On the Complexity of Barrier Resilience for Fat Regions

Matias Korman[1(✉)], Maarten Löffler[2],
Rodrigo I. Silveira[1,3], and Darren Strash[4]

[1] Departmento de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain
{matias.korman,rodrigo.silveira}@upc.edu
[2] Department of Computing and Information Sciences, Utrecht University, Utrecht, The Netherlands
m.loffler@uu.nl
[3] Departmento de Matemática, Universidade de Aveiro, Aveiro, Portugal
rodrigo.silveira@ua.pt
[4] Computer Science Department, University of California, Irvine, USA
dstrash@uci.edu

Abstract. In the *barrier resilience* problem (introduced by Kumar *et al.*, Wireless Networks 2007), we are given a collection of regions of the plane, acting as obstacles, and we would like to remove the minimum number of regions so that two fixed points can be connected without crossing any region. In this paper, we show that the problem is NP-hard when the regions are arbitrarily fat regions (even when they are axis-aligned rectangles of aspect ratio $1 : (1 + \varepsilon)$). We also show that the problem is fixed-parameter tractable (FPT) for such regions. Using our FPT algorithm, we show that if the regions are β-fat and their arrangement has bounded ply Δ, there is a $(1 + \varepsilon)$-approximation that runs in $O(2^{f(\Delta,\varepsilon,\beta)} n^7)$ time, where $f \in O(\frac{\Delta^2\beta^6}{\varepsilon^4} \log(\beta\Delta/\varepsilon))$.

1 Introduction

The *barrier resilience* problem asks for the minimum number of spatial regions from a collection $\mathcal{D}$ that need to be removed, such that two given points p and q are in the same connected component of the complement of the union of the remaining regions. This problem was posed originally in 2005 by Kumar et al [11,12], motivated from sensor networks. In their formulation, the regions are unit disks (sensors) in some rectangular strip $B \subset \mathbb{R}^2$, where each sensor is able to detect movement inside its disk. The question is then how many sensors need to fail before an entity can move undetected from one side of the strip to the other (that is, how *resilient* to failure the sensor system is). Kumar *et al.* present a polynomial time algorithm to compute the resilience in this case. They also consider the case where the regions are disks in an annulus, but their approach cannot be used in that setting.

P. Flocchini et al. (Eds.): ALGOSENSORS 2013, LNCS 8243, pp. 201–216, 2014.
DOI: 10.1007/978-3-642-45346-5_15,

1.1 Related Work

Despite the seemingly small change from a rectangular strip to an annulus, the second problem still remains open, even for the case in which regions are unit disks in $\mathbb{R}^2$. There has been partial progress towards settling the question: Bereg and Kirkpatrick [2] present a factor 5/3-approximation algorithm for the unit disk case. Afterwards, Alt *et al.* [1] and Tseng and Kirkpatrick [16] independently showed that if the regions are line segments in $\mathbb{R}^2$, the problem is NP-hard. We also note that Tseng and Kirkpatrick [16] also sketched how to extend their proof for the case in which the input consists of (translated and rotated) copies of a fixed square or ellipse, but no formal proof was given.

The problem of covering barriers with sensors is very current and has received a lot of attention in the sensor network community (e.g. [3,4,8]). In the algorithms community, closely related problems involving region intersection graphs have also became quite popular. Gibson *et al.* [7] study the opposite problem of ours: compute the maximum number of disks one can remove such that p and q are still separated.

1.2 Results

We present constructive algorithms for two natural restricted variants of the problem. In Sect. 3 we show that the problem is fixed-parameter tractable on the resilience when the regions satisfy an upper bound on the *fatness* [6] (intuitively speaking, the regions must have some resemblance to a unit disk). In Sect. 4 we also show that if the collection of regions has bounded *ply* [14] (that is, sensors are more or less evenly distributed in the plane), the FPT result can be used to obtain an approximation scheme. In particular, the constructive results apply to the original unit disk coverage setting (formal definitions of fatness and ply are given in the corresponding sections).

As a complement to these algorithms, in Sect. 5 we show that the problem is NP-hard even when the input is a collection of arbitrary fat regions in $\mathbb{R}^2$. The result holds even if regions consist of axis-aligned rectangles of aspect ratio $1 : 1 + \varepsilon$ and $1 + \varepsilon : 1$.

Our results rely on tools and techniques from both computational geometry and graph theory.

Due to lack of space, several proofs and extensions have been omitted. A full version of the paper containing all the omitted details can be found in [10].

2 Preliminaries

We denote with p and q the points that need to be connected, and with $\mathcal{D}$ the set of regions that represent the sensors. To simplify the presentation of our results, we make the following general position assumption: all intersections between boundaries of regions in $\mathcal{D}$ consist of isolated points. We say that a collection of objects in the plane are *pseudodisks* if the boundaries of any two of them intersect at most twice.

We formally define the concepts of *resilience* and *thickness* introduced in [2]. The *resilience of a path* π between two points p and q, denoted $r(\pi)$, is the number of regions of $\mathcal{D}$ intersected by π. Given two points p and q, the *resilience of* p *and* q, denoted $r(p, q)$, is the minimum resilience over all paths connecting p and q. In other words, the resilience between p and q is the minimum number of regions of $\mathcal{D}$ that need to be removed to have a path between p and q that does not intersect any region of $\mathcal{D}$. From now on, we assume that neither p nor q are contained in any region of $\mathcal{D}$. Note that such regions must always be counted in the minimum resilience paths, hence we can ignore them (and update the resilience we obtain accordingly).

Often it will be useful to refer to the arrangement (i.e., the subdivision of the plane into faces, see e.g. [5] for a formal definition) induced by the regions of $\mathcal{D}$, which we denote by $\mathcal{A}(\mathcal{D})$. Based on this arrangement we define a weighted dual graph $G_{\mathcal{A}(\mathcal{D})}$ as follows. There is one vertex for each cell (i.e., face) of $\mathcal{A}(\mathcal{D})$. Each pair of neighboring cells A, B is connected in $G_{\mathcal{A}(\mathcal{D})}$ by two directed edges, (A, B) and (B, A). The weight of an edge is 1 if, when traversing from the starting cell to the destination one, we enter a region of $\mathcal{D}$ (or 0 if we leave a region[1]).

The *thickness* of a path π between p and q, denoted $t(\pi)$, equals the number of sensor region intersections of π (possibly counting the same region multiple times). Given two points p and q, the *thickness of* p *and* q, denoted $t(p, q)$, is the value $|\wp_{G_{\mathcal{A}(\mathcal{D})}}(p, q)| + \Delta(p)$, where $\wp_{G_{\mathcal{A}(\mathcal{D})}}(p, q)$ is a shortest path in $G_{\mathcal{A}(\mathcal{D})}$ from the cell of p to the cell of q, and $\Delta(p)$ equals the number of regions that contain p. Also note that the resilience (or thickness) between two points only depends on the cells to which the points belong to. Hence, we can naturally extend the definitions of thickness to encompass two cells of $\mathcal{A}(\mathcal{D})$, or a cell and a point.

Note that thickness and resilience can be different (since entering the same region several times has no impact on the resilience, but is counted every time for the thickness). In fact, the thickness between two points can be efficiently computed in polynomial time using any shortest path algorithm for weighted graphs (for example, using Dijkstra's algorithm). However, as we will see later, the thickness (and the associated shortest path) will help us find a path of low resilience.

Throughout the paper we often use the following fundamental property of disks, already observed in [2]. In the statement below, "well-separated" is in the sense used in [2]—i.e., the distance between p and q is at least $2\sqrt{3}$.[2]

Lemma 1 ([2], Lemma 1). *Let $\mathcal{D}$ be a set of unit disks, and let π^* be a path from p to q of minimum resilience. If p, q are well-separated, then π^* encounters no disk of $\mathcal{D}$ more than twice.*

Corollary 1 ([2]). *When the regions of $\mathcal{D}$ are unit disks, the thickness between two well-separated points is at most twice their resilience.*

[1] Note that no other option is possible under our general position assumption.

[2] Note that the well-separatedness of p and q is used to prove a factor 2 instead of 3. Everything still works for ill-separated points, at a slight increase of the constants. Our most general statements for β-fat regions do not make this requirement.

3 Fixed-Parameter Tractability

In this section we introduce a single-exponential fixed-parameter tractable (FPT) algorithm, where the parameter is the length of the optimal solution. Thus, our aim is to obtain an algorithm that given a problem instance, determines whether or not there is a path of resilience r between p and q, and runs in $O(2^{f(r)}n^c)$ time for some constant c and some polynomial function f.

For clarity we first explain the algorithm for the special case of unit disks. Afterwards, in Sect. 3.2, we show how to adapt the solution to the case in which $\mathcal{D}$ is a collection of β-fat objects. Note that for treating the case of unit disk regions we assume that p and q are well-separated, so we can apply Lemma 1. This requirement is afterwards removed in Sect. 3.2.

First we give a quick overview of the method of Kumar *et al.*[11] for open belt regions. Their idea consists in considering the intersection graph of $\mathcal{D}$ together with two additional artificial vertices s,t with some predefined adjacencies. There is a path from the bottom side to the top side of the belt if and only if there is no path between s and t in the graph. Hence, computing the resilience of the network is equal to finding a minimum vertex cut between s and t.

Our approach is to find a low thickness path that passes through p and q, cut open through it, and transform the problem instance into one with something similar to an open belt region. We then follow the approach of Kumar *et al.* taking into account that the right and left boundaries of our region correspond to the same point. Hence, instead of using a regular vertex cut, we will use a vertex multicut [17].

Consider the shortest path τ between the cells containing p and q in $G_{\mathcal{A}(\mathcal{D})}$, let t be the number of traversed disks (recall that we assumed that p is not contained in any region, hence this number is exactly the *thickness* of p and q). We observe that cells with high thickness to p or q can be ignored when we look for low resilience paths.

Lemma 2. *The minimum resilience path between p and q cannot traverse cells whose thickness to p or q is larger than* $1.5t$.

Proof. We argue about thickness to p; the argument with respect to q is analogous. Let ρ be a path of minimum resilience between p and q, and let r be the resilience of ρ. Recall that ρ does not enter a disk more than twice, hence the thickness of ρ is at most $2r \leq 2t$. Assume, for the sake of contradiction, that the thickness of some cell C traversed by ρ is greater than $1.5t$. Let ρ_C be the portion of ρ from C to q. By the triangle inequality, the thickness of ρ_C is less than $0.5t$. However, by concatenating τ and ρ_C we would obtain a path that connects p with C whose thickness is less than $1.5t$, giving a contradiction. ⊠

Let R be the union of the cells of the arrangement that have thickness from p at most $1.5t$; we call R the *domain* of the problem. Observe that R is connected, but need not be simple (see Fig. 1(a)). By the previous result, cells that do not belong to R can be discarded, since they will never belong to a path of resilience r. Note that the number of cells remaining in R might still be quadratic, hence

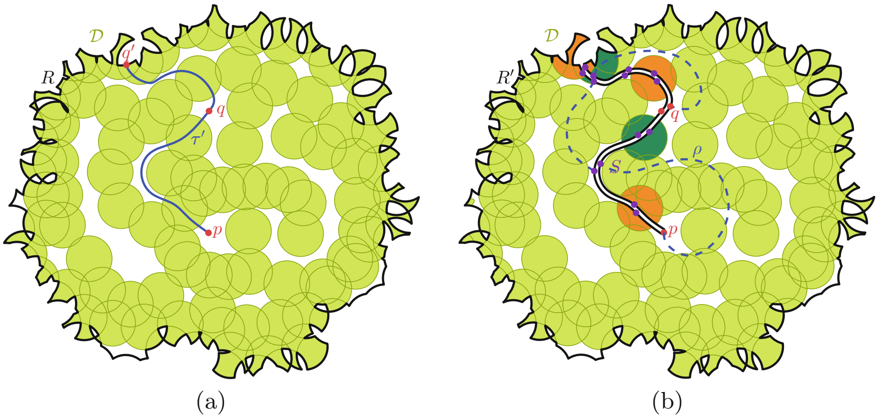


Fig. 1. (a) The reduced domain R, and a path τ' from p to q' via q. (b) After cutting along τ', we get the domain R'. We add a set S of extra vertices on the boundary of R', and we now have two copies of q. A crossing pattern, consisting of a topological path ρ (defined by the sequence of points of S it passes) and a binary assignment to the disks of $\mathcal{D}$ intersected by τ', is also shown.

asymptotically speaking the instance size has not decreased (the purpose of this pruning will become clear later). We extend the minimum resilience path τ from q until a point q' on the boundary of the domain. Let τ' denote the extended path (Fig. 1(a)).

Lemma 3. *There exists a path τ' of minimum resilience from p to a point q' on the boundary of R via q, whose thickness is at most* $1.5t$.

Proof. Consider any shortest path tree from p in the dual graph of the reduced domain R, defined as the corresponding subgraph of $G_{\mathcal{A}(\mathcal{D})}$. All leaves of the tree correspond to cells on the boundary of R, which by definition have thickness at most $1.5t$ from p. Therefore all other cells in the tree lie on a path from p to a boundary cell that has length exactly $1.5t$, including q. ⊠

We "cut open" through τ', removing the cut region from our domain. Note that cells that are traversed by τ' are split by two copies of the same Jordan curve (Fig. 1(b)). After this cut we have two paths from p to q. We arbitrarily call them the left and right paths. Consider now a minimum resilience path denoted ρ; let $r = r(\rho)$ denote its resilience. This path can cross τ' several times, and it can even coincide with τ' in some parts (shared subpaths). Although we do not know how and where these crossings occur, we can *guess* (i.e., try all possibilities) the topology of ρ with respect to τ'. For each disk that τ' passes through, we either remove it (at a cost of 1) or we make it an obstacle. That way we explicitly know which of the regions traversed by τ' could be traversed by ρ. Additionally, we guess how many times ρ and τ' share part of their paths (either for a single crossing in one cell, or for a longer shared subpath). For each shared subpath, we guess from which cell ρ arrives and leaves (and if the entry or exit was from the left or right path). We call each such configuration a *crossing pattern* between τ' and ρ. Figure 1(b) illustrates a crossing pattern.

Lemma 4. *For any problem instance $\mathcal{D}$, there are at most $2^{4r \log r + o(r \log r)}$ crossing patterns between τ' and ρ, where $r = r(\rho)$.*

Proof. First, for all disks in τ', we guess whether or not they are also traversed by ρ. By Lemma 3, τ' has thickness at most $1.5t$, there are at most such many disks (hence up to $2^{1.5t}$ choices for which disks are traversed by ρ).

We now bound the number of (maximal) shared subpaths between ρ and τ': recall that ρ passes through exactly $r = r(\rho)$ disks, and visits each disk at most twice. Hence, there cannot be more than $2r$ shared subpaths. Observe that τ' cannot traverse many cells of $\mathcal{A}(\mathcal{D})$: when moving from a cell to an adjacent one, we either enter or leave a disk of $\mathcal{D}$. Since we cannot leave a disk we have not entered and τ' has thickness at most $1.5t$, we conclude that at most $3t$ cells will be traversed by τ' (other than the starting and ending cells).

For each shared subpath we must pick two of the cells traversed in τ' (as candidates for first and last cell in the subpath). By the previous observation there are at most $3t$ candidates for first and last cell (since that is the number of cells traversed by τ'). Additionally, for each shared subpath we must determine from which side ρ entered and left the subpath (four options in total). Since these choices are independent, in total we have at most $2r \times (3t \times 3t \times 4)^{2r} = 2r \cdot 36^{2r} \cdot t^{4r}$ options for the number of crossing patterns. Combining both bounds and using the fact that $t \leq 2r$, we obtain:

$$\begin{aligned} 2^{1.5t} \cdot 2r \cdot 36^{2r} \cdot t^{4r} &\leq 2^{5r} \cdot 2r \cdot 36^{2r} \cdot (2r)^{4r} \\ &= 2^{9r+1+\log r+2r\log 36+4r\log r} \\ &= 2^{4r\log r+o(r\log r)} \end{aligned}$$

⊠

Note that the bound is very loose, since most of the choices will lead to an invalid crossing pattern. However, the importance of the lemma is in the fact that the total number of crossing patterns only depends on r.

Our FPT algorithm consists in considering all possible crossing patterns, finding the optimal solution for a fixed crossing pattern, and returning the solution of smallest resilience. From now on, we assume that a given pattern has been fixed, and we want to obtain the path of smallest resilience that satisfies the given pattern. If no path exists, we simply discard it and associate infinite resilience to it.

3.1 Solving the Problem for a Fixed Crossing Pattern

Recall that the crossing pattern gives us information on how to deal with the disks traversed by τ'. Thus, we remove all cells of the arrangement that contain

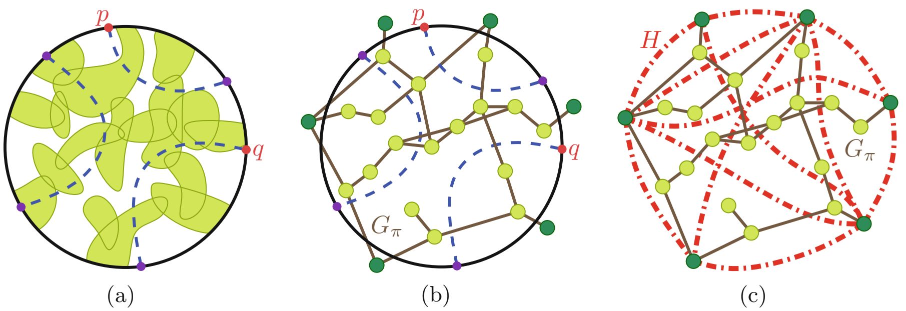


Fig. 2. (a) We may schematically represent W as a circle, since the geometry no longer plays a role. Partial paths are shown dashed. (b) The intersection graph of the regions after adding extra vertices for boundary pieces between points of $S \cup \{p, q\}$, shown green (Colour figure online). (c) The secondary graph H, representing the forbidden pairs.

one or more disks that are forbidden to ρ. Similarly, we remove from $\mathcal{D}$ the disks that ρ must cross. After this removal, several cells of our domain may be merged.

Since we do not use the geometry, we may represent our domain by a disk W (possibly with holes). After the transformation, each remaining region of $\mathcal{D}$ becomes a pseudodisk, and ρ becomes a collection of disjoint partial paths, each of which has its endpoints on the boundary of W (see Fig. 2(a)). To solve the subproblem associated with the crossing pattern we must remove the minimum number of disks so that all partial paths are feasible.

We consider the intersection graph G_I between the remaining regions of $\mathcal{D}$. That is, each vertex represents a region of $\mathcal{D}$, and two vertices are adjacent if and only if their corresponding regions intersect. Similarly to [11], we must augment the graph with boundary vertices. The partial paths split the boundary of R into several components. We add a vertex for each component (these vertices are called *boundary vertices*). We connect each such vertex to vertices corresponding to pseudodisks that are adjacent to that piece of boundary (Fig. 2(b)). Let $G_\mathcal{X} = (V_\mathcal{X}, E_\mathcal{X})$ be the resulting graph associated to crossing pattern $\mathcal{X}$. Note that no two boundary vertices are adjacent.

We now create a secondary graph H as follows: the vertices of H are the boundary vertices of $G_\mathcal{X}$. We add an edge between two vertices if there is a partial path that separates the vertices in $G_\mathcal{X}$ (Fig. 2(c)). Two vertices connected by an edge of H are said to form a *forbidden pair* (each partial path that would create the edge is called a *witness* partial path). We first give a bound on the number of forbidden pairs that H can have.

Lemma 5. *Any crossing pattern has at most $2r^2 + r$ forbidden pairs.*

Proof. Observe that $G_\mathcal{X}$ only adds edges between boundary vertices. Thus, it suffices to show that $G_\mathcal{X}$ has at most $2r + 1$ boundary vertices. Since partial paths cannot cross, each such path creates a single cut of the domain. This cut introduces a single additional boundary vertex (except the first partial path that introduces two vertices). Recall that we can map the partial paths to crossings between paths τ' and ρ and, by Lemma 4, these paths can cross at most $2r$ times. Thus, we conclude that there cannot be more than $2r + 1$ boundary vertices. ⊠

The following lemma shows the relationship between the vertex multicut problem and the minimum resilience path for a fixed pattern.

Lemma 6. *There are k vertices of $G_{\mathcal{X}}$ whose removal disconnects all forbidden pairs if and only if there are k disks in $\mathcal{D}$ whose removal creates a path between p and q that obeys the crossing pattern $\mathcal{X}$.*

Proof. Let $\mathcal{A}'$ be the regions of $\mathcal{A}(\mathcal{D})$ inside R that are not covered by any disk after the k disks have been removed and let R' be their union. By definition, there is a path between p and q with the fixed crossing pattern if all partial paths are feasible (i.e., there exists a path connecting the two endpoints that is totally within R'). The reasoning for each partial path is analogous to the one used by Kumar *et al.* [11]. If all partial paths are possible, then no forbidden pair can remain connected in $G_{\mathcal{X}}$, since—by definition—each forbidden pair disconnects at least one partial path (the witness path). On the other hand, as soon as one forbidden pair remains connected, there must exist at least one partial path (the witness path) that crosses the forbidden pair. Thus if a forbidden path is not disconnected, there can be no path connecting p and q for that crossing pattern. ⊠

That is, thanks to Lemma 6, we can transform the barrier resilience problem to the following one: given two graphs $G = (V, E)$, and $H = (V, E')$ on the same vertex set, find a set $D \subset V$ of minimum size so that no pair $(u, v) \in E'$ is connected in $G \setminus D$. This problem is known as the (vertex) *multicut* problem [17]. Although the problem is known to be NP-hard if $|E'| > 2$ [9], there exist several FPT algorithms on the size of the cut and on the size of the set E' [13,17]. Among others, we distinguish the method of Xiao ([17], Theorem 5) that solves the vertex multicut problem in roughly $O((2k)^{k+\ell/2}n^3)$ time, where k is the number of vertices to delete, $\ell = |E'|$, and n is the number of vertices of G.

Theorem 1. *Let $\mathcal{D}$ be a collection of unit disks in $\mathbb{R}^2$, and let p and q be two well-separated points. There exists an algorithm to test whether $r(p, q) \leq r$, for any value r, and if so, to compute a path with that resilience, in $O(2^{f(r)}n^3)$ time, where $f(r) = r^2 \log r + o(r^2 \log r)$.*

Proof. Recall that our algorithm considers all possible crossings between ρ and τ'. For any fixed crossing pattern $\mathcal{X}$, our algorithm computes $G_{\mathcal{X}}$, and all associated forbidden pairs. We then execute Xiao's FPT algorithm [17] for solving the vertex multicut problem. By Lemma 6, the number of removed vertices (plus the number of disks that were forced to be deleted by $\mathcal{X}$) will give the minimum resilience associated with $\mathcal{X}$.

Regarding the running time, the most expensive part of the algorithm is running an instance of the vertex multicut problem for each possible crossing pattern. Observe that the parameters k and ℓ of the vertex multicut problem are bounded by functions of r as follows: $k \leq r$ and $\ell \leq 2r^2 + r$ (the first claim is direct from the definition of resilience, and the second one follows from Lemma 5). Hence, a single instance of the vertex multicut problem will need $O((2r)^{r+(2r^2+r)/2}n^3) = O(2^{(1+\log r)(r^2+1.5r)}n^3) = O(2^{r^2 \log r + o(r^2 \log r)}n^3)$ time. By Lemma 4 the number of crossing patterns is bounded by $2^{4r \log r + o(r \log r)}$. Thus, by multiplying both expressions we obtain the bound on the running time, and the theorem is shown. ⊠

We remark that the importance of this result lies in the fact that an FPT algorithm exists. Hence, although the dependency on r is high, we emphasize that the bounds are rather loose. We also note that both the minimum resilience path and the disks to be deleted can be reported.

3.2 Extension to Fat Regions

We now generalize the algorithm to similarly-sized β-fat regions. A region D is β-fat if there exist two concentric disks C and C' whose radii differ by at most a factor β, such that $C \subseteq D \subseteq C'$ (whenever the constant β is not important, the region D is simply called *fat*). Since we need the regions to be of similar size, we assume without loss of generality that the radius of C is 1 and the radius of C' is β; in this case we will call D a *β -fat unit region*. For the purpose, we must extend Lemmas 1, 2, 3, 4 and 5 to consider β-fat unit regions. The dependency of β in most of the Lemmas is quadratic (see details in the full version [10]), but the rest of the algorithm remains unchanged: the only property of unit disks that is still used is the fact that they are connected, to be able to phrase the problem as a vertex cut in the region intersection graph.

Theorem 2. *Let $\mathcal{D}$ be a collection of n connected β-fat unit regions in $\mathbb{R}^2$, and let p and q be two points. Let r be a parameter. There exists an algorithm to test whether $r(p,q) \leq r$, and if so, to compute a path with that resilience, in $O(2^{f(\beta,r)}n^3)$ time, where $f(\beta,r) \in O(\beta^4 r^2 \log(\beta r))$.*

4 $(1+\varepsilon)$-Approximation

The arrangement formed by a collection of regions $\mathcal{D}$ is said to have bounded ply Δ if no point $p \in \mathbb{R}^2$ is contained in more than Δ elements of $\mathcal{D}$. In this section we present an efficient polynomial-time approximation scheme (EPTAS) for computing the resilience of an arrangement of disks of bounded ply Δ.

The general idea of the algorithm is very simple: first, we compute all pairs of regions that can be reached by removing at most k disks, for $k = \lceil 4\Delta/\varepsilon^2 \rceil$. Then, we compute a shortest path in the dual graph of the arrangement of regions, augmented with extra edges. We prove that the length of the resulting path is a $(1+\varepsilon)$-approximation of the resilience.

As in the previous section, we first consider the case in which $\mathcal{D}$ is a set of n unit disks in $\mathbb{R}^2$ of ply Δ. Let $\mathcal{A}(\mathcal{D})$ be the arrangement induced by the regions of $\mathcal{D}$, and let $G_{\mathcal{A}(\mathcal{D})}$ be the dual graph of $\mathcal{A}(\mathcal{D})$. Recall that $G_{\mathcal{A}(\mathcal{D})}$ has a vertex for every cell of $\mathcal{A}(\mathcal{D})$, and a directed edge between all pairs of adjacent cells of cost 1 when entering a disk, and cost 0 when leaving a disk. For any given k, let G_k be the graph obtained from $G_{\mathcal{A}(\mathcal{D})}$ by adding, for each pair of cells $A, B \in \mathcal{A}(\mathcal{D})$ with resilience at most k, a *shortcut edge* $\overrightarrow{AB}$ of cost $r(A, B)$.

For a pair of cells of $\mathcal{A}(\mathcal{D})$, we can test whether $r(A, B)$ is smaller than k, and if it is, compute it, in $O(2^{f(k)}n^3)$ time (where $f(k) = r^2 \log r + o(r^2 \log r)$) by applying Theorem 1 to a point $p \in A$ and a point $q \in B$. Since there are $O(n^2)$ cells in $\mathcal{A}(\mathcal{D})$, we can compute G_k by doing this $O(n^4)$ times, leading to a total running time of $O(2^{f(k)}n^7)$. Observe that this running time is polynomial in n, and exponential in k. In particular, it is an EPTAS since $k = 4\Delta/\varepsilon^2$. Again, we emphasize that the bounds are loose, and that our objective is to show the existence of an EPTAS to the resilience problem.

4.1 Analysis

Lemma 7. *Let $D \in \mathcal{D}$, where $\mathcal{A}(\mathcal{D})$ has ply Δ, and let s,t be any two points inside D. Then the resilience between s and t in $\mathcal{D}$ is at most Δ.*

Proof. Let c be the number of disks containing either s or t ($c \geq 1$, since D contains both points). These c disks clearly must be removed. Now we analyze what other disks, not containing neither s nor t, may need to be removed too. For each other disk D_1 (not containing both s and s) that needs to be removed in an optimal solution, there must be another disk D_2 that intersects D_1 and, together, separate s and t inside D. We call such a pair of disks a *separating pair*.

Thus if the resilience is $(c + c')$, there must be at least c' *disjoint* separating pairs intersecting D. Moreover, since disks have unit-size, if two disks form a separating pair, at least one of them must intersect the center of D. Figure 3 illustrates the argument.

Since the ply of $\mathcal{A}(\mathcal{D})$ is Δ, this implies that there can be at most $\Delta - c$ separating pairs, and thus the resilience is at most Δ. ⊠

The previous lemma implies that in an optimal resilience path, if a disk appears twice, its two occurrences cannot be more than 2Δ apart (when counting the cells traversed by the path between the two occurrences of the disk).

To prove our result it will be convenient to focus on the sequence of disks encountered by a path when going from p to q. It turns out that such problem is essentially a string problem, where each symbol represents a disk encountered by the path. In that context, the thickness will be equivalent to the number of symbols of the string (recall that we assume that p is not contained in any disk), and the resilience to the number of distinct symbols.

Let $S = \langle s_1 \dots s_n \rangle$ be a string of n symbols from some alphabet $\mathfrak{A}$, such that no symbol appears more than twice. Let T be a substring of S. We define $\ell(T)$ to be the length of T, and $d(T)$ to be the number of distinct symbols in T. Clearly,

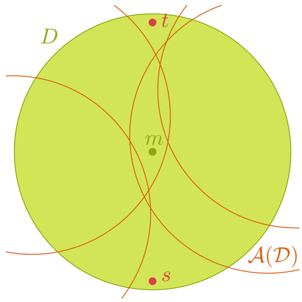


Fig. 3. The point m is contained in at least $c' + 1$ disks of $\mathcal{A}(\mathcal{D})$. Therefore, one of the neighboring cells has ply at least $c' + c + 1$

$\frac{1}{2}\ell(T) \leq d(T) \leq \ell(T)$. Let σ and k be two fixed integers such that $\sigma < k$. We define the *cost* of a substring T of S to be:

$$\psi(T) = \begin{cases} \sigma & \text{if } T = \langle \mathfrak{a}\lambda\mathfrak{a} \rangle \text{ for some } \mathfrak{a} \in \mathfrak{A}, \text{ string } \lambda \text{ s. t.} \mathfrak{a} \notin \lambda, \text{ and } \ell(T) > \sigma \\ d(T) & \text{if } \ell(T) \leq k \\ \ell(T) & \text{otherwise} \end{cases}$$

Note that, in the string context, d acts as the resilience, ℓ as the thickness, and ψ is the approximation we compute. Intuitively, if T is short (i.e., length at most k) we can compute the exact value $d(T)$. If T has a symbol whose two appearances are far away we will use a "shortcut" and pay σ (i.e., for unit disk regions, by Lemma 7, we will take $\sigma = \Delta$). Otherwise, we will approximate d by ℓ. Given a long string, we wish to subdivide S into a *segmentation* $\mathcal{T}$, composed of m disjoint segments (i.e. substrings of S) $T_1, \ldots, T_m$, that minimize the total cost $\psi(\mathcal{T}) = \sum_i \psi(T_i)$. Clearly, $\psi(\mathcal{T}) \leq \ell(S)$.

Lemma 8. *Let S be a sequence. There exists a segmentation $\mathcal{T}$ such that $\psi(\mathcal{T}) \leq (1 + \varepsilon)d(S)$, where $\varepsilon = 2\sqrt{\sigma/k}$.*

Proof. Let λ be an integer such that $\sigma < \lambda < k$, of exact value to be specified later. First, we consider all pairs of equal symbols in S that are more than λ apart. We would like to take all of these pairs as separate segments; however, we cannot take segments that are not disjoint. So, we greedily take the leftmost symbol $\mathfrak{s}$ whose partner is more than λ further to the right, and mark this as a segment. We recurse on the substring remaining to the right of the rightmost $\mathfrak{s}$.[3] Finally, we segment the remaining pieces greedily into pieces of length k. Figure 4 illustrates the resulting segmentation.

Now, we prove that the resulting segmentation has a cost of at most $(1 + \varepsilon)d(S)$. First, consider a symbol to be *counted* if it appears in only one short

[3] In fact, we could choose any disjoint collection such that after their removal there are no more segments of this type longer than λ.

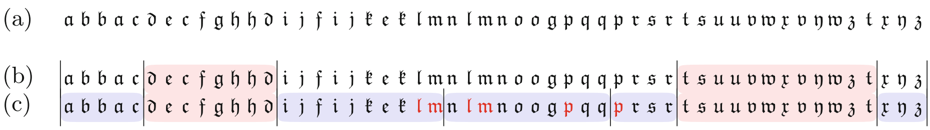


Fig. 4. (a) A string of 52 symbols, each appearing twice. (b) First, we identify a maximal set of segments bounded by equal symbols, and longer than $\lambda = 4$. (c) Then, we segment the remaining pieces into segments of length $k = 10$. Red symbols are double-counted (Colour figure online).

(blue) segment, and to be *double-counted* if it appears in two different short segments. Suppose $\mathfrak{s}$ is double-counted. Then the distance between its two occurrences must be smaller than λ, otherwise it would have formed a long (red) segment. Therefore, it must appear in two adjacent short segments. The leftmost of these two segments has length exactly k, but only λ of these can have a partner in the next segment. So, at most a fraction λ/k symbols are double-counted.

Second, we need to analyze the cost of the long (red) segments. In the worst case, all symbols in the segment also appear in another place, where they were already counted. In this case, the true cost would be 0, and we pay σ too much. However, we can assign this cost to the at least λ symbols in the segment; since each symbol appears only twice they can be charged at most once. So, we charge at most σ/λ to each symbol. The total cost is then bounded by $(1 + \lambda/k + \sigma/\lambda)d(S)$. To optimize the approximation factor, we choose λ such that $\lambda/k = \sigma/\lambda$; more precisely we take $\lambda = \lceil\sqrt{k\sigma}\rceil$. Recall that we initially set $k = \lceil 4\sigma/\varepsilon^2 \rceil$. ⊠

4.2 Application to Resilience Approximation

We now show that the shortest path between any p, q in G_k is a $(1 + \varepsilon)$-approximation of their resilience. Let π be a path from p to q in $\mathbb{R}^2$, and let $S(\pi)$ be the sequence that records every disk of $\mathcal{D}$ we enter along π, plus the disks that contain the start point of π, added at the beginning of the sequence, in any order. Then we have $|S(\pi)| = t(\pi)$.

Lemma 9. *For every path π from p to q and every segmentation $\mathcal{T}$ of $S(\pi)$, there exists a path from p to q in G_k of cost at most $\psi(\mathcal{T})$.*

Lemma 10. *For any $p, q \in \mathbb{R}^2$, it holds $r(\wp_{G_k}(p, q)) \leq (1 + \varepsilon) r(p, q)$.*

Theorem 3. *Let $\mathcal{D}$ be a set of unit disks of ply Δ in $\mathbb{R}^2$. We can compute a path π between any two given points $p, q \in \mathbb{R}^2$ whose resilience is at most $(1+\varepsilon)r(p, q)$ in $O(2^{f(\Delta,\varepsilon)} n^7)$ time, where $f(\Delta, \varepsilon) = 16\frac{\Delta^2 \log(\Delta/\varepsilon)}{\varepsilon^4} + o(\frac{\Delta^2 \log(\Delta/\varepsilon)}{\varepsilon^4})$.*

4.3 Extension to Fat Regions

As in Sect. 3.2, we now generalize the result to arbitrary β-fat unit regions.

Lemma 11. *Let $D \in \mathcal{D}$, where $\mathcal{A}(\mathcal{D})$ has ply Δ, and let p,q be any two points inside D. Then the resilience between p and q in $\mathcal{D}$ is at most $(2\beta + 1)^2\Delta$.*

As before, the rest of the arguments do not rely on the geometry of the regions anymore, and we can proceed as in the disk case. The only difference is that the value σ of doing a shortcut has increased to $(2\beta + 1)^2\Delta$.

Theorem 4. *Let $\mathcal{D}$ be a set of unit disks of ply Δ in $\mathbb{R}^2$. We can compute a path π between any two points $p, q \in \mathbb{R}^2$ whose resilience is at most $(1+\varepsilon)r(p, q)$ in $O(2^{f(\Delta,\beta,\varepsilon)}n^7)$ time, where $f \in O(\frac{\Delta^2\beta^6}{\varepsilon^4} \log(\beta\Delta/\varepsilon))$.*

5 NP-Hardness

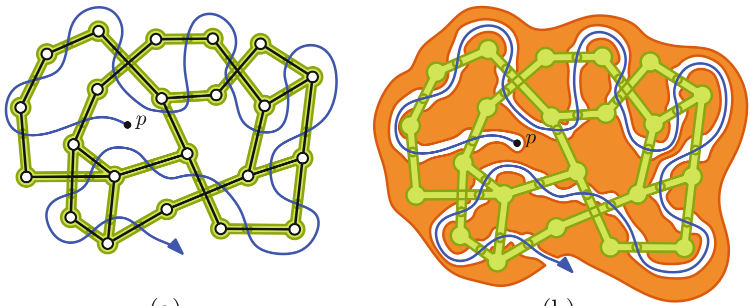
In this section we show that computing the resilience of certain types of fat regions is NP-hard. We recall that NP-hardness was shown in [1] and [16] for the case in which regions are line segments in $\mathbb{R}^2$. In this section we show hardness extends for the case in which ranges have bounded fatness (i.e., ranges are not skinny). We note that Tseng [15] sketched how to extend the proof given in [16] for the case in which $\mathcal{D}$ is a collection of (translated and rotated) copies of a fixed square or ellipse. Although the spirit of the construction is clear, no details and no formal proof of correctness were given.

In addition to providing completeness to the rest of our results, our construction is of independent interest, since it is completely different from those given in [1] and [16]. Moreover, our proof has the advantage of being very easy to extend to other shapes. We also note that the construction of Tseng uses several rotations of a fixed shape (i.e., 3 for a square, 4 for an ellipse), whereas our construction only needs two different rotations of the same shape.

First we show NP-hardness for general connected regions, and later we extend it to axis-aligned rectangles of aspect ratio $1 : 1 + \varepsilon$ and $1 + \varepsilon : 1$. We start the section establishing some useful graph-theoretical results.

Let G be a graph, and let p be a point in the plane. Let Γ be an embedding of G into the plane, which behaves properly (vertices go to distinct points, edges are curves that don't meet vertices other than their endpoints and don't triple cross), and such that p is not on a vertex or edge of the embedding. We say Γ is an *odd* embedding around p if it has the following property: every cycle of G has odd length if and only if the winding number of the corresponding closed curve in the plane in Γ around p is odd. We say a graph G is *oddly embeddable* if there exists an odd embedding Γ for it. We begin by proving that vertex cover is still NP-hard for this constrained class of graphs. (Omitted proofs can be found in the full version [10].)

Lemma 12. *Minimum vertex cover on oddly embeddable graphs of maximum degree 3 is NP-hard.*



(a) (b)

Fig. 5. Creating regions to follow Γ and T.

Given an embedded graph Γ, we say that a curve in the plane is an *odd Euler path* if it does not go through any vertex of Γ and it crosses every edge of Γ an odd number of times.

Lemma 13. *Let p be a point in the plane, and Γ an oddly embedded graph around p. Then there exists an odd Euler path for Γ that starts at p and ends in the outer face. Moreover, such path can be computed in polynomial time.*

Lemma 14. *Let p be a given point in the plane, and Γ an oddly embedded graph (not necessarily planar) around p. Furthermore, let T be a curve that forms an odd Euler path from p to the outer face. Then we can construct a set $\mathcal{D}$ of connected regions such that a minimum set of regions from $\mathcal{D}$ to remove corresponds exactly to a minimum vertex cover in Γ.*

Proof. If T is self-intersecting, then we can rearrange the pieces between self-intersections to remove all self-intersections. Thus we assume that T is a simple path.

If T crosses any edge of Γ more than once, we insert an even number of extra vertices on that edge such that afterwards, every edge is crossed exactly once. Let Γ' be the resulting graph. Since we inserted an even number of vertices on every edge, finding a minimum vertex cover in Γ' will give us a minimum vertex cover in Γ.

Now, for each vertex v in Γ', we create one region D_v in $\mathcal{D}$. This region consists of the point where v is embedded, and the pieces of the edges adjacent to v up to the point where they cross T. Figure 5(a) shows an example (the regions have been dilated by a small amount for visibility; if the embedding Γ has enough room this does not interfere with the construction). Note that all regions are simply connected.

Finally, we create one more special region W in $\mathcal{D}$ that forms a corridor for T. Then W is duplicated at least n times to ensure that crossing this "wall" will always be more expensive than any other solution. Figure 5(b) shows this.

Now, in order to escape, anyone starting at p must roughly follow T in order to not cross the wall. This means that for every edge of Γ' that T passes, one of the regions blocking the path (one of the vertices incident to the edge) must be disabled. The smallest number of regions to disable to achieve this corresponds to a minimum vertex cover in Γ'. ⊠

Combining this result with Lemma 12, we obtain our first hardness result for the barrier resilience problem. As mentioned before, our construction can be modified so that it works for a much more restricted class of regions: axis-aligned rectangles of sizes $1 \times (1+\varepsilon)$ and $(1+\varepsilon) \times 1$ for any $\varepsilon > 0$ (as long as ε depends polynomially on n). Details on the necessary changes can be found in the full version [10].

Theorem 5. *The barrier resilience problem for regions that are axis-aligned rectangles of aspect ratio* $1 : (1+\varepsilon)$ *is NP-hard.*

A similar approach can likely be used to show NP-hardness for other specific shapes of regions. However, it seems that a vital property is that they need to be able to completely cross each other: that is, the regions in $\mathcal{D}$ should not be pseudodisks.[4] Thus, if one were to prove that vertex cover for oddly embeddable graphs of bounded degree is NP-hard would also imply that the barrier resilience problem for unit disks is also NP-hard.

Acknowledgements. M.K was partially supported by the Secretary for Universities and Research of the Ministry of Economy and Knowledge of the Government of Catalonia and the European Union. M.L. was supported by the Netherlands Organisation for Scientific Research (NWO) under grant 639.021.123. R.S. was partially supported by FP7 Marie Curie Actions Individual Fellowship PIEF-GA-2009-251235 and by FCT through grant SFRH/BPD/88455/2012. M.K and R.S. were also supported by projects MINECO MTM2012-30951 and Gen. Cat. DGR2009SGR1040 and by ESF EUROCORES program EuroGIGA-ComPoSe IP04-MICINN project EUI-EURC-2011-4306.

References

1. Alt, H., Cabello, S., Giannopoulos, P., Knauer, C.: On some connection problems in straight-line segment arrangements. In: Proceedings of the EuroCG, pp. 27–30 (2011)
2. Bereg, S., Kirkpatrick, D.: Approximating barrier resilience in wireless sensor networks. In: Dolev, S. (ed.) ALGOSENSORS 2009. LNCS, vol. 5804, pp. 29–40. Springer, Heidelberg (2009)
3. Chang, C.-Y.: The k-barrier coverage mechanism in wireless visual sensor networks. In: Proceedings of the WCNC, pp. 2318–2322 (2012)
4. Chen, D.Z., Gu, Y., Li, J., Wang, H.: Algorithms on minimizing the maximum sensor movement for barrier coverage of a linear domain. In: Fomin, F.V., Kaski, P. (eds.) SWAT 2012. LNCS, vol. 7357, pp. 177–188. Springer, Heidelberg (2012). http://arxiv.org/abs/1207.6409

[4] A similar fact was also observed in [16].

5. de Berg, M., Cheong, O., van Kreveld, M., Overmars, M.: Computational Geometry: Algorithms and Applications, 3rd edn. Springer, Heidelberg (2008)
6. de Berg, M., Katz, M.J., van der Stappen, A.F., Vleugels, J.: Realistic input models for geometric algorithms. In: Proceedings of the SoCG, pp. 294–303 (1997)
7. Gibson, M., Kanade, G., Varadarajan, K.: On isolating points using disks. In: Demetrescu, C., Halldórsson, M. (eds.) ESA 2011. LNCS, vol. 6942, pp. 61–69. Springer, Heidelberg (2011)
8. He, S., Chen, J., Li, X., Shen, X., Sun, Y.: Cost-effective barrier coverage by mobile sensor networks. In: Proceedings of the INFOCOM, pp. 819–827 (2012)
9. Hu, T.C.: Multi-commodity network flows. Oper. Res. **11**(3), 344–360 (1963)
10. Korman, M., Löffler, M., Silveira, R.I., Strash, D.: On the complexity of barrier resilience for fat regions. CoRR, abs/1302.4707 (2013)
11. Kumar, S., Lai, T.-H., Arora, A.: Barrier coverage with wireless sensors. In: Proceedings of the MOBICOM, pp. 284–298 (2005)
12. Kumar, S., Lai, T.-H., Arora, A.: Barrier coverage with wireless sensors. Wirel. Netw. **13**(6), 817–834 (2007)
13. Marx, D.: Parameterized graph separation problems. Theor. Comput. Sci. **351**(3), 394–406 (2006)
14. Miller, G., Teng, S., Thurston, W., Vavasis, S.: Separators for sphere-packings and nearest neighbor graphs. J. ACM **44**(1), 1–29 (1992)
15. Tseng, K.-C.R.: Resilience of wireless sensor networks. Master's thesis, University of British Columbia (2011)
16. Tseng, K.-C.R., Kirkpatrick, D.: On barrier resilience of sensor networks. In: Erlebach, T., Nikoletseas, S., Orponen, P. (eds.) ALGOSENSORS 2011. LNCS, vol. 7111, pp. 130–144. Springer, Heidelberg (2012)
17. Xiao, M.: Simple and improved parameterized algorithms for multiterminal cuts. Theor. Comput. Syst. **46**(4), 723–736 (2010)