COLUMBIA UNIVERSITY EEME E6911 FALL '25

ADV. TOPICS IN CONTROL : PROBABILISTIC ROBOTICS

BAYES FILTER

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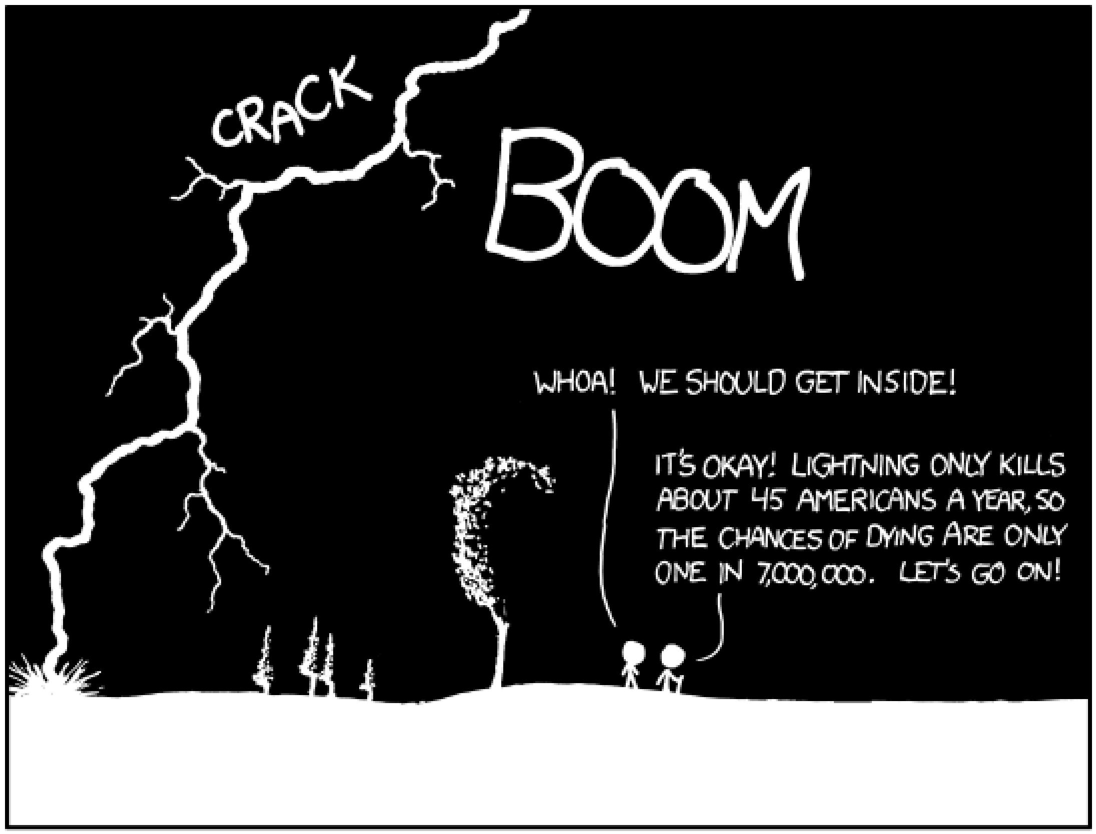
Joint and Conditional Probability



THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT STATISTIC IS ONE IN SIX.

Joint and Conditional Probability

$$P(x, z) = P(X = x, Z = z)$$

$$P(x|z) = \frac{P(x, z)}{P(z)}$$

Independence:

$$P(x, z) = P(x)P(z)$$

$$P(x|z) = P(x)$$

Belief and Prediction

- State given all past inputs and measurements

$$bel(x_t) = P(x_t | z_{1..t}, u_{1..t})$$

- State before applying the very last measurement

$$\overline{bel}(x_{t+1}) = P(x_{t+1} | z_{1..t}, u_{1..t+1})$$

Markov Assumption

- Last known state contains all history

$$bel(x_t) = P(x_t | x_{t-1}, z_t, u_t)$$

- Prediction stems from last known state

$$\overline{bel}(x_t) = P(x_t | x_{t-1}, u_t)$$

Bayesian Inference

$$P(x|z) = \frac{P(z|x)P(x)}{P(z)}$$

Bayesian Inference

Measurement model

Prior knowledge about the system state

$$P(x|z) = \frac{P(z|x)P(x)}{P(z)} = \eta P(z|x)P(x)$$

New knowledge about the system state, after incorporating the measurement

Normalization factor

Example: Just Measurements

- Measurements are unreliable.
- We measure multiple times.
- How does the belief converge?

Is the door open or closed?

Model Through Reasoning

- Door is made of glass
- Vision algorithm can be confused

- Door is actually closed:
  - Can be easily confused for open.
- Door is actually open:
  - Likely to be perceived as open.

Quantify Your Reasoning

- Make a model:

$$P(z = open|x = closed) = 0.5$$

$$P(z = closed|x = closed) = 0.5$$

$$P(z = open|x = open) = 0.9$$

$$P(z = closed|x = open) = 0.1$$

- In practice:
  - Guess from common sense.
  - Measure and characterize.

Set Initial Conditions

- No idea about the door state
- Prior:

$$P(x = open) = 0.5$$
$$P(x = closed) = 0.5$$

- Belief we are after:

$$bel(x_t) = P(x_t | z_{1..t})$$

Apply measurements

- Example measurement sequence:
  - Open
  - Open
  - Open
  - Open
  - Open
  - Closed
  - Open
  - Open

Apply measurements

$$P(x = open|z = open) = \eta P(z = open|x = open)bel(open)$$
$$P(x = closed|z = open) = \eta P(z = open|x = closed)bel(closed)$$

$$P(x = open|z = open) = \eta \cdot 0.9 \cdot 0.5 = 0.45\eta$$
$$P(x = closed|z = open) = \eta \cdot 0.5 \cdot 0.5 = 0.25\eta$$

$$\eta = \frac{1}{0.45 + 0.25} = 1.429$$

$$bel(open) = 0.45 \cdot 1.429 = 0.643$$
$$bel(closed) = 1 - 0.643 = 0.357$$

Decide

- Are we confident enough after single sensor reading?
- Can the robot safely proceed through the door?
- What should the robot decide to do?
- How would the decision differ from deterministic robotics algorithms?

Apply Measurements Recursively

```
import numpy as np

# index 0 means open
# index 1 means closed

# rows are measurements, columns are priors
measurement_model = np.array(
   [ [ 0.9, 0.5 ],
     [ 0.1, 0.5 ] ]
)

bel = np.array(
    [ [ 0.5 ],
      [ 0.5 ] ]
)

measurements = [ 0, 0, 0, 0, 0, 1, 0, 0 ]

for measurement in measurements:
    unnormalized_posterior = (
        measurement_model * \
        np.repeat(bel, 2, axis=1).transpose())[measurement]
    posterior = unnormalized_posterior / sum(unnormalized_posterior)
    print(round(posterior[0], 3))
    bel = np.array([ posterior ]).transpose()
```

Element-wise multiplication

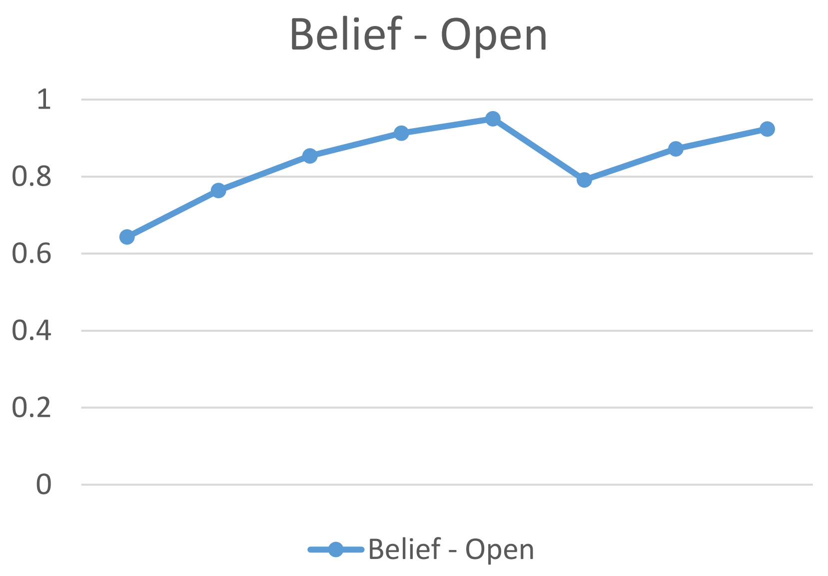
$$\begin{bmatrix} P(0|0) & P(0|1) \\ P(1|0) & P(1|1) \end{bmatrix} * \begin{bmatrix} bel(0) & bel(1) \\ bel(0) & bel(1) \end{bmatrix}$$

$$\begin{bmatrix} P(0|0)bel(0) & P(0|1)bel(1) \\ P(1|0)bel(0) & P(1|1)bel(1) \end{bmatrix}$$

Pick the row that corresponds to the current measurement

Results

Belief – Door open
0.643
0.764
0.854
0.913
0.95
0.791
0.872
0.924



Example: Input and Measurements

- Robot can push the door
- Door opens (maybe?)
- We have the model
- Use the model for prediction
- Measure
- How does the belief converge now?

Quantify your reasoning

- Door already open stays open if pushed:

$$P(x_{t+1} = open | x_t = open, u = push) = 1$$

$$P(x_{t+1} = closed | x_t = open, u = push) = 0$$

- Closed door opens with 60% chance when pushed:

$$P(x_{t+1} = open | x_t = closed, u = push) = 0.6$$

$$P(x_{t+1} = closed | x_t = closed, u = push) = 0.4$$

Quantify your reasoning

- Nothing changes to the door if we do nothing:

$$P(x_{t+1} = open | x_t = open, u = none) = 1$$

$$P(x_{t+1} = closed | x_t = open, u = none) = 0$$

$$P(x_{t+1} = open | x_t = closed, u = none) = 0$$

$$P(x_{t+1} = closed | x_t = closed, u = none) = 1$$

Predict from model

$$\overline{bel}(open) = P(open|open, push)bel(open) + P(open|closed, push)bel(closed)$$

$$\overline{bel}(open) = 1 \cdot 0.5 + 0.6 \cdot 0.5 = 0.8$$
$$\overline{bel}(closed) = 0 \cdot 0.5 + 0.4 \cdot 0.5 = 0.2$$

- Feed prediction into measurement equations
- We now start with stronger belief that the door is open

Prediction model – Matrix form

$$\boldsymbol{M} = \begin{bmatrix} P(open|open) & P(open|closed) \\ P(closed|open) & P(closed|closed) \end{bmatrix}$$

$$\begin{bmatrix} \overline{bel}(open) \\ \overline{bel}(closed) \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} bel(open) \\ bel(closed) \end{bmatrix}$$

- Push

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0.6 \\ 0 & 0.4 \end{bmatrix}$$

- No action

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Apply measurements

$$P(x = open|z = open) = \eta P(z = open|x = open)\ \overline{bel}(open)$$
$$P(x = closed|z = open) = \eta P(z = open|x = closed)\ \overline{bel}(closed)$$

$$P(x = open|z = open) = \eta \cdot 0.9 \cdot 0.8 = 0.72\eta$$
$$P(x = closed|z = open) = \eta \cdot 0.5 \cdot 0.2 = 0.1\eta$$

$$\eta = \frac{1}{0.72 + 0.1} = 1.2195$$

$$bel(open) = 0.72 \cdot 1.2195 = 0.878$$
$$bel(closed) = 1 - 0.878 = 0.122$$