COLUMBIA UNIVERSITY EEME E6911 FALL '25

TOPICS IN CONTROL : PROBABILISTIC ROBOTICS

SIMULTANEOUS LOCALIZATION AND MAPPING (SLAM)

Instructor: Ilija Hadzic

Mobile Robotics Problems

- Localization
- Mapping
- Simultaneous Localization and Mapping (SLAM)
- Navigation
- Exploration

Mapping, Localization and SLAM

- Localization:
  - Map provided
  - Use sensors
  - Determine the pose
- Mapping
  - Pose provided
  - Use sensors
  - Determine the map
- SLAM
  - Nothing provided
  - Use sensors
  - Determine the map and pose

SLAM Problem

- Find the belief

$$p(x_t, m_t \mid z_{0..t}, u_{0..t})$$

- Robot only knows its inputs and sensors.
- Map is part of the state.
- Map belief changes over time as the robot learns it.

Online vs. Full SLAM

- Online SLAM.
- Only recovers present pose belief

$$p(x_t, m_t \mid z_{0..t}, u_{0..t})$$

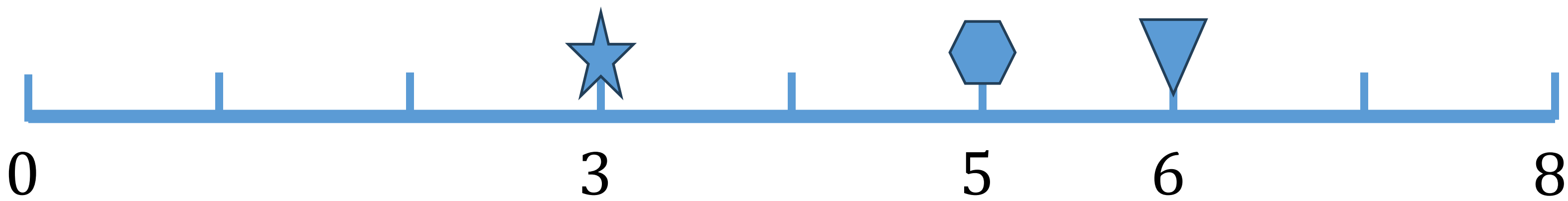
- Full SLAM.
- Retrofits past beliefs as the map updates

$$p(x_{0..t}, m_t \mid z_{0..t}, u_{0..t})$$

EKF SLAM

- Solves Online SLAM
- Needs landmarks
- State is the pose of the robot and landmark poses
- Landmarks added as they are discovered
- State converges after repeated landmark observations (loop closure)
- Occupancy grid can be constructed afterwards

Simple 1D Example



- Robot moves on a loop (shown unrolled).
- True velocity is $\mathcal{N}(\mu_v = 1\, m/s, \sigma_v^2 = 0.05\, m^2/s^2)$.
- Robot's velocity belief is constant $1\, m/s$.
- Sampling period is $\Delta t = 0.1s$.
- Landmark is observed only when closer than $0.1\, m$.
- Measurement variance is $\sigma_z^2 = 0.01$.
- Loop length is $8m$ an it is known to the system.

Motion Model

- Just like Kalman Filter:

$$x[n] = x[n-1] + \Delta t \cdot v[n]$$

$$\bar{\sigma}_x^2\,[n] = \sigma_x^2[n-1] + \Delta t^2 \sigma_v^2[n]$$

- Cumulative motion after k iterations:

$$x[n] = x[n-k] + \Delta t \cdot \sum_{i=n-k+1}^{n} v[i]$$

$$\bar{\sigma}_x^2\,[n] = \sigma_x^2[n-k] + k\Delta t^2 \sigma_v^2[n]$$

Motion Model

- Just like Kalman Filter:

$$x[n] = x[n-1] + \Delta t \cdot v[n]$$

$$\bar{\sigma}_x^2\,[n] = \sigma_x^2[n-1] + \Delta t^2 \sigma_v^2[n]$$

- Cumulative motion after k iterations:

Gaussian realizations

$$x[n] = x[n-k] + \Delta t \cdot \sum_{i=n-k+1}^{n} v[i]$$

$$\bar{\sigma}_x^2\,[n] = \sigma_x^2[n-k] + k\Delta t^2 \sigma_v^2[n]$$

Measurement Model

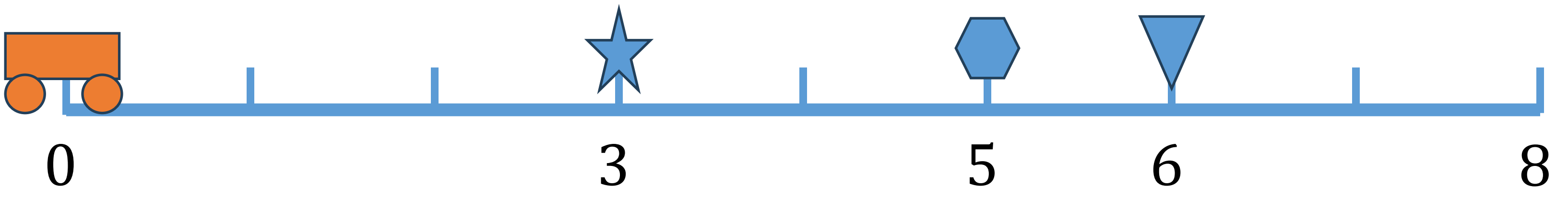
- For *i*th landmark, measure the distance:

$$z[n] = m_i[n-1] - x[n]$$

- Measurement Jacobian:

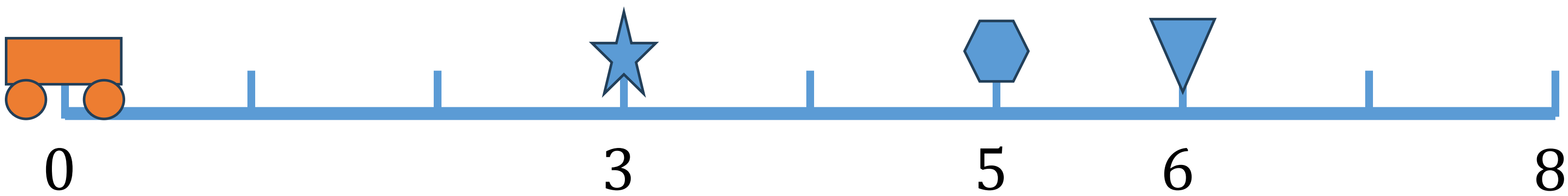
$$\boldsymbol{J}_z = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial m_i} \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

Initialization



- Robot declares the starting point the to be the origin
  - Arbitrary choice
- No landmarks have been spotted
  - Robot does not even know if they are there
- What is the state and covariance?

Initialization



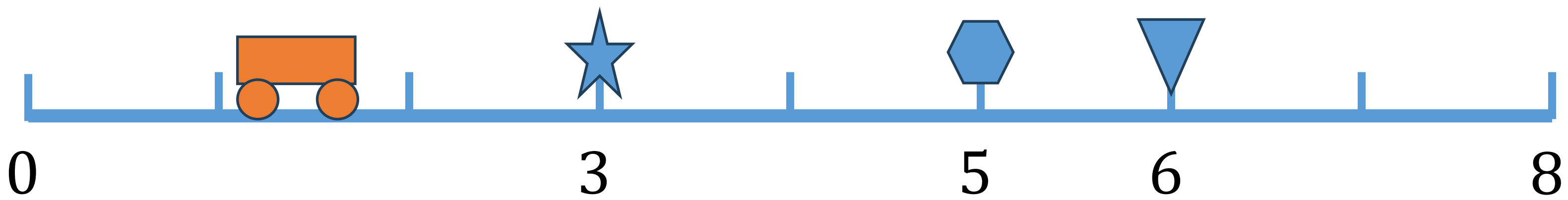
$$\boldsymbol{x} = [0]$$

Origin picked by convention

$$\Sigma = [0]$$

No randomness in picking the origin

Moving towards first landmark

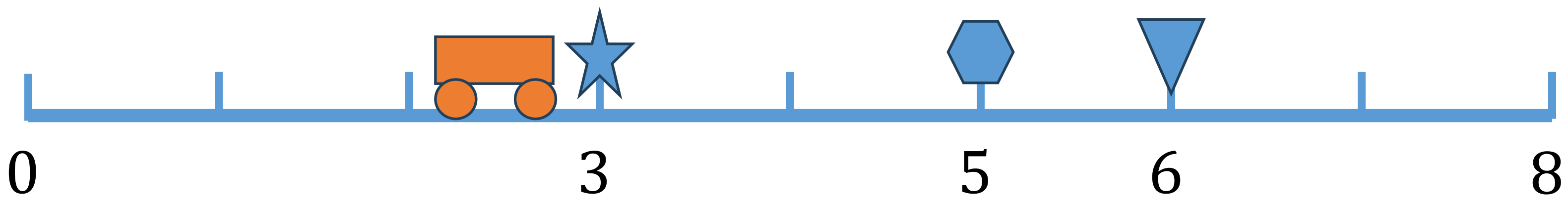


- Nothing interesting happens
- Robot accumulates odometry
- Still no idea that any landmark exists

$$\boldsymbol{x} = [x > 0]$$

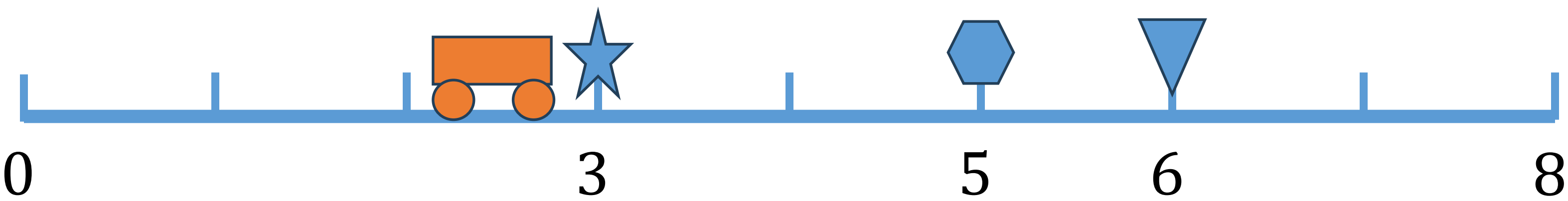
$$\Sigma = [\sigma^2 > 0]$$

Reached first landmark (within 0.1m)



- What are the true positions of:
  - Robot?
  - Landmark?
- What happens to the state?
- What is the new covariance?

Reached first landmark



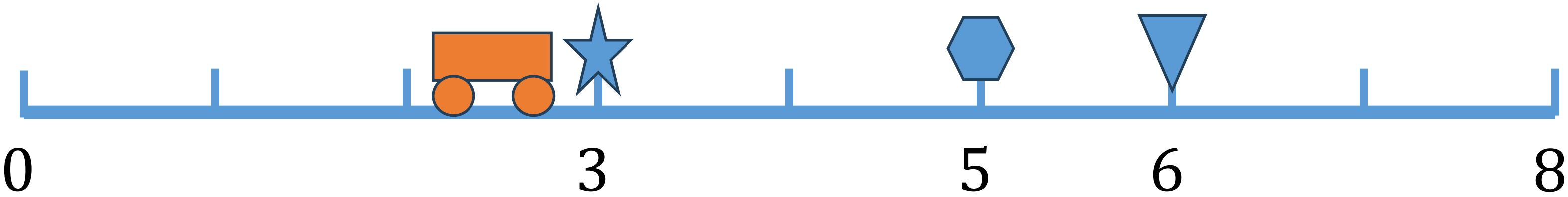
- Landmark observed after N samples, where N satisfies:

$$2.9 < 0.1 \cdot \sum_{i=1}^{N} x_i : \mathcal{N}(v = 1, \sigma_v^2 = 0.05) \leq 3$$

- Example realization: $N = 29, x_{\text{GT}} = 2.94$.
- This is our ground truth position at the first landmark.

```
0.1 * sum(numpy.random.normal(1, math.sqrt(0.05), N))
```

Reached first landmark



- Position ground truth:

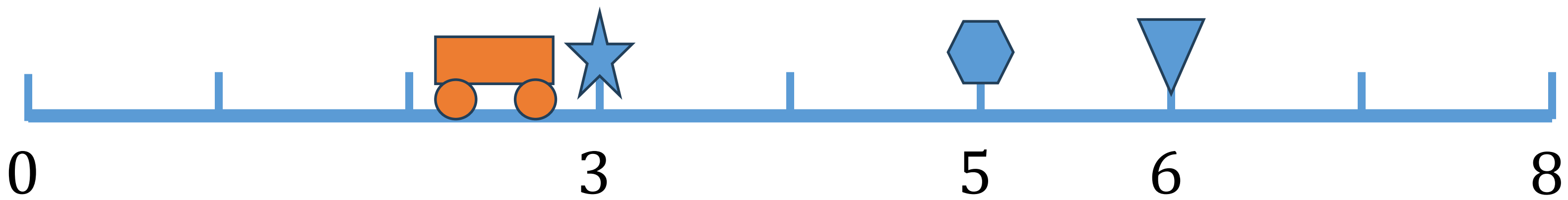
$$x_{\mathrm{GT}} = 2.94\, m$$

- Position belief:

$$x = 29 \cdot 0.1s \cdot 1\, m/s = 2.9\, m$$

$$\sigma_x^2 = 29 \cdot 0.1^2 \cdot 0.05 = 0.0145$$

Reached first landmark



- Distance to the landmark ground truth:

$$d_{\mathrm{GT}} = m_{1\mathrm{GT}} - x_{\mathrm{GT}} = 3 - 2.94 = 0.06$$

- Measured distance:

$$z_1 = d \colon \mathcal{N}\left(d = 0.06, \sigma_d^2 = 0.01\right)$$

- Example realization:

$$z_1 = 0.048, \qquad \sigma_z^2 = 0.01$$

Landmark Addition

- Landmark has never been observed
- Add measured position to the prior state
- Set prior covariance to $\sigma_m \rightarrow \infty$
- Calculate the EKF posterior

Landmark Addition

- Prior

$$\boldsymbol{x} = \begin{bmatrix} x \\ x + z_1 \end{bmatrix} = \begin{bmatrix} 2.9 \\ 2.9 + 0.048 \end{bmatrix} = \begin{bmatrix} 2.9 \\ 2.948 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_m^2 \to \infty \end{bmatrix} = \begin{bmatrix} 0.0145 & 0 \\ 0 & \sigma_m^2 \to \infty \end{bmatrix}$$

- What is the expected measurement?

Landmark Addition

- Kalman Gain

$$\boldsymbol{H} = \boldsymbol{J}_z$$

$$\boldsymbol{K} = \boldsymbol{\Sigma}\boldsymbol{H}^T(\boldsymbol{H}\boldsymbol{\Sigma}\boldsymbol{H}^T + \sigma_z^2)^{-1}$$

$$\mathbf{\Sigma H}^T = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_m^2 \to \infty \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sigma_x^2 \\ \sigma_m^2 \end{bmatrix}$$

$$\mathbf{H\Sigma H}^T = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -\sigma_x^2 \\ \sigma_m^2 \end{bmatrix} = [\sigma_x^2 + \sigma_m^2]$$

Landmark Addition

- Kalman Gain

$$\boldsymbol{K} = \begin{bmatrix} \dfrac{-\sigma_x^2}{\sigma_x^2 + \sigma_m^2 + \sigma_z^2} \\ \dfrac{\sigma_m^2}{\sigma_x^2 + \sigma_m^2 + \sigma_z^2} \end{bmatrix}$$

$$\lim_{\sigma_m^2 \to \infty} \boldsymbol{K} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Landmark Addition

- Update mean posterior (remains unchanged)

$$\boldsymbol{x}[n] = \boldsymbol{x}[n-1] + \boldsymbol{K}(z_1 - h(\boldsymbol{x}[n-1]))$$

$$\boldsymbol{x}[n] = \begin{bmatrix} x \\ x + z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z_1 - z_1) = \begin{bmatrix} x \\ x + z_1 \end{bmatrix} = \begin{bmatrix} 2.9 \\ 2.948 \end{bmatrix}$$

- Is unchanged posterior expected? Why?
- Will covariance posterior also remain unchanged?

Landmark Addition

- Update covariance

$$\mathbf{\Sigma}[\mathrm{n}] = (\boldsymbol{I} - \boldsymbol{KH})\mathbf{\Sigma}[n-1]$$

$$\mathbf{\Sigma}[\mathrm{n}] = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \dfrac{-\sigma_x^2}{\sigma_x^2 + \sigma_m^2 + \sigma_z^2} \\ \dfrac{\sigma_m^2}{\sigma_x^2 + \sigma_m^2 + \sigma_z^2} \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \right) \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_m^2 \end{bmatrix}$$

- Exercise: Simplify the above expression

Landmark Addition

- Update covariance

$$\boldsymbol{\Sigma}[\mathrm{n}] = \begin{bmatrix} \frac{\sigma_x^2(\sigma_m^2+\sigma_z^2)}{\sigma_x^2+\sigma_m^2+\sigma_z^2} & \frac{\sigma_x^2\sigma_m^2}{\sigma_x^2+\sigma_m^2+\sigma_z^2} \\ \frac{\sigma_x^2\sigma_m^2}{\sigma_x^2+\sigma_m^2+\sigma_z^2} & \frac{\sigma_m^2(\sigma_x^2+\sigma_z^2)}{\sigma_x^2+\sigma_m^2+\sigma_z^2} \end{bmatrix}$$

$$\lim_{\sigma_m^2\to\infty} \boldsymbol{\Sigma}[\mathrm{n}] = \begin{bmatrix} \sigma_x^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2+\sigma_z^2 \end{bmatrix}$$

Landmark Addition

- Making sense out of the result:

$$\lim_{\sigma_m^2 \to \infty} \boldsymbol{\Sigma}[\mathrm{n}] = \begin{bmatrix} \sigma_x^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2 + \sigma_z^2 \end{bmatrix}$$

Landmark Addition

- Making sense out of the result:

$$\lim_{\sigma_m^2 \to \infty} \mathbf{\Sigma}[\mathrm{n}] = \begin{bmatrix} \sigma_x^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2 + \sigma_z^2 \end{bmatrix}$$

Simply observing the landmark adds no new information about the location.

Landmark Addition

- Making sense out of the result:

$$\lim_{\sigma_m^2 \to \infty} \boldsymbol{\Sigma}[\mathrm{n}] = \begin{bmatrix} \sigma_x^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2 + \sigma_z^2 \end{bmatrix}$$

But we learned new information about the landmark!

Landmark Addition

- Making sense out of the result:

$$\lim_{\sigma_m^2 \to \infty} \mathbf{\Sigma}[\mathrm{n}] = \begin{bmatrix} \sigma_x^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2 + \sigma_z^2 \end{bmatrix}$$

Landmark and position information is highly correlated.

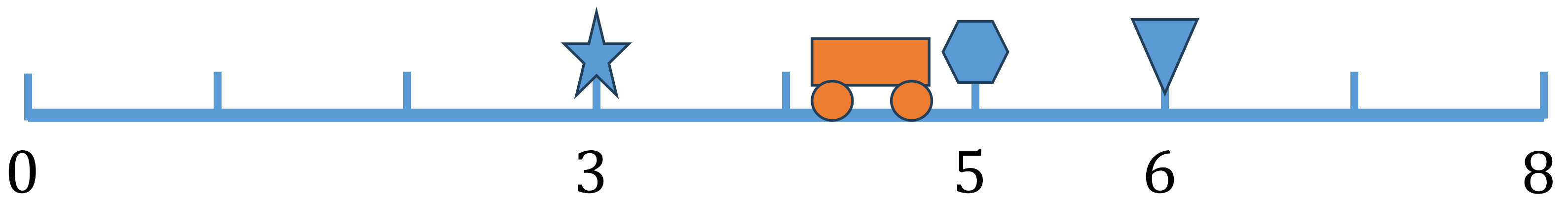
Landmark Addition

- Posterior

$$\boldsymbol{x} = \begin{bmatrix} 2.9 \\ 2.948 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_x^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2 + \sigma_z^2 \end{bmatrix} = \begin{bmatrix} 0.0145 & 0.0145 \\ 0.0145 & 0.0245 \end{bmatrix}$$

Move to the second landmark



$$x_{\mathrm{GT}} = 4.92, N = 19$$

$$d_{\mathrm{GT}} = m_{2\mathrm{GT}} - x_{\mathrm{GT}} = 5 - 4.92 = 0.08$$

$$z_1 = d: \mathcal{N}\left(d = 0.08, \sigma_d^2 = 0.01\right)$$

$$z_1 = 0.135, \qquad \sigma_z^2 = 0.01$$

$$x = 4.8, \qquad \sigma_x^2 = 0.024$$

Adding the second landmark

- Prior

Robot moved

Landmark 1 belief unchanged

Landmark 2 prior added

$$\boldsymbol{x} = \begin{bmatrix} 4.8 \\ 2.948 \\ 4.935 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.024 & 0.0145 & 0 \\ 0.0145 & 0.0245 & 0 \\ 0 & 0 & \sigma_m^2 \to \infty \end{bmatrix}$$

Dimensionality adaptation

- Jacobian dimension is 1x2, but we need 1x3
- We need Jacobian with one extra dimension
- Jacobian must operate on position and landmark 2

$$\boldsymbol{x} = \begin{bmatrix} 4.8 \\ 2.948 \\ 4.935 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.024 & 0.0145 & 0 \\ 0.0145 & 0.0245 & 0 \\ 0 & 0 & \sigma_m^2 \rightarrow \infty \end{bmatrix}$$

Filtering/sifting matrix

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{H} = \boldsymbol{J}_z\boldsymbol{F} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

Filtering/sifting matrix

$$\begin{array}{rccc} & \text{Position/ pose} & \text{Landmark 1} & \text{Landmark 2} \\ \boldsymbol{F} = & \left[\begin{matrix}1\\0\end{matrix}\right. & \begin{matrix}0\\0\end{matrix} & \left.\begin{matrix}0\\1\end{matrix}\right] \end{array}$$

$$\boldsymbol{H} = \boldsymbol{J}_z\boldsymbol{F} = \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 0 & 1\end{bmatrix}$$

Prior Covariance Pattern

$$\boldsymbol{\Sigma} = \begin{bmatrix} a & x & 0 \\ x & b & 0 \\ 0 & 0 & d \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.024 & 0.0145 & 0 \\ 0.0145 & 0.0245 & 0 \\ 0 & 0 & \sigma_m^2 \to \infty \end{bmatrix}$$

Prior Covariance Pattern

$$\boldsymbol{\Sigma} = \begin{bmatrix} a & x & 0 \\ x & b & 0 \\ 0 & 0 & d \end{bmatrix}$$

Crossterms

Position/ pose

Landmark 1

Landmark 2

Calculating Posterior

$$\boldsymbol{\Sigma}\boldsymbol{H}^T = \begin{bmatrix} a & x & 0 \\ x & b & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -a \\ -x \\ d \end{bmatrix}$$

$$\boldsymbol{H}\boldsymbol{\Sigma}\boldsymbol{H}^T = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -a \\ -x \\ d \end{bmatrix} = a + d$$

$$\boldsymbol{K} = \boldsymbol{\Sigma}\boldsymbol{H}^T(\boldsymbol{H}\boldsymbol{\Sigma}\boldsymbol{H}^T + e)^{-1} = \frac{1}{a + d + e} \begin{bmatrix} -a \\ -x \\ d \end{bmatrix}$$

$$\lim_{d \to \infty} \boldsymbol{K} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Calculating Posterior

$$\boldsymbol{\Sigma H}^T = \begin{bmatrix} a & x & 0 \\ x & b & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -a \\ -x \\ d \end{bmatrix}$$

$$\boldsymbol{H\Sigma H}^T = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -a \\ -x \\ d \end{bmatrix} = a + d$$

$$\boldsymbol{K} = \boldsymbol{\Sigma H}^T (\boldsymbol{H\Sigma H}^T + e)^{-1} = \frac{1}{a + d + e} \begin{bmatrix} -a \\ -x \\ d \end{bmatrix}$$

$$\lim_{d \to \infty} \boldsymbol{K} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

New landmark added as is, previous landmarks and pose unchanged

Posterior Covariance

$$\mathbf{\Sigma}[\mathrm{n}] = (\boldsymbol{I} - \boldsymbol{KH})\mathbf{\Sigma}[\mathrm{n}-1]$$

$$\cdots$$

$$\cdots$$

$$\mathbf{\Sigma}[\mathrm{n}] = \frac{1}{a+d+e}\begin{bmatrix} a(d+e) & x(d+e) & ad \\ x(d+e) & -x^2 + b(a+d+e) & xd \\ ad & xd & d(a+e) \end{bmatrix}$$

$$\lim_{\boldsymbol{d}\to\infty} \mathbf{\Sigma}[\mathrm{n}] = \begin{bmatrix} a & x & a \\ x & b & x \\ a & x & a+e \end{bmatrix}$$

$$a = \sigma_x^2$$

$$b = \sigma_{m1}^2$$

$$e = \sigma_z^2$$

New state

- Posterior

$$\boldsymbol{x} = \begin{bmatrix} 4.8 \\ 2.948 \\ 4.935 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.024 & 0.0145 & 0.024 \\ 0.0145 & 0.0245 & 0.0145 \\ 0.024 & 0.0145 & 0.034 \end{bmatrix}$$

Making sense out of the result

$$\boldsymbol{x} = \begin{bmatrix} 4.8 \\ 2.948 \\ 4.935 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.024 & 0.0145 & 0.024 \\ 0.01\text{[illegible]}5 & 0.0245 & 0.0145 \\ 0\text{[illegible]} & 0.0145 & 0.034 \end{bmatrix}$$

We are less confident about the position because it’s just odometry.

Making sense out of the result

$$\boldsymbol{x} = \begin{bmatrix} 4.8 \\ 2.948 \\ 4.935 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.024 & 0.0145 & 0.024 \\ 0.0145 & 0.0245 & 0.0\text{[illegible]}45 \\ 0.024 & 0.0145 & 0.\text{[illegible]} \end{bmatrix}$$

Landmarks and position are coorelated.

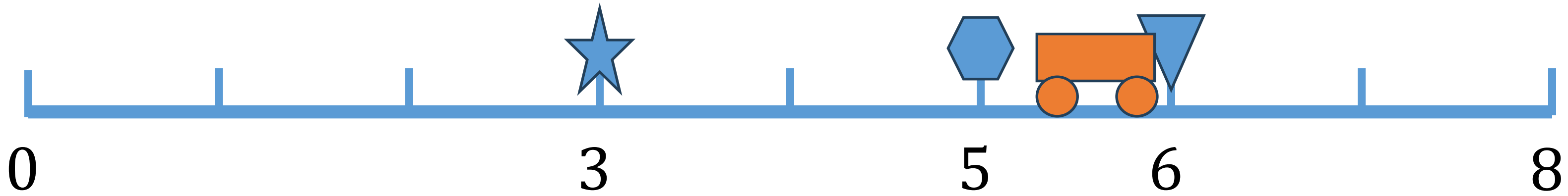
Making sense out of the result

$$\boldsymbol{x} = \begin{bmatrix} 4.8 \\ 2.948 \\ 4.935 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.024 & 0.0145 & 0.024 \\ 0.0145 & 0.0245 & 0.0145 \\ 0.024 & 0.0145 & 0.034 \end{bmatrix}$$

We are less confident about the second landmark. Why?

Move to the third landmark



$$x_{\mathrm{GT}} = 5.97, N = 10$$

$$d_{\mathrm{GT}} = m_{3\mathrm{GT}} - x_{\mathrm{GT}} = 6 - 5.97 = 0.03$$

$$z_1 = d : \mathcal{N}\left(d = 0.03, \sigma_d^2 = 0.01\right)$$

$$z_1 = 0.017, \qquad \sigma_z^2 = 0.01$$

$$x = 5.8, \qquad \sigma_x^2 = 0.029$$

Add the third landmark

- Prior

$$\boldsymbol{x} = \begin{bmatrix} 5.8 \\ 2.948 \\ 4.935 \\ 5.829 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.029 & 0.0145 & 0.024 & 0 \\ 0.0145 & 0.0245 & 0.0145 & 0 \\ 0.024 & 0.0145 & 0.034 & 0 \\ 0 & 0 & 0 & \sigma_m^2 \to \infty \end{bmatrix}$$

Prior and posterior pattern

$$\boldsymbol{\Sigma} = \begin{bmatrix} a & x & w & 0 \\ x & b & x & 0 \\ w & x & c & 0 \\ 0 & 0 & 0 & d \to \infty \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} a & x & w & a \\ x & b & x & x \\ w & x & c & w \\ a & x & w & a + e \end{bmatrix}$$

$a = \sigma_x^2$

$b = \sigma_{m1}^2$

$c = \sigma_{m2}^2$

$e = \sigma_z^2$

x, w Pre-existing crossterms

- Exercise: Prove this.

Prior and posterior pattern

$$\mathbf{\Sigma} = \begin{bmatrix} \boxed{\begin{matrix} a & x & w \\ x & b & x \\ w & x & c \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & d \to \infty \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{\Sigma} = \begin{bmatrix} \boxed{\begin{matrix} a & x & w \\ x & b & x \\ w & x & c \end{matrix}} & \begin{matrix} a \\ x \\ w \end{matrix} \\ \begin{matrix} a & x & w \end{matrix} & a + e \end{bmatrix}$$

$a = \sigma_x^2$

$b = \sigma_{m1}^2$

$c = \sigma_{m2}^2$

$e = \sigma_z^2$

x, w Pre-existing crossterms

- Exercise: Prove this.

Prior-posterior pattern

$$\mathbf{\Sigma} = \begin{bmatrix} a & x & w & 0 \\ x & b & x & 0 \\ w & x & c & 0 \\ 0 & 0 & 0 & d \to \infty \end{bmatrix}$$

$$\mathbf{\Sigma} = \begin{bmatrix} a & x & w & a \\ x & b & x & x \\ w & x & c & w \\ a & x & w & a + e \end{bmatrix}$$

$a = \sigma_x^2$

$b = \sigma_{m1}^2$

$c = \sigma_{m2}^2$

$e = \sigma_z^2$

x, w Pre-existing crossterms

- Exercise: Prove this.

Prior and posterior pattern

$$\boldsymbol{\Sigma} = \begin{bmatrix} a & x & w & 0 \\ x & b & x & 0 \\ w & x & c & 0 \\ 0 & 0 & 0 & d \to \infty \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} a & x & w & a \\ x & b & x & x \\ w & x & c & w \\ a & x & w & a + e \end{bmatrix}$$

$a = \sigma_x^2$

$b = \sigma_{m1}^2$

$c = \sigma_{m2}^2$

$e = \sigma_z^2$

x, w Pre-existing crossterms

- Exercise: Prove this.

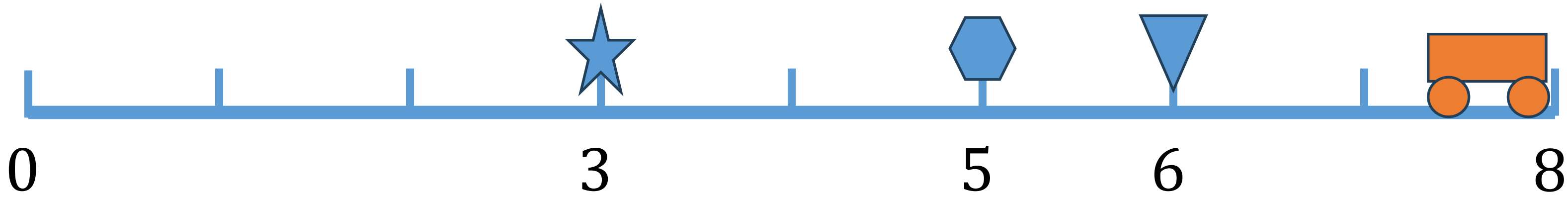
Add the third landmark

- Prior

$$\boldsymbol{x} = \begin{bmatrix} 5.8 \\ 2.948 \\ 4.935 \\ 5.829 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.029 & 0.0145 & 0.024 & 0.029 \\ 0.0145 & 0.0245 & 0.0145 & 0.0145 \\ 0.024 & 0.0145 & 0.034 & 0.024 \\ 0.029 & 0.0145 & 0.024 & 0.039 \end{bmatrix}$$

Move to the end of the road …

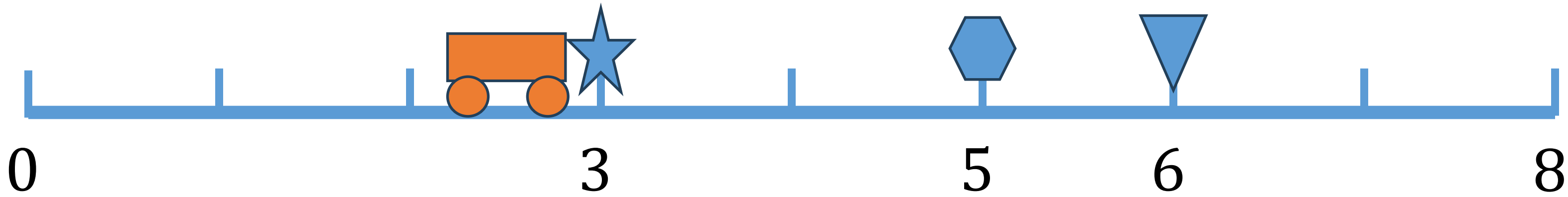


$$x_{\mathrm{GT}} = 8.03, N = 20$$

$$x = 7.8, \qquad \sigma_x^2 = 0.039$$

- The loop wraps around

… and wrap around



$$x_{\mathrm{GT}} = 2.94, N = 29$$

$$d_{\mathrm{GT}} = m_{1\mathrm{GT}} - x_{\mathrm{GT}} = 3 - 2.94 = 0.06$$

$$z_1 = d : \mathcal{N}\left(d = 0.06, \sigma_d^2 = 0.01\right)$$

$$z_1 = 0.097, \qquad \sigma_z^2 = 0.01$$

$$x = 2.7, \qquad \sigma_x^2 = 0.053$$

Advance the filter state

- Prior

$$\boldsymbol{x} = \begin{bmatrix} 2.7 \\ 2.948 \\ 4.935 \\ 5.829 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.053 & 0.0145 & 0.024 & 0.029 \\ 0.0145 & 0.0245 & 0.0145 & 0.0145 \\ 0.024 & 0.0145 & 0.034 & 0.024 \\ 0.029 & 0.0145 & 0.024 & 0.039 \end{bmatrix}$$

Advance the filter state

- Prior

$$\boldsymbol{x} = \begin{bmatrix} 2.7 \\ 2.948 \\ 4.935 \\ 5.829 \end{bmatrix}$$

Predicted position

New uncertainty

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.053 & 0.0145 & 0.024 & 0.029 \\ 0.0145 & 0.0245 & 0.0145 & 0.0145 \\ 0.024 & 0.0145 & 0.034 & 0.024 \\ 0.029 & 0.0145 & 0.024 & 0.039 \end{bmatrix}$$

Fuse measurement

$$\boldsymbol{H} = \boldsymbol{J}_z\boldsymbol{F} = [-1 \quad 1 \quad 0 \quad 0]$$

$$\boldsymbol{\Sigma H}^T = \begin{bmatrix} -0.0385 \\ 0.01 \\ -0.0095 \\ -0.0145 \end{bmatrix}$$

$$\mathbf{H}\boldsymbol{\Sigma H}^T = 0.0485$$

$$\boldsymbol{K} = \boldsymbol{\Sigma H}^T(\boldsymbol{H\Sigma H}^T + \sigma_z^2)^{-1} = \begin{bmatrix} -0.658 \\ 0.171 \\ -0.162 \\ -0.248 \end{bmatrix}$$

Fuse measurement – mean update

$$\boldsymbol{x}[n] = \boldsymbol{x}[n-1] + \boldsymbol{K}(z_1 - h(\boldsymbol{x}[n-1]))$$

$$\boldsymbol{x}[n] = \begin{bmatrix} 2.7 \\ 2.948 \\ 4.935 \\ 5.829 \end{bmatrix} + \begin{bmatrix} -0.658 \\ 0.171 \\ -0.162 \\ -0.248 \end{bmatrix} (0.097 - 0.248)$$

Ground truth difference plus noise

Beliefs difference

$$\boldsymbol{x}[n] = \begin{bmatrix} 2.799 \\ 2.922 \\ 4.959 \\ 5.866 \end{bmatrix}$$

- Did the estimate improve?

Fuse measurement – covariance

$$\boldsymbol{\Sigma}[\mathrm{n}] = (\boldsymbol{I} - \boldsymbol{KH})\boldsymbol{\Sigma}[\mathrm{n}-1]$$

$$\boldsymbol{I} - \boldsymbol{KH} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -0.969 \\ 0.015 \\ -0.014 \\ -0.022 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{I} - \boldsymbol{KH} = \begin{bmatrix} 0.342 & 0.658 & 0 & 0 \\ 0.171 & 0.829 & 0 & 0 \\ -0.162 & 0.162 & 1 & 0 \\ -0.248 & 0.248 & 0 & 1 \end{bmatrix}$$

Fuse measurement – covariance

$$\boldsymbol{\Sigma}[\mathrm{n}] = \begin{bmatrix} 0.342 & 0.658 & 0 & 0 \\ 0.171 & 0.829 & 0 & 0 \\ -0.162 & 0.162 & 1 & 0 \\ -0.248 & 0.248 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.053 & 0.0145 & 0.024 & 0.029 \\ 0.0145 & 0.0245 & 0.0145 & 0.0145 \\ 0.024 & 0.0145 & 0.034 & 0.024 \\ 0.029 & 0.0145 & 0.024 & 0.039 \end{bmatrix}$$

$$\boldsymbol{\Sigma}[\mathrm{n}] = \begin{bmatrix} 0.028 & 0.021 & 0.018 & 0.019 \\ 0.021 & 0.023 & 0.016 & 0.017 \\ 0.018 & 0.016 & 0.032 & 0.021 \\ 0.019 & 0.017 & 0.021 & 0.035 \end{bmatrix}$$

- Did the confidence improve?
- For which landmarks?

Covariance before closing the loop

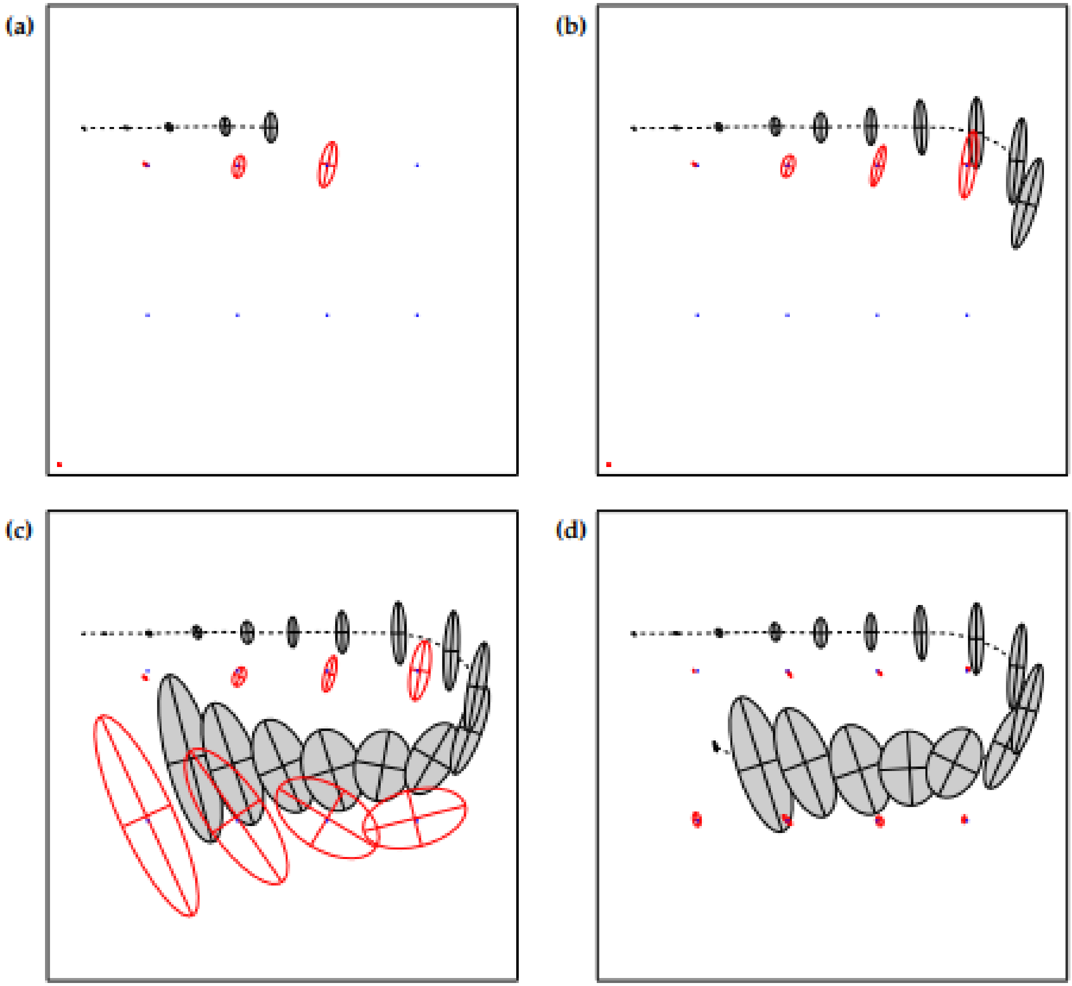
$$\mathbf{\Sigma}[\mathrm{n}] = \begin{bmatrix} 0.053 & 0.0145 & 0.024 & 0.029 \\ 0.0145 & 0.0245 & 0.0145 & 0.0145 \\ 0.024 & 0.0145 & 0.034 & 0.024 \\ 0.029 & 0.0145 & 0.024 & 0.039 \end{bmatrix}$$

- Did the confidence improve?
- For which landmarks?

Key takeaways

- Initially, landmark confidence just reflects the pose confidence
- Initially, pose is based on odometry only
- When the loop closes, estimate adjusts for *all* landmarks and pose
- When the loop closes, confidence improves for *all* landmarks and pose.

Key takeaways



Source: `www.probabilistic-robotics.org`