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Write down all necessary steps in solutions in order to receive as many points as possible.

1 (3 points)

Canary Islands are favorite holiday destinations in winter for Swedes. Tenerife and Gran Canaria are two main islands in Canary Islands. Suppose that 70% Swedes have been to Tenerife, 80% Swedes have been to Gran Canaria, and 60% Swedes have been to both Tenerife and Gran Canaria.

(1.1) (1p) Find the probability that a randomly selected Swede has been to at least one of the two main islands (Tenerife and Gran Canaria)?

(1.2) (1p) Find the probability that a randomly selected Swede has been to exactly one of the two main islands (Tenerife and Gran Canaria)?

(1.3) (1p) Given that a randomly selected Swede has been to Tenerife, find the probability that the Swede has been to Gran Canaria as well.

Solution. (1.1) Let

$$T = \{\text{a randomly selected Swede has been to Tenerife}\}$$

$$G = \{\text{a randomly selected Swede has been to Gran Canaria}\}$$

It is from the conditions that $P(T) = 70\%$, $P(G) = 80\%$ and $P(T \cap G) = 60\%$. Therefore

$$\begin{aligned} &P(\text{a randomly selected Swede has been to at least one of the two main islands}) \\ &= P(T \cup G) = P(T) + P(G) - P(T \cap G) = 70\% + 80\% - 60\% = 90\%. \end{aligned}$$

(1.2)

$$\begin{aligned} &P(\text{a randomly selected Swede has been to exactly one of the two main islands}) \\ &= P(T \cap G') + P(T' \cap G) = [P(T) - P(T \cap G)] + [P(G) - P(T \cap G)] \\ &= [70\% - 60\%] + [80\% - 60\%] = 10\% + 20\% = 30\%. \end{aligned}$$

(1.3)

$$P(G | T) = \frac{P(G \cap T)}{P(T)} = \frac{60\%}{70\%} = \frac{6}{7} = 0.857.$$

□

2 (3 points)

Let (X, Y) be a two dimensional random variable with a joint probability density function $f(x, y)$ as follows:

$$f(x, y) = \begin{cases} x - y, & \text{if } 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(2.1) (1p) Find the probability $P(X - Y \leq 1)$.

(2.2) (1p) Find the mean $E(X)$ of X .

(2.3) (1p) Are X and Y independent? Why?

Solution. (2.1) Better to draw a graph in order to see clearly the integral bounds.

$$\begin{aligned} P(X - Y \leq 1) &= \int_1^2 \left(\int_{x-1}^1 (x - y) dy \right) dx = \int_1^2 \left[xy - \frac{1}{2}y^2 \right]_{y=x-1}^{y=1} dx = \int_1^2 (x - \frac{1}{2}x^2) dx \\ &= \left[\frac{1}{2}x^2 - \frac{1}{6}x^3 \right]_1^2 = \frac{1}{3}. \end{aligned}$$

(2.2) We first find the probability density function $f_X(x)$ of X as follows

$$f_X(x) = \int_0^1 f(x, y) dy = \int_0^1 (x - y) dy = \left[xy - \frac{1}{2}y^2 \right]_{y=0}^{y=1} = x - \frac{1}{2}.$$

Therefore,

$$E(X) = \int_1^2 x f_X(x) dx = \int_1^2 x(x - \frac{1}{2}) dx = \int_1^2 (x^2 - \frac{1}{2}x) dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^2 \right]_1^2 = \frac{19}{12} = 1.583.$$

(2.3) The two marginal probability density functions $f_X(x)$ and $f_Y(y)$ can be found as follows:

$$f_X(x) = \int_0^1 f(x, y) dy = \int_0^1 (x - y) dy = \left[xy - \frac{1}{2}y^2 \right]_{y=0}^{y=1} = x - \frac{1}{2},$$

$$f_Y(y) = \int_1^2 f(x, y) dx = \int_1^2 (x - y) dx = \left[\frac{1}{2}x^2 - xy \right]_{x=1}^{x=2} = \frac{3}{2} - y.$$

It is clear that $f(x, y) \neq f_X(x) \cdot f_Y(y)$, therefore X and Y are NOT independent. \square

3 (3 points)

Suppose that the tips (Svenska: dricks) that a waiter in a certain restaurant follow a distribution X with a mean 20 SEK and a standard deviation 10 SEK. Assume that tips from customers are independent. Find the probability that a waiter receives at most 2200 SEK tips for the next 100 customers.

Solution. Let $X_i, 1 \leq i \leq 100$ denote the tips from the next 100 customers. According to the conditions, $\mu = E(X_i) = 10$ and $\sigma^2 = V(X_i) = 10^2$.

$$\begin{aligned} & P(\text{a waiter receives at most 2200 SEK tips for the next 100 customers}) \\ &= P(X_1 + X_2 + \dots + X_{100} \leq 2200) = P(\bar{X} \leq \frac{2200}{100}) \\ &= P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{22 - 20}{10/\sqrt{100}}\right) = P(N(0, 1) \leq 2) = \Phi(2) = 0.9772. \end{aligned}$$

\square

4 (3 points)

Let a population X be continuous with a probability density function $f(x) = \theta \cdot x^{\theta-1}$ for $0 < x < 1$, where $\theta > 0$ is an unknown parameter. Suppose that a sample $\{0.51, 0.18, 0.63\}$ is taken from this population.

(4.1) (1p) Find a point estimate $\hat{\theta}_{MM}$ of θ based on the method of moments.

(4.2) (2p) Find a point estimate $\hat{\theta}_{ML}$ of θ based on the maximum-likelihood method.

Solution. (4.1) We can easily compute $\bar{x} = \frac{0.51+0.18+0.63}{3} = 0.44$. The first equation is $E(X) = \bar{x}$, where the population mean $E(X)$ can be computed as

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_0^1 x \cdot \theta \cdot x^{\theta-1} dx = \theta / (\theta + 1).$$

Therefore $E(X) = \bar{x}$ directly gives

$$\hat{\theta}_{MM} = \bar{x} / (1 - \bar{x}) = 0.44 / (1 - 0.44) = 0.7857.$$

(4.2) The likelihood function is

$$L(\theta) = f(x_1) \cdot f(x_2) \cdot \dots \cdot f(x_n) = \theta \cdot x_1^{\theta-1} \cdot \dots \cdot \theta \cdot x_n^{\theta-1} = \theta^n \cdot (x_1 x_2 \dots x_n)^{\theta-1}.$$

In order to find a point $\hat{\theta}_{ML}$ maximizing $L(\theta)$, let us first take

$$\ln L(\theta) = n \ln \theta + (\theta - 1) \ln(x_1 x_2 \dots x_n).$$

Now taking derivative and setting it as zero give

$$0 = \ln' L(\theta) = n/\theta + \ln(x_1 x_2 \dots x_n) \implies \hat{\theta}_{ML} = -n/\ln(x_1 x_2 \dots x_n) = -3/\ln(0.51 \cdot 0.18 \cdot 0.63) = 1.0526.$$

□

5 (3 points)

The distribution X of temperatures (unit: °C degree Celsius) in a certain region on 13 January 2025 at 8:00 is assumed to be a normal random variable $X \sim N(\mu, \sigma^2)$. In order to estimate the average temperature μ in this region, a sample of measured temperatures is given with: sample size 16, sample mean -5 °C and sample standard deviation 1.2 °C. Construct a two-sided 95% confidence interval of μ .

Solution. This is Case 1.3, and the confidence interval is

$$I_\mu = \bar{x} \mp t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}} = -5 \mp t_{0.025}(15) \frac{1.2}{\sqrt{16}} = -5 \mp 2.13 \frac{1.2}{\sqrt{16}} = -5 \mp 0.639 = (-5.639, -4.361).$$

□

6 (3 points)

A sample with sample size n is to be taken from a population $X \sim N(\mu, 2.1^2)$, and one wants to test the hypotheses $H_0 : \mu = 6.2$ against $H_a : \mu < 6.2$. For this test, in order to make Type-I error α to be 0.025, and Type-II error $\beta(4.2)$ to be at most 0.01 for $\mu = 4.2$, how large should n be?

Solution. Since the population is $X \sim N(\mu, 2.1^2)$, it follows from $H_a : \mu < 6.2$ and Type-I error $\alpha = 0.025$ that the rejection region C is

$$C = (-\infty, -z_\alpha) = (-\infty, -1.96).$$

Then the Type-II error $\beta(4.2)$ can be computed as follows:

$$\begin{aligned} \beta(4.2) &= P(H_0 \text{ is not rejected when } H_0 \text{ is false and } \mu = 4.2) = P(TS \notin C, \text{ when } \mu = 4.2) \\ &= P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > -1.96, \text{ when } \mu = 4.2\right) \\ &= P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > -1.96 + \frac{(\mu_0 - \mu)}{\sigma/\sqrt{n}}, \text{ when } \mu = 4.2\right) \\ &= P(N(0, 1) > -1.96 + \frac{2\sqrt{n}}{2.1}). \end{aligned}$$

The Type-II error $\beta(4.2)$ is at most 0.01, therefore

$$\begin{aligned} \beta(4.2) &= P(N(0, 1) > -1.96 + \frac{2\sqrt{n}}{2.1}) \leq 0.01 \\ \iff 1 - \beta(4.2) &= 1 - P(N(0, 1) > -1.96 + \frac{2\sqrt{n}}{2.1}) = P(N(0, 1) \leq -1.96 + \frac{2\sqrt{n}}{2.1}) \geq 0.99 \\ \iff -1.96 + \frac{2\sqrt{n}}{2.1} &\geq 2.33 \quad (\text{because } P(N(0, 1) \leq 2.33) = 0.99) \\ \iff n &\geq \left((2.33 + 1.96) \cdot \frac{2.1}{2}\right)^2 = 4.5045^2 = 20.2905. \end{aligned}$$

□

1. Basic probability

(1.1) Conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

(1.2) Total probability $P(B) = \sum_{i=1}^k P(B|A_i)P(A_i)$ where $\{A_i\}$ are disjoint and $\cup_{i=1}^k A_i = S$.

(1.3) Bayes' Theorem $P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^k P(B|A_i)P(A_i)}$ where $\{A_i\}$ are in (1.2).

2. Random variables (r.v.s)

(2.1) Discrete r.v. X has a pmf $p(x) = P(X = x)$ satisfying $p(x) \geq 0$ and $\sum p(x_i) = 1$,

X	x_1	x_2	\cdots	x_n	\cdots
$p(x)$	$p(x_1)$	$p(x_2)$	\cdots	$p(x_n)$	\cdots

Expectation (or *Expected value* or *mean*) $\mu_X = E(X) = \sum x_i p(x_i)$;
 Variance $\sigma_X^2 = V(X) = E(X - \mu_X)^2 = E(X^2) - \mu_X^2 = \sum x_i^2 p(x_i) - (\sum x_i p(x_i))^2$.

(2.2) Continuous r.v. X has a pdf $f(x)$ satisfying $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$,

$$P(a < X < b) = \int_a^b f(x)dx.$$

Expectation (or *Expected value* or *mean*) $\mu_X = E(X) = \int_{-\infty}^{\infty} xf(x)dx$;
 Variance $\sigma_X^2 = V(X) = E(X - \mu_X)^2 = E(X^2) - \mu_X^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - (\int_{-\infty}^{\infty} xf(x)dx)^2$.

(2.3) Cumulative distribution function (cdf) of a r.v. X is $F(x) = P(X \leq x)$.

(2.4) X and Y are r.v.s, a , b and c are scalars, then

$$E(aX + bY + c) = aE(X) + bE(Y) + c,$$

$$V(aX + bY + c) = a^2V(X) + b^2V(Y) + 2ab \text{cov}(X, Y),$$

$$E(g(X, Y)) = \begin{cases} \sum_{i,j} g(x_i, y_j) \cdot p(x_i, y_j), & \text{for discrete } (X, Y), \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f(x, y)dxdy, & \text{for continuous } (X, Y). \end{cases}$$

(2.5) • Discrete r.v. (X, Y) has a joint pmf $p(x, y)$ satisfying $p(x, y) \geq 0$ and $\sum_{x_i} \sum_{y_i} p(x_i, y_i) = 1$.

The marginal pmf of X is $p_X(x) = \sum_y p(x, y)$;
 The marginal pmf of Y is $p_Y(y) = \sum_x p(x, y)$;

X and Y are *independent* if $p(x, y) = p_X(x) \cdot p_Y(y)$.

The marginal pdf of X is $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$;

The marginal pdf of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$;
 X and Y are *independent* if $f(x, y) = f_X(x) \cdot f_Y(y)$.

3. Several special r.v.s

(3.1) $X \sim Bin(n, p)$ has a pmf $p(x) = P(X = x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$, $x = 0, 1, 2, \dots, n$.

$$E(X) = n \cdot p, \quad V(X) = n \cdot p \cdot (1-p).$$

(3.2) $X \sim Po(\lambda)$ has a pmf $p(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0, 1, 2, \dots$.

$$E(X) = \lambda, \quad V(X) = \lambda.$$

(3.3) $X \sim Hypergeometric$ has a pmf $p(x) = P(X = x) = \frac{\binom{M}{x} \binom{N-M}{n-x}}{\binom{N}{n}}$.

(3.4) $X \sim Exp(\lambda)$ has a pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(3.5) $X \sim N(\mu, \sigma^2)$ has a pdf

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty.$$

$E(X) = \mu, \quad V(X) = \sigma^2$

(3.6) $X \sim U(a, b)$ has a pdf

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

$$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{(b-a)^2}{12}.$$

4. Central Limit Theorem (CLT)

Suppose that a population has mean = μ and variance = σ^2 . A random sample $\{X_1, X_2, \dots, X_n\}$ from this population is given. Then for large $n \geq 30$,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

- If the population is normal, then (1) holds for any n .
- Note that $\mu = E(\bar{X})$ and $(\sigma/\sqrt{n})^2 = V(\bar{X})$.

5. Several notations in statistics

(5.1) Sample mean: $\bar{X} = \frac{X_1+X_2+\dots+X_n}{n} = \frac{\sum X_i}{n}$; $\bar{x} = \frac{x_1+x_2+\dots+x_n}{n} = \frac{\sum x_i}{n}$.

(5.2) Sample variance:

$$S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1} = \frac{1}{n-1} \left(\sum X_i^2 - \frac{(\sum X_i)^2}{n} \right); \quad s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} = \frac{1}{n-1} \left(\sum x_i^2 - \frac{(\sum x_i)^2}{n} \right).$$

• Capital letters \bar{X} and S^2 refer to the objects based on random sample (therefore they are in general r.v.s), while small letters \bar{x} and s^2 are the objects based on observations (so they are scalars).

(5.3) A point estimator of θ obtained by Maximum Likelihood method is denoted as $\hat{\theta}_{ML}$.

6. Confidence Interval (CI)

In this course, three types of confidence intervals are studied depending on the unknown population parameter(s): CI-1 (confidence intervals for population mean(s)), CI-2 (confidence intervals for population variance(s)), and CI-3 (confidence intervals for population proportion(s)).

CI-1: $(1 - \alpha)$ CI of a population mean μ

case 1.1 (any n) If population $X \sim N(\mu, \sigma^2)$ and σ^2 is known, then $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ and

$$I_\mu = \left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \right) := \bar{x} \mp z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}.$$

case 1.2 ($n \geq 30$) For any population X , it holds that $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ and

$$I_\mu = \bar{x} \mp z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \text{ or } I_\mu = \bar{x} \mp z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}.$$

case 1.3 (any n) If population $X \sim N(\mu, \sigma^2)$ and σ^2 is unknown, then $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim T(n-1)$ and

$$I_\mu = \bar{x} \mp t_{\alpha/2}(n-1) \cdot \frac{s}{\sqrt{n}}.$$

CI-1': $(1 - \alpha)$ CI of the difference of two population means $\mu_X - \mu_Y$

case 1.1' (any n_1, n_2) If independent populations $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, and σ_X^2, σ_Y^2 are known, then $\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}} \sim N(0, 1)$, and $I_{\mu_X - \mu_Y} = (\bar{x} - \bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}$.

case 1.2' ($n_1, n_2 \geq 30$) For any independent populations X and Y , it holds that

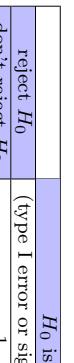
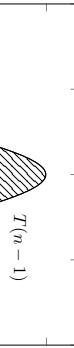
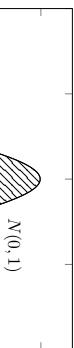
$$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}} \sim N(0, 1) \text{ and}$$

$$I_{\mu_X - \mu_Y} = (\bar{x} - \bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}} \text{ or } I_{\mu_X - \mu_Y} = (\bar{x} - \bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{\sigma}_X^2}{n_1} + \frac{\hat{\sigma}_Y^2}{n_2}}.$$

case 1.3' (any n_1, n_2) If independent populations $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, where σ_X^2, σ_Y^2 are unknown but $\sigma_X^2 = \sigma_Y^2$, then

$$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim T(n_1 + n_2 - 2), \text{ where } S^2 = \frac{(n_1 - 1)S_X^2 + (n_2 - 1)S_Y^2}{n_1 + n_2 - 2}, \text{ and}$$

$$I_{\mu_X - \mu_Y} = (\bar{x} - \bar{y}) \mp t_{\alpha/2}(n_1 + n_2 - 2) \cdot s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}.$$



CI-2: $(1 - \alpha)$ CI of population variance(s) σ^2

- If a population $X \sim N(\mu, \sigma^2)$ and σ^2 is unknown, then $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$, and

$$I_{\sigma^2} = \left(\frac{(n-1)s^2}{\chi_{\alpha/2}^2(n-1)}, \frac{(n-1)s^2}{\chi_{1-\alpha/2}^2(n-1)} \right).$$

If two independent populations $X \sim N(\mu_X, \sigma^2)$ and $Y \sim N(\mu_Y, \sigma^2)$, and σ^2 is unknown, then $\frac{(n_1+n_2-2)S^2}{\sigma^2} \sim \chi^2(n_1 + n_2 - 2)$, and

$$I_{\sigma^2} = \left(\frac{(n_1 + n_2 - 2)s^2}{\chi_{\alpha/2}^2(n_1 + n_2 - 2)}, \frac{(n_1 + n_2 - 2)s^2}{\chi_{1-\alpha/2}^2(n_1 + n_2 - 2)} \right),$$

where $S^2 = \frac{(n_1-1)S_X^2 + (n_2-1)S_Y^2}{n_1 + n_2 - 2}$.

CI-3: $(1 - \alpha)$ CI of population proportion(s)

- If a (large) population has an unknown proportion p , then $\frac{\hat{p} - p}{\sqrt{(1-p)/n}} \sim N(0, 1)$ if $n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$ with $\hat{p} = x/n$, and $I_p = \hat{p} \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.
- If two independent (large) populations have unknown proportions p_1 and p_2 , then

$$\frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}} \sim N(0, 1)$$

if $n_i\hat{p}_i \geq 10$ and $n_i(1 - \hat{p}_i) \geq 10$ for $i = 1, 2$, and $I_{p_1 - p_2} = (\hat{p}_1 - \hat{p}_2) \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$.

7. Hypothesis Test (HT)

	H_0 is true	H_0 is false and $\theta = \theta_1$
reject H_0	(type I error or significance level) α	(power) $h(\theta_1) = 1 - h(\theta_1)$
don't reject H_0	1 - α	(type II error) $\beta(\theta_1) = 1 - h(\theta_1)$

reject $H_0 \Leftrightarrow TS \in C \Leftrightarrow p\text{-value} < \alpha$

χ^2 tests for populations (non-parametric tests)

Suppose that for a random sample of a population X the n elements of it are classified into k disjoint groups $A_i, 1 \leq i \leq k$. For each group $A_i, 1 \leq i \leq k$, suppose that there are $N_i, 1 \leq i \leq k$ elements inside. Let $p_i = P(A_i)$ assuming a given distribution of X . Note that $p_1 + p_2 + \dots + p_k = 1$ and $N_1 + N_2 + \dots + N_k = n$. One wants to test the hypotheses

$$H_0 : P(A_i) = p_i, \quad 1 \leq i \leq k, \quad H_a : P(A_i) \neq p_i \text{ for some } 1 \leq i \leq k.$$

If n is large in the sense that $np_i \geq 5$ for all $1 \leq i \leq k$, then the test statistic is

$$\sum_{i=1}^k \frac{(N_i - np_i)^2}{np_i} \approx \chi^2(k-1).$$

Therefore the observation of the test statistic is

$$TS = \sum_{i=1}^k \frac{(n_i - np_i)^2}{np_i}, \text{ where } n_i \text{ is the observation of } N_i, 1 \leq i \leq k.$$

For the critical region C , one can take (note that if H_0 is true, then TS should be close to zero)

$$C = (\chi_\alpha^2(k-1), \infty).$$

The conclusion would be $TS \in C \iff H_0$ is rejected.

8. Linear and logistic regression

(Multiple) linear regression: $Y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + \varepsilon, \varepsilon \sim N(0, \sigma^2)$.

\bullet Y : response variable (which is normal r.v.), $\{x_1, \dots, x_k\}$: predictors (which are scalars).

\bullet sample: $\{(x_{11}, \dots, x_{1k}; y_1), (x_{21}, \dots, x_{2k}; y_2), \dots, (x_{n1}, \dots, x_{nk}; y_n)\}$.
 \bullet how to estimate $\beta_j \approx \hat{\beta}_j$: least square method, that is, to minimize $\sum_{i=1}^n (\hat{y}_i - y_i)^2$, where the estimated (multiple) linear regression line \hat{y} is

$$\hat{y} = \beta_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k.$$

\bullet $\frac{\hat{\beta}_j - \beta_j}{s.e(\hat{\beta}_j)} \sim T(n-k-1)$, this helps determine whether or not the real $\beta_j = 0$?

\bullet $\sigma^2 \approx \frac{SSE}{n-k-1}$, this gives an estimation of the size of the error.

$\bullet R^2 = \frac{SSE}{SS_T}$ this gives how well the model is (if $R^2 \approx 1$, then the model fits the sample very well).

\bullet How to test $\beta_1 = \dots = \beta_k = 0$? Use the random variable $\frac{SS_R/k}{SSE/(n-k-1)} \sim F(k, n-k-1)$.

Logistic regression: Let Y can only take 0 or 1 with $P(Y=1) = p$ and $P(Y=0) = 1-p$,

$$E(Y) = p(x_1, \dots, x_k) = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}.$$

$\bullet Y$: response variable (which is Bernoulli r.v. $P(Y=1) = p$ and $P(Y=0) = 1-p$, so $E(Y) = p$), $\{x_1, \dots, x_k\}$: predictors (which are scalars).

\bullet sample: $\{(x_{11}, \dots, x_{1k}; y_1), (x_{21}, \dots, x_{2k}; y_2), \dots, (x_{n1}, \dots, x_{nk}; y_n)\}$.

\bullet how to estimate $\beta_j \approx \hat{\beta}_j$: maximal likelihood method (maximize $\prod_{i=1}^n p(x_{i1}, \dots, x_{ik})^{y_i} (1 - p(x_{i1}, \dots, x_{ik}))^{1-y_i}$).

\bullet $\frac{\hat{\beta}_j - \beta_j}{s.e(\hat{\beta}_j)} \approx N(0, 1)$ for large $n \geq 30$, this helps determine whether or not the real $\beta_j = 0$?

\bullet Classification of a new object $Y(x_1, \dots, x_k)$ as 1 or 0 according

$$Y(x_1, \dots, x_k) = \begin{cases} 1, & \text{if } \hat{p}(x_1, \dots, x_k) \geq 0.5, \\ 0, & \text{if } \hat{p}(x_1, \dots, x_k) < 0.5, \end{cases}$$

where the estimated logit function $\hat{p}(x_1, \dots, x_k)$ is

$$\hat{p}(x_1, \dots, x_k) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k}}.$$

9. Tables

(9.1) Table for $N(0,1)$ standard normal random variable $\Phi(x) = P(N(0,1) \leq x)$, $x \geq 0$.

There is an important relation $\Phi(-x) = 1 - \Phi(x)$, $x \geq 0$.

x	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7957	0.8023	0.8051	0.8078	0.8106	0.8133	0.8160
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9222	0.9256	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	0.9330
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0.9965
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0.9981	0.9981
2.9	0.9981	0.9982	0.9983	0.9984	0.9984	0.9985	0.9986	0.9986	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9990	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9993	0.9993	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(9.2) Table for $T(f)$ random variable $F(x) = P(T(f) \leq x)$, where f is a parameter called 'degrees of freedom'.

f	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.9995	$F(x)$
1	1.00	3.08	6.31	12.71	31.82	63.66	127.32	636.62	
2	0.82	1.89	2.92	4.30	6.96	9.92	14.09	31.60	
3	0.76	1.64	2.35	3.18	4.54	5.84	7.45	12.92	
4	0.74	1.53	2.13	2.78	3.75	4.60	5.60	8.61	
5	0.73	1.48	2.02	2.57	3.36	4.03	4.77	6.87	
6	0.72	1.44	1.94	2.45	3.14	3.71	4.32	5.96	
7	0.71	1.41	1.89	2.36	3.00	3.50	4.03	5.41	
8	0.71	1.40	1.86	2.31	2.90	3.36	3.83	5.04	
9	0.70	1.38	1.83	2.26	2.82	3.25	3.69	4.78	
10	0.70	1.37	1.81	2.23	2.76	3.17	3.58	4.59	
11	0.70	1.36	1.80	2.20	2.72	3.11	3.50	4.44	
12	0.70	1.36	1.78	2.18	2.68	3.05	3.43	4.32	
13	0.69	1.35	1.77	2.16	2.65	3.01	3.37	4.22	
14	0.69	1.35	1.76	2.14	2.62	2.98	3.33	4.14	
15	0.69	1.34	1.75	2.13	2.60	2.95	3.29	4.07	
16	0.69	1.34	1.75	2.12	2.58	2.92	3.25	4.01	
17	0.69	1.33	1.74	2.11	2.57	2.90	3.22	3.97	
18	0.69	1.33	1.73	2.10	2.55	2.88	3.20	3.92	
19	0.69	1.33	1.73	2.09	2.54	2.86	3.17	3.88	
20	0.69	1.33	1.72	2.09	2.53	2.85	3.15	3.85	
21	0.69	1.32	1.72	2.08	2.52	2.83	3.14	3.82	
22	0.69	1.32	1.72	2.07	2.51	2.82	3.12	3.79	
23	0.69	1.32	1.71	2.07	2.50	2.81	3.10	3.77	
24	0.68	1.32	1.71	2.06	2.49	2.80	3.09	3.75	
25	0.68	1.32	1.71	2.06	2.49	2.79	3.08	3.73	
26	0.68	1.31	1.71	2.06	2.48	2.78	3.07	3.71	
27	0.68	1.31	1.70	2.05	2.47	2.77	3.06	3.69	
28	0.68	1.31	1.70	2.05	2.47	2.76	3.05	3.67	
29	0.68	1.31	1.70	2.05	2.46	2.76	3.04	3.66	
30	0.68	1.31	1.70	2.04	2.46	2.75	3.03	3.65	
40	0.68	1.30	1.68	2.02	2.42	2.70	2.97	3.55	
50	0.68	1.30	1.68	2.01	2.40	2.68	2.94	3.50	
60	0.68	1.30	1.67	2.00	2.39	2.66	2.91	3.46	
100	0.68	1.29	1.66	1.98	2.36	2.63	2.87	3.39	
∞	0.67	1.28	1.65	1.96	2.33	2.58	2.81	3.29	

(9.3) Table for $\chi^2(f)$ random variable $F(x) = P(\chi^2(f) \leq x)$, where f is a parameter.

f	$F(x)$										
	0.005	0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.30	0.40	0.50
1	0.00	0.00	0.00	0.00	0.00	0.02	0.06	0.15	0.27	0.45	1
2	0.00	0.00	0.01	0.02	0.05	0.10	0.21	0.45	0.71	1.39	2
3	0.02	0.02	0.07	0.11	0.22	0.35	0.58	1.01	1.42	2.37	3
4	0.06	0.09	0.21	0.30	0.48	0.71	1.06	1.65	2.19	2.75	4
5	0.16	0.21	0.41	0.55	0.83	1.15	1.61	2.34	3.00	3.66	5
6	0.30	0.38	0.68	0.87	1.24	1.64	2.20	3.07	3.83	4.57	6
7	0.48	0.60	0.99	1.24	1.69	2.17	2.83	3.82	4.67	5.49	7
8	0.71	0.86	1.34	1.65	2.18	2.73	3.49	4.59	5.53	6.42	8
9	0.97	1.15	1.73	2.09	2.70	3.33	4.17	5.38	6.39	7.36	9
10	1.26	1.48	2.16	2.56	3.25	3.94	4.87	6.18	7.27	8.30	10
11	1.59	1.83	2.60	3.05	3.82	4.57	5.58	6.99	8.15	9.24	11
12	1.93	2.21	3.07	3.57	4.40	5.23	6.30	7.81	9.03	10.18	12
13	2.31	2.62	3.57	4.11	5.01	5.89	7.04	8.63	9.93	11.13	13
14	2.70	3.04	4.07	4.66	5.63	6.57	7.79	9.47	10.82	12.08	14
15	3.11	3.48	4.60	5.23	6.26	7.26	8.55	10.31	11.72	13.03	15
16	3.54	3.94	5.14	5.81	6.91	7.96	9.31	11.15	12.62	13.98	16
17	3.98	4.42	5.70	6.41	7.56	8.67	10.09	12.00	13.53	14.94	17
18	4.44	4.90	6.26	7.01	8.23	9.39	10.86	12.86	14.44	15.89	18
19	4.91	5.41	6.84	7.63	8.91	10.12	11.65	13.72	15.35	16.85	19
20	5.40	5.92	7.43	8.26	9.59	10.85	12.44	14.58	16.27	17.81	20
21	5.90	6.45	8.03	8.90	10.28	11.59	13.24	15.44	17.18	18.77	21
22	6.40	6.98	8.64	9.54	10.98	12.34	14.04	16.31	18.10	19.73	22
23	6.92	7.53	9.26	10.20	11.69	13.09	14.85	17.19	19.02	20.69	23
24	7.45	8.08	9.89	10.86	12.40	13.85	15.66	18.06	19.94	21.65	24
25	7.99	8.65	10.52	11.52	13.12	14.61	16.47	18.94	20.87	22.62	25
26	8.54	9.22	11.16	12.20	13.84	15.38	17.29	19.82	21.79	23.58	26
27	9.09	9.80	11.81	12.88	14.57	16.15	18.11	20.70	22.72	24.54	27
28	9.66	10.39	12.46	13.56	15.31	16.93	18.94	21.59	23.65	25.51	28
29	10.23	10.99	13.12	14.26	16.05	17.71	19.77	22.48	24.58	26.48	29
30	10.80	11.59	13.79	14.95	16.79	18.49	20.60	23.36	25.51	27.44	30
40	16.91	17.92	20.71	22.16	24.43	26.51	29.05	32.34	34.87	37.13	40
50	23.46	24.67	27.99	29.71	32.36	34.76	37.69	41.45	44.31	46.86	50
60	30.34	31.74	35.53	37.48	40.48	43.19	46.46	50.64	53.81	56.62	60
100	59.90	61.92	67.33	70.06	74.22	77.93	82.36	87.95	92.13	95.81	100

Table for $\chi^2(f)$ random variable $F(x) = P(\chi^2(f) \leq x)$, where f is a parameter.

f	$F(x)$									
	0.60	0.70	0.80	0.90	0.95	0.975	0.99	0.995	0.999	0.9995
1	0.71	1.07	1.64	2.71	3.84	5.02	6.63	7.88	10.83	12.12
2	1.83	2.41	3.22	4.61	5.99	7.38	9.21	10.60	13.82	15.20
3	2.95	3.66	4.64	6.25	7.81	9.35	11.34	12.84	16.27	17.73
4	4.04	4.88	5.99	7.78	9.49	11.14	13.28	14.86	18.47	20.00
5	5.13	6.06	7.29	9.24	11.07	12.83	15.09	16.75	20.52	22.11
6	6.21	7.23	8.56	10.64	12.59	14.45	16.81	18.55	22.46	24.10
7	7.28	8.38	9.80	12.02	14.07	16.01	18.48	20.28	24.32	26.02
8	8.35	9.52	11.03	13.36	15.51	17.53	20.09	21.95	26.12	27.87
9	9.41	10.66	12.24	14.68	16.92	19.02	21.67	23.59	27.88	29.67
10	10.47	11.78	13.44	15.99	18.31	20.48	23.21	25.19	29.59	31.42
11	11.53	12.90	14.63	17.28	19.68	21.92	24.72	26.76	31.14	33.14
12	12.58	14.01	15.81	18.55	21.03	23.34	26.22	28.30	32.91	34.82
13	13.64	15.12	16.98	19.81	22.36	24.74	27.69	29.82	34.53	36.48
14	14.69	16.22	18.15	21.06	23.68	26.12	29.14	31.32	36.12	38.11
15	15.73	17.32	19.31	22.31	25.00	27.49	30.58	32.80	37.70	39.79
16	16.78	18.42	20.47	23.54	26.30	28.85	32.00	34.27	39.25	41.31
17	17.82	19.51	21.61	24.77	27.59	30.19	33.41	35.72	40.79	42.88
18	18.87	20.60	22.76	25.99	28.87	31.53	34.81	37.16	42.31	44.43
19	19.91	21.69	23.90	27.20	30.14	32.85	36.19	38.58	43.82	45.97
20	20.95	22.77	25.04	28.41	31.41	34.17	37.57	40.00	45.31	47.50
21	21.99	23.86	26.17	29.62	32.67	35.48	38.93	41.40	46.80	49.01
22	23.03	24.94	27.30	30.81	33.92	36.78	40.29	42.80	48.27	50.51
23	24.07	26.02	28.43	32.01	35.17	38.08	41.64	44.18	49.73	52.00
24	25.11	27.10	29.55	33.20	36.42	39.36	42.98	45.56	51.18	53.48
25	26.14	28.17	30.68	34.38	37.65	40.65	44.31	46.93	52.62	54.95
26	27.18	29.25	31.79	35.56	38.89	41.92	45.64	48.29	54.05	56.41
27	28.21	30.32	32.91	36.74	40.11	43.19	46.96	49.64	55.48	57.86
28	29.25	31.39	34.03	37.92	41.34	44.46	48.28	50.99	56.89	59.30
29	30.28	32.46	35.14	39.09	42.56	45.72	49.59	52.34	58.30	60.73
30	31.32	33.53	36.25	40.26	43.77	46.98	50.89	53.67	59.70	62.16
40	41.62	44.16	47.27	51.81	55.76	59.34	63.69	66.77	73.40	76.09
50	51.89	54.72	58.16	63.17	67.50	71.42	76.15	79.49	86.66	89.56
60	62.13	65.23	68.97	74.40	79.08	83.30	88.38	91.95	99.61	102.69
100	102.95	106.91	111.67	118.50	124.34	129.56	135.81	140.17	149.45	153.17

(9.4) Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
 If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1-p) \geq n-k)$.

n	k	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	p
2	0	0.9025	0.8100	0.7225	0.6400	0.5625	0.4900	0.4225	0.3600	0.3025	0.2500	
3	0	0.9575	0.9900	0.9775	0.9600	0.9375	0.9100	0.8775	0.8400	0.7975	0.7500	
4	0	0.8574	0.7290	0.6141	0.5120	0.4219	0.3430	0.2746	0.2160	0.1664	0.1250	
5	0	0.9928	0.9720	0.9392	0.8960	0.8438	0.7840	0.7183	0.6480	0.5747	0.5000	
6	0	0.9999	0.9990	0.9966	0.9920	0.9844	0.9730	0.9571	0.9360	0.9089	0.8750	
7	0	0.8145	0.6561	0.5220	0.4096	0.3164	0.2401	0.1785	0.1296	0.0915	0.0625	
8	0	0.9860	0.9477	0.8905	0.8192	0.7383	0.6517	0.5630	0.4752	0.3910	0.3125	
9	0	0.9995	0.9963	0.9880	0.9728	0.9492	0.9163	0.8735	0.8208	0.7585	0.6875	
10	0	0.7738	0.5905	0.4437	0.3277	0.2373	0.1681	0.1160	0.0778	0.0503	0.0313	
11	0	0.9774	0.9185	0.8352	0.7373	0.6328	0.5282	0.4284	0.3370	0.2562	0.1875	
12	0	0.9988	0.9914	0.9734	0.9421	0.8965	0.8369	0.7648	0.6826	0.5931	0.5000	
13	0	0.9999	0.9995	0.9978	0.9933	0.9844	0.9692	0.9460	0.9130	0.8688	0.8125	
14	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
15	0	0.7351	0.5314	0.3771	0.2621	0.1780	0.1176	0.0754	0.0467	0.0277	0.0156	
16	0	0.9672	0.8857	0.7765	0.6554	0.5339	0.4202	0.3191	0.2333	0.1636	0.1094	
17	0	0.9978	0.9842	0.9527	0.9011	0.8306	0.7443	0.6471	0.5443	0.4415	0.3438	
18	0	0.9999	0.9987	0.9941	0.9830	0.9624	0.9295	0.8826	0.8208	0.7447	0.6563	
19	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
20	0	0.6983	0.4783	0.3206	0.2097	0.1335	0.0824	0.0490	0.0280	0.0152	0.0078	
21	0	0.9556	0.8503	0.7166	0.5767	0.4449	0.3294	0.2338	0.1586	0.1024	0.0625	
22	0	0.9962	0.9743	0.9262	0.8520	0.7564	0.6471	0.5323	0.4199	0.3124	0.2266	
23	0	0.9998	0.9973	0.9879	0.9667	0.9294	0.8740	0.8002	0.7102	0.6083	0.5000	
24	0	1.0000	0.9998	0.9988	0.9953	0.9871	0.9712	0.9444	0.9037	0.8471	0.7734	
25	0	1.0000	1.0000	0.9999	0.9996	0.9987	0.9962	0.9910	0.9812	0.9643	0.9375	
26	0	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9984	0.9963	0.9922	0.9900	
27	0	0.6634	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084	0.0039	
28	0	1.0000	0.8131	0.6572	0.5033	0.3671	0.2553	0.1691	0.1064	0.0632	0.0352	
29	0	2.0000	0.9619	0.8948	0.7969	0.6785	0.5518	0.4278	0.3154	0.2201	0.1445	
30	0	3.0000	0.9950	0.9786	0.8862	0.8059	0.7064	0.5941	0.4770	0.3633	0.2448	
31	0	4.0000	0.9996	0.9971	0.9896	0.9727	0.9420	0.8939	0.8263	0.7396	0.6367	
32	0	5.0000	1.0000	0.9998	0.9988	0.9887	0.9747	0.9502	0.9115	0.8555	0.7733	
33	0	6.0000	1.0000	1.0000	0.9999	0.9987	0.9964	0.9915	0.9819	0.9643	0.9399	
34	0	7.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9993	0.9983	0.9961	0.9941	
35	0	8.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
36	0	9.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
37	0	10.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
38	0	11.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
39	0	12.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	

Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
 If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1-p) \geq n-k)$.

n	k	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	p
11	0	0.5688	0.3138	0.1673	0.0859	0.0422	0.0198	0.0088	0.0036	0.0014	0.0005	
12	0	0.5404	0.2824	0.1422	0.0687	0.0317	0.0138	0.0057	0.0022	0.0008	0.0002	
13	0	0.5040	0.2242	0.1087	0.0544	0.0274	0.0121	0.0055	0.0023	0.0009	0.0003	
14	0	0.4816	0.1659	0.0843	0.0443	0.0274	0.0158	0.0085	0.0042	0.0016	0.0003	
15	0	0.4540	0.1122	0.0567	0.0317	0.0187	0.0103	0.0057	0.0022	0.0008	0.0002	
16	0	0.4204	0.0788	0.0419	0.0247	0.0145	0.0085	0.0042	0.0016	0.0003	0.0001	
17	0	0.3984	0.0581	0.0316	0.0187	0.0111	0.0065	0.0034	0.0013	0.0003	0.0001	
18	0	0.3774	0.0419	0.0247	0.0145	0.0085	0.0042	0.0022	0.0008	0.0002	0.0001	
19	0	0.3563	0.0316	0.0187	0.0111	0.0065	0.0034	0.0013	0.0003	0.0001	0.0001	
20	0	0.3352	0.0247	0.0145	0.0085	0.0042	0.0022	0.0011	0.0003	0.0001	0.0001	
21	0	0.3142	0.0187	0.0111	0.0065	0.0034	0.0013	0.0004	0.0001	0.0001	0.0001	
22	0	0.2932	0.0145	0.0085	0.0042	0.0022	0.0011	0.0004	0.0001	0.0001	0.0001	
23	0	0.2722	0.0111	0.0065	0.0034	0.0013	0.0004	0.0001	0.0001	0.0001	0.0001	
24	0	0.2512	0.0085	0.0042	0.0022	0.0011	0.0004	0.0001	0.0001	0.0001	0.0001	
25	0	0.2302	0.0065	0.0034	0.0013	0.0004	0.0001	0.0001	0.0001	0.0001	0.0001	
26	0	0.2092	0.0042	0.0022	0.0011	0.0004	0.0001	0.0001	0.0001	0.0001	0.0001	
27	0	0.1882	0.0022	0.0011	0.0004	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
28	0	0.1672	0.0011	0.0004	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
29	0	0.1462	0.0004	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
30	0	0.1252	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
31	0	0.1042	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
32	0	0.0832	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
33	0	0.0622	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
34	0	0.0412	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
35	0	0.0202	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
36	0	0.0092	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
37	0	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	

Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1-p) \geq n-k)$.

n	k	0.05	0.10	0.15	0.20	0.25	p	0.30	0.35	0.40	0.45	0.50
14	0	0.4877	0.2288	0.1028	0.0440	0.0178	0.0068	0.0024	0.0008	0.0002	0.0001	
1	0.8470	0.5284	0.3567	0.1979	0.1010	0.0475	0.0205	0.0081	0.0009			
2	0.9699	0.8416	0.6479	0.4481	0.2811	0.1608	0.0839	0.0398	0.0170	0.0065		
3	0.9558	0.9559	0.8535	0.6982	0.5213	0.3552	0.2205	0.1243	0.0632	0.0287	0.0123	
4	0.9996	0.9908	0.9533	0.8702	0.7415	0.5842	0.4227	0.2793	0.1672	0.0898	0.0464	
5	1.0000	0.9985	0.9885	0.9561	0.8883	0.7805	0.6405	0.4859	0.3373	0.2120	0.1260	0.0596
6	1.0000	0.9998	0.9978	0.9884	0.9617	0.9067	0.8164	0.6925	0.5461	0.3953	0.2348	0.0245
7	1.0000	1.0000	0.9997	0.9976	0.9897	0.9685	0.9247	0.8499	0.7414	0.6047	0.4478	0.2902
8	1.0000	1.0000	1.0000	0.9996	0.9978	0.9917	0.9757	0.8811	0.7830	0.6405	0.4743	0.3145
9	1.0000	1.0000	1.0000	1.0000	0.9997	0.9983	0.9940	0.9825	0.9574	0.9102	0.8011	0.6626
10	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9989	0.9961	0.9886	0.9713	0.9177	0.6855
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9978	0.9935	0.9777	0.8338
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9894	0.9283
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9755
14	0	0.4633	0.2059	0.0874	0.0352	0.0134	0.0047	0.0016	0.0005	0.0001	0.0000	
1	0.8290	0.5490	0.3186	0.1671	0.0802	0.0353	0.0142	0.0052	0.0017	0.0005		
2	0.9638	0.8159	0.6042	0.3980	0.2361	0.1268	0.0617	0.0271	0.0107	0.0037		
3	0.9445	0.9444	0.8227	0.6482	0.4613	0.2969	0.1727	0.0905	0.0424	0.0176		
4	0.9994	0.9873	0.9383	0.8358	0.6865	0.5155	0.3519	0.2173	0.1204	0.0592		
5	0.9999	0.9978	0.9832	0.9389	0.8516	0.7216	0.5643	0.4032	0.2608	0.1509		
6	1.0000	0.9997	0.9964	0.9819	0.9434	0.8689	0.7548	0.6098	0.4522	0.3036		
7	1.0000	1.0000	0.9994	0.9958	0.9827	0.9500	0.8868	0.7869	0.6535	0.5000		
8	1.0000	1.0000	0.9999	0.9992	0.9958	0.9848	0.9578	0.9050	0.8182	0.6964		
9	1.0000	1.0000	1.0000	0.9990	0.9992	0.9963	0.9876	0.9662	0.9231	0.8491		
10	1.0000	1.0000	1.0000	0.9999	0.9993	0.9972	0.9907	0.9745	0.9408	0.8737		
11	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9981	0.9937	0.9824	0.8807		
12	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9989	0.9963	0.9790		
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9939		
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
15	0	0.4401	0.1853	0.0743	0.0281	0.0100	0.0033	0.0010	0.0003	0.0001	0.0000	
1	0.8108	0.5147	0.2839	0.1407	0.0635	0.0261	0.0098	0.0033	0.0010	0.0003		
2	0.9571	0.7892	0.5614	0.3518	0.1971	0.0994	0.0451	0.0183	0.0066	0.0021		
3	0.9360	0.9316	0.7899	0.5981	0.4050	0.2459	0.1339	0.0651	0.0281	0.0106		
4	0.9991	0.9830	0.9209	0.7982	0.6302	0.4499	0.2892	0.1666	0.0853	0.0384		
5	0.9999	0.9967	0.9765	0.9183	0.8103	0.6598	0.4900	0.3288	0.1976	0.1051		
6	1.0000	0.9995	0.9944	0.9733	0.9204	0.8247	0.6881	0.5272	0.3660	0.2272		
7	1.0000	0.9999	0.9930	0.9729	0.9256	0.8406	0.7161	0.5629	0.4018	0.2409		
8	1.0000	1.0000	0.9998	0.9985	0.9795	0.9743	0.9329	0.8579	0.7441	0.5982		
9	1.0000	1.0000	1.0000	0.9998	0.9924	0.9771	0.9417	0.8759	0.7728	0.6454		
10	1.0000	1.0000	1.0000	0.9997	0.9984	0.9938	0.9809	0.9514	0.8949	0.7439		
11	1.0000	1.0000	1.0000	1.0000	0.9997	0.9987	0.9951	0.9851	0.9616	0.8180		
12	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9991	0.9965	0.9894	0.8988		
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998		
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		

Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1-p) \geq n-k)$.

n	k	0.05	0.10	0.15	0.20	0.25	p	0.30	0.35	0.40	0.45	0.50
17	0	0.4181	0.1668	0.0631	0.0225	0.0075	0.0023	0.0007	0.0002	0.0000	0.0000	
1	0.7922	0.4818	0.2525	0.1182	0.0501	0.0193	0.0067	0.0021	0.0006	0.0001		
2	0.9497	0.7618	0.5198	0.3096	0.1637	0.0774	0.0327	0.0123	0.0041	0.0012		
3	0.9912	0.9174	0.7556	0.5489	0.3530	0.2019	0.1028	0.0464	0.0184	0.0064		
4	0.9988	0.9779	0.9013	0.7582	0.5739	0.3887	0.2348	0.1260	0.0596	0.0245		
5	0.9999	0.9953	0.9681	0.8943	0.7653	0.5968	0.4197	0.2639	0.1471	0.0717		
6	1.0000	0.9992	0.9623	0.8929	0.7752	0.6188	0.4478	0.2902	0.1662			
7	1.0000	0.9999	0.9883	0.9598	0.8787	0.7872	0.6405	0.4743	0.3145			
8	1.0000	1.0000	0.9997	0.9974	0.9876	0.9597	0.9006	0.8011	0.6626	0.5000		
9	1.0000	1.0000	1.0000	0.9999	0.9999	0.9970	0.9894	0.9699	0.9283	0.7597		
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999		
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
16	0	0.3774	0.1351	0.0456	0.0144	0.0042	0.0011	0.0003	0.0001	0.0000		
1	0.7547	0.4203	0.1985	0.0829	0.0310	0.0104	0.0031	0.0008	0.0002	0.0000		
2	0.9335	0.7054	0.4413	0.2369	0.1113	0.0462	0.0170	0.0055	0.0015	0.0004		
3	0.9836	0.8850	0.6841	0.4551	0.2631	0.1332	0.0591	0.0230	0.0077	0.0022		
4	0.9980	0.9648	0.8556	0.6733	0.4654	0.2822	0.1500	0.0696	0.0280	0.0096		
5	0.9998	0.9914	0.9463	0.8369	0.6678	0.4739	0.2968	0.1629	0.0777	0.0318		
6	1.0000	0.9983	0.9837	0.9324	0.8251	0.6655	0.4812	0.3081	0.1727	0.0835		
7	1.0000	0.9997	0.9959	0.9767	0.9225	0.8180	0.6656	0.4878	0.3169	0.1796		
8	1.0000	1.0000	0.9992	0.9933	0.9713	0.9161	0.8145	0.6675	0.4940	0.3238		
9	1.0000	1.0000	0.9999	0.9974	0.9674	0.9125	0.8139	0.6710	0.4917	0.3238		
10	1.0000	1.0000	1.0000	0.9997	0.9797	0.9563	0.9115	0.8159	0.6762	0.4917		
11	1.0000	1.0000	1.0000	0.9995	0.9972	0.9886	0.9648	0.9129	0.8204	0.6822		
12	1.0000	1.0000	1.0000	1.0000	0.9999	0.9969	0.9884	0.9658	0.9165	0.7473		
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999		
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
16	0	0.3774	0.1351	0.0456	0.0144	0.0042	0.0011	0.0003	0.0001	0.0000		

(9.5) Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	μ									
k	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725	0.7358
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371	0.9197
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9865	0.9810
4	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977	0.9963	0.9956
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019	0.1827	0.1653	0.1496	0.1353
1	0.6890	0.6626	0.6268	0.5918	0.5578	0.5249	0.4932	0.4628	0.4337	0.4060
2	0.904	0.8795	0.8571	0.8335	0.8088	0.7834	0.7572	0.7306	0.7037	0.6767
3	0.9743	0.9662	0.9569	0.9463	0.9344	0.9212	0.9068	0.8913	0.8747	0.8571
4	0.9946	0.9923	0.9893	0.9857	0.9814	0.9763	0.9704	0.9636	0.9559	0.9473
5	0.9990	0.9985	0.9978	0.9968	0.9955	0.9940	0.9920	0.9896	0.9834	0.9775
6	0.9999	0.9997	0.9996	0.9994	0.9991	0.9987	0.9981	0.9974	0.9966	0.9955
7	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997	0.9996	0.9994	0.9992	0.9989
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
0	0.0408	0.0334	0.0273	0.0224	0.0183	0.0150	0.0123	0.0101	0.0082	0.0067
1	0.1712	0.1468	0.1257	0.1074	0.0916	0.0780	0.0663	0.0563	0.0477	0.0404
2	0.3799	0.3397	0.3027	0.2689	0.2381	0.2102	0.1851	0.1626	0.1425	0.1247
3	0.6025	0.5584	0.5152	0.4735	0.4335	0.3954	0.3594	0.3257	0.2942	0.2650
4	0.7806	0.7442	0.7064	0.6678	0.6288	0.5898	0.5512	0.5132	0.4763	0.4405
5	0.8946	0.8705	0.8441	0.8156	0.7851	0.7531	0.7199	0.6858	0.6510	0.6160
6	0.9554	0.9421	0.9267	0.9091	0.8893	0.8675	0.8436	0.8180	0.7908	0.7622
7	0.9832	0.9769	0.9692	0.9559	0.9489	0.9361	0.9214	0.9049	0.8867	0.8666
8	0.9943	0.9917	0.9883	0.9840	0.9786	0.9721	0.9642	0.9549	0.9442	0.9319
9	0.9995	0.9992	0.9987	0.9981	0.9972	0.9959	0.9943	0.9922	0.9896	0.9863
k	5.2	5.4	5.6	5.8	6.0	6.5	7.0	7.5	8.0	8.5
0	0.0055	0.0045	0.0037	0.0030	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002
1	0.0342	0.0289	0.0244	0.0206	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019
2	0.1088	0.0948	0.0824	0.0715	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093
3	0.2381	0.2133	0.1906	0.1700	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301
4	0.4061	0.3733	0.3422	0.3127	0.2851	0.2237	0.1730	0.1321	0.0966	0.0744
5	0.5809	0.5461	0.5119	0.4783	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496
6	0.7324	0.7017	0.6703	0.6384	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562
7	0.8449	0.8149	0.7970	0.7710	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856
8	0.9181	0.9027	0.8857	0.8672	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231
9	0.9603	0.9512	0.9409	0.9292	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530
10	0.9823	0.9775	0.9718	0.9651	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634
11	0.9904	0.9875	0.9841	0.9799	0.9761	0.9467	0.9208	0.8881	0.8487	0.8047
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	0.1225	0.1108	0.1003	0.0907	0.0821	0.0743	0.0672	0.0608	0.0550	0.0498
1	0.3796	0.3546	0.3309	0.3084	0.2873	0.2674	0.2487	0.2311	0.2146	0.1991
2	0.6496	0.6227	0.5960	0.5597	0.5348	0.5184	0.4936	0.4695	0.4460	0.4232
3	0.8386	0.8194	0.7787	0.7576	0.7360	0.7141	0.6919	0.6696	0.6474	0.6141
4	0.9379	0.9275	0.9162	0.9041	0.8912	0.8774	0.8477	0.8318	0.8153	0.7894
5	0.9796	0.9751	0.9700	0.9643	0.9580	0.9510	0.9433	0.9349	0.9258	0.9161
6	0.9941	0.9925	0.9884	0.9828	0.9794	0.9756	0.9713	0.9665	0.9623	0.9573
7	0.9985	0.9974	0.9967	0.9958	0.9947	0.9934	0.9919	0.9901	0.9881	0.9850
8	0.9997	0.9995	0.9994	0.9989	0.9985	0.9981	0.9976	0.9969	0.9962	0.9953
9	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996	0.9995	0.9994	0.9993	0.9992
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	9.0	9.5	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0	μ
0	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.0012	0.0008	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0062	0.0042	0.0028	0.0012	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
3	0.0212	0.0149	0.0103	0.0049	0.0023	0.0011	0.0005	0.0002	0.0001	0.0000	0.0000
4	0.0550	0.0403	0.0293	0.0151	0.0076	0.0037	0.0018	0.0009	0.0004	0.0002	0.0000
5	0.1157	0.0885	0.0671	0.0375	0.0203	0.0107	0.0055	0.0028	0.0014	0.0007	0.0000
6	0.2068	0.1649	0.1301	0.0786	0.0458	0.0259	0.0142	0.0076	0.0040	0.0021	0.0000
7	0.3239	0.2687	0.2202	0.1432	0.0895	0.0540	0.0316	0.0180	0.0100	0.0054	0.0000
8	0.4557	0.3918	0.3328	0.2320	0.1550	0.0998	0.0621	0.0374	0.0220	0.0126	0.0000
9	0.5874	0.5218	0.4579	0.3405	0.2424	0.1658	0.1094	0.0699	0.0433	0.0261	0.0000
10	0.7060	0.6453	0.5830	0.4599	0.3472	0.2517	0.1757	0.1185	0.0774	0.0491	0.0000
11	0.8030	0.7520	0.6968	0.5793	0.4616	0.3532	0.2600	0.1848	0.1270	0.0847	0.0000
12	0.8758	0.8364	0.7916	0.6887	0.5760	0.4631	0.3585	0.2676	0.1931	0.1350	0.0000
13	0.9261	0.8981	0.8645	0.7813	0.6815	0.5730	0.4644	0.3632	0.2745	0.2009	0.0000
14	0.9585	0.9400	0.9165	0.8540	0.7720	0.6751	0.5704	0.4657	0.3675	0.2808	0.0000
15	0.9780	0.9665	0.9513	0.9074	0.8444	0.7636	0.6694	0.5681	0.4667	0.3715	0.0000
16	0.9889	0.9823	0.9730	0.9441	0.8987	0.8355	0.7559	0.6641	0.5660	0.4677	0.0000
17	0.9947	0.9911	0.9857	0.9678	0.9370	0.8905	0.8272	0.7489	0.6593	0.5640	0.0000
18	0.9976	0.9957	0.9928	0.9823	0.9626	0.9302	0.8826	0.8195	0.7423	0.6550	0.0000
19	0.9989	0.9980	0.9965	0.9907	0.9787	0.9573	0.9235	0.8752	0.8122	0.7363	0.0000
20	0.9996	0.9991	0.9984	0.9953	0.9884	0.9750	0.9521	0.9170	0.8682	0.8055	0.0000
21	0.9998	0.9996	0.9993	0.9977	0.9939	0.9859	0.9712	0.9469	0.9108	0.8615	0.0000
22	0.9999	0.9999	0.9997	0.9990	0.9970	0.9924	0.9833	0.9673	0.9418	0.9047	0.0000
23	1.0000	0.9999	0.9999	0.9995	0.9985	0.9960	0.9907	0.9805	0.9633	0.9367	0.0000
24	1.0000	1.0000	1.0000	0.9998	0.9993	0.9980	0.9950	0.9888	0.9777	0.9594	0.0000
25	1.0000	1.0000	1.0000	0.9999	0.9997	0.9974	0.9938	0.9869	0.9748	0.9548	0.0000
26	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9987	0.9967	0.9925	0.9848	0.0000
27	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9983	0.9959	0.9912	0.0000
28	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9991	0.9978	0.9950	0.9900	0.0000
29	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9989	0.9973	0.9947	0.0000
30	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9986	0.9976	0.0000
31	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9993	0.9980	0.0000
32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.0000
33	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.0000
34	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.0000
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000