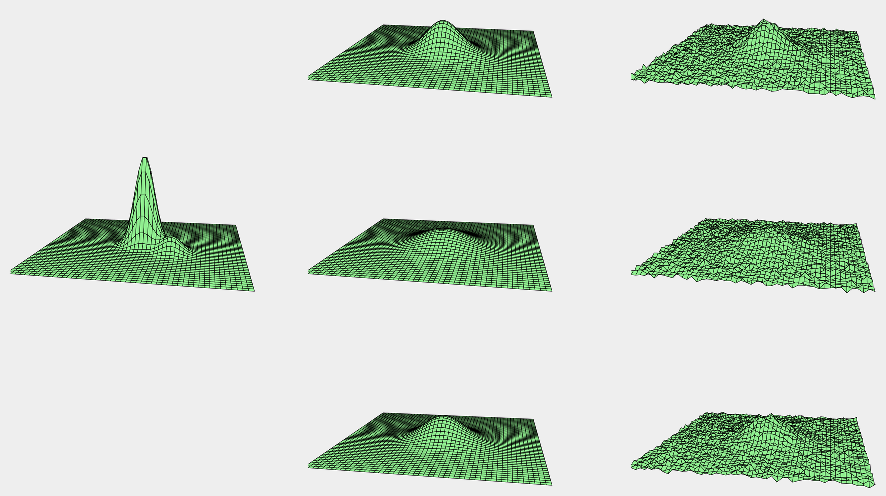
Efficient Estimators for Sequential and Resolution-Limited Inverse Problems

Darren Homrighausen   Christopher R. Genovese
Department of Statistics, Carnegie Mellon University

Sequential Inverse Problem: Notation

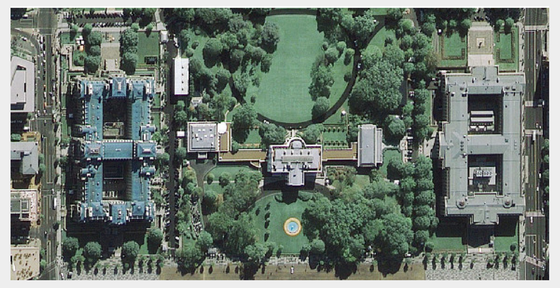


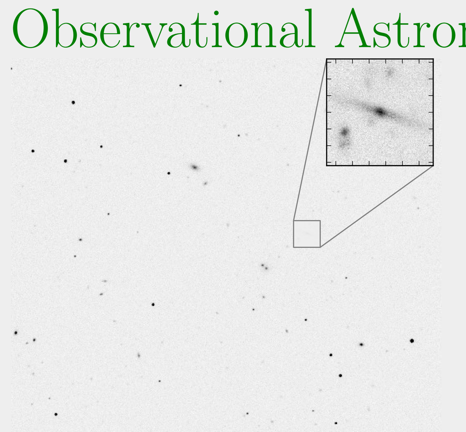
θ $K_i\theta$ $Y_i = K_i\theta + \epsilon Z$

Two Data Analysis Examples:

Satellite Imaging   Observational Astronomy

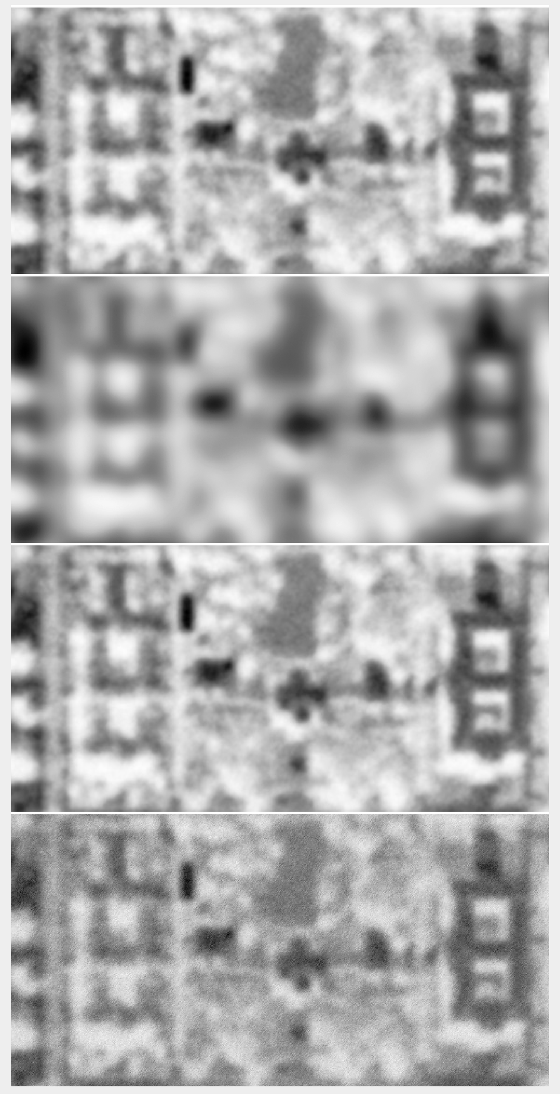
θ

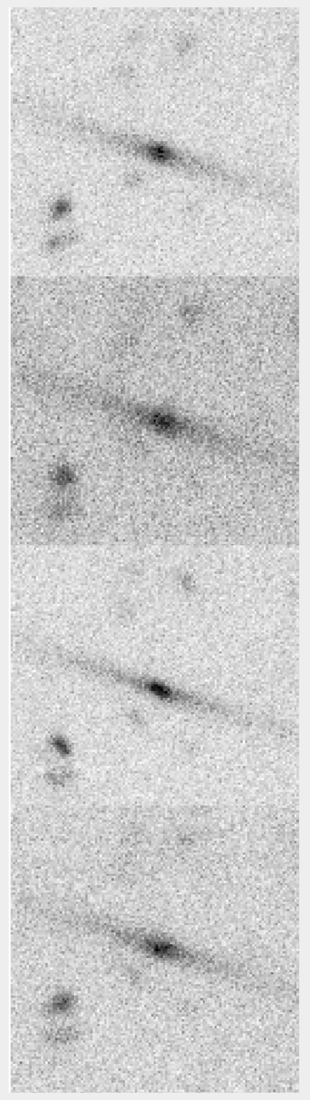




(The data are sequential and low quality).

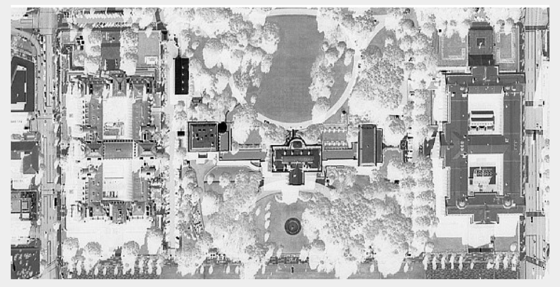
Y_1, Y_2, Y_3, Y_2

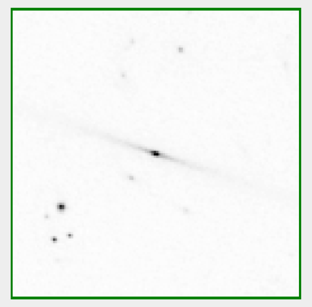




Reconstruction with Our Estimator:

$\hat{\theta}_n$





Scientific Goals

Formulate an estimator that:

- Has good theoretical properties.
- Requires no user defined tuning parameters.
- Can be efficiently updated after recording a new observation.

Statistical Model:

For $i = 1, \ldots, n, \ldots$

$$Y_i = K_i\theta + \epsilon Z_i$$

Assumptions:

1. $\theta \in \Theta = \{\theta : ||\theta||_2^2 \leq T^2, \theta \in \mathbb{R}^p\}$.
2. ϵ is known

Suppose $(K_i)_{i=1}^n$ are

3a. Known

3b. Simultaneously diagonalizable

($\exists$ unitary $\Psi \in \mathbb{C}^{p \times p}$ s.t. $K_i = \Psi D_i \Psi^*$ for some diagonal D_i).

(This includes circulant matrices and hence convolutions.)

Manipulations:

Begin by rotating by eigenvectors Ψ:

- $\beta := \Psi^*\theta \in \mathbb{C}^p$.
- $\mathcal{B} := \Psi^*\Theta$
- $X_i := \Psi^* Y_i$

Define

- $\Delta_n := \sum_{i=1}^n D_i^* D_i = \sum_{i=1}^n |D_i|^2$
- $B_n = \Delta_n^{-1} \sum_{i=1}^n D_i^* X_i$

Our Estimator:

Chosen estimator is:

$$\hat{\beta}_{nj} = \left(1 - \frac{\Omega_n^2 \epsilon^2}{\Delta_{nj}|B_{nj}|^2}\right)_+ \qquad (1)$$

for $\Omega_n^2 := (p-2)\left(1 + \frac{\max_j \Delta_{nj}}{\min_j \Delta_{nj}}\right)$.

An estimator of θ is formed via $\hat{\theta}_n = \Psi\hat{\beta}_n$.

Definitions:

Let the risk be:

$$R(\hat{\theta}) = \mathbb{E}||\hat{\theta} - \theta||_2^2$$

and define

$$\mathcal{E} := \{\hat{\theta} = \Psi\lambda(B_n) : \lambda \in \mathbb{C}^p\}.$$

Uniformly Consistent:

Let $\hat{\beta}_n$ be defined as in equation (1) and $\hat{\theta}_n = \Psi\hat{\beta}_n$. Then:

$$\limsup_{n\to\infty} \sup_{\theta\in\Theta} \gamma_n R\left(\hat{\theta}_n, \theta\right) < C < \infty$$

where $\gamma_n = \max_j \Delta_{nj}$.

Satisfies an Oracle Inequality:

Let

$$R(\theta_*, \theta) = \underset{\hat{\theta}\in\mathcal{E}}{\operatorname{argmin}}\, R(\hat{\theta}, \theta)$$

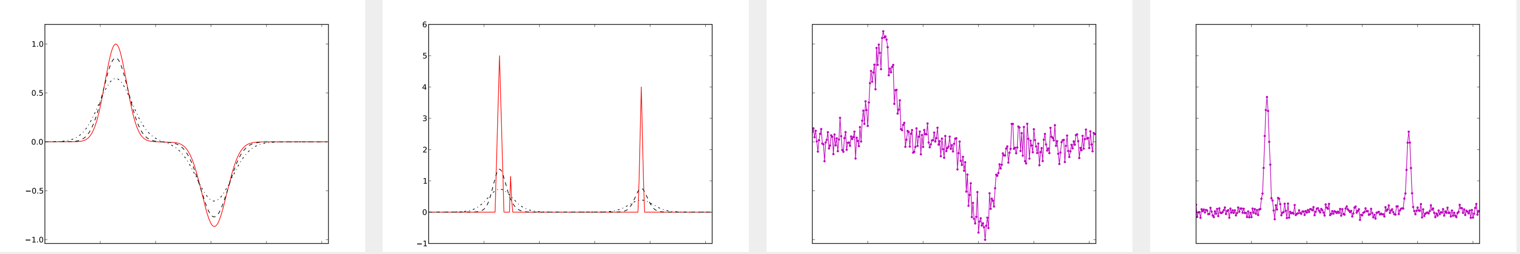
be the risk of the $\mathcal{E}$ oracle. Then

$$R\left(\hat{\theta}_n, \theta\right) \leq R(\theta_*, \theta)(1 + O(1))$$

for a term $O(1)$ that does not depend on θ.

Simulations:

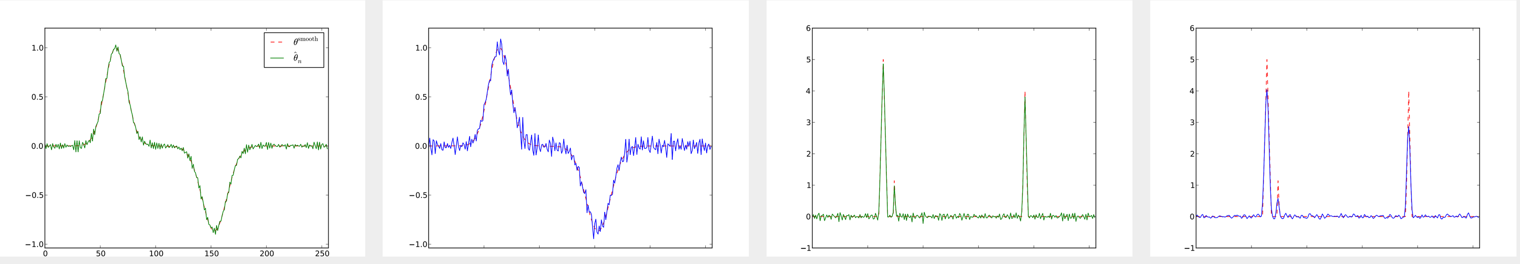
θ_1 and $K_i\theta_1$ θ_2 and $K_i\theta_2$ Examples of Observations



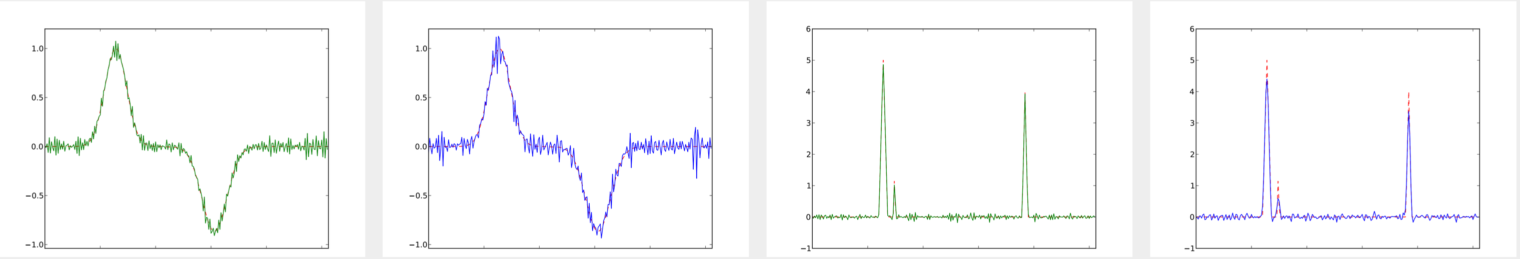
Reconstructions:

$\hat{\theta}_n$ $\hat{\theta}_{\mathrm{GCV}}$ $\hat{\theta}_n$ $\hat{\theta}_{\mathrm{GCV}}$

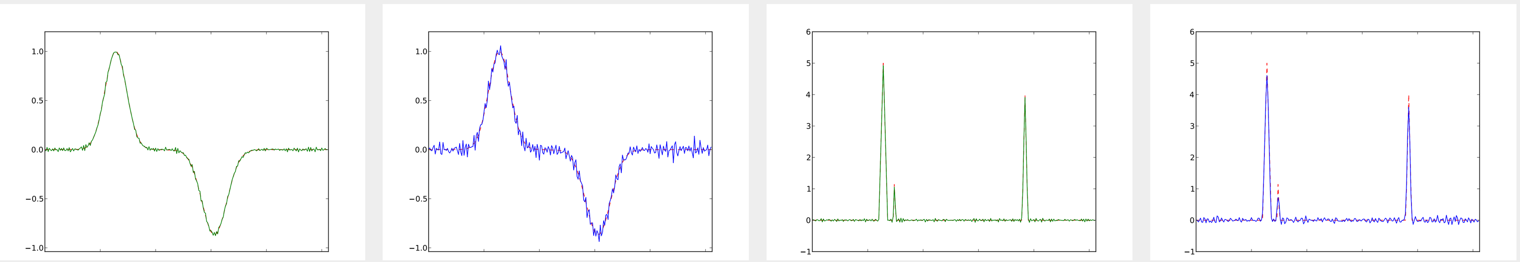
$n = 50$



$n = 100$



$n = 200$



	$RR(\hat{\theta}_n, \theta_1)$	$RR(\hat{\theta}_{\mathrm{GCV}}, \theta_1)$	$RR(\hat{\theta}_n, \theta_2)$	$RR(\hat{\theta}_{\mathrm{GCV}}, \theta_2)$
$n = 50$	0.210	0.223	0.116	0.171
$n = 100$	0.149	0.199	0.092	0.149
$n = 200$	0.120	0.173	0.079	0.141

$$RR(\hat{\theta}, \theta) = \sqrt{\frac{R(\hat{\theta}, \theta)}{||\theta||^2}}$$

http://www.stat.cmu.edu/~dhomrigh   dhomrigh@stat.cmu.edu