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2. Linear Algebra

- Prove or disprove: Empty set is a vector space.

Sol^n

A vector space V, should hold additive identity property.

Additive identity :- there exists an element $0 \in V$ such that $v + 0 = v \ \forall v \in V$

$\because$ Empty set is empty (no element) hence it fails to have the zero vector as an element.

$\therefore$ empty set violates additive identity property

$\Rightarrow$ Empty set is not a vector space

//.

- – Show that the inverse of $M = I + (\mathbf{u}\mathbf{v}^T)$ is of the type $I + \alpha(\mathbf{u}\mathbf{v}^T)$, where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{v}^T\mathbf{u} \neq 0$.
  – Continuing from previous question, find α.
  – For what $\mathbf{u}$ and $\mathbf{v}$ is M singular?
  – Find the null space of M, if it is singular.

Sol^n

$M = I + (uv^T)$ where $u, v \in \mathbb{R}^n$

As $u, v \in \mathbb{R}^n$ $\therefore$ $\dim(u) = \dim(v) = n \times 1$

Say we want to solve,

$Mx = y$ where $x \in \mathbb{R}^n$

$\Rightarrow (I + uv^T)x = y$

$\Rightarrow x + uv^Tx = y$

$\Rightarrow x = y - uv^Tx$

Now $v^Tx \to (1\times n) \times (n\times 1) \to (1\times 1)$

So v^Tx is a scaler, say r

$x = y - ur$, where $r = v^Tx$

$r = v^Tx = v^T(y - ur) = v^Ty - v^Tur$

$\Rightarrow r = \dfrac{v^Ty}{1 + v^Tu}$ Put to find x

$\Rightarrow x = y - u\left(\dfrac{v^Ty}{1 + v^Tu}\right)$

$\Rightarrow x = y\left(I - \dfrac{uv^T}{1 + v^Tu}\right)$

We had $Mx = y$

$\Rightarrow \quad x = yM^{-1}$

compare with $x = y\left(I - \frac{uv^T}{1+v^Tu}\right)$

$\Rightarrow \quad M^{-1} = I - \frac{uv^T}{1+v^Tu}$

$$\boxed{M^{-1} = I + \alpha(uv^T) \text{ where } \alpha = \frac{-1}{1+v^Tu}}$$

— For what values of u and v M is singular

- Null space of M

If M is singular then, there exists $x \in \mathbb{R}^n$ such that $Mx = 0$

$\Rightarrow \quad Mx = 0$

$\Rightarrow \quad (I + uv^T)x = 0$

$\Rightarrow \quad x + u(v^Tx) = 0$

$\Rightarrow \quad x = -u(v^Tx)$

$\hookrightarrow$ 1×1 scaler let s.

$\Rightarrow \quad x = -su$ Put in above eq.

$-su = -u(v^T(-su))$

$\Rightarrow \quad -u = u(v^Tu)$

$\Rightarrow \quad \boxed{v^Tu = -1}$

So for $v^Tu = -1$, M is singular.

$x = -u(v^Tx)$

$\Rightarrow$ $\boxed{x = u \text{ i.e, Null space of } M \text{ is spanned by the vector } u.}$

- – Consider the 2×2 matrix:

$$A = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}$$

There exists vectors such that when the matrix A acts on those vectors, the subspace of the vectors does not change. Mathematically, $Ax = \lambda x$. The vectors x are called *eigenvectors* and the values λ are the corresponding *eigenvalues*. Find the eigenvalues and corresponding eigenvectors for the matrix A.

Sol^n :-

$$Ax = \lambda x$$

$$\Rightarrow \quad (A - \lambda I)x = 0$$

Above eq has non zero solutions if and only if $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -2-\lambda & 2 \\ -6 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$$

$$\Rightarrow \quad \lambda = 1, 2 \quad \text{\# Eigen values}$$

For $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$$

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{aligned} -3x_1 + 2x_2 &= 0 \\ -6x_1 + 4x_2 &= 0 \end{aligned} \quad \Rightarrow \quad x_1 = \frac{2}{3} x_2$$

$$\Rightarrow \quad x = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} x_2$$

$$\therefore \boxed{\text{Eigen vector } \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} x \text{ for } \lambda = 1}$$

For $\lambda = 2$

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{array}{l} -4x_1 + 2x_2 = 0 \\ -6x_1 + 3x_2 = 0 \end{array} \Rightarrow x_1 = \tfrac{1}{2}x_2$$

$$\Rightarrow x = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} x_2$$

$\therefore$ $\boxed{\text{Eigenvector } \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} x \text{ for } \lambda = 2}$ //

– Consider a diagonal matrix Λ which has the eigenvalues of A as its diagonal entries. Find the matrix U such that the equation $AU = U\Lambda$ holds.

Sol^n Let $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$AU = U\Lambda$$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2a + 2c & -2b + 2d \\ -6a + 5c & -6b + 5d \end{bmatrix} = \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3a + 2c & -4b + 2d \\ -6a + 4c & -6b + 3d \end{bmatrix} = 0$$

$$\Rightarrow \quad a = \frac{2}{3}c \quad , \quad b = \frac{d}{2}$$

$$\Rightarrow \quad \boxed{U = \begin{bmatrix} \frac{2}{3}c & \frac{d}{2} \\ c & d \end{bmatrix} \quad \text{for} \quad c, d \in \mathbb{R}}$$

– Note that we can write the matrix A as $A = U\Lambda U^{-1}$. Find the inverse of the matrix U computed in the previous question and verify.

Sol^n

$$U = \begin{bmatrix} \frac{2}{3}c & \frac{d}{2} \\ c & d \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$|U| = \frac{2}{3}cd - \frac{cd}{2} = \frac{cd}{6} \quad , \quad A = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}$$

$$U^{-1} = \frac{1}{(cd/6)} \begin{bmatrix} d & -c \\ -d/2 & \frac{2}{3}c \end{bmatrix}^T$$

$$= \frac{1}{(cd/6)} \begin{bmatrix} d & -d/2 \\ -c & \frac{2}{3}c \end{bmatrix}$$

$$U\Lambda U^{-1} = \frac{1}{(cd/6)} \begin{bmatrix} \frac{2}{3}c & \frac{d}{2} \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} d & -d/2 \\ -c & \frac{2}{3}c \end{bmatrix}$$

$$= \frac{1}{(cd/6)} \begin{bmatrix} \frac{2}{3}c & d \\ c & 2d \end{bmatrix} \begin{bmatrix} d & -d/2 \\ -c & \frac{2}{3}c \end{bmatrix}$$

$$= \frac{1}{(cd/6)} \begin{bmatrix} \frac{2}{3}cd - cd & -cd/3 + 2cd/3 \\ cd - 2cd & -cd/2 + 4cd/3 \end{bmatrix}$$

$$= \frac{1}{(cd/6)} \begin{bmatrix} -cd/3 & cd/3 \\ -cd & 5cd/6 \end{bmatrix}$$

$$\boxed{U\Lambda U^{-1} = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix} = A}$$

- Show that for any square matrix A, the eigenvectors of A are also eigenvectors of A^2. What are the eigenvalues for A^2?

Sol^n

Let λ be an eigenvalue of A.

$\Rightarrow \quad A v = \lambda v$ for some $v \neq 0$

Multiplying A both sides

$$\Rightarrow \quad A.(Av) = A.(\lambda v)$$

$$\Rightarrow \quad A^2 v = \lambda A v = \lambda (\lambda v).v$$

$$\Rightarrow \quad A^2 v = \lambda^2 v$$

$\therefore$ λ^2 is an ei value of A^2 and eigenvector of A i.e. v is also eigenvector of A^2.

3. Probability

- If two binary random variables X and Y are independent, are $\bar{X}$ ($\bar{X}$ is the complement of X) and Y also independent? Prove your claim.

Sol^n Given: X and Y are independent

$$\Rightarrow \quad P(X \cap Y) = P(X) \cdot P(Y) \longrightarrow (1)$$

To show, $\bar{X}$ and Y are also independent.

i.e, $P(\bar{X} \cap Y) = P(\bar{X}) \cdot P(Y)$

$$\bar{X} \cap Y = (1 - X) \cap Y = Y - X \cap Y$$

$$\therefore \quad P(\bar{X} \cap Y) = P(Y) - P(X \cap Y)$$

$$\Rightarrow \quad P(\bar{X} \cap Y) = P(Y) - P(X) \cdot P(Y)$$

$$= (1 - P(X)) \cdot P(Y)$$

$$\therefore \quad \boxed{P(\bar{X} \cap Y) = P(\bar{X}) \cdot P(Y)}$$

$\therefore$ $\bar{X}$ & Y are independent.

- Show that if two variables x and y are independent, then their covariance is zero.

Sol^n

$$\begin{aligned}
cov(X,Y) &= E[(X-E[X])(Y-E[Y])] \\
&= E[XY - XE[Y] - YE[X] + E[X]E[Y]] \\
&= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

We know that for independent RV x, y

$E[xy] = E[x]E[y]$

$\Rightarrow cov(X,Y) = E[XY] - E[XY] = 0$

- By using a change of variables, verify that the univariate Gaussian distribution given by:

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

satisfies the equation:

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)x\,dx = \mu$$

Next, by differentiating both sides of the normalization condition:

$$\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)dx = 1$$

with respect to σ^2, verify that the Gaussian satisfies the equation:

$$\mathbb{E}[x^2] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)x^2dx = \mu^2 + \sigma^2$$

Sol^n

$$E(x) = \int p(x)\, x\, dx$$

$$\boxed{\text{To verify:- } E[x] = \int_{-\infty}^{\infty} N(x|\mu,\sigma^2)x\,dx = \mu}$$

$$E[x] = \int_{-\infty}^{\infty} N(x|\mu,\sigma^2)\, x\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right) x\, dx$$

Put $y = x - \mu$

$$\Rightarrow E(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}y^2\right)(y+\mu)\,dy$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}y^2\right)dy + \cancel{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}y^2\right)y\,dy} \rightarrow 0$$

Odd function integrated over real line is zero

$$E(x) = \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}y^2\right)dy$$

Put $z = \frac{y}{\sqrt{2\sigma^2}}$

$$\Rightarrow \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} y^2\right) dy = \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-z^2) \sqrt{2\sigma^2}\, dz$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-z^2)\, dz$$

$$= \frac{\mu}{\sqrt{\pi}} \times \sqrt{\pi}$$

$$\boxed{E[x] = \mu}$$

To verify:- $E[x^2] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\, x^2\, dx = \mu^2 + \sigma^2$

We know that,

$$\frac{d(fg)}{dx} = f \frac{dg}{dx} + g \frac{df}{dx}$$

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Let $f(\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}}$, $g(\sigma^2) = \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$

$$\mathcal{N}(x|\mu,\sigma^2) = f(\sigma^2)\, g(\sigma^2)$$

We know that $\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\, dx = 1$

Differentiating above eq we get,

$$\frac{d}{d\sigma^2}\left(\int f(\sigma^2)\, g(\sigma^2)\, dx = 1\right)$$

$$\Rightarrow \int\left[f(\sigma^2)\frac{d}{d\sigma^2}g(\sigma^2) + g(\sigma^2)\frac{d}{d\sigma^2}f(\sigma^2)\right]dx = 0$$

$$\frac{d}{d\sigma^2}g(\sigma^2) = \frac{(x-\mu)^2}{2\sigma^4}\, g(\sigma^2)$$

$$\frac{d}{d\sigma^2}f(\sigma^2) = -(2\pi\sigma^2)^{-3/2}\pi = -(f(\sigma^2))^3\pi$$

$$\int\left[f(\sigma^2)\, g(\sigma^2)\frac{(x-\mu)^2}{2\sigma^4} - g(\sigma^2)(f(\sigma^2))^3\pi\right]dx = 0$$

$$\Rightarrow \int f(\sigma^2)\, g(\sigma^2)\left[\frac{(x-\mu)^2}{2\sigma^4} - \pi(f(\sigma^2))^2\right]dx = 0$$

$$\Rightarrow \int \frac{N(x|\mu,\sigma^2)\,(x-\mu)^2\,dx}{2\sigma^4} - \frac{1}{2\sigma^2}\overbrace{\int N(x|\mu,\sigma^2)\,dx}^{\to 1} = 0$$

[Eq ①]

We know that $\int_{-\infty}^{\infty} N(x|\mu,\sigma^2)\,dx = 1$

$$\frac{1}{2\sigma^4}\left[\int N(x|\mu,\sigma^2)\,x^2\,dx + \mu^2\underbrace{\int N(x|\mu,\sigma^2)\,dx}_{\Downarrow\ 1} - 2\mu\underbrace{\int N(x|\mu,\sigma^2)\,x\,dx}_{\Downarrow\ \mu}\right]$$

$$\Rightarrow \frac{1}{2\sigma^4}\left[\int N(x|\mu\sigma^2)\,x^2\,dx + \mu^2 - 2\mu^2\right]$$

$$\Rightarrow \frac{1}{2\sigma^4}\left[\int N(x|\mu,\sigma^2)\,x^2\,dx - \mu^2\right]$$

Put in eq ①

$$\Rightarrow \frac{1}{2\sigma^4}\left[\int N(x|\mu,\sigma^2)\,x^2\,dx - \mu^2\right] - \frac{1}{2\sigma^2} = 0$$

$$\Rightarrow \int N(x|\mu,\sigma^2)\,x^2\,dx = \mu^2 + \sigma^2$$

$$\Rightarrow \boxed{\mathbb{E}(x^2) = \mu^2 + \sigma^2}$$

- There are two coins $C1$ and $C2$. $C1$ has a equal prior on a head ($H = 1$) or tail ($T = 0$) and the fate of $C2$ is dependent on $C1$. If $C1$ is a head, $C2$ will be a head with probability 0.7. If $C1$ is a tail, $C2$ will be a head with probability 0.5. $C1$ and $C2$ are tossed in sequence once, and the observed sum of the two coins, $S = C1 + C2$, is 1. What is the probability that $C1 = T$ and $C2 = H$ (*Hint: use Bayes theorem*)?

Solⁿ To find, $P(C_1 = T, C_2 = H \mid S = 1)$

$$P(C_1, C_2 \mid S) = \frac{P(C_1, C_2, S)}{P(S)}$$

$$= \frac{P(S \mid C_1, C_2) \cdot P(C_1, C_2)}{P(S)}$$

$$= \frac{P(S \mid C_1, C_2) \cdot P(C_2 \mid C_1) \cdot P(C_1)}{P(S)}$$

So,

$$P(C_1 = T, C_2 = H \mid S = 1) = \frac{P(S=1 \mid C_1 = T, C_2 = H) \cdot P(C_2 = H \mid C_1 = T) \cdot P(C_1 = T)}{P(S=1)}$$

$$P(S=1 \mid C_1 = T, C_2 = H) = 1$$
$$P(C_2 = H \mid C_1 = T) = 0.5$$
$$P(C_1 = T) = 0.5$$

$$P(S=1) = \sum_{C_1} \sum_{C_2} P(S, C_1, C_2)$$

$$= \sum_{C_1} \sum_{C_2} P(S \mid C_1, C_2) \cdot P(C_2 \mid C_1) \cdot P(C_1)$$

$C_1 = T, C_2 = H$ $\quad$ $C_1 = H, C_2 = T$

$1 \cdot 0.5 \cdot 0.5$ $\quad$ $1 \cdot 0.5 \cdot 0.3$

$$P(S=1) = 0.5 \times 0.8$$

$$\boxed{P(C_1 = T, C_2 = H \mid S = 1) = \frac{1 \cdot 0.5 \cdot 0.5}{0.5 \cdot 0.8} = \frac{5}{8}}$$

- Suppose that we have three coloured boxes r (red), b (blue), and g (green). Box r contains 3 apples, 4 oranges, and 3 limes, box b contains 1 apple, 1 orange, and 0 limes, and box g contains 3 apples, 3 oranges, and 4 limes. If a box is chosen at random with probabilities $p(r) = 0.2$, $p(b) = 0.2$, $p(g) = 0.6$, and a piece of fruit is removed from the box (with equal probability of selecting any of the items in the box), then what is the probability of selecting an apple? If we observe that the selected fruit is in fact an orange, what is the probability that it came from the green box?

Sol^n

$P(\text{apple}) = \sum_{\text{box}} P(\text{apple}|\text{box}) \cdot P(\text{box})$

$\Rightarrow P(\text{apple}|r) \cdot P(r) + P(\text{apple}|g) \cdot P(g) + P(\text{apple}|b) \cdot P(b)$

$\Rightarrow 3/10 \cdot 0.2 + 3/10 \cdot 0.6 + 1/2 \cdot 0.2$

$\Rightarrow \boxed{P(\text{apple}) = 0.34}$

$P(\text{green box}|\text{orange}) = ??$

$$P(g|\text{orange}) = \frac{P(\text{orange}|g) \cdot P(g)}{P(\text{orange})}$$

$P(\text{orange}) = P(\text{orange}|r) \cdot r) + P(\text{orange}|g) \cdot P(g) + P(\text{oran}|b) \cdot P(b)$

$= 4/10 \cdot 0.2 + 3/10 \cdot 0.6 + 1/2 \cdot 0.2$

$= 0.36$

$$\boxed{P(g|\text{orange}) = \frac{3/10 \cdot 0.6}{0.36} = \frac{1}{2} = 0.5}$$