# IASc-INSA-NASI SRFP 2023
## Four-week Report

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## Introduction

This is the four-week report of my summer research project under Prof L. Sunil Chandran at Indian Institute of Science, Bengaluru.
I have primarily worked for the past four weeks trying to learn and understand the nature and number of monochromatic perfect matchings in graphs and $n$-uniform hypergraphs, and the main topics that I have learnt and worked on over the past four weeks are summarized below.

## Perfect matchings in graphs

In a graph $G = (V, E)$, a *perfect matching* is a matching $M$ of the graph such that $\forall\, v \in V(G)$, $v$ is a part of some matched edge in $M$.
For a perfectly monochromatic colouring $c$ of $G$, the number of normal colour classes containing at least one perfect matching is represented by $\mu(G, c)$. The *matching index* of $G$, denoted by $\mu(G)$ is defined as $\mu(G) = \max\limits_{c \in \mathcal{C}(G)} \mu(G, c)$, where $\mathcal{C}(G)$ is the set of all perfectly monochromatic colourings of $G$.

In a graph, Bogdanov[3] proved that $\forall\, G$ isomorphic to $K_4$, the number of *disjoint* perfect matchings of $G = 3$, and for all other graphs non-isomorphic to $K_4$, the number of disjoint perfect matchings is at most 2. This proof was used in [2] to theorize that $\forall\, G$ isomorphic to $K_4$, $\mu(G) = 3$, otherwise $\mu(G) \leq 2$.

## Hypergraphs

According to [1], a *hypergraph* is a pair $H = (V, E)$, where $V$ is a finite set whose elements are called *vertices*, and $E$ is a family of subsets of $V$, called *edges*.
A hypergraph $H$ is said to be *$n$-uniform* if $\forall\, e \in E$, $e$ contains precisely $n$ vertices.

A *perfect matching*, $\mathcal{M}$ of a $n$-uniform hypergraph $H = (V, E)$ can be defined as a matching $M$ of $H$ such that $\forall\, v \in V$, $v$ is part of some edge in $M$. It is trivial to note that for a $n$-uniform hypergraph to have a perfect matching, $|V| \equiv 0 \pmod{n}$.

A $n$-uniform hypergraph $H$ is said to be perfectly monochromatic if the following properties hold:

1. $\forall\, e \in E$, $e$ must be part of some perfect matching $\mathcal{M}$.

2. Every perfect matching, $\mathcal{M}$ of $H$ is monochromatic.

3. Let $\mathcal{C}$ be set of colour classes such that $\exists$ a perfect matching $\mathcal{M}$ of $H$ such that $\mathcal{M}$ belongs to a colour class $c \in \mathcal{C}$. Then $\forall\, c \in \mathcal{C}$, $c$ must belong to *exactly* 1 perfect matching $\mathcal{M}$ of $H$.

Now, for a perfectly monochromatic hypergraph $H$, we call the dimension $D = |\mathcal{C}|$ of the hypergraph to be the number of colours in it. We are intereseted in finding the bounds on $D$ for a $n$-uniform hypergraph containing $k \cdot n$ vertices.

## Perfect matchings in 3-uniform hypergraphs

Since we are interested in finding the bounds on $D$, we start with trying to find the bounds on $D$ for 3-uniform hypergraphs. We analyzed some particular cases and the results we came up with are:

- For $|V| = 3$, it can be trivially observed that $D = 1$.

- For $|V| = 6$, the upper bound on $D = 10$.
  This can be verified by observing that in a complete 3-uniform hypergraph with $|V| = 6$, there are $\binom{6}{3} = 20$ edges. We can colour each edge and its complement with the same colour, forming a monochromatic perfect matching, and thus the number of monochromatic perfect matchings for a 3-uniform hypergraph with $|V| = 6$ is $20/2 = 10$.

- For $|V| = 9$, the upper bound on $D$ turned out to be 13. I verified it independently by writing a program which found out the value of $D$ by combining randomization and brute force techniques.

- For $|V| = 12$, I developed a partly randomized program to compute the bounds on $D$, and the upper bound from that program turned out to be 10, which surprisingly enough, turns out to be the same value as for $|V| = 6$.

Besides trying to analyze and compute the bounds for $D$ for 3-uniform hypergraphs, I have also tried implementing Algorithm 1 in [2], in the process of which I learnt about the Edmonds' blossom algorithm for finding the maximum number of perfect matchings in a graph, and the Edmonds-Karp algorithm for finding the maximum flow in a graph.

## References

[1] N. Alon and J. H. Spencer. *The Probabilistic Method.* Wiley Publishing, 4th edition, 2016.

[2] L. S. Chandran and R. Gajjala. Perfect matchings and quantum physics: Progress on krenn's conjecture, 2022.

[3] I. B. (https://mathoverflow.net/users/17581/ilya bogdanov). Graphs with only disjoint perfect matchings. MathOverflow. URL:https://mathoverflow.net/q/267013 (version: 2017-04-12).