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Four-week Report

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Introduction

This is the four-week report of my summer research project under Prof L. Sunil Chandran at Indian Institute of Science, Bengaluru.
I have primarily worked for the past four weeks trying to learn and understand the nature and number of monochromatic perfect matchings in regular graphs and n-uniform hypergraphs, and the main topics that I have learnt and worked on over the past four weeks are summarized below.

Perfect matchings in regular graphs

In a regular graph $G = (V, E)$, a *perfect matching* is a matching M of the graph such that $\forall\, v \in V(G)$, v is a part of some matched edge in M.
For a perfectly monochromatic colouring c of G, the number of normal colour classes containing at least one perfect matching is represented by $\mu(G, c)$. The *matching index* of G, denoted by $\mu(G)$ is defined as $\mu(G) = \max\limits_{c \in \mathcal{C}(G)} \mu(G, c)$, where $\mathcal{C}(G)$ is the set of all perfectly monochromatic colourings of G.

In a regular graph, Bogdanov[3] proved that $\forall\, G$ isomorphic to K_4, the number of *disjoint* perfect matchings of $G = 3$, and for all other regular graphs non-isomorphic to K_4, the number of disjoint perfect matchings is at most 2. This proof was used in [2] to theorize that $\forall\, G$ isomorphic to K_4, $\mu(G) = 3$, otherwise $\mu(G) \leq 2$.

Hypergraphs

According to [1], a *hypergraph* is a pair $H = (V, E)$, where V is a finite set whose elements are called *vertices*, and E is a family of subsets of V, called *edges*.
A hypergraph H is said to be *n-uniform* if $\forall\, e \in E$, e contains precisely n vertices.

A *perfect matching*, $\mathcal{M}$ of a n-uniform hypergraph $H = (V, E)$ can be defined as a matching M of H such that $\forall\, v \in V$, v is part of some edge in M. It is trivial to note that for a n-uniform hypergraph to have a perfect matching, $|V| \equiv 0 \pmod{n}$.

A n-uniform hypergraph H is said to be perfectly monochromatic if the following properties hold:

1. $\forall\, e \in E$, e must be part of some perfect matching $\mathcal{M}$.

2. Every perfect matching, $\mathcal{M}$ of H is monochromatic.

3. Let $\mathcal{C}$ be set of colour classes such that $\exists$ a perfect matching $\mathcal{M}$ of H such that $\mathcal{M}$ belongs to a colour class $c \in \mathcal{C}$. Then $\forall\, c \in \mathcal{C}$, c must belong to *exactly* 1 perfect matching $\mathcal{M}$ of H.

Now, for a perfectly monochromatic hypergraph H, we call the dimension $D = |\mathcal{C}|$ of the hypergraph to be the number of colours in it. We are intereseted in finding the bounds on D for a n-uniform hypergraph containing $k \cdot n$ vertices.

Perfect matchings in 3-uniform hypergraphs

Since we are interested in finding the bounds on D, we start with trying to find the bounds on D for 3-uniform hypergraphs. We analyzed some particular cases and the results we came up with are:

- For $|V| = 3$, it can be trivially observed that $D = 1$.

- For $|V| = 6$, the upper bound on $D = 10$.
  This can be verified by observing that in a complete 3-uniform hypergraph with $|V| = 6$, there are $\binom{6}{3} = 20$ edges. We can colour each edge and its complement with the same colour, forming a monochromatic perfect matching, and thus the number of monochromatic perfect matchings for a 3-uniform hypergraph with $|V| = 6$ is $20/2 = 10$.

- For $|V| = 9$, the upper bound on D turned out to be 13. I verified it independently by writing a program which found out the value of D by combining randomization and brute force techniques.

- For $|V| = 12$, I developed a partly randomized program to compute the bounds on D, and the upper bound from that program turned out to be 10, which surprisingly enough, turns out to be the same value as for $|V| = 6$.

Besides trying to analyze and compute the bounds for D for 3-uniform hypergraphs, I have also tried implementing Algorithm 1 in [2], in the process of which I learnt about the Edmonds' blossom algorithm for finding the maximum number of perfect matchings in a regular graph, and the Edmonds-Karp algorithm for finding the maximum flow in a regular graph.

References

[1] N. Alon and J. H. Spencer. *The Probabilistic Method*. Wiley Publishing, 4th edition, 2016.

[2] L. S. Chandran and R. Gajjala. Perfect matchings and quantum physics: Progress on krenn's conjecture, 2022.

[3] I. B. (https://mathoverflow.net/users/17581/ilya bogdanov). Graphs with only disjoint perfect matchings. MathOverflow. URL:https://mathoverflow.net/q/267013 (version: 2017-04-12).