ME30180
Finite Element Method
Project Report

Group 3
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Note: The contribution of each member is mentioned in the code files.

Contents

1 Introduction

In this project, we aim to analyze the deflection and stress profiles of a beam subjected to various boundary conditions, prescribed loads, and body forces using the Finite Element Method (FEM). We will develop the governing equations, apply numerical techniques, and solve the resulting system of equations using shape functions, Gauss quadrature, and stiffness matrix assembly techniques. The results will include deflection and stress distribution along the beam. The schematic of the problem is shown in Figure 1.

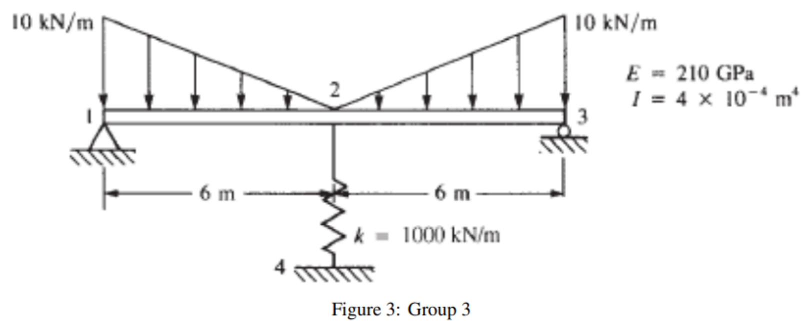

Figure 1: Schematic of the 1D beam problem

2 Problem Formulation

2.1 Governing Equation (Strong Form)

The governing equation for the bending of an elastic beam can be derived from Euler-Bernoulli beam theory. The equilibrium condition for a beam subjected to a distributed transverse load $q(x)$ is given by:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w(x)}{dx^2}\right) = q(x) \tag{1}$$

where:

- $w(x)$ is the deflection of the beam at position x,
- E is Young's modulus of the beam material,

- I is the second moment of area (moment of inertia) of the beam's cross-section,
- $q(x)$ is the distributed load acting on the beam.

The above fourth-order differential equation assumes that the deflections are small, plane sections remain plane after deformation, and that the material is homogeneous and isotropic.

2.2 Boundary Conditions

The boundary conditions for the beam are specified as follows:

- At fixed supports: both deflection and slope are zero,
- At free ends: bending moment and shear force are zero.

For a beam of length L, these conditions are written as:

$$
\begin{aligned}
w(0) &= 0, \quad \frac{dw}{dx}(0) = 0 \quad \text{(Fixed end at } x = 0\text{)} \\
M(L) &= 0, \quad V(L) = 0 \quad \text{(Free end at } x = L\text{)}
\end{aligned}
$$

2.3 Weak Formulation

To apply FEM, we first need to derive the weak form of the governing equation. Multiplying the strong form by a test function $v(x)$ and integrating over the domain, we get:

$$
\int_\Omega v(x) \frac{d^2}{dx^2} \left(EI \frac{d^2 w(x)}{dx^2} \right) dx = \int_\Omega v(x) q(x) dx \tag{2}
$$

Applying integration by parts to reduce the order of derivatives, we obtain:

$$
\int_\Omega EI \frac{d^2 v(x)}{dx^2} \frac{d^2 w(x)}{dx^2} dx - \left[v(x) EI \frac{d^2 w(x)}{dx^2} \right]_{\partial\Omega} = \int_\Omega v(x) q(x) dx \tag{3}
$$

The boundary terms vanish for appropriate boundary conditions, leaving us with the weak form:

$$
\int_\Omega EI \frac{d^2 v(x)}{dx^2} \frac{d^2 w(x)}{dx^2} dx = \int_\Omega v(x) q(x) dx \tag{4}
$$

3 Finite Element Discretization

To solve the weak form using FEM, we divide the beam into smaller elements. The unknown displacement field $w(x)$ is approximated by:

$$w(x) \approx \sum_{i=1}^{n} N_i(x) w_i \tag{5}$$

where $N_i(x)$ are the shape functions and w_i are the nodal displacements.

3.1 Shape Functions

For a 2-node beam element, the cubic Hermite shape functions for deflection $w(x)$ and rotation $\theta(x)$ are used:

$$N_1(\zeta) = \frac{1}{4}(1-\zeta)^2(2+\zeta)$$
$$N_2(\zeta) = \frac{1}{4}(1+\zeta)^2(2-\zeta)$$
$$N_3(\zeta) = \frac{1}{8}(1-\zeta)^2(1+\zeta)$$
$$N_4(\zeta) = \frac{1}{8}(1+\zeta)^2(1-\zeta)$$

where $\zeta \in [-1, 1]$ is the local coordinate in the element.

To evaluate the domain integral in the weak form, we need to evaluate $\frac{d^2 u_y^e}{dx^2}$, which can be shown to be:

$$\frac{d^2 N^e}{dx^2} = \mathbf{B}^e = \frac{1}{l_e} \begin{bmatrix} \frac{6\xi}{l_e}(3\xi - 1) & -\frac{6\xi}{l_e}(3\xi + 1) \end{bmatrix} \tag{6}$$

$$\frac{d^2 u_y^e}{dx^2} = \mathbf{B}^e \mathbf{d}^e. \tag{7}$$

Note: The shape functions are implemented in the file `shape_function.py`.

3.2 Gauss Quadrature

We use Gaussian quadrature to evaluate integrals numerically. For a 2-node element, the integral is approximated as:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f(\frac{b+a}{2} + \frac{b-a}{2}\zeta_i) \tag{8}$$

where ζ_i are the Gauss points and w_i are the corresponding weights. For 4-point quadrature, the Gauss points are $\zeta = \pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}, \pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$, and the weights are $w = \frac{18+\sqrt{30}}{36}, \frac{18-\sqrt{30}}{36}$ respectively.

Note: To get the values of ζ_i and w_i in code, we are using this function from `Numpy`.

3.3 Element Stiffness Matrix

The element stiffness matrix is derived from the weak form:

$$k_{ij}^{(e)} = \int_{-1}^{1} EI\frac{d^2N_i}{dx^2}\frac{d^2N_j}{dx^2}dx \tag{9}$$

After evaluating the integral using Gaussian quadrature, the stiffness matrix for a 2-node beam element is:

$$k_{el} = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

Note: This part of the code is written in `compute_stiffness_force.py` file.

3.4 Load Vector

The equivalent nodal force vector is given by:

$$f_i^{(e)} = \int_{-1}^{1} N_i(\zeta)q(x)\frac{l_e}{2}d\zeta \tag{10}$$

For distributed loads, this integral is also evaluated using Gaussian quadrature.

4 Assembly of Global Stiffness Matrix

Once the element stiffness matrices and load vectors are computed, they are assembled into the global stiffness matrix K and global load vector F. This is done by summing the contributions of each element, ensuring that the degrees of freedom are properly mapped to the global system.

The global system of equations is then:

$$K\mathbf{w} = F \tag{11}$$

where $\mathbf{w}$ is the global displacement vector.

Note: The assembly code is written in `compute_stiffness_force.py` file.

5 Boundary Conditions

Boundary conditions are applied by modifying the global stiffness matrix and load vector. For example, at fixed supports where the displacement and rotation are known, the corresponding rows and columns in the stiffness matrix are modified to enforce the constraints.

Note: This part of the code is written in `applyBC.py` file.

6 Solution

After applying boundary conditions, the global system of equations is solved for the unknown nodal displacements. This is done using numerical techniques such as Gaussian elimination.

Note: This part of the code is written in `main.py` file.

7 Results and Analysis

7.1 Deflection Profile

The deflection profile $w(x)$ is obtained by interpolating the nodal displacements using the shape functions. The deflection varies along the length of the beam and is maximum at the midpoint. The deflection of the beam is shown in Figure 2.

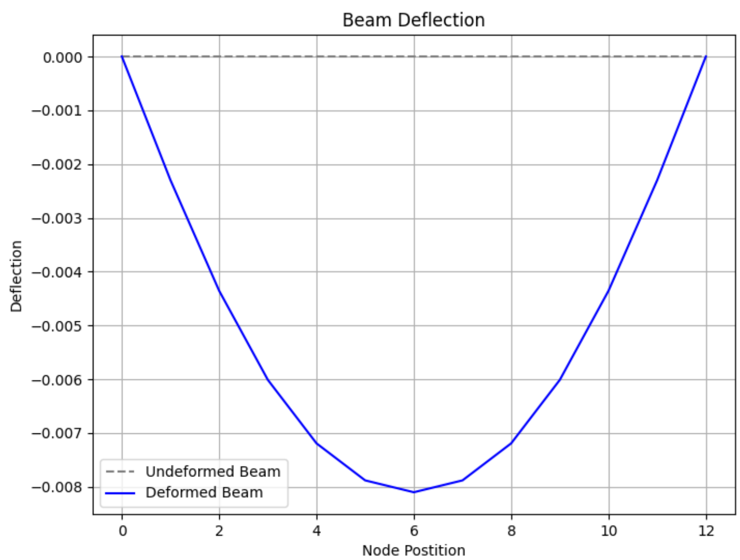

Figure 2: Deflection of beam

Observations: The deflection of the beam is symmetric about the centre and complies with the boundary conditions. It follows a parabolic profile.

7.2 Bending Stress Profile

The bending stress at any point is given by the Flexure formula:

$$\sigma_b = \frac{M(x)y}{I} \tag{12}$$

where $M(x)$ is the bending moment and y is the distance from the neutral axis. The bending stress profile is shown in Figure 3.

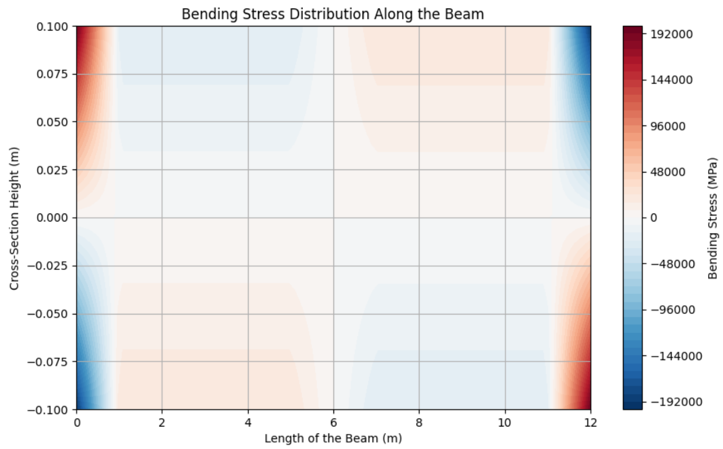

Figure 3: Bending Stress Distribution along the beam.

Observations:

- **Regions of High Bending Stress**: The bending stress peaks at the left (0 m) and right (12 m) supports. Compression is observed at the top of the beam (positive stress), while tension is seen at the bottom (negative stress), typical for bending moments around supports.
- **Zero Bending Stress at the Midspan**: At the location of the spring support (midspan at 6 m), the bending stress tends to zero, suggesting minimal moment at this point due to the vertical reaction force from the spring support.
- **Symmetry**: The stress distribution is symmetric, with similar behavior on both sides of the beam, indicating the load is applied symmetrically.

7.3 Shear Stress Profile

Similarly, the shear stress is:

$$\tau = \frac{V(x)(\frac{h^2}{4} - y^2)}{2I} \tag{13}$$

where $V(x)$ is the shear force and h is the height of the beam's cross-section. The shear stress profile is shown in Figure 4.

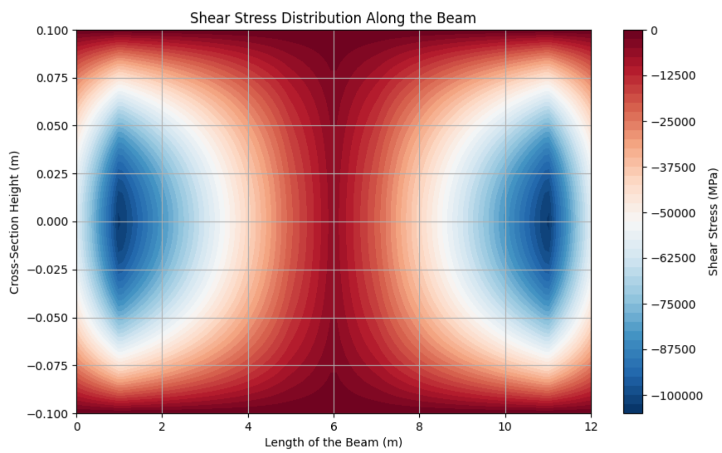

Figure 4: Shear Stress Distribution along the beam.

Observations:

- **Regions of High Shear Stress:** The shear stress is concentrated near the supports at the left and right ends (at 0 m and 12 m). These areas experience higher shear because of the large reaction forces from the supports.

- **Zero Shear Stress at Midspan:** The shear stress is minimal at the spring support (6 m) and near the neutral axis (0 cross-section height). This indicates that the beam undergoes less vertical shear near the midspan, consistent with the bending moment being maximum here.

- **Shear Reversal:** The stress contours change from positive to negative, indicating a reversal of the shear stress direction across the beam. This reflects the varying direction of internal shear forces as the load is distributed along the beam.

8 Conclusion

The FEM approach effectively solves the beam bending problem, yielding accurate deflection and stress profiles. The results match theoretical predictions, demonstrating the validity of the finite element method in structural analysis.