COT 3100: Spring 2012 Exam 2, 50 minutes. **PROBLEM 1**. Name:

There are 4 problems on this exam on 4 pages. Write your name on each page. Keep your answers to each problem on its own page. You may write on the back, if needed.

Explain each of your answers below. For each set A and B you state, you don't have to prove that A and B are uncountable. But you do need to explain why $A - B$ or $A \cap B$ has the desired property.

- Give an example of two uncountable sets A and B such that $A - B$ is finite.

  Solution: there are of course many solutions to each problem. For this one, let A and B be the same set: the set of real numbers, $\mathbb{R}$. Then $A - B$ is empty, of size zero.

- Give an example of two uncountable sets A and B such that $A - B$ is countably infinite.

  Solution: Let A be the set of real numbers $\mathbb{R}$. Let $B = \mathbb{R} - \mathbb{Z}^+$. That is, B is the set of all numbers that are real but that are not positive integers. Then $A - B = \mathbb{Z}^+$ which is countably infinite.

- Give an example of two uncountable sets A and B such that $A - B$ is uncountable.

  Solution: Let A be the positive real numbers (greater than zero) and B be the negative real numbers. Then $A - B = A$ which is uncountable.

- Give an example of two uncountable sets A and B such that $A \cap B$ is finite.

  Solution: Let A be the positive real numbers (greater than zero) and B be the negative real numbers. Then $A \cap B$ is empty, which has size zero.

- Give an example of two uncountable sets A and B such that $A \cap B$ is countably infinite.

  Solution: Let A be the set of all positive real numbers. Let B be the set of all negative real numbers and all integers. Then $A \cap B = \mathbb{Z}^+$ which is countably infinite.

- Give an example of two uncountable sets A and B such that $A \cap B$ is uncountable.

  Solution: Let A and B both be the same set, $\mathbb{R}$, the set of real numbers. Then $A \cap B = \mathbb{R}$ which is uncountable.

COT 3100: Spring 2012 Exam 2, **PROBLEM 2**. Name:

- Prove or disprove the following statement: if x and y are irrational, then $x - y$ is irrational. What kind of proof or disproof did you use?

  Solution: the statement is false. Let x be any irrational number, and let $y = -x$. Then $x - y = 0$, which is rational. This is a counter-example.

- Let n be an integer. Prove the following statement: n is odd if and only if n^2 is odd.

  Solution:

  – n odd implies that n^2 is odd:
  If n is odd then $n = 2k+1$ for some integer k. Then $n^2 = (2k+1)^2 = 4k^2+4k+1$. This can be written as $2(2k^2+2k)+1$, which is of the form $2c+1$ for some integer c. Thus n^2 is odd.

  – n^2 odd implies that n is odd:
  For this one, we must use a proof by contrapositive. That is, we need to prove that n even implies that n^2 is even. If n is even then $n = 2k$ for some integer k, and thus $n^2 = (2k)^2 = 2(2k^2)$, which is of the form $2c$ for some integer c. Thus n^2 is even.

COT 3100: Spring 2012 Exam 3, **PROBLEM 3**. Name:

Show that x^3 is $O(x^4)$ but x^4 is not $O(x^3)$. Be precise, show the constants c and k for the first part, and showing that c and k cannot exist for the second part.

Solution:

- show that x^3 is $O(x^4)$: If we assume $1 \leq x$, then we can take this inequality and multiply both sides by x^3, obtaining $x^3 \leq x^4$. Thus, x^3 is $O(x^4)$ with $c = 1$ and $k = 1$.
- show that x^4 is not $O(x^3)$: For this to hold, we must have $x^4 \leq cx^3$ for some constant c and for all $x > k$. But if we divide this inequality by x^3, we get $x \leq c$ for all large x. This cannot hold.

COT 3100: Spring 2012 Exam 2, **PROBLEM 4**. Name:

Show that if n is an integer, then n^2 mod 4 is equal to 0 or 1.

Solution: There are 2 cases to consider:

- n even: then $n = 2k$ for some integer k, and $n^2 = 4k^2$. This is congruent to zero (mod 4).
- n odd: then $n = 2k+1$ for some integer k, and $n^2 = (2k+1)^2 = 4k^2+4k+1$. We can write this as $n^2 = 4(k^2+4k)+1$ and thus n^2 mod 4 is equal to one.