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Write down all necessary steps in solutions in order to receive as many points as possible.

1 (3 points)

A box contains 10 balls, of which 4 are black and 6 are white. Now 3 balls are randomly taken out from the box at the same time. The following three events are defined:

$$A = \{\text{there is exactly 1 black ball in these 3 chosen balls}\},$$

$$B = \{\text{there is at least 1 black ball in these 3 chosen balls}\},$$

$$C = \{\text{there is at least 1 white ball in these 3 chosen balls}\}.$$

(1.1) (1p) Find the probability $P(A)$.(1.2) (1p) Find the probability $P(B)$.(1.3) (1p) Find the probability $P(A \cup B \cup C)$.*Solution.* (1.1) It is from combination that

$$P(A) = P(1 \text{ black and } 2 \text{ white}) = \frac{\binom{4}{1} \cdot \binom{6}{2}}{\binom{10}{3}} = \frac{4 \cdot 15}{120} = \frac{1}{2} = 0.5.$$

(1.2)

$$P(B) = P(1 \text{ black and } 2 \text{ white}) + P(2 \text{ black and } 1 \text{ white}) + P(3 \text{ black and } 0 \text{ white})$$

$$= \frac{\binom{4}{1} \cdot \binom{6}{2}}{\binom{10}{3}} + \frac{\binom{4}{2} \cdot \binom{6}{1}}{\binom{10}{3}} + \frac{\binom{4}{3} \cdot \binom{6}{0}}{\binom{10}{3}} = \frac{60 + 36 + 4}{120} = \frac{5}{6} = 0.8333.$$

(1.3) It is clear that A is contained in B , therefore $A \cup B = B$. It implies that

$$P(A \cup B \cup C) = P(B \cup C) = 1 - P(B' \cap C') = 1 - P(\text{no black} \cap \text{no white}) = 1 - 0 = 1.$$

□

2 (3 points)

Let (X, Y) be a two dimensional discrete random variable with the following joint probability mass function $p(x, y)$:

$X \setminus Y$	-1	1
-1	0.2	a
1	0.1	0.3

The table tells that both X and Y take values -1 and 1 .(2.1) (0.5p) Find the value of a .(2.2) (1p) Find the conditional probability $P(X > 0 \mid Y < 0)$.(2.3) (0.5p) Find the mean $E(X)$ of X .(2.4) (1p) Are X and Y independent? Why?*Solution.* (2.1) Since the sum of $p(x, y)$ should be equal to 1, it holds that

$$1 = 0.2 + 0.1 + 0.3 + a \implies a = 0.4.$$

(2.2)

$$P(X > 0 \mid Y < 0) = \frac{P(X > 0 \cap Y < 0)}{P(Y < 0)} = \frac{P(X = 1 \cap Y = -1)}{P(Y = -1)} = \frac{0.1}{0.2 + 0.1} = \frac{1}{3}.$$

(2.3) The table for only X can be obtained as follows

X	-1	1
$p_X(x)$	0.6	0.4

The mean is

$$E(X) = -1 \cdot 0.6 + 1 \cdot 0.4 = -0.2.$$

(2.4) X and Y are NOT independent because $p(x, y) \neq p_X(x) \cdot p_Y(y)$. For example,

$$0.2 = p(-1, -1) \neq p_X(-1) \cdot p_Y(-1) = 0.6 \cdot 0.3 = 0.18.$$

□

3 (3 points)

Suppose that there is a program \mathbb{S} at LiU which has 1000 students in total. Now a professor in this program \mathbb{S} is planning to give a new AI course which is open to all these 1000 students. This AI course will be given only when there are at least 50 registered students. According to a survey, each student will register for this AI course with probability 4%. Assume that all these 1000 students are independent. What is the probability that the AI course will be given? (Namely, what is the probability that there are at least 50 registered students?).

Solution. **Method-1 (CLT):** Let $X_1, X_2, \dots, X_{1000}$ denote “register/non-register” for these 1000 students, then one can assume that $X_1, X_2, \dots, X_{1000}$ are independent and each has the same distribution as follows

X_i	0	1
$p(x)$	96%	4%

Then our aim is to find the probability $P(X_1 + X_2 + \dots + X_{1000} \geq 50)$. To this end, one needs to find the individual mean μ and individual variance σ^2 :

$$\mu = E(X_i) = 0 \cdot 96\% + 1 \cdot 4\% = 4\% = 0.04,$$

$$\sigma^2 = V(X_i) = E(X_i^2) - \mu^2 = (0^2 \cdot 96\% + 1^2 \cdot 4\%) - 4\%^2 = 0.04 - 0.0016 = 0.0384.$$

Therefore it is from CLT that

$$\begin{aligned} P(X_1 + X_2 + \dots + X_{1000} \geq 50) &= P\left(\frac{X_1 + X_2 + \dots + X_{1000}}{1000} \geq 50/1000\right) = P(\bar{X} \geq 0.05) \\ &= P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq \frac{0.05 - 0.04}{\sqrt{0.0384}/\sqrt{1000}}\right) = P(N(0, 1) \geq 1.61) = 1 - \Phi(1.61) = 1 - 0.9463 = 0.0537. \end{aligned}$$

Method-2 (Binomial): If we use X to denote the number of registered students, then X is a Binomial random variable: $X \sim \text{Bin}(1000, 4\%)$. Then we aim to find $P(X \geq 50)$. To this end, one uses normal approximation to Binomial (which is actually CLT) as follows:

$$\begin{aligned} P(X \geq 50) &= P(\text{Bin}(1000, 4\%) \geq 50) = P(N(1000 \cdot 4\%, 1000 \cdot 4\% \cdot (1 - 4\%)) \geq 50) = P(N(40, 38.4) \geq 50) \\ &= P(N(0, 1) \geq \frac{50 - 40}{\sqrt{38.4}}) = P(N(0, 1) \geq 1.61) = 1 - \Phi(1.61) = 1 - 0.9463 = 0.0537. \end{aligned}$$

□

4 (3 points)

Suppose that a population X has a probability density function as follows

$$f(x) = \begin{cases} \theta, & \text{if } 0 < x < 1, \\ 1 - \theta, & \text{if } 1 \leq x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

where $0 < \theta < 1$ is an unknown parameter. A sample $\{0.8, 1.6, 0.9\}$ is taken from this population.

(4.1) (1p) Find a point estimate $\hat{\theta}_{MM}$ of θ based on the method of moments.

(4.2) (2p) Find a point estimate $\hat{\theta}_{ML}$ of θ based on the maximum-likelihood method.

Solution. (4.1) We first compute the mean $E(X)$ of X as follows

$$E(X) = \int_{-\infty}^{+\infty} x \cdot f(x) dx = \int_0^1 x \cdot \theta dx + \int_1^2 x \cdot (1 - \theta) dx = \frac{1}{2}\theta + \frac{3}{2}(1 - \theta) = 1.5 - \theta.$$

Therefore, the equation $E(X) = \bar{x}$ gives (with $\bar{x} = \frac{0.8+1.6+0.9}{3} = 1.1$)

$$1.5 - \theta = \bar{x} \implies \hat{\theta}_{MM} = 1.5 - \bar{x} = 1.5 - 1.1 = 0.4.$$

(4.2) The likelihood function is

$$L(\theta) = f(x_1) \cdot f(x_2) \cdots \cdot f(x_n) = f(0.8) \cdot f(1.6) \cdot f(0.9) = \theta \cdot (1 - \theta) \cdot \theta = \theta^2 - \theta^3.$$

The derivative $L'(\theta) = 0$ gives

$$0 = L'(\theta) = 2\theta - 3\theta^2 \implies 0 = 2 - 3\theta \implies \hat{\theta}_{ML} = \frac{2}{3} = 0.6667.$$

□

5 (3 points)

Assume that starting salaries X of all LiU year-2025 graduates follow a normal distribution $X \sim N(\mu, \sigma^2)$, where μ is the average starting salary of all LiU year-2025 graduates. To study μ , a sample of 16 LiU year-2025 graduates is chosen. The average salary of these 16 graduates is 51000 SEK/month and the salary standard deviation of these 16 graduates is 2000 SEK/month. Construct a 95% two-sided confidence interval I_μ of μ based on this sample.

Solution. This is Case 1.3:

$$I_\mu = \bar{x} \mp t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}} = 51000 \mp t_{0.025}(15) \frac{2000}{\sqrt{16}} = 51000 \mp 2.13 \frac{2000}{4} = 51000 \mp 1065 = (49935, 52065).$$

□

6 (3 points)

Recently, a 30-day training program is launched in a Fitness Center claiming that any adult attending this program can loss weight more than 5 kg. To study such a claim, 3 persons are randomly chosen from this program and their weights (before and after the program) are recorded as follows:

	Person 1	Person 2	Person 3
weight before	91.6	96.4	88.2
weight after	82.6	91.7	86.1
difference (weight before - weight after)	9.0	4.7	2.1

Assume that the differences $\{9.0, 4.7, 2.1\}$ are regarded as a sample from a normal population $X \sim N(\mu, \sigma^2)$. Therefore, the claim seems to be correct if $\mu > 5$. To check whether or not $\mu > 5$, hypothesis testing (HT) will be used with a significance level $\alpha = 5\%$.

(6.1) (1p) Write down H_0 and H_a .

(6.2) (2p) Is H_0 rejected? Why?

Solution. (6.1)

$$\begin{cases} H_0 : \mu = 5 \\ H_a : \mu > 5 \end{cases}$$

(6.2) First, it can be obtained that

$$\bar{x} = \frac{9.0 + 4.7 + 2.1}{3} = 5.2667, \quad s^2 = \frac{1}{3-1} ((9.0 - 5.2667)^2 + (4.7 - 5.2667)^2 + (2.1 - 5.2667)^2) = 12.1433.$$

The Test Statistic and the Rejection Region can be computed as follows:

$$TS = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{5.2667 - 5}{\sqrt{12.1433}/\sqrt{3}} = 0.1326,$$

$$C = (t_{\alpha}(n-1), +\infty) = (t_{0.05}(3-1), +\infty) = (2.92, +\infty).$$

Since $TS \notin C$, we do NOT reject H_0 (that is, the sample does not provide any evidence that $\mu > 5$, namely the claim is NOT supported by the sample). \square

1. Basic probability

(1.1) Conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

(1.2) Total probability $P(B) = \sum_{i=1}^k P(B|A_i)P(A_i)$ where $\{A_i\}$ are disjoint and $\cup_{i=1}^k A_i = S$.

(1.3) Bayes' Theorem $P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^k P(B|A_i)P(A_i)}$ where $\{A_i\}$ are in (1.2).

2. Random variables (r.v.s)

(2.1) Discrete r.v. X has a pmf $p(x) = P(X = x)$ satisfying $p(x) \geq 0$ and $\sum p(x_i) = 1$,

X	x_1	x_2	\cdots	x_n	\cdots
$p(x)$	$p(x_1)$	$p(x_2)$	\cdots	$p(x_n)$	\cdots

Expectation (or *Expected value* or *mean*) $\mu_X = E(X) = \sum x_i p(x_i)$;
 Variance $\sigma_X^2 = V(X) = E(X - \mu_X)^2 = E(X^2) - \mu_X^2 = \sum x_i^2 p(x_i) - (\sum x_i p(x_i))^2$.

(2.2) Continuous r.v. X has a pdf $f(x)$ satisfying $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$,

$$P(a < X < b) = \int_a^b f(x)dx.$$

Expectation (or *Expected value* or *mean*) $\mu_X = E(X) = \int_{-\infty}^{\infty} xf(x)dx$;

Variance $\sigma_X^2 = V(X) = E(X - \mu_X)^2 = E(X^2) - \mu_X^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - (\int_{-\infty}^{\infty} xf(x)dx)^2$.

(2.3) Cumulative distribution function (cdf) of a r.v. X is $F(x) = P(X \leq x)$.

(2.4) X and Y are r.v.s, a , b and c are scalars, then

$$E(aX + bY + c) = aE(X) + bE(Y) + c,$$

$$V(aX + bY + c) = a^2V(X) + b^2V(Y) + 2ab \text{cov}(X, Y),$$

$$E(g(X, Y)) = \begin{cases} \sum_{i,j} g(x_i, y_j) \cdot p(x_i, y_j), & \text{for discrete } (X, Y), \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f(x, y)dxdy, & \text{for continuous } (X, Y). \end{cases}$$

(2.5) • Discrete r.v. (X, Y) has a joint pmf $p(x, y)$ satisfying $p(x, y) \geq 0$ and $\sum_{x_i} \sum_{y_i} p(x_i, y_i) = 1$.

The marginal pmf of X is $p_X(x) = \sum_y p(x, y)$;
 The marginal pmf of Y is $p_Y(y) = \sum_x p(x, y)$;

X and Y are *independent* if $p(x, y) = p_X(x) \cdot p_Y(y)$.

• Continuous r.v. (X, Y) has a joint pdf $f(x, y)$ satisfying $f(x, y) \geq 0$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$.

The marginal pdf of X is $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$;

The marginal pdf of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$;

X and Y are *independent* if $f(x, y) = f_X(x) \cdot f_Y(y)$.

3. Several special r.v.s

(3.1) $X \sim Bin(n, p)$ has a pmf $p(x) = P(X = x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$, $x = 0, 1, 2, \dots, n$.

$E(X) = n \cdot p$, $V(X) = n \cdot p \cdot (1-p)$.

(3.2) $X \sim Po(\lambda)$ has a pmf $p(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0, 1, 2, \dots$.

$E(X) = \lambda$, $V(X) = \lambda$.

(3.3) $X \sim Hypergeometric$ has a pmf $p(x) = P(X = x) = \frac{\binom{M}{x} \binom{N-M}{n-x}}{\binom{N}{n}}$.

(3.4) $X \sim Exp(\lambda)$ has a pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(3.5) $X \sim N(\mu, \sigma^2)$ has a pdf

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty.$$

$E(X) = \mu$, $V(X) = \sigma^2$

(3.6) $X \sim U(a, b)$ has a pdf

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

$$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{(b-a)^2}{12}.$$

4. Central Limit Theorem (CLT)

Suppose that a population has mean = μ and variance = σ^2 . A random sample $\{X_1, X_2, \dots, X_n\}$ from this population is given. Then for large $n \geq 30$,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

- If the population is normal, then (1) holds for any n .
- Note that $\mu = E(\bar{X})$ and $(\sigma/\sqrt{n})^2 = V(\bar{X})$.

5. Several notations in statistics

(5.1) Sample mean: $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n} = \frac{\sum X_i}{n}$; $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\sum x_i}{n}$.

(5.2) Sample variance:

$$S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1} = \frac{1}{n-1} \left(\sum X_i^2 - \frac{(\sum X_i)^2}{n} \right); \quad s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} = \frac{1}{n-1} \left(\sum x_i^2 - \frac{(\sum x_i)^2}{n} \right).$$

- Capital letters X and S^2 refer to the objects based on random sample (therefore they are in general r.v.s), while small letters \bar{x} and s^2 are the objects based on observations (so they are scalars).

(5.3) A point estimator of θ obtained by Maximum Likelihood method is denoted as $\hat{\theta}_{ML}$.

6. Confidence Interval (CI)

In this course, three types of confidence intervals are studied depending on the unknown population parameter(s): CI-1 (confidence intervals for population mean(s)), CI-2 (confidence intervals for population variance(s)), and CI-3 (confidence intervals for population proportion(s)).

CI-1: $(1 - \alpha)$ CI of a population mean μ

case 1.1 (any n) If population $X \sim N(\mu, \sigma^2)$ and σ^2 is known, then $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ and

$$I_\mu = \left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \right) := \bar{x} \mp z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}.$$

case 1.2 ($n \geq 30$) For any population X , it holds that $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ and

$$I_\mu = \bar{x} \mp z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \text{ or } I_\mu = \bar{x} \mp z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}.$$

case 1.3 (any n) If population $X \sim N(\mu, \sigma^2)$ and σ^2 is unknown, then $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim T(n-1)$ and

$$I_\mu = \bar{x} \mp t_{\alpha/2}(n-1) \cdot \frac{s}{\sqrt{n}}.$$

CI-1': $(1 - \alpha)$ CI of the difference of two population means $\mu_X - \mu_Y$

case 1.1' (any n_1, n_2) If independent populations $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, and σ_X^2, σ_Y^2 are known, then $\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}} \sim N(0, 1)$, and $I_{\mu_X - \mu_Y} = (\bar{x} - \bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}$.

case 1.2' ($n_1, n_2 \geq 30$) For any independent populations X and Y , it holds that

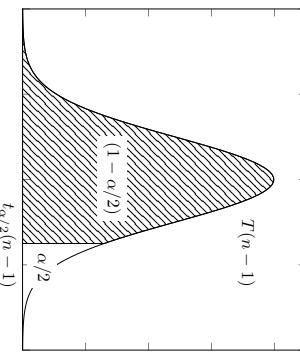
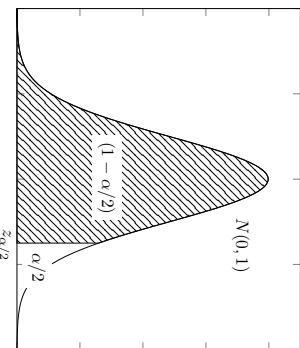
$$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}} \sim N(0, 1) \text{ and}$$

$$I_{\mu_X - \mu_Y} = (\bar{x} - \bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}} \text{ or } I_{\mu_X - \mu_Y} = (\bar{x} - \bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{\sigma}_X^2}{n_1} + \frac{\hat{\sigma}_Y^2}{n_2}}.$$

case 1.3' (any n_1, n_2) If independent populations $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, where σ_X^2, σ_Y^2 are unknown but $\sigma_X^2 = \sigma_Y^2$, then

$$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim T(n_1 + n_2 - 2), \text{ where } S^2 = \frac{(n_1 - 1)S_X^2 + (n_2 - 1)S_Y^2}{n_1 + n_2 - 2}, \text{ and}$$

$$I_{\mu_X - \mu_Y} = (\bar{x} - \bar{y}) \mp t_{\alpha/2}(n_1 + n_2 - 2) \cdot s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}.$$



CI-2: $(1 - \alpha)$ CI of population variance(s) σ^2

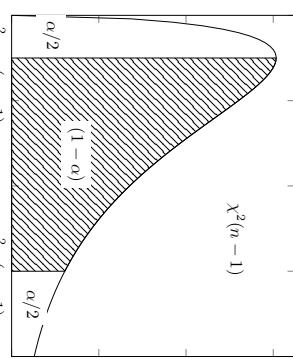
- If a population $X \sim N(\mu, \sigma^2)$ and σ^2 is unknown, then $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$, and

$$I_{\sigma^2} = \left(\frac{(n-1)s^2}{\chi_{\alpha/2}^2(n-1)}, \frac{(n-1)s^2}{\chi_{1-\alpha/2}^2(n-1)} \right).$$

- If two independent populations $X \sim N(\mu_X, \sigma^2)$ and $Y \sim N(\mu_Y, \sigma^2)$, and σ^2 is unknown, then $\frac{(n_1+n_2-2)S^2}{\sigma^2} \sim \chi^2(n_1 + n_2 - 2)$, and

$$I_{\sigma^2} = \left(\frac{(n_1 + n_2 - 2)s^2}{\chi_{\alpha/2}^2(n_1 + n_2 - 2)}, \frac{(n_1 + n_2 - 2)s^2}{\chi_{1-\alpha/2}^2(n_1 + n_2 - 2)} \right),$$

where $S^2 = \frac{(n_1-1)S_X^2 + (n_2-1)S_Y^2}{n_1 + n_2 - 2}$.



CI-3: $(1 - \alpha)$ CI of population proportion(s)

- If a (large) population has an unknown proportion p , then $\frac{\hat{p} - p}{\sqrt{p(1-p)/n}} \sim N(0, 1)$ if $n\hat{p} \geq 10$ and $n(1 - \hat{p}) \geq 10$ with $\hat{p} = x/n$, and $I_p = \hat{p} \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.
- If two independent (large) populations have unknown proportions p_1 and p_2 , then

$$\frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}} \sim N(0, 1)$$

if $n_i\hat{p}_i \geq 10$ and $n_i(1 - \hat{p}_i) \geq 10$ for $i = 1, 2$, and $I_{p_1 - p_2} = (\hat{p}_1 - \hat{p}_2) \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$.

7. Hypothesis Test (HT)

	H_0 is true	H_0 is false and $\theta = \theta_1$
reject H_0	(type I error or significance level) α	(power) $h(\theta_1) = 1 - h(\theta_1)$
don't reject H_0	1 - α	(type II error) $\beta(\theta_1) = 1 - h(\theta_1)$

reject $H_0 \Leftrightarrow TS \in C \Leftrightarrow p\text{-value} < \alpha$

χ^2 tests for populations (non-parametric tests)

Suppose that for a random sample of a population X , the n elements of it are classified into k disjoint groups $A_i, 1 \leq i \leq k$. For each group $A_i, 1 \leq i \leq k$, suppose that there are $N_i, 1 \leq i \leq k$ elements inside. Let $p_i = P(A_i)$ assuming a given distribution of X . Note that $p_1 + p_2 + \dots + p_k = 1$ and $N_1 + N_2 + \dots + N_k = n$. One wants to test the hypotheses

$$H_0 : P(A_i) = p_i, \quad 1 \leq i \leq k, \quad H_a : P(A_i) \neq p_i \text{ for some } 1 \leq i \leq k.$$

If n is large in the sense that $np_i \geq 5$ for all $1 \leq i \leq k$, then the test statistic is

$$\sum_{i=1}^k \frac{(N_i - np_i)^2}{np_i} \approx \chi^2(k-1).$$

Therefore the observation of the test statistic is

$$TS = \sum_{i=1}^k \frac{(n_i - np_i)^2}{np_i}, \text{ where } n_i \text{ is the observation of } N_i, 1 \leq i \leq k.$$

For the critical region C , one can take (note that if H_0 is true, then TS should be close to zero)

$$C = (\chi_\alpha^2(k-1), \infty).$$

The conclusion would be $TS \in C \iff H_0$ is rejected.

8. Linear and logistic regression

(Multiple) linear regression: $Y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + \varepsilon, \varepsilon \sim N(0, \sigma^2)$.

\bullet Y : response variable (which is normal r.v.), $\{x_1, \dots, x_k\}$: predictors (which are scalars).

\bullet sample: $\{(x_{11}, \dots, x_{1k}; y_1), (x_{21}, \dots, x_{2k}; y_2), \dots, (x_{n1}, \dots, x_{nk}; y_n)\}$.
 \bullet how to estimate $\beta_j \approx \hat{\beta}_j$: least square method, that is, to minimize $\sum_{i=1}^n (\hat{y}_i - y_i)^2$, where the estimated (multiple) linear regression line \hat{y} is

$$\hat{y} = \beta_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k.$$

$\bullet \frac{\hat{\beta}_j - \beta_j}{s.e(\hat{\beta}_j)} \sim T(n-k-1)$, this helps determine whether or not the real $\beta_j = 0$?

$\bullet \sigma^2 \approx \frac{SSE}{n-k-1}$, this gives an estimation of the size of the error.

$\bullet R^2 = \frac{SSE}{SS_T}$ this gives how well the model is (if $R^2 \approx 1$, then the model fits the sample very well).

\bullet How to test $\beta_1 = \dots = \beta_k = 0$? Use the random variable $\frac{SS_R/k}{SSE/(n-k-1)} \sim F(k, n-k-1)$.

Logistic regression: Let Y can only take 0 or 1 with $P(Y=1) = p$ and $P(Y=0) = 1-p$,

$$E(Y) = p(x_1, \dots, x_k) = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}.$$

$\bullet Y$: response variable (which is Bernoulli r.v. $P(Y=1) = p$ and $P(Y=0) = 1-p$, so $E(Y) = p$), $\{x_1, \dots, x_k\}$: predictors (which are scalars).

\bullet sample: $\{(x_{11}, \dots, x_{1k}; y_1), (x_{21}, \dots, x_{2k}; y_2), \dots, (x_{n1}, \dots, x_{nk}; y_n)\}$.

\bullet how to estimate $\beta_j \approx \hat{\beta}_j$: maximal likelihood method (maximize $\prod_{i=1}^n p(x_{i1}, \dots, x_{ik})^{y_i} (1 - p(x_{i1}, \dots, x_{ik}))^{1-y_i}$).

$\bullet \frac{\hat{\beta}_j - \beta_j}{s.e(\hat{\beta}_j)} \approx N(0, 1)$ for large $n \geq 30$, this helps determine whether or not the real $\beta_j = 0$?

\bullet Classification of a new object $Y(x_1, \dots, x_k)$ as 1 or 0 according

$$Y(x_1, \dots, x_k) = \begin{cases} 1, & \text{if } \hat{p}(x_1, \dots, x_k) \geq 0.5, \\ 0, & \text{if } \hat{p}(x_1, \dots, x_k) < 0.5, \end{cases}$$

where the estimated logit function $\hat{p}(x_1, \dots, x_k)$ is

$$\hat{p}(x_1, \dots, x_k) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k}}.$$

9. Tables

(9.1) Table for $N(0,1)$ standard normal random variable $\Phi(x) = P(N(0,1) \leq x)$, $x \geq 0$.

There is an important relation $\Phi(-x) = 1 - \Phi(x)$, $x \geq 0$.

x	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7957	0.8023	0.8051	0.8078	0.8106	0.8133	0.8160
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9222	0.9256	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	0.9330
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0.9965
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0.9981	0.9981
2.9	0.9981	0.9982	0.9983	0.9984	0.9984	0.9985	0.9986	0.9986	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9990	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9993	0.9993	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(9.2) Table for $T(f)$ random variable $F(x) = P(T(f) \leq x)$, where f is a parameter called 'degrees of freedom'.

f	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.9995	$F(x)$
1	1.00	3.08	6.31	12.71	31.82	63.66	127.32	636.62	
2	0.82	1.89	2.92	4.30	6.96	9.92	14.09	31.60	
3	0.76	1.64	2.35	3.18	4.54	5.84	7.45	12.92	
4	0.74	1.53	2.13	2.78	3.75	4.60	5.60	8.61	
5	0.73	1.48	2.02	2.57	3.36	4.03	4.77	6.87	
6	0.72	1.44	1.94	2.45	3.14	3.71	4.32	5.96	
7	0.71	1.41	1.89	2.36	3.00	3.50	4.03	5.41	
8	0.71	1.40	1.86	2.31	2.90	3.36	3.83	5.04	
9	0.70	1.38	1.83	2.26	2.82	3.25	3.69	4.78	
10	0.70	1.37	1.81	2.23	2.76	3.17	3.58	4.59	
11	0.70	1.36	1.80	2.20	2.72	3.11	3.50	4.44	
12	0.70	1.36	1.78	2.18	2.68	3.05	3.43	4.32	
13	0.69	1.35	1.77	2.16	2.65	3.01	3.37	4.22	
14	0.69	1.35	1.76	2.14	2.62	2.98	3.33	4.14	
15	0.69	1.34	1.75	2.13	2.60	2.95	3.29	4.07	
16	0.69	1.34	1.75	2.12	2.58	2.92	3.25	4.01	
17	0.69	1.33	1.74	2.11	2.57	2.90	3.22	3.97	
18	0.69	1.33	1.73	2.10	2.55	2.88	3.20	3.92	
19	0.69	1.33	1.73	2.09	2.54	2.86	3.17	3.88	
20	0.69	1.33	1.72	2.09	2.53	2.85	3.15	3.85	
21	0.69	1.32	1.72	2.08	2.52	2.83	3.14	3.82	
22	0.69	1.32	1.72	2.07	2.51	2.82	3.12	3.79	
23	0.69	1.32	1.71	2.07	2.50	2.81	3.10	3.77	
24	0.68	1.32	1.71	2.06	2.49	2.80	3.09	3.75	
25	0.68	1.32	1.71	2.06	2.49	2.79	3.08	3.73	
26	0.68	1.31	1.71	2.06	2.48	2.78	3.07	3.71	
27	0.68	1.31	1.70	2.05	2.47	2.77	3.06	3.69	
28	0.68	1.31	1.70	2.05	2.47	2.76	3.05	3.67	
29	0.68	1.31	1.70	2.05	2.46	2.76	3.04	3.66	
30	0.68	1.31	1.70	2.04	2.46	2.75	3.03	3.65	
40	0.68	1.30	1.68	2.02	2.42	2.70	2.97	3.55	
50	0.68	1.30	1.68	2.01	2.40	2.68	2.94	3.50	
60	0.68	1.30	1.67	2.00	2.39	2.66	2.91	3.46	
100	0.68	1.29	1.66	1.98	2.36	2.63	2.87	3.39	
∞	0.67	1.28	1.65	1.96	2.33	2.58	2.81	3.29	

(9.3) Table for $\chi^2(f)$ random variable $F(x) = P(\chi^2(f) \leq x)$, where f is a parameter.

f	$F(x)$										
	0.005	0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.30	0.40	0.50
1	0.00	0.00	0.00	0.00	0.00	0.02	0.06	0.15	0.27	0.45	1
2	0.00	0.00	0.01	0.02	0.05	0.10	0.21	0.45	0.71	1.39	2
3	0.02	0.02	0.07	0.11	0.22	0.35	0.58	1.01	1.42	2.37	3
4	0.06	0.09	0.21	0.30	0.48	0.71	1.06	1.65	2.19	2.75	4
5	0.16	0.21	0.41	0.55	0.83	1.15	1.61	2.34	3.00	3.66	5
6	0.30	0.38	0.68	0.87	1.24	1.64	2.20	3.07	3.83	4.57	6
7	0.48	0.60	0.99	1.24	1.69	2.17	2.83	3.82	4.67	5.49	7
8	0.71	0.86	1.34	1.65	2.18	2.73	3.49	4.59	5.53	6.42	8
9	0.97	1.15	1.73	2.09	2.70	3.33	4.17	5.38	6.39	7.36	9
10	1.26	1.48	2.16	2.56	3.25	3.94	4.87	6.18	7.27	8.30	10
11	1.59	1.83	2.60	3.05	3.82	4.57	5.58	6.99	8.15	9.24	11
12	1.93	2.21	3.07	3.57	4.40	5.23	6.30	7.81	9.03	10.18	12
13	2.31	2.62	3.57	4.11	5.01	5.89	7.04	8.63	9.93	11.13	13
14	2.70	3.04	4.07	4.66	5.63	6.57	7.79	9.47	10.82	12.08	14
15	3.11	3.48	4.60	5.23	6.26	7.26	8.55	10.31	11.72	13.03	15
16	3.54	3.94	5.14	5.81	6.91	7.96	9.31	11.15	12.62	13.98	16
17	3.98	4.42	5.70	6.41	7.56	8.67	10.09	12.00	13.53	14.94	17
18	4.44	4.90	6.26	7.01	8.23	9.39	10.86	12.86	14.44	15.89	18
19	4.91	5.41	6.84	7.63	8.91	10.12	11.65	13.72	15.35	16.85	19
20	5.40	5.92	7.43	8.26	9.59	10.85	12.44	14.58	16.27	17.81	20
21	5.90	6.45	8.03	8.90	10.28	11.59	13.24	15.44	17.18	18.77	21
22	6.40	6.98	8.64	9.54	10.98	12.34	14.04	16.31	18.10	19.73	22
23	6.92	7.53	9.26	10.20	11.69	13.09	14.85	17.19	19.02	20.69	23
24	7.45	8.08	9.89	10.86	12.40	13.85	15.66	18.06	19.94	21.65	24
25	7.99	8.65	10.52	11.52	13.12	14.61	16.47	18.94	20.87	22.62	25
26	8.54	9.22	11.16	12.20	13.84	15.38	17.29	19.82	21.79	23.58	26
27	9.09	9.80	11.81	12.88	14.57	16.15	18.11	20.70	22.72	24.54	27
28	9.66	10.39	12.46	13.56	15.31	16.93	18.94	21.59	23.65	25.51	28
29	10.23	10.99	13.12	14.26	16.05	17.71	19.77	22.48	24.58	26.48	29
30	10.80	11.59	13.79	14.95	16.79	18.49	20.60	23.36	25.51	27.44	30
40	16.91	17.92	20.71	22.16	24.43	26.51	29.05	32.34	34.87	37.13	40
50	23.46	24.67	27.99	29.71	32.36	34.76	37.69	41.45	44.31	46.86	50
60	30.34	31.74	35.53	37.48	40.48	43.19	46.46	50.64	53.81	56.62	60
100	59.90	61.92	67.33	70.06	74.22	77.93	82.36	87.95	92.13	95.81	100

Table for $\chi^2(f)$ random variable $F(x) = P(\chi^2(f) \leq x)$, where f is a parameter.

f	$F(x)$									
	0.60	0.70	0.80	0.90	0.95	0.975	0.99	0.995	0.999	0.9995
1	0.71	1.07	1.64	2.71	3.84	5.02	6.63	7.88	10.83	12.12
2	1.83	2.41	3.22	4.61	5.99	7.38	9.21	10.60	13.82	15.20
3	2.95	3.66	4.64	6.25	7.81	9.35	11.34	12.84	16.27	17.73
4	4.04	4.88	5.99	7.78	9.49	11.14	13.28	14.86	18.47	20.00
5	5.13	6.06	7.29	9.24	11.07	12.83	15.09	16.75	20.52	22.11
6	6.21	7.23	8.56	10.64	12.59	14.45	16.81	18.55	22.46	24.10
7	7.28	8.38	9.80	12.02	14.07	16.01	18.48	20.28	24.32	26.02
8	8.35	9.52	11.03	13.36	15.51	17.53	20.09	21.95	26.12	27.87
9	9.41	10.66	12.24	14.68	16.92	19.02	21.67	23.59	27.88	29.67
10	10.47	11.78	13.44	15.99	18.31	20.48	23.21	25.19	29.59	31.42
11	11.53	12.90	14.63	17.28	19.68	21.92	24.72	26.76	31.14	33.14
12	12.58	14.01	15.81	18.55	21.03	23.34	26.22	28.30	32.91	34.82
13	13.64	15.12	16.98	19.81	22.36	24.74	27.69	29.82	34.53	36.48
14	14.69	16.22	18.15	21.06	23.68	26.12	29.14	31.32	36.12	38.11
15	15.73	17.32	19.31	22.31	25.00	27.49	30.58	32.80	37.70	39.79
16	16.78	18.42	20.47	23.54	26.30	28.85	32.00	34.27	39.25	41.31
17	17.82	19.51	21.61	24.77	27.59	30.19	33.41	35.72	40.79	42.88
18	18.87	20.60	22.76	25.99	28.87	31.53	34.81	37.16	42.31	44.43
19	19.91	21.69	23.90	27.20	30.14	32.85	36.19	38.58	43.82	45.97
20	20.95	22.77	25.04	28.41	31.41	34.17	37.57	40.00	45.31	47.50
21	21.99	23.86	26.17	29.62	32.67	35.48	38.93	41.40	46.80	49.01
22	23.03	24.94	27.30	30.81	33.92	36.78	40.29	42.80	48.27	50.51
23	24.07	26.02	28.43	32.01	35.17	38.08	41.64	44.18	49.73	52.00
24	25.11	27.10	29.55	33.20	36.42	39.36	42.98	45.56	51.18	53.48
25	26.14	28.17	30.68	34.38	37.65	40.65	44.31	46.93	52.62	54.95
26	27.18	29.25	31.79	35.56	38.89	41.92	45.64	48.29	54.05	56.41
27	28.21	30.32	32.91	36.74	40.11	43.19	46.96	49.64	55.48	57.86
28	29.25	31.39	34.03	37.92	41.34	44.46	48.28	50.99	56.89	59.30
29	30.28	32.46	35.14	39.09	42.56	45.72	49.59	52.34	58.30	60.73
30	31.32	33.53	36.25	40.26	43.77	46.98	50.89	53.67	59.70	62.16
40	41.62	44.16	47.27	51.81	55.76	59.34	63.69	66.77	73.40	76.09
50	51.89	54.72	58.16	63.17	67.50	71.42	76.15	79.49	86.66	89.56
60	62.13	65.23	68.97	74.40	79.08	83.30	88.38	91.95	99.61	102.69
100	102.95	106.91	111.67	118.50	124.34	129.56	135.81	140.17	149.45	153.17

(9.4) Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
 If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1-p) \geq n-k)$.

n	k	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	p
2	0	0.9025	0.8100	0.7225	0.6400	0.5625	0.4900	0.4225	0.3600	0.3025	0.2500	
3	0	0.9575	0.9900	0.9775	0.9600	0.9375	0.9100	0.8775	0.8400	0.7975	0.7500	
4	0	0.8574	0.7290	0.6141	0.5120	0.4219	0.3430	0.2746	0.2160	0.1664	0.1250	
5	0	0.9928	0.9720	0.9392	0.8960	0.8438	0.7840	0.7183	0.6480	0.5747	0.5000	
6	0	0.9999	0.9990	0.9966	0.9920	0.9844	0.9730	0.9571	0.9360	0.9089	0.8750	
7	0	0.8145	0.6561	0.5220	0.4096	0.3164	0.2401	0.1785	0.1296	0.0915	0.0625	
8	0	0.9860	0.9477	0.8905	0.8192	0.7383	0.6517	0.5630	0.4752	0.3910	0.3125	
9	0	0.9995	0.9963	0.9880	0.9728	0.9492	0.9163	0.8735	0.8208	0.7585	0.6875	
10	0	0.7738	0.5905	0.4437	0.3277	0.2373	0.1681	0.1160	0.0778	0.0503	0.0313	
11	0	0.9774	0.9185	0.8352	0.7373	0.6328	0.5282	0.4284	0.3370	0.2562	0.1875	
12	0	0.9988	0.9914	0.9734	0.9421	0.8965	0.8369	0.7648	0.6826	0.5931	0.5000	
13	0	0.9999	0.9995	0.9978	0.9933	0.9844	0.9692	0.9460	0.9130	0.8688	0.8125	
14	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
15	0	0.7351	0.5314	0.3771	0.2621	0.1780	0.1176	0.0754	0.0467	0.0277	0.0156	
16	0	0.9672	0.8857	0.7765	0.6554	0.5339	0.4202	0.3191	0.2333	0.1636	0.1094	
17	0	0.9978	0.9842	0.9527	0.9011	0.8306	0.7443	0.6471	0.5443	0.4415	0.3438	
18	0	0.9999	0.9987	0.9941	0.9830	0.9624	0.9295	0.8826	0.8208	0.7447	0.6563	
19	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
20	0	0.6983	0.4783	0.3206	0.2097	0.1335	0.0824	0.0490	0.0280	0.0152	0.0078	
21	0	0.9556	0.8503	0.7166	0.5767	0.4449	0.3294	0.2338	0.1586	0.1024	0.0625	
22	0	0.9962	0.9743	0.9262	0.8520	0.7564	0.6471	0.5323	0.4199	0.3124	0.2266	
23	0	0.9998	0.9973	0.9879	0.9667	0.9294	0.8740	0.8002	0.7102	0.6083	0.5000	
24	0	1.0000	0.9998	0.9988	0.9953	0.9871	0.9712	0.9444	0.9037	0.8471	0.7734	
25	0	1.0000	1.0000	0.9999	0.9996	0.9987	0.9962	0.9910	0.9812	0.9643	0.9375	
26	0	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9984	0.9963	0.9922	0.9900	
27	0	0.6634	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084	0.0039	
28	0	1.0000	0.8131	0.6572	0.5033	0.3671	0.2553	0.1691	0.1064	0.0632	0.0352	
29	0	2.0000	0.9619	0.8948	0.7969	0.6785	0.5518	0.4278	0.3154	0.2201	0.1445	
30	0	3.0000	0.9950	0.9786	0.8862	0.8059	0.7064	0.5941	0.4770	0.3633	0.2700	
31	0	4.0000	0.9996	0.9971	0.9896	0.9727	0.9420	0.8939	0.8263	0.7396	0.6367	
32	0	5.0000	1.0000	0.9998	0.9988	0.9887	0.9747	0.9502	0.9115	0.8555	0.7700	
33	0	6.0000	1.0000	1.0000	0.9999	0.9987	0.9964	0.9915	0.9819	0.9643	0.9300	
34	0	7.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9993	0.9983	0.9961	0.9900	
35	0	8.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
36	0	9.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
37	0	10.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
38	0	11.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
39	0	12.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	

Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
 If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1-p) \geq n-k)$.

n	k	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	p
11	0	0.5688	0.3138	0.1673	0.0859	0.0422	0.0198	0.0088	0.0036	0.0014	0.0005	
12	0	0.5404	0.2824	0.1422	0.0687	0.0317	0.0138	0.0057	0.0022	0.0008	0.0002	
13	0	0.5040	0.2242	0.1087	0.0544	0.0274	0.0121	0.0055	0.0023	0.0009	0.0003	
14	0	0.4816	0.1659	0.0843	0.0443	0.0274	0.0158	0.0085	0.0042	0.0016	0.0003	
15	0	0.4552	0.1174	0.0574	0.0317	0.0197	0.0102	0.0053	0.0026	0.0011	0.0003	
16	0	0.4256	0.0982	0.0496	0.0285	0.0158	0.0085	0.0043	0.0021	0.0009	0.0003	
17	0	0.3971	0.0778	0.0389	0.0206	0.0113	0.0060	0.0030	0.0015	0.0006	0.0002	
18	0	0.3744	0.0648	0.0297	0.0163	0.0085	0.0045	0.0022	0.0010	0.0004	0.0002	
19	0	0.3531	0.0567	0.0268	0.0134	0.0067	0.0034	0.0017	0.0007	0.0003	0.0001	
20	0	0.3323	0.0468	0.0233	0.0117	0.0057	0.0028	0.0013	0.0005	0.0002	0.0001	
21	0	0.3125	0.0378	0.0208	0.0107	0.0052	0.0024	0.0011	0.0004	0.0002	0.0001	
22	0	0.2960	0.0288	0.0180	0.0088	0.0044	0.0021	0.0009	0.0004	0.0002	0.0001	
23	0	0.2823	0.0208	0.0120	0.0058	0.0028	0.0013	0.0006	0.0003	0.0001	0.0001	
24	0	0.2660	0.0188	0.0100	0.0048	0.0022	0.0010	0.0005	0.0002	0.0001	0.0001	
25	0	0.2500	0.0167	0.0080	0.0038	0.0017	0.0007	0.0003	0.0001	0.0001	0.0001	
26	0	0.2379	0.0146	0.0060	0.0028	0.0012	0.0005	0.0002	0.0001	0.0001	0.0001	
27	0	0.2279	0.0126	0.0048	0.0020	0.0008	0.0003	0.0001	0.0001	0.0001	0.0001	
28	0	0.2167	0.0106	0.0038	0.0016	0.0006	0.0002	0.0001	0.0001	0.0001	0.0001	
29	0	0.2062	0.0086	0.0028	0.0011	0.0004	0.0001	0.0001	0.0001	0.0001	0.0001	
30	0	0.1943	0.0067	0.0020	0.0007	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	
31	0	0.1823	0.0048	0.0015	0.0004	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
32	0	0.1719	0.0030	0.0010	0.0003	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
33	0	0.1673	0.0020	0.0006	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
34	0	0.1633	0.0010	0.0003	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
35	0	0.1594	0.0006	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
36	0	0.1554	0.0003	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
37	0	0.1514	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
38	0	0.1474	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	
39	0	0.1434	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	

Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1-p) \geq n-k)$.

n	k	0.05	0.10	0.15	0.20	0.25	p	0.30	0.35	0.40	0.45	0.50
14	0	0.4877	0.2288	0.1028	0.0440	0.0178	0.0068	0.0024	0.0008	0.0002	0.0001	
1	0.8470	0.5284	0.3567	0.1979	0.1010	0.0475	0.0205	0.0081	0.0009			
2	0.9699	0.8416	0.6479	0.4481	0.2811	0.1608	0.0839	0.0398	0.0170	0.0065		
3	0.9558	0.9559	0.8535	0.6982	0.5213	0.3552	0.2205	0.1243	0.0632	0.0287	0.0123	
4	0.9996	0.9908	0.9533	0.8702	0.7415	0.5842	0.4227	0.2793	0.1672	0.0898	0.0464	
5	1.0000	0.9985	0.9885	0.9561	0.8883	0.7805	0.6405	0.4859	0.3373	0.2120	0.1260	0.0596
6	1.0000	0.9998	0.9978	0.9884	0.9617	0.9067	0.8164	0.6925	0.5461	0.3953	0.2348	0.0245
7	1.0000	1.0000	0.9997	0.9976	0.9897	0.9685	0.9247	0.8499	0.7414	0.6047	0.4478	0.2902
8	1.0000	1.0000	1.0000	0.9996	0.9978	0.9917	0.9757	0.8811	0.7830	0.6405	0.4743	0.3145
9	1.0000	1.0000	1.0000	1.0000	0.9997	0.9983	0.9940	0.9825	0.9574	0.9102	0.8011	0.6626
10	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9989	0.9961	0.9886	0.9713	0.9177	0.6855
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9978	0.9935	0.9777	0.8338
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9894	0.9283
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9755
14	0	0.4633	0.2059	0.0874	0.0352	0.0134	0.0047	0.0016	0.0005	0.0001	0.0000	
1	0.8290	0.5490	0.3186	0.1671	0.0802	0.0353	0.0142	0.0052	0.0017	0.0005		
2	0.9638	0.8159	0.6042	0.3980	0.2361	0.1268	0.0617	0.0271	0.0107	0.0037		
3	0.9445	0.9444	0.8227	0.6482	0.4613	0.2969	0.1727	0.0905	0.0424	0.0176		
4	0.9994	0.9873	0.9383	0.8358	0.6865	0.5155	0.3519	0.2173	0.1204	0.0592		
5	0.9999	0.9978	0.9832	0.9389	0.8516	0.7216	0.5643	0.4032	0.2608	0.1509		
6	1.0000	0.9997	0.9964	0.9819	0.9434	0.8689	0.7548	0.6098	0.4522	0.3036		
7	1.0000	1.0000	0.9994	0.9958	0.9827	0.9500	0.8868	0.7869	0.6535	0.5000		
8	1.0000	1.0000	0.9999	0.9992	0.9958	0.9848	0.9578	0.9050	0.8182	0.6964		
9	1.0000	1.0000	1.0000	0.9990	0.9992	0.9963	0.9876	0.9662	0.9231	0.8491		
10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9972	0.9907	0.9745	0.9408		
11	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9981	0.9937	0.9824	0.9524		
12	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9989	0.9963	0.9790		
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9999		
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
15	0	0.4401	0.1853	0.0743	0.0281	0.0100	0.0033	0.0010	0.0003	0.0001	0.0000	
1	0.8108	0.5147	0.2839	0.1407	0.0635	0.0261	0.0098	0.0033	0.0010	0.0003		
2	0.9571	0.7892	0.5614	0.3518	0.1971	0.0994	0.0451	0.0183	0.0066	0.0021		
3	0.9360	0.9316	0.7899	0.5981	0.4050	0.2459	0.1339	0.0651	0.0281	0.0106		
4	0.9991	0.9830	0.9209	0.7982	0.6302	0.4499	0.2892	0.1666	0.0853	0.0384		
5	0.9999	0.9967	0.9765	0.9183	0.8103	0.6598	0.4900	0.3288	0.1976	0.1051		
6	1.0000	0.9995	0.9944	0.9733	0.9204	0.8247	0.6881	0.5272	0.3660	0.2272		
7	1.0000	0.9999	0.9930	0.9729	0.9256	0.8406	0.7161	0.5629	0.4018	0.2409		
8	1.0000	1.0000	0.9998	0.9985	0.9795	0.9743	0.9329	0.8579	0.7441	0.5982		
9	1.0000	1.0000	1.0000	0.9998	0.9984	0.9929	0.9771	0.9417	0.8759	0.7728		
10	1.0000	1.0000	1.0000	1.0000	0.9997	0.9984	0.9809	0.9514	0.8949	0.8287		
11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9951	0.9851	0.9616	0.9214		
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9965	0.9894	0.9529		
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9999		
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		

Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1-p) \geq n-k)$.

n	k	0.05	0.10	0.15	0.20	0.25	p	0.30	0.35	0.40	0.45	0.50
17	0	0.4181	0.1668	0.0631	0.0225	0.0075	0.0023	0.0007	0.0002	0.0000	0.0000	
1	0.7922	0.4818	0.2525	0.1182	0.0501	0.0193	0.0067	0.0021	0.0006	0.0001		
2	0.9497	0.7618	0.5198	0.3096	0.1637	0.0774	0.0327	0.0123	0.0041	0.0012		
3	0.9912	0.9174	0.7556	0.5489	0.3530	0.2019	0.1028	0.0464	0.0184	0.0064		
4	0.9988	0.9779	0.9013	0.7582	0.5739	0.3887	0.2348	0.1260	0.0596	0.0245		
5	0.9999	0.9953	0.9681	0.8943	0.7653	0.5968	0.4197	0.2639	0.1471	0.0717		
6	1.0000	0.9992	0.9623	0.8929	0.7752	0.6188	0.4478	0.2902	0.1662			
7	1.0000	0.9999	0.9883	0.9598	0.8787	0.7872	0.6405	0.4743	0.3145			
8	1.0000	1.0000	0.9997	0.9974	0.9876	0.9597	0.9006	0.8011	0.6626	0.5000		
9	1.0000	1.0000	1.0000	0.9999	0.9999	0.9970	0.9894	0.9699	0.9283	0.7597		
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
16	0	0.3774	0.1351	0.0456	0.0144	0.0042	0.0011	0.0003	0.0001	0.0000		
1	0.7547	0.4203	0.1985	0.0829	0.0310	0.0104	0.0031	0.0008	0.0002	0.0000		
2	0.9335	0.7054	0.4413	0.2369	0.1113	0.0462	0.0170	0.0055	0.0015	0.0004		
3	0.9836	0.8850	0.6841	0.4551	0.2631	0.1332	0.0591	0.0230	0.0077	0.0022		
4	0.9980	0.9648	0.8556	0.6733	0.4654	0.2822	0.1500	0.0696	0.0280	0.0096		
5	0.9998	0.9914	0.9463	0.8369	0.6678	0.4739	0.2968	0.1629	0.0777	0.0318		
6	1.0000	0.9983	0.9837	0.9324	0.8251	0.6655	0.4812	0.3081	0.1727	0.0835		
7	1.0000	0.9997	0.9959	0.9767	0.9225	0.8180	0.6656	0.4878	0.3169	0.1796		
8	1.0000	1.0000	0.9992	0.9933	0.9713	0.9161	0.8145	0.6675	0.4940	0.3238		
9	1.0000	1.0000	0.9999	0.9974	0.9674	0.9125	0.8139	0.6710	0.4917	0.3238		
10	1.0000	1.0000	1.0000	0.9997	0.9797	0.9563	0.9115	0.8159	0.6762	0.4917		
11	1.0000	1.0000	1.0000	0.9995	0.9972	0.9886	0.9648	0.9129	0.8204	0.6822		
12	1.0000	1.0000	1.0000	1.0000	0.9999	0.9969	0.9884	0.9658	0.9165	0.7597		
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000		

(9.5) Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	μ									
k	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725	0.7358
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371	0.9197
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9865	0.9810
4	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977	0.9963	0.9956
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9994
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019	0.1827	0.1653	0.1496	0.1353
1	0.6890	0.6626	0.6268	0.5918	0.5578	0.5249	0.4932	0.4628	0.4337	0.4060
2	0.904	0.8795	0.8571	0.8335	0.8088	0.7834	0.7572	0.7306	0.7037	0.6767
3	0.9743	0.9662	0.9569	0.9463	0.9344	0.9212	0.9068	0.8913	0.8747	0.8571
4	0.9946	0.9923	0.9893	0.9857	0.9814	0.9763	0.9704	0.9636	0.9559	0.9473
5	0.9990	0.9985	0.9978	0.9968	0.9955	0.9940	0.9920	0.9896	0.9834	0.9775
6	0.9999	0.9997	0.9996	0.9994	0.9991	0.9987	0.9981	0.9974	0.9966	0.9955
7	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997	0.9996	0.9994	0.9992	0.9989
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
0	0.0408	0.0334	0.0273	0.0224	0.0183	0.0150	0.0123	0.0101	0.0082	0.0067
1	0.1712	0.1468	0.1257	0.1074	0.0916	0.0780	0.0663	0.0563	0.0477	0.0404
2	0.3799	0.3397	0.3027	0.2689	0.2381	0.2102	0.1851	0.1626	0.1425	0.1247
3	0.6025	0.5584	0.5152	0.4735	0.4335	0.3954	0.3594	0.3257	0.2942	0.2650
4	0.7806	0.7442	0.7064	0.6678	0.6288	0.5898	0.5512	0.5132	0.4763	0.4405
5	0.8946	0.8705	0.8441	0.8156	0.7851	0.7531	0.7199	0.6858	0.6510	0.6160
6	0.9554	0.9421	0.9267	0.9091	0.8893	0.8675	0.8436	0.8180	0.7908	0.7622
7	0.9832	0.9769	0.9692	0.9559	0.9489	0.9361	0.9214	0.9049	0.8867	0.8666
8	0.9943	0.9917	0.9883	0.9840	0.9786	0.9721	0.9642	0.9549	0.9442	0.9319
9	0.9995	0.9992	0.9987	0.9981	0.9972	0.9959	0.9943	0.9922	0.9896	0.9863
k	5.2	5.4	5.6	5.8	6.0	6.5	7.0	7.5	8.0	8.5
0	0.0055	0.0045	0.0037	0.0030	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002
1	0.0342	0.0289	0.0244	0.0206	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019
2	0.1088	0.0948	0.0824	0.0715	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093
3	0.2381	0.2133	0.1906	0.1700	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301
4	0.4061	0.3733	0.3422	0.3127	0.2851	0.2237	0.1730	0.1321	0.0966	0.0744
5	0.5809	0.5461	0.5119	0.4783	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496
6	0.7324	0.7017	0.6703	0.6384	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562
7	0.8449	0.8149	0.7970	0.7710	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856
8	0.9181	0.9027	0.8857	0.8672	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231
9	0.9603	0.9512	0.9409	0.9292	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530
10	0.9823	0.9775	0.9718	0.9651	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634
11	0.9904	0.9875	0.9841	0.9799	0.9761	0.9467	0.9208	0.8881	0.8487	0.8047
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	0.1225	0.1108	0.1003	0.0907	0.0821	0.0743	0.0672	0.0608	0.0550	0.0498
1	0.3796	0.3546	0.3309	0.3084	0.2873	0.2674	0.2487	0.2311	0.2146	0.1991
2	0.6496	0.6227	0.5960	0.5597	0.5348	0.5184	0.4936	0.4695	0.4460	0.4232
3	0.8386	0.8194	0.7787	0.7576	0.7360	0.7141	0.6919	0.6696	0.6474	0.6142
4	0.9379	0.9275	0.9162	0.9041	0.8912	0.8774	0.8477	0.8318	0.8153	0.7915
5	0.9796	0.9751	0.9700	0.9643	0.9580	0.9510	0.9433	0.9349	0.9258	0.9161
6	0.9941	0.9925	0.9884	0.9828	0.9794	0.9756	0.9713	0.9665	0.9623	0.9573
7	0.9985	0.9974	0.9967	0.9958	0.9947	0.9934	0.9919	0.9901	0.9881	0.9850
8	0.9997	0.9995	0.9994	0.9989	0.9985	0.9981	0.9976	0.9969	0.9962	0.9953
9	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996	0.9995	0.9994	0.9993	0.9992
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	9.0	9.5	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0	μ
0	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.0012	0.0008	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0062	0.0042	0.0028	0.0012	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
3	0.0212	0.0149	0.0103	0.0049	0.0023	0.0011	0.0005	0.0002	0.0001	0.0000	0.0000
4	0.0550	0.0403	0.0293	0.0151	0.0076	0.0037	0.0018	0.0009	0.0004	0.0002	0.0000
5	0.1157	0.0885	0.0671	0.0375	0.0203	0.0107	0.0055	0.0028	0.0014	0.0007	0.0000
6	0.2068	0.1649	0.1301	0.0786	0.0458	0.0259	0.0142	0.0076	0.0040	0.0021	0.0000
7	0.3239	0.2687	0.2202	0.1432	0.0895	0.0540	0.0316	0.0180	0.0100	0.0054	0.0000
8	0.4557	0.3918	0.3328	0.2320	0.1550	0.0998	0.0621	0.0374	0.0220	0.0126	0.0000
9	0.5874	0.5218	0.4579	0.3405	0.2424	0.1658	0.1094	0.0699	0.0433	0.0261	0.0000
10	0.7060	0.6453	0.5830	0.4599	0.3472	0.2517	0.1757	0.1185	0.0774	0.0491	0.0000
11	0.8030	0.7520	0.6968	0.5793	0.4616	0.3532	0.2600	0.1848	0.1270	0.0847	0.0000
12	0.8758	0.8364	0.7916	0.6887	0.5760	0.4631	0.3585	0.2676	0.1931	0.1350	0.0000
13	0.9261	0.8981	0.8645	0.7813	0.6815	0.5730	0.4644	0.3632	0.2745	0.2009	0.0000
14	0.9585	0.9400	0.9165	0.8540	0.7720	0.6751	0.5704	0.4657	0.3675	0.2808	0.0000
15	0.9780	0.9665	0.9513	0.9074	0.8444	0.7636	0.6694	0.5681	0.4667	0.3715	0.0000
16	0.9889	0.9823	0.9730	0.9441	0.8987	0.8355	0.7559	0.6641	0.5660	0.4677	0.0000
17	0.9947	0.9911	0.9857	0.9678	0.9370	0.8905	0.8272	0.7489	0.6593	0.5640	0.0000
18	0.9976	0.9957	0.9928	0.9823	0.9626	0.9302	0.8826	0.8195	0.7423	0.6550	0.0000
19	0.9989	0.9980	0.9965	0.9907	0.9787	0.9573	0.9235	0.8752	0.8122	0.7363	0.0000
20	0.9996	0.9991	0.9984	0.9953	0.9884	0.9750	0.9521	0.9170	0.8682	0.8055	0.0000
21	0.9998	0.9996	0.9993	0.9977	0.9939	0.9859	0.9712	0.9469	0.9108	0.8615	0.0000
22	0.9999	0.9999	0.9997	0.9990	0.9970	0.9924	0.9833	0.9673	0.9418	0.9047	0.0000
23	1.0000	0.9999	0.9999	0.9995	0.9985	0.9960	0.9907	0.9805	0.9633	0.9367	0.0000
24	1.0000	1.0000	1.0000	0.9998	0.9993	0.9980	0.9950	0.9888	0.9777	0.9594	0.0000
25	1.0000	1.0000	1.0000	0.9999	0.9997	0.9974	0.9938	0.9869	0.9748	0.9548	0.0000
26	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9987	0.9967	0.9925	0.9848	0.0000
27	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9983	0.9959	0.9912	0.0000
28	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9991	0.9978	0.9950	0.9900	0.0000
29	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9989	0.9973	0.9947	0.0000
30	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9986	0.9976	0.0000
31	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9993	0.9980	0.0000
32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.0000
33	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.0000
34	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.0000
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000