Que 1

a) $u_{xx} + u_{yy} = 0$

Solⁿ:- $D(u) = f(x,y)$

Here $D(u) = u_{xx} + u_{yy}$

as $f(x,y) = 0$ $\therefore$ $\boxed{\text{Homogeneous}}$

$\boxed{\text{Order :- } 2} \rightarrow (u_{xx}, u_{yy})$

$\boxed{\text{Linear}}$ as coefficients of terms involving u is not function of u.

b) $u x u_x + u^2 x y u_{xy} = 0$

Solⁿ:- $D(u) = u x u_x + u^2 x u_{xy} = 0$

$\therefore$ $\boxed{\text{Homogeneous}}$

$\boxed{\text{Order :- } 2} \rightarrow (u_{xy})$.

$\boxed{\text{Non-linear}}$ due to, $u x u_x$ & $u^2 x y u_{xy}$ present.

c) $(u_t)^2 + f(x,y)u = g(x,y)$

Solⁿ:- $D(u) = (u_t)^2 + f(x,y)u = g(x,y) \neq 0$

$\therefore$ $\boxed{\text{Non-Homogeneous / Hetrogeneous}}$

$\boxed{\text{Order - } 1} \rightarrow (u_t, u)$

$\boxed{\text{Non-linear}} \rightarrow$ due to $(u_t)^2$.

d) $u_t + u u_x = \nu u_{xx}$

Solⁿ:- $D(u) = u_t + u u_x - \nu u_{xx} = 0$

$\therefore$ $\boxed{\text{Homogeneous}}$

$\boxed{\text{Order - } 2} \rightarrow (u_{xx})$

$\boxed{\text{Non-linear}} \rightarrow$ due to $u u_x$

e) $U_t + CU_x = 0$

Sol^n: $D(U) = U_t + CU_x = 0$

$\therefore$ $\boxed{\text{Homogeneous}}$

$\boxed{\text{Order-1}} \rightarrow (U_t, U_x)$

$\boxed{\text{Linear}}$ → as coefficient of terms involving u are function of x, y, z, t.

Que 2

a) $x^2 U_{xx} - 2xy U_{xy} + y^2 U_{yy} = 0$

Sol^n:- Comparing with $AU_{xx} + 2BU_{xy} + CU_{yy}$

$$A = x^2, \quad 2B = -2xy, \quad C = y^2$$

$$\Delta = B^2 - AC = (-xy)^2 - x^2y^2 = 0$$

$\therefore$ $\boxed{\text{Parabolic} \rightarrow 2^{nd} \text{ Order PDE}}$

$\therefore$ $\boxed{\text{1 real Characteristic}}$

$$\frac{dy}{dx} = \frac{-B \pm \sqrt{\Delta}}{A} = \frac{-B}{A} \pm 0 = \frac{xy}{x^2} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\rightarrow \ln y = \ln cx \Rightarrow \boxed{y = cx}$$

b) $U_{xx} + (1+y^2)^2 U_{yy} - 2y(1+y^2) U_y = 0$

Sol^n:- Comparing with $AU_{xx} + 2BU_{xy} + CU_{yy}$

$$A = 1, \quad 2B = 0, \quad C = (1+y^2)^2$$

$$\Rightarrow \Delta = B^2 - AC = -(1+y)^2 < 0$$

$\therefore$ $\boxed{\text{Elliptic } 2^{nd} \text{ Order PDE}}$

c) $x^2 u_{xx} - 2xy\, u_{xy} - 3y^2 u_{yy} = u_y - u_x + f(x,y)u$

Solⁿ:- Comparing with $Au_{xx} + 2Bu_{xy} + Cu_{yy}$

$A = x^2 \,, 2B = -2xy \,, C = -3y^2$

$\Delta = B^2 - AC = 4x^2y^2 + 3x^2y^2 = 7y^2x^2$

$\Delta = 7x^2y^2$

$\Delta = 0 \,, x = 0 \,, y = 0$

Parabolic

1 real characteristic

$$\frac{dy}{dx} = \frac{-B \pm \sqrt{\Delta}}{A} = \frac{xy}{x^2} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \ln y = \ln cx$$

$$\Rightarrow \boxed{y = cx}$$

$\Delta \neq 0 \,, x \neq 0 \,, y \neq 0$

Hyperbolic

2 real characteristics

$$\frac{dy}{dx} = \frac{-B \pm \sqrt{\Delta}}{A}$$

$$\frac{dy}{dx} = \frac{xy \pm \sqrt{7}\, xy}{x^2}$$

$$\frac{dy}{dx} = \frac{y}{x}(1+\sqrt{7})$$

$$\frac{dy}{y} = \frac{dx}{x}(1+\sqrt{7})$$

$$\boxed{y = cx^{\sqrt{7}+1}}$$

$$\frac{dy}{dx} = \frac{y}{x}(1-\sqrt{7})$$

$$\frac{dy}{y} = -\frac{dx}{x}(\sqrt{7}+1)$$

$$\ln y = -\ln x + \ln c$$

$$y = cx^{1-\sqrt{7}}$$

$$\Rightarrow \boxed{y x^{1+\sqrt{7}} = c}$$

Que 3 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(x)\frac{\partial u}{\partial x} = f(x,y)$

transform to $\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} = A(x) f(x,y)$, $a(x) = \frac{d A(x)}{dx}$

Sol^n:-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(x)\frac{\partial u}{\partial x} = f(x,y)$$

$$\Rightarrow A(x)\frac{\partial^2 u}{\partial x^2} + A(x)\frac{\partial^2 u}{\partial y^2} + A(x)a(x)\frac{\partial u}{\partial x} = \underbrace{A(x) f(x,y)}_{\substack{\text{Substitute} \\ \tilde{u}\text{ terms}}}$$

$$\Rightarrow A(x)\frac{\partial^2 u}{\partial x^2} + A(x)\frac{\partial^2 u}{\partial y^2} + A(x)a(x)\frac{\partial u}{\partial x} = \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2}$$

Here ~~doing some~~ by some intuition,

$$\frac{\partial^2 \tilde{u}}{\partial y^2} = A(x)\frac{\partial^2 u}{\partial y^2}$$

We can assume

$$\tilde{u} = A(x) u + \text{some function of } x.$$

let $F(x)$ be that function

$$\tilde{u} = A(x) u + F(x)$$

$$\Rightarrow \frac{\partial^2 \tilde{u}}{\partial y^2} = A(x)\frac{\partial^2 u}{\partial y^2}$$

It should then also satisfy other half

$$\Rightarrow A(x)\frac{\partial^2 u}{\partial x^2} + A(x)a(x)\frac{\partial u}{\partial x} = \frac{\partial^2}{\partial x^2}\left(A(x)u + F(x)\right)$$

$$\Rightarrow \cancel{A(x)\frac{\partial^2 u}{\partial x^2}} + A(x)a(x)\frac{\partial u}{\partial x} = \cancel{A(x)\frac{\partial^2 u}{\partial x^2}} + 2A'(x)\frac{\partial u}{\partial x} + A''(x)u + F''(x)$$

$$\Rightarrow \boxed{F''(x) = A(x)a(x)\frac{\partial u}{\partial x} - \frac{d^2 A(x)}{dx^2} - 2\left(\frac{dA}{dx}\right)u}$$

$\therefore$ $\tilde{u}(x,y) = A(x)u(x,y) + F(x)$ where $F(x)$ can be found from above eq^n.

Que 4 :- $T_{xx} + T_{yy} = 0 \qquad 0 \le x \le a, \quad 0 \le y \le b, \quad a = 1m, \ b = 0.5m$

$$\frac{\partial T}{\partial y}(x,0) = 0, \quad \frac{\partial T}{\partial y}(x,b) = 0, \quad T(0,y) = T_0 = 300K$$

$$T(a,y) = T_0\left[1 + 0.2\left(\frac{y}{b} - 0.5\right)\right]$$

Sol^n :- We try to make left boundary $T(0,y) = T_0$ to 0 by some transformation.

let $T(x,y) = T_0 + t(x,y)$

Here, $T(0,y) = T_0 = T_0 + t(0,y) \Rightarrow \boxed{t(0,y) = 0}$

$$T(a,y) = T_0\left[1 + 0.2\left(\frac{y}{b} - 0.5\right)\right] = T_0 + t(a,y)$$

$$\Rightarrow \boxed{t(a,y) = 0.2T_0\left(\frac{y}{b} - 0.5\right)}$$

$$\boxed{\frac{\partial T}{\partial y}(x,0) = \frac{\partial t}{\partial y}(x,0) = 0}$$

$$\boxed{\frac{\partial T}{\partial y}(x,b) = \frac{\partial t}{\partial y}(x,b) = 0}$$

$$\boxed{T_{xx} + T_{yy} = t_{xx} + t_{yy} = 0}$$

As for $t(x,y)$ laplace's equation and one of the boundary condition is linear and homogenous, the method of separation of variables can be applied.

let $t(x,y) = F(x)G(y)$

$t_{xx} + t_{yy} = 0$

$\Rightarrow F''G + G''F = 0$

$\Rightarrow \dfrac{F''}{F} = -\dfrac{G''}{G} = \lambda$ (let)

$\Rightarrow \dfrac{F''}{F} = \lambda \qquad | \qquad G'' = -G\lambda$

Three cases $\lambda > 0, \ \lambda = 0, \ \lambda < 0$.

Case I :- $\lambda > 0$

$G'' = -\lambda G$

$\Rightarrow$ Solution :- $G(y) = c_1 \cos\sqrt{\lambda}y + c_2 \sin\sqrt{\lambda}y$

$\frac{\partial t}{\partial y} = F(x)\, G'(y)$

$\frac{\partial t}{\partial y}(x,0) = F(x)\, G'(0) = 0 \Rightarrow G'(0) = 0$

similarly $G'(b) = 0$

$G'(y) = \sqrt{\lambda}(-c_1 \sin\sqrt{\lambda}y + c_2 \cos\sqrt{\lambda}y)$

$G'(0) = 0 \Rightarrow c_2 = 0$

$G'(b) = 0 \Rightarrow -c_1\sqrt{\lambda}\sin\sqrt{\lambda}b = 0$

$\Rightarrow \sqrt{\lambda}b = n\pi$

$\Rightarrow \lambda = \frac{n^2\pi^2}{b^2}$

positive eigenvalues

$\Rightarrow \boxed{G(y) = c_1 \cos\left(\frac{n\pi y}{b}\right)}$

$F'' = \lambda F$

$\Rightarrow$ Solution :- $F(x) = c_3 \cosh\left(\frac{n\pi x}{b}\right) + c_4 \sinh\left(\frac{n\pi x}{b}\right)$

$t(0,y) = 0$

$\Rightarrow F(0)\, G(y) = 0 \Rightarrow F(0) = 0 \Rightarrow c_3 = 0$

$\Rightarrow \boxed{F(x) = c_4 \sinh\left(\frac{n\pi x}{b}\right)}$

Case II :- $\lambda = 0$

$G'' = 0$

$\Rightarrow G'(y) = c_1$

As $G'(0) = 0 \Rightarrow c_1 = 0$

$\Rightarrow G(y) = c$

But $G(y)$ is nonzero, for zero eigenvalue

$\therefore$ Eigenfunction

$\Rightarrow \boxed{G_0(y) = 1}$

$F''(x) = 0$

$F(x) = c_5 + c_6 x$

$F(0) = 0 \Rightarrow c_5 = 0$

$\Rightarrow \boxed{F(x) = c_6 x}$

Case III :- $\lambda < 0$

$\Rightarrow G'' = G\lambda$

$\Rightarrow$ Solution:- $G(y) = k_1 \cosh\sqrt{\lambda}y + k_2 \sinh\sqrt{\lambda}y$

$G'(y) = \sqrt{\lambda}\,(k_1 \sinh\sqrt{\lambda}y + k_2 \cosh\sqrt{\lambda}y)$

$G'(0) = 0 \Rightarrow k_2 = 0$

$G'(b) = 0 \Rightarrow \sqrt{\lambda}\,k_1 \sinh(\sqrt{\lambda}\,b) = 0$

$\Rightarrow k_1 = 0 \rightarrow$ trivial sol^n.

~~As no nonzero value of~~

As trivial sol^n $\rightarrow$ so no negative eigen value.

By principle of superposition, general sol^n :-

$$t(x,y) = a_1(C_6 x)\cdot(1) + \sum_{n=1}^{\infty} a_2 C_4 \sinh\left(\frac{n\pi}{b}x\right) C_1 \cos\left(\frac{n\pi}{b}y\right)$$

$$\Rightarrow t(x,y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{b}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

For ~~fixing A_0~~ :

$$t(a,y) = A_0 a + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{b}a\right)\cos\left(\frac{n\pi y}{b}\right) = 0.2\,T_0\left(\frac{y}{b} - 0.5\right) \quad -(1)$$

Integrate w.r.t to y from $(0,b)$

$$\Rightarrow \int_0^b A_0 a\,dy + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi a}{b}\right)\underbrace{\cancel{\int_0^b \cos\left(\frac{n\pi y}{b}\right)dy}}_{0} = \underbrace{\cancel{\int_0^b 0.2T_0\left(\frac{y}{b} - 0.5\right)dy}}_{0}$$

$$\Rightarrow A_0 ab + 0 = 0$$

$$\Rightarrow \boxed{A_0 = 0}$$

Multiply eq (1) by $\cos\left(\frac{m\pi y}{b}\right)$ and integrate w.r.t y from $(0,b)$

$$\Rightarrow \int_0^b \underbrace{\cancel{A_0}}_{0} a \cos\left(\frac{m\pi y}{b}\right)dy + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi a}{b}\right)\underbrace{\int_0^b \cos\frac{m\pi y}{b}\cos\frac{n\pi y}{b}\,dy}_{\text{Non zero for } m=n} = \int_0^b 0.2T_0\left(\frac{y}{b} - 0.5\right)\cos\left(\frac{m\pi y}{b}\right)dy$$

$$\Rightarrow A_n \sinh\left(\frac{n\pi a}{b}\right)\underbrace{\int_0^b \cos^2\frac{n\pi y}{b}\,dy}_{\text{Avg value } = b/2} = \cancel{\left[\frac{\sin\left(\frac{n\pi y}{b}\right)0.2T_0\left(\frac{y}{b} - 0.5\right)}{\frac{n\pi}{b}}\right]_0^b} - \frac{1}{n\pi}\int_0^b \sin\frac{n\pi}{b}\,dy$$

$$\Rightarrow A_n \sinh\left(\frac{n\pi a}{b}\right) \times \frac{b}{2} = \frac{b}{n^2\pi^2}(\cos n\pi - 1) \times 0.2T_0$$

$$\Rightarrow A_n = \frac{0.4T_0}{\pi^2} \frac{(\cos n\pi - 1)}{n^2 \sin\left(\frac{n\pi a}{b}\right)}$$

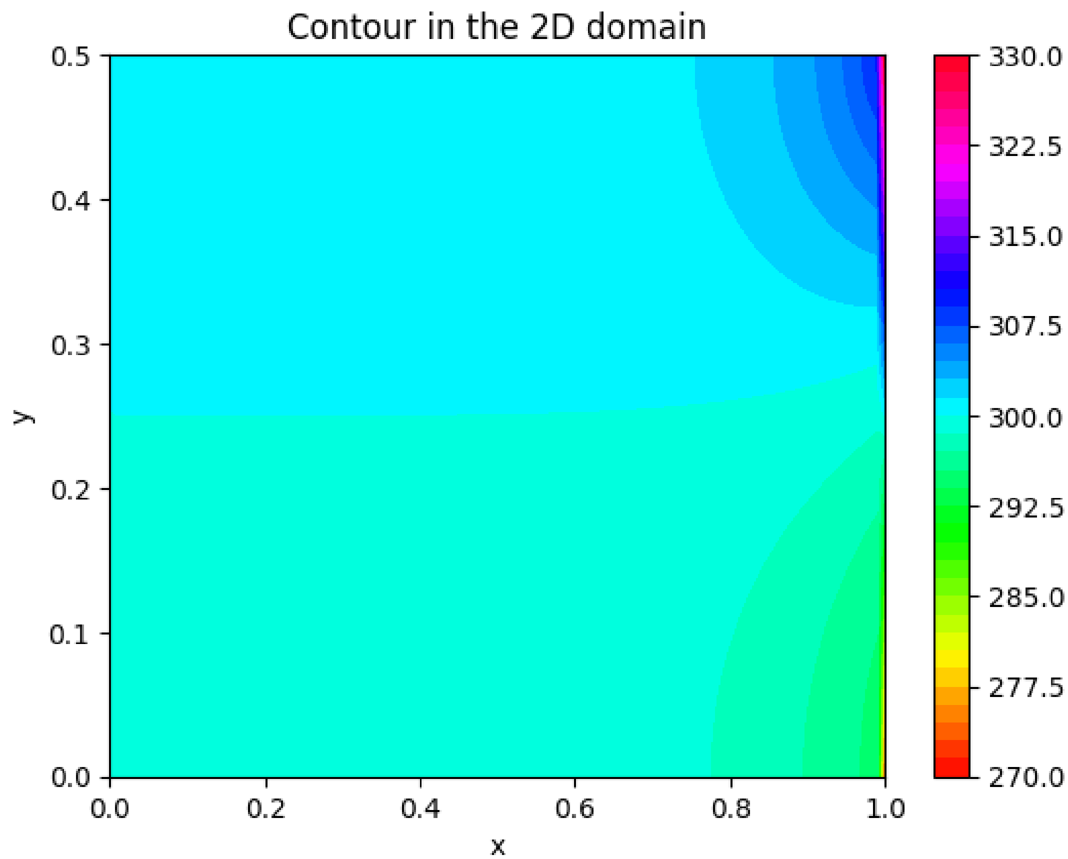
$$\Rightarrow t(x,y) = 0 + \frac{0.4T_0}{\pi^2}\sum_{n=1}^{\infty} \frac{(\cos n\pi - 1)}{\sin\left(\frac{n\pi a}{b}\right) \times n^2} \sinh\left(\frac{n\pi x}{b}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow T(x,y) = T_0 + t(x,y)$$

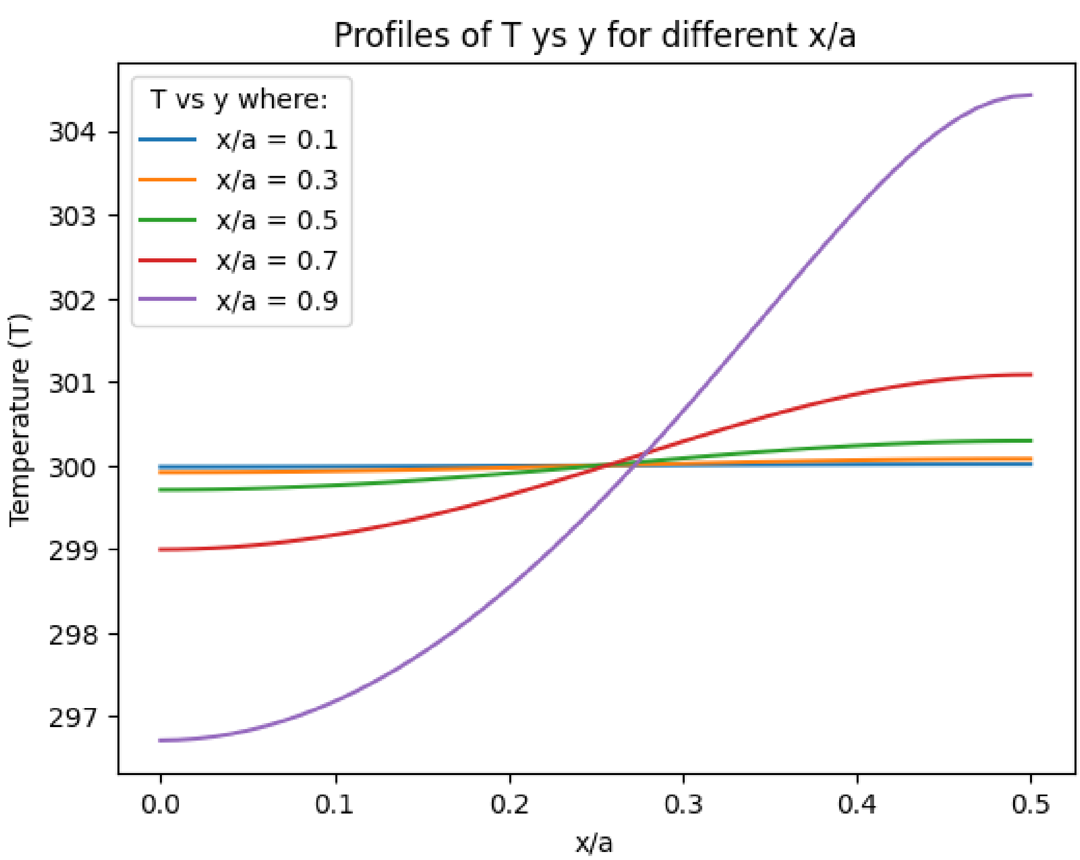
$$\boxed{T(x,y) = T_0 + \frac{0.4T_0}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(n\pi - 1)}{n^2 \sin\left(\frac{n\pi a}{b}\right)} \sinh\left(\frac{n\pi x}{b}\right) \cos\left(\frac{n\pi y}{b}\right)}$$

Que 4) Analytic Solution plots: -

(a) contours in the 2D domain;



(b) profiles of T vs y for x/a = 0.1, 0.3, 0.5, 0.7 and 0.9;



(c) profiles of T vs x for y/b = 0.1, 0.5 and 0.9.