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FOUR-WEEK REPORT

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Introduction

This is the four-week report of my summer research project under Prof L. Sunil Chandran at Indian Institute of Science, Bengaluru.

I have primarily worked for the past four weeks trying to learn and understand the nature and number of monochromatic perfect matchings in graphs and n -uniform hypergraphs, and the main topics that I have learnt and worked on over the past four weeks are summarized below.

Perfect matchings in graphs

In a graph $G = (V, E)$, a *perfect matching* is a matching M of the graph such that $\forall v \in V(G)$, v is a part of some matched edge in M .

For a perfectly monochromatic colouring c of G , the number of normal colour classes containing at least one perfect matching is represented by $\mu(G, c)$. The *matching index* of G , denoted by $\mu(G)$ is defined as $\mu(G) = \max_{c \in \mathcal{C}(G)} \mu(G, c)$, where $\mathcal{C}(G)$ is the set of all perfectly monochromatic colourings of G .

In a graph, Bogdanov[3] proved that $\forall G$ isomorphic to K_4 , the number of *disjoint* perfect matchings of $G = 3$, and for all other graphs non-isomorphic to K_4 , the number of disjoint perfect matchings is at most 2. This proof was used in [2] to theorize that $\forall G$ isomorphic to K_4 , $\mu(G) = 3$, otherwise $\mu(G) \leq 2$.

Hypergraphs

According to [1], a *hypergraph* is a pair $H = (V, E)$, where V is a finite set whose elements are called *vertices*, and E is a family of subsets of V , called *edges*.

A hypergraph H is said to be *n-uniform* if $\forall e \in E$, e contains precisely n vertices.

A *perfect matching*, \mathcal{M} of a n -uniform hypergraph $H = (V, E)$ can be defined as a matching M of H such that $\forall v \in V$, v is part of some edge in M . It is trivial to note that for a n -uniform hypergraph to have a perfect matching, $|V| \equiv 0 \pmod{n}$.

A n -uniform hypergraph H is said to be perfectly monochromatic if the following properties hold:

1. Every perfect matching, \mathcal{M} of H is monochromatic.

2. Let \mathcal{C} be set of colour classes such that \exists a perfect matching \mathcal{M} of H such that \mathcal{M} belongs to a colour class $c \in \mathcal{C}$. Then $\forall c \in \mathcal{C}$, c must belong to *exactly* 1 perfect matching \mathcal{M} of H .

Now, for a perfectly monochromatic hypergraph H , we call the dimension $D = |\mathcal{C}|$ of the hypergraph to be the number of colours in it. We are interested in finding the bounds on D for a n -uniform hypergraph containing $k \cdot n$ vertices.

Perfect matchings in 3-uniform hypergraphs

Since we are interested in finding the bounds on D , we start with trying to find the bounds on D for 3-uniform hypergraphs. We analyzed some particular cases and the results we came up with are:

- For $|V| = 3$, it can be trivially observed that $D = 1$.
- For $|V| = 6$, the upper bound on $D = 10$.
This can be verified by observing that in a complete 3-uniform hypergraph with $|V| = 6$, there are $\binom{6}{3} = 20$ edges. We can colour each edge and its complement with the same colour, forming a monochromatic perfect matching, and thus the number of monochromatic perfect matchings for a 3-uniform hypergraph with $|V| = 6$ is $20/2 = 10$.
- For $|V| = 9$, the upper bound on D turned out to be 13. I verified it independently by writing a program which found out the value of D by combining randomization and brute force techniques.
- For $|V| = 12$, I developed a partly randomized program to compute the bounds on D , and the upper bound from that program turned out to be 10, which surprisingly enough, turns out to be the same value as for $|V| = 6$.

Besides trying to analyze and compute the bounds for D for 3-uniform hypergraphs, I have also tried implementing *Algorithm 1 - To decide whether a non-trivial matching covered graph non-isomorphic to K_4 is Type 1 or Type 2* in [2], in the process of which I learnt about the Edmonds' blossom algorithm for finding the maximum number of perfect matchings in a graph, and the Edmonds-Karp algorithm for finding the maximum flow in a graph.

References

- [1] N. Alon and J. H. Spencer. *The Probabilistic Method*. Wiley Publishing, 4th edition, 2016.
- [2] L. S. Chandran and R. Gajjala. Perfect matchings and quantum physics: Progress on krenn's conjecture, 2022.
- [3] I. B. (<https://mathoverflow.net/users/17581/ilya-bogdanov>). Graphs with only disjoint perfect matchings. MathOverflow. URL:<https://mathoverflow.net/q/267013> (version: 2017-04-12).