \|B^j f\|_{\partial W^{k-r_j,p}(\partial_0 B_\epsilon^+)} + \|f\|_{W^{l,p}(B_\epsilon^+)} \right)$ *for all $f \in W^{k,p}(B_\epsilon^+)$.*

2. *If $f \in W^{l,p}(B_\epsilon^+)$ and $Af \in W^{k-r,p}(B_\epsilon^+)$ and $B^j f \in \partial W^{k-r_j,p}(\partial_0 B_\epsilon^+)$ then actually $f \in W^{k,p}(B_\delta^+)$.*

3. *If $g \in W^{l-r,p}(B_\delta^+ / \partial_e)$ and $h_j \in \partial W^{l-r_j,p}(\partial_0 B_\delta^+ / \partial)$ then there exists $f \in W^{l,p}(B_\epsilon^+, \partial_e)$ with*

$$g - Af \in W^{k-r,p}(B_\epsilon^+, \partial_e), \quad h_j - B^j f \in \partial W^{k-r_j,p}(\partial_0 B_\epsilon^+ / \partial).$$

The generalisation of Theorem 4 on manifold with variable coefficients is now straight-forward. The only nontrivial issue is the definition of semi-elliptic operator A on manifold. This requires a Riemannian metric g and ellipticity is naturally defined at every point, viewed in a chart, as we did before. But this only defines the action of A on $C^\infty(M)$ (or $C^r(M)$ if regularity is important), but not on $W^{k,p}(M/\mathcal{A})$ where $\mathcal{A} \subset \partial M$ is a connected component.

The action of a differential operator A can be defined to be component-wise on $W^{k,p}(M/\mathcal{A}) \hookrightarrow \bigoplus_i W^{k,p}(\mathcal{R}_i / \mathcal{A}_i)$ where $\mathcal{R}_i$ is an Euclidean plan or a half-plan and $\mathcal{A}_i$ the corresponding boundary part, i.e.

$$
\begin{array}{ccc}
W^{k,p}(M/\mathcal{A}) & \xrightarrow{\iota} & \bigoplus_i W^{k,p}(\mathcal{R}_i / \mathcal{A}_i) \\
\scriptstyle A \downarrow & & \downarrow \scriptstyle A \\
W^{l,p}(M/\mathcal{A}) & \xrightarrow{\iota} & \bigoplus_i W^{l,p}(\mathcal{R}_i, \mathcal{A}_i)
\end{array}
$$

It remains to check that the component-wise operation of A maps an element in the image on $W^{k,p}(M/\mathcal{A})$ to an element in the image of $W^{l,p}(M/\mathcal{A})$. This can be done using the projection as we did when defining trace operator on manifold, but the situation is much simpler here since we can differentiate directly an element in $\mathcal{S}^*(M)$.

Theorem 21 (Elliptic equation on manifold). *Let M be a compact manifold possibly with boundary (and a compatible foliation if the weight is not uniform). Let A be a general semi-elliptic operator of weight r, of variable coefficients and $\{B^j\}_j$ be a set boundary operators of weight r_j satisfying CBC with respect to A. Then for all $\frac{\sigma}{\rho p} + \max_j r_j < l < k < +\infty$, the square*

$$
\begin{array}{ccc}
W^{k,p}(M) & \xrightarrow{\mathcal{C}} & W^{k-r,p}(M) \oplus_{j=1}^m \partial W^{k-r_j,p}(\partial M) \\
\scriptstyle \iota \uparrow\downarrow & & \uparrow\downarrow \scriptstyle \iota \\
W^{l,p}(M) & \xrightarrow{\mathcal{C}} & W^{l-r,p}(M) \oplus_{j=1}^m \partial W^{l-r_j,p}(\partial M)
\end{array}
$$

is exact where $\mathcal{C} = (A, B^j)$.

Proof. We can suppose $l \geq k - 1$, the general case follows using

Lemma 22. *If the two following squares are exact*

$$
\begin{array}{ccc}
E \xrightarrow{\ l\ } F & , & G \xrightarrow{\ q\ } H \\
\scriptstyle m \downarrow \quad \downarrow \scriptstyle p & & \scriptstyle r \downarrow \quad \downarrow \scriptstyle s \\
G \xrightarrow{\ q\ } H & & K \xrightarrow{\ t\ } L
\end{array}
$$

then

$$
\begin{array}{c}
E \xrightarrow{\ l\ } F \\
\scriptstyle rm \downarrow \quad \downarrow \scriptstyle sp \\
K \xrightarrow{\ t\ } L
\end{array}
$$

is exact.

Now covering M by finitely many charts of type $B_\delta \subset B_\epsilon$ and $B_\delta^+ \subset B_\epsilon^+$ such that the interior of B_δ and of B_δ^+ cover M. Also, choose a partition of unity $\sum \psi = 1$ subordinated to B_δ and B_δ^+. The exactness will be established if we can prove the analogue of the 2 last statements of Theorem 4

For the regularity statement: If $f \in W^{l,p}(M)$, $Af \in W^{k-r,p}(M)$ and $B^j f \in W^{k-r_j,p}(\partial M)$ then the same holds for ψf in B_ϵ and B_ϵ^+ since

$$
[A, \psi]f \in W^{l-r+1,p} \subset W^{k-r,p}, \quad [B^j, \psi]f \in \partial W^{l-r_j+1,p} \subset \partial W^{k-r_j,p}
$$

Therefore $\psi f \in W^{k,p}(B_\delta)$ or $W^{k,p}(B_\delta^+)$ hence $f \in W^{k,p}(M)$.

For the approximation: If $g \in W^{l-r,p}(M)$ and $h_j \in \partial W^{l-r_j,p}(\partial M)$ then $\psi g \in W^{l-r,p}(B_\delta/\partial)$ or $W^{l-r,p}(B_\delta^+/\partial_e)$ and $\psi h_j \in \partial W^{l-r_j,p}(\partial_0 B_\delta^+/\partial)$. Then by Lemma 20, we can find $\tilde{f} \in W^{l,p}(B_\epsilon/\partial)$ with $\psi g - A\tilde{f} \in W^{k-r,p}(B_\epsilon/\partial)$ or in a boundary cube $\tilde{f} \in W^{l,p}(B_\epsilon^+/\partial_e)$ with $\psi g - A\tilde{f} \in W^{k-r,p}(B_\epsilon^+/\partial_e)$ with $\psi h_j - B^j\tilde{f} \in \partial W^{k-r_j,p}(\partial_0 B_\epsilon^+/\partial)$. Then $f := \sum \tilde{f}$ makes

sense and satisfies $\begin{cases} g - Af = \sum(\psi g - A\tilde{f}) & \text{is in } W^{k-r,p}(M) \\ h - B^j f = \sum(\psi h_j - B^j\tilde{f}) & \text{is in } \partial W^{k-r_j,p}(\partial M) \end{cases}$ $\qquad \square$

3.2 Consequences of Theorem 7.

Under the same setup as Theorem 21, one has

Theorem 23 (Regularity of kernel and cokernel)**.** *The map* $\mathcal{C} = (A, B) : W^{k,p}(M) \longrightarrow W^{k-r,p}(M) \bigoplus_{j=1}^{m} \partial W^{k-r_j,p}(\partial M)$ *has closed range, finite dimensional kernel and cokernel and the kernel and cokernel are independent of k in the sense of Theorem 7. In particular,* $\ker \mathcal{C} \subset C^\infty(M)$

The analoguous regularity for cokernel is less straightforward. We resume here the result.

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Theorem 24 (Regularity of cokernel)**.** *If $r > \max r_j$ then the image of $\mathcal{C}$ can be represented by finitely many linear relations: $(g, h) \in \operatorname{Im} \mathcal{C}$ if and only if it satisfies finitely many equations of type:*

$$\langle g, \gamma \rangle_M + \sum_{j=1}^{m} \langle h_j, \eta_j \rangle_{\partial M} = 0$$

with $\gamma \in C^\infty(M)$ and $\eta_j \in C^\infty(\partial M)$.

If $\max r_j - r = k \geq 0$ then for all $g \in W^{k-r,p}(M)$, the normal derivatives $\frac{\partial g}{\partial v^i}$ are well defined if $\frac{\sigma_i}{\rho} \leq k$. The cokernel is then given by the relations

$$\langle g, \gamma \rangle_M + \sum_{\sigma_i/\rho \leq k} \langle \frac{\partial g}{\partial v^i}, \chi_i \rangle_{\partial M} + \sum_{j=1}^{m} \langle h_j, \eta_j \rangle_{\partial M} = 0$$

with $\gamma \in C^\infty(M)$, $\chi_i \in C^\infty(\partial M)$, $\eta_j \in C^\infty(\partial M)$.

4 Parabolic equation on manifold.

4.1 Parabolicity and local results.

Definition 25. *The constant coefficient differential operator $A(D_x, D_t) = \sum_{\|(\alpha,\beta)\| \leq r} a_{\alpha\beta} D_x^\alpha D_t^\beta$ is called **parabolic** if its symbol $A(\xi, \theta) := \sum_{\|(\alpha,\beta)\| = r} a_{\alpha\beta} \xi^\alpha \theta^\beta$ has no zero when $\xi \in \mathbb{R}$ and $\operatorname{Im} \theta \leq 0$ except $\xi = \theta = 0$.*

Example 26. *Take $A = \partial_t - \partial_x^2 - \partial_y^2 - \partial_z^2 = iD_t + D_x^2 + D_y^2 + D_z^2$, the symbol is $i\theta + \sum \xi_i^2$ has no zero $\xi \in \mathbb{R}^3, \operatorname{Im} \theta \leq 0$ except 0. Generally, the operator $\partial_t + A(D_{x^i})$ is parabolic if A is an elliptic operator with the symbol $A(\xi) \geq 0$ for all $\xi \in \mathbb{R}$ with equality only at $\xi = 0$.*

Remark 27.   *1. If $\sigma = \operatorname{lcm}(\sigma_1, \ldots, \sigma_n)$ is the lcm of weights of variable x_i and τ is the weight of t, then parabolicity implies $2\tau \mid \sigma$. Therefore if the weights of x_i are uniform, one can suppose that $\tau = 1$.*

    2. Parabolicity implies ellipticity.

Similarly to the elliptic case, we attempt to define an approximate inverse G of A, of the form

$$G(\xi, \theta) = (1 - \psi(\xi, \theta))/A(\xi, \theta)$$

such that $G(D_x, D_t) : W^{k-r,p}(X \times T^+/0) \longrightarrow W^{k,p}(X \times T^+/0)$ and $\psi(D_x, D_t) : W^{k,p}(X \times T^+/0) \longrightarrow W^{k,p}(X \times T^+/0)$ for all $k, l \in \mathbb{R}$.

The sufficient condition for this is that $\psi(\xi, \theta) = \psi(\xi)\hat{\chi}(\theta)$ where ψ is compactly support and $\hat{\chi} \in \mathcal{S}(T)$ with $\hat{\chi} - 1$ having a zero of infinite order at $\theta = 0$, and $\hat{\chi}$ extends to a

holomorphic function in $\operatorname{Im}\theta \leq 0$. The function $\hat{\chi}$ used in section 2 suffices. We then have the following exact square

$$
\begin{array}{ccc}
W^{k,p}(X \times T^+/0) & \underset{A}{\overset{G}{\rightleftarrows}} & W^{k-r,p}(X \times T^+/0) \\
\psi \Big\uparrow\Big\downarrow \iota & & \iota \Big\uparrow\Big\downarrow \psi \\
W^{l,p}(X \times T^+/0) & \overset{A}{\underset{G}{\rightleftarrows}} & W^{l-r,p}(X \times T^+/0)
\end{array}
$$

The theory in section 2 also allows us to treat spatial boundary condition, that is, to replace the Euclidean plan X by the half-plan $X \times Y^+$. The analog of **CBC condition** for boundary operators

$$
B^j(D_x, D_y, D_t) = \sum_{\|(\alpha,\beta,\gamma)\| \leq r_j} b^j_{\alpha\beta\gamma} D_x^\alpha D_y^\beta D_t^\gamma
$$

is that the symbols

$$
B^j(\xi, \eta, \theta) = \sum_{\|(\alpha,\beta,\gamma)\| = r_j} b^j_{\alpha\beta\gamma} \xi^\alpha \eta^\beta \theta^\gamma
$$

are linearly independent modulo $A^+(\xi, \eta, \theta)$ as polynomial in η for all $\xi \in \mathbb{R}^n$ **and** for all $\operatorname{Im}\theta \leq 0$ except when $\xi = \theta = 0$. In that case we have the exactness of

$$
\begin{array}{ccc}
W^{k,p}(X \times Y^+ \times T^+/0) & \overset{(A,B^j)}{\longrightarrow} & W^{k-r,p}(X \times Y^+ \times T^+/0) \oplus_{j=1}^m \partial W^{k-r_j,p}(X \times T^+/0) \\
\Big\downarrow \iota & & \iota \Big\uparrow \\
W^{l,p}(X \times Y^+ \times T^+/0) & \overset{(A,B^j)}{\longrightarrow} & W^{l-r,p}(X \times Y^+ \times T^+/0) \oplus_{j=1}^m \partial W^{l-r_j,p}(X \times T^+/0)
\end{array}
$$

4.2 Global results and causality.

We will use the following setup. Let M be a compact manifold (possibly with boundary), of the form $N \times [\alpha, \omega] \ni (x,t)$. The global product gives a foliation that allows us to set the spatial weight to be uniformly σ and the temporal weight to be τ. The boundary of M has 3 parts: $\partial_\alpha M := N \times \alpha$, $\partial_\omega M := N \times \omega$ and $\partial_S M := \partial N \times [\alpha, \omega]$.

Let A be a parabolic operator, meaning that A is parabolic at every point and $B^j, j = \overline{1, m}$ be a set of boundary operator satisfying CBC condition at every point on $\partial_S M$. We take into account the initial condition by only considering the space $W^{k,p}(M/\partial_\alpha)$ of function vanishing before time $t = \alpha$. As before the operator

$$
\mathcal{C} := (A, B^j) : W^{k,p}(M/\partial_\alpha) \longrightarrow W^{k-r,p}(M/\partial_\alpha) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(\partial_S M/\partial_\alpha)
$$

has closed range, finite dimensional kernel and cokernel which are independent of $k > \frac{1}{p} + \max r_j$.

The same method allows us to conclude that $\ker \mathcal{C} \subset C^\infty(M)$ and the cokernel is given by finitely many linear relations of type

$$\langle g, \gamma \rangle_M + \sum_j \langle h_j, \eta_j \rangle_{\partial M} + \sum_i \langle \frac{\partial}{\partial \nu^i} g, \chi_i \rangle_{\partial_S M}$$

where $\gamma \in C^\infty(M/\partial_\omega)$, $\chi_i \in C^\infty(\partial_S M/\partial_\omega)$ and $\eta_j \in C^\infty(\partial_S M/\partial_\omega)$.

The difference with elliptic equation is that the kernel and cokernel of $\mathcal{C}$ are not only of finite dimension, but are zero.

Theorem 28 (Causality). *With the previous setup, the operator $\mathcal{C} = (A, B^j)$ defines an isomorphism*

$$W^{k,p}(M/\partial_\alpha) \xrightarrow{\quad \mathcal{C} \quad} W^{k-r,p}(M/\partial_\alpha) \bigoplus_{j=1}^m \partial W^{k-r_j,p}(\partial_S M/\partial_\alpha)$$

for all $k > \frac{1}{p} + \max_j r_j$, and therefore an isomorphism

$$C^\infty(M/\partial_\alpha) \xrightarrow{\quad \mathcal{C} \quad} C^\infty(M/\partial_\alpha) \bigoplus_{j=1}^m C^\infty(\partial_S M/\partial_\alpha)$$

Proof. Let $\beta \leq \gamma$ be real numbers in $[\alpha, \omega]$ and let $\ker(\beta, \gamma)$ and $\mathrm{coker}(\beta, \gamma)$ be the kernel and cokernel of operator $\mathcal{C}$ on $N \times [\beta, \gamma]$ with vanishing initial condition at β. Since $\dim \ker(\beta, \gamma)$ and $\dim \mathrm{coker}(\beta, \gamma)$ are integer-valued, using the fact that $\dim \ker(\beta, \omega)$ is decreasing in β and $\dim \mathrm{coker}(\alpha, \gamma)$ is increasing in γ, one can easily check that it suffices to show that the two functions are continuous in (β, γ) to prove that they are identically 0.

The following statements can be verified mechanically:

1. *Monotonicity:* $\dim \ker(\beta, \gamma)$ is decreasing in β, $\dim \mathrm{coker}(\beta, \gamma)$ is increasing in γ.

2. *One-sided continuity:* $\dim \ker(\beta, \gamma)$ is left-continuous in β, $\dim \mathrm{coker}(\beta, \gamma)$ is right-continuous in γ.

3. *One-sided semi-continuity:* $\dim \ker(\beta, \gamma)$ is left upper semi-continuous in γ, i.e.

$$\lim_{\gamma_1 \to \gamma_2^-} \inf \dim \ker(\beta, \gamma_1) \geq \dim \ker(\beta, \gamma_2)$$

This is due to the left-continuity in first variable of $\dim \ker$ and the exact sequence

$$0 \longrightarrow \ker(\gamma_1, \gamma_2) \longrightarrow \ker(\beta, \gamma_2) \longrightarrow \ker(\beta, \gamma_1)$$

where the last arrow is the restriction. Similar statement for coker:

$$\lim_{\beta_2 \to \beta_1^+} \dim \inf \mathrm{coker}(\beta_2, \gamma) \geq \dim \mathrm{coker}(\beta_1, \gamma)$$

This 3 statements suffice to finish the proof in the case where boundary conditions B^j on $\partial_S M$ are of constant coefficients since ker, coker only depend on the difference $\gamma - \beta$, up to a translation in time of the solutions.

In case B^j are of variable coefficients, the idea of making translation in time can be formulated using Index theory for Fredholm maps:

We recall that Fredholm maps between Banach spaces E, F are those in $L(E, F)$ with closed image and finite dimensional kernel and cokernel. It is a classical result that

1. The set $\mathcal{F}$ of Fredholm maps are open in $L(E, F)$.

2. The index $i(l) := \dim \ker l - \dim \operatorname{coker} l$ is continuous in $\mathcal{F}$.

The difference $\dim \ker(\beta, \gamma) - \dim \operatorname{coker}(\beta, \gamma)$ can be regarded as the index of a continuous family $\mathcal{C}_{(\beta, \gamma)}$ of operators on the same space $N \times [0, 1]$ using the diffeomorphism

$$N \times [0, 1] \xrightarrow{\sim} N \times [\beta, \gamma].$$

Hence $\dim \ker(\beta, \gamma) - \dim \operatorname{coker}(\beta, \gamma)$ is constant. It follows that $\dim \ker(\beta, \gamma)$ is both increasing and one-sided semi-continuous in γ hence is right-continuous in γ, hence $\dim \operatorname{coker}(\beta, \gamma)$ is continuous in γ. Other continuities follows similarly. $\qquad \square$

Remark 29. *To take into account the initial condition $f\big|_\alpha = f_\alpha$ smooth, one looks for solution of the form $f = f_b + f_\#$ where f_b satisfies the initial condition and $f_\# \in W^{k,p}(N \times [\alpha, \omega]/\alpha)$ satisfying a parabolic equation $(A f_\#, B^j f) = (g, h)$ where g, h and the coefficients of A and B^j depend smoothly on f_b, and therefore still C^∞ in (x, t).*

4.3 Regularisation effect and Gårding inequality.

With the same technique used for elliptic equation, one can also prove regularity result for parabolic equation. There are 2 different points, in comparison with the elliptic case:

1. There is a regularisation effect of parabolic equation: An arbitrarily weak estimate in the past gives an arbitrarily strong estimate in the future. We will see that this is in fact a consequence of the causality of parabolic equation (Theorem 28) and Kondrachov's theorem.

2. The temporal boundary condition is thicken: We will look at the norm on $N \times [\alpha, \pi]$ rather than the restriction to $\partial_\alpha M$.

Theorem 30 (Regularity and Garding inequality)**.** *Under the same setup and notation as Section 4.2, let $p \in (1, +\infty)$ and $k > l > \frac{1}{p} + \max r_j$. We denote by $W^{k,p}([\beta, \gamma])$ the Sobolev space $W^{k,p}(N \times [\beta, \gamma])$. Suppose that*

$$f \in W^{l,p}([\alpha, \omega]), \quad A f \in W^{k-r,p}([\alpha, \omega]), \quad B^j f \in \partial W^{k-r_j,p}([\alpha, \omega])$$

then $f \in W^{k,p}([\pi, \omega])$ for all $\pi \in (\alpha, \omega)$. Also, for all $l' > -\infty$, there exists a constant $C > 0$ such that

$$\|f\|_{W^{k,p}([\pi,\omega])} \leq C \left(\|Af\|_{W^{k-r,p}([\alpha,\omega])} + \|B^j f\|_{\partial W^{k-r_j,p}([\alpha,\omega])} + \|f\|_{W^{l',p}([\alpha,\pi])} \right).$$

In particular, for homogeneous equation, i.e. $Af = 0, B^j f = 0$, the solution is C^∞ and an arbitrarily weak estimate in the past gives an arbitrarily strong estimate in the future.

Proof. Let us explain why the theorem is true in the case of no spatial boundary $\partial N = \emptyset$. In this case, there is no distinction between l and l'. Consider A as an elliptic operator on $N \times [\tilde{\pi}, \omega]$ with $\tilde{\pi} = \frac{\alpha + \pi}{2}$ and with no boundary operator, one has the following exact diagram:

$$
\begin{array}{ccc}
W^{k,p}([\tilde{\pi},\omega]) & \xrightarrow{\ A\ } & W^{k-r,p}([\tilde{\pi},\omega]) \\
\cup \downarrow & & \cup \downarrow \\
W^{l,p}([\tilde{\pi},\omega]) & \xrightarrow{\ A\ } & W^{l-r,p}([\tilde{\pi},\omega])
\end{array}
$$

Therefore the if $f \in W^{l,p}([\alpha,\omega])$ and $Af \in W^{k-r,p}([\alpha,\omega])$ then $f \in W^{k,p}([\tilde{\pi},\omega]) \subset W^{k,p}([\pi,\omega])$ and

$$\|f\|_{W^{k,p}([\pi,\omega])} \leq \|f\|_{W^{k,p}([\tilde{\pi},\omega])} \leq C \left(\|Af\|_{W^{k-r,p}([\alpha,\omega])} + \|f\|_{W^{l,p}([\alpha,\omega])} \right) \tag{4}$$

$$\leq C \left(\|Af\|_{W^{k-r,p}([\alpha,\omega])} + \|f\|_{W^{l,p}([\alpha,\pi])} + \|f\|_{W^{l,p}([\tilde{\pi},\omega])} \right) \tag{5}$$

It remains to check that we can get rid of the $\|f\|_{W^{l,p}([\tilde{\pi},\omega])}$ term on the right hand side. Suppose not, then there exists a sequence $\{f_i\} \subset W^{l,p}([\alpha,\omega])$ such that $Af_i \to 0$ in $W^{k-r,p}([\alpha,\omega])$ and $f_i \to 0$ in $W^{l,p}([\alpha,\pi])$ but $\|f_i\|_{W^{l,p}([\tilde{\pi},\omega])} = 1$. Then by (5), $\{f_i\}$ is a bounded sequence in $W^{k,p}([\tilde{\pi},\omega])$ and, by Kondrachov's theorem, can be supposed to converge in $W^{l,p}([\tilde{\pi},\omega])$ to a function $\tilde{f}$ which has $\|\tilde{f}\|_{W^{l,p}([\tilde{\pi},\omega])} = 1$ and $A\tilde{f} = 0$ on $[\tilde{\pi}, \omega]$ because A commutes with the restriction. Moreover, since $\|f_i\|_{W^{l,p}([\alpha,\pi])} \to 0$, one has $\tilde{f} \in W^{l,p}([\tilde{\pi},\omega]/\tilde{\pi})$ and the fact that $\tilde{f} \neq 0$ contradicts Theorem 28). $\qquad \square$

Remark 31. *The proof of Theorem 30 in the general case, with spatial boundary taken into account requires the notion of bigraded Sobolev spaces on half-plan, see [**?**, page 97-100]. This is also how the regularity result for cokernel of elliptic operator, Theorem 24, is proved.*

5 Example: Linear heat equation.

We use the same setup of M, N, α, ω as Section 4.2. Let Δ be the (geometer's) Laplacian

$$-\Delta f := g^{ij}(x) \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k(x) \frac{\partial f}{\partial x^k} \right)$$

It is easy to check that Δ is an elliptic operator with symbol $\Delta \geq 0$ (there is a factor i when passing from $\frac{\partial}{\partial x^i}$ to D_{x^i}). Hence on $M = N \times [\alpha, \omega]$ the operator $\frac{\partial}{\partial t} + \Delta$ is parabolic.

5.1 Linear system.

We will look at the linear parabolic system of equations for $F = (f^1, \ldots, f^n) : M \longrightarrow \mathbb{R}^n$:

$$\frac{\partial F}{\partial t} + \Delta F + a\nabla F + bF = G \tag{6}$$

where in local coordinates $(a\nabla F)^\alpha = a_\beta^{\alpha i} \frac{\partial f^\beta}{\partial x^i}$ and $(bF)^\alpha = b_\beta^\alpha f^\beta$ and $(\Delta F)^\alpha = \Delta f^\alpha$ and the coefficients $a_\beta^{\alpha i}$ and b_β^α are smooth.

We will say that a function $F = (f^1, \ldots, f^n) : M \longrightarrow \mathbb{R}^n$ of class $W^{k,p}$ if it is $W^{k,p}$ component-wise. We also denote abusively by $W^{k,p}(M)$ the direct sum $W^{k,p}(M)^{\oplus n}$ where F belongs to.

Theorem 32 (Linear heat equation)**.** *Let $p > \dim M + 1 = \dim N + 2$ and $k \geq 0$, then for all $G \in W^{k,p}(N \times [\alpha, \omega]/\alpha)$, there exists a unique $F \in W^{k+2,p}(N \times [\alpha, \omega]/\alpha)$ that solves (6). Moreover, the operator*

$$F \longmapsto \frac{\partial F}{\partial t} + \Delta F + a\nabla F + bF$$

is an isomorphism between Banach spaces $W^{k+2,p}(N \times [\alpha, \omega]/\alpha) \longrightarrow W^{k,p}(N \times [\alpha, \omega]/\alpha)$.

Proof. Note that

$$H : W^{k+2,p}(N \times [\alpha, \omega]/\alpha) \longrightarrow W^{k,p}(N \times [\alpha, \omega]/\alpha)$$

$$F \longmapsto \frac{\partial F}{\partial t} + \Delta F$$

is a direct sum of parabolic operators in each component, and hence an isomorphism, and

$$K : W^{k+2,p}(N \times [\alpha, \omega]/\alpha) \longrightarrow W^{k,p}(N \times [\alpha, \omega]/\alpha)$$

$$F \longmapsto a\nabla F + bF$$

is a compact operator because it factors through $W^{k+1}(N \times [\alpha, \omega]/\alpha)$. Therefore $H + K$ is a Fredholm map with the same index as H, which is 0. It is sufficient to check that the kernel of $H + K$ is trivial.

Suppose that $F = (f^1, \ldots, d^n) \in \ker(H + K)$ then $f^\alpha \in W^{2,p}(N \times [\alpha, \omega]/\alpha)$, so f^α and $\frac{\partial f^\alpha}{\partial x^i}$ are continuous function on $N \times [\alpha, \omega]$. Since

$$\frac{\partial f^\alpha}{\partial t} + \Delta f^\alpha = -a_\beta^{\alpha i} \frac{\partial f^\beta}{\partial x^i} - b_\beta^\alpha f^\beta,$$

by repeated use of Theorem 30 the f^α are smooth for $t > \alpha$.

Let $e := \frac{1}{2}|F|^2 := \frac{1}{2}\sum_\alpha |f_\alpha|^2$, then e is continuous on $N \times [\alpha, \omega]$, vanishes on $N \times \{\alpha\}$ and one has

$$\frac{de}{dt} = -\Delta e - |\nabla F|^2 - a_\beta^{\alpha i} f^\alpha \frac{\partial f^\beta}{\partial x^i} - b_\beta^\alpha f^\alpha f^\beta$$

$$\leq -\Delta e + \frac{1}{2}C|F|^2 = -\Delta e + Ce$$

where we used the inequality $-u^2 - 2uv \le v^2$ to bound the second and third terms. We conclude that $F = 0$ since $e = 0$ by the following Maximum principle. $\qquad\square$

5.2 Maximum principle and L^∞-Comparison theorem.

With the same proof as for open set in $\mathbb{R}^n$, one has the maximum principle for parabolic equation on manifolds. The constant C in the following Theorem 33 can depend on the point $x \in M$, but will be most of the time globally constant, since the manifold M is compact. The following statement of Maximum principle will be sufficient for most of our application.

Theorem 33 (Maximum principle). *Let $f : M \longrightarrow \mathbb{R}$ be a continuous function on $M = N \times [\alpha, \omega]$ with $f|_{\partial_\alpha M} \le 0$ and $f|_{\partial_S M} \le 0$. Suppose that whenever $f > 0$, f is smooth satisfies*

$$\frac{\partial f}{\partial t} \le -\Delta f + Cf$$

Then in fact $f \le 0$.

With the same proof as Theorem 33, one can prove the following L^∞ Comparison theorem.

Theorem 34 (L^∞-Comparison theorem). *Let $f : M = N \times [\alpha, \omega] \longrightarrow \mathbb{R}$ be a continuous function on M, smooth for time $t > 0$ such that*

$$\frac{df}{dt} = -\Delta f + a\nabla f + bf \text{ on } N \times (\alpha, \omega] \tag{7}$$

where a is a smooth vector field and b is a smooth function on N. Then there exists $B = B(a,b)$ depending only on a and b such that

$$\||f|_\omega\|_{L^\infty} \le e^{B(\omega-\alpha)}\||f|_\alpha\|_{L^\infty}$$

Proof. We can suppose $b \le -1$ and prove that $\|f\|_{L^\infty(\partial_\omega M)} \le \|f\|_{L^\infty(\partial_\alpha M)}$. Intuitively, this means that since heat spreads out, the largest density must be attained at time $t = \alpha$. In fact, choose $B = \max_M b + 1$ and define $\tilde{f} = fe^{-B(t-\alpha)}$ then $\||\tilde{f}|_\alpha\|_{L^\infty} = \||f|_\alpha\|_{L^\infty}$ and $\||\tilde{f}|_\omega\|_{L^\infty} = e^{-B(\omega-\alpha)}\||f|_\omega\|_{l^\infty}$. The function $\tilde{f}$ satisfies the same heat equation (7) as f, with b replaced by $b - B \le -1$.

Now let us prove that under this supposition, $|f|$ attains it maximum at time $t = \alpha$. Since we can replace the solution f of (7) by $-f$, we can suppose, for sake of contradiction, that $|f|$ attains it maximum on $N \times [\alpha, \omega]$ at (x^*, t^*) with $|f(x^*, t^*)| = f(x^*, t^*) > 0$ and $t^* > \alpha$. Then one has

$$\begin{cases} \nabla f(x^*, t^*) = 0, \\ \frac{df}{dt}(x^*, t^*) \ge 0, \text{ (this is not true if } t^* = \alpha) \\ \Delta f(x^*, t^*) \ge 0, \\ f(x^*, t^*) > 0 \end{cases}$$

Plugging these in (7), one has a contradiction. $\qquad\square$

5.3   Backwards heat equation and L^1-Comparison theorem.

We will use backwards heat equation, which is just heat equation with the reversed sense of time (so with the reversed sign for Δ as well), in order to dualise the estimate of Theorem 34 and obtain a L^1 estimate of f at time $t = \omega$ in term of its L^1 norm at $t = \alpha$. In particular, we prove the following theorem.

Theorem 35 (L^1-comparison theorem). *Let a be a smooth, divergence-free vector field on a Riemannian manifold N and b be a smooth function on N. Let $f : N \times [\alpha, \omega] \longrightarrow \mathbb{R}$ be a continuous function on M such that*

$$\frac{df}{dt} = -\Delta f + a\nabla f + bf \text{ on } N \times (\alpha, \omega]. \tag{8}$$

Then there exists $B = B(a, b)$ depending only on a and b such that

$$\|f|_\omega\|_{L^1} \leq e^{B(\omega - \alpha)}\|f|_\alpha\|_{L^1}$$

Proof. Since L^1 is the dual space of L^∞, it is sufficient to prove that for all $h \in C^\infty(N)$, one has

$$\int_{N \times \{\omega\}} fh \leq e^{B(\omega - \alpha)}\left\|f|_\alpha\right\|_{L^1} \cdot \|h\|_{L^\infty}.$$

Consider the backwards heat equation $\begin{cases} \frac{dg}{dt} = \Delta g - \tilde{a}\nabla g - \tilde{b}g, & \text{on } N \times [\alpha, \omega] \\ g|_\omega = h, \end{cases}$ which is just a heat equation on $N \times [\alpha, \omega]$ with initial condition at α if we pose $\tilde{g}(t) := g(\omega + \alpha - t)$. The solution g exists and is smooth on $N \times [\alpha, \omega]$. One has, at any time t

$$\int_N g\Delta f = \int_N g\left(-\frac{df}{dt} + a\nabla f + bf\right)$$

$$\int_N f\Delta g = \int_N f\left(\frac{dg}{dt} + \tilde{a}\nabla g + \tilde{b}g\right)$$

Therefore

$$\int_N f\frac{dg}{dt} + g\frac{df}{dt} = \int_N (a\nabla f)g - (\tilde{a}\nabla g)f + (b - \tilde{b})fg$$

Choose $b = \tilde{b}$ and $\tilde{a} = -a$ then the term $(b - \tilde{b})fg$ vanishes and the two first terms become $\int_N \nabla_a(fg) = -\int_N fg \operatorname{div} a = 0$ where $\operatorname{div} a := \frac{\partial}{\partial x^i}a^i$ is the divergence. Therefore one has $\frac{d}{dt}\int_N fg = 0$, meaning that

$$\int_N f|_\omega \cdot h = \int_{N \times \omega} fg = \int_{N \times \alpha} fg \leq \|f|_\alpha\|_{L^1} \cdot \|h\|_{L^\infty} \leq e^{B(\omega - \alpha)}\|f|_\alpha\|_{L^1} \cdot \|h\|_{L^\infty}$$

where we applied Theorem 34 to g (strictly speaking, to $\tilde{g}$) and B only depends on $\tilde{a} = -a$ and $\tilde{b} = b$. $\qquad\square$