COLUMBIA UNIVERSITY EEME E6911 FALL '25

TOPICS IN CONTROL : PROBABILISTIC ROBOTICS

PROBABILITY REVIEW

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The Proverbial Glass

- This glass is half full.
- This glass is half empty.
- Not a philosophy class!

The Proverbial Glass

- The fluid level is at about 50%.
- How good is this information?
- Between 49% and 51% full.
- Between 49% and 51%, with 90% confidence.
- Between 49% and 51%, with 90% confidence, Gaussian.

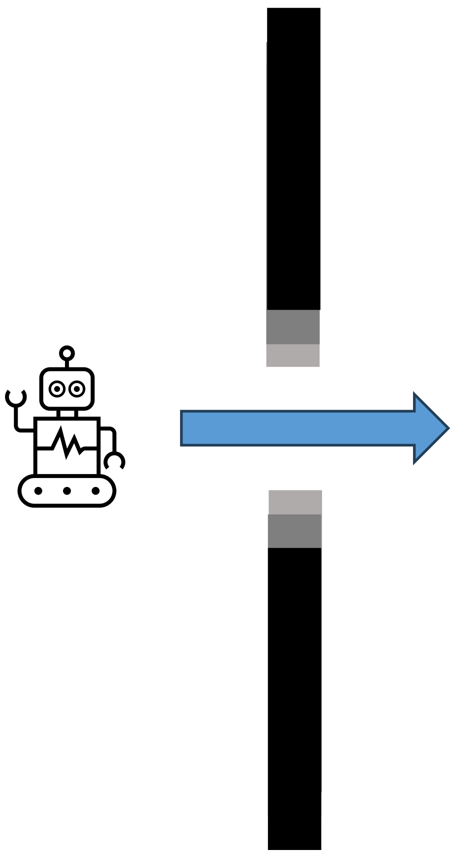
Fundamental Principle

- No sensor can measure the signal with full accuracy.
- No actuation results in perfectly accurate motion.

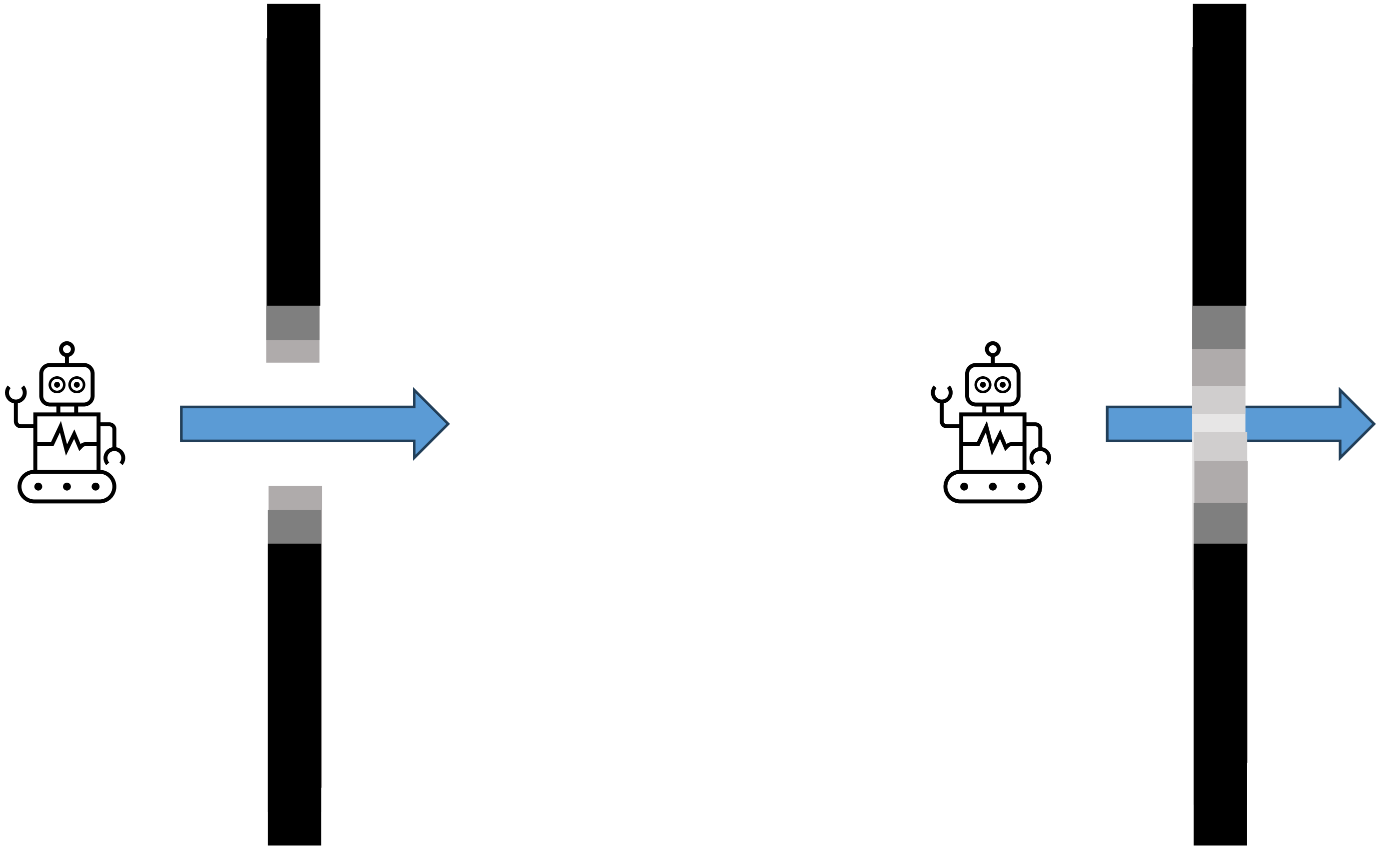
- Sensing and motion are **uncertain.**
- Signals are **random variables**.

- Robot must account for uncertainty when taking actions.
- We must understand random processes when designing the system.

Importance of Uncertainty



Importance of Uncertainty



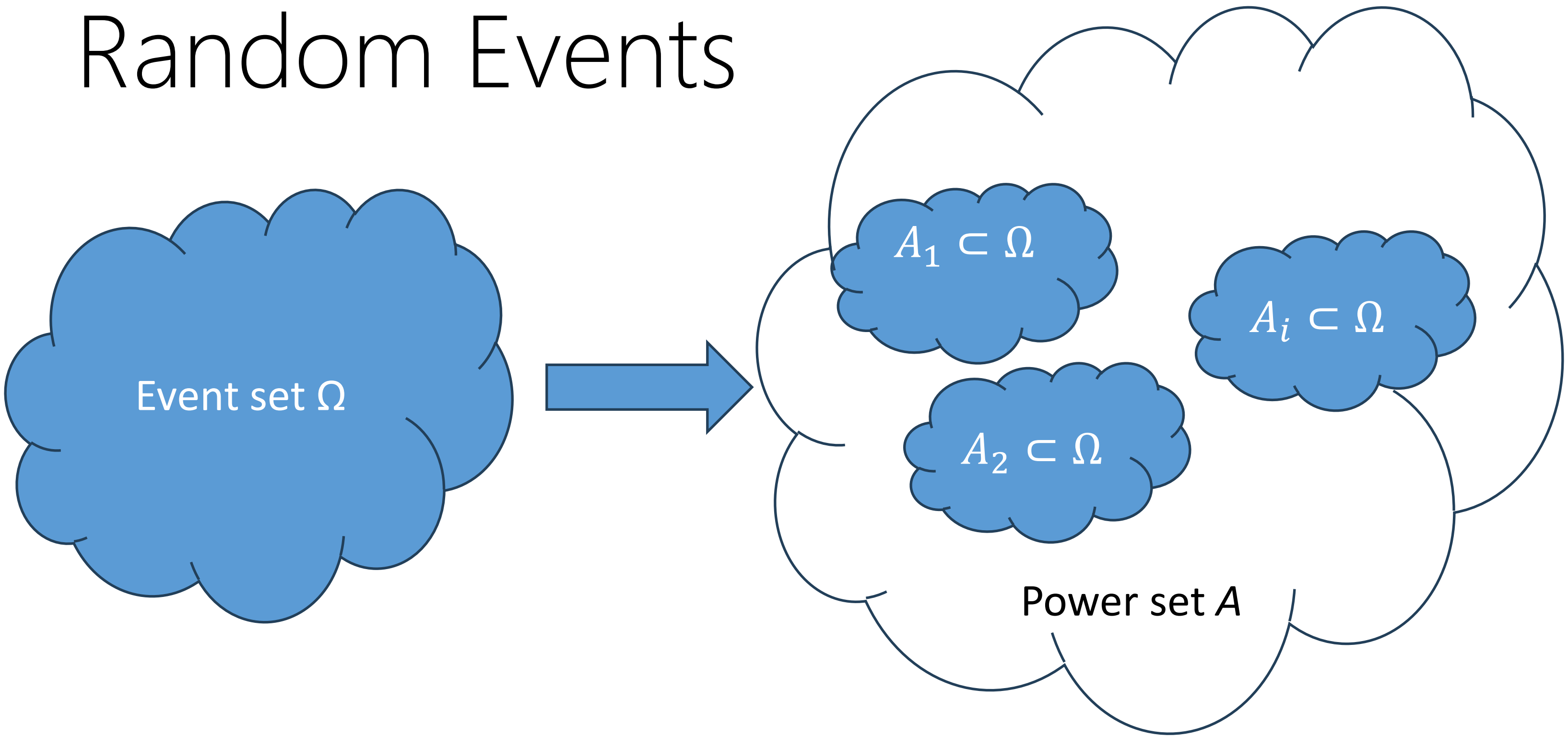
Importance of Uncertainty

- Robot actions depend on perception confidence:
  - Take the action, accept the risk.
  - Do not take the action and remain safe.
  - Take (suboptimal) action based on higher-confidence data.
  - Take another action to first reduce uncertainty.

Characterizing Uncertainty

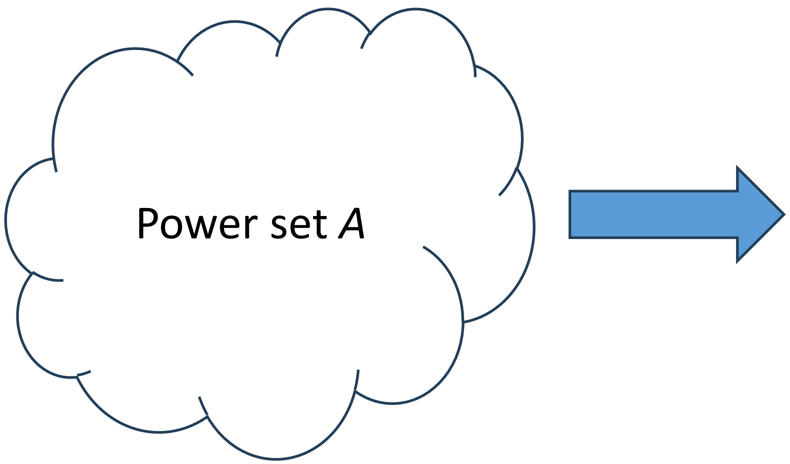
- Parametric:
  - Distribution
  - Mean
  - Covariance
- Non-parametric (Monte-Carlo):
  - Particle Cloud
  - Particle Importance

Random Events



- Start with event set
- Construct a set of all subsets of events set
- These are all events that can happen in our "world"
- Example: rolled 3, rolled an odd number …

Probability



$$[0,1] \subset R$$

- Probability is a function that maps *A* to interval $[0,1]$

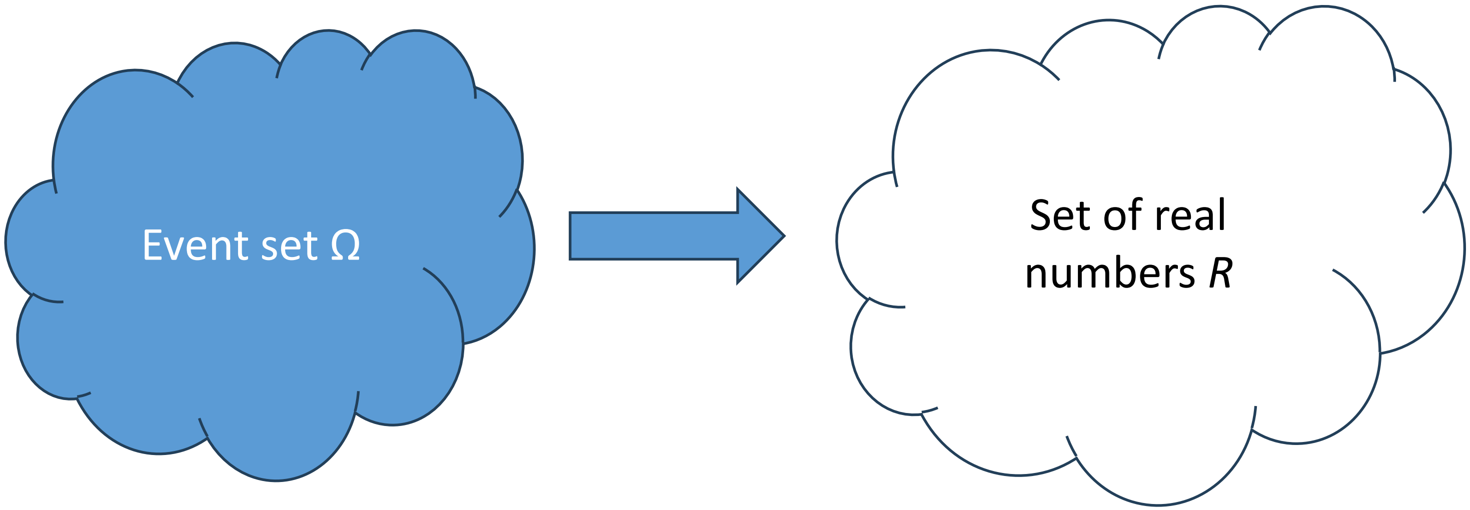
$$P(\Omega) = 1 \qquad P(\emptyset) = 0$$

$$A_i \cap A_j = \emptyset, \forall i, j \Rightarrow P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

Calculating the Probability

- Combinatorics:
  - Count the events
- Geometry:
  - Represent events as volume in space
- Calculus:
  - Integrate the PDF over the events of interest

Random Variable



- Start with event set
- Assign numeric value to each (elementary) event
- Result: mapping from numeric values to probability

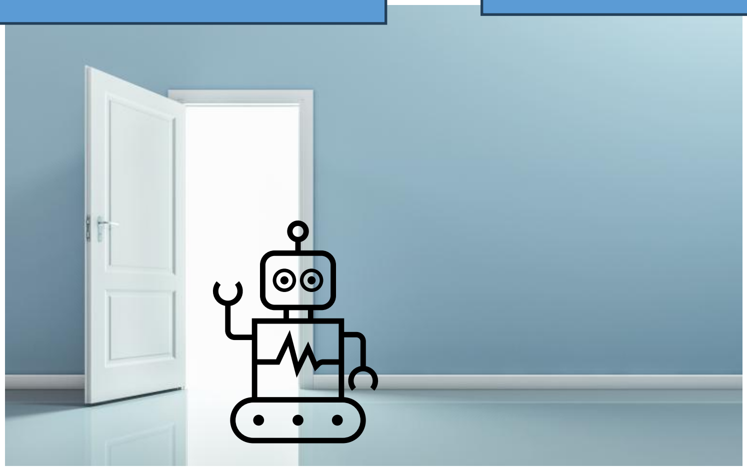
Discrete Random Variables

Specific realization

$$0 < P(X = x) \leq 1$$

Probability

Variable (some signal)



$$\sum_{x} P(X = x) = 1$$

Continuous Random Variables

Specific realization threshold

$$0 < P(X < x) \leq 1$$

Cumulative probability

Variable (some signal)

$$\lim_{x \to \infty} P(X < x) = 1$$

$$f(x) = \frac{d}{dx} P(X < x)$$

Mean, Variance, Standard Deviation

$$E[X] = \sum_{x} xP(x) \qquad E[X] = \int_{x} xf(x)dx$$

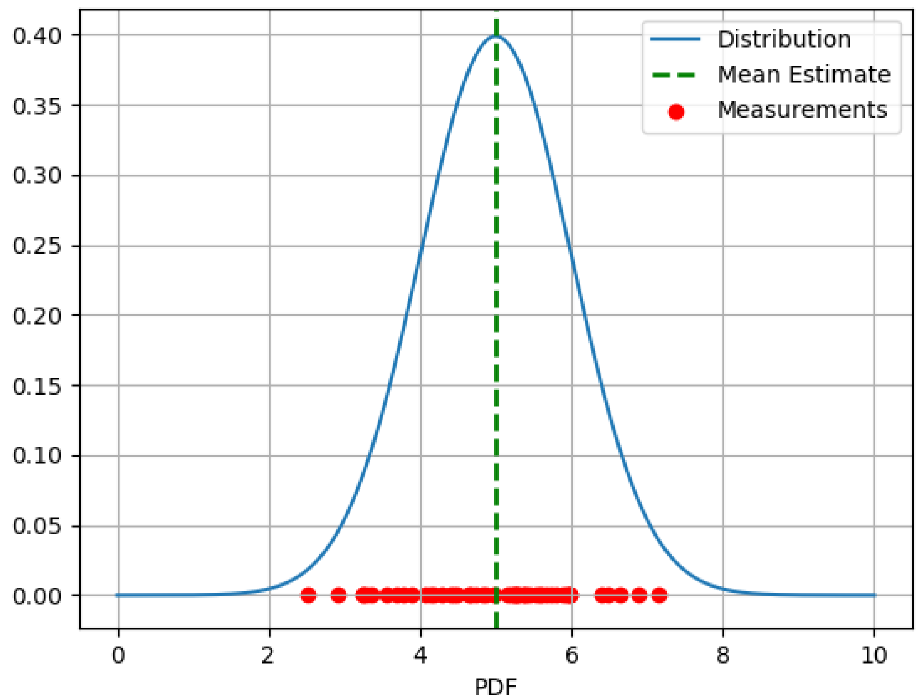
$$V[X] = E\left[\left(X - E(X)\right)^2\right] = E[X^2] - E^2\,[X]$$

$$\sigma[X] = \sqrt{V[X]}$$

Example: 1D Gaussian

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

- Measurement consists of (μ, σ) pair.
- Distribution is implied.
- Higher σ means less confident measurement.
- Both parameters used in decision.



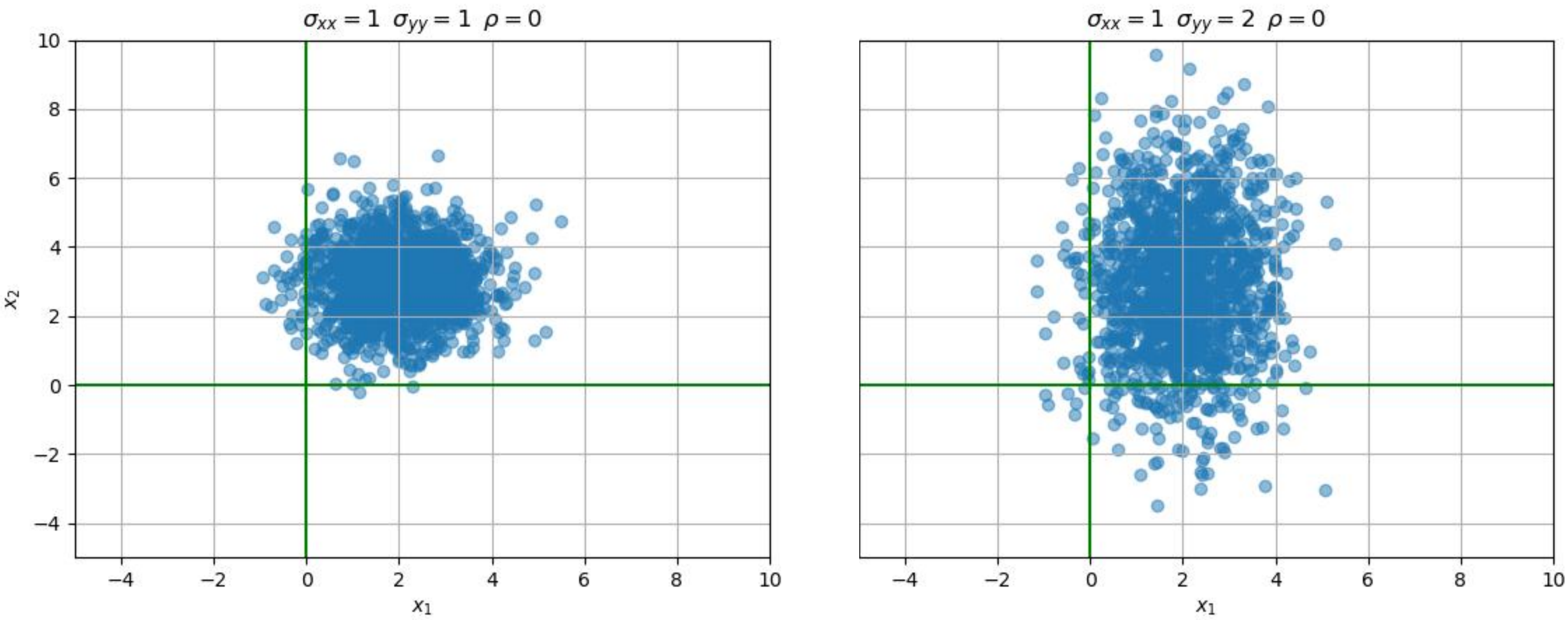
Multiple dimensions

- Each dimension has its own variance.
- But there is more: correlation.

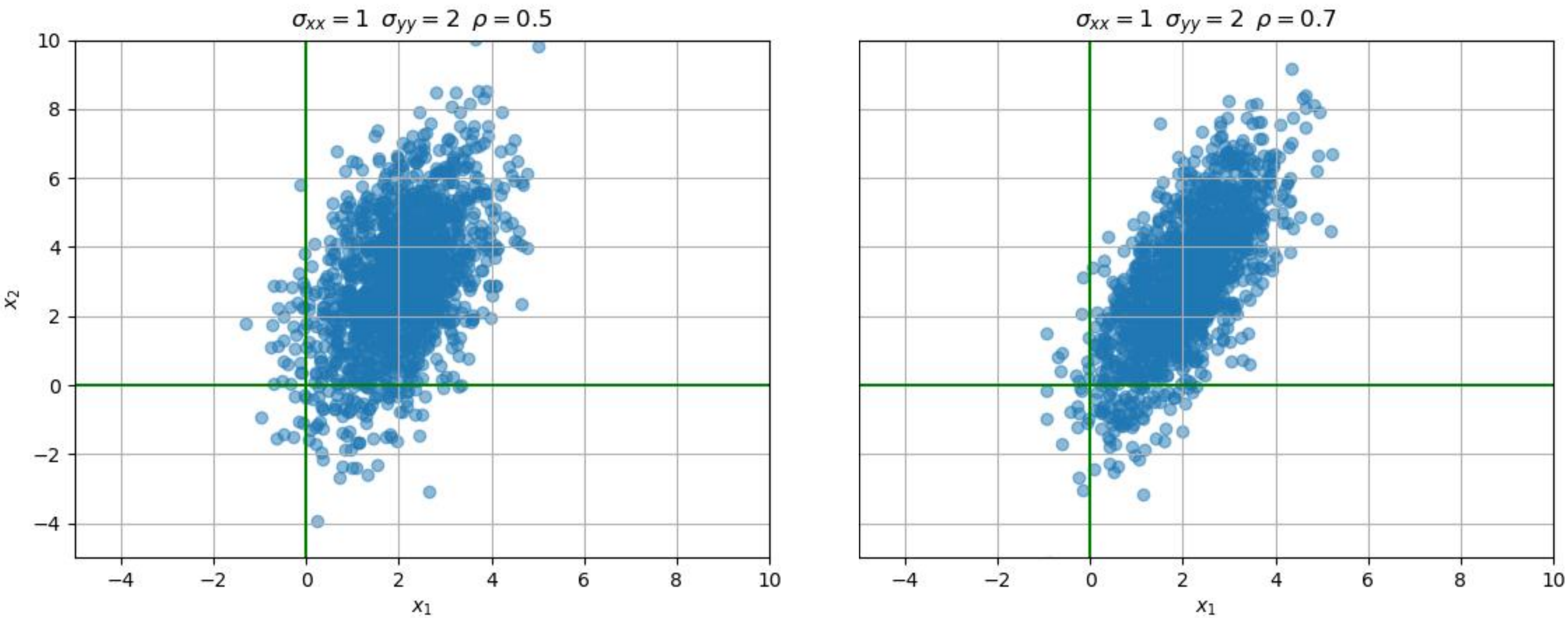
$$f(x) = \frac{1}{(2\pi)^{\frac{k}{2}}\sqrt{|\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11}^2 & \cdots & \rho_{1k}\sigma_{11}\sigma_{kk} \\ \vdots & \ddots & \vdots \\ \rho_{1k}\sigma_{11}\sigma_{kk} & \cdots & \sigma_{kk}^2 \end{bmatrix}$$

Example: 2D Gaussian, uncorrelated

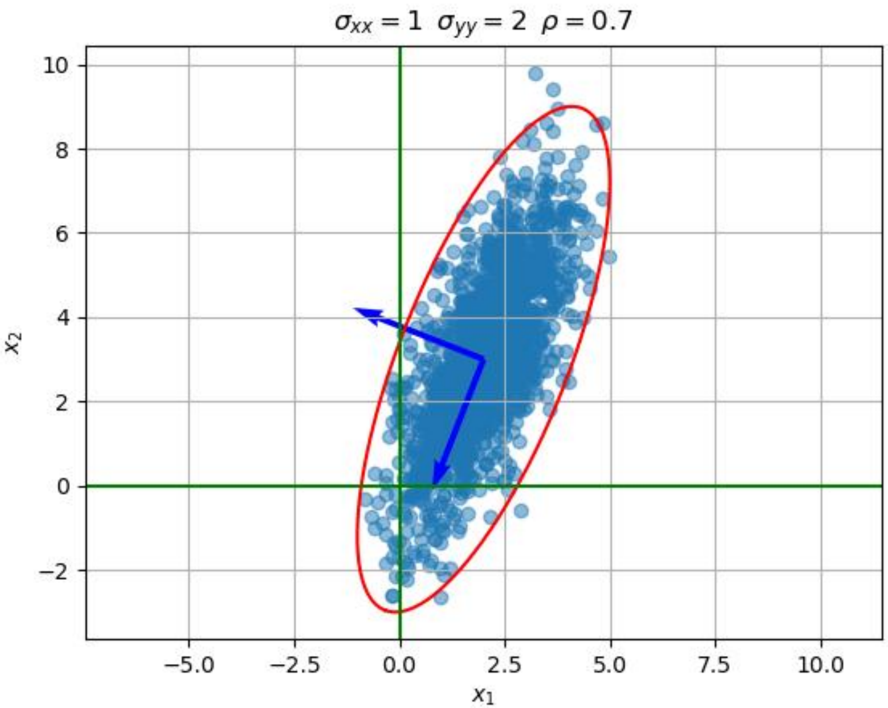


Example: 2D Gaussian, correlated



Uncertainty Ellipse

- Center of the ellipse is at the mean.
- Radii align with eigenvectors.
- Radii are proportional to eigenvalues $3\sqrt{\lambda}$ → 99.7%



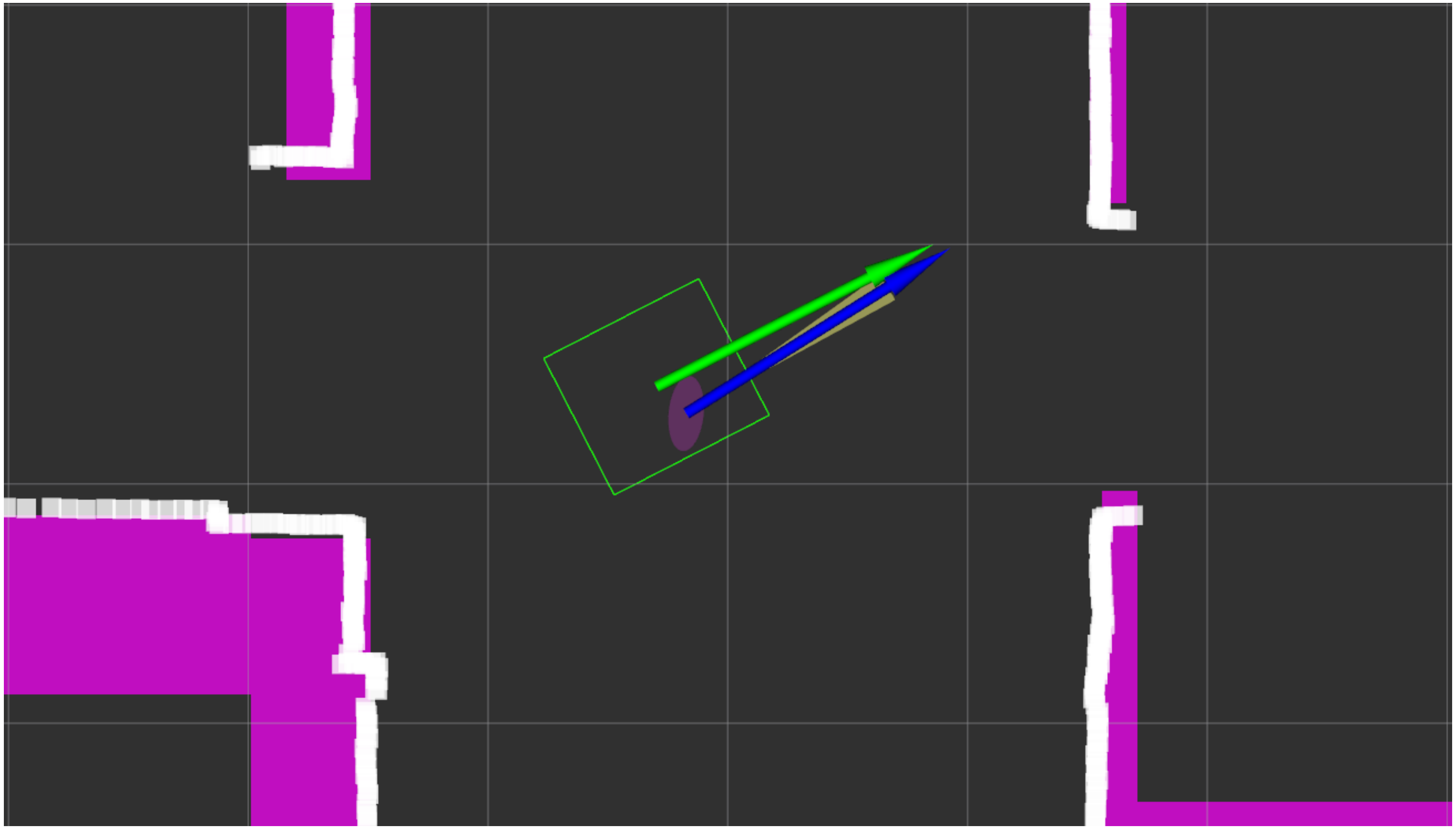
Aside: Eigenvectors and Eigenvalues

$$\lambda \boldsymbol{v} = \mathbf{A}\boldsymbol{v}$$

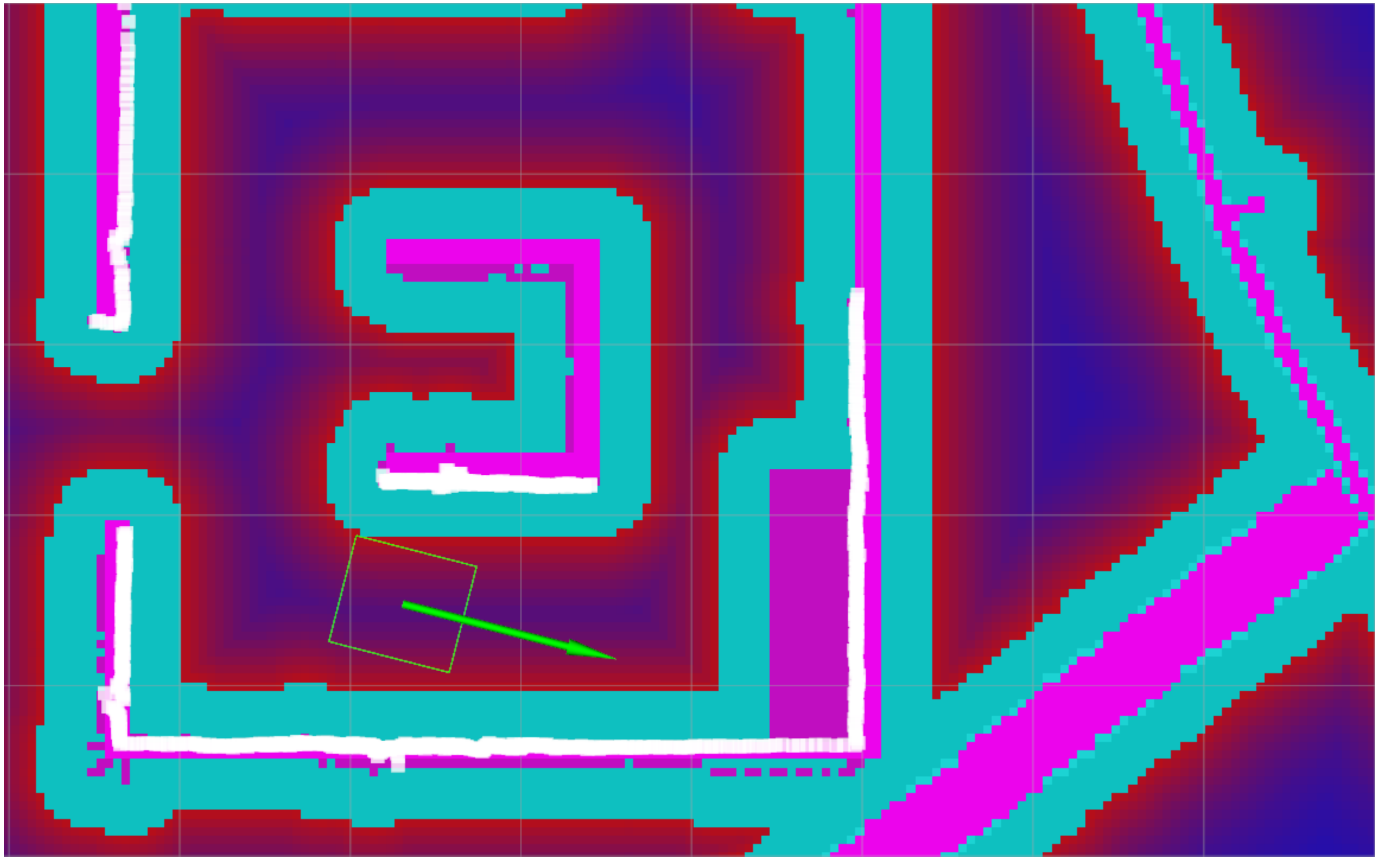
$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

Example: AGV Localization



Example: Cost Map Inflation



Propagating Covariance through Computation

- Linear Transformation

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x}$$

$$\boldsymbol{\Sigma}_y = \mathbf{A}\boldsymbol{\Sigma}_x\mathbf{A}^T$$

- General Transformation

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$$

$$\boldsymbol{\Sigma}_y = \mathbf{J}\boldsymbol{\Sigma}_x\mathbf{J}^T$$

$$\mathrm{J} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_m} \end{bmatrix}$$