1. Write weak forms, using the weighted residual method, for the following differential equations and boundary conditions. The primary variable in each case is u and the domain for each one-dimensional differential equation is $0 < x < 1$.

(a) $a\dfrac{du}{dx} + cu + q = 0$, $u(0) = u_0$

Natural BC [As u at $x=0$ known] $\therefore w(0) = 0$

Sol^n

Here $R = a\dfrac{du}{dx} + cu + q$

Let weights be arbitrary function of x, $w(x)$ for $0<x<1$

We have $\int_0^1 w\, R\, dx = 0$

$$\Rightarrow \int_0^1 w\left[a\frac{du}{dx} + cu + q\right]dx = 0$$

$$\Rightarrow \int_0^1 w\left(a\frac{du}{dx}\right)dx + \int_0^1 w\,(cu+q)\,dx = 0$$

$$\Rightarrow w\,a\,u\Big|_0^1 - \int_0^1 w'\,a\,u\,dx + \int_0^1 w\,(cu+q)\,dx = 0$$

$$\Rightarrow w(1)\,a\,u(1) - \cancel{w(0)}\,a\,u(0) + \int_0^1 \left[w\,(cu+q) - w'\,a\,u\right]dx = 0$$

$$\Rightarrow \boxed{w(1)\,a\,u(1) + \int_0^1 \left[w\,[cu+q] - w'au\right]dx = 0}$$

Weak form

(b) $\dfrac{d}{dx}\left(a\dfrac{du}{dx}\right) + q = 0, \quad u(0) = u_0, \quad \left(a\dfrac{du}{dx} + ku\right)_{x=1} = \theta_0$

Sol^n

$$R = \frac{d}{dx}\left(a\frac{du}{dx}\right) + q$$

Let weights be arbitrary function of x

$w(x), \quad 0 < x < 1$

We have $\displaystyle\int_0^1 w R \, dx = 0$

$$\Rightarrow \int_0^1 w\left[\frac{d}{dx}\left(a\frac{du}{dx}\right) + q\right] dx = 0$$

$$\Rightarrow w\left(a\frac{du}{dx}\right)\Big|_0^1 - \int_0^1 w'\left(a\frac{du}{dx}\right) dx + \int_0^1 wq \, dx = 0$$

$$\Rightarrow w\, a\frac{du}{dx}\Big|_{x=1} - \cancel{w\, a}\frac{du}{dx}\Big|_{x=0} - \int_0^1 w'\left(a\frac{du}{dx}\right) dx + \int_0^1 wq\, dx = 0$$

Natural BC

At $x = 0$, $u(0) = u_0$ $\quad u$ is known $\therefore w(0) = 0$

At $x = 1$ $\quad \left(a\dfrac{du}{dx} + ku\right) = \theta_0$

$$\boxed{w(1)\left[\theta_0 - ku(1)\right] - \int_0^1 w'\left(a\frac{du}{dx}\right) dx + \int_0^1 wq\, dx = 0}$$

Weak form

(C) $\quad -\frac{d}{dx}\left(a\frac{du}{dx}\right) + b\frac{du}{dx} + q = 0, \quad u(0) = u_0, \quad u(1) = u_1$

$\underline{\underline{Sol^n}}$ $\quad R = -\frac{d}{dx}\left(a\frac{du}{dx}\right) + b\frac{du}{dx} + q$

Let weights be arbitrary function of x
$w(x)$ for $0 < x < 1$

As $u(0) = u_0$, $u(1) = u_1$ i.e, u is known at $x=0,1$

$\therefore$ $w(0) = 0$, $w(1) = 1$ [Natural BC]

We have $\int_0^1 w R \, dx = 0$

$$\Rightarrow \int_0^1 w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) + b\frac{du}{dx} + q\right] dx = 0$$

$$\Rightarrow -w\, a\frac{du}{dx}\Big|_0^1 + \int_0^1 w'\left(a\frac{du}{dx}\right)dx + \int_0^1 w\left[b\frac{du}{dx} + q\right]dx = 0$$

$$-\cancel{w(1)}^{\,0}\, a\frac{du}{dx}\Big|_{x=1} + \cancel{w(0)}^{\,0}\, a\frac{du}{dx}\Big|_{x=0}$$

$$\boxed{\int_0^1 w'\left(a\frac{du}{dx}\right)dx + \int_0^1 wb\frac{du}{dx}\,dx + \int_0^1 wq\,dx = 0}$$

Weak form

$$\left(\frac{dw}{dx}\right)_{x=0} = \varphi_0$$

(d) $\frac{d}{dx}\left(a\frac{d^2 v}{dx^2}\right) + f = 0$, $v(0)=v_0$, $v(1)=v_1$, $\wedge$

<u>Sol^n</u>

$$R = \frac{d}{dx}\left(a\frac{d^2 v}{dx^2}\right) + f$$

Let weights be arbitrary function of x $w(x)$ for $0 < x < 1$

We know $\int_0^1 w R\, dx = 0$

$$\Rightarrow \int_0^1 w\left[\frac{d}{dx}\left(a\frac{d^2 v}{dx^2}\right) + f\right] dx = 0$$

$$\Rightarrow w a \frac{d^2 v}{dx^2}\Big|_0^1 - \int_0^1 w' a \frac{d^2 v}{dx^2} dx + \int_0^1 w f\, dx = 0$$

$$\cancel{w(1)}\, a\left(\frac{d^2 v}{dx^2}\right)_{x=1} - \cancel{w(0)}\, a\left(\frac{d^2 v}{dx^2}\right)_{x=0}$$

0 0

As $v(0)=0$, $v(1)=0$ $\therefore$ Natural BC

$\therefore$ $w(0) = 0$, $w(1)=0$

$$\boxed{-\int_0^1 w' a\frac{d^2 v}{dx^2} dx + \int_0^1 w f\, dx = 0}$$

Weak form

(e) $-\frac{d}{dx}\left((1+2x^2)\frac{du}{dx}\right) + u = x^2, \quad u(0)=1, \quad \left(\frac{du}{dx}\right)_{x=1} = 2$

Sol^n $R = -\frac{d}{dx}\left((1+2x^2)\frac{du}{dx}\right) + u - x^2$

Let weights be arbitrary function of x
$w(x)$ for $0 < x < 1$

We know $\int_0^1 wR\,dx = 0$

$\Rightarrow \int w\left(-\frac{d}{dx}\left((1+2x^2)\frac{du}{dx}\right) + u - x^2\right)dx = 0$

$\Rightarrow -w(1+2x^2)\frac{du}{dx}\Big|_0^1 + \int_0^1 w(1+2x^2)\frac{du}{dx}\,dx + \int_0^1 w(u-x^2)\,dx = 0$

As $u(0)=1$, Natural BC

$\therefore w(0)=0$

$$\boxed{-6\,w(1) + \int_0^1 w(1+2x^2)\frac{du}{dx}\,dx + \int_0^1 w(u-x^2)\,dx = 0}$$

Weak form

(f) $-\frac{d}{dx}\left(v\frac{dv}{dx}\right) + f = 0\;,\quad v(1) = \sqrt{2}\;,\quad \left(v\frac{dv}{dx}\right)_{x=0} = 0$

Solⁿ

$R = -\frac{d}{dx}\left(v\frac{dv}{dx}\right) + f$

Let weights be arbitrary function of x
$w(x)$ for $0 < x < 1$

We know $\int_0^1 wR\,dx = 0$

$$\Rightarrow \int_0^1 w\left(-\frac{d}{dx}\left(v\frac{dv}{dx}\right) + f\right)dx = 0$$

$$\Rightarrow -w\left(v\frac{dv}{dx}\right)\Big|_0^1 + \int_0^1 w'\left(v\frac{dv}{dx}\right)dx + \int_0^1 wf\,dx = 0$$

$$-\cancel{w(1)}^{\,0}\left(v\frac{dv}{dx}\right)_{x=1} + w(0)\cancel{\left(v\frac{dv}{dx}\right)_{x=0}}^{\,0} \qquad \left(v\frac{dv}{dx}\right)_{x=0} = 0$$

$v(1) = \sqrt{2}$ Natural BC
$\therefore\; w(1) = 0$

$$\boxed{\int_0^1 w'\left(v\frac{dv}{dx}\right)dx + \int_0^1 wf\,dx = 0}$$

Weak form

(g) $-\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right]\right\}=f$, and $\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)-\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right]\right\}=q$, with $u(0)=u(1)=w(0)=w(1)=0$, $\left(\frac{dw}{dx}\right)_{x=0}=0$ and $\left(EI\frac{d^2w}{dx^2}\right)_{x=1}=M_0$

Sol^n

$$R_1 = -\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right]\right\}-f$$

Let weights be arbitrary function of x

$v(x)$ for $0<x<1$

$$\int_0^1 vR\,dx=0$$

$$\Rightarrow \int_0^1 v\left[-\frac{d}{dx}\left(EA\left[\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right)\right)-f\right]dx=0$$

$$\Rightarrow \boxed{-vEA\left(\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right)\Big|_0^1}+\int_0^1 v'EA\left(\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right)dx-\int_0^1 vf\,dx=0$$

$$-\cancel{v(1)}^{0}EA\left(\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right)_{x=1}+\cancel{v(0)}^{0}EA\left(\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right)_{x=0}$$

$$\boxed{\int_0^1 v'EA\left(\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right)dx-\int_0^1 vf\,dx=0}$$

Weak form

$$R_2=\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)\underbrace{-\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right]\right\}}_{f}-q$$

$$\int_0^1 R_2 v\,dx=0$$

Same as previous part

$$\Rightarrow \int_0^1 v\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)dx=v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\Big|_0^1-\int_0^1 v'\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)dx$$

$$\Rightarrow \cancel{v(1)}^{0} \frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]_{x=1} - \cancel{v(0)}^{0} \frac{d}{dx}\left(EI\left(\frac{d^2w}{dx^2}\right)\right)_{x=0} - \int_0^1 v'\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)dx$$

$$\Rightarrow -\int_0^1 v'\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)dx$$

$$\Rightarrow -v' EI\left(\frac{d^2w}{dx^2}\right)\Big|_0^1 + \int_0^1 v'' EI\frac{d^2w}{dx^2}dx$$

$$\Rightarrow \underbrace{-v'(1)EI\left(\frac{d^2w}{dx^2}\right)_{x=1}}_{M_0} + \cancel{v'(0)}^{0} EI\left(\frac{d^2w}{dx^2}\right)_{x=0} + \int_0^1 v'' EI\frac{d^2w}{dx^2}dx$$

As $\left(\frac{dw}{dx}\right)_{x=0} = 0 \quad \therefore v'(0) = 0$

$$\Rightarrow \int_0^1 v\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)dx = -v'(1)M_0 + \int_0^1 v'' EI\frac{d^2w}{dx^2}dx$$

$\therefore$ Combing all we, get

$$\int_0^1 R_2 v\,dx = \boxed{-v'(1)M_0 + \int_0^1 v'' EI\frac{d^2w}{dx^2}dx + \int_0^1 v' EA\left[\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right]dx - \int_0^1 vq\,dx = 0}$$

Weak form

2. Write weak forms, using the variational principle, for the above differential equations and boundary conditions.

(a) $\partial\Pi = \int_{\Omega} \delta x^T A(x)\, dx = 0$ $\quad v(0) = v_0 \therefore w(0) = 0$

NBC

$\Rightarrow \partial\Pi = \int_0^1 \underbrace{\delta x^T}_{w} \left[a\frac{dv}{dx} + cv + q \right] dx = 0$

$\Rightarrow w\, a\, v \Big|_0^1 - \int_0^1 w' \, a\, v\, dx + \int_0^1 w\,(cv+q)\, dx = 0$

$\Rightarrow w(1)\, a\, v(1) - \cancel{w(0)}\, a\, v(0) + \int_0^1 [w\,(cv+q) - w'\, av]\, dx = 0$

NBC

$$\Rightarrow \boxed{w(1)\, a\, v(1) + \int_0^1 [w\,[cv+q] - w' a v]\, dx = 0}$$

Weak form

(b) $\partial\Pi = \int_{\Omega} \delta x^T A(x)\, dx = 0$ $\quad v(0) = v_0 \therefore w(0) = 0$

$\partial\Pi = \int_0^1 \underbrace{\delta x^T}_{w} \left[\frac{d}{dx}\left(a\frac{dv}{dx}\right) + q \right] dx = 0$

$\Rightarrow w\left(a\frac{dv}{dx}\right)\Big|_0^1 - \int_0^1 w' \left(a\frac{dv}{dx}\right) dx + \int_0^1 wq\, dx = 0$

$\Rightarrow w\, a\frac{dv}{dx}\Big|_{x=1} - \cancel{w}\, a\frac{dv}{dx}\Big|_{x=0} - \int_0^1 w' \left(a\frac{dv}{dx}\right) dx + \int_0^1 wq\, dx = 0$

Natural BC

At $x=0$, $v(0) = v_0$ $\quad v$ is known $\therefore w(0) = 0$

At $x=1$ $\quad \left(a\frac{dv}{dx} + kv\right) = \theta_0$

$$\boxed{w(1)[\theta_0 - k v(1)] - \int_0^1 w' \left(a\frac{dv}{dx}\right) dx + \int_0^1 wq\, dx = 0}$$

Weak form

(c) $\partial\Pi = \int_\Omega \delta x^T A(x)\, dx$ Weak form $v(0)=0 \therefore w(0)=0$ [NBC]

$v(1)=0 \therefore w(1)=0$

$\partial\Pi = \int_0^1 \underbrace{\delta x^T}_{w}\left(-\frac{d}{dx}\left(a\frac{dv}{dx}\right) + b\frac{dv}{dx} + q\right) dx$

$\Rightarrow -w\, a\frac{dv}{dx}\Big|_0^1 + \int_0^1 w'\left(a\frac{dv}{dx}\right)dx + \int_0^1 w\left[b\frac{dv}{dx} + q\right]dx = 0$

$-\cancel{w(1)}\, a\frac{dv}{dx}\Big|_{x=1} + \cancel{w(0)}\, a\frac{dv}{dx}\Big|_{x=0}$ — NBC, NBC ($\to 0$)

$$\boxed{\int_0^1 w'\left(a\frac{dv}{dx}\right)dx + \int_0^1 wb\frac{dv}{dx}\,dx + \int_0^1 wq\,dx = 0}$$

Weak form

(d) $\partial\Pi = \int_\Omega \delta x^T A(x)\, dx = 0$

$\partial\Pi = \int_0^1 \underbrace{\delta x^T}_{w}\left(\frac{d}{dx}\left(a\frac{d^2v}{dx^2}\right) + f\right) dx = 0$

$\Rightarrow \int_0^1 w\left[\frac{d}{dx}\left(a\frac{d^2v}{dx^2}\right) + f\right] dx = 0$

$\Rightarrow w\, a\frac{d^2v}{dx^2}\Big|_0^1 - \int_0^1 w'\, a\frac{d^2v}{dx^2}\, dx + \int_0^1 wf\, dx = 0$

$\cancel{w(1)}\, a\left(\frac{d^2v}{dx^2}\right)_{x=1} - \cancel{w(0)}\, a\left(\frac{d^2v}{dx^2}\right)_{x=0}$ ($\to 0$)

As $v(0)=0$, $v(1)=0$ $\therefore$ Natural BC

$\therefore w(0)=0$, $w(1)=0$

$$\boxed{-\int_0^1 w'\, a\frac{d^2v}{dx^2}\, dx + \int_0^1 wf\, dx = 0}$$

Weak form

(e)

$$\partial\Pi = \int_\Omega \partial x^T A(x)\, dx = 0$$

$$\partial\Pi = \int_0^1 \underbrace{\partial x^T}_{\omega} \left[-\frac{d}{dx}\left[(1+2x^2)\frac{dv}{dx}\right] + v - x^2 \right] dx = 0$$

$$\Rightarrow \int \omega \left(-\frac{d}{dx}\left((1+2x^2)\frac{dv}{dx}\right) + v - x^2\right) dx = 0$$

$$\Rightarrow -\omega(1+2x^2)\frac{dv}{dx}\Big|_0^1 + \int_0^1 \omega(1+2x^2)\frac{dv}{dx}\,dx + \int_0^1 \omega(v-x^2)\,dx = 0$$

As $v(0) = 1$, Natural BC

$\therefore\ \omega(0) = 0$

$$\boxed{-6\,\omega(1) + \int_0^1 \omega(1+2x^2)\frac{dv}{dx}\,dx + \int_0^1 \omega(v-x^2)\,dx = 0}$$

Weak form

(f)

$$\partial\Pi = \int_\Omega \partial x^T A(x)\, dx = 0$$

$$\partial\Pi = \int_0^1 \underbrace{\partial x^T}_{\omega} \left[-\frac{d}{dx}\left(v\frac{dv}{dx}\right) + f \right] dx = 0$$

$$\Rightarrow \int_0^1 \omega\left(-\frac{d}{dx}\left(v\frac{dv}{dx}\right) + f\right) dx = 0$$

$$\Rightarrow -\omega\left(v\frac{dv}{dx}\right)\Big|_0^1 + \int_0^1 \omega'\left(v\frac{dv}{dx}\right) dx + \int_0 \omega f\, dx = 0$$

$$-\cancel{\omega(1)}^{\,0}\left(v\frac{dv}{dx}\right)_{x=1} + \cancel{\omega(0)\left(v\frac{dv}{dx}\right)}^{\,0}_{x=0} \qquad \left(v\frac{dv}{dx}\right)_{x=0} = 0$$

$v(1) = \sqrt{2}$ Natural BC

$\therefore\ \omega(1) = 0$

$$\boxed{\int_0^1 \omega\left(v\frac{dv}{dx}\right) dx + \int_0^1 \omega f\, dx = 0}$$

Weak form

(g)

$$\partial\Pi = \int_\Omega \partial x^T A(x)\, dx = 0$$

$$\partial\Pi = \int_0^1 \partial x^T \left[-\frac{d}{dx}\left\{ EA\left[\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right]\right\} - f \right] dx = 0 \qquad (\partial x^T \to v)$$

$$\Rightarrow \boxed{-vEA\left(\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right)\Big|_0^1} + \int_0^1 v' EA\left(\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right) dx - \int_0^1 uf\,dx = 0$$

$$\to -\cancel{v(1)}^{0} EA\left(\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right)_{x=1} + \cancel{v(0)}^{0} EA\left(\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right)_{x=0}$$

$$\to \boxed{\int_0^1 v' EA\left(\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right) dx - \int_0^1 vf\,dx = 0}$$

Weak form

$$\partial\Pi = \int_0^1 \partial x^T \left[\frac{d}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left\{EA\left[\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right]\right\} - q \right] dx = 0 \qquad (\partial x^T \to v)$$

$$\Rightarrow \int_0^1 v\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) dx = v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\Big|_0^1 - \int_0^1 v'\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) dx$$

$$\Rightarrow \cancel{v(1)}^{0}\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]_{x=1} - \cancel{v(0)}^{0}\frac{d}{dx}\left(EI\left(\frac{d^2w}{dx^2}\right)\right)_{x=0} - \int_0^1 v'\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) dx$$

$$\Rightarrow -\int_0^1 v'\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) dx$$

$$\Rightarrow -v' EI\left(\frac{d^2w}{dx^2}\right)\Big|_0^1 + \int_0^1 v'' EI\frac{d^2w}{dx^2}\, dx$$

$$\Rightarrow -\underbrace{v'(1)EI\left(\frac{d^2w}{dx^2}\right)_{x=1}}_{M_0} + \cancel{v'(0)}^{0} EI\left(\frac{d^2w}{dx^2}\right)_{x=0} + \int_0^1 v'' EI\frac{d^2w}{dx^2}\, dx$$

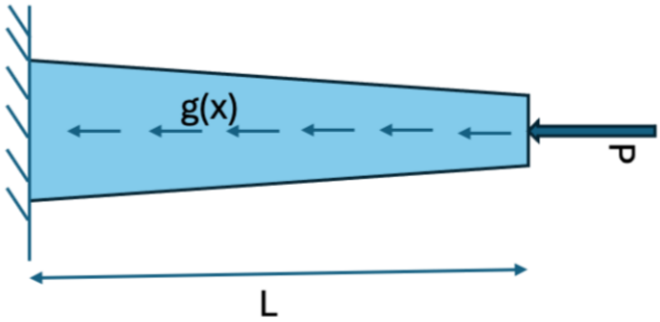
As $\left(\frac{dw}{dx}\right)_{x=0} = 0 \quad \therefore v'(0) = 0$

$$\Rightarrow \int_0^1 v\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) dx = -v'(1)M_0 + \int_0^1 v'' EI\frac{d^2w}{dx^2}\, dx$$

∴ Combing all we, get

$$\boxed{= -v'(1)M_0 + \int_0^1 v'' EI\frac{d^2w}{dx^2}\, dx + \int_0^1 v' EA\left[\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right] dx - \int_0^1 vq\,dx = 0}$$

3. Using the variational principle, write the weak form for the bar shown in the figure below. The cross-sectional area at the left is A_0 and on the right is A_1. The Young's modulus for the bar material is E.



Sol^n

$$\Pi = \Pi_{int} - \Pi_{ext}$$

$$\Pi_{int} = \frac{1}{2}\int_0^L AE\left(\frac{dv}{dx}\right)^2 dx$$

$$\Pi_{ext} = P\,v(L) + \int_0^L g(x)\cdot v(x)\,dx$$

Say, infinitesimal change in function, $\delta v(x) = \varepsilon\, w(x)$ where $w(x)$ is arbitrary function & $0<\varepsilon<1$ is a very small positive number

$$\delta\Pi_{init} = \frac{1}{2}\int_0^1 AE\; 2\left(\frac{dv}{dx}\right)\cdot\frac{d\,\delta v}{dx}\,dx$$

$$\delta\Pi_{ext} = P\,\delta v(L) + \int_0^L g(x)\,\delta v(x)\,dx$$

$\delta v(x) = \varepsilon\, w(x)$

$\rightarrow \frac{\delta\Pi}{\varepsilon} = 0$

$$\delta\Pi = \int_0^L AE\,\frac{dv}{dx}\,\frac{d\,\delta v}{dx}\,dx - P\,\delta v(L) - \int_0^L g(x)\,\delta v(x)\,dx = 0$$

$$\Rightarrow \boxed{\int_0^L AE\,\frac{dv}{dx}\,\frac{dw}{dx}\,dx - P\,\delta_v(L) - \int_0^L g(x)\,w\,dx = 0}$$

Weak form

4. Transient heat conduction in one-dimension is given by.

$$-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+Q+c\frac{\partial T}{\partial t}=0$$

where T is temperature, k is thermal conductivity, Q is heat generation per unit length, and c is specific heat. Boundary conditions may be given as $T_{\Gamma_1}=T_0$ and $\left(-k\frac{\partial T}{\partial x}\right)_{\Gamma_2}=Q_0$

(a) Construct a weak form for the problem.

(b) Using shape functions, $N_1=1-\frac{x}{l_e}$ and $N_2=\frac{x}{l_e}$, construct the semi-discrete form for a typical element of length l_e.

(c) Consider a region of length 10, with properties $k=5$, $c=1$, $Q=0$. Divide the region into four equal-length elements and establish the set of global semi-discrete equations.

Sol^n (a) $R=-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+Q+c\frac{\partial T}{\partial t}$

Let weights be arbitrary function of x $w(x)$

$\int_\Omega R\,w\,dx=0$

$\Rightarrow \int_\Omega w\left(-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+Q+c\frac{\partial T}{\partial t}\right)dx=0$

$\Rightarrow -wk\frac{\partial T}{\partial x}\Big|_{\Gamma_1}^{\Gamma_2}+\int_\Omega w'k\frac{\partial T}{\partial x}dx+\int_\Omega\left(Q+c\frac{\partial T}{\partial t}\right)dx=0$

As $T_{\Gamma_1}=T_0$ $\therefore$ $w(\Gamma_1)=0$

$$\boxed{w(\Gamma_2)Q_0+\int_\Omega w'k\frac{\partial T}{\partial x}dx+\int_\Omega w\left(Q+c\frac{\partial T}{\partial t}\right)dx=0}$$

Weak form

(b) $T=[N]\{c\}$ weights

$T(x)=N_1T_1+N_2T_2$, $w(x)=N_i(x)$

$N_1=1-\frac{x}{l_e}$, $N_2=\frac{x}{l_e}$, $w'(x)=\frac{\partial N_i(x)}{\partial x}$

$$T(x) = \left(1 - \frac{x}{l_e}\right) T_1 + \frac{x}{l_e} T_2$$

$$\frac{\partial T}{\partial x} = -\frac{T_1}{l_e} + \frac{T_2}{l_e} = \sum_{j=1}^{2} \frac{\partial N_j}{\partial x} T_j$$

$$\frac{\partial T}{\partial t} = \sum_{j=1}^{2} N_j \frac{\partial T_j}{\partial x}$$

From weak form

$$\int_\Omega w' k \frac{\partial T}{\partial x}\, dx = \underbrace{\int_\Omega k \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x}\, dx}_{K \to \text{global stiffness Matrix}}\, T_j$$

$$\int_\Omega c\, w \frac{\partial T}{\partial t}\, dx = \underbrace{\int_\Omega c\, N_i N_j\, dx}_{C \to \text{damping matrix}} \frac{\partial T}{\partial t}$$

$$\int_\Omega w Q\, dx - w(T_2) Q_0 = \underbrace{\int_\Omega N_i Q\, dx - w(T_2) Q_0}_{-F}$$

So weak form becomes,

$$\boxed{K\, T(t) + C \frac{\partial T}{\partial t} = F}$$

$\hookrightarrow$ semi-discrete form

(c) $l = 10$, $k = 5$, $c = 1$, $f = 0$, 4 elements

$$K^e = \int_0^{l_e} \frac{\partial N_i}{\partial x} k \frac{\partial N_j}{\partial x} dx$$

$$= k\int_0^{l_e} \begin{bmatrix} -1/l_e \\ 1/l_e \end{bmatrix} \begin{bmatrix} -1/l_e & 1/l_e \end{bmatrix} dx$$

$$= k\int_0^{l_e} \begin{bmatrix} 1/l_e^2 & -1/l_e^2 \\ -1/l_e^2 & 1/l_e^2 \end{bmatrix} dx$$

$$K^e = \frac{k}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ Here $l_e = \frac{l}{4} = \frac{10}{4} = \frac{5}{2}$

$$K^e = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

As 4 equal length elements ∴ $K \to 5\times5$

$$K = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix}$$

$$C^e = \int_0^{l_e} N_i \, c \, N_j \, dx$$

$$= c\int_0^{l_e} \left[\begin{bmatrix} 1-x/l_e \\ x/l_e \end{bmatrix} \begin{bmatrix} \frac{1-x}{l_e} & \frac{x}{l_e} \end{bmatrix} dx\right]$$

$$= c\int_0^{l_e} \begin{bmatrix} (1-x/l_e)^2 & x/l_e\,(1-x/l_e) \\ \frac{x}{l_e}\left(1-\frac{x}{l_e}\right. & \left(\frac{x}{l_e}\right)^2 \end{bmatrix} dx$$

$$C_e = \frac{c\, l_e}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{5}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

As 4 equal length elements $\therefore C \to 5\times5$

$$C = \frac{5}{12}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$F = -\cancelto{0}{\int N_i Q\, dx} + \omega(T_2)Q_0 = \omega(T_2)Q_0$$

$$KT + C\frac{\partial T}{\partial t} = F$$

$$\boxed{\begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} + \frac{5}{12}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} T_1' \\ T_2' \\ T_3' \\ T_4' \\ T_5' \end{bmatrix} = F}$$

5. The differential equations for the bending of a beam are given by

(a) $a\frac{dV}{dx} + q = 0$
(b) $a\frac{dM}{dx} + V = 0$
(c) $a\frac{d\theta}{dx} - \frac{M}{EI} = 0$
(d) $a\frac{dw}{dx} - \theta - \frac{V}{GA} = 0$

in which V is shear force, M is moment, θ is section rotation, w is displacement, EI is bending stiffness, GA is shear stiffness, and q is load as shown in Figure below.

Boundary conditions are given by

(a) $V = V_0$ or $w = w_0$
(b) $M = M_0$ or $\theta = \theta_0$

(a) Construct weak forms for equations.

$\rightarrow$ $a\frac{dV}{dx} + q = 0$

$R = a\frac{dV}{dx} + q \,,\quad \int_0^L \delta w \, R \, dx = 0$

$\Rightarrow \int_0^L \delta w \left[a\frac{dV}{dx} + q \right] dx = 0$

$\Rightarrow \boxed{\delta w\, a V \Big|_0^L - \int_0^L \frac{d\,\delta w}{dx}\, a \frac{dV}{dx} dx + \int_0^L \delta w\, q\, dx = 0}$

$\rightarrow$ $a\frac{dM}{dx} + V = 0$

$R = a\frac{dM}{dx} + V \,,\quad \int_0^L \delta\theta \, R \, dx = 0$

$\Rightarrow \int_0^L \delta\theta \left[a\frac{dM}{dx} + V \right] dx = 0$

$\Rightarrow \boxed{\delta\theta\, a M \Big|_0^L - \int_0^L \frac{d\,\delta\theta}{dx}\, a \frac{dM}{dx} dx + \int_0^L \delta\theta\, V\, dx = 0}$

$$\frac{a\,d\theta}{dx} - \frac{M}{EI} = 0$$

$$R = \frac{a\,d\theta}{dx} - \frac{M}{EI} \qquad \int_0^L \delta M\, R\, dx = 0$$

$$\Rightarrow \int_0^L \delta M \left[a\frac{d\theta}{dx} - \frac{M}{EI} \right] dx = 0$$

$$\Rightarrow \boxed{\delta M\, a\, \theta \Big|_0^L - \int_0^L \frac{d\,\delta M}{dx}\, a\, \frac{d\theta}{dx}\, dx - \int_0^L \delta M \frac{M}{EI}\, dx = 0}$$

$$a\frac{dW}{dx} - \theta - \frac{V}{GA} = 0$$

$$R = a\frac{dW}{dx} - \theta - \frac{V}{GA}, \qquad \int_0^L \delta V\, R\, dx = 0$$

$$\Rightarrow \int_0^L \delta V \left[a\frac{dW}{dx} - \theta - \frac{V}{GA} \right] dx = 0$$

$$\Rightarrow \boxed{\delta V\, a\, W \Big|_0^L - \int_0^L \frac{d\,\delta V}{dx}\, a\, \frac{dW}{dx}\, dx - \int_0^L \delta V \left[\theta + \frac{V}{GA} \right] dx = 0}$$

(b) For $GA = \infty$, deduce irreductible differential equation in terms of w. Express BC in terms of w.

Solⁿ :-

For $GA \to \infty$

Eq ① $\frac{dV}{dx} = -q$

Eq ② $\frac{dM}{dx} = -V$

By eq ① & ②

$\Rightarrow \frac{d^2M}{dx^2} = q$ —— (v)

Eq ③ $\frac{d\theta}{dx} = \frac{M}{EI} \Rightarrow M = EI\frac{d\theta}{dx}$

Eq ④ $\frac{dw}{dx} - \theta - \cancel{\frac{V}{GA}}^{0} = 0$ [As $GA \to \infty$]

$\Rightarrow \theta = \frac{dw}{dx}$ Put in ③

$M = EI\frac{d^2w}{dx^2}$ Put (v)

$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q$

$$\Rightarrow \boxed{\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] - q = 0}$$

Given $w(0) = w_0$, $w(L) = w_0$

$\theta(0) = \theta_0$, $\theta(L) = \theta_0$

$\Rightarrow \left.\frac{dw}{dx}\right|_{x=0} = \theta_0$, $\left.\frac{dw}{dx}\right|_{x=L} = \theta_0$

(c) Construct a weak form for above problem.

Sol^n

$$\underbrace{\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right]-q}_{R}=0$$

Let weights $\rightarrow$ $v(x)$

$$\int_0^L v(x)\,R\,dx=0$$

$$\Rightarrow \int_0^L v\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right]dx-\int_0^L vq=0$$

$$\Rightarrow \cancel{v\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]\Bigg|_0^L}^{\,0}-\int_0^L v'\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]dx-\int_0^L vq=0$$

As $w(0)=w_0$ $\therefore v(0)=0$ [Natural BC]

$w(L)=w_0$ $\therefore v(L)=0$

$$-\cancel{v'EI\frac{d^2w}{dx^2}\Bigg|_0^L}^{\,0}+\int_0^L v''EI\frac{d^2w}{dx^2}dx-\int_0^L vq=0$$

As $w'(0)=\theta_0$ $\therefore v'(0)=0$ [Natural BC]

$w'(L)=\theta_0$ $\therefore v'(L)=0$

$$\boxed{\int_0^L v''EI\frac{d^2w}{dx^2}dx-\int_0^L vq=0}$$

Weak form

(d) $l=10$, $EI=3$, $q=1$

Sol^n Deriving eq^n for 2 nodes, General sol^n.
for more nodes refer code [q5d.py].

Weak Form:- $\int_0^L v'' EI \frac{d^2 w}{dx^2}\, dx - \int_0^L vq = 0$

let $v(x) = c_0 + c_1\left(\frac{x}{L}\right) + c_2\left(\frac{x}{L}\right)^2 + c_3\left(\frac{x}{L}\right)^3$

$v(0) = v_1 = c_0$

$v(L) = v_2 = c_0 + c_1 + c_2 + c_3$

$w'(0) = \theta_1 = c_1/L$

$w'(L) = \theta_2 = c_1/L + 2c_2/L + 3c_3/L$

let $\frac{\partial \bar{w}(x)}{\partial v_1} = N_1$, $\frac{\partial \bar{w}(x)}{\partial v_2} = N_3$

$\frac{\partial \bar{w}(x)}{\partial \theta_1} = N_2$, $\frac{\partial \bar{w}(x)}{\partial \theta_2} = N_4$

where $\bar{w}(x) = v_1 N_1 + \theta_1 N_2 + v_2 N_3 + \theta_2 N_4$

But putting boundary values we get

$N_1(x) = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$ | $N_3(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$

$N_2(x) = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$ | $N_4(x) = \frac{-x^2}{L} + \frac{x^3}{L^2}$

$$w''(x) = \begin{bmatrix} N_1'' & N_2'' & N_3'' & N_4'' \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = Bd$$

Substituting N to weak form

$$\int_0^L \left(EI\ Bd\ N_i''(x) - q N_i\right) dx = 0 \quad \text{for } i = 1,2,3,4$$

$$\Rightarrow \underbrace{\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}}_{K} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \frac{qL}{12}\begin{bmatrix} 6 \\ L \\ 6 \\ -L \end{bmatrix}$$

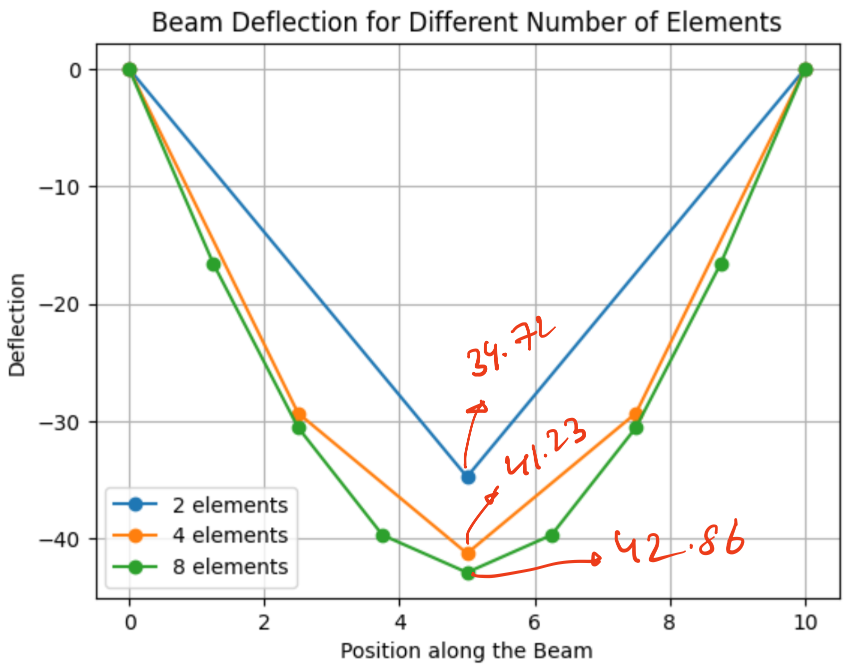
Using this equation, used code to find deflection for 2, 4, 8 element.

Exact solution for deflection at center:-

$$\Delta_{max} = \frac{q l^4}{384 EI}$$

$$= \frac{1 \times 10^4}{384 \times 3}$$

$$\boxed{\Delta_{max} \sim 43.29}$$



Our solⁿ for 8 element → 42.86

∴ As no. of elements increases, error decreases

Que 6 (a) $\Pi(v) = \int_a^b \left[EI\left(\frac{dv}{dx}\right)^2 - Pv^2 \right] dx - (vg)\big|_{x=b}$

Sol^n $\partial\Pi = \int_a^b EI\, 2\frac{dv}{dx}\,\delta\left(\frac{dv}{dx}\right) - \int_a^b P.\,2v\,\delta v\,dx - \delta v g\big|_{x=b}$

As, $\delta\left(\frac{dv}{dx}\right) = \frac{d}{dx}(\delta v)$ $\qquad v.dv$

$\int_a^b v\,dv = -\int_a^b v\,du + uv\big|_a^b$

Integrate by parts & as $\delta v = 0$ on a to b.

$\partial\Pi = \int_a^b \delta v\left[-\frac{d}{dx}\left(2EI\frac{dv}{dx}\right) - 2Pv\right] dx = 0$

We know $\partial\Pi = \int_\Omega \delta v^T A(v)\, d\Omega$

As above true for any variations eq $\rightarrow A(v)=0$

$-\frac{d}{dx}\left(2EI\frac{dv}{dx}\right) - 2Pv = 0$

$\Rightarrow \boxed{EI\frac{d^2v}{dx^2} + Pv = 0}$

Boundary Conditions:- $uv\big|_a^b = EI\frac{dv}{dx}\Big|_a^b = -vg\big|_{x=b}$

(a) $v(a) = 0$

(b) $vg\big|_{x=b} \Rightarrow$ At $x=b$, $EI\frac{dv}{dx} = g$

Que 6 (b) $\Pi(v) = \int_a^b [EA(\frac{dv}{dx})^2 + kv^2 - 2qv]dx + \alpha[v(a)^2 + v(b)^2]$

Soln

$$\partial\Pi = \int_a^b \left[2EA\frac{dv}{dx}\,\delta\left(\frac{dv}{dx}\right) + 2kv\,\delta v - 2q\,\delta v\right]dx + 2\alpha\,\delta v[v(a) + v(b)]$$

As $\delta\frac{dv}{dx} = \frac{d}{dx}(\delta v)$,

$$\int_a^b u\,dv = -\int_a^b v\,du + uv\Big|_a^b$$

Integrating by parts & as $\delta v = 0$ over a to b

$$\partial\Pi = \int_a^b \delta v\left[\frac{d}{dx}\left(-2EA\frac{dv}{dx}\right) + 2kv - 2q\right]dx = 0$$

We know $\partial\Pi = \int_\Omega \delta v^T A(v)\,d\Omega$

As above true for any variation, eq $\rightarrow A(v) = 0$

$$\Rightarrow \frac{d}{dx}\left(-2EA\frac{dv}{dx}\right) + 2kv - 2q = 0$$

$$\Rightarrow \boxed{EA\frac{d^2v}{dx^2} - kv + q = 0}$$

Boundary conditions :- $-2EA\frac{dv}{dx}\Big|_a^b + 2\alpha v(a) + 2\alpha v(b) = 0$

At $x = a$, $EA\frac{dv}{dx} = \alpha v(a)$

At $x = b$, $EA\frac{dv}{dx} = -\alpha v(b)$

Que 6 (c) $\Pi\left[v, \lambda_a, \lambda_b\right] = \int_a^b \left[EA\left(\frac{dv}{dx}\right)^2 + kv^2 - 2qv\right]dx + \lambda_a v(a) + \lambda_b v(b)$

Sol^n $\delta\Pi = \int_a^b \left[2EA\frac{dv}{dx}\frac{\delta dv}{dx} + 2kv\,\delta v - 2q\,\delta v\right]dx + \lambda_a \delta v(a) + \lambda_b \delta v(b)$

As $\delta\frac{dv}{dx} = \frac{d}{dx}(\delta v)$,

$$\int_a^b u\,dv = -\int_a^b v\,du + uv\Big|_a^b$$

Integrating by parts we get,

$$\delta\Pi = \int_a^b \delta v\left[-\frac{d}{dx}\left(2EA\frac{dv}{dx}\right) + 2kv - 2q\right]dx$$

We know $\partial\Pi = \int_\Omega \delta v^T A(v)\, d\Omega$

As above true for any variation, eq $\to A(v) = 0$

$\Rightarrow$ $-\frac{d}{dx}\left(2EA\frac{dv}{dx}\right) + 2kv - 2q = 0$

$\Rightarrow$ $\boxed{EA\frac{d^2v}{dx^2} - kv + q = 0}$

Boundary Conditions:- $\delta v\, 2EA\frac{dv}{dx}\Big|_a^b + \lambda_a \delta v(a) + \lambda_b \delta v(b) = 0$

$\Rightarrow$ At $x = a$ $\quad 2EA\frac{dv}{dx}\Big|_a = \lambda_a$

At $x = b$ $\quad -2EA\frac{dv}{dx}\Big|_b = \lambda_b$

7. Find a two-parameter Galerkin solution to the problem

$-\frac{d^2u}{dx^2} = f$ for $0 < x < L$

with $\left(\frac{du}{dx}\right)_{x=0} = \left(\frac{du}{dx}\right)_{x=L} = 0$

Using the trigonometric approximation function if

(a) $f = f_0$

(b) $f = f_0 cos(\pi x/L)$

(a) $f = f_0$

Sol^n Let $v(x) = c_1 \cos\left(\frac{\pi x}{L}\right) + c_2 \cos\left(\frac{2\pi x}{L}\right)$

Let, $w_1(x) = \frac{\partial v}{\partial c_1} = \cos\left(\frac{\pi x}{L}\right)$

$w_2(x) = \frac{\partial v}{\partial c_2} = \cos\left(\frac{2\pi x}{L}\right)$

By Galerkin conditions

$\int w_1(x) R(x)\, dx = 0$; $\int w_2(x) R(x)\, dx = 0$

$R(x) = -\frac{\partial^2 v}{\partial x^2} - f_0$

Case I $\int_0^L w_1(x) R(x)\, dx = 0$

$\Rightarrow \int_0^L \cos\frac{\pi x}{L}\left[\frac{c_1\pi^2}{L^2}\cos\left(\frac{\pi x}{L}\right) + \frac{4c_2\pi^2}{L^2}\cos\left(\frac{2\pi x}{L}\right) - f_0\right] dx = 0$

$\Rightarrow \frac{c_1\pi^2}{L^2}\underbrace{\int_0^L \cos^2\frac{\pi x}{L}\, dx}_{-L} + \frac{4c_2\pi^2}{L^2}\int_0^L \cos\frac{\pi x}{L}\cos\frac{2\pi x}{L}\, dx - f_0\int_0^L \cos\frac{\pi x}{L}\, dx = 0$

$\int_0^L \left[\cos\frac{2\pi x}{L} - 1\right] dx$ | $\int_0^L \left(1 - 2\sin^2\frac{\pi x}{L}\right)\cos\frac{\pi x}{L}\, dx$ | $\frac{1}{\pi/L}\left[\sin\frac{\pi x}{L}\right]_0^L$

$\Rightarrow \left[\frac{1}{\frac{2\pi}{L}}\sin\frac{2\pi x}{L} - x\right]_0^L$ | $\Rightarrow \left[k_1 \sin\frac{\pi x}{L} + k_2 \sin^3\left(\frac{\pi x}{L}\right)\right]_0^L$ | 0

$\Rightarrow -L$ | $\Rightarrow 0$

$\Rightarrow \frac{c_1\pi^2}{L^2}(-L) = 0 \Rightarrow \boxed{c_1 = 0}$

Case II $\int_0^L w_2(x)\,R(x)\,dx = 0$

$$\Rightarrow \int_0^L \cos\frac{2\pi x}{L}\left[\frac{c_1\pi^2}{L^2}\cos\frac{\pi x}{L} + \frac{4c_2\pi^2}{L^2}\cos\frac{2\pi x}{L} - f_0\right]dx = 0$$

$$\Rightarrow \frac{c_1\pi^2}{L^2}\int_0^L \cancel{\cos\frac{2\pi x}{L}\cos\frac{\pi x}{L}\,dx}^{\,0} + \frac{4c_2\pi^2}{L^2}\underbrace{\int_0^L \cos^2\frac{2\pi x}{L}\,dx}_{-L} - f_0\int_0^L \cancel{\cos\frac{2\pi x}{L}\,dx}^{\,0} = 0$$

$$\Rightarrow \frac{4c_2\pi^2}{L^2}(-L) = 0 \Rightarrow c_2 = 0$$

Put c_1 & c_2 in $v(x)$, we get $\boxed{v(x)=0}$

(b) $f = f_0\cos(\pi x/L)$

Sol^n Let $v(x) = c_1\cos\left(\frac{\pi x}{L}\right) + c_2\cos\left(\frac{2\pi x}{L}\right)$

Let, $w_1(x) = \frac{\partial v}{\partial c_1} = \cos\left(\frac{\pi x}{L}\right)$

$w_2(x) = \frac{\partial v}{c_2} = \cos\left(\frac{2\pi x}{L}\right)$

By Galerkin conditions

$\int w_1(x)\,R(x)\,dx = 0$; $\int w_2(x)\,R(x)\,dx = 0$

$$R(x) = -\frac{\partial^2 v}{\partial x^2} - f_0\cos\frac{\pi x}{L}$$

Case I $\int_0^L w_1(x)\,R(x)\,dx = 0$

$$\Rightarrow \int_0^L \cos\frac{\pi x}{L}\left[\frac{c_1\pi^2}{L^2}\cos\left(\frac{\pi x}{L}\right) + \frac{4c_2\pi^2}{L^2}\cos\left(\frac{2\pi x}{L}\right) - f_0\cos\left(\frac{\pi x}{2}\right)\right]dx = 0$$

$$\Rightarrow \left(\frac{c_1\pi^2}{L^2} - f_0\right) \underbrace{\int_0^L \cos^2\left(\frac{\pi x}{L}\right) dx}_{-L} + \frac{4c_2\pi^2}{L^2} \int_0^L \cancel{\cos\frac{\pi x}{L} \cos\left(\frac{2\pi x}{L}\right) dx}^{0} = 0$$

$$\int_0^L \left[\cos\frac{2\pi x}{L} - 1\right] dx$$

$$\Rightarrow \left[\frac{1}{\frac{2\pi}{L}} \sin\frac{2\pi x}{L} - x\right]_0^L$$

$$\Rightarrow -L$$

$$\int_0^L \left[1 - 2\sin^2\left(\frac{\pi x}{L}\right)\right] \cos\frac{\pi x}{L} dx$$

$$\Rightarrow \left[k_1 \sin\left(\frac{\pi x}{L}\right) + k_2 \sin^3\left(\frac{\pi x}{L}\right)\right]_0^L$$

$$\Rightarrow 0$$

$$\Rightarrow \left[c_1\frac{\pi^2}{L^2} - f_0\right] = 0 \Rightarrow \boxed{c_1 = \frac{f_0}{(\pi/L)^2}}$$

Case II $\int_0^L w_2(x) R(x)\, dx = 0$

$$\Rightarrow \int_0^L \cos\frac{2\pi x}{L}\left[\frac{c_1\pi^2}{L^2}\cos\frac{\pi x}{L} + \frac{4c_2\pi^2}{L^2}\cos\frac{2\pi x}{L} - f_0\cos\frac{\pi x}{L}\right] dx = 0$$

$$\Rightarrow \left(\frac{c_1\pi^2}{L^2} - f_0\right) \int_0^L \cancel{\cos\frac{2\pi x}{L}\cos\frac{\pi x}{L}}^{0}\, dx + \frac{4c_2\pi^2}{L^2} \underbrace{\int_0^L \cos^2\frac{2\pi x}{L}\, dx}_{-L} = 0$$

$$\Rightarrow \frac{4c_2\pi^2}{L^2}(-L) = 0$$

$$\Rightarrow c_2 = 0$$

Put c_1 & c_2 in $v(x)$, we get

$$\boxed{v(x) = \frac{f_0}{\left(\frac{\pi}{L}\right)^2} \cos\frac{\pi x}{L}}$$

8. Find a one-parameter Galerkin solution to the problem

$$-2u\frac{d^2u}{dx^2} + \left(\frac{du}{dx}\right)^2 = 4 \text{ for } 0 < x < 1$$

with $u(0) = 1$ and$u(1) = 0$

Sol^n

Let's assume solution

$U(x) = C_1 + C_2 x + C_3 x^2$

Apply boundary solution,

At $x=0$, $U=1$

$\Rightarrow U(0) = C_1 = 1$

At $x=1$, $U=0$

$\Rightarrow 1 + C_2 + C_3 = 0 \Rightarrow C_2 = -(1+C_3)$

$\Rightarrow U(x) = 1 - (1+C_3)x + C_3 x^2$

$\Rightarrow U(x) = (1-x) + C_3(x^2 - x)$

This is one parameter solution

Let $w(x) = \frac{\partial U}{\partial C_3} = x^2 - x$

$$R = -2U\frac{d^2U}{dx^2} + \left(\frac{dU}{dx}\right)^2 - 4$$

$$\left(\frac{dU}{dx}\right)^2 = \left(-(1+C_3) + 2C_3 x\right)^2$$

$$= (1+C_3)^2 + 4C_3^2 x^2 - 4(1+C_3)C_3 x$$

$$\frac{d^2U}{dx^2} = 2C_3$$

$$\therefore \quad R = -2\left[1 - (1+c_3)x + c_3x^2\right]2c_3 + \left[(1+c_3)^2 - 4(1+c_3)c_3x + 4c_3^2x^2\right] - 4$$

$$R = \left[-4c_3 + (1+c_3)^2 - 4\right] + \left[\cancel{4c_3(1+c_3)} - 4\cancel{(1+c_3)c_3}\right]x + \left[-4\cancel{c_3x^2} + 4\cancel{c_3x^2}\right]x^2$$

$$\Rightarrow \quad R = \left[c_3^2 - 2c_3 - 4\right]$$

$$\int_0^1 R\,w\,dx = 0$$

$$\Rightarrow \quad c_3^2 - 2c_3 - 4 \int_0^1 (x^2 - x)\,dx = 0$$

$$\Rightarrow \quad (c_3 - 3)(c_3 + 1) = 0$$

$$\Rightarrow \quad c_3 = -1, 3$$

$$\therefore \quad \boxed{u(x) = 1 - x^2 \quad \text{or} \quad u(x) = 1 - 4x + 3x^2}$$