COSC 302: Analysis of Algorithms — Spring 2018
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Worksheet 2 — Asymptotic Analysis and Proofs (solutions)

1. Prove that $n^2 + n + 5 = \Theta(n^2)$.

   We show that $n^2 + n + 5 = O(n^2)$ and that $n^2 + n + 5 = \Omega(n^2)$. By Theorem 3.1, this shows that $n^2 + n + 5 = \Theta(n^2)$.

   First we show that $n^2 + n + 5 = O(n^2)$.

   Proof. We show that $\exists c > 0, n_0 > 0$ s.t. $0 \le n^2 + n + 5 \le cn^2$, $\forall n \ge n_0$.

$$
\begin{aligned}
n^2 + n + 5 &\le n^2 + n + 5n^2 && n \ge 1\\
&\le n^2 + n^2 + 5n^2 && n \ge 1\\
&= 7n^2 && n \ge 1
\end{aligned}
$$

   Furthermore, we have that $n^2 + n + 5 \ge 0$ for $n \ge 1$. Therefore for $c = 7$, and all $n \ge n_0 = 1$, we have that $0 \le n^2 + n + 5 \le cn^2$ and by definition $n^2 + n + 5 = O(n^2)$. □

   Next, we show that $n^2 + n + 5 = \Omega(n^2)$.

   Proof. We show that $\exists c > 0, n_0 > 0$ s.t. $0 \le cn^2 \le n^2 + n + 5$, $\forall n \ge n_0$.

$$
\begin{aligned}
n^2 + n + 5 &\ge n^2 + n && \forall n\\
&\ge n^2 + n^2 && n \ge 1\\
&= 2n^2 && n \ge 1\\
&\ge 0 && n \ge 1
\end{aligned}
$$

   Furthermore, we have that $n^2 + n + 5 \ge 0$ for $n \ge 1$. Therefore for $c = 2$, and all $n \ge n_0 = 1$, we have that $0 \le cn^2 \le n^2 + n + 5$ and by definition $n^2 + n + 5 = \Omega(n^2)$. □

2. *(Reflexivity)* Prove that if $f(n)$ is asymptotically non-negative, then $f(n) = O(f(n))$.
   Solution:

   Proof. First note that since $f(n)$ is asymptotically non-negative, $\exists n_1 > 0$ such that $f(n) \ge 0$ $\forall n \ge n_1$. Then,

$$
\begin{aligned}
0 &\le f(n) && n \ge n_1\\
&\le 1 \cdot f(n) && n \ge n_1
\end{aligned}
$$

   Therefore for $c = 1$ and $n_0 = n_1$, $0 \le f(n) \le cf(n)$ for all $n \ge n_0$ and $f(n) = O(f(n))$. □

3. Prove that $\lg(n!) = \Theta(n \lg n)$.
   Solution:

   Proof. We show that $\lg(n!) = O(n \lg n)$ and $\lg(n!) = \Omega(n \lg n)$. By Theorem 3.1, this shows that $\lg(n!) = \Theta(n \lg n)$.

   ($\lg(n!) = O(n \lg n)$):

$$\begin{aligned} \lg(n!) &= \lg(n \cdot (n-1) \cdots 2 \cdot 1) \\ &\leq \lg(n \cdot n \cdots n \cdot n) && \text{for } n \geq 1 \\ &= \lg(n^n) \\ &= n \lg n. \end{aligned}$$

   Therefore for $c = 1$, and all $n \geq n_0 = 1$, we have that $0 \leq \lg(n!) \leq cn \lg n$, and $\lg(n!) = O(n \lg n)$.

   ($\lg(n!) = \Omega(n \lg n)$):

$$\begin{aligned} \lg(n!) &= \lg(n \cdot (n-1) \cdots (n/2+1) \cdots 2 \cdot 1) \\ &\geq \lg(n \cdot (n-1) \cdots (n/2+1)) && \text{when } n \geq 4 \\ &\geq \lg(\frac{n}{2} \cdot \frac{n}{2} \cdots \frac{n}{2}) && n/2 \text{ times} \\ &= \frac{n}{2} \lg(\frac{n}{2}) \\ &= \frac{n}{2} \lg n - \frac{n}{2} \\ &\geq \frac{n}{2} \lg n - \frac{n}{2} \lg \sqrt{n} && 1 \leq \lg \sqrt{n}, \text{ when } n \geq 4 \\ &= \frac{n}{2} \lg n - \frac{n}{4} \lg n \\ &= \frac{n}{4} \lg n \end{aligned}$$

   Therefore for $c = 1/4$, and all $n \geq n_0 = 4$, we have that $0 \leq c \lg(n!) \leq n \lg n$, and $\lg(n!) = \Omega(n \lg n)$. □

4. *(Transitivity)* Let $f(n)$, $g(n)$, and $h(n)$ be asymptotically non-negative functions. Prove that if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.
   Solution:

   Proof. Let $f(n) = O(g(n))$, then there are positive constants c_1, n_1 such that $0 \leq f(n) \leq c_1 g(n)$ for all $n \geq n_1$. Further, let $g(n) = O(h(n))$, then there are positive constants c_2, n_2 such that $0 \leq g(n) \leq c_2 h(n)$ for all $n \geq n_2$.

   Then,

$$\begin{aligned} 0 &\leq f(n) && n \geq n_1 \quad (\text{since } f(n) = O(g(n))) \\ &\leq c_1 g(n) && n \geq n_1 \quad (\text{since } f(n) = O(g(n))) \\ &\leq c_1 c_2 h(n) && n \geq \max\{n_1, n_2\} \quad (\text{since } g(n) = O(h(n))) \end{aligned}$$

   Therefore, for $c = c_1 c_2$ and $n_0 = \max\{n_1, n_2\}$ we have that $0 \leq f(n) \leq ch(n)$ and $f(n) = O(h(n))$. □

5. *(Reflexivity)* Prove that if $f(n)$ is asymptotically non-negative, then $f(n) = O(f(n))$. *(Edit: This is a repeat of question 2. Instead we will show that $f(n) = \Omega(f(n))$.)*

   Proof. First note that since $f(n)$ is asymptotically non-negative, $\exists n_1 > 0$ such that $f(n) \geq 0$ $\forall n \geq n_1$. Then,

$$\begin{aligned} 0 &\leq 1 \cdot f(n) && n \geq n_1 \\ &\leq f(n) && n \geq n_1 \end{aligned}$$

   Therefore for $c = 1$ and $n_0 = n_1$, $0 \leq cf(n) \leq f(n)$ for all $n \geq n_0$ and $f(n) = \Omega(f(n))$. □

6. *(Transpose symmetry)* Let $f(n)$ and $g(n)$ be two asymptotically non-negative functions. Prove that if $f(n) = O(g(n))$ then $g(n) = \Omega(f(n))$.
   Solution:

   Proof. We first show the forward direction, that is: if $f(n) = O(g(n))$ then $g(n) = \Omega(f(n))$.

   ($\Rightarrow$) Suppose $f(n) = O(g(n))$, then there are positive constants c_1, n_1 such that $0 \leq f(n) \leq c_1 g(n)$ for all $n \geq n_1$.

   Since $c_1 > 0$, it is also the case that $0 \leq \frac{1}{c_1} f(n) \leq g(n)$ for all $n \geq n_1$. Therefore, for constants $c = \frac{1}{c_1} > 0, n_0 = n_1 > 0$, we have that $0 \leq cf(n) \leq g(n)$. Hence, $g(n) = \Omega(f(n))$.

   Now we show the reverse direction, that is: if $g(n) = \Omega(f(n))$, then $f(n) = O(g(n))$. The proof is similar.

   ($\Leftarrow$) Suppose $g(n) = \Omega(f(n))$, then there are positive constants c_1, n_1 such that $0 \leq c_1 f(n) \leq g(n)$ for all $n \geq n_1$.

   Since $c_1 > 0$, it is also the case that $0 \leq f(n) \leq \frac{1}{c_1} g(n)$ for all $n \geq n_1$. Therefore, for constants $c = \frac{1}{c_1} > 0, n_0 = n_1 > 0$, we have that $0 \leq f(n) \leq cg(n)$. Hence, $f(n) = O(g(n))$. □

7. Prove that $n^3 \neq O(n^2)$ by contradiction.

   Solution:

   Proof. Suppose, for sake of contradiction that $n^3 = O(n^2)$. That is, there are constants $c > 0, n_0 > 0$ such that $0 \leq n^3 \leq cn^2$ for all $n \geq n_0$.

   Let $x = \max\{n_0, c+1\}$. For $n \geq x$, $xn^2 \leq n^3$, and by definition of $O(\cdot)$, $n^3 \leq cn^2$. This implies that $xn^2 \leq cn^2$, but $xn^2 > cn^2$—a contradiction. $\square$

8. Prove that, for $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ by induction.
   (Edit: The original version of this question said "for $i \in \mathbb{N}$")

   Solution:

   Proof. Base case: $n = 0$, then both $\sum_{i=0}^{0} i = 0$ and $\frac{0(0+1)}{2} = 0$.

   Inductive step: Assume that $\sum_{i=0}^{k-1} i = \frac{(k-1)k}{2}$. Then

$$\begin{aligned}
\sum_{i=0}^{k} i &= \sum_{i=0}^{k-1} i + k \\
&= \frac{(k-1)k}{2} + k && \text{I.H.} \\
&= \frac{k^2 - k + 2k}{2} \\
&= \frac{k(k+1)}{2}.
\end{aligned}$$

$\square$

9. Prove that, for $c \neq 1$ and $n \in \mathbb{N}$, $\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}$ by induction.

Solution:

Proof. Base case: For $n = 0$, then

$$\sum_{i=0}^{0} c^i = c^0 = 1 = \frac{c-1}{c-1} = \frac{c^1-1}{c-1}.$$

Inductive step: Assume that $\sum_{i=0}^{k-1} c^i = \frac{c^k-1}{c-1}$. Then,

$$\begin{aligned}
\sum_{i=0}^{k} i &= \sum_{i=0}^{k} c^i \\
&= \sum_{i=0}^{k-1} c^i + c^k \\
&= \frac{c^k-1}{c-1} + c^k && \text{I.H.} \\
&= \frac{c^k-1}{c-1} + c^k \frac{c-1}{c-1} \\
&= \frac{c^k-1}{c-1} + \frac{c^{k+1}-c^k}{c-1} \\
&= \frac{c^{k+1}+c^k-c^k-1}{c-1} \\
&= \frac{c^{k+1}-1}{c-1}.
\end{aligned}$$

□

10. Prove the following by induction: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any amount of postage larger than 23 cents.

Solution:

Proof. Let p be the amount of postage we can make. We have two cases: $23 < p \leq 28$, and $p > 28$.

Case 1: We begin with $p > 28$. We inductively assume that we can make $p - 5$ cents worth of postage (which is greater than 23). Then we add a 5-cent stamp to get $p = (p - 5) + 5$ cents of postage.

Case 2: We now show it is true for $23 < p \leq 28$, by manually showing that each value p is a sum of 5's and 7's.

$p = 24 = 7 + 7 + 5 + 5$

$p = 25 = 5 + 5 + 5 + 5 + 5$

$p = 26 = 7 + 7 + 7 + 5$

$p = 27 = 7 + 5 + 5 + 5 + 5$

$p = 28 = 7 + 7 + 7 + 7$ $\square$