Finite sample corrections for parameters estimation and significance testing

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1. Mathematical derivations related to Power Law and Exponential Distribution

The following are the notations use in this Section:

Table 1. Notations use in Section: Mathematical derivations related to Power Law (PL) and Exponential Distribution (EXP)

Notation	Description	Example: PL, EXP
$f_*(x)$	Probability density function (PDF) for $*$ distribution	$f_{PL}(x)$, $f_{EXP}(x)$
$F_*(x)$	Cumulative density function (CDF) for $*$ distribution	$F_{PL}(x)$, $F_{EXP}(x)$
$F_*^{-1}(u)$	Inverse cumulative density function for $*$ distribution	$F_{PL}^{-1}(u)$, $F_{EXP}^{-1}(u)$
$\mathbb{L}_*$	Likelihood function for $*$ distribution	$\mathbb{L}_{PL}$, $\mathbb{L}_{EXP}$
$\hat{P^*}$	Estimated parameter(s) for $*$ distribution	$\hat{\alpha}$, $\hat{\beta}$
P_T^*	True parameter(s) for $*$ distribution	α_T, β_T
RE_*	Relative estimation error, $\frac{\sqrt{\langle(\hat{P^*}-P_T^*)^2\rangle}}{P_T^*}$	RE_{PL}, RE_{EXP}
x_{max}	Postulated largest element, $x_{max} = F^{-1}(1-\delta)$	X_{max}^{PL}, X_{max}^{EXP}

Review of Probability Distributions and Parameters Estimation. The PDF for Power law (PL) and exponential (EXP) distributions are

$$f_{PL}(x) = \frac{\alpha-1}{x_{min}^{1-\alpha}}x^{-\alpha}; \qquad f_{EXP}(x) = \beta\exp(-\beta(x-x_{min})) \tag{S1}$$

for $x \in [x_{min}, \infty)$, and their cumulative distribution functions are

$$F_{PL}(x) = 1-(\frac{x}{x_{min}})^{1-\alpha}; \qquad F_{EXP}(x) = 1-\exp(-\beta(x-x_{min})). \tag{S2}$$

Hence, their inverse cumulative distribution functions are

$$F_{PL}^{-1} = x_{min}(1-u)^{\frac{1}{\alpha-1}}; \qquad F_{EXP}^{-1} = x_{min} - \frac{\ln(1-u)}{\beta}, \tag{S3}$$

which maps $u \sim U(0,1)$ to $x \in [x_{min}, \infty)$ for the PL and EXP distribution respectively.

We use the maximum likelihood method to estimate the parameters in the distribution. To do so, we write down the log-likelihood function:

$$\begin{gathered}\ln\mathbb{L} = \ln\left[\prod_{i=1}^{N} f\left(x_i|\hat{P}\right)\right] \\ \ln\mathbb{L}_{PL} = N\left(\ln(\alpha-1)-\ln(x_{min})\right) - \alpha\sum_{i=1}^{N}\ln\left(\frac{x_i}{x_{min}}\right); \qquad \ln\mathbb{L}_{EXP} = N\ln(\beta) - \beta\sum_{i=1}^{N}(x_i-x_{min}).\end{gathered} \tag{S4}$$

for both distributions. Applying the maximization condition $\frac{\partial[\ln L])}{\partial\alpha} = 0$ for the PL and $\frac{\partial[\ln L]}{\partial\beta} = 0$ for the EXP distribution, yield the estimated parameters,

$$\hat{\alpha} = 1 - \frac{1}{\langle\ln\left(\frac{x}{x_{min}}\right)\rangle}; \qquad \hat{\beta} = \frac{1}{\langle x\rangle - x_{min}}. \tag{S5}$$

where the angled brackets denote the mean. Note that the above condition is necessary but not sufficient to show that $\mathbb{L}$ is maximized. The reader can check that the sufficient condition $\frac{\partial^2[\ln\mathbb{L}]}{\partial\alpha^2} < 0$ and $\frac{\partial^2[\ln\mathbb{L}]}{\partial\beta^2} < 0$ are also satisfied by $\hat{\alpha}$ and $\hat{\beta}$.

Adjustment for Finite Largest Element. In the main paper, we have demonstrated how the presence of the finite largest element effect (FLE) causes us to underestimate the value of $\langle \ln(x) \rangle$ (for PL) and $\langle x \rangle$ (for EXP). This results in an overestimated $\hat{\alpha}$ (for PL) and $\hat{\beta}$ (for EXP). We correct for this by adding back the truncated parts (from x_{max} to ∞) into $\langle \ln(x) \rangle$ and $\langle x \rangle$,

$$\begin{aligned} \left\langle \ln\left(\frac{x}{x_{min}}\right)\right\rangle_{adj} &= \left\langle \ln\left(\frac{x}{x_{min}}\right)\right\rangle_{data} \int_{x_{min}}^{x_{max}} f_{PL}(x)\,dx + \int_{x_{max}}^{\infty} \ln\left(\frac{x}{x_{min}}\right) f_{PL}(x)\,dx; \\ \langle x\rangle_{adj} &= \langle x\rangle_{data} \int_{x_{min}}^{x_{max}} f_{EXP}(x)\,dx + \int_{x_{max}}^{\infty} x f_{EXP}(x)\,dx. \end{aligned} \tag{S6}$$

This yields the adjusted values,

$$\begin{aligned} \left\langle \ln\left(\frac{x}{x_{min}}\right)\right\rangle_{adj} &= \left\langle \ln\left(\frac{x}{x_{min}}\right)\right\rangle_{data} \left[1 - \left(\frac{x_{max}}{x_{min}}\right)^{1-\alpha}\right] + \frac{1}{\alpha-1}\left(\frac{x_{max}}{x_{min}}\right)^{1-\alpha}\left[(\alpha-1)\ln\left(\frac{x_{max}}{x_{min}}\right)+1\right] \\ \langle x\rangle_{adj} &= \langle x\rangle_{data}\left[1-\exp\left(-\beta(x_{max}-x_{min})\right)\right] + \frac{1}{\beta}\left(\beta x_{max}+1\right)\exp\left(-\beta(x_{max}-x_{min})\right), \end{aligned} \tag{S7}$$

for the PL and EXP distribution respectively. By inserting Eq. (S7) into Eq. (S5), we obtain the nonlinear equations

$$\begin{aligned} &\left\langle \ln\left(\frac{x}{x_{min}}\right)\right\rangle_{data} + \frac{1}{1-\hat{\alpha}} + \left(\frac{x_{max}}{x_{min}}\right)^{1-\hat{\alpha}}\left[\ln\left(\frac{x_{max}}{x_{min}}\right) - \left\langle \ln\left(\frac{x}{x_{min}}\right)\right\rangle_{data} + \frac{1}{\hat{\alpha}-1}\right] = 0; \\ &\left[\hat{\beta}_{adj}\left(\langle x\rangle_{data} - x_{min}\right)+1\right]\exp\left[-\hat{\beta}_{adj}\left(x_{max}-x_{min}\right)\right] + \hat{\beta}_{adj}\left(\langle x\rangle_{data}-x_{min}\right) - 1 = 0 \end{aligned} \tag{S8}$$

that can be solved to obtain the adjusted estimated $\hat{\alpha}$ and $\hat{\beta}$.

Approximate the Relative Error. To estimate the errors in P^* arising due to the FLE effect, x_{max}, we define the relative estimated parameter error as

$$RE = \frac{\sqrt{\left\langle \left(\hat{P} - P_T\right)^2 \right\rangle}}{P_T}. \tag{S9}$$

The finite largest element x_{max} can be expressed in term of δ using Eq. (S3),

$$X_{max}^{PL} = x_{min}\delta^{\frac{1}{\alpha_T - 1}}; \qquad X_{max}^{EXP} = x_{min} - \frac{\ln(\delta)}{\beta_T}, \tag{S10}$$

the unadjusted terms $\langle \ln(x) - \ln(x_{min})\rangle_{unadj}$ and $\langle x\rangle_{unadj}$ only contain $x \in [x_{min}, x_{max}]$, so they can be expressed as,

$$\left\langle \ln\left(\frac{x}{x_{min}}\right)\right\rangle_{unadj} = \int_{x_{min}}^{X_{max}^{PL}} \ln\left(\frac{x}{x_{min}}\right) f_{PL}(x|\alpha_T)\,dx; \qquad \langle x\rangle_{unadj} = \int_{x_{min}}^{X_{max}^{EXP}} x f_{EXP}(x|\beta_T)\,dx. \tag{S11}$$

Performing the integral for both equations gives,

$$\begin{aligned} \left\langle \ln\left(\frac{x}{x_{min}}\right)\right\rangle_{unadj} &= \frac{1}{\alpha_T - 1}\left\{1 - \left(\frac{X_{max}^{PL}}{x_{min}}\right)^{1-\alpha_T}\left[(\alpha_T-1)\ln\left(\frac{X_{max}^{PL}}{x_{min}}\right)+1\right]\right\} \\ \langle x\rangle_{unadj} &= \frac{1}{\beta_T}\left\{(\beta_T x_{min}+1) - \left(\beta_T X_{max}^{EXP}+1\right)\exp\left(-\beta_T(X_{max}^{EXP} - x_{min})\right)\right\}. \end{aligned} \tag{S12}$$

To check the impact this has on the estimated parameters, we substitute Eq. (S12) into Eq. (S5) to obtain the unadjusted estimated parameter for PL distribution,

$$\begin{aligned} \hat{\alpha}_{unadj} &= 1 + \frac{\alpha_T - 1}{1 - \left(\frac{X_{max}^{PL}}{x_{min}}\right)^{1-\alpha_T}\left[(\alpha_T-1)\ln\left(\frac{X_{max}^{PL}}{x_{min}}\right)+1\right]} \\ &\approx \alpha_T + (\alpha_T - 1)\left(\frac{X_{max}^{PL}}{x_{min}}\right)^{1-\alpha_T}\left[(\alpha_T-1)\ln\left(\frac{X_{max}^{PL}}{x_{min}}\right)+1\right] + \mathcal{O}\left\{\left(\frac{X_{max}^{PL}}{x_{min}}\right)^{2(1-\alpha_T)}\left[(\alpha_T-1)\ln\left(\frac{X_{max}^{PL}}{x_{min}}\right)+1\right]^2\right\} \end{aligned} \tag{S13}$$

and for EXP distribution,

$$\begin{aligned} \hat{\beta}_{unadj} &= \frac{\beta_T}{1 - (\beta_T X_{max}^{EXP}+1)\exp\left(-\beta_T(X_{max}^{EXP} - x_{min})\right)} \\ &\approx \beta_T + \beta_T\left[\beta_T x_{max}+1\right]\exp\left(-\beta_T(x_{max}-x_{min})\right) + \mathcal{O}\left\{\left[\beta_T x_{max}+1\right]^2\exp\left(-2\beta_T(x_{max}-x_{min})\right)\right\}. \end{aligned} \tag{S14}$$

The approximations in Eq. (S13) and Eq. (S14) are estimated using first order Taylor expansion. Rewriting $\hat{\alpha}_{unadj}$ and $\hat{\beta}_{unadj}$ in terms of δ from Eq. (S10) will lead to

$$\begin{aligned} \alpha_{unadj} &\approx \alpha_T + (\alpha_T - 1)\,\delta\left[1 - \ln(\delta)\right] \\ \beta_{unadj} &\approx \beta_T + \beta_T\delta\left[1 - \ln(\delta) + \beta_T x_{min}\right]. \end{aligned} \tag{S15}$$

Rewriting the relative error for PL and EXP distribution in term of δ, we get

$$\begin{aligned} RE_{PL}(\alpha_{unadj}) &= \left(1-\frac{1}{\alpha_T}\right)\delta\left[1-\ln(\delta)\right], \qquad \delta\in[0,1] \\ RE_{EXP}(\beta_{unadj}) &= \delta\left[1-\ln(\delta)+\beta_T x_{min}\right], \qquad \delta\in[0,1]. \end{aligned} \tag{S16}$$

While it is useful to note that both relative errors are monotonic increasing function with $RE_{PL}\in[0,1]$ and $RE_{EXP}\in[0,\beta_T x_{min}]$, it is more important to observe that for a given δ, the RE_{PL} is always smaller or equal to RE_{EXP}. This sensitivity of the estimated parameter (relative error) on finite largest element ($x_{max}(\delta)$) provides us an insight on answering why power law distribution pass the significance testing more often compared to exponential distribution on unadjusted parameters.

2. Relationship between KS distance, Distribution Noise, and Sample Size

For a data set $X=\{x_1^{(em)},x_2^{(em)},\ldots,x_N^{(em)}\}$ that is f_x distributed has CDF F_x, the probability integral transform transforms X to the standard uniform distribution. Mathematically, this is the transformation $X=\{x_1^{(em)},x_2^{(em)},\ldots x_N^{(em)}\}\rightarrow\{F_x^{-1}(x_1^{(em)}),F_x^{-1}(x_2^{(em)}),\ldots,F_x^{-1}(x_N^{(em)})\}=\{u_1^{(em)},u_2^{(em)},\ldots,u_N^{(em)}\}$ (). Thereafter, statistical significance testing is done by examining how plausible the empirically transformed set $U^{(em)}=\{u_1^{(em)},u_2^{(em)},\ldots u_N^{(em)}\}$ can be obtained by chance, from the standard uniform distribution, $U^{(s)}=\{u_1^{(s)},u_2^{(s)},\ldots u_N^{(s)}\}$. This can be done by comparing the KS distance and distribution noise value between $U^{(em)}$ and $U^{(s)}$. Since $U^{(em)}$ is fixed and depends only on the sample, we only need to focus on the samples drawn from standard uniform distribution ($U^{(s)}$) to understand the relationship between KS distance (d_{KS}), distribution noise (d_{DN}) and sample size (N).

We simulated 10^6 samples of various sizes N from $U(0,1)$. Thus, for each N we have 10^6 simulated sample $U^{(s)}=\{u_1^{(s)},u_1^{(s)},\ldots,u_N^{(s)}\}$ for $s=[1,10^6]$ ordered such that $u_i^{(s)}\leq u_{i+1}^{(s)}$. For each sample then we calculate its KS distance d_{KS} as

$$d_{KS}^{(s)}=\forall_{i=1}^{N}\sup\left(\left|u_i^{(s)}-\frac{i}{N}\right|\right), \tag{S17}$$

and also its distribution noise d_{DN} using

$$\begin{aligned} d_{DN} &= \sqrt{\frac{\sum_{i=1}^{N}\left(u_i-u_{i-1}\right)^2\left[f(u_{i-1},u_i)-1\right]^2}{\sum_{i=1}^{N}\left(u_i-u_{i-1}\right)^2}} \\ &= \sqrt{\frac{\sum_{i=1}^{N}\left(u_i-u_{i-1}\right)^2\left(\frac{1}{N(u_i-u_{i-1})}-1\right)^2}{\sum_{i=1}^{N}\left(u_i-u_{i-1}\right)^2}}, \end{aligned} \tag{S18}$$

Hence, for each sample size N, we will obtain 10^6 pairs of d_{KS} and d_{DN}.

KSdistance and Sample Size. The d_{KS} value at different deciles exhibits the power law relationship as $d_{KS}\propto A_{KS}N^{-\alpha_{KS}}$ for $N>50$, shown in Fig. S 1. By assuming that the proportional constant A_{KS} depends only on the percentile level $\wp_{KS}$, we begin experimenting with several functional forms and eventually settled for

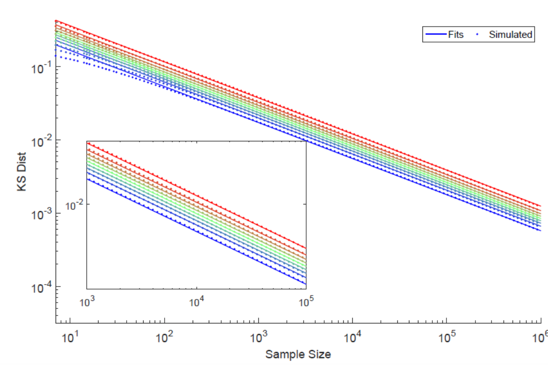
$$A_{KS}=\left(\frac{100}{\wp_{KS}}-1\right)^{-0.176}\exp(-0.274). \tag{S19}$$

The value of the exponent α_{KS} can be determined by the slopes of the different deciles. For simplicity, we assume constant exponent,

$$\alpha_{KS}=0.492, \tag{S20}$$

which is the mean of the slopes. In Fig. S2, we fit the A and α_{KS} to the different percentiles $\wp_{KS}$, which provided insight when choosing the functional form for α_{KS} and A. Combining Eq. (S19) with Eq. (S20), we obtain the d_{KS} and N at percentile $\wp_{KS}$,

$$d_{KS}(\wp_{KS},N)=\frac{\left(\frac{100}{\wp_{KS}}-1\right)^{-0.176}\exp(-0.274)}{N^{0.492}}, \quad N>50. \tag{S21}$$



S Fig. 1. Log-log plot of d_{KS} against N for different deciles going from the 10^{th} percentile (blue) to the 90^{th} (red), obtained from 10^6 simulations.

Analytic Solution relating Distribution Noise to Sample Size. For a sample distributed by $U(0,1)$ we are able to sort it to get $U^{(s)}$, where the superscript (s) refers to the ordered sample. We now define $\Delta=\{\Delta_1=u_1^{(s)},\Delta_2=u_2^{(s)}-u_1^{(s)},\Delta_3=u_3^{(s)}-u_2^{(s)},\ldots,\Delta_N=u_N^{(s)}-u_{N-1}^{(s)}\}$. Each of the Δ_k follows a similar distribution $f_{\Delta_1}(\Delta)=N(1-\Delta)^{N-1}$, which is the distribution of first uniform spacing (1). Rewriting Eq. (S18) in term of Δ yield

$$d_{DN}=\sqrt{\frac{\sum_{i=1}^{N}\left(\Delta_i(\frac{1}{N\Delta_i}-1)\right)^2}{\sum_{i=1}^{N}\Delta_i^2}}, \tag{S22}$$

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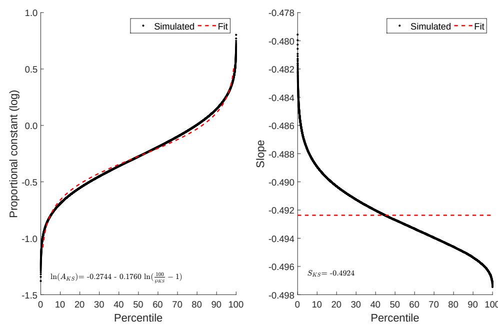
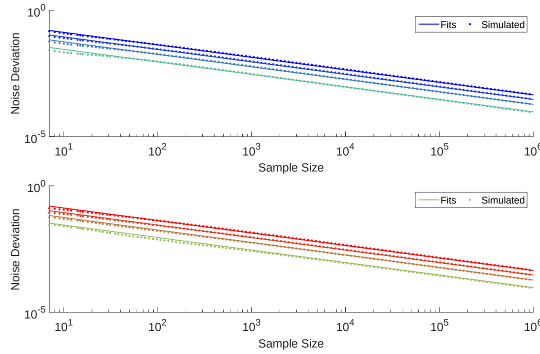


S Fig. 2. Fits for A_{KS} and α_{KS} at different percentile.

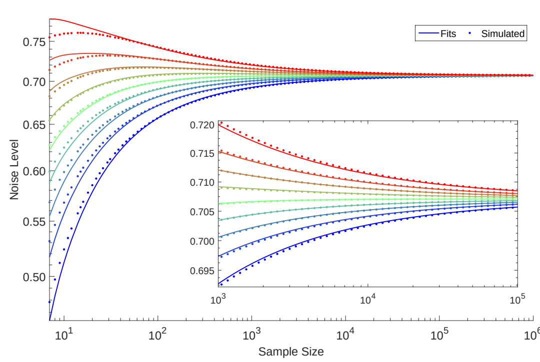


S Fig. 4. Relationship between absolute deviation of distribution noise from expected value, $|(d_{DN} - d_{DN}^{(*)}|$ and sample size N at various deciles obtained from 10^6 simulations. The top figure shows deciles from the 10^{th} percentile (blue) to to the 40^{th} (green), while the bottom shows decile from the 60^{th} (green) to the 90^{th} (red).

in the limit where N is large $N > 50$, we can apply the continuous approximation,

$$\begin{aligned} d_{DN}^{(*)} &= \sqrt{\frac{\int_0^1 \left(\Delta_i(\frac{1}{N\Delta_i} - 1)\right)^2 N(1-\Delta)^{N-1}\, d\Delta}{\int_0^1 \Delta^2 N(1-\Delta)^{N-1}\, d\Delta}} \\ &= \sqrt{\frac{1}{2} + \frac{2-N}{2N^2}}. \end{aligned} \qquad \text{[S23]}$$

where the $(*)$ indicates the analytically derived distribution noise, we have performed the integration in the surd, using the properties of the Beta distribution. Eq. (S23) shows $d_{DN}^{(*)} \to 1/\sqrt{2}$ as $N \to \infty$ as illustrated in Fig. S 3.



S Fig. 3. Relationship between distribution noise d_{DN} and sample size N at deciles going from the 10^{th} percentile (blue) to the 90^{th} (red), obtained from 10^6 simulations. The d_{DN} value converges to $1/\sqrt{2}$ as N increases.

Distribution Noise and Sample Size. We show the absolute deviation of d_{DN} value for the expected value $d_{DN}^{(*)}$ at different deciles Fig. S 4. Again there are straight lines on log-log plot, which indicate the power law relationship as $d_{DN} - d_{DN}^{(*)} \propto A_{DN}N^{-\alpha_{DN}}$ for $N > 50$.

Thus, by assuming the proportional constant A_{DN} depends only the percentile level $\wp_{DN}$, after experimenting with several functional forms, we write down the relationship between d_{DN} and N at percentile $\wp_{DN}$ as

$$A_{DN} = \Phi(\wp_{DN} - 50) \exp\left(-\frac{[50 - |\wp_{DN} - 50|]^{0.430}}{|\wp_{DN} - 50|^{0.302}}\right), \qquad \text{[S24]}$$

where $\Phi(x)$ represents the sign of x. For simplicity we assume constant power law exponent $\alpha_{DN} = 0.495$, which is average of the slopes in log-log plot excluding $\wp_{DN} \in [40, 60]$. Refer to Fig. S 5 for the fits of proportional constant and power law exponent at different percentile. Hence, the equation that relate d_{DN} and N at percentile $\wp_{DN}$ is

$$\begin{aligned} d_{DN}(\wp_{DN}, N) =& \langle d_{DN}\rangle + \\ & \Phi(\wp_{DN} - 50)\frac{\exp\left(-\frac{[50-|\wp_{DN}-50|]^{0.430}}{|\wp_{DN}-50|^{0.302}}\right)}{N^{0.495}}, \end{aligned} \qquad \text{[S25]}$$

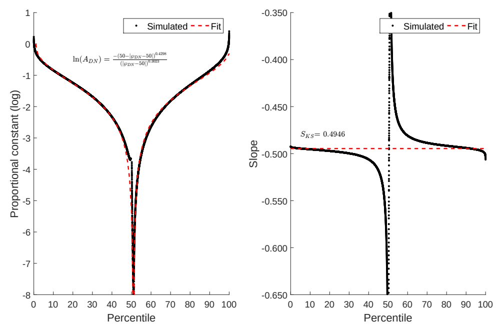
where $\Phi(x)$ represents the sign of x, and

$$\langle d_{DN}\rangle = \sqrt{\frac{1}{2} + \frac{2-N}{2N^2}}\left(\frac{N}{N+0.5}\right) \qquad \text{[S26]}$$

is the mean of d_{DN} that converges to $1/\sqrt{2}$ as $N \to \infty$, and $N/(N+0.5)$ is added in as a correction factor for small N. Again this result also suggests that if we have two samples with sizes N_1 and N_2 with $N_2 > N_1$ from the same distribution, we should compare $N_1^{0.495}(d_{DN}^{(1)} - 1/\sqrt{2})$ against $N_2^{0.495}(d_{DN}^{(2)} - 1/\sqrt{2})$. Otherwise, we will make wrong conclusion that the N_2 sample fits the distribution better.

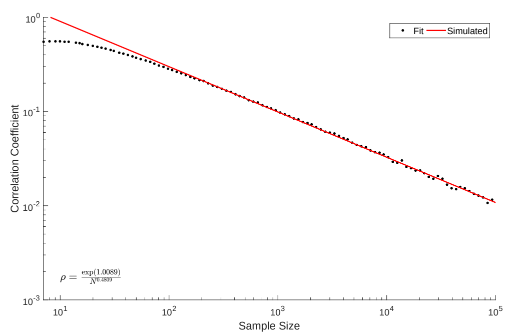
Correlation between KS distance and Distribution Noise. Since d_{KS} and d_{DN} are both measure of deviations, we need to account for the inherent correlation between these two variables, when applying them simultaneously to significance testing. To do this, we compute the Pearson correlation between d_{KS} and d_{DN} obtained from the 10^6 simulated samples for various sample sizes. Fig. S 6 shows the fits of the correlation coefficients,

$$\rho_{d_{KS}, d_{DN}}(N) = \frac{e}{N^{0.481}}. \qquad \text{[S27]}$$



S Fig. 5. Fits for A_{DN} and α_{DN} at different percentile.

and confirms our intuition that d_{KS} is positively correlated with d_{DN}. Since d_{KS} is a measure at the cumulative level, the correlation between d_{KS} and d_{DN} vanishes as $N \to \infty$, since the distribution noises cancel each other in this limit.



S Fig. 6. Correlation between d_{KS} and d_{DN} at different sample sizes, obtained from the 10^6 simulated samples.

3. Real Data Description

Taiwan Housing Data. The Taiwanese housing price shown in text is obtained from the Department of Land Administration (http://plvr.land.moi.gov.tw/DownloadOpenData). All housing transaction from 2010 are available. Following a previous study (2), we used only data from the Greater Taipei Area, containing Taipei city, New Taipei city, Keelung City and Taoyuan city, from 2010 to 2012 for the Zhu Zai Da Lous (which are condominium type housing). We understand from (2) that the distribution contains statistically significant outliers at the tail and choose to truncate the data set at 4.5×10^5, which is the $x_m^{(PL)}in$, to avoid the scenario where the distribution is polluted by outliers.

Taiwan Income Data. The Taiwan income data set contains information on the survey of household income in Taiwan. The summary of its findings can be found at http://win.dgbas.gov.tw/fies/index.asp while the raw data set is available through a paid subscription which can be obtained by completing a form at http://win.dgbas.gov.tw/fies/order.asp. Following a previous studies reviewed in (3), the income distribution generally has an exponential body and power-law tail. We use the Clauset-Shalizi-Newman method for PL (4) to obtain the x_{min} of the PL part of the Taiwan Income distribution. Next, we to truncate the full data set at this value $x_{min}^{(PL)}$, which leaves us with only the exponential part of the distribution. In the main text, we work with data containing on the EXP part of the distribution.

SGX Normalized Returns. It is well known from previous works that the short interval ($\sim$ 5mins) normalized returns of stock price indices follow a power law distribution (5–7). We check this result on the the Straits Times Index (STI), which tracks the Singapore Stock Exchange (SGX), using the methods discussed in this paper. Since most of the public sources only provide historical data for the days' close, we need to constructed the index using the tick-historical data in order to get fine grained, short interval ($\sim$ 5mins) data. To construct the STI price time series, we choose the price for a particular interval by considering the tick price that is closest the end of the 5min interval. We do this for all component stocks of the STI during the period 2009 to 2016. We note that our index price time series is not entirely similar to commonly published STI, since the STI is capital weighted. Since we do not have the capital information of the underlying company, we used a simple average weighting formula to construct the index. However, we do not expect this difference in weighting to change the power-law characteristics of its return distribution, as many of the stock indices returns (cited above) is power-law distributed but generally have different weighting formulas.

Now we obtain the set of returns $\mathbf{r} = r_1, r_2, \ldots, r_N$, by calculating for returns each 5min interval using:

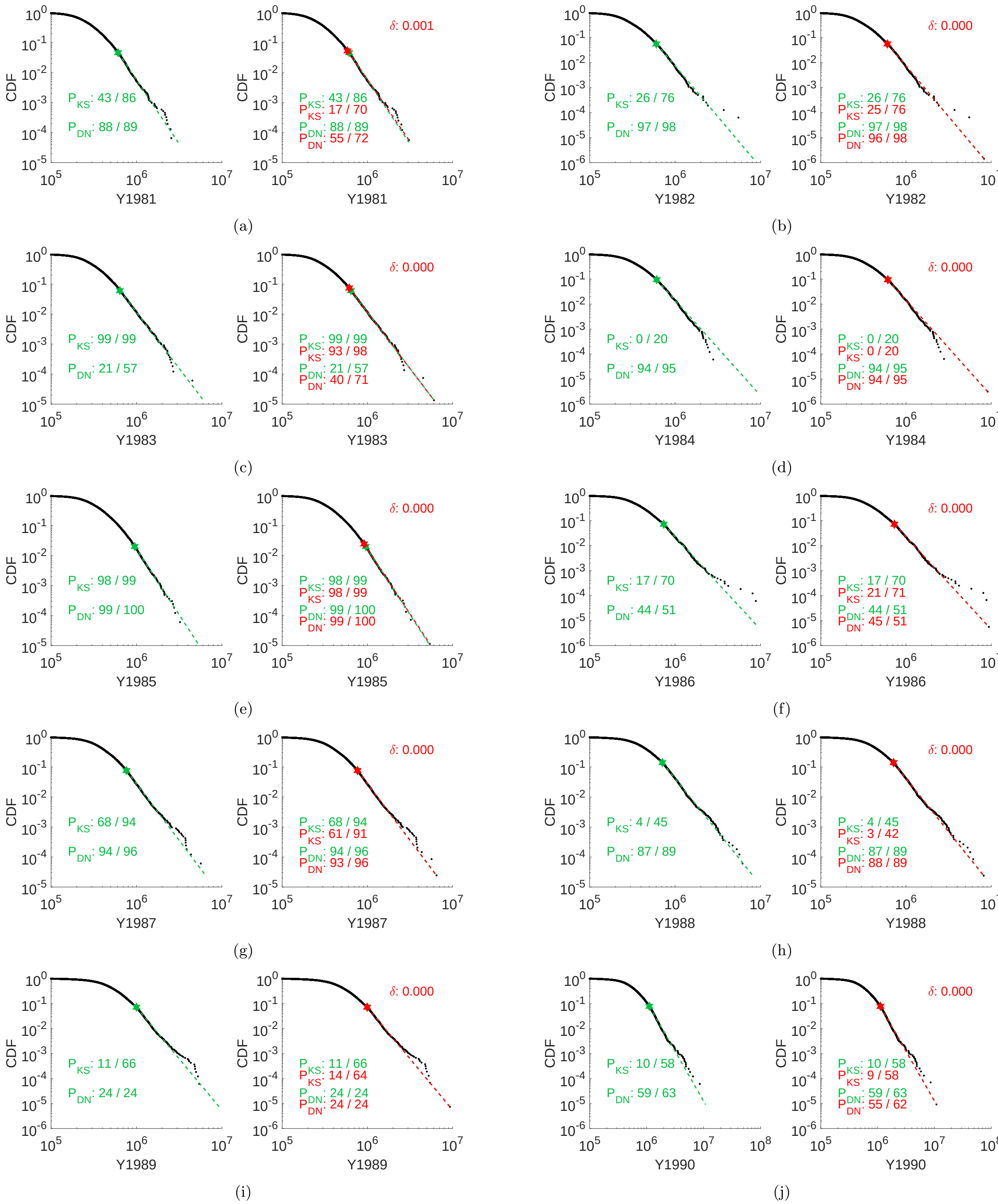
$$r_t = \frac{p_t}{p_{t-1}}. \qquad \text{[S28]}$$

Where r_t and p_t are the return and index price at interval t, and p_{t-1} is index price of the previous $(t-1)$ interval. The normalized return $\mathbf{r}^{(n)}$ is then calculated using

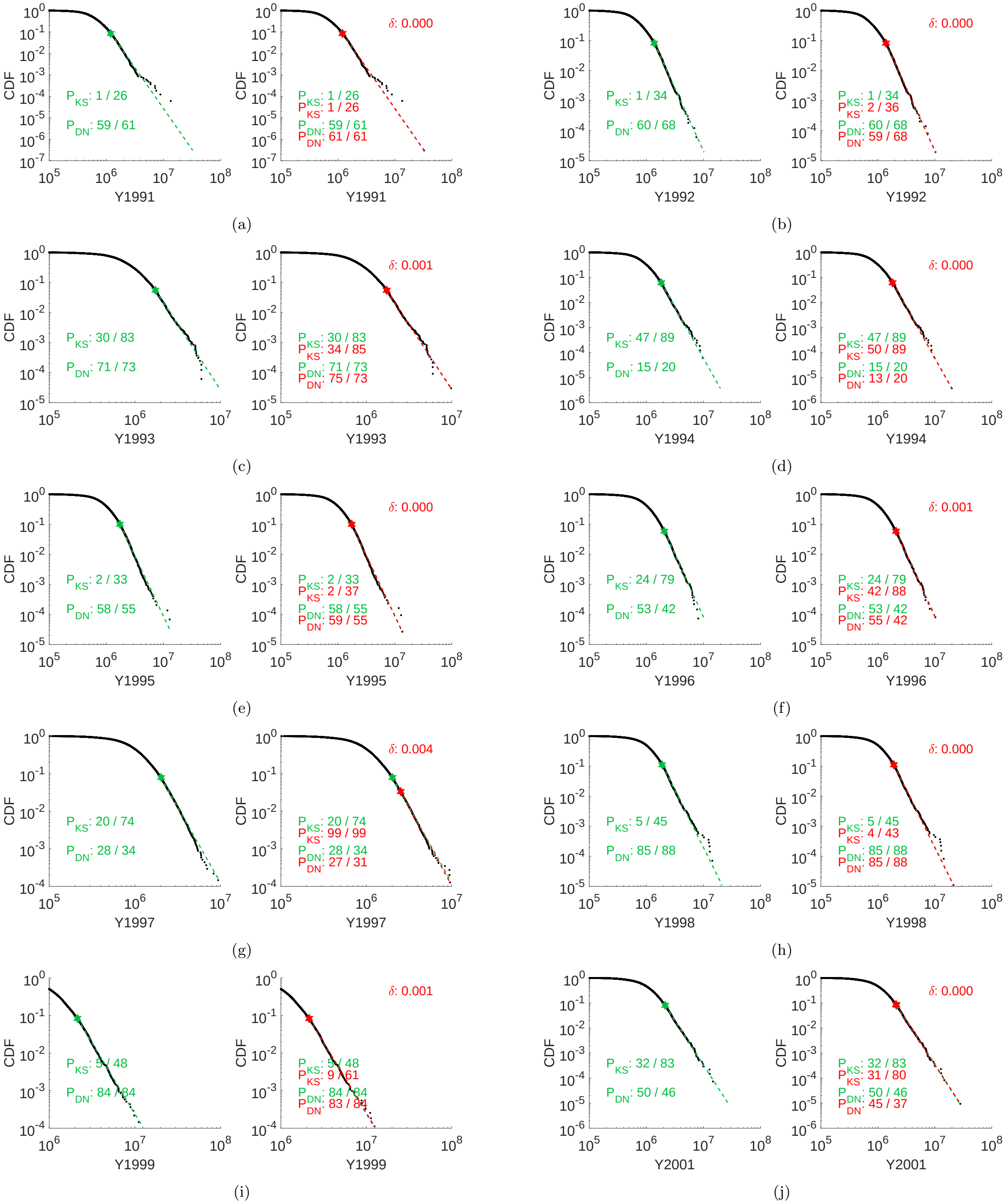
$$r_t^{(n)} = \frac{r_t - \langle \mathbf{r} \rangle}{\sigma(\mathbf{r})}, \qquad \text{[S29]}$$

where $\langle \mathbf{r} \rangle$ and $\sigma(\mathbf{r})$ are respectively, the mean and standard deviation of $\mathbf{r}$.

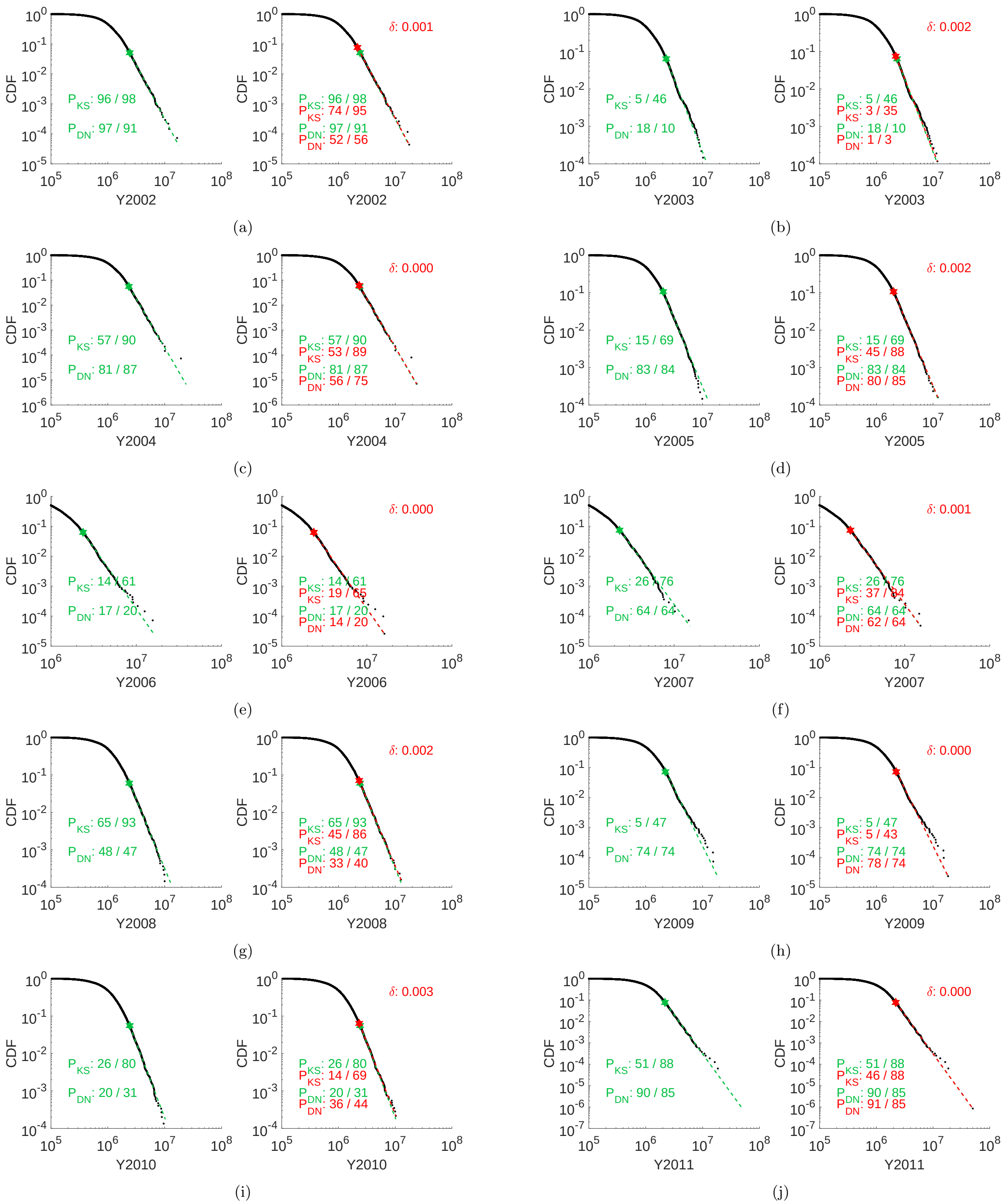
4. Fits to Real Data



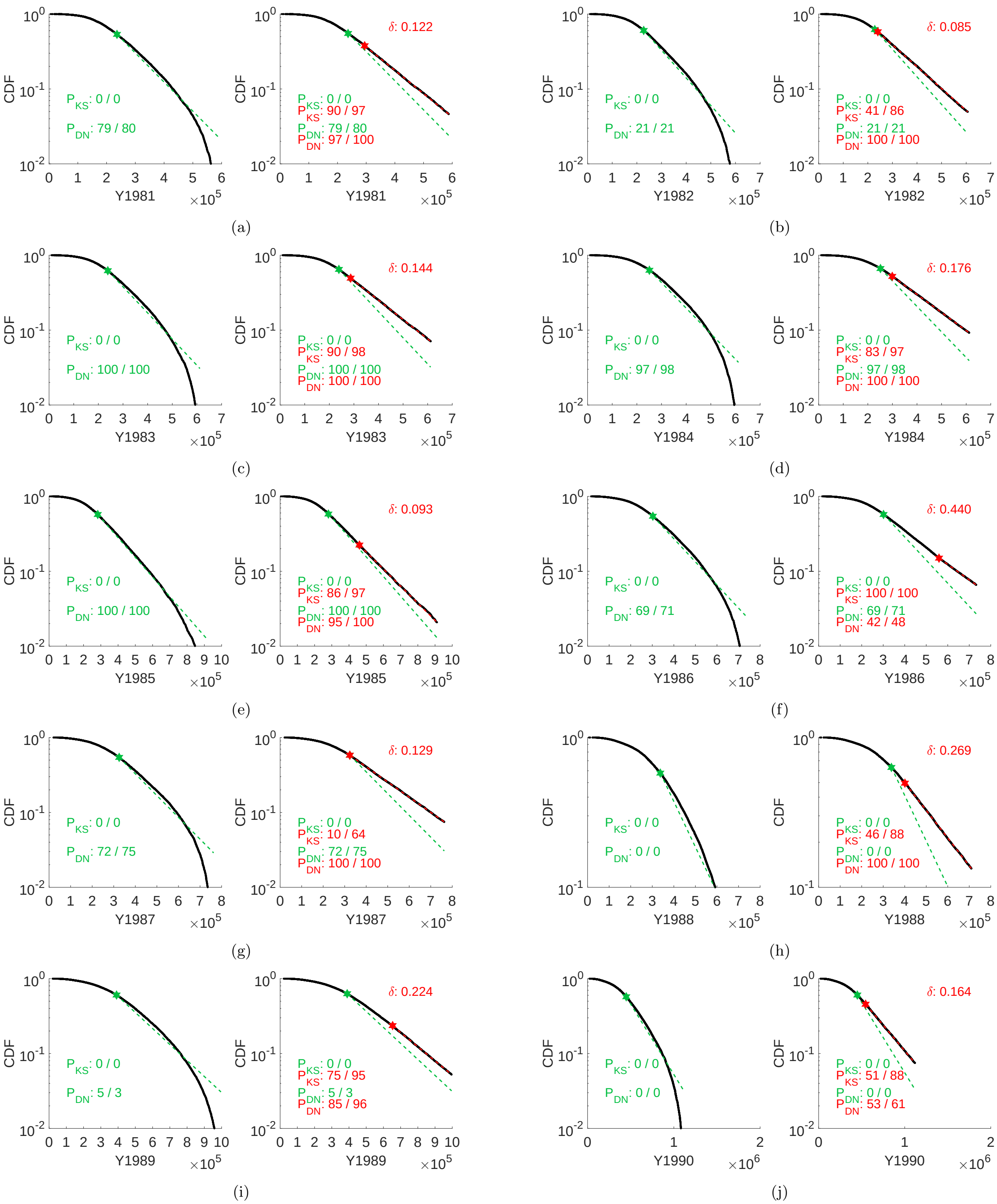
S Fig. 7. Year by year Taiwan income data from 1981 to 1990, fitted to power law distributions.



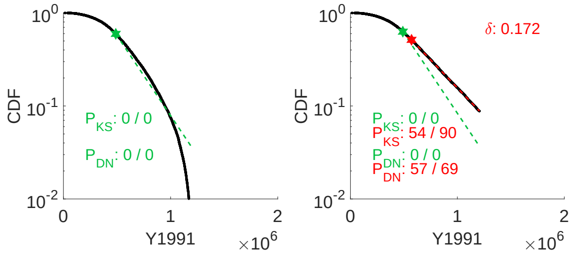
S Fig. 8. Year by year Taiwan income data from 1991 to 2001, fitted to power law distributions.



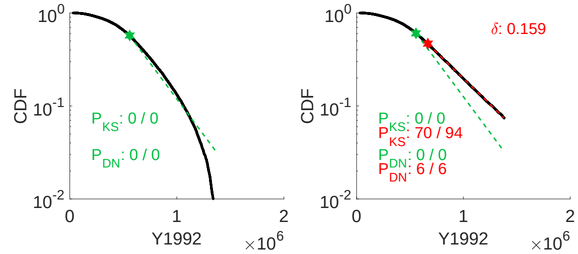
S Fig. 9. Year by year Taiwan income data from 2002 to 2011, fitted to power law distributions.



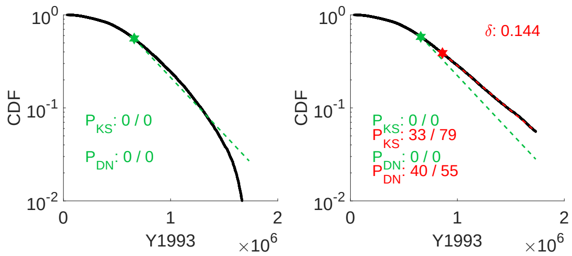
S Fig. 10. Year by year Taiwan income data from 1981 to 1990, fitted to exponential distributions.



(a)



(b)



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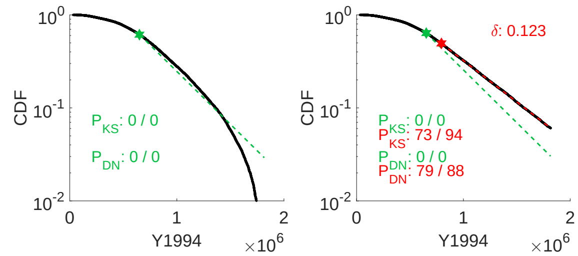

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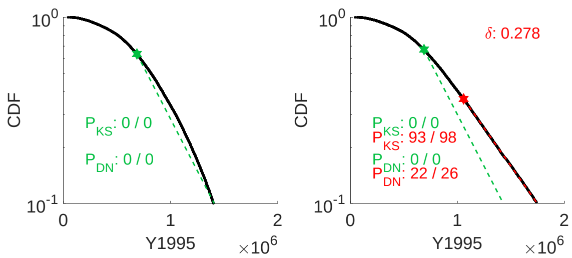

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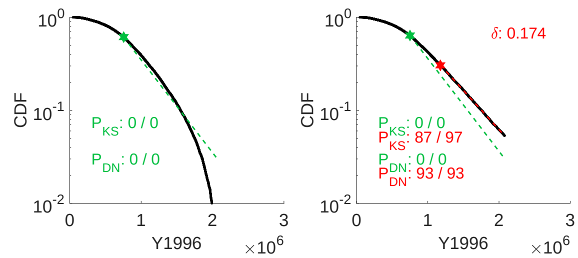

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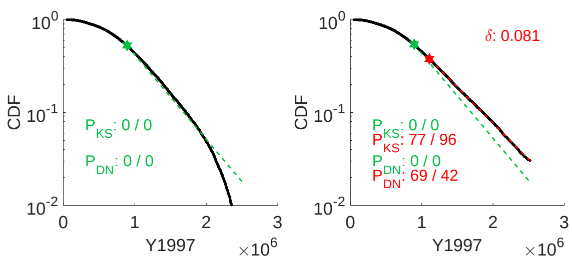
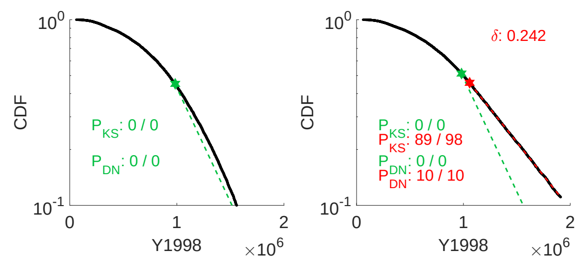


(g)



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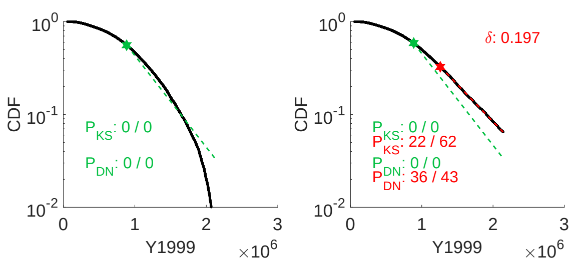
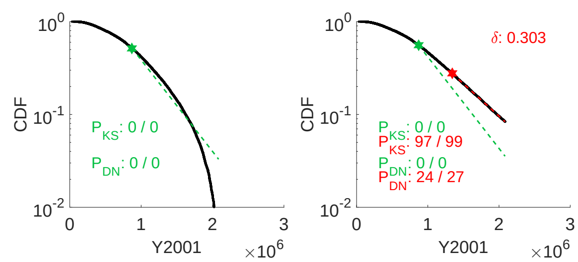
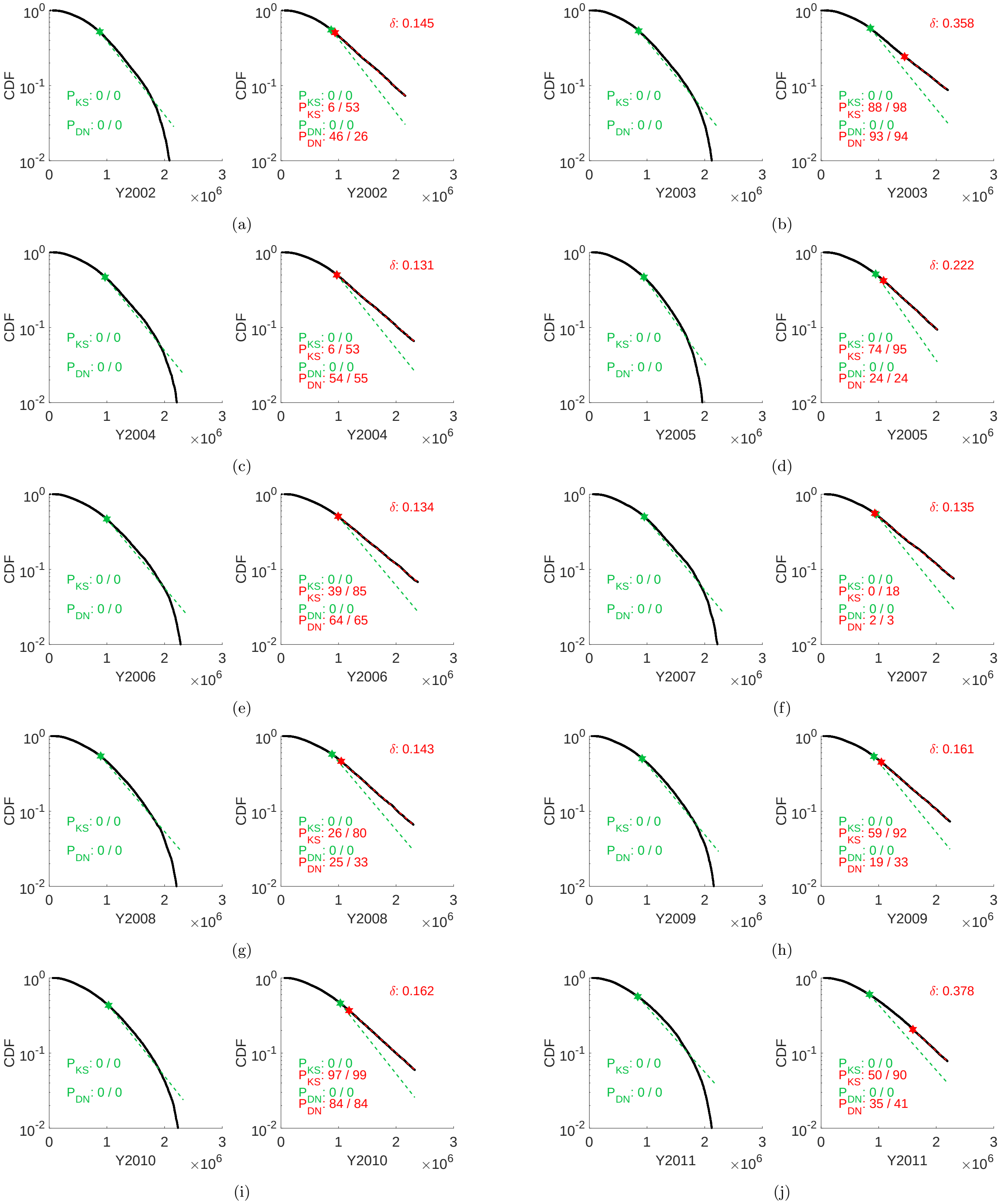


(i)

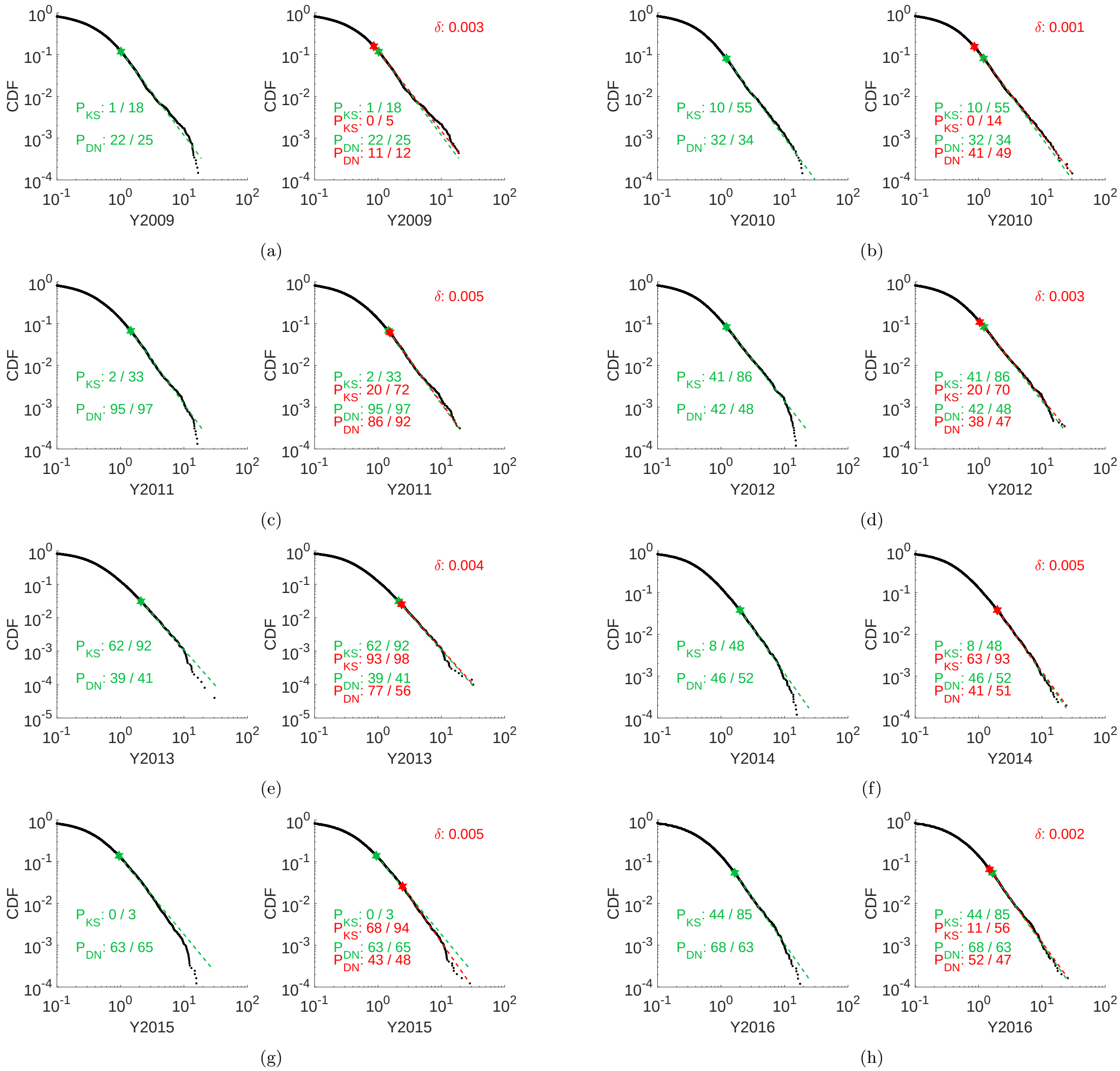


(j)

S Fig. 11. Year by year Taiwan income data from 1991 to 2001, fitted to exponential distributions.



S Fig. 12. Year by year Taiwan income data from 2002 to 2011, fitted to exponential distributions.



S Fig. 13. Year by year SGX normalized returns from 2009 to 2016, fitted to power law distributions.

1. Pyke R (1965) Spacings. *Journal of the Royal Statistical Society. Series B (Methodological)* pp. 395–449.
2. Tay DJ, Chou CI, Li SP, Tee SY, Cheong SA (2016) Bubbles are departures from equilibrium housing markets: Evidence from singapore and taiwan. *PloS one* 11(11):e0166004.
3. Yakovenko VM, Rosser Jr JB (2009) Colloquium: Statistical mechanics of money, wealth, and income. *Reviews of Modern Physics* 81(4):1703.
4. Clauset A, Shalizi CR, Newman ME (2009) Power-law distributions in empirical data. *SIAM review* 51(4):661–703.
5. Mantegna RN, Stanley HE (1995) Scaling behaviour in the dynamics of an economic index. *Nature* 376(6535):46–49.
6. Gopikrishnan P, Plerou V, Amaral LAN, Meyer M, Stanley HE (1999) Scaling of the distribution of fluctuations of financial market indices. *Physical Review E* 60(5):5305.
7. Teh BK, Cheong SA (2016) The asian correction can be quantitatively forecasted using a statistical model of fusion-fission processes. *PloS one* 11(10):e0163842.