The Application of Nonparametric Statistical Tests in Geography

50

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CONCEPTS AND TECHNIQUES IN MODERN GEOGRAPHY No. 50

DIE APPLICATION QE NONPARAMETRIC STATISTICAL TESTS IM GEOGRAPHY

by

John Coshall

(Polytechnic of North London)

CONTENTS

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Dedication

To Catherine

I INTRODUCTION

Relatively recent developments in the field of Statistics have involved many techniques of hypothesis testing that do not make implicit assumptions about the nature of the populations from which samples have been drawn. Such statistical tests are referred to as nonparametric. Parametric tests, on the other hand, do make such assumptions typically that samples have been drawn from normal populations. In that nonparametric tests tend not to possess the power of their parametric counterparts (Labovitz, 1970), geographers generally prefer parametric methods of analysis, although their data may not meet the necessary assumptions underlying such procedures (Pringle, 1976). The geographical literature is dominated by parametric, as opposed to nonparametric methods of statistical inference (Vincent and Haworth, 1984). However, it is unlikely that behavioural, attitudinal and socio-economic data gathered in geographical surveys, or information collected from official sources, for example, conform to the normality assumption as required by parametric forms of analysis (Ilbery, 1977).

This is not to say that when a geographer's data fail to meet the assumptions of parametric tests, the alternative should be the immediate use of a particular nonparametric technique. Parametric tests are the more powerful form of analysis if the assumptions underlying them are met. However, there are several instances when the use of parametric techniques is questionable. Firstly, parametric t-tests for both one- and two-samples require sampling from normal population(s) and in the latter case, equality of population variances. When samples are small, as for example in geographical pilot studies, the researcher's assumption of normality is most debatable (Blalock, 1979) and this could lead to fallacious conclusions. Secondly, the parametric F statistic employed to test for equality of population variances and also used in analysis of variance, is highly sensitive to departures from normality (Siegel and Tukey, 1960; Coshall, 1986). Box (1953) cited examples wherein the significance level of the F statistic is specified as $\alpha = 0.05$, but in fact the actual level

is as large as 0.166 or as small as 0.0056. Thirdly, there are situations when a researcher's data may not constitute numerical measurement. For example, the data might be categories like male/female, or consist of ordered items such as regions ranked according to their perceived benefits to potential migrants. Even if numerical measurement is achieved, non-normality makes it difficult to assess the true level of significance of parametric test statistics (Conover, 1980). In such situations, a nonparametric form of analysis may be preferable.

Major texts that describe statistical techniques specifically for geographers focus for the main part on parametric forms of analysis (see for example King, 1969; Cole and King, 1970; Yeates, 1974; Gregory, 1978; Silk, 1979; Shaw and Wheeler, 1985). Often the emphasis is on the computational procedures involved in parametric tests, rather than why and how they should be applied. The present monograph describes a series of nonparametric tests that are likely to have application in geographical analyses when the use of parametric techniques is questionable for the reasons just discussed, or indeed impossible due to the level of measurement achieved (see Section I ii). The tests are illustrated by a wide variety of geographical case studies and emphasis is placed on the sorts of data to which they are applicable and on conveying the rationale underlying them. The computational process is explained in depth.

Naturally, the choice of tests to include had to be selective. Besides having to possess utility in the context of geographical problem solving, the nonparametric tests described tend to be the most statistically efficient of those available (see Section I iii). Sections II to IV respectively describe nonparametric tests for one sample, two related samples and two independent samples. Sections V and VI respectively describe a test for each of k related and k independent samples. In these latter two sections, there are detailed discussions of the ordered alternative hypothesis and multiple comparisons procedure associated with these tests. These are not featured in standard geographical texts and are useful if the geographer wishes to pinpoint significant differences between groups of study objects or to seek trends between them. Section VII presents an assessment of the tests described in this monograph in relation to nonparametric tests not so included. The vast majority of the tests described may be performed without recourse to a computer. Nonetheless, reference is made to the wide-ranging selection of nonparametric tests available in the SPSS^{x} (Nie et al., 1983) and MINITAB computer packages (Ryan et al., 1985). Also in this section, recent applications of nonparametric methods to multivariate analyses in geography are described. Appendix 1 presents a summary of the mathematical notation used throughout this text. Statistical tables for the nonparametric methods described are not housed in one collection. These are therefore collected together in Appendices 2 to 13.

(i) Prerequisites and terminology

The only assumptions made by the author are that the reader is acquainted with basic ideas of probability, a working knowledge of the normal distribution and Spearman's rank correlation coefficient. Simple and thorough explanations of these and other terminologies discussed in this sub-section are presented in Francis (1979).

The form of statistical inference that has received most attention from developers and users of nonparametric statistics is called *hypothesis* testing. A hypothesis is a proposition about a population(s), for example, concerning the numerical value of a population parameter (such as the mean or median) or the distributional form of that population. Hypothesis testing is the process of inferring from a sample whether or not the proposition about the population may be accepted. If evidence from the sample casts doubt on the hypothesis, then the hypothesis is rejected, otherwise it is not *rejected.* The hypothesis that is actually tested is called the *null hypothesis* (denoted by H_0). This is usually the hypothesis that the researcher sets out to disprove (or "nullify"). The inherent working logic of a statistical test assumes that H_o is true and on the basis of this assumption, a *test statistic* is computed. If the numerical value of this test statistic is improbable under the assumption that H_o is true, we

conclude that the proposition involved in H_o is unlikely and it is thus rejected.

If H_o is rejected, an *alternative* (or *research) hypothesis* (denoted by H_1) is accepted in its stead. The alternative hypothesis usually refers to an effect that the researcher wishes to demonstrate, for example, a difference in central tendency between two or more groups of study objects. Alternative hypotheses are of two types, one- or two-tailed. A two-tailed hypothesis considers any change in the value of a parameter, be it either an increase or decrease. A one-tailed hypothesis looks strictly for an increase, or alternatively, for a decrease. The setting up of a one-tailed alternative thus involves more a *priori* information than does a two-tailed alternative.

Before conducting a statistical test, the researcher selects a *level* at *significance* (denoted by a) for the test statistic. Statistical tables present the probabilities of various test statistics taking specific numerical values under the assumption that H_o is true. The level of significance of a test is the lowest value of this probability which will be tolerated before H_o becomes so improbable as to be rejected. The common value for a is 0.05. This means that if a test statistic takes a value whose probability of occurrence under H_o is less than one in twenty, then H_o is rejected in favour of H_1.

(ii) Levels at measurement

The process of selecting an appropriate statistical test involves consideration of the *level* at *measurement* of the researcher's data. The most basic level of measurement involves classification of items into two or more groups that are as homogeneous as possible. This level of measurement is referred to as a *nominal scale.* For example, people may be classified according to religion (Methodist, Catholic etc.). *Ordinal scales* involve ordering individuals according to the degree to which they possess a characteristic. This does not always imply that the researcher knows how much of that characteristic the items possess (Blalock, 1979). For example, a behavioural geographer may employ attitude measurement scales to group individuals into classes of people who are favourably, neutrally or unfavourably disposed towards a scheme of urban renewal.

If it is possible to rank items according to the degree to which they possess a characteristic, then an <u>interval scale</u> of measurement is attained. This requires a physical unit of measurement that can be agreed upon as a common standard (Blalock, 1979), such as the Fahrenheit or Celsius scales. The unit of measurement and the zero point in measuring temperature are arbitary; they are different for the two scales. When we add a true zero point as the origin of an interval scale, we.have a <u>ratio scale.</u> The ratio of any two scale points is independent of the unit of measurement. If two objects are measured in pounds and grams, the ratio of the two pound weights would equal that of the two gram weights.

Generally speaking, nonparametric tests do not require levels of measurement as strong as those required by parametric tests. The latter require measurement at least at an interval scale. Most nonparametric tests apply to the analysis of nominal and ordinal data, or to interval/ratio data that has been degraded to ordinal scale data (such as categories of disposable income) and for which there are no parametric equivalents.

(iii) <u>Statistical efficiency</u>

When there is a parametric equivalent to a nonparametric test, it is possible to compare the relative efficiency of the two tests under various conditions. This is achieved by considering the <u>asymptotic relative efficiency</u> (ARE). The word 'asymptotic' refers to infinitely large samples which obviously cannot arise in practice. Asymptotic theory does, however, permit the researcher to make approximate conclusions concerning the relative efficiency of two tests for finite sample sizes.

If we assume that two tests I and II have the same level of significance a, the ARE of test II with respect to test I is the limiting ratio of the sample sizes n_1/n_2, such that both tests achieve the same <u>power.</u> The power of a test is the probability of correctly rejecting H_o, that is rejecting H_o when indeed it should be rejected. If the ARE of test II with respect to test I is less

than 100%, we say that test II is less efficient than test I. Conversely, it is more efficient if the ARE exceeds 100%. A nonparametric test may be compared with a parametric equivalent for sampling from different populations. Throughout this text, the ARE of the nonparametric tests are discussed with respect to parametric alternatives when they are available.

ONE-SAMPLE TESTS

Geographers frequently encounter situations that involve drawing a sample and testing if it could have been drawn from a population with certain specified characteristics. Such tests commonly examine whether a set of observed frequencies are sufficiently close to frequencies that would be expected under some contention or null hypothesis. These are called goodness of fit tests.

Three goodness of fit tests are described. The binomial test is used for dichotomous nominal data (i.e. data may be grouped into two classes) to determine whether the proportion of cases in the classes is as would be expected under some criterion - often that of equality. The chi-square test is used when nominal data are in several discrete classes. A major use of this test is to examine if observed frequencies are close to those that would be specified by a particular statistical distribution, such as the normal, uniform or Poisson. Thirdly, there is the Kolmogorov-Smirnov test, which is applicable to ordinal data and treats the individual readings separately and does not lose information by grouping data.

(1) The binomial test

In this test, the population is conceived of as comprising two mutually exclusive classes, such as male/female, married/single or urban/rural. The null hypothesis is usually that the proportion in one class equals that in another. The appropriateness of H_0 or otherwise is based on a sample of n independent observations of the dichotomous items.

For small samples, the binomial distribution:

$$P(X = r) = {}^{n}C_{r}p^{r}q^{n-r},$$

where ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$ is used to assess the probability of observing the sample results if H_0 is true. In this distribution, X represents the variable or event of interest, r is the number of occurrences of that event, n is the sample size, p is the probability of an item being in one of the classes (usually 0.5 under Ho) and q = 1 - p; n and p are called the parameters of the distribution.

An example of the binomial test being applied to a small sample can be illustrated by employing data originally collected by Potter (1986), where 10 individuals' perceptions of spatial disparities between the 11 parishes of Barbados, West Indies were examined. Potter used a set of 8 adjectival pairs such as rich/poor, developed/underdeveloped etc., to measure the demographic, social and economic attributes of the parishes. Respondents were asked to consider whether the positive pole of each adjectival pair applied to each of the parishes in turn, giving in all cases a "yes" or "no" answer. By employing traditional repertory grid techniques, Potter validated the choice of scales. The results for the parish of St. Michael are presented in Table 1.

From this table, it can be seen that seven respondents felt that St. Michael is "rich" rather than "poor", but is it significantly more than the 5 respondents that would be expected if the number of people in the population who thought St. Michael was "rich" equalled the number who thought it was "poor"? We set up the null hypothesis that the probability of an individual regarding St. Michael as a "rich" parish is p = 0.5, against an alternative that $p \neq 0.5$. The appropriate binomial distribution is thus:

$$P(X = r) = {}^{10}C_{r}(0.5)^{r}(0.5)^{10-r}$$

and if we let X represent the event of a parish being perceived as "rich", we derive:

$$P(X \geq 7) = P(X = 10) + P(X = 9) + P(X = 8) + P(X = 7)$$

$$P(X \geq 7) = (.5)^{10} + 10(.5)^{9}(.5) + 45(.5)^{8}(.5)^{2} + 120(.5)^{7}(.5)^{3}$$

so, $P(X \geq 7) = 176(0.5)^{10} = 0.1719.$

Table 1 Frequency with which respondents in = 10) felt that the list of adjectives applied to um parish 2L at_ Michael. Barbados

Adjectives			
Rich	Agricultural	Populated	Tourist
7	2	10	9
Developed	Urban	Traditional	Growing
10	10	3	6

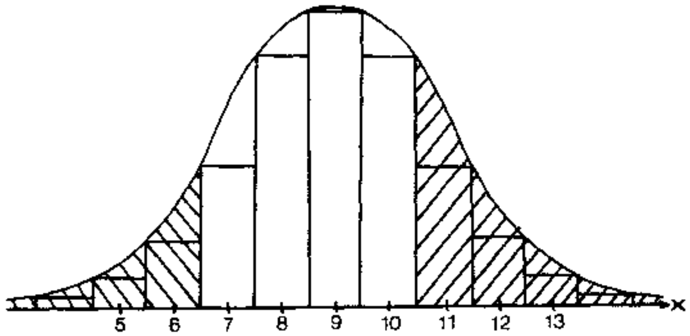
Source: Potter, 1986, p. 186

Thus, the probability of 7 or more people regarding St. Michael as "rich" is 0.1719 under H_o. If we adopt a conventional significance level of $\alpha = 0.05$ for this two-tailed test, we fail to reject H_o and the observed frequency of 7 leads us to infer that the numbers of people in the population regarding the parish as "rich" and "poor" are equal, i.e. there is insufficient evidence to conclude that $p \neq 0.5$. If this binomial distribution is investigated further, it is found that only $P(X \geq 9)$ and conversely $P(X \leq 1)$ are less than $\alpha/2 = 0.025$. Hence from Table 1, a significant proportion of people perceive St. Michael as populated, tourist, developed and urban.

For large n, computation via the above procedure can be tedious. It has been suggested that if either $np > 5$ when $p \leq 0.5$ or $nq > 5$ when $p > 0.5$, then the binomial distribution is adequately approximated by the normal distribution with the same mean and variance (Noel, 1971). An appropriate continuity correction of ±0.5 has to be used as is explained in Figure 1, because a discrete distribution is being approximated by a continuous one.

The large sample approximation is illustrated with reference to data reported in Knoke and Burke (1980, p.23). The data, taken from the 1977 General Social Survey by the National Opinion Research Center in Chicago, report voting turnout in the 1976

Figure i use U a continuity correction in the normal approximation S. fiat binomial distribution



Whenever a discrete distribution is approximated by a continuous one, a continuity correction of ±0.5 is required. Suppose in the above figure, we require $P(X \geq 11)$. Given the continuous scale of the horizontal axis, this is treated as $P(X \geq 10.5)$, as shown by the shading. Similarly, $P(X > 11)$ is treated as $P(X \geq 11.5)$. For the left tail, $P(X \leq 6)$ is regarded as $P(X \leq 6.5)$ as also shown by the shading. Similarly, $P(X < 6)$ is treated as $P(X \leq 5.5)$.

Presidential election. The binomial test is used to examine the voting turnout of blacks. Suppose we wish to examine if, for example, more than half of the blacks voted, then the null hypothesis to be tested is that p, the probability of a black voting, is 0.5. A one-tailed alternative is adopted to the effect that more than one half of the blacks voted. Of 169 blacks in the survey, 105 voted and 64 did not. We use the fact that the mean or expected value of the binomial distribution is given by $E(x) = np$ and the variance by $var(x) = npq$. The test statistic with a continuity correction is:

$$\frac{(x \pm 0.5) - E(x)}{\sqrt{var(x)}}$$

which is approximately standard normally distributed. Let X represent the event of a black voting, so under H_o, $E(x) = 169(.5) = 84.5$ and $var(x) = 169(.5)(.5) = 42.25$, so our test statistic is:

$$\frac{(105 - 0.5) - 84.5}{\sqrt{42.25}} = 3.08 \in N(0,1).$$

(The notation e means "belongs to"). From tables of standard normal deviates in Appendix 2, P(N(0,1) > 1.645] = 0.05, so we reject H_o and conclude that significantly more than 50% of the blacks voted.

The parameter p in the null hypothesis need not always be one half. Suppose we wish to test, for example, whether more than 60% of the whites voted in the 1976 Presidential election. Of 1304 whites interviewed, 882 voted and 422 did not. Under H_o, p = 0.6 and q = 0.4, so E(x) = 1304(0.6) = 782.4 and var(x) = 1304(.6)(.4) = 312.96 and the test statistic becomes:

$$\frac{(882 - 0.5) - 782.4}{\sqrt{312.96}} = 5.60 \in N(0,1),$$

which is very highly significant (Appendix 2), so we conclude with a high degree of confidence that more than 60% of the whites voted in the election.

(ii) The chi-square

This is a test of goodness of fit. One of the major uses of the chi-square (x^2) is to determine if there is a significant difference between an observed set of frequencies falling in particular categories and those frequencies that would be expected under a null hypothesis. It is common for the null hypothesis to be that the observed data belong to one of the classical distributions in Statistics, such as the normal, Poisson, binomial or uniform. The null hypothesis is assessed by how close the observed frequencies are to those that would be expected if H_o was true.

The test statistic is:

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \text{ ......(1)},$$

where O_i and E_i are the observed and expected frequencies respectively in category i, and k is the number of categories. If the observed frequencies are close to those expected under H_o, then the numerical value of X^2 given by equation (1) will be close

to zero. If there is a divergence between the O_i and E_i, then X^2 will be large, indicative that the observed frequencies are unlikely to have come from the population on which H_o is based.

A common application of the test in geography has been the comparison of an observed frequency distribution of points in space, such as settlements, with some distribution postulated by theory. In particular, the Poisson distribution represents a useful benchmark against which empirical patterns may be compared (King, 1969), in that this statistical distribution provides a good description of random phenomena. Beyond the random (Poisson) distribution in one direction lie point distributions that are more and more clustered and in the other direction, point distributions that are more and more regular.

Haggett et al. (1977, p.416) considered a hypothetical map that had been exhaustively divided by a regular lattice of square cells into small quadrats. A frequency distribution of the number of quadrats with x = 0, 1, 2... points in them was constructed and is reproduced in the first two columns of Table 2. The chi-square test is used to examine if the points are located so as to form a random point pattern - i.e. if the frequencies of points in the cells are close to those that would be expected under a Poisson distribution. To conduct the chi-square test, we have to generate expected frequencies under H_o and compare them with those observed. From Table 2, it may be computed that the mean is λ = 0.52 observed points per quadrat. The null hypothesis is therefore that the observed frequencies in column 2 are Poisson distributed with parameter λ = 0.52. If x is the number of points per quadrat, then we may generate expected frequencies under H_o by using the Poisson distribution:

Table 2. Number of points per quadrat from a hypothetical Mg

No. of points/ quadrat	Obs. freq of quad-rats (O_i)	Prob. under H_0, $P(X = x)$	Expected freq. (E_i)	$O_i - E_i$	$\frac{(O_i - E_i)^2}{E_i}$
0	59	0.5945	59.45	-0.45	0.003
1	32	0.3092	30.92	1.08	0.038
2	9 { 7	0.0804	9.63 { 8.04		
3	2	0.0139	1.39	-0.63	0.041
≥ 4	0	0.0020	0.20		
					χ^2 = 0.082

Source: Haggett et al., 1977, p.415

$$P(X = x) = \frac{e^{-\lambda}.\lambda^x}{x!}, \text{ for } x = 0,1,2\ldots\ldots$$

In this distribution, λ represents the mean (0.52 points per quadrat) and e is the exponential constant 2.71828 These probabilities, $P(X = x)$ for $x = 0,1,2$ ...... , are shown in column 3 of Table 2. The expected frequencies in column **4** are simply found by multiplying each of the $P(X = x)$ by $n = 100$. When conducting the chi-square test, expected frequencies that are less than 5 should be combined with a neighbouring category or categories until a total of more than 5 is achieved (Cochran, 1954). This is necessary for the last three categories of column 4 in Table 2. Their partners in the column of observed frequencies are also combined, before one computes the $(O_i - E_i)$ as required by the test statistic represented by equation (1). The computation of the test statistic is completed in column 6.

There are many different sampling distributions of the chi-square statistic, and they depend on the decrees of freedom (df) in an experiment. The size of the **df** (the parameter of this distribution) depends upon the number of categories of observations that are free to vary after certain restrictions have been placed on the data. For example, in column 4 of Table 2, we know that the sum of the expected frequencies must be 100, so knowing the values in the first two categories, we automatically know that the third is 100 - 90.37 = 9.63. Often this is referred

to as the loss of a **df** due **to** the 'last cell' and this is always lost in chi-square tests of goodness of fit. One df is also lost for each parameter calculated from the data, and here we have **computed λ, the mean. Generally, the df of a chi-square goodness of fit test is given by:**

$$df = k - 1 - m,$$

where k **is** the number of categories of data after grouping (if necessary) and m is the number of parameters computed from the data. In the present example, k = 3 and m = 1, so df **=** 1. If we were fitting a normal distribution, m = 2 as the mean and variance are often computed from the data to generate expected frequencies.

Selecting a conventional significance level of $\alpha = 0.05$, we refer to Appendix 3 of chi-square variates to find that with df = 1, $P(\chi^2_1 > 3.841) = 0.05$. Our value of 0.082 in Table 2 is thus not significant, **so** we fail to reject H_0 and conclude that the observed frequencies of points in the quadrats are Poisson **distributed with parameter value $\lambda = 0.52$. We decide that the** pattern of points in the quadrats follows a Poisson distribution with this parameter value. It should be noted that this provides us with a measure of the degree of order present in the spatial pattern, rather than offering an explanation of the underlying determinants of its form.

(iii) The Kolmogorov-Smirnov (K-S) test

This is also a test of goodness of fit, but unlike the chi-square test it uses the cumulative frequency (or probability) distribution, rather than the frequency distribution which is used in the chi-square. The Kolmogorov-Smirnov test makes more complete use of the available data than the X^2 test in that **it** does not require the lumping of categories and because **it** considers the order of the categories or observations. Hays (1980) is of the opinion that the K-S test is superior to the chi-square test, in that the latter requires large samples, is always approximate and " the goodness of the approximation varies with a number of factors, not all of which can be taken into account in a simple rule of thumb" (Hays, 1980, p.752).

The null hypothesis of the K-S test is that a random sample has been drawn from a specified population of measurements (Lewis, 1971), for example, the normal or Poisson distributions. Suppose a sample of n items has a cumulative probability distribution (cpd) represented by $S_n(x)$ and assume that S(x) is a consistent estimator of F(x), the unknown cpd of the population from which the sample was randomly drawn. Let G(x) be some hypothetical cod, such as that of the normal or Poisson distributions. The null hypothesis is that F(x) and G(x) are equal and the K-S test statistic is the largest absolute difference between the empirical Old $S_n(x)$, and the hypothetical cpd G(x).

The data in Table 3 are used to show how the K-S test is employed to examine if a sample may be considered to have been drawn from a normal population. The data are rainfall (inches) in Derby for the 50 years between 1917 and 1966 inclusive (Hammond and McCullagh, 1974). The mean and variance of these annual rainfall figures are 25.2 and 19.24 inches respectively. The null hypothesis is that the frequencies in the second column of Table 3 are normally distributed with these parameter values, against an alternative that the frequencies are non-normal.

The 02d, $S_n(x)$, of the n = 50 sample values is shown in column 3 of Table 3. Further columns in Table 3 generate the expected cpd of the normal distribution with a mean and variance of 25.2 and 19.24 respectively. The first step in performing this task is to standardise the ranges of column 1. For example, in standardising the range 16 - 18 inches, we obtain:

$$z_1 = \frac{16 - 25.2}{\sqrt{19.24}} = -2.10 \quad \text{and} \quad z_2 = \frac{18 - 25.2}{\sqrt{19.24}} = -1.64.$$

Appendix 2 is used to evaluate the probabilities that a standard normal variable would lie in each of these ranges. These probabilities are shown in the second half of Table 3. In the

Table 3. Annual rainfall 41 Derby sewage works. 1917-66

Annual rainfall (ins)	Number of years (f)	Cumulative proportion of years Sn(x)	Standardised ranges
Below 16	0	0	Below -2.10
16 - 18	1	0.02	-2.10 to -1.64
18 - 20	4	0.10	-1.64 to -1.18
20 - 22	8	0.26	-1.18 to -0.73
22 - 24	7	0.40	-0.73 to -0.27
24 - 26	12	0.64	-0.27 to 0.18
26 - 28	6	0.76	0.18 to 0.64
28 - 30	5	0.86	0.64 to 1.09
30 - 32	4	0.94	1.09 to 1.55
32 - 34	1	0.96	1.55 to 2.00
34 - 36	0	0.96	2.00 to 2.46
36 - 38	2	1.00	2.46 to 2.92
Above 38	0	1.00	Above 2.92

Annual rainfall (ins)	Probability	Cumulative prob. G(x)	|Sn(x) - G(x)|
Below 16	.01786	.01786	.018
16 - 18	.03264	.05050	.031
18 - 20	.06850	.11900	.019
20 - 22	.11370	.23270	.027
22 - 24	.16088	.39358	.006.
24 - 26	.17784	.57142	.069
26 - 28	.16749	.73891	.021
28 - 30	.12323	.86214	.002
30 - 32	.07729	.93943	.001
32 - 34	.03782	.97725	.017
34 - 36	.01580	.99305	.033
36 - 38	.00520	.99825	.002
Above 38	.00175	1	0

Source: Hammond and McCullagh, 1974, p.96

penultimate column of this table, these probabilities are cumulated to derive the cpd, G(x), under H. The test statistic of the K-S test is the maximum of the absolute differences between $S_n(x)$ and G(x), starred in the final column of Table 3. The test statistic is denoted by K and in symbols:

$$K = \max |S_n(x) - G(x)| = 0.069, \text{ in the present example.}$$

The exact sampling distribution of K is known and tabulated for n ≤ 40. Statistical tables are presented in Appendix 4. As is shown in Appendix 4, for n 40 an approximation is used, based on the asymptotic distribution of the test statistic. In the present example, and adopting a significance level of $\alpha = 0.05$ for this two-tailed test, we compute that for n = 50, H_o should be rejected for $K > \frac{1.36}{\sqrt{50}} = 0.192$.

Our result is not significant and we conclude that the rainfall data are normally distributed with the computed parameter values. It should be noted that IS .(x) - G(x)I could be obtained graphically, by determining the maximum vertical displacement on the cumulative frequency graphs of the two distributions, S(x) and 0(x). The goodness of fit tests presented here represent one of the few instances in which the researcher is seeking confirmation of H_o rather than nullification of it.

The above procedure may readily be applied to the case of two independent samples. The test statistic is again the maximum absolute difference, K, between cumulative frequency distributions, but this time of <u>two observed</u> variables X and Y (Bradley, 1968). In this instance, the K-S test examines if the populations from which the samples have been drawn differ in any respect at all, such as central tendency, dispersion or skewness. It has been used in this fashion to compare longitudinal stream profiles obtained from Ordnance Survey 1:25000 series, in order to assess the accuracy of contour maps and their usefulness to geomorphologists (Wheeler, 1979). It might be argued, however, that in this case the samples are not truly independent. Bennett (1977) used the K-S test to compare differences in component scores obtained from principal components analysis of socio-demographic data, where raw data and transformed data acted as input.

(iv)

The choice between the use of the binomial, chi-square and K-S tests for goodness of fit is determined by a) the number of categories in the data, b) the level of measurement, c) the sample

size and d) the power of the statistical test (Siegel, 1956). The binomial test may be used when there are just two categories involved in the classification of data. It is also useful when the sample size is too small to justify the application of the chi-square test. Both the binomial and chi-square tests may be used with nominal or ordinal data. When nominal measurement is attained, there is no parametric alternative to the use of X^2, so the concept of asymptotic relative efficiency is meaningless. The Kolmogorov-Smirnov test treats individual categories or observations separately and it does not lose information by grouping the data, which is sometimes necessary for the X^2 test. In such instances, X^2 is less powerful than the K-S test. Also, the K-S test is a conservative one, i.e. if H_o is rejected by the test, then we can have real confidence in that decision (Goodman, 1954).

III TESTS FOR TWO RELATED SAMPLES

The tests presented in this section are designed to establish whether two treatments are different, or whether one treatment has differential effects. A treatment is the variable of interest in an experiment and the researcher wishes to determine if it has had some significant effect on the items in the sample(s). For example, a treatment might be the application of a new type of fertiliser designed to effect an increase in crop yield. On the other hand, the researcher may also wish to compare two treatments, for example, two methods of measuring hillslopes in order to assess errors.

There is the problem, however, that an observed effect may be ascribed to a treatment, when in fact it is due to one or more variables that are extraneous to the experiment. One method of measuring hillslope profiles may be deemed superior to another method simply because of the skill, expertise or patience of the individual using that measuring device. One method to overcome this difficulty is to use two related or matched samples in the research. This is achieved either by using each item as its own control or by pairing items and then randomly assigning the members of each pair to two different treatments. When an item acts as its own control, it is exposed to both treatments at different times. When items are paired, they should be as alike as possible in respect of any extraneous variables likely to influence the outcome of the experiment (an example of this is psychologists using twins in learning experiments).

Three tests that examine treatment effects are described in this section. When ordinal measurement within the data items is possible, to the extent that one member of a pair can be ranked in relation to the size of the other member, the two-sample sign test is applicable. The test is analagous to the parametric paired t-test, which strictly speaking assumes normality. The Wilcoxon test uses the magnitude and direction of differences between pairs and is thus more powerful than the sign test. If measurement is at least at an interval scale, the more powerful Walsh test may be applicable to the data.

1.) The sign test

This, the oldest of all the nonparametric tests, is designed to examine a difference between two conditions. It is particularly useful in research in which quantitative measurement is impossible, but in which it is possible to rank one item in a matched pair with another. The only assumption is that the variable under consideration has a continuous distribution.

Consider the data in the first three columns of Table 4. Mean levels of pollution in the river Trent have been measured at 21 stations over two different time periods (Trent River Authority, 1969). The level of pollution can be measured in a number of ways, but one of the more reliable scales is the measurement of ammonical nitrogen (AN), which is the indicator reported in this table. The samples in the second and third columns of Table 4 are matched, in that the same n = 21 stations appear in both samples and thus act as their own control. The sign test is used here to see if legislation has led to a lower level of AN in the period 1965-67 than in the period 1959-61.

The fourth column of Table 4 reports the signs of the differences between AN levels at these two time periods. If the values in columns two and three had been equal, it is conventional to omit that pair from further analysis and reduce n accordingly. We set up the null hypothesis of no difference, in which case the probabilities of observing a plus or minus are equal and we may state:

$$H_0: P(+) = P(-).$$

We adopt a one-tailed alternative hypothesis to test if the AN levels at 1959-61 are greater than those at 1965-67, namely:

$$H_1: P(+) > P(-).$$

Under H_o, $P(+) = P(-)$ 0.5, and we evaluate the probability of observing 19 or more plusses in Table 4 out of a maximum possible

Table 4. Mean levels 2f ammonical nitrogen at sample stations an the River Trent

Station	1959-61 (A)	1965-67 (B)	Sign of difference (A) - (B)
Milton	1.1	0.6	+
Hanley	0.7	0.8	-
Stoke-on-Trent	5.6	3.1	+
Hanford Bridge	11.1	5.6	+
Stone	9.7	6.6	+
Great Haywood	6.6	5.4	+
Handsacre	2.9	2.2	+
Yoxall	2.4	1.8	+
Wychnor	1.6	1.4	+
Walton-on-Trent	7.7	6.7	+
Burton-on-Trent	7.3	4.6	+
Willington	5.6	3.8	+
Swarkestone	4.8	2.9	+
Shardlow	3.6	3.0	+
Sawley	2.7	2.4	+
Nottingham	2.4	2.2	+
Gunthorpe	2.9	2.4	+
Kelham	2.5	2.2	+
Dunham	2.3	1.7	+
Gainsborough	1.8	1.5	+
Keadby	1.4	1.5	-

Source: Trent River Authority, 1969

of 21, under the assumption of no difference. This is achieved by using the binomial distribution ${}^{21}C_r p^r q^{21-r}$, (see Section II 1), where $P(+) = p$ and $P(-) = q$ and $p = q = 0.5$. Hence:

$$\begin{aligned} P(19 \text{ or more plusses}) &= P(21+) + P(20+) + P(19+) \\ &= (.5)^{21} + 21(.5)^{20}(.5) + 210(.5)^{19}(.5)^2 \\ &= 232(.5)^{21} \\ &= 0.00011. \end{aligned}$$

If we adopt the conventional significance level of $\alpha = 0.05$, we reject H_o in favour of H_1 and conclude that the levels of AN at 1959-61 were significantly greater than the AN levels at 1965-67. Having matched samples, we may conclude that legislation has been effective, assuming that the effects of any extraneous factors are cancelled out.

As Hoel's (1971) criterion is met, namely if np > 5 when p 0.5, we could have approximated P(19 or more plusses) by using the

normal distribution with an appropriate continuity correction. If

:1(.5) and var(x) = npq = 21(.5)(.5), thus:

$$\frac{(x \pm 0.5) - E(x)}{\sqrt{var(x)}} = \frac{(19 - 0.5) - 10.5}{\sqrt{5.25}} = 3.49 \in N(0,1).$$

Referring to Appendix 2, we find that P[N(0,1) > 3.49] = 0.00024 <).05, which approximates well with the binomial result.

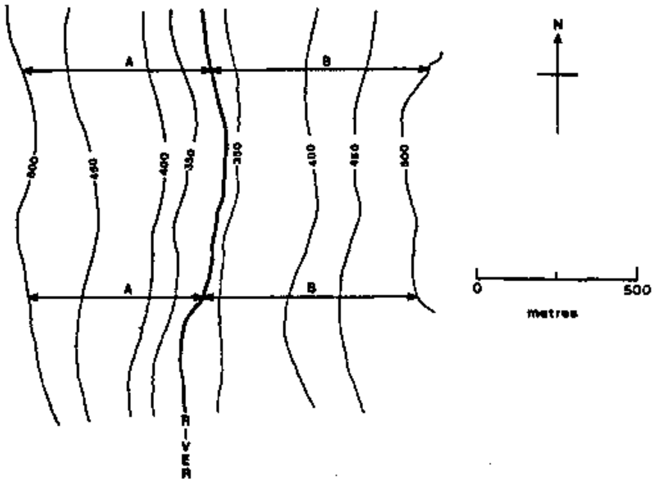
11] The Wilcoxon test

The sign test uses information about the direction of differences between pairs. The Wilcoxon matched-pairs incorporates the magnitude as well as direction of the differences into the analysis.

Figure 2 illustrates the west and east slopes on opposing rides of a short stretch of the River Leadon near Castle Frome in ierefordshire. The Wilcoxon matched-pairs test is used to see if there is any significant difference between the gradients of the west and east slopes. The null hypothesis is that the slopes are equal against the two-tailed alternative that they are not. Measurements of contour spacing along cross-profiles have been nade at regular intervals. Two such pairs of measurements are shown in Figure 2. The first two columns of Table 5 record the number of millimetres on Figure 2 from the valley axis to the Fourth contour line above it for the west (A) and east (B) slopes. Eight such regularly spaced measurements were taken up the two slopes, of which those in Figure 2 are the first and last.

Each value of A is matched with a B value - opposing slopes on the same profile - and the differences, d_i, between the west and east distances to the fourth contour line are shown in the third column of Table 5. The fourth column gives the rank of the d_i from lowest to highest, regardless of their signs. Equal

Figure 2 Contour spacing on the west and east slopes of. the valley of the River Leadon. Herefordshire



values of d_i are given the average value of those ranks that would have been ascribed to them. If the scores of any pair are equal, then $d_i = 0$. It is conventional to omit such pairs from the analysis. The next step in the Wilcoxon test is to affix to each rank in column 4 of Table 5 the sign of the difference. This is shown in column 5 and indicates which ranks arose from positive and negative differences, d_i. If the west and east slopes are equal as posited under H_o, we would expect an equal spread of the larger ranks favouring treatments A and B. This implies that the sum of positive and negative ranks should be equal under H_o. The test statistic, T, is the smaller sum of like-signed ranks, here the positive ones. Thus from column 5 of Table 5, T = 6.5 + 4 + 4 = 14.5.

A table for the significance of T for n up to 25 is presented

= 0.05, we find that P(T

and conclude that the west and east slopes of this stretch of the river valley are equal in gradient.

Table 5. Contour spacing 211 the west dad east slopes of the valley 2f the River Leadon. Herefordshire

Contour spacing on slope: West (A)	East (B)	d_i = (A) - (B)	Rank of $\lvert d_i \rvert$	Sign & rank of d_i
35	39	-4	6.5	-6.5
35	36	-1	1.5	-1.5
36	32	4	6.5	6.5
34	32	2	4	4
37	35	2	4	4
37	38	-1	1.5	-1.5
35	37	-2	4	-4
32	40	-8	8	-8

Source: Figure 2

When n is larger than 25, it may be shown that the smaller sum of like-signed ranks, T, is closely normally distributed with parameters $E(T) = n(n + 1)/4$ and $var(T) = n(n + 1)(2n + 1)/24$, where n is the number of matched pairs after any deletion of pairs for which $d_i = 0$. (In fact Siegel (1956) has shown the approximation to be excellent for smaller values of n than 25). The large sample application of the Wilcoxon test is illustrated by comparing one aspect of the building morphology of 25 town centres in southern Scotland over different time periods, 1919-45 and 1946-75. (Appendix 5 may be used in this instance, but the data are utilised here to illustrate the large scale approximation). In the post-war period, this region as a whole experienced appreciable economic growth (Whitehand, 1979) and this may be expected to have had a direct effect on urban rebuilding. Among other variables, Whitehand recorded the percentage of ground-floor façades (most of which were shopfronts) that had been altered in the 25 town centres during the two time periods. These percentages are reproduced in columns 2 and 3 of Table 6. The null hypothesis is of no difference in these percentage figures for the pre- and post-war years. one-tailed alternative is that the post-war percentages of façade conversions exceed the pre-war

Table 6 Percentages 2E facade conversions in 25 southern Scottish towns. 1919-45 And 1946-75

Town	% of facade conversions: 1919-1945 (A)	1946-1975 (B)	Sign d1 = (A) - (B)	Ranks of $|d_1|$	& rank of d1
Biggar	6	3	3	7.5	7.5
Carluke	11	8	3	7.5	7.5
Cumnock	12	3	9	21	21
Douglas	0	2	-2	5	-5
Dumfries	6	10	-4	11	-11
Galashields	14	7	7	18.5	18.5
Hawick	7	11	-4	11	-11
Innerleithen	0	6	-6	16	-16
Jedburgh	6	13	-7	18.5	-18.5
Kelso	6	4	2	5	5
Kilmarnock	13	12	1	2	2
Lanark	10	11	-1	2	-2
Langholm	8	4	4	11	11
Larkhall	6	6	0	-	-
Lauder	0	0	0	-	-
Lesmahagow	3	3	0	-	-
Lochmaben	10	2	-8	20	-20
Lockerbie	4	8	-4	11	-11
Melrose	8	2	6	16	16
Moffat	1	5	-4	11	-11
Peebles	10	11	-1	2	-2
Penicuik	7	17	-10	22	-22
Sanquhar	8	3	5	14	14
Selkirk	12	10	2	5	5
Strathaven	7	13	-6	16	-16

Source: Whitehand, 1979, p.567

percentages.

T is computed as before, with omission of the towns of Larkhall, Lauder and Lesmahagow as the percentages of facade conversions are equal during both time periods and $d_i = 0$. The value of n is thus reduced from 25 to 22. The fourth column of Table 6 gives the differences, d_i, between the 1919-45 and 1946-75 percentages. The fifth column of this table indicates the ranks of the d_i ignoring their sign. Column 6 reports the signs of these ranks, the sum of the positive ranks being less than the negative ones. From column 6, we compute that T = 7.5 + 7.5 + 21 + + 5 = 107.5, so the test statistic under H_o is:

$$\frac{T - E(T)}{\sqrt{var(T)}} = \frac{T - n(n+1)/4}{\sqrt{n(n+\ )(2n+1)/24}} \in N(0,1), \text{ hence}$$

$$\frac{107.5 - 22(23)/4}{\sqrt{22(23)(45)/24}} = \frac{-19}{\sqrt{948.75}} = -0.62 \in N(0,1).$$

From Appendix 2, P[N(0,1) < -1.645] = 0.05, so we fail to reject H_o and conclude that the pre- and post-war percentages of ground-floor facade alterations are not significantly different. (From Appendix 5, we would have required T s 66 for rejection of H_o at a 5% significance level in this instance). Being matched samples, it is assumed that extraneous factors such as fluctuations in the building industry or population growth tend to be cancelled out.

(iii) The Walsh test

This very powerful test examines central tendency based on two related samples. It requires measurement in at least an interval scale and is useful if the researcher can assume that the populations from which the samples are drawn are symmetrical. This does not mean that the populations are normally distributed, otherwise the parametric t-test should be used. Walsh,s test assumes that if the populations are symmetrical, then the mean and median are equal.

The test is illustrated by analysing errors in surveying hillslopes using an Abney level (Abrahams and Melville, 1975). It seems reasonable that the errors may come from symmetrical populations. The true slopes of 10 profiles were measured upslope and downslope, and average downslope readings (DR) and upslope readings (UR) were obtained. Errors in the slopes of the profiles were found by subtracting the true slopes from DR and UR. These results are presented in Table 7.

Consider firstly the values of ($\overline{DR}$ - true slope). The null hypothesis is that the median difference is zero against the two-tailed alternative that it is non-zero. The first step is to

Table 7 Errors in average downslope readings ($\overline{DR}$) and upslope readings ($\overline{UR}$) by Abney level

$\overline{DR}$ - true slope	$\overline{UR}$ - true slope
-0.20	-0.20
+0.20	+0.10
+0.10	0
+0.10	0
+0.10	-0.10
0	-0.10
0	-0.10
-0.10	-0.20
-0.10	-0.20
0	-0.20

Units are degrees and minutes of an arc

Source: Abrahams and Melville, 1975, p.300

order the differences from lowest to highest, in the form:

$$d_1 \le d_2 \le d_3 \le \ldots\ldots \le d_n,$$

where n is the number of differences, here 10. Thus for the downslope errors in Table 7:

$$d_1 = -.2,\ d_2 = -.1,\ d_3 = -.1,\ d_4 = 0,\ d_5 = 0,\ d_6 = 0,\ d_7 = .1,$$
$$d_8 = .1,\ d_9 = .1 \text{ and } d_{10} = .2 \ldots\ldots(2).$$

Tables of significance for this test are presented in Appendix 6. We find that for a two-tailed test and with n = 10, H_o is rejected at a significance level of m = 0.051 (the closest we may get to α = 0.05), if either:

$$\max[d_7,\ .5(d_5 + d_{10})] < 0 \text{ \underline{or} } \min[d_4,\ .5(d_1 + d_8)] > 0 \ldots\ldots(3).$$

This means that H_0 should be rejected at α = 0.051 if either a) the larger value of d_7 or $.5(d_5 + d_{10})$ is negative, or b) the smaller value of d_4 or $.5(d_1 + d_8)$ is positive. In the present example and using (2) and (3), we may reject H_0 at this significance level if either:

$$\max[.1,\ .5(0 + .2)] < 0 \text{ \underline{or} } \min[0,\ .5(-.2 + 0)] > 0,$$

neither of which are here the case. We fail to reject H_o and conclude that the median of the d_i for the downslope errors is not significantly different from zero.

In a similar vein, we may test whether the median of the d_i

for the upslope errors is zero. From Table 7 we derive:

$$d_1 = -.2,\ d_2 = -.2,\ d_3 = -.2,\ d_4 = -.2,\ d_5 = -.1,\ d_6 = -.1,$$
$$d_7 = -.1,\ d_8 = 0,\ d_9 = 0 \text{ and } d_{10} = .1.$$

Using equation (3), we reject H_0 if either

$$\max[-.1,\ .5(-.1 + .1)] < 0 \text{ \underline{or} } \min[-.2,\ .5(-.2 - .1)] > 0,$$

and now the first of these criteria is met. We thus reject H_o in favour of H_1 and conclude that the median of the upslope errors is significantly different from zero.

(iv) Discussion

The sign test and Wilcoxon's signed-rank test are generally both useful in the same experimental situations for paired samples. Neither test is particularly restricted by a moderate number of ties. The Wilcoxon test, however, requires more information about relative magnitudes as well as directions of differences. In that it uses more information, it is more powerful than the sign test. If the populations are in fact normal, the ARE of the sign test is about 95% for n = 6 pairs of readings, but it drops to a lower bound of 63% as n increases (Hodges and Lehmann, 1956). In similar circumstances, the ARE of the Wilcoxon test is in the region of 95% for all n. The latter test has higher ARE than the sign test for sampling from non-normal populations (it never falls below 86.4%) and is thus the preferable test if the data permit. This is not surprising in that the sign test is unaffected by the relative magnitudes of the d_i and uses less of the information in the data.

The Walsh test may be used if measurement is at least at an interval scale. If the populations are normal, the Walsh test has ARE of 95% for most values of n and α and it can reach 99% (for n = 9, α = 0.01 and a one-tailed test). The lower bound of the ARE of this test is 87.5%. The major problem in using the Walsh test is that tables of significance are not available for n > 15.

IV TESTS FOR TWO INDEPENDENT SAMPLES

When the use of two related samples is impractical or inappropriate, the researcher may use two independent samples. For example, samples could be obtained randomly from two populations. A second method of obtaining two independent samples could be the random assignment of two treatments to the items of a sample whose origins are arbitrary. The tests presented here, like those of the previous section, examine whether differences in samples evidence differences in the processes applied to them.

Three procedures are described that test for the significance of a difference in the populations from which two independent samples have been drawn. A common problem in geography is to examine two independent samples drawn from two different populations to see if the populations have the same or different proportions of elements in the various categories of a study variable. For example, samples of north and south facing slopes may be taken to examine whether equal proportions of the two slopes are covered by various types of plant communities. A set of frequencies formed by classifying sample items into categories constitutes a contingency table, and the chi-square test may be applied to its analysis. The populations and categories of the study variable may be defined by measurement as weak as nominal in this test.

If the researcher is testing if two populations differ in central tendency, then an adaption of Fisher,s exact test (sometimes referred to as the median test) is applicable. This test is used when it is possible only to dichotomise items above or below the combined sample median. The Mann-Whitney test is used to examine if two independent samples have been drawn from the same population or populations with the same distribution. It is often used, therefore, to test for differences in central tendency. It is applicable when ordinal measurement is achieved and the data in both samples may be ranked. The latter test has been used in geography in such diverse areas as examination of spatial variations in water quality for different groups of rock-type and

land use (Prowse, 1984) and an analysis of the University Grants Committee,s cuts in student numbers and grants (Hoare, 1981).

(i) The chi-square test

This test as described in the present section is used to determine the significance of differences between two independent groups according to some criterion of relevance. For example, White and Watts (1977) employed the chi-square test to see if two groups of early phase (1953-60) and intermediate phase (1961-70) of broiler producing plants in the East Midlands differed according to the frequencies of their types of ownership (corporate organisation or individually owned). To conduct the test, sample items are cross-classified according to their group membership and levels of the criterion measured. The resultant frequencies constitute the contingency table.

In the present monograph, the test is illustrated by reference to Herbert,s (1976) socio-geographical study of delinquency areas in Cardiff. Areas of relatively high juvenile crime delinquency were identified from data collected from the police, social services and probation office. Delinquency areas were to be compared with areas of similar general characteristics but with low delinquency rates (called ,non-delinquent, areas). The research strategy required that sampled individuals in delinquent and non-delinquent areas should have equal amounts of awareness of their neighbourhoods. A social survey produced frequencies of varying levels of knowledge of 600 respondents for each of three delinquent and non-delinquent areas. These are reported in the 3X2 contingency table in Table 8A. (To conduct this test, it is not necessary for the sample sizes to be equal).

The null hypothesis is that the proportion of delinquency areas that are "very well known" equals the proportion of non-delinquency areas that are "very well known", that the proportion of delinquency areas that are "quite well known" equals the proportion of non-delinquent areas that are "quite well known" etc. In effect, H_o is that individuals, levels of knowledge are independent of the type of neighbourhood.

Table 8 The chi-square test awned la levels a knowledge af neighbourhoods in Cardiff

(A) Original frequencies

How well neighbourhood is known	Neighbourhood: Delinquent	Non-delinquent	Total
Very well	126	108	234
Quite well	121	148	269
Not well	53	44	97
Total	300	300	600

Source: Herbert, 1976, p.482

(B) Expected frequencies under H_0

How well neighbourhood is known	Neighbourhood: Delinquent	Non-delinquent	Total
Very well	117	117	234
Quite well	134.5	134.5	269
Not well	48.5	48.5	97
Total	300	300	600

(C) Computation of the chi-square statistic

O_i	E_i	$(O_i - E_i)$	$\frac{(O_i - E_i)^2}{E_i}$
126	117	9	0.692
121	134.5	-13.5	1.355
53	48.5	4.5	0.418
108	117	-9	0.692
148	134.5	13.5	1.355
44	48.5	-4.5	0.418
			χ^2 = 4.930

Under the, null hypothesis of independence and by the multiplication law of probabilities, consider one particular cell in the contingency table (Table 8A):

P(area is delinquent and very well known)

= P(area is delinquent).P(area is very well known)

$$= \frac{300}{600} \times \frac{234}{600}$$

Thus, under H_o, the expected number of delinquent areas that are "very well known" is:

$$E(\text{area is delinquent and very well known}) = 600 \times \frac{300}{600} \times \frac{234}{600} = 117.$$

Expected frequencies under H_o for the remaining five cells are computed in a similar manner. Note, however, that to simplify the arithmetic, the expected frequencies may be computed as the product of the row and column totals associated with a cell (300 X 234) divided by the overall sample size (600). The expected frequencies under H_o are shown in Table 8B.

The closeness of the observed (O_i) and expected frequencies (E_i) is tested by the chi-square statistic in the manner described in Section II ii. The computations are shown in Table 8C. the larger is X^2, the more likely it is that the levels of knowledge and the types of neighbourhood are not independent. Given a contingency table with r rows and c columns, the number of degrees of freedom associated with the chi-square statistic is always:

$$(r - 1).(c - 1)$$

Our result of 4.93 is thus distributed as chi-square with df = 2. From Appendix 3, we find that $P(\chi^2_2 > 5.991) = 0.05$. Our result is thus not significant. We conclude that levels of knowledge and neighbourhood type are independent and that there is no significant difference between levels of knowledge of delinquency and non-delinquency areas.

When df > 1, the X^2 test for contingency tables should only be used if no cell has an expected frequency of less than 1 and less than 20% of the cells have an expected frequency of less than 5. Should either of these requirements not be met, it is necessary to combine adjacent categories (Cochran, 1954).

In the case of a 2X2 contingency table (a = 1), there is a numerically simpler and equivalent method for computing the chi-square statistic. Given a 2X2 table in which A, B, C and D are observed frequencies of the form:

	Group 1	Group 2
Class 1	A	B
Class 2	C	D

it may be shown that:

$$\chi^2 = \frac{N(|A.D - B.C|)^2}{(A+B)\ (C+D)(A+C)(B+D)} \quad ......(4),$$

whereN=A+B+C+D and IA.D - B.CI is the modulus notation (see Appendix 1). Equation (4) avoids computing expected frequencies and in the 212 case yields an identical result to the formula for X^2 already discussed. Some authors advocate the incorporation of Yates, continuity correction (N/2) in equation (4). However, this is overly conservative and equation (4) is preferable (Pearson, 1947; Conover, 1974). In the 2X2 case, equation (4) may be used if $N \geq 20$ and all the expected frequencies are 5 or more. If these criteria are not met, the 2X2 contingency table should be analysed by Fisher,s exact probability test, as is discussed at the beginning of the next subsection.

Both approaches to the computation of X^2 for a 2X2 contingency table are illustrated using a small part of the data collected by Dean and James (1981) in their study of schizophrenia in Plymouth. They argued that an examination of case notes *was* a necessary extension to the study of areal differences in the degree to which hospital admission occurs in the management of schizophrenia. Table 9A presents a breakdown of male and female admissions in relation to whether psychiatrists perceived that domestic circumstances were relevant to the decision to admit a patient. The null hypothesis is that perceived importance of domestic circumstances and the sex of the admitted patient are independent. Tables 9B and 9C compute the chi-square statistic in the manner described at the start of this section. From Appendix 3, $P(\chi^2_1 > 3.841) = 0.05$. We thus reject H_0 and conclude that perceptions of the importance of domestic circumstances depends on the patient,s sex. The largest values in column 4 of Table 9C indicate that domestic circumstances are perceived as being important in the decision to admit significantly less men and more women than would be expected by chance. Applying equation (4), we

Table 9 The perceived importance of domestic circumstances and

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(A) Original frequencies

Importance of domestic circumstances	Males	Females	Total
Unimportant	69 (A)	95 (B)	164
Important	21 (C)	57 (D)	78
Total	90	152	242

Source: Dean and James, 1981, p.48

(B) Expected frequencies under H_0

Importance of domestic circumstances	Males	Females	Total
Unimportant	60.99	103.01	164
Important	29.01	48.99	78
Total	90	152	242

(C) Computation of the chi-square statistic

O_i	E_i	$(O_i - E_i)$	$\frac{(O_i - E_i)^2}{E_i}$
69	60.99	8.01	1.052
21	29.01	-8.01	2.212
95	103.01	-8.01	0.623
57	48.99	8.01	1.310
			$\chi^2 = 5.197$

derive the same value of the test statistic, allowing for decimal rounding error:

$$\chi^2 = \frac{242.\{|(69)(57) - (95)(21)|\}^2}{164 \times 78 \times 90 \times 152} = \frac{242(1938)^2}{12792 \times 13680} = 5.194$$

A final point is that the contingency procedure may be readily extended to k independent samples. Thus, for example, a 3X3 contingency table could be constructed by considering the attitudes (favourable, indifferent, unfavourable) towards an urban renewal scheme of three groups of residents (young, middle aged, elderly), in an attempt to determine whether attitudes depend on age. The degrees of freedom are still computed by df. = (r - 1).(c - 1).

(ii) Fisher,s exact test

Fisher,s test is used to determine if two groups differ in the proportion with which they fall into two categories. It is a particularly useful nonparametric technique for examining discrete ordinal or nominal data when two independent samples are small in size. This section describes the application of Fisher,s test to an examination of whether two independent samples evidence a difference in central tendency, in this case the median. Some authors refer to this application under another name - the median test (Siegel, 1956; Conover, 1980). The data in Table 10 are used to illustrate the test. Field measurements of soil pH were carried out near the rims and at the bottoms of hollows in the Dorset heathlands (Sperling et al., 1977). If chemical solution leads to progressive deepening of the hollows, pH values at the bottoms would be expected to be less than those on the rims. The null hypothesis of no difference in the median pH soil contents at the rims and bottoms of the hollows is set up. The one-tailed alternative is that the median pH value at the *rims* exceeds that at the bottom. The median score for all pH values in both samples may be derived from their ordered values, shown in Table 10B. The median of 16 readings is the mean of the 8th and 9th readings, namely 3.7. All 16 readings may now be grouped with respect to the overall median, as shown in Table 10c.

Under H_o of no difference, we would expect the frequencies lying above and below the overall median to be equal for both samples. The probability, p, of observing a particular set of frequencies in a 2X2 table (regarding the marginal totals as

Table 10 Fisher,s test applied to pH soil determination al the rims and bottoms of hollows in Dorset heathlands

(A) pH soil determinations at the rim and bottom of hollows

pH value Rim	Bottom
3.8	4.0
3.6	-
4.7	3.7
3.8	3.1
3.8	3.7
3.2	3.2
3.8	4.0
3.4	3.0
3.3	

Source: Sperling et al., 1977, p.215

(B) Ordered pH determinations

3.0 3.1 3.2 3.2 3.3 3.4 3.6 3.7 3.7 3.8 3.8 3.8 3.8 4.0 4.0 4.7

(C) Fisher's test: form for the data

	Pos'n in hollow Rim	Bottom	Total
No. of values exceeding overall median	5(A)	2(B)	7(A+B)
No. of values not exceeding overall median	4(C)	5(D)	9(C+D)
Total	9(A+C)	7(B+D)	16(N)

fixed) is given by the hypergeometric distribution. Using the notation of Table 10C, p is given by:

$$p = \frac{{}^{A+C}C_A \cdot {}^{B+D}C_B}{{}^{N}C_{A+B}}, \text{ where } {}^{X}C_Y = \frac{X!}{Y!(X - Y)!}.$$

Here,

$$p = \frac{{}^{9}C_5 \cdot {}^{7}C_2}{{}^{16}C_7} = \frac{(126)(21)}{11440} = 0.231.$$

Table 10 (continued)

(D) More extreme frequency occurrences than those obtained in Table 10C

(i)

	Pos'n in hollow		
	Rim	Bottom	Total
No. of values exceeding overall median	6	1	7
No. of values not exceeding overall median	3	6	9
Total	9	7	16

$$p = \frac{{}^{9}C_{8} \cdot {}^{7}C_{1}}{{}^{16}C_{7}} = \frac{(84)(7)}{11440} = 0.051$$

(ii)

	Pos'n in hollow		
	Rim	Bottom	Total
No. of values exceeding overall median	7	0	7
No. of values not exceeding overall median	2	7	9
Total	9	7	16

$$p = \frac{{}^{9}C_{7} \cdot {}^{7}C_{0}}{{}^{16}C_{7}} = \frac{(36)(1)}{11440} = 0.003$$

We have thus computed that the probability of such a distribution of frequencies in Table 10C is 0.231. However, more extreme deviations from the distribution under H_o could occur with the same marginal totals. This should be considered because a statistical test of H_o asks "what is the probability under H_o of such an occurrence or one even more extreme?" (Siegel, 1956, p. 98). More extreme occurrences would be those shown in Table 10D. The probability of these two frequency distributions occurring under H_o may be computed as before to be 0.051 and 0.003 as shown in this table. Hence the probability of occurrence of the frequency distribution in Table 10C or more extreme occurrences is

0.231 + 0.051 + 0.003 = 0.285, under H_o. This value is greater than the conventional significance level of $\alpha = 0.05$, so we fail to reject H_o and conclude that the median pH soil determination at the rims and bottoms of these hollows are equal. This does not support the notion that higher rates of chemical solution lead to progressive deepening of the hollows.

Consideration of the more extreme occurrences above was obviously unnecessary in the present example, as the probability of occurrence of the frequency distribution in Table 10C under H_o already exceeded cc 0.05. However, when such computations are required, they can become tedious. For larger samples, ($N = n_i + n_2 >=$ be analysed by the contingency chi-square statistic, represented by equation (4) and discussed in the last section. If, however, any cell has an expected frequency less than 5, the procedure just outlined involving the hypergeometric distribution should be used. The large scale application of Fisher,s test is illustrated by comparing urban population densities in a zone immediately surrounding the central business district with those densities found in more peripheral zones. Everson and Fitzgerald (1973, p.120-1) compiled data pertaining to population densities (excluding agricultural land) per acre in the urban wards in the city of Norwich. Random samples of $n_1 = 11$ wards between 0.5 and 1.5 Kms. from the city centre and $n_2 = 13$ wards between 3 and 5 Kms. from the centre were drawn. The densities in these N = 24 wards are reported in Table 11A.

The null hypothesis is that there is no difference between the median population densities of these two zones. Well known geographical theory would postulate that the densities would be higher in the zone nearer the CBD. We proceed as before and find that the median score for the combined samples is 11.93 persons per acre. This enables construction of Table 11B, which is similar in construction to Table 10C. From equation (4), the appropriate test statistic is:

$$\frac{N(|A.D - B.C|)^2}{(A + B)(C + D)(A + C)(B + D)}$$

Table 11 Fisher's test applied population densities and varying distances from the city centre of Norwich

(A) Population densities per acre

Population densities:	
.5 to 1.5 Kms from city centre	3 to 5 Kms from city centre
4.59	0.22
10.86	0.71
11.10	2.31
14.19	3.65
16.39	4.15
19.27	4.23
22.30	5.80
26.15	7.29
28.35	11.07
31.44	12.76
48.74	18.56
	23.26
	25.93

Source: Everson and Fitzgerald, 1973, p.120-1

(B) Fisher's test: form for the data

	Distance from city centre		
	.5 to 1.5 Kms	3 to 5 Kms	Total
No. of densities exceeding overall median	8 (A)	4 (B)	12
No. of densities not exceeding overall median	3 (C)	9 (D)	12
Total	11	13	24

where N is the total number of readings in both samples. This is the test statistic described in the previous section for a 2X2 contingency table. This test statistic is distributed as X^2, with one degree of freedom. Therefore,

$$\frac{24(|72 - 12|)^2}{12.12.11.13} = 4.196 \in \chi^2_1$$

From Appendix 3, $P(\chi^2_1 > 3.841) = 0.05$, so we reject H_0 and conclude that the median population density/acre is higher in the

zone nearer the CBD. This procedure may be extended to k independent samples and then the test statistic would be gathered from the conventional chi-square analysis of the 2Xk contingency table formed by dichotomising the data with respect to the overall median.

(iii) The Mann-Whitney test

The Mann-Whitney test examines if two independent samples have been drawn from the same population or populations with the same distribution. When the latter is the null hypothesis, failure to reject H_0 would infer that populations are not significantly different in respect of central tendency. For this reason, many authors feel that the Mann-Whitney test is a useful alternative to the parametric t-test when the assumptions underlying the latter are not met. However, care should be taken when one fails to reject H_0, for although it can be asserted that the two samples are from different populations, it may not be straightforward to say in what specific way(s) the populations differ.

Table 12 Segregation indices (SI) in 10 selected American SMSA, 1970

Southern U.S.A.			Northern U.S.A.		
SMSA	SI (X)	Rank	SMSA	SI (Y)	Rank
Atlanta	0.51	6	New York	0.29	1
Birmingham	0.30	2	Detroit	0.58	8
Greensboro-Winston-Salem	0.40	3	Chicago	0.45	5
			Philadelphia	0.44	4
St. Louis	0.57	7	Pittsburgh	0.59	9
			Boston	0.66	10
Sum of ranks		$R_x = 18$			$R_y = 37$

Source: Glantz and Delaney, 1973, p.6

The test is illustrated with reference to residential segregation of blacks in selected American Standard Metropolitan Statistical Areas (SMSA) in 1970. Glantz and Delaney (1973) computed segregation indices (SI) for the SMSA; the higher the value of the index, the greater is the degree of residential concentration. Indices for 10 SMSA in northern and southern U.S.A. are reported in Table 12. The null hypothesis is that the SI values in both regions have the same distribution, against a two-tailed alternative that they have not. Failure to reject H_o would suggest that the northern and southern SI values do not significantly differ.

The inherent logic of the Mann-Whitney test is that if the sample evidence favours the alternative hypothesis, then a majority of one of the populations of SI values will exceed a majority of the SI values in the other population. As shown in Table 12, the first step is to rank all the combined sample items from highest to lowest. We next arbitrarily select one of the samples. Taking each value in the selected sample in turn, we count the number of items in the other sample that have higher ranks. The total of this count is usually denoted by U. Ties between items contribute 0.5 to the value of U. Selecting the southern SMSA sample in Table 12, we find that three Y scores have higher ranks than the X score of 0.51, five Y scores have higher ranks than has X = 0.30, five Y scores have higher ranks than has X = 0.40 and three Y scores have higher ranks than has X = 0.57. We thus compute that U = 3 + 5 + 5 + 3 = 16. A quicker method of computing U is given by:

$$U = n_x n_y + (0.5)n_x(n_x + 1) - R_x \ldots\ldots (5),$$

where n_x is the number of readings in the arbitrarily selected sample, R_x is the sum of the ranks given to the items in that sample and n_y is the size of the other sample. From Table 12, $n_x = 4$, $n_y = 6$ and $R_x = 18$, so $U = (4)(6) + (0.5)(4)(5) - 18 = 16$.

Suppose that we had carried out the same procedure, but had initially arbitrarily selected the northern sample. In this instance, it would be found that U = 4 + 0 + 2 + 2 + 0 = 8. Either U or U' could be used as the test statistic. Now under H_o that the samples were drawn from populations with the same distributions, one would expect by chance that for each of the n_x items in the arbitrarily selected sample, $n_y/2$ items in the other

sample would have higher ranks. Thus under H_o, the expected value of U, namely E(U), would have value $(n_x n_y)/2$. In the present example, $E(U) = (4)(6)/2 = 12$. For the two-tailed alternative hypothesis, we could thus examine the computed U = 16 in relation to the upper critical value or U'= 8 in relation to the lower critical region of the distribution of U with mean (or expected value) of 12. Appendix 7 presents critical values of the U statistic, but only for the upper critical region. It is not necessary, however, to compute both U and U' to see which is numerically larger, for they are related by:

$$U = n_x n_y - U' \text{ or } U' = n_x n_y - U,$$

as may be readily verified in the present example. In Appendix 7, the values given should be included within the critical region, so in the present example, and selecting a = 0.05, we reject H_o if U 22. Hence the value of U = 16 is not significant and we conclude that the segregation indices in the northern and southern SMSA are equal.

When either n or n are greater than 20, a large scale approximation may be used. In this circumstance, we use the facts that under H_o, U is approximately normally distributed with parameters $E(U) = n_x n_y/2$ and $var(U) = n_x n_y(N + 1)/12$ where $N = n_x + n_y$ (Gibbons, 1971). The application of this procedure is illustrated in Table 13, which shows the percentages of total cropland in the 19 most western and the 24 most eastern counties of Ohio given over to wheat cultivation in 1949 (King, 1969). The null hypothesis is that the distributions of the percentages of land given over to wheat are equal, against the alternative hypothesis that they are not. As before, the combined data are ranked from lowest to highest in this table. Now by equation (5):

$$U = (19)(24) + (0.5)(24)(25) - 386 = 370.$$

Therefore, standardising and employing a continuity correction because U is discrete, the test statistic under H_o is:

$$\frac{(U \pm 0.5) - E(U)}{\sqrt{var(U)}} \in N(0,1), \text{ so}$$

Table 13 percentage 2f harvested cropland given to wheat cultivation j west and east Ohio. 1949

Western counties			Eastern counties		
County	Wheat %(X)	Rank	County	Wheat %(Y)	Rank
Williams	21.76	29	Ashtabula	13.52	9
Defiance	22.05	32	Trumbull	16.53	14
Paulding	20.04	22	Mahoning	20.13	24
Van West	18.55	17	Columbiana	19.22	20
Mercer	20.48	25	Jefferson	15.61	13
Darke	21.09	28	Belmont	10.87	4
Preble	26.77	42	Monroe	9.75	2
Butler	24.76	37	Washington	13.43	8
Hamilton	13.86	10	Lake	9.77	3
Fulton	20.97	27	Geauga	12.68	6
Henry	20.09	23	Portage	19.13	19
Putnam	24.16	36	Stark	22.53	34
Allen	23.76	35	Carroll	17.73	16
Auglaze	22.02	31	Harrison	13.00	7
Shelby	22.33	33	Guernsey	14.14	11
Miami	26.59	41	Noble	9.24	1
Montgomery	25.88	39	Morgan	14.94	12
Warren	24.97	38	Muskinghum	19.58	21
Clermont	16.58	15	Coshacton	21.96	30
		$R_x = 560$	Holmes	26.38	40
			Wayne	27.10	43
			Summit	19.08	18
			Medina	20.86	26
			Cuyahoga	12.52	5
					$R_r = 386$

Source: King, 1969, p.175-6

$$\frac{(370 - 0.5) - (19)(24)/2}{\sqrt{(19)(24)(44)/12}} = \frac{141.5}{\sqrt{1672}} = 3.46 \in N(0,1).$$

Unless ties are very extensive, no adjustment to this test statistic is necessary (Siegel, 1956). It should be noted that we could compute U' as before, by $U' = n_x n_y - U = (19)(24) - 370 = 86$ and $E(U')$ and $var(U')$ are equal to $E(U)$ and $var(U)$ respectively. The test statistic under H_o will be found to have value -3.46 e N(0,1) - the same in magnitude but different in sign to the result found previously. This illustrates that the arbitrary selection of the X or Y readings as a basis for the computation of U or U' in

no way affects the level of significance of the derived test statistic. In this problem given a two-tailed test, we find from Appendix 1 that P[N(0,1| > 1.961 = 0.025 so our result is significant. We conclude that the percentages of cropland given over to wheat in the western and eastern counties of Ohio have been drawn from different populations.

Under the null hypothesis, the probability, p, of the percentage cropland devoted to wheat in a western county exceeding that in an eastern county is one half. In symbols, P((> Y) = p = 0.5. Having rejected H_o, it may be shown that an unbiassed and consistent estimator of p, written as $\hat{p}$, is given by $\hat{p} = U/n_x n_y$ (Gibbons, 1971). In the present example, $\hat{p} = 370/(19)(24) = 0.811$. Thus we estimate that 81.1% of the western counties exceed eastern counties in terms of the percentage of harvested cropland given over to wheat.

(iv) Discussion

In that the adaption of Fisher's test to the analysis of population medians does not require the populations to be identical when H_o is true, the test may be applied in situations where the Mann-Whitney test is invalid. The ARE of the median test relative to the t-test when the populations are in fact normal is 63.7%, which is a relatively poor level (Gibbons, 1971), but thereagain, nonparametric tests are specifically designed to analyse samples from non-normal populations. The ARE of the Mann-Whitney test when compared with the t-test is computed under the assumption that the distributions of X and Y are identical except for their means. If the populations are normal, the ARE is 95.5%, if they are uniform it is 100% and if the populations have a symmetric non-normal distribution the ARE is 150%. The lowest value of the ARE if the populations only differ in location is never lower than 86.4% and it can be infinite. In the cases of normal and uniform populations, the ARE's of the Mann-Whitney test relative to the median test are above 100%, but below 100% for symmetric non-normal populations and large samples.

The chi-square test as described in this section is used if the objective is to see if the populations differ in any respect location, dispersion, skewness etc. When N < 20 and the data in a 2X2 contingency table, Fisher,s test should be used. The X^2 test has been criticised in that it does not make efficient use of all the properties of the data and it is recommended that if the populations are continuously distributed then the Kolmogorov-Smirnov test for two independent samples (discussed at the end of Section II iii) should be employed.

V A TEST FOR k RELATED SAMPLES

This section presents the Friedman test which examines differences among three or more related samples. It is useful when measurement is at least an ordinal scale and does have the advantage over other applicable tests of having tables of exact probabilities for significance available for very small samples. The application of such a procedure for an overall difference is often a precursor to trying to identify the populations between which the greatest differences occur. This is called a multiple comparisons procedure. If the researcher is able to set up an alternative hypothesis in which ordering between the populations is possible (e.g. a temporal increase or decrease), then Page,s (1963) adaption of the Friedman test is applicable. Both procedures are described for the Friedman test in this section. They have also been illustrated in an analysis of crude birth rates over time of several less developed countries (Coshall, 1988).

Friedman,s test parallels conventional parametric analysis of variance, wherein the test statistic is F. However, as noted in the introduction to this monograph, the F-test is particularly sensitive to departures from normality (Box, 1953; Siegel and Tukey, 1960; Conover, 1980). If the normality assumption is untenable, the Friedman test is a safer way of seeking significant differences between populations among k related samples.

(i) I Friedman test

If the data from k related samples are at least ordinal, this test examines if the k samples could have been drawn from the same population. The test is illustrated by means of data in Table 14A, which shows male employment change (000,s) in service industries in Great Britain, 1961-80 (Daniels, 1983).

In Table 14A, there are k = 3 samples being analysed over n = 6 blocks. The null hypothesis is that the employment levels in service industries in each of the three years tabulated are equal, against the alternative that they are not. If one hypothesised that the employment levels decreased from 1961 to 1980, this would

Table 14 The service industries in Great Britain. 1961-80

(A) Employment

Sector	Employment (000's) 1961	1971	1980
Transport/communications	1438	1307	121[illegible]
Distributive trades	1340	1180	124[illegible]
Insurance/banking/finance	377	479	58[illegible]
Professional/scientific services	737	1002	117[illegible]
Miscellaneous services	819	893	106[illegible]
Public administration/defence	916	996	971

Source: Daniels, 1983, p.303

(B) General form of the data for the Friedman test

Blocks	Treatments 1	2	.....	k	Row totals
1	r_{11}	r_{12}	.....	r_{1k}	$k(k + 1)/2$
2	r_{21}	r_{22}	.....	r_{2k}	$k(k + 1)/2$
.	.	.	.....	.	.
.	.	.	.....	.	.
n	r_{n1}	r_{n2}	.....	r_{nk}	$k(k + 1)/2$
Totals	R_1	R_2	.....	R_k	$nk(k + 1)/2$

(C) The Friedman test: form for the data from Table 14A

Sector	Ranks of employment levels 1961	1971	1980
Transport/communications	3	2	1
Distributive trades	3	1	2
Insurance/banking/finance	1	2	3
Professional/scientific services	1	2	3
Miscellaneous services	1	2	3
Public administration/defence	1	3	2
Totals R_j, j = 1, 2, 3	10	12	14

be an ordered alternative hypothesis, to be discussed later in this section.

Within each block, the k observations are ranked from least to greatest. If the employment figure in the ith. block under the jth. condition is X_{ij}, then let r_{ij} be the rank of X_{ij} in the joint ranking of X_{i1}, X_{12} X_{ik}. Further, we define $R_j = \sum_{i=1}^{n} r_{ij}$. **The general form of the data for analysis by the Friedman test is shown in Table 14B and the numerical values of R_j in the** present example are shown in Table 14C. If the null hypothesis of no differences is true, then the ranks r_{ij} = 1, 2, 3 would be expected to appear in all the columns with about the same frequency. Thus the column totals would be the same and equal to the mean column total, $n(k + 1)/2$, from Table 14B. The sum of the deviations of the observed column totals about this mean is zero, but the sum of squares of these deviations would be indicative of differences between the k samples (Gibbons, 1971). Therefore, we could employ:

$$T = \sum_{j=1}^{k} \{R_j - n(k + 1)/2\}^2$$

as a test statistic under H_0, as it would be sensitive to differences in locations. From Table 14C, it is found that $n(k + 1)/2 = 12$, so:

$$T = (10 - 12)^2 + (12 - 12)^2 + (14 - 12)^2 = 8.$$

Unfortunately, statistical tables for the significance of T, such as those of Owen (1962), are restricted in the values of n and k for which values are reported.

However, tables are presented in Appendix 8 for the statistic:

$$S = \frac{12T}{nk(k + 1)} \text{ ......(6).}$$

In the present example, $S = (12)(8)/(18)(4) = 1.333$. From Appendix 8, $P(S \geq 1.333) = 0.57$, so we fail to reject H_0 and conclude that male employment levels in each of the three years are equal. It might be noted that for female employment levels in the same service industries (Daniels, 1983), it is found that S =

12 and from Appendix 8, P(S

significant differences in the three years (in fact, a generally decreasing trend over time).

It has been shown that the computation of S as presented in equation (6) may be derived from the more often quoted and equivalent formula (Gibbons, 1971):

$$S = \frac{12}{nk(k+1)} \sum_{j=1}^{k} R_j^2 - 3n(k+1) \text{ ......(7).}$$

Equation (7) is more convenient for large data matrices than equation (6). Thus from Table 14C:

$$S = [12(100 + 144 + 196)/18(4)] - 3(6)(4) = 1.333.$$

The tables in Appendix 8 are somewhat restricted in the situations to which they are applicable. For larger samples than those recorded in this table, S is approximately distributed as chi-square with (k - 1) degrees of freedom.

The data in Table 15 are used to illustrate the large scale approximation. This table presents employment (000,s) by industry in Great Britain 1977-81 (Daniels, 1983). The ranked values across each treatment are shown in brackets, along with their totals R for j = 1, 2, 5 From equation (7) it is computed that:

$$S = [12(13^2 + 30^2 + .. + 24^2)/9(5)(6)] - 3(9)(6) = 16.1,$$

which is approximately distributed as χ^2 with 4 degrees of freedom. From Appendix 3, $P(\chi_4^2 > 9.488) = 0.05$, so our result is highly significant and we conclude that employment levels differ over the five years.

(ii) The ordered alternative hypothesis

Often there is sufficient *a priori* evidence to adopt an ordered alternative hypothesis, for example, that the employment levels for the industries in Table 15 have been decreasing annually between 1977 and 1981. In this situation, Friedman,s test may be adapted by a method suggested by Page (1963). The null hypothesis of no difference remains the same. Let the unknown treatment effects be denoted by t , j = 1, 2 5 then

the ordered alternative above may be written as:

$$H_A: t_1 \le t_2 \le t_3 \le t_4 \le t_5.$$

We use the sum of the ranks R_j, j = 1, 2,.....,5 from Table 15 to compute Page's L statistic, given by:

$$L = \sum_{j=1}^{k} j(R_j).$$

Hence, from Table 15 with k = 5 treatments, $L = R_1 + 2R_2 + 3R_3 + 4R_4 + 5R_5$ = 13 + 2(30) + 3(39) + 4(29) + 5(24) = 426. Tables for the significance of L are presented in Appendix 9, whence it may be found that with n = 9 and k = 5, P(L = 431) = 0.05. Our observed value of L = 426 is not significant and we cannot accept the ordered alternative hypothesis. For values of n and k outside the table in Appendix 9, we may use the fact that the statistic:

$$L' = \frac{L - E(L)}{\sqrt{var(L)}} \text{ ......(8)}$$

is approximately standard normally distributed with parameters:

$$E(L) = \frac{nk}{4}(k+1)^2 \text{ and } var(L) = \frac{n(k^3 - k)^2}{144(k-1)}.$$

In the present example, E(L) = 9(5)(36)/4 = 405 and var(L) = $9(120)^2/4(144)$ = 225, so in equation (8), L = (426 - 405)/15 = 1.4, which again is not significant at the a = 0.05 level (Appendix 2).

It is possible to compute Spearman,s rank correlation coefficient, r_s, between the rank ordering expected under H_A and the ordering observed for each industry in Table 15. This would be indicative of the strength of agreement between each observed ordering and that ordering expected by the ordered alternative hypothesis. Spearman,s r_s is given by:

$$r_s = 1 - \frac{6\Sigma d_i^2}{k(k^2 - 1)} \text{ ......(9),}$$

where k is the number of pairs of readings and d_i, i = 1, 2 5 are the differences in the ranks of the pairs. For example, to compute r_s between the ranking expected under H_A and

Table 15 Employment by industry in Great Britain, 1977-81

	1981	1980	1979	1978	1977
Manufacturing	5917(1)	6633(2)	7067(3)	7144(4)	7185(5)
Construction	1077(1)	1219(2)	1262(5)	1234(4)	1223(3)
Gas/elect/water	330(1)	340(5)	338(4)	335(2)	337(3)
Transport/comm'ns	1417(1)	1475(4)	1485(5)	1472(3)	1455(2)
Distributive trades	2576(1)	2685(2)	2780(5)	2738(4)	2706(3)
Insurance/bank/fin	1220(3)	1254(5)	1236(4)	1201(2)	1159(1)
Prof'nal/scientific	3532(2)	3556(4)	3573(5)	3551(3)	3506(1)
Miscellaneous serv's	2350(2)	2440(4)	2441(5)	2372(3)	2317(1)
Public admin/defence	1523(1)	1543(2)	1560(3)	1561(4)	1564(5)
R_j j = 1, 2,....,5	13	30	39	29	24

Source: Daniels, 1983, p.302

the ranking observed in the construction industry, examine Table 16. It is thus derived:

Rank under H_A	1	2	3	4	5	
Rank for construction ind.	1	2	5	4	3	
d_i	0	0	-2	0	2	
d_i^2	0	0	4	0	4	$\Sigma d_i^2 = 8.$

Thus in equation (9), $r_s = 1 - 6(8)/5(24) = 0.6$, as shown in the right hand column of Table 16. The average of all these nine rank coefficients is $\bar{r}_s = 2.1/9 = 0.2333$. This figure is indicative of the overall weakness or agreement between the expected ordering under H_A and that which is observed for all nine industries. Hollander and Wolfe (1973) have shown that:

$$\bar{r}_s = \frac{12L}{nk^2(k-1)} - \frac{3(k+1)}{k-1},$$

where L is Page,s statistic. Thus in the present example, using the value of L = 426 derived previously:

$$\bar{r}_s = 12(426)/45(24) - 3(6)/4 = 0.2333, \text{ as above.}$$

Having computed $\bar{r}_s$, it is thereby possible to derive Page's L and vice versa. As was concluded by examination of Page's L, $\bar{r}_s$ = 0.2333 evidences only slight agreement in the observed ordering for the nine industries and that expected under H_A.

Table 16 Bank correlation between the hypothetical ordered alternative and the observed orderings

	Employment year					
	1981	1980	1979	1978	1977	
Ordering expected under H_A	1	2	3	4	5	
	Observed ordering					
Industry	1981	1980	1979	1978	1977	r_s
Manufacturing	1	2	3	4	5	1.0
Construction	1	2	5	4	3	0.6
Gas/elect/water	1	5	4	2	3	0.1
Transport/comm'ns	1	4	5	3	2	0.1
Distributive trades	1	2	5	4	3	0.6
Insurance/bank/fin	3	5	4	2	1	-0.7
Prof'nal/scientific	2	4	5	3	1	-0.3
Miscellaneous serv's	2	4	5	3	1	-0.3
Public admin/defence	1	2	3	4	5	1.0
					Σr_s =	2.1

Source: Table 15

(iii) Multiple comparisons

Originally, the null hypothesis of no differences in employment levels for the five years in question was rejected in favour of the alternative that one or more of these employment levels are not equal. It is possible to use the procedure of multiple comparisons to identify between which years the greatest differences lie.

The first step is to compute the k(k - 1)/2 absolute differences $|R_u - R_v|$, u < v, where the R_j, j = 1, 2,.....,k are found from Table 15. For example, $|R_1 - R_2| = |13 - 30| = 17$, $|R_1 - R_3| = |13 - 39| = 26$ etc. These absolute differences are shown in Table 17. The significance of these differences depends on the experimentwise error rate (EER), α, involved in the multiple comparisons. Suppose we have to make N_d individual decisions during the multiple comparisons. Let N_f be the number of incorrect decisions made. The error rate for the multiple comparisons procedure is the random variable N_f/N_d. The EER is the probability that under H_0, N_f/N_d is greater than zero; in symbols $P(N_f/N_d > 0) = \alpha$. If the unknown effects of treatments U and V are denoted by t_u and t_v, then it is decided that $t_u \neq t_v$ if:

$$|R_u - R_v| \geq r(\alpha, k, n),$$

where α is the EER and n and k are as before. Statistical tables for $r(\alpha, k, n)$ are presented in Appendix 10. Selecting $\alpha = 0.05$ and with $n = 9$ and $k = 5$ from Table 15, the closest we may get to this EER from Appendix 10 is $P[r(\alpha, 5, 9) > 19] = 0.037$. At an EER of 0.01, the closest we may get is $P[r(\alpha, 5, 9) > 22] = 0.008$. From Table 17, only $|R_1 - R_3| > 22$. Therefore, the principal reason we rejected the null hypothesis of no difference in employment levels over the five years is due to the significant difference in employment levels of 1979 and 1981, as indicated in Tables 15 and 17.

For large samples beyond the scope of Appendix 10, we decide that $t_u \neq t_v$ if:

Table 17 The differences $|R_u - R_v|$ in the Friedman test

V	U: 1	2	3	4	5
1	-				
2	17*	-			
3	26	9	-		
4	16	1	10	-	
5	11	6	15	5	-

Source: Table 15

*Significant at $p < 0.01$

$$|R_u - R_v| \geq q(\alpha, k)\sqrt{\frac{nk(k+1)}{12}} \quad ......(10),$$

where a is again the EER, and q(a, k) is the upper a percentile of the range of k independent N(0,1) variables and which is presented in Appendix 11. From Appendix 11, it is found that q(0.05, 5) = 3.858. Hence the right side of the inequality (10) becomes:

$$3.858\sqrt{(45)(6)/12} = 18.3.$$

As before, only $|R_1 - R_3|$ exceeds this value. Adopting $\alpha = 0.01$, we find from Appendix 11 that $q(0.01, 5) = 4.603$, so the right side of equation (10) is $4.603\sqrt{(45)(6)/12} = 21.8$, therefore the difference in employment levels between 1979 and 1981 is significant at the same level as before.

(iv) Discussion

The ARE of the Friedman test with respect to the parametric F-test in the cases of normal, uniform and double exponential populations for various values of k - the number of treatments - are given in Table 18. The third row of Table 18 is also applicable to the ARE of Page,s L statistic, relative to the parametric F test when the populations are normal (Hollander, 1967). The lower bound of the ARE of the Friedman test is 57.6% and the ARE can be infinite (Hollander and Wolfe, 1973). An alternative to Friedman,s test when measurement is nominal or at a dichotomised ordinal scale is Cochran,s Q statistic. However, if

Table 18 The asymptotic relative efficiency (7.) 2f the Friedman test

Population distribution	No. of treatments, k: 3	4	5	10
Double exponential	112.5	120	125	136.4
Uniform	75	80	83.3	90.9
Normal	71.6	76.4	79.6	86.9

Source: Hollander and Wolfe, 1973, p.183

the data are at least ordinal, Friedman's test is preferable, as it has greater ARE and tables of exact significance are available for small samples.

1a A TEST FOR k INDEPENDENT SAMPLES

The Kruskal-Wallis test is described in this section. It is designed to determine if k independent samples could have been drawn from the same population or from k identical populations. It again parallels conventional parametric analysis of variance and is an alternative to the F-test when the normality assumption is untenable. It is a test based on ranks, requiring at least ordinal measurement. As in the previous section, the methods for ordered alternatives and multiple comparisons are described. The test has been used to examine differences in the chemical concentrations present in water for major rock types (Prowse, 1984). Lewis (1971) used the test to compare the organic content of soil at six inches depth in three location classes; high forests of mixed woodland, oak and pine. The ordered alternatives and multiple comparisons procedures have been illustrated in an analysis of the numbers of retail outlets in a sample of southern English towns of varying population sizes (Coshall, 1988).

(i) Bat Kruskal-Wallis test

This test is based on ranks, as was the Friedman test and it has the null hypothesis that there is no overall difference between the k independent populations. To illustrate the test, data are used from a study of microspatial consumer cognition (Coshall, 1984). Shops that are regularly part of the consumer's comparison shopping have been referred to as the evoked set (Gronhaug, 1973; Coshall, 1985). It has been hypothesised that the size of the buyer,s evoked set may be constrained by psychological and personality-related variables, among others (Potter and Coshall, 1985). In particular, risk perception is such a constraining influence and refers to uncertainty about product requirements and uncertainty as to possible purchase consequences in terms of levels of satisfaction. In the above mentioned study, levels of perceived risk were measured in a questionnaire by 3 five-point scales, which factor analysis showed to be internally consistent. Consumers were grouped into three approximately equal sized classes, reflecting perceptions of low,

moderate and high risk. Also in the questionnaire, the numbers of shops in consumers, evoked sets were gathered. The null hypothesis to be examined by the Kruskal-Wallis test is that the sizes of consumers, evoked sets are equal over the three levels of perceived risk. The alternative is simply that the evoked set sizes differ. The ordered alternative hypothesis is discussed later in this section.

Table 19 presents the evoked set sizes of 22 randomly selected consumers in terms of these individuals, perceptions of risk. Also the evoked set sizes have been ranked from lowest to highest, independently of which sample the item is a member. As in the Friedman test, k is the number of treatments and R , j = 1, 2, ,k represents the sum of the ranks attributed to each of the k treatments. Additionally, define $n_1, n_2, \dots n_k$ to represent the number of blocks in which each of the k treatments is measured. Let the total number of observations be $N = \sum_{j=1}^{k} n_j$.

Under the null hypothesis of no differences, one would expect the ranks to be distributed randomly and evenly throughout the data matrix. In this case, the total sum of the ranks allocated from 1 to N inclusive, namely N(N + 1)/2, would be divided proportionally according to the sample size among the k samples. For the jth. sample, the expected sum of ranks would be:

$$\frac{n_j}{N}.(0.5)N(N + 1) = n_j(0.5)(N + 1).$$

The R , j = 1, 2 k are the actual sums of ranks assigned to the k treatments. Following a similar line of argument as discussed in the Friedman test, a reasonable test statistic could be based on:

$$K = \sum_{j=1}^{k} \left\{R_j - n_j(0.5)(N + 1)\right\}^2.$$

H_o would be rejected for large K.

Table 19 The evoked set sizes of consumers perceiving various levels of risk

Evoked set sizes of consumers perceiving:					
Low risk	Rank	Medium risk	Rank	High risk	Rank
0	3.5	0	3.5	4	20
0	3.5	2	13	4	20
1	9	1	9	0	3.5
3	16.5	3	16.5	3	16.5
1	9	2	13	1	9
2	13			6	22
0	3.5			4	20
0	3.5			3	16.5
1	9				
Sum	70.5		55		127.5

Source: Coshall, 1984

The calculations involved in deriving the significance of K are cumbersome and tedious. Statistical tables for testing K are available for k = 3, 4 and 5, but only with the n equal and very small (Rijkoort, 1952). More practical as a test statistic is a weighted sum of squares of the deviations defined in its simplest form by Kruskal and Wallis (1952) as:

$$H = \frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} - 3(N+1) \text{ ......(11).}$$

When ties are involved in the ranking procedure, H is divided by:

$$1 - \frac{\Sigma T}{N^3 - N} \text{ ......(12),}$$

where $T = t^3 - t$ and t is the number of tied observations in a tied group of scores. As before, $N = \sum_{j=1}^{k} n_j$. From Table 19, 6 readings are tied on evoked set sizes of 0 shops, 5 readings on sizes of 1 shop, 3 readings on 2 shops, 4 readings on 3 shops and 3 readings on 4 shops. Therefore:

$\Sigma T = (6^3 - 6) + (5^3 - 5) + (3^3 - 3) + (4^3 - 4) + (3^3 - 3) = 438.$

Thus from equations (11) and (12):

$$H = \left\{1 - \frac{438}{22^3 - 22}\right\}^{-1} \left[\frac{12}{(22)(23)}\left(\frac{70.5^2}{9} + \frac{55^2}{5} + \frac{127.5^2}{8}\right) - 3(23)\right],$$

$H = (0.9588)^{-1}[75.635 - 69] = 6.92.$

Tables for the significance of H for k = 3 and n up to 5 for j = 1, 2, 3 are presented in Appendix 12. In other cases, we use the fact that for larger n , H is approximately distributed as chi-squared with (k - 1) degrees of freedom. Therefore, in the present example and from Appendix 3 with 2 degrees of freedom, we find that $P(\chi_2^2 > 5.991) = 0.05$, so the value of H = 6.92 is significant. We reject H_0 and conclude that the evoked set sizes of consumers shopping under perceptions of low, moderate and high risk are different.

(ii) The ordered alternative hypothesis

It is possible to test H_0 against an ordered alternative using Jonckheere,s (1954) J statistic. Suppose we adopt an ordered alternative hypothesis that consumers, evoked set sizes become larger as they perceive greater levels of risk. This would be in an attempt to alleviate the effects of the constraint upon spatial behaviour. Let the unknown treatment effects be denoted by t , j = 1, 2, k then in the present example, this ordered alternative may be written as:

$$H_A: t_1 \le t_2 \le t_3.$$

To conduct the test with an ordered alternative hypothesis, k(k - 1)/2 Mann-Whitney counts are computed between the samples in the manner described in Section IV iii. Let U_{xy} be the number of items in sample X that precede items in sample Y, letting U_{xy} 0.5 if items are equal. From Table 19, it may be computed that $U_{12} = 31$, $U_{13} = 60.5$ and $U_{23} = 31$. When an ordered alternative hypothesis is being examined, the test statistic is the total of these U_{xy} values, namely:

$$J = \sum_{u=1}^{k-1} \sum_{v=u+1}^{k} U_{uv}.$$

Here, J = 122.5 and critical values for J are tabulated for small n_j, j = 1, 2,....k in Hollander and Wolfe (1973, p.311-27). For larger n_j, it can be shown that:

$$J' = \frac{J - E(J)}{\sqrt{var(J)}},$$

where $E(J) = (N^2 - \sum_{j=1}^{k} n_j^2)/4$, and

$var(J) = [N^2(2N + 3) - \sum_{j=1}^{k} n_j^2(2n_j + 3)]/72$ is approximately standard normally distributed (Hollander and Wolfe, 1973). In the present example, $N = 22$, $n_1 = 9$, $n_2 = 5$, $n_3 = 8$, so:

$$E(J) = [22^2 - (81 + 25 + 64)]/4 = 78.5, \text{ and}$$

$$var(J) = [484(47) - \{81(21) + 25(13) + 64(19)\}]/72 = 270.92.$$

The test statistic under H_0 is therefore:

$$\frac{122.5 - 78.5}{\sqrt{270.92}} = 2.67 \in N(0,1).$$

From Appendix 2, we find that our result is significant, so we reject H_0 and conclude that consumers, evoked sets increase in size as does the perceived level of risk that they attach to the purchase.

(Iii) Multiple comparisons

This result is also apparent if we conduct a multiple comparisons procedure based on an experimentwise error rate of a = 0.05. As in the Friedman test, we compute the k(k - 1)/2 absolute differences $|R_u - R_v|$ $u < v$, where the R_j, $j = 1, 2, \ldots, k$ are found from Table 19. These differences are compared with certain critical values. In the case of unequal sample sizes, Dunn (1964) offered a method of conducting multiple comparisons associated with the Kruskal-Wallis test. By this method, it is concluded that the effects of treatments U and V, denoted by t_u and t_v are unequal if:

$$|R_u - R_v| \geq z_\beta \sqrt{\frac{N(N+1)}{12}} \sqrt{\frac{1}{n_u} + \frac{1}{n_v}},$$

where $\beta = \frac{\alpha}{k(k-1)}$, z_β is a standard normal variable, $N = \sum_{j=1}^{k} n_j$ and n_u and n_v are the sample sizes. In the present problem, β =

0.05/3(2) = 0.00833, so from Appendix 2, we find $z_{0.00833} = 2.394$. The pertinent computations for the data in Table 19 are illustrated in Table 20. All these three absolute differences exceed their critical values. Therefore, we reject H_0 in favour of the ordered alternative. The evoked sets show a significant increase in size as consumers perceive more risk.

If the k sample sizes are equal $n_1 = n_2 = \ldots\ldots = n_k = n$ say, then for small n it is decided that the treatment effects t_u and t_v are unequal if:

$$|R_u - R_v| \geq y(\alpha, k, n),$$

where tables of $y(a, k, n)$ are provided in Appendix 13. For larger, but equal sample sizes, Miller (1966) suggested that $t_u \neq t_v$ if:

$$|R_u - R_v| \geq q(\alpha, k)\sqrt{\frac{k(nk+1)}{12}},$$

where $q(\alpha, k)$ is the upper α percentile of the range of k independent normal variables, tabulated in Appendix 11.

(iv) Discussion

The Kruskal-Wallis test is the most efficient of all non-parametric tests for k independent samples. If the normality assumptions of parametric analysis of variance are met, then the ARE of the Kruskal-Wallis test with respect to the F-test is 95.5%. Jonckheere,s J statistic also has an ARE of 95.5% compared to the F statistic if the populations are normal. With uniform and double exponential populations, the ARE of the Kruskal-Wallis test relative to the F-test are 100% and 150% respectively (Andrews, 1954). It is thus more powerful in these circumstances than its parametric counterpart.

It should be noted that the adaption of Fisher,s test referred to as the median test and discussed in Section IV (ii) may be extended to test if k independent samples have been drawn from the same population or from populations with equal medians. Both this procedure and the Kruskal-Wallis test require at least ordinal measurement. However, in that the latter test uses more information from the sampled data, it is the more powerful method.

Table 20 Multiple comparisons in the Kruskal-Wallis test

$\lvert R_u - R_v \rvert$	$2.394\sqrt{N(N+1)/12}\ \sqrt{1/n_u + 1/n_v}$
$\lvert R_1 - R_2 \rvert = 15.5$	$2.394\sqrt{(22)(23)/12}\ \sqrt{1/9 + 1/5} = 8.67$
$\lvert R_1 - R_3 \rvert = 57.0$	$2.394\sqrt{(22)(23)/12}\ \sqrt{1/9 + 1/8} = 7.56$
$\lvert R_2 - R_3 \rvert = 72.5$	$2.394\sqrt{(22)(23)/12}\ \sqrt{1/5 + 1/8} = 8.86$

Source: Table 19

VII DISCUSSION

The preceding sections have described some nonparametric statistical tests that may be used in geographical situations where either only nominal or ordinal levels of measurement have been achieved or there is doubt as to whether interval or ratio data meet the normality assumption which, strictly speaking, is required by parametric methods of analysis. Generally, the inherent logic of the nonparametric tests described in the previous sections has the pedagogic virtue of being easy to comprehend, as well as being straightforward to perform without recourse to a computer.

The choice of presented tests here had to be selective and was based on their relative power and readiness of application to geographical problems. In the one-sample case, the runs test is omitted in the present monograph. This test examines if events of interest occur in random order. Geographical applications are likely to be in the context of randomness of temporal occurrences or in testing whether a gathered sample is random. However, it is only applicable to dichotomous events in that "+" or "-" are allocated to items (such as male or female respondents to a survey) and sequences of plusses and minuses are examined. This restriction (together with the fact that it is less powerful than the tests herein included) led to its omission. A general omission has been methods based on 'randomisation, (see Bradley, 1968, p.88-141), which examine if treatments may be regarded as equivalent. Often, such tests have an ARE of 100% with respect to their parametric equivalents when sampling is from normal populations. However, nonparametric tests are designed to examine non-normal situations and for large sample sizes, randomisation procedures are not computationally feasible without recourse to a computer. Alternatives to the Friedman and Kruskal-Wallis tests were cited in the appropriate sections, but the former two tests are the most powerful for examining central tendency in the cases of k related and independent samples respectively.

A majority of the tests described in this monograph are part of the widely available SPSS[x] and MINITAB computer packages. Not available in either of these packages are Walsh,s test and the ordered alternative and multiple comparisons procedures for k samples. In the SPSSx package, the procedure NPAR TESTS performs the tests (Nie et al., 1983, p.671-695) except that the CROSSTABS procedure analyses 2 to k sample contingency tables. In both MINITAB and SPSSx, the particular nonparametric tests are accessed by subcommand names. These names are shown in capital letters in Table 21, together with notes concerning the output or computations involved. All the subcommands produce the numerical value of the test statistic and its level of significance. Table 21 illustrates that SPSSx is the more extensive, but both packages are user friendly and easy to learn with readily comprehensible manuals.

A final point concerns recent research which has suggested that a nonparametric approach may be adopted to the analysis of multivariate phenomena in geography. In particular, a non-parametric alternative to conventional factor analysis of repertory grids gathered in behavioural studies has been put forward (Potter and Coshall, 1984), and later suggested as a general method of factor analysis in urban geography (Potter and Coshall, 1986; Coshall and Potter, 1986). Possibly the greatest advantage of the nonparametric approach is that it may be performed by hand, enabling the researcher to retain a feel for the data. As with conventional parametric factor analysis, a series of variables that are to be collapsed into a smaller set of factors or dimensions are measured across a number of cases or study objects. Briefly, the nonparametric method of factor analysis involves dichotomising the values obtained for each variable according to whether they are above or below the overall mean value of that variable. This dichotomisation is usually represented by 0,s and 1,s. Similar patterns of 0,s and 1,s suggest that the variables in question are correlated. The statistical significance of the number of common 0,s and 1,s is established by the binomial distribution discussed in Section II

Variables showing statistically significantly similar patterns of dichotomisation are grouped together to form the factors. A lengthy discussion of this method is provided by

Table 21 Nonoarametric tests available in the MINITAB and SPSS computer packages

Test	MINITAB	SPSSx
One-sample		
Binomial	The BINOMIAL subcommand with user-specified n & p computes the cumulative density function of the binomial distribution. The test per se is not available.	BINOMIAL. It has a default setting of p = 0.5, but the user may specify other values of p according to H_o. Output includes the nos. of cases in the two categories.
Chi-square	There is no specific subcommand to perform this test. However, the E_i may be found for the uniform, normal, exponential and Poisson dist'ns, if the user specifies the parameter values (Ryan et al., 1985, p.152-4). Then the x^2 statistic has to be programmed by the user.	CHISQUARE. Expected values are input by the user, unless use is made of the K-S sub-command (see next test). Output includes O_i, and $O_i - E_i$.
Kolmogorov-Smirnov	n.a.	K-S. Tests if a sample could have been drawn from uniform, normal, Poisson populations. The user specifies the parameters of these dist'ns. The most extreme +/- and the absolute differences are reported.

Table 21 (continued)

Test	MINITAB	SPSSx
<u>Two related samples</u>		
Sign test	STEST. If the median of the pop'n of all changes is M, SINT finds an ca. confidence interval for M.	SIGN. The no. of +/- and tied differences are reported. The binomial dist'n tests for significance if n s 25, otherwise the normal is used.
Wilcoxon	WTEST Only available for the one-sample version of the test that examines $H_o: M = M_o$.	WILCOXON The no. of +/- and tied differences are reported. Binomial or normal dist'ns are used for significance as appropriate.
Walsh	n.a.	n.a.
<u>Two Independent Samples</u>		
Chi-square	CHISQUARE Expected frequencies reported if desired.	Use the CROSSTABS procedure. Fisher's exact test is used if n < 20 in 2X2 tables. Numerous contingency coefficients are available (Nie et al., 1983, p.294).

Table 21 (continued)

Test	MINITAB	SPSSx
Fisher's exact test for the median	n.a.	MEDIAN Displays the 2X2 table of readings above & below the median. Fisher's test is used if n s 30, otherwise the x^2 test.
Mann-Whitney	MANN-WHITNEY Output includes the no. of cases & the sample median values of the two groups. A confidence interval & point estimator for the difference between the pop'n medians is computed.	M-W Output includes the no. of cases & the mean rank of the two groups. U is corrected for tied readings.
k <u>related samples</u>		
Friedman	n.a.	FRIEDMAN Output includes the no. of cases and mean rank of the k groups.
k <u>independent samples</u>		
Kruskal-Wallis	KRUSKAL-WALLIS	K-W Output includes the no. of cases and mean rank of k groups. X^2 is corrected for ties.

Potter and Coshall (1986). Very often, simple dichotomisation of data, or indeed ranking methods as described in the present monograph, give an immediate indication of the structure inherent in data matrices. Above all, the nonparametric approach to factor analysis produces essentially the same answer as conventional computer dependent procedures, whilst having the pedagogic virtue of demonstrating to students how factor analysis works.

The nonparametric method of multivariate data analysis based on the dichotomisation of data has also been extended to offer an alternative to conventional canonical correlation analysis (Coshall and Potter, 1987). Canonical correlation examines inter-relationships between two sets of data and is one of the most complex multivariate techniques to use and understand. However, the nonparametric method of canonical correlation possesses the advantages mentioned in the previous paragraph. The simple expedient of data dichotomisation often reveals the major inter-relationships between two data sets and the approach readily illustrates the underlying mechanics of canonical correlation.

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APPENDIX 1

A SUMMARY OF THE NOTATION USED

ARE - asymptotic relative efficiency (see Section I (iii)).

α - The significance level of a test. A one-tailed test (involving an inequality in H_1) has one critical region of size α. A two-tailed test (involving $\neq$ in H_1) has two critical regions each of size $\alpha/2$.

df - degrees of freedom (see Section II (ii)).

${}^nC_r$ - this notation represents $\frac{n!}{r!(n-r)!}$ (see Section II i).

E(x) - the expected value or mean of a random variable X.

N(0,1) - the normal distribution with zero mean and unit variance. This is called the standard normal distribution.

$\hat{p}$ - an estimator of a population proportion p (see Section IV (iii)).

P(X = x) - the probabilty of x occurrences of the random variable X.

t_i - the effect of the ith. treatment (see Section V (ii)).

var(x) - the variance of the random variable X.

$|a - b|$ - the modulus notation, indicating that only the magnitude of the difference a - b is required, rather than its sign, e.g. $|7 - 10| = 3$. This is also called the absolute difference.

! - the factorial notation. Generally, $n! = n(n - 1)(n - 2)\ldots2.1$, e.g. $5! = 120$.

ϵ - is distributed as a specified statistic, so $X \epsilon N(0,1)$ means that the random variable X follows a standard normal distribution.

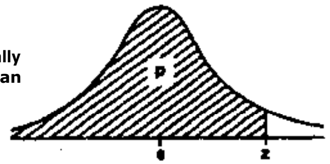
χ^2_ν - a chi-square variable with ν degrees of freedom.

APPENDIX 2

PROBABILITIES ASSOCIATED WITH THE STANDARD NORMAL DISTRIBUTION

Source: Dunstan et al., 1983, 7

The table gives the probability p that a normally distributed random variable Z with zero mean and unit variance is less than or equal to z.



2	AO	A1	22	.03	.84	211	A8	27	20	A*
0.0	.50000	.50399	.50798	.51197	.S1595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.S5962	.56356	.56749	.57142	.57535
02	.57926	.58317	.58706	.59095	.59843	.S9871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
OA	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
OA	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.0	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
8.8	.78814	.79103	.79389	.79673	.79955	.00234	.00511	.80785	.81057	.81327
OA	.81594	.01859	.02121	.82381	.82639	.02894	.03147	.83398	.83646	.83891
10	.84134	.94375	.84614	.841149	.05083	.05314	.85543	.05769	.85993	.86214
1.1	.06433	.86650	.86864	.87076	.87286	.07493	.87698	.87900	.08100	.08298
12	.88493	.08686	.88877	.89065	.89251	.89435	.89617	.09796	.89973	.90147
1.2	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
IA	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
10	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
14	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1A	.96407	.96485	.96562	.96638	.96712	.96784	.96056	.96926	.96995	.97062
1.8	.97128	.97193	.97257	.97320	.973e1	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
21	.98214	.99257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
22	.98610	.98645	.98679	.98713	.98745	.90778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2A	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.6	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.0	.99534	.99547	.99560	.99573	.99505	.99598	.99609	.99621	.99632	.99643
27	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
LS	.99744	.99752	.99760	.99767	.99774	.99701	.99780	.99795	.99801	.99807
22	.99813	.99919	.99825	.99031	.99836	.99841	.99846	.99951	.99856	.99961
2^2	.99865	.99869	.99874	.99878	.99082	.99686	.99889	.99093	.99896	.99900
2^1	.99903	.99906	.09910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
32	.99931	:99934	.99936	.99930	.99940	.99942	.99944	.99946	.99948	.99950
22	.99952	.99953	.99955	.99957	.99958	.99960	'.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
22	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99903	.99983
3.s	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989•
^{21}	.99989	.99990	.99990	.99990	.99991	.99991'	.99992	.99992	.99992	.99992
^{32}	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
2^2	.99995	.99995•	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

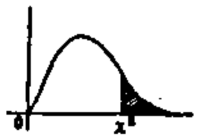
APPENDIX 3

PROBABILITIES ASSOCIATED WITH THE CHI-SQUARE DISTRIBUTION

For more than 30 degrees of freedom, the expression $\sqrt{2\chi^2} - \sqrt{2\nu - 1}$, where 1/ Is the no. of degrees of freedom, Is approximately standard normally distributed.

Source: noel, 1971, 392

TABLE III. χ^2 Distribution



Degrees of freedom	P = 0.99	0.98	0.95	0.90	0.80	0.70	0.50	0.30	0.20	0.10	0.05	0.02	0.01
1	0.000157	0.000628	0.00393	0.0158	0.0642	0.148	0.455	1.074	1.642	2.706	3.841	5.412	6.635
2	0.0201	0.0404	0.103	0.211	0.446	0.713	1.386	2.408	3.219	4.605	5.991	7.824	9.210
3	0.115	0.185	0.352	0.584	1.005	1.424	2.366	3.665	4.642	6.251	7.815	9.837	11.341
4	0.297	0.429	0.711	1.064	1.649	2.195	3.357	4.878	5.989	7.779	9.488	11.668	13.277
5	0.554	0.752	1.145	1.610	2.343	3.000	4.351	6.064	7.289	9.236	11.070	13.388	15.086
6	0.872	1.134	1.635	2.204	3.070	3.828	5.348	7.231	8.558	10.645	12.592	15.033	16.812
7	1.239	1.564	2.167	2.833	3.822	4.671	6.346	8.383	9.803	12.017	14.067	16.622	18.475
8	1.646	2.032	2.733	3.490	4.594	5.527	7.344	9.524	11.030	13.362	15.507	18.168	20.090
9	2.088	2.532	3.325	4.168	5.380	6.393	8.343	10.656	12.242	14.684	16.919	19.679	21.666
10	2.558	3.059	3.940	4.865	6.179	7.267	9.342	11.781	13.442	15.987	18.307	21.161	23.209
11	3.053	3.609	4.575	5.578	6.989	8.148	10.341	12.899	14.631	17.275	19.675	22.618	24.725
12	3.571	4.178	5.226	6.304	7.807	9.034	11.340	14.011	15.812	18.549	21.026	24.054	26.217
13	4.107	4.765	5.892	7.042	8.634	9.926	12.340	15.119	16.985	19.812	22.362	25.472	27.688
14	4.660	5.368	6.571	7.790	9.467	10.821	13.339	16.222	18.151	21.064	23.685	26.873	29.141
15	5.229	5.985	7.261	8.547	10.307	11.721	14.339	17.322	19.311	22.307	24.996	28.259	30.578
16	5.812	6.614	7.962	9.312	11.152	12.624	15.338	18.418	20.465	23.542	26.296	29.633	32.000
17	6.408	7.255	8.672	10.085	12.002	13.531	16.338	19.511	21.615	24.769	27.587	30.995	33.409
18	7.015	7.906	9.390	10.865	12.857	14.440	17.338	20.601	22.760	25.989	28.869	32.346	34.805
19	7.633	8.567	10.117	11.651	13.716	15.352	18.338	21.689	23.900	27.204	30.144	33.687	36.191
20	8.260	9.237	10.851	12.443	14.578	16.266	19.337	22.775	25.038	28.412	31.410	35.020	37.566
21	8.897	9.915	11.591	13.240	15.445	17.182	20.337	23.858	26.171	29.615	32.671	36.343	38.932
22	9.542	10.600	12.338	14.041	16.314	18.101	21.337	24.939	27.301	30.813	33.924	37.659	40.289
23	10.196	11.293	13.091	14.848	17.187	19.021	22.337	26.018	28.429	32.007	35.172	38.968	41.638
24	10.856	11.992	13.848	15.659	18.062	19.943	23.337	27.096	29.553	33.196	36.415	40.270	42.980
25	11.524	12.697	14.611	16.473	18.940	20.867	24.337	28.172	30.675	34.382	37.652	41.566	44.314
26	12.198	13.409	15.379	17.292	19.820	21.792	25.336	29.246	31.795	35.563	38.885	42.856	45.642
27	12.879	14.125	16.151	18.114	20.703	22.719	26.336	30.319	32.912	36.741	40.113	44.140	46.963
28	13.565	14.847	16.928	18.939	21.588	23.647	27.336	31.391	34.027	37.916	41.337	45.419	48.278
29	14.256	15.574	17.708	19.768	22.475	24.577	28.336	32.461	35.139	39.087	42.557	46.693	49.588
30	14.953	16.306	18.493	20.599	23.364	25.508	29.336	33.530	36.250	40.256	43.773	47.962	50.892

APPENDIX 4

QUANTILES OF THE K–S STATISTIC

Source; Conover, 1980, 462

One-Sided Test	$p = .90$	.95	.975	.99	.995		$p = .90$	.95	.975	.99	.995
Two-Sided Test	$p = .80$	.90	.95	.98	.99		$p = .80$	.90	.95	.98	.99
$n = 1$	.900	.950	.975	.990	.995	$n = 21$	.226	.259	.287	.321	.344
2	.684	.776	.842	.900	.929	22	.221	.253	.281	.314	.337
3	.565	.636	.708	.785	.829	23	.216	.247	.275	.307	.330
4	.493	.565	.624	.689	.734	24	.212	.242	.269	.301	.323
5	.447	.509	.563	.627	.669	25	.208	.238	.264	.295	.317
6	.410	.468	.519	.577	.617	26	.204	.233	.259	.290	.311
7	.381	.436	.483	.538	.576	27	.200	.229	.254	.284	.305
8	.358	.410	.454	.507	.542	28	.197	.225	.250	.279	.300
9	.339	.387	.430	.480	.513	29	.193	.221	.246	.275	.295
10	.323	.369	.409	.457	.489	30	.190	.218	.242	.270	.290
11	.308	.352	.391	.437	.468	31	.187	.214	.238	.266	.285
12	.296	.338	.375	.419	.449	32	.184	.211	.234	.262	.281
13	.285	.325	.361	.404	.432	33	.182	.208	.231	.258	.277
14	.275	.314	.349	.390	.418	34	.179	.205	.227	.254	.273
15	.266	.304	.338	.377	.404	35	.177	.202	.224	.251	.269
16	.258	.295	.327	.366	.392	36	.174	.199	.221	.247	.265
17	.250	.286	.318	.355	.381	37	.172	.196	.218	.244	.262
18	.244	.279	.309	.346	.371	38	.170	.194	.215	.241	.258
19	.237	.271	.301	.337	.361	39	.168	.191	.213	.238	.255
20	.232	.265	.294	.329	.352	40	.165	.189	.210	.235	.252
					Approximation for $n > 40$		$\frac{1.07}{\sqrt{n}}$	$\frac{1.22}{\sqrt{n}}$	$\frac{1.36}{\sqrt{n}}$	$\frac{1.52}{\sqrt{n}}$	$\frac{1.63}{\sqrt{n}}$

APPENDIX 5

CRITICAL VALUES OF T IN THE WILCOXON HATCHED-PAIRS SIGNED RANKS TEST

Source: Siegel, 1956, 254

N	Level of significance for one-tailed test		
	.025	.01	.005
	Level of significance for two-tailed test		
	.05	.02	.01
6	0	—	—
7	2	0	—
8	4	2	0
9	6	3	2
10	8	5	3
11	11	7	5
12	14	10	7
13	17	13	10
14	21	16	13
15	25	20	16
16	30	24	20
17	35	28	23
18	40	33	28
19	46	38	32
20	52	43	38
21	59	49	43
22	66	56	49
23	73	62	55
24	81	69	61
25	89	77	68

APPENDIX 6

CRITICAL VALUES FOR THE WALSH TEST

Source: Siegel, 1956, 255, (adapted from Walsh, 1949)

N	Significance level of tests		Tests	
			Two-tailed: accept $\mu_1 \neq 0$ if either	
	One-tailed	Two-tailed	One-tailed: accept $\mu_1 < 0$ if	One-tailed: accept $\mu_1 > 0$ if
4	.062	.125	$d_4 < 0$	$d_1 > 0$
5	.062	.125	$\frac{1}{2}(d_4 + d_5) < 0$	$\frac{1}{2}(d_1 + d_2) > 0$
	.031	.062	$d_5 < 0$	$d_1 > 0$
6	.047	.094	$\max[d_5, \frac{1}{2}(d_4 + d_6)] < 0$	$\min[d_2, \frac{1}{2}(d_1 + d_3)] > 0$
	.031	.062	$\frac{1}{2}(d_5 + d_6) < 0$	$\frac{1}{2}(d_1 + d_2) > 0$
	.016	.031	$d_6 < 0$	$d_1 > 0$
7	.055	.109	$\max[d_5, \frac{1}{2}(d_4 + d_7)] < 0$	$\min[d_3, \frac{1}{2}(d_1 + d_4)] > 0$
	.023	.047	$\max[d_6, \frac{1}{2}(d_5 + d_7)] < 0$	$\min[d_2, \frac{1}{2}(d_1 + d_3)] > 0$
	.016	.031	$\frac{1}{2}(d_6 + d_7) < 0$	$\frac{1}{2}(d_1 + d_2) > 0$
	.008	.016	$d_7 < 0$	$d_1 > 0$
8	.043	.086	$\max[d_6, \frac{1}{2}(d_4 + d_8)] > 0$	$\min[d_3, \frac{1}{2}(d_1 + d_5)] > 0$
	.027	.055	$\max[d_6, \frac{1}{2}(d_5 + d_8)] < 0$	$\min[d_3, \frac{1}{2}(d_1 + d_4)] > 0$
	.012	.023	$\max[d_7, \frac{1}{2}(d_6 + d_8)] < 0$	$\min[d_2, \frac{1}{2}(d_1 + d_3)] > 0$
	.008	.016	$\frac{1}{2}(d_7 + d_8) < 0$	$\frac{1}{2}(d_1 + d_2) > 0$
	.004	.008	$d_8 < 0$	$d_1 > 0$
9	.051	.102	$\max[d_6, \frac{1}{2}(d_4 + d_9)] < 0$	$\min[d_4, \frac{1}{2}(d_1 + d_6)] > 0$
	.022	.043	$\max[d_7, \frac{1}{2}(d_5 + d_9)] < 0$	$\min[d_3, \frac{1}{2}(d_1 + d_5)] > 0$
	.010	.020	$\max[d_8, \frac{1}{2}(d_5 + d_9)] < 0$	$\min[d_2, \frac{1}{2}(d_1 + d_5)] > 0$
	.006	.012	$\max[d_8, \frac{1}{2}(d_7 + d_9)] < 0$	$\min[d_2, \frac{1}{2}(d_1 + d_3)] > 0$
	.004	.008	$\frac{1}{2}(d_8 + d_9) < 0$	$\frac{1}{2}(d_1 + d_2) > 0$
10	.056	.111	$\max[d_6, \frac{1}{2}(d_4 + d_{10})] < 0$	$\min[d_5, \frac{1}{2}(d_1 + d_7)] > 0$
	.025	.051	$\max[d_7, \frac{1}{2}(d_5 + d_{10})] < 0$	$\min[d_4, \frac{1}{2}(d_1 + d_6)] > 0$
	.011	.021	$\max[d_8, \frac{1}{2}(d_6 + d_{10})] < 0$	$\min[d_3, \frac{1}{2}(d_1 + d_5)] > 0$
	.005	.010	$\max[d_9, \frac{1}{2}(d_6 + d_{10})] < 0$	$\min[d_2, \frac{1}{2}(d_1 + d_5)] > 0$
11	.048	.097	$\max[d_7, \frac{1}{2}(d_4 + d_{11})] < 0$	$\min[d_5, \frac{1}{2}(d_1 + d_8)] > 0$
	.028	.056	$\max[d_7, \frac{1}{2}(d_5 + d_{11})] < 0$	$\min[d_5, \frac{1}{2}(d_1 + d_7)] > 0$
	.011	.021	$\max[\frac{1}{2}(d_6 + d_{11}), \frac{1}{2}(d_8 + d_9)] < 0$	$\min[\frac{1}{2}(d_1 + d_6), \frac{1}{2}(d_3 + d_4)] > 0$
	.005	.011	$\max[d_9, \frac{1}{2}(d_7 + d_{11})] < 0$	$\min[d_3, \frac{1}{2}(d_1 + d_5)] > 0$
12	.047	.094	$\max[\frac{1}{2}(d_4 + d_{12}), \frac{1}{2}(d_5 + d_{11})] < 0$	$\min[\frac{1}{2}(d_1 + d_9), \frac{1}{2}(d_2 + d_8)] > 0$
	.024	.048	$\max[d_8, \frac{1}{2}(d_5 + d_{12})] < 0$	$\min[d_5, \frac{1}{2}(d_1 + d_8)] > 0$
	.010	.020	$\max[d_9, \frac{1}{2}(d_6 + d_{12})] < 0$	$\min[d_4, \frac{1}{2}(d_1 + d_7)] > 0$
	.005	.011	$\max[\frac{1}{2}(d_7 + d_{12}), \frac{1}{2}(d_9 + d_{10})] < 0$	$\min[\frac{1}{2}(d_1 + d_6), \frac{1}{2}(d_3 + d_4)] > 0$
13	.047	.094	$\max[\frac{1}{2}(d_4 + d_{13}), \frac{1}{2}(d_5 + d_{12})] < 0$	$\min[\frac{1}{2}(d_1 + d_{10}), \frac{1}{2}(d_2 + d_9)] > 0$
	.023	.047	$\max[\frac{1}{2}(d_5 + d_{13}), \frac{1}{2}(d_6 + d_{12})] < 0$	$\min[\frac{1}{2}(d_1 + d_9), \frac{1}{2}(d_2 + d_8)] > 0$
	.010	.020	$\max[\frac{1}{2}(d_6 + d_{13}), \frac{1}{2}(d_9 + d_{10})] < 0$	$\min[\frac{1}{2}(d_1 + d_8), \frac{1}{2}(d_4 + d_5)] > 0$
	.005	.010	$\max[d_{10}, \frac{1}{2}(d_7 + d_{13})] < 0$	$\min[d_4, \frac{1}{2}(d_1 + d_7)] > 0$
14	.047	.094	$\max[\frac{1}{2}(d_4 + d_{14}), \frac{1}{2}(d_5 + d_{13})] < 0$	$\min[\frac{1}{2}(d_1 + d_{11}), \frac{1}{2}(d_2 + d_{10})] > 0$
	.023	.047	$\max[\frac{1}{2}(d_5 + d_{14}), \frac{1}{2}(d_6 + d_{13})] < 0$	$\min[\frac{1}{2}(d_1 + d_{10}), \frac{1}{2}(d_2 + d_9)] > 0$
	.010	.020	$\max[d_{10}, \frac{1}{2}(d_6 + d_{14})] < 0$	$\min[d_5, \frac{1}{2}(d_1 + d_9)] > 0$
	.005	.010	$\max[\frac{1}{2}(d_7 + d_{14}), \frac{1}{2}(d_{10} + d_{11})] < 0$	$\min[\frac{1}{2}(d_1 + d_8), \frac{1}{2}(d_4 + d_5)] > 0$
15	.047	.094	$\max[\frac{1}{2}(d_4 + d_{15}), \frac{1}{2}(d_5 + d_{14})] < 0$	$\min[\frac{1}{2}(d_1 + d_{12}), \frac{1}{2}(d_2 + d_{11})] > 0$
	.023	.047	$\max[\frac{1}{2}(d_5 + d_{15}), \frac{1}{2}(d_6 + d_{14})] < 0$	$\min[\frac{1}{2}(d_1 + d_{11}), \frac{1}{2}(d_2 + d_{10})] > 0$
	.010	.020	$\max[\frac{1}{2}(d_6 + d_{15}), \frac{1}{2}(d_{10} + d_{11})] < 0$	$\min[\frac{1}{2}(d_1 + d_{10}), \frac{1}{2}(d_5 + d_6)] > 0$
	.005	.010	$\max[d_{11}, \frac{1}{2}(d_7 + d_{15})] < 0$	$\min[d_5, \frac{1}{2}(d_1 + d_9)] > 0$

Appendix 7

CRITICAL THE MANN-WHITNEY STATISTIC

Source: Dunstan et al., 1983, 17-8

The table gives the upper tail critical values u_c of the statistic

$$U = \sum_{i=1}^{m}\sum_{j=1}^{n} Z_{ij}$$

where $Z_{ij} = 1$ if $X_i < Y_j$ and $Z_{ij} = 0$ if $X_i > Y_j$ given the independent samples $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$.
The lower tail critical values are given by $mn - u_c$.

Since U is discrete, exact significance levels cannot in general be achieved. The critical values given are those whose significance levels are nearest to those stated.

One tail 0.5%   Two tail 1%

n\m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1																														
2													26	28	30	32	34	36	38	40	42	44	45	47	49	51	53	55	57	59
3							21	24	27	30	32	35	38	40	43	46	48	51	54	57	59	62	65	67	70	73	75	78	81	83
4					20	24	27	31	34	38	41	45	48	51	55	58	62	65	69	72	75	79	82	86	89	93	96	99	103	106
5				20	25	29	33	37	41	46	50	54	58	62	66	70	74	79	83	87	91	95	99	103	107	112	116	120	124	128
6				24	29	34	39	44	48	53	58	63	68	73	77	82	87	92	97	101	106	111	116	120	125	130	135	140	144	149
7			21	27	33	39	44	50	55	61	66	72	77	83	88	94	99	105	110	116	121	127	132	137	143	148	154	159	165	170
8			24	31	37	44	50	56	62	69	75	81	87	93	99	105	111	118	124	130	136	142	148	154	160	166	172	178	185	191
9			27	34	41	48	55	62	69	76	83	90	97	103	110	117	124	130	137	144	151	157	164	171	177	184	191	198	204	211
10			30	38	46	53	61	69	76	84	91	99	106	113	121	128	136	143	150	158	165	172	180	187	195	202	209	217	224	231
11			32	41	50	58	66	75	83	91	99	107	115	123	131	140	148	156	164	172	180	188	196	204	212	220	227	235	243	251
12			35	45	54	63	72	81	90	99	107	116	125	133	142	151	159	168	177	185	194	203	211	220	228	237	246	254	263	271
13		26	38	48	58	68	77	87	97	106	115	125	134	143	153	162	171	181	190	199	208	218	227	236	245	254	264	273	282	291
14		28	40	51	62	73	83	93	103	113	123	133	143	153	163	173	183	193	203	213	223	233	242	252	262	272	282	292	301	311
15		30	43	55	66	77	88	99	110	121	131	142	153	163	174	184	195	206	216	226	237	247	258	268	279	289	300	310	320	331
16		32	46	58	70	82	94	105	117	128	140	151	162	173	184	196	207	218	229	240	251	262	273	284	295	307	318	329	340	351
17		34	48	62	74	87	99	111	124	136	148	159	171	183	195	207	219	230	242	254	265	277	289	300	312	324	335	347	359	370
18		36	51	65	79	92	105	118	130	143	156	168	181	193	206	218	230	243	255	267	280	292	304	316	329	341	353	365	378	390
19		38	54	69	83	97	110	124	137	150	164	177	190	203	216	229	242	255	268	281	294	307	320	332	345	358	371	384	397	409
20		40	57	72	87	101	116	130	144	158	172	185	199	213	226	240	254	267	281	294	308	321	335	348	362	375	389	402	416	429
21		42	59	75	91	106	121	136	151	165	180	194	208	223	237	251	265	280	294	308	322	336	350	364	378	392	406	420	435	449
22		44	62	79	95	111	127	142	157	172	188	203	218	233	247	262	277	292	307	321	336	351	366	380	395	410	424	439	453	468
23		45	65	82	99	116	132	148	164	180	196	211	227	242	258	273	289	304	320	335	350	366	381	396	411	427	442	457	472	487
24		47	67	86	103	120	137	154	171	187	204	220	236	252	268	284	300	316	332	348	364	380	396	412	428	444	459	475	491	507
25		49	70	89	107	125	143	160	177	195	212	228	245	262	279	295	312	329	345	362	378	395	411	428	444	461	477	494	510	526
26		51	73	93	112	130	148	166	184	202	220	237	254	272	289	307	324	341	358	375	392	410	427	444	461	478	495	512	529	546
27		53	75	96	116	135	154	172	191	209	227	246	264	282	300	318	335	353	371	389	406	424	442	459	477	495	512	530	548	565
28		55	78	99	120	140	159	178	198	217	235	254	273	292	310	329	347	365	384	402	420	438	457	475	494	512	530	548	566	584
29		57	81	103	124	144	165	185	204	224	243	263	282	301	320	340	359	378	397	416	435	453	472	491	510	529	548	566	585	604
30		59	83	106	128	149	170	191	211	231	251	271	291	311	331	351	370	390	409	429	449	468	487	507	526	546	565	584	604	623

One tail 1%   Two tail 2%

n\m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1																														
2									18	20	22	24	26	28	30	31	33	35	37	39	41	43	44	46	48	50	52	54	56	58
3					15	18	21	23	26	29	31	34	36	39	42	44	47	50	52	55	57	60	63	65	68	70	73	76	78	81
4				16	20	23	26	30	33	36	40	43	46	50	53	56	60	63	66	70	73	76	79	83	86	89	93	96	99	103
5			15	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96	100	104	108	112	116	120	124
6			18	23	28	33	37	42	47	51	56	61	65	70	75	79	84	89	93	98	102	107	112	116	121	126	130	135	139	144
7			21	26	32	37	43	48	53	59	64	69	75	80	85	91	96	101	106	112	117	122	128	133	138	143	149	154	159	164
8			23	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	125	131	137	143	149	155	161	167	173	178	184
9		18	26	33	40	47	53	60	67	73	80	87	93	100	106	113	119	126	133	139	146	152	159	165	172	178	185	191	198	204
10		20	29	36	44	51	59	66	73	81	88	95	102	110	117	124	131	138	145	153	160	167	174	181	188	195	203	210	217	224
11		22	31	40	48	56	64	72	80	88	96	104	112	119	127	135	143	151	158	166	174	182	189	197	205	213	220	228	236	244
12		24	34	43	52	61	69	78	87	95	104	112	121	129	138	146	154	163	171	180	188	196	205	213	221	230	238	246	255	263
13		26	36	46	56	65	75	84	93	102	112	121	130	139	148	157	166	175	184	193	202	211	220	229	238	247	256	265	274	283
14		28	39	50	60	70	80	90	100	110	119	129	139	148	158	168	177	187	197	206	216	225	235	245	254	264	273	283	292	302
15		30	42	53	64	75	85	96	106	117	127	138	148	158	169	179	189	199	209	220	230	240	250	260	270	281	291	301	311	321
16		31	44	56	68	79	91	102	113	124	135	146	157	168	179	190	200	211	222	233	244	254	265	276	287	297	308	319	330	340
17		33	47	60	72	84	96	108	119	131	143	154	166	177	189	200	212	223	235	246	257	269	280	292	303	314	326	337	348	360
18		35	50	63	76	89	101	114	126	138	151	163	175	187	199	211	223	235	247	259	271	283	295	307	319	331	343	355	367	379
19		37	52	66	80	93	106	120	133	145	158	171	184	197	209	222	235	247	260	273	285	298	310	323	335	348	360	373	385	398
20		39	55	70	84	98	112	125	139	153	166	180	193	206	220	233	246	259	273	286	299	312	325	338	351	365	378	391	404	417
21		41	57	73	88	102	117	131	146	160	174	188	202	216	230	244	257	271	285	299	313	326	340	354	368	381	395	409	422	436
22		43	60	76	92	107	122	137	152	167	182	196	211	225	240	254	268	283	298	312	326	341	355	369	384	398	412	427	441	455
23		44	63	79	96	112	128	143	159	174	189	205	220	235	250	265	280	295	310	325	340	355	370	385	400	415	430	444	459	474
24		46	65	83	100	116	133	149	165	181	197	213	229	245	260	276	292	307	323	338	354	369	385	400	416	431	447	462	478	493
25		48	68	86	104	121	138	155	172	188	205	221	238	254	270	287	303	319	335	351	368	384	400	416	432	448	464	480	496	512
26		50	70	89	108	126	143	161	178	195	213	230	247	264	281	297	314	331	348	365	381	398	415	431	448	465	481	498	515	531
27		52	73	93	112	130	149	167	185	203	220	238	256	273	291	308	326	343	360	378	395	412	430	447	464	481	499	516	533	550
28		54	76	96	116	135	154	173	191	210	228	246	265	283	301	319	337	355	373	391	409	427	444	462	480	498	516	534	551	569
29		56	78	99	120	139	159	178	198	217	236	255	274	292	311	330	348	367	385	404	422	441	459	478	496	515	533	551	570	588
30		58	81	103	124	144	164	184	204	224	244	263	283	302	321	340	360	379	398	417	436	455	474	493	512	531	550	569	588	607

One tail 2.5%   Two tail 5%

n\m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1																				20	21	22	23	24	25	26	27	28	29	30
2					10	12	14	16	18	19	21	23	25	26	28	30	32	34	35	37	39	41	42	44	46	48	50	51	53	55
3				12	15	17	19	22	24	27	29	32	34	37	39	42	44	47	49	52	54	57	59	62	64	67	69	72	74	76
4			12	15	18	22	25	28	31	34	37	40	44	47	50	53	56	59	62	66	69	72	75	78	81	84	87	91	94	97
5		10	15	18	22	26	30	34	38	41	45	49	53	56	60	64	68	72	75	79	83	87	90	94	98	102	105	109	113	117
6		12	17	22	26	31	35	40	44	48	53	57	62	66	70	75	79	84	88	92	97	101	106	110	114	119	123	128	132	136
7		14	19	25	30	35	40	45	50	55	60	66	71	76	81	86	91	96	101	106	111	116	121	126	131	136	141	146	151	156
8		16	22	28	34	40	45	51	57	62	68	74	79	85	91	96	102	108	113	119	124	130	136	141	147	152	158	164	169	175
9		18	24	31	38	44	50	57	63	69	76	82	88	94	101	107	113	119	126	132	138	144	151	157	163	169	175	182	188	194
10		19	27	34	41	48	55	62	69	76	83	90	97	104	111	117	124	131	138	145	152	158	165	172	179	186	193	199	206	213
11		21	29	37	45	53	60	68	76	83	91	98	106	113	121	128	135	143	150	158	165	173	180	187	195	202	210	217	224	232
12		23	32	40	49	57	66	74	82	90	98	106	114	122	130	139	147	155	163	171	179	187	195	203	211	219	227	235	243	251
13		25	34	44	53	62	71	79	88	97	106	114	123	132	140	149	158	166	175	183	192	201	209	218	226	235	244	252	261	269
14		26	37	47	56	66	76	85	94	104	113	122	132	141	150	159	169	178	187	196	205	215	224	233	242	251	261	270	279	288
15		28	39	50	60	70	81	91	101	111	121	130	140	150	160	170	180	189	199	209	219	229	238	248	258	268	277	287	297	307
16		30	42	53	64	75	86	96	107	117	128	139	149	159	170	180	191	201	211	222	232	243	253	263	274	284	294	305	315	325
17		32	44	56	68	79	91	102	113	124	135	147	158	169	180	191	202	213	224	235	246	256	267	278	289	300	311	322	333	344
18		34	47	59	72	84	96	108	119	131	143	155	166	178	189	201	213	224	236	247	259	270	282	293	305	316	328	339	351	362
19		35	49	62	75	88	101	113	126	138	150	163	175	187	199	211	224	236	248	260	272	284	296	308	320	333	345	357	369	381
20	20	37	52	66	79	92	106	119	132	145	158	171	183	196	209	222	235	247	260	273	285	298	311	323	336	349	361	374	387	399
21	21	39	54	69	83	97	111	124	138	152	165	179	192	205	219	232	246	259	272	285	299	312	325	338	352	365	378	391	404	418
22	22	41	57	72	87	101	116	130	144	158	173	187	201	215	229	243	256	270	284	298	312	326	340	353	367	381	395	409	422	436
23	23	42	59	75	90	106	121	136	151	165	180	195	209	224	238	253	267	282	296	311	325	340	354	368	383	397	411	426	440	454
24	24	44	62	78	94	110	126	141	157	172	187	203	218	233	248	263	278	293	308	323	338	353	368	383	398	413	428	443	458	473
25	25	46	64	81	98	114	131	147	163	179	195	211	226	242	258	274	289	305	320	336	352	367	383	398	414	429	445	460	476	491
26	26	48	67	84	102	119	136	152	169	186	202	219	235	251	268	284	300	316	333	349	365	381	397	413	429	445	461	477	493	510
27	27	50	69	87	105	123	141	158	175	193	210	227	244	261	277	294	311	328	345	361	378	395	411	428	445	461	478	495	511	528
28	28	51	72	91	109	128	146	164	182	199	217	235	252	270	287	305	322	339	357	374	391	409	426	443	460	477	495	512	529	546
29	29	53	74	94	113	132	151	169	188	206	224	243	261	279	297	315	333	351	369	387	404	422	440	458	476	493	511	529	547	564
30	30	55	76	97	117	136	156	175	194	213	232	251	269	288	307	325	344	362	381	399	418	436	454	473	491	510	528	546	564	583

One tail 5%   Two tail 10%

n\m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1										10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	28	29
2			6	8	10	12	13	15	17	18	20	22	23	25	27	28	30	32	33	35	37	39	40	42	44	45	47	49	50	52
3		6	9	11	14	16	18	21	23	25	28	30	32	35	37	39	42	44	46	49	51	53	56	58	60	63	65	67	70	72
4		8	11	14	17	20	23	26	29	32	35	38	41	44	47	50	53	56	59	62	65	68	71	74	77	80	83	85	88	91
5		10	14	17	21	25	28	32	35	39	43	46	50	53	57	61	64	68	71	75	78	82	86	89	93	96	100	103	107	111
6		12	16	20	25	29	33	37	42	46	50	54	58	63	67	71	75	79	83	88	92	96	100	104	109	113	117	121	125	129
7		13	18	23	28	33	38	43	48	52	57	62	67	72	76	81	86	91	96	100	105	110	115	119	124	129	134	138	143	148
8		15	21	26	32	37	43	48	54	59	64	70	75	81	86	91	97	102	108	113	118	124	129	134	140	145	150	156	161	167
9		17	23	29	35	42	48	54	60	66	72	78	84	90	96	102	108	114	120	125	131	137	143	149	155	161	167	173	179	185
10	10	18	25	32	39	46	52	59	66	72	79	86	92	99	105	112	118	125	131	138	145	151	158	164	171	177	184	190	197	203
11	11	20	28	35	43	50	57	64	72	79	86	93	100	108	115	122	129	136	143	150	158	165	172	179	186	193	200	207	214	221
12	12	22	30	38	46	54	62	70	78	86	93	101	109	117	124	132	140	147	155	163	171	178	186	194	201	209	217	224	232	240
13	13	23	32	41	50	58	67	75	84	92	100	109	117	125	134	142	150	159	167	175	183	192	200	208	217	225	233	241	250	258
14	14	25	35	44	53	63	72	81	90	99	108	117	125	134	143	152	161	170	179	188	196	205	214	223	232	241	249	258	267	276
15	15	27	37	47	57	67	76	86	96	105	115	124	134	143	153	162	172	181	190	200	209	219	228	238	247	256	266	275	285	294
16	16	28	39	50	61	71	81	91	102	112	122	132	142	152	162	172	182	192	202	212	222	232	242	252	262	272	282	292	302	312
17	17	30	42	53	64	75	86	97	108	118	129	140	150	161	172	182	193	203	214	225	235	246	256	267	277	288	298	309	319	330
18	18	32	44	56	68	79	91	102	114	125	136	147	159	170	181	192	203	215	226	237	248	259	270	281	292	303	315	326	337	348
19	19	33	46	59	71	83	96	108	120	131	143	155	167	179	190	202	214	226	237	249	261	272	284	296	307	319	331	342	354	366
20	20	35	49	62	75	88	100	113	125	138	150	163	175	188	200	212	225	237	249	261	274	286	298	310	323	335	347	359	371	384
21	21	37	51	65	78	92	105	118	131	145	158	171	183	196	209	222	235	248	261	274	286	299	312	325	338	350	363	376	389	402
22	22	39	53	68	82	96	110	124	137	151	165	178	192	205	219	232	246	259	272	286	299	313	326	339	353	366	379	393	406	419
23	23	40	56	71	86	100	115	129	143	158	172	186	200	214	228	242	256	270	284	298	312	326	340	354	368	382	396	409	423	437
24	24	42	58	74	89	104	119	134	149	164	179	194	208	223	238	252	267	281	296	310	325	339	354	368	383	397	412	426	441	455
25	25	44	60	77	93	109	124	140	155	171	186	201	217	232	247	262	277	292	307	323	338	353	368	383	398	413	428	443	458	473
26	26	45	63	80	96	113	129	145	161	177	193	209	225	241	256	272	288	303	319	335	350	366	382	397	413	428	444	460	475	491
27	27	47	65	83	100	117	134	150	167	184	200	217	233	249	266	282	298	315	331	347	363	379	396	412	428	444	460	476	492	509
28	28	49	67	85	103	121	138	156	173	190	207	224	241	258	275	292	309	326	342	359	376	393	409	426	443	460	476	493	510	526
29	28	50	70	88	107	125	143	161	179	197	214	232	250	267	285	302	319	337	354	371	389	406	423	441	458	475	492	510	527	544
30	29	52	72	91	111	129	148	167	185	203	221	240	258	276	294	312	330	348	366	384	402	419	437	455	473	491	509	526	544	562

APPENDIX 8

UPPER TAIL PROBABILITIES FOR FRIEDMAN'S S STATISTIC

Source: Hollander and Wolfe, 1973, 366-71

$k = 3, n = 2(1)13$; $k = 4, n = 2(1)8$; $k = 5, n = 3, 4, 5$

For given k and n, the tabled entry for the point x is $P_0\{S \geq x\}$. Under these conditions, if x is such that $P_0\{S \geq x\} = \alpha$, then $s(\alpha, k, n) = x$. For given k and n, the entries are terminated at $x_{k,n}$, where $x_{k,n}$ is the smallest value of x such that $P_0\{S \geq x\}$ is zero to three decimal places.

$k = 3, n = 2$

x	$P_0\{S \geq x\}$
0	1.000
1	.833
3	.500
4	.167

$k = 3, n = 3$

x	$P_0\{S \geq x\}$
.000	1.000
.667	.944
2.000	.528
2.667	.361
4.667	.194
6.000	.028

$k = 3, n = 4$

x	$P_0\{S \geq x\}$
.0	1.000
.5	.931
1.5	.653
2.0	.431
3.5	.273
4.5	.125
6.0	.069
6.5	.042
8.0	.005

$k = 3, n = 5$

x	$P_0\{S \geq x\}$
.0	1.000
.4	.954
1.2	.691
1.6	.522
2.8	.367
3.6	.182
4.8	.124
5.2	.093
6.4	.039
7.6	.024
8.4	.008
10.0	.001

$k = 3, n = 6$

x	$P_0\{S \geq x\}$
.000	1.000
.333	.956
1.000	.740
1.333	.570
2.333	.430
3.000	.252
4.000	.184
4.333	.142
5.333	.072
6.333	.052
7.000	.029
8.333	.012
9.000	.008
9.333	.006
10.333	.002
12.000	.000

$k = 3, n = 7$

x	$P_0\{S \geq x\}$
.000	1.000
.286	.964
.857	.768
1.143	.620
2.000	.486
2.571	.305
3.429	.237
3.714	.192
4.571	.112
5.429	.085
6.000	.051
7.143	.027
7.714	.021
8.000	.016
8.857	.008
10.286	.004
10.571	.003
11.143	.001
12.286	.000

$k = 3, n = 8$

x	$P_0\{S \geq x\}$
.00	1.000
.25	.967
.75	.794
1.00	.654
1.75	.531
2.25	.355
3.00	.285
3.25	.236
4.00	.149
4.75	.120

$k = 3, n = 8$

x	$P_0\{S \geq x\}$
5.25	.079
6.25	.047
6.75	.038
7.00	.030
7.75	.018
9.00	.010
9.25	.008
9.75	.005
10.75	.002
12.00	.001
12.25	.001
13.00	.000

$k = 3, n = 9$

x	$P_0\{S \geq x$
.000	1.000
.222	.971
.667	.814
.889	.685
1.556	.569
2.000	.398
2.667	.328
2.889	.27[illegible]
3.556	.18[illegible]
4.222	.15[illegible]
4.667	.10[illegible]
5.556	.069
6.000	.05[illegible]
6.222	.048
6.889	.031
8.000	.019

$k = 3, n = 9$

x	$P_0\{S > x\}$
8.222	.016
8.667	.010
9.556	.006
10.667	.004
10.889	.003
11.556	.001
12.667	.001
13.556	.000

$k = 3, n = 10$

x	$P_0\{S > x\}$
.0	1.000
.2	.974
.6	.830
.8	.710
1.4	.601
1.8	.436
2.4	.368
2.6	.316
3.2	.222
3.8	.187
4.2	.135
5.0	.092
5.4	.078
5.6	.066
6.2	.046
7.2	.030
7.4	.026
7.8	.018
8.6	.012
9.6	.007
9.8	.006
10.4	.003
11.4	.002
12.2	.001
12.6	.001
12.8	.001
13.4	.000

$k = 3, n = 11$

x	$P_0\{S > x\}$
.000	1.000
.182	.976
.545	.844
.727	.732
1.273	.629
1.636	.470
2.182	.403
2.364	.351
2.909	.256
3.455	.219
3.818	.163
4.545	.116
4.909	.100
5.091	.087
5.636	.062
6.545	.043
6.727	.038
7.091	.027
7.818	.019
8.727	.013
8.909	.011
9.455	.006
10.364	.004
11.091	.003
11.455	.002
11.636	.001
12.182	.001
13.273	.001
13.636	.000

$k = 3, n = 12$

x	$P_0\{S > x\}$
1.167	.654
1.500	.500
2.000	.434
2.167	.383
2.667	.287
3.167	.249
3.500	.191
4.167	.141
4.500	.123
4.667	.108
5.167	.080
6.000	.058
6.167	.051
6.500	.038
7.167	.027
8.000	.020
8.167	.017
8.667	.011
9.500	.007
10.167	.005
10.500	.004
10.667	.003
11.167	.002
12.167	.002
12.500	.001
12.667	.001
13.167	.001
13.500	.000

$k = 3, n = 13$

x	$P_0\{S > x\}$
1.385	.527
1.846	.463
2.000	.412
2.462	.316
2.923	.278
3.231	.217
3.846	.165
4.154	.145
4.308	.129
4.769	.098
5.538	.073
5.692	.065
6.000	.050
6.615	.037
7.385	.028
7.538	.025
8.000	.016
8.769	.012
9.385	.009
9.692	.007
9.846	.005
10.308	.004
11.231	.003
11.538	.002
11.692	.002
12.154	.001
12.462	.001
12.923	.001
14.000	.001
14.308	.000

$k = 3, n = 12$

x	$P_0\{S > x\}$
.000	1.000
.167	.978
.500	.856
.667	.751

$k = 3, n = 13$

x	$P_0\{S > x\}$
.000	1.000
.154	.980
.462	.866
.615	.767
1.077	.675

$k = 4, n = 2$

x	$P_0\{S > x\}$
.0	1.000
.6	.958
1.2	.833

$k = 4, n = 2$

x	$P_0\{S > x\}$
1.8	.792
2.4	.625
3.0	.542
3.6	.458
4.2	.375
4.8	.208
5.4	.167
6.0	.042

$k = 4, n = 3$

x	$P_0\{S > x\}$
.2	1.000
.6	.958
1.0	.910
1.8	.727
2.2	.608
2.6	.524
3.4	.446
3.8	.342
4.2	.300
5.0	.207
5.4	.175
5.8	.148
6.6	.075
7.0	.054
7.4	.033
8.2	.017
9.0	.002

$k = 4, n = 4$

x	$P_0\{S > x\}$
.0	1.000
.3	.992
.6	.928
.9	.900
1.2	.800
1.5	.754
1.8	.677

$k = 4, n = 4$

x	$P_0\{S > x\}$
2.1	.649
2.4	.524
2.7	.508
3.0	.432
3.3	.389
3.6	.355
3.9	.324
4.5	.242
4.8	.200
5.1	.190
5.4	.158
5.7	.141
6.0	.105
6.3	.094
6.6	.077
6.9	.068
7.2	.054
7.5	.052
7.8	.036
8.1	.033
8.4	.019
8.7	.014
9.3	.012
9.6	.007
9.9	.006
10.2	.003
10.8	.002
11.1	.001
12.0	.000

$k = 4, n = 5$

x	$P_0\{S > x\}$
.12	1.000
.36	.975
.60	.944
1.08	.857
1.32	.771
1.56	.709
2.04	.652
2.28	.561
2.52	.521

$k = 4, n = 5$

x	$P_0\{S > x\}$
3.00	.445
3.24	.408
3.48	.372
3.96	.298
4.20	.260
4.44	.226
4.92	.210
5.16	.162
5.40	.151
5.88	.123
6.12	.107
6.36	.093
6.84	.075
7.08	.067
7.32	.055
7.80	.044
8.04	.034
8.28	.031
8.76	.023
9.00	.020
9.24	.017
9.72	.012
9.96	.009
10.20	.007
10.68	.005
10.92	.003
11.16	.002
11.64	.002
11.88	.002
12.12	.001
12.60	.001
12.84	.000

$k = 4, n = 6$

x	$P_0\{S > x\}$
.0	1.000
.2	.996
.4	.957
.6	.940
.8	.874
1.0	.844
1.2	.789

$k = 4, n = 6$

x	$P_0\{S > x\}$
1.4	.772
1.6	.679
1.8	.668
2.0	.609
2.2	.574
2.4	.541
2.6	.512
3.0	.431
3.2	.386
3.4	.375
3.6	.338
3.8	.317
4.0	.270
4.2	.256
4.4	.230
4.6	.218
4.8	.197
5.0	.194
5.2	.163
5.4	.155
5.6	.127
5.8	.114
6.2	.108
6.4	.089
6.6	.088
6.8	.073
7.0	.066
7.2	.060
7.4	.056
7.6	.043
7.8	.041
8.0	.037
8.2	.035
8.4	.032
8.6	.029
8.8	.023
9.0	.022
9.4	.017
9.6	.014
9.8	.013
10.0	.010
10.2	.010
10.4	.009
10.6	.007

$k = 4, n = 6$

x	$P_0\{S \geqslant x\}$
10.8	.006
11.0	.006
11.4	.004
11.6	.003
11.8	.003
12.0	.002
12.2	.002
12.6	.001
12.8	.001
13.0	.001
13.2	.001
13.4	.001
13.6	.000

$k = 4, n = 7$

x	$P_0\{S \geqslant x\}$
.086	1.000
.257	.984
.429	.963
.771	.906
.943	.845
1.114	.800
1.457	.757
1.629	.685
1.800	.652
2.143	.590
2.314	.557
2.486	.524
2.829	.456
3.000	.418
3.171	.382
3.514	.366
3.686	.310
3.857	.297
4.200	.262
4.371	.239
4.543	.220
4.886	.195
5.057	.180

$k = 4, n = 7$

x	$P_0\{S \geqslant x\}$
5.229	.161
5.571	.143
5.743	.122
5.914	.118
6.257	.100
6.429	.093
6.600	.085
6.943	.073
7.114	.063
7.286	.056
7.629	.052
7.800	.041
7.971	.038
8.314	.035
8.486	.033
8.657	.030
9.000	.023
9.171	.020
9.343	.017
9.686	.015
9.857	.013
10.029	.012
10.371	.010
10.543	.009
10.714	.008
11.057	.007
11.229	.005
11.400	.004
11.743	.004
11.914	.003
12.086	.003
12.429	.002
12.600	.002
12.771	.002
13.114	.001
13.286	.001
13.457	.001
13.800	.001
13.971	.001
14.143	.001
14.486	.000

$k = 4, n = 8$

x	$P_0\{S \geqslant x\}$
.00	1.000
.15	.998
.30	.971
.45	.959
.60	.912
.75	.890
.90	.849
1.05	.837
1.20	.765
1.35	.757
1.50	.710
1.65	.681
1.80	.654
1.95	.629
2.25	.558
2.40	.517
2.55	.507
2.70	.471
2.85	.450
3.00	.404
3.15	.389
3.30	.362
3.45	.350
3.60	.326
3.75	.323
3.90	.287
4.05	.278
4.20	.242
4.35	.226
4.65	.219
4.80	.193
4.95	.191
5.10	.168
5.25	.158
5.40	.148
5.55	.141
5.70	.121
5.85	.117
6.00	.110
6.15	.106
6.30	.100
6.45	.094

$k = 4, n = 8$

x	$P_0\{S \geqslant x\}$
6.60	.081
6.75	.079
7.05	.068
7.20	.060
7.35	.058
7.50	.051
7.65	.049
7.80	.046
7.95	.042
8.10	.038
8.25	.037
8.55	.031
8.70	.028
8.85	.025
9.00	.023
9.15	.022
9.45	.019
9.60	.016
9.75	.015
9.90	.014
10.05	.014
10.20	.011
10.35	.011
10.50	.009
10.65	.009
10.80	.008
10.95	.008
11.10	.006
11.25	.006
11.40	.005
11.55	.005
11.85	.004
12.00	.004
12.15	.004
12.30	.003
12.45	.003
12.60	.002
12.75	.002
12.90	.002
13.05	.002
13.20	.002
13.35	.001
13.50	.001

$k = 4, n = 8$

x	$P_0\{S \geqslant x\}$
13.65	.001
13.80	.001
13.95	.001
14.25	.001
14.40	.001
14.55	.001
14.70	.001
14.85	.000

$k = 5, n = 3$

x	$P_0\{S \geqslant x\}$
.000	1.000
.267	1.000
.533	.988
.800	.972
1.067	.941
1.333	.914
1.600	.845
1.867	.831
2.133	.768
2.400	.720
2.667	.682
2.933	.649
3.200	.595
3.467	.559
3.733	.493
4.000	.475
4.267	.432
4.533	.406
4.800	.347
5.067	.326
5.333	.291
5.600	.253
5.867	.236
6.133	.213
6.400	.172
6.667	.163
6.933	.127
7.200	.117
7.467	.096
7.733	.080

$k = 5, n = 3$

x	$P_0\{S \geqslant x\}$
8.000	.063
8.267	.056
8.533	.045
8.800	.038
9.067	.028
9.333	.026
9.600	.017
9.867	.015
10.133	.008
10.400	.005
10.667	.004
10.933	.003
11.467	.001
12.000	.000

$k = 5, n = 4$

x	$P_0\{S \geqslant x\}$
.0	1.000
.2	.999
.4	.991
.6	.980
.8	.959
1.0	.940
1.2	.906
1.4	.895
1.6	.850
1.8	.815
2.0	.785
2.2	.759
2.4	.715
2.6	.685
2.8	.630
3.0	.612
3.2	.579
3.4	.552
3.6	.500
3.8	.479
4.0	.442
4.2	.413
4.4	.395
4.6	.370

$k = 5, n = 4$

x	$P_0\{S \geqslant x\}$
4.8	.329
5.0	.317
5.2	.286
5.4	.275
5.6	.249
5.8	.227
6.0	.205
6.2	.197
6.4	.178
6.6	.161
6.8	.143
7.0	.136
7.2	.121
7.4	.113
7.6	.095
7.8	.086
8.0	.080
8.2	.072
8.4	.063
8.6	.060
8.8	.049
9.0	.043
9.2	.038
9.4	.035
9.6	.028
9.8	.025
10.0	.021
10.2	.019
10.4	.017
10.6	.014
10.8	.011
11.0	.010
11.2	.008
11.4	.007
11.6	.006
11.8	.005
12.0	.004
12.2	.004
12.4	.003
12.6	.002
12.8	.002
13.0	.001
13.2	.001
13.4	.001

$k = 5, n = 4$

x	$P_0\{S \geqslant x\}$
13.6	.001
13.8	.000

$k = 5, n = 5$

x	$P_0\{S \geqslant x\}$
.00	1.000
.16	1.000
.32	.994
.48	.986
.64	.972
.80	.958
.96	.932
1.12	.925
1.28	.891
1.44	.865
1.60	.842
1.76	.823
1.92	.789
2.08	.765
2.24	.721
2.40	.707
2.56	.679
2.72	.657
2.88	.613
3.04	.594
3.20	.562
3.36	.535
3.52	.518
3.68	.494
3.84	.454
4.00	.443
4.16	.410
4.32	.398
4.48	.371
4.64	.349
4.80	.325
4.96	.316
5.12	.295
5.28	.275
5.44	.255
5.60	.246

$k = 5, n = 5$		$k = 5, n = 5$		$k = 5, n = 5$		$k = 5, n = 5$	
x	$P_0\{S > x\}$	x	$P_0\{S > x\}$	x	$P_0\{S > x\}$	x	$P_0\{S > x\}$
5.76	.227	8.16	.077	10.56	.019	12.96	.003
5.92	.218	8.32	.073	10.72	.018	13.12	.003
6.08	.195	8.48	.066	10.88	.015	13.28	.003
6.24	.183	8.64	.058	11.04	.013	13.44	.002
6.40	.174	8.80	.056	11.20	.012	13.60	.002
6.56	.164	8.96	.049	11.36	.012	13.76	.002
6.72	.151	9.12	.046	11.52	.010	13.92	.002
6.88	.146	9.28	.042	11.68	.009	14.08	.001
7.04	.130	9.44	.038	11.84	.008	14.24	.001
7.20	.121	9.60	.035	12.00	.007	14.40	.001
7.36	.112	9.76	.032	12.16	.006	14.56	.001
7.52	.107	9.92	.029	12.32	.006	14.72	.001
7.68	.094	10.08	.026	12.48	.005	14.88	.001
7.84	.089	10.24	.024	12.64	.004	15.04	.000
8.00	.082	10.40	.022	12.80	.004		

APPENDIX 9

CRITICAL VALUES FUR PAGE'S L STATISTIC

Source: Rollander and Wolfe, 1973, 372

$k = 3, \quad n = 2(1)20;$
$k = 4(1)8, \quad n = 2(1)12$

For given k, n, and α, the tabled entry is $\ell(\alpha, k, n)$ satisfying $P_0\{L \geq \ell(\alpha, k, n)\} \approx \alpha$.

	k								
	3			4			5		
	α			α			α		
n	.001	.01	.05	.001	.01	.05	.001	.01	.05
2			28		60	58	109	106	103
3		42	41	89	87	84	160	155	150
4	56	55	54	117	114	111	210	204	197
5	70	68	66	145	141	137	259	251	244
6	83	81	79	172	167	163	307	299	291
7	96	93	91	198	193	189	355	346	338
8	109	106	104	225	220	214	403	393	384
9	121	119	116	252	246	240	451	441	431
10	134	131	128	278	272	266	499	487	477
11	147	144	141	305	298	292	546	534	523
12	160	156	153	331	324	317	593	581	570
13	172	169	165						
14	185	181	178						
15	197	194	190						
16	210	206	202						
17	223	218	215						
18	235	231	227						
19	248	243	239						
20	260	256	251						

	k								
	6			7			8		
	α			α			α		
n	.001	.01	.05	.001	.01	.05	.001	.01	.05
2	178	173	166	269	261	252	388	376	362
3	260	252	244	394	382	370	567	549	532
4	341	331	321	516	501	487	743	722	701
5	420	409	397	637	620	603	917	893	869
6	499	486	474	757	737	719	1,090	1,063	1,037
7	577	563	550	876	855	835	1,262	1,232	1,204
8	655	640	625	994	972	950	1,433	1,401	1,371
9	733	717	701	1,113	1,088	1,065	1,603	1,569	1,537
10	811	793	777	1,230	1,205	1,180	1,773	1,736	1,703
11	888	869	852	1,348	1,321	1,295	1,943	1,905	1,868
12	965	946	928	1,465	1,437	1,410	2,112	2,072	2,035

APPENDIX 10

CRITICAL VALUES FOR ALL TREATMENTS MULTIPLE COMPARISONS BASED ON FRIEDMAN RANK SUMS

Source: Hollander and Wolfe, 1973, 373-78

$k = 3, n = 3(1)15;\ k = 4(1)15, n = 2(1)15$

For a given k and n, the entries in the table correspond to $P_0\left\{|R_u - R_v| < r(\alpha, k, n),\ u = 1, \ldots, k-1,\ v = u+1, \ldots, k\right\} \approx 1 - \alpha$.

$k = 3$

n	$r(\alpha, 3, n)$	α
3	6*	.028
4	7*	.042
	8*	.005
5	8*	.039
	9*	.008
6	9*	.029
	10*	.009
7	9*	.051
	10	.023
	11*	.008
8	10*	.039
	11	.018
	12*	.007
9	10*	.048
	11	.026
	12*	.013
10	11*	.037
	12	.019
	13*	.010
11	11*	.049
	12	.028
	14*	.008
12	12*	.038
	13	.022
	14*	.012
13	12*	.049
	13	.030
	15*	.009
14	13*	.038
	14	.023
	16*	.007
15	13*	.047
	14	.028
	16*	.010

$k = 4$

n	$r(\alpha, 4, n)$	α
2	6*	.083
3	8*	.049
	9*	.007
4	10*	.026
	11*	.005
5	11*	.037
	12*	.013
6	12*	.037
	13	.018
	14*	.006
7	13*	.037
	14	.020
	15*	.008
8	14*	.034
	15	.019
	16*	.009
9	15	.032
	17*	.010
10	15*	.046
	16	.029
	18*	.010
11	16	.041
	17	.026
	19	.009
12	17	.038
	18	.023
	20	.008
13	18	.032
	19	.021
	21	.008
14	18	.042
	19	.028
	21	.011
15	19	.037
	20	.024
	22	.010

$k = 5$

n	$r(\alpha, 5, n)$	α
2	8	.050
3	10	.067
	11	.018
	12	.002
4	12	.054
	13	.020
	14	.006
5	14	.040
	16	.006
6	15	.049
	16	.028
	17	.013
7	16	.052
	17	.033
	19	.009
8	18	.036
	19	.022
	20	.012
9	19	.037
	20	.024
	22	.008
10	20	.038
	21	.025
	23	.009
11	21	.038
	22	.025
	24	.010
12	22	.038
	23	.025
	25	.011
13	23	.035
	24	.024
	26	.011
14	24	.034
	25	.024
	27	.011
15	24	.045
	26	.022
	28	.010

$k = 6$

n	$r(\alpha, 6, n)$	α
2	10	.033
3	13	.030
	14	.008
4	15	.047
	16	.018
	17	.006
5	17	.047
	18	.022
	19	.010
6	19	.040
	20	.021
	21	.010
7	20	.049
	21	.032
	23	.010
8	22	.039
	23	.026
	25	.008
9	23	.043
	24	.030
	26	.012
10	24	.047
	26	.023
	28	.009
11	26	.036
	27	.026
	29	.012
12	27	.039
	28	.028
	31	.009
13	28	.039
	29	.028
	32	.010
14	29	.040
	30	.030
	33	.011
15	30	.040
	32	.023
	34	.012

$k = 7$

n	$r(\alpha, 7, n)$	α
2	12	.024
3	15	.048
	16	.016
4	18	.040
	20	.007
5	20	.052
	21	.028
	22	.014
6	22	.050
	23	.032
	25	.009
7	24	.047
	25	.032
	27	.011
8	26	.041
	27	.030
	29	.011

k = 7

n	r(α, 7, n)	α
9	27	.050
	29	.026
	31	.011
10	29	.042
	30	.031
	33	.010
11	30	.049
	32	.027
	35	.009
12	32	.040
	33	.030
	36	.011
13	33	.043
	35	.025
	38	.009
14	34	.047
	36	.028
	39	.011
15	36	.038
	37	.030
	41	.009

k = 8

n	r(α, 8, n)	α
2	14	.018
3	17	.067
	18	.027
	19	.009
4	21	.036
	23	.007
5	23	.057
	24	.034
	26	.009
6	26	.045
	27	.027
	29	.009
7	28	.048
	29	.032
	31	.012
8	30	.046
	31	.033
	34	.009
9	32	.043
	33	.032
	36	.010
10	34	.040
	35	.031
	38	.010
11	35	.048
	37	.028
	40	.010
12	37	.042
	39	.026
	42	.010
13	39	.039
	40	.030
	44	.009
14	40	.042
	42	.027
	45	.012
15	42	.037
	43	.030
	47	.011

k = 9

n	r(α, 9, n)	α
2	15	.069
	16	.014
3	20	.041
	22	.005
4	23	.064
	24	.034
	26	.008
5	27	.040
	28	.023
	29	.013
6	29	.058
	30	.038
	33	.008
7	32	.046
	33	.032
	36	.008
8	34	.049
	36	.026
	38	.012
9	36	.050
	38	.030
	41	.010
10	38	.050
	40	.031
	43	.011
11	40	.048
	42	.030
	46	.009
12	42	.046
	44	.029
	48	.009
13	44	.042
	46	.027
	50	.009
14	46	.041
	48	.026
	52	.009
15	47	.046
	50	.025
	54	.009

k = 10

n	r(α, 10, n)	α
2	17	.056
	18	.011
3	22	.057
	23	.026
	24	.010
4	26	.060
	27	.033
	29	.009
5	30	.047
	31	.029
	33	.010
6	33	.051
	34	.033
	37	.008
7	36	.047
	37	.033
	40	.010
8	38	.052
	40	.031
	43	.010
9	41	.046
	43	.027
	46	.009
10	43	.047
	45	.030
	49	.009
11	45	.049
	47	.032
	51	.010
12	48	.040
	50	.027
	54	.009
13	50	.039
	52	.026
	56	.009
14	52	.039
	54	.026
	58	.010
15	53	.045
	56	.026
	60	.010

k = 11

n	r(α, 11, n)	α
2	19	.045
	20	.009
3	25	.038
	27	.007
4	29	.057
	30	.033
	32	.010
5	33	.055
	34	.035
	37	.008
6	37	.045
	38	.030
	41	.008
7	40	.049
	41	.035
	44	.011
8	43	.046
	44	.035
	48	.009
9	46	.043
	47	.034
	51	.009
10	48	.047
	50	.031
	54	.009
11	51	.040
	53	.027
	57	.009
12	53	.043
	55	.029
	59	.011
13	55	.046
	57	.031
	62	.010
14	57	.045
	60	.026
	64	.011
15	59	.046
	62	.027
	67	.009

$k = 12$

n	$r(\alpha, 12, n)$	α
2	21	.038
	22	.008
3	27	.053
	28	.027
	29	.012
4	32	.055
	33	.033
	35	.011
5	37	.042
	38	.027
	40	.011
6	40	.059
	42	.028
	45	.008
7	44	.050
	46	.026
	49	.009
8	47	.050
	49	.030
	52	.011
9	50	.048
	52	.032
	56	.010
10	53	.047
	55	.032
	59	.010
11	56	.043
	58	.029
	62	.011
12	58	.048
	61	.027
	65	.011

$k = 12$

n	$r(\alpha, 12, n)$	α
13	61	.043
	63	.030
	68	.010
14	63	.046
	66	.027
	71	.009
15	66	.040
	68	.028
	73	.011

$k = 13$

n	$r(\alpha, 13, n)$	α
2	23	.032
	24	.006
3	30	.038
	32	.009
4	35	.054
	36	.033
	38	.012
5	40	.049
	41	.033
	44	.009
6	44	.054
	46	.027
	49	.009
7	48	.051
	50	.028
	53	.010
8	52	.046
	53	.035
	57	.010

$k = 13$

n	$r(\alpha, 13, n)$	α
9	55	.048
	57	.030
	61	.010
10	58	.047
	60	.032
	65	.009
11	61	.046
	63	.032
	68	.010
12	64	.045
	66	.032
	71	.010
13	67	.041
	69	.030
	74	.011
14	69	.046
	72	.028
	77	.010
15	72	.040
	74	.030
	80	.010

$k = 14$

n	$r(\alpha, 14, n)$	α
2	25	.027
	26	.005
3	32	.052
	33	.028
	35	.006
4	38	.053
	39	.034
	41	.013

$k = 14$

n	$r(\alpha, 14, n)$	α
5	43	.057
	45	.027
	47	.012
6	48	.050
	50	.026
	53	.009
7	52	.053
	54	.030
	57	.012
8	56	.051
	58	.031
	62	.010
9	60	.047
	62	.029
	66	.010
10	63	.048
	65	.033
	70	.010
11	66	.049
	69	.029
	74	.009
12	69	.048
	72	.030
	77	.010
13	72	.047
	75	.030
	80	.011

$k = 14$

n	$r(\alpha, 14, n)$	α
14	75	.045
	78	.028
	84	.009
15	78	.043
	81	.028
	87	.010

$k = 15$

n	$r(\alpha, 15, n)$	α
2	26	.071
	27	.024
	28	.005
3	35	.039
	37	.010
4	41	.053
	42	.035
	45	.008
5	47	.046
	48	.033
	51	.010
6	52	.047
	53	.035
	57	.009
7	56	.055
	58	.032
	62	.010

$k = 15$

n	$r(\alpha, 15, n)$	α
8	60	.056
	63	.027
	67	.009
9	64	.052
	67	.028
	71	.011
10	68	.049
	71	.028
	75	.011
11	72	.043
	74	.032
	79	.011
12	75	.045
	78	.028
	83	.010
13	78	.046
	81	.030
	87	.009
14	81	.046
	84	.030
	90	.010
15	84	.043
	87	.029
	94	.009

APPENDIX 11

CRITICAL VALUES FOR THE RANGE OF k INDEPENDENT N(0,1) VARIABLES

Source: Hollander and Wolfe, 1973, 330

For a given k and α, the tabled entry is $q(\alpha, k, \infty)$.

	α								
k	.0001	.0005	.001	.005	.01	.025	.05	.10	.20
2	5.502	4.923	4.6S4	3.970	3.643	3.170	2.772	2.326	1.812
3	5.864	5.316	5.063	4.424	4.120	3.682	3.314	2.902	2.424
4	6.083	5.553	5.309	4.694	4.403	3.984	3.633	3.240	2.784
5	6.240	5.722	5.484	4.886	4.603	4.197	3.858	3.478	3.037
6	6.362	5.853	5.619	5.033	4.757	4.361	4.030	3.661	3.232
7	6.461	5.960	5.730	5.154	4.882	4.494	4.170	3.808	3.389
8	6.546	6.050	5.823	5.255	4.987	4.605	4.286	3.931	3.520
9	6.618	6.127	5.903	5.341	5.078	4.700	4.387	4.037	3.632
10	6.682	6.196	5.973	5.418	5.157	4.784	4.474	4.129	3.730
11	6.739	6.257	6.036	5.485	5.227	4.858	4.552	4.211	3.817
12	6.791	6.311	6.092	5.546	5.290	4.925	4.622	4.285	3.895
13	6.837	6.361	6.144	5.602	5.348	4.985	4.685	4.351	3.966
14	6.880	6.407	6.191	5.652	5.400	5.041	4.743	4.412	4.030
15	6.920	6.449	6.234	5.699	5.448	5.092	4.796	4.468	4.089
16	6.957	6.488	6.274	5.742	5.493	5.139	4.845	4.519	4.144
17	6.991	6.525	6.312	5.783	5.535	5.183	4.891	4.568	4.195
18	7.023	6.559	6.347	5.820	*5.574*	5.224	4.934	4.612	4.242
19	7.054	6.591	6.380	5.856	5.611	5.262	4.974	4.654	4.287
20	7.082	6.621	6.411	5.889	5.645	5.299	5.012	4.694	4.329
22	7.135	6.677	6.469	5.951	5.709	5.365	5.081	4.767	4.405
24	7.183	6.727	6.520	6.006	5.766	5.425	5.144	4.832	4.475
26	7.226	6;773	6.568	6.057	5.818	5.480	5.201	4.892	4.537
28	7.266	6.816	6.611	6.103	5.866	5.530	5.253	4.947	4.595
30	7.303	6.855	6.651	6.146	5.911	5.577	5.301	4.997	4.648
32	7.337	6.891	6.689	6.186	5.952	5.620	5.346	5.044	4.697
34	7.370	6.925	6.723	6.223	5.990	5.660	5.388	5.087	4.743
36	7.400	6.957	6.756	6.258	6.026	5.698	S.427	5.128	4.786
38	7.428	6.987	6.787	6.291	6.060	5.733	5.463	5.166	4.826
40	7.455	7.015	6.816	6.322	6.092	5.766	5.498	5.202	4.864
50	7.571	7.137	6.941	6.454	6.228	5.909	5.646	5.357	5.026
60	7.664	7.235	7.041	6.561	6.338	6.023	5.764	5.480	5.155
70	7.741	7.317	7.124	6.649	6.429	6.118	5.863	5.582	5.262
80	7.808	7.387	7.1%	6.725	6.507	6.199	5.947	S.669	5.353
90	7.866	7.448	7.259	6.792	6.575	6.270	6.020	5.745	5.433
100	7.918	7.502	7.314	6.850	6.636	6.333	6.085	5.812	5.503

APPENDIX 12

UPPER TAIL PROBABILITIES OF THE KRUSKAL -WALLIS H STATISTIC

Source: Hollander and Wolfe, 1973, 294-310

$k = 3, n_1 = 1(1)5, n_2 = n_1(1)5, 2 \leq n_3 = n_2(1)5$

For $k = 3$ and sample sizes n_1, n_2, n_3, the tabled entry for the point x is $P_0\{H \geq x\}$. Thus if x is such that $P_0\{H \geq x\} = \alpha$, then $h(\alpha, 3, (n_1, n_2, n_3)) = x$.

$n_1 = 1, n_2 = 1, n_3 = 2$

x	$P_0\{H \geq x\}$
.300	1.000
1.800	.833
2.700	.500

$n_1 = 1, n_2 = 1, n_3 = 3$

x	$P_0\{H \geq x\}$
.533	1.000
.800	.800
2.133	.700
3.200	.300

$n_1 = 1, n_2 = 1, n_3 = 4$

x	$P_0\{H \geq x\}$
.143	1.000
.786	.933
1.000	.800
1.286	.667
2.143	.600
2.500	.467
3.571	.200

$n_1 = 1, n_2 = 1, n_3 = 5$

x	$P_0\{H \geq x\}$
.257	1.000
.429	.905
1.029	.857
1.114	.762
1.457	.667
1.714	.571

$n_1 = 1, n_2 = 1, n_3 = 5$

x	$P_0\{H \geq x\}$
2.314	.524
2.829	.333
3.857	.143

$n_1 = 1, n_2 = 2, n_3 = 2$

x	$P_0\{H \geq x\}$
.000	1.000
.400	.933
.600	.867
1.400	.733
2.000	.600
2.400	.467
3.000	.333
3.600	.200

$n_1 = 1, n_2 = 2, n_3 = 3$

x	$P_0\{H \geq x\}$
.095	1.000
.238	.933
.429	.900
.810	.833
.857	.800
1.238	.700
1.381	.600
1.952	.567
2.143	.533
2.381	.433
3.095	.267
3.524	.200
3.857	.133
4.286	.100

$n_1 = 1, n_2 = 2, n_3 = 4$

x	$P_0\{H \geq x\}$
.000	1.000
.161	.971
.268	.933
.321	.895
.536	.857
.643	.819
.696	.800
1.018	.781
1.071	.743
1.125	.705
1.286	.667
1.393	.629
1.446	.590
1.875	.533
2.036	.495
2.143	.476
2.250	.457
2.411	.400
2.571	.305
2.786	.286
2.893	.267
3.161	.190
3.696	.171
3.750	.133
4.018	.114
4.500	.076
4.821	.057

$n_1 = 1, n_2 = 2, n_3 = 5$

x	$P_0\{H \geq x\}$
.050	1.000
.133	.964
.200	.940
.450	.905
.467	.845

$n_1 = 1, n_2 = 2, n_3 = 5$

x	$P_0\{H \geq x\}$
.583	.821
.667	.798
.717	.774
1.000	.750
1.117	.738
1.200	.714
1.250	.655
1.383	.619
1.533	.583
1.783	.560
1.800	.536
1.917	.488
2.050	.464
2.333	.429
2.450	.393
2.717	.298
2.800	.286
2.867	.214
3.133	.202
3.333	.190
3.383	.179
3.783	.131
4.050	.119
4.200	.095
4.450	.071
5.000	.048
5.250	.036

$n_1 = 1, n_2 = 3, n_3 = 3$

x	$P_0\{H \geq x\}$
.000	1.000
.143	.986
.286	.957
.571	.871
1.000	.771

$n_1 = 1, n_2 = 3, n_3 = 3$

x	$P_0\{H > x\}$
1.143	.743
1.286	.600
1.571	.571
2.000	.514
2.286	.486
2.571	.329
3.143	.243
3.286	.157
4.000	.129
4.571	.100
5.143	.043

$n_1 = 1, n_2 = 3, n_3 = 4$

x	$P_0\{H > x\}$
.056	1.000
.097	.971
.208	.950
.333	.921
.431	.900
.500	.871
.556	.843
.764	.786
.875	.743
1.097	.721
1.208	.707
1.222	.679
1.389	.629
1.431	.557
1.764	.536
1.833	.514
1.875	.471
2.097	.457
2.208	.443
2.333	.429
2.431	.371
2.722	.300
2.764	.229
3.000	.221
3.097	.214
3.208	.200
3.222	.157

$n_1 = 1, n_2 = 3, n_3 = 4$

x	$P_0\{H > x\}$
3.764	.136
3.889	.129
4.056	.093
4.097	.086
4.208	.079
4.764	.071
5.000	.057
5.208	.050
5.389	.036
5.833	.021

$n_1 = 1, n_2 = 3, n_3 = 5$

x	$P_0\{H > x\}$
.000	1.000
.071	.992
.160	.972
.178	.952
.284	.929
.338	.889
.551	.869
.604	.853
.640	.833
.711	.770
.818	.750
.960	.730
1.084	.694
1.138	.683
1.351	.651
1.404	.611
1.440	.591
1.511	.571
1.600	.560
1.671	.520
1.778	.488
1.884	.480
1.938	.468
2.044	.452
2.204	.437
2.400	.413
2.418	.405
2.560	.341

$n_1 = 1, n_2 = 3, n_3 = 5$

x	$P_0\{H > x\}$
2.844	.258
2.951	.218
3.040	.210
3.218	.190
3.271	.183
3.378	.143
3.484	.135
3.804	.131
3.840	.123
4.018	.095
4.284	.083
4.338	.079
4.551	.075
4.711	.056
4.871	.052
4.960	.048
5.404	.044
5.440	.036
5.760	.028
6.044	.020
6.400	.012

$n_1 = 1, n_2 = 4, n_3 = 4$

x	$P_0\{H > x\}$
.000	1.000
.067	.987
.167	.968
.267	.930
.300	.911
.567	.873
.600	.835
.667	.803
.867	.759
.967	.721
1.067	.689
1.200	.676
1.367	.644
1.500	.600
1.667	.537
1.767	.498
2.167	.460

$n_1 = 1, n_2 = 4, n_3 = 4$

x	$P_0\{H > x\}$
2.267	.410
2.400	.384
2.467	.349
2.667	.305
2.700	.260
2.967	.235
3.000	.222
3.267	.178
3.367	.171
3.467	.152
3.867	.121
3.900	.108
4.067	.102
4.167	.083
4.267	.070
4.800	.067
4.867	.054
4.967	.048
5.100	.041
5.667	.035
6.000	.029
6.167	.022
6.667	.010

$n_1 = 1, n_2 = 4, n_3 = 5$

x	$P_0\{H > x\}$
.033	1.000
.060	.983
.104	.968
.186	.952
.273	.938
.278	.922
.295	.906
.360	.890
.409	.875
.540	.848
.622	.821
.731	.806
.758	.794
.796	.778
.818	.762

$n_1 = 1, n_2 = 4, n_3 = 5$

x	$P_0\{H > x\}$
.906	.730
.933	.719
.976	.690
1.151	.676
1.167	.665
1.195	.651
1.233	.640
1.342	.625
1.369	.614
1.495	.606
1.500	.589
1.587	.562
1.604	.535
1.669	.517
1.778	.498
1.806	.483
1.849	.468
1.931	.460
2.040	.441
2.067	.432
2.106	.419
2.242	.406
2.286	.400
2.455	.394
2.460	.354
2.504	.346
2.591	.300
2.651	.286
2.896	.251
2.913	.222
2.940	.216
3.000	.208
3.087	.194
3.158	.187
3.240	.183
3.349	.151
3.524	.146
3.595	.138
3.682	.132
3.813	.110
3.960	.102
3.987	.098
4.206	.095
4.222	.087

$n_1 = 1, n_2 = 4, n_3 = 5$

x	$P_0\{H > x\}$
4.287	.071
4.549	.067
4.636	.063
4.724	.060
4.833	.059
4.860	.056
4.986	.044
5.078	.041
5.160	.038
5.515	.037
5.558	.035
5.596	.033
5.733	.027
5.776	.025
5.858	.024
5.864	.022
5.967	.021
6.431	.019
6.578	.016
6.818	.013
6.840	.011
6.954	.008
7.364	.005

$n_1 = 1, n_2 = 5, n_3 = 5$

x	$P_0\{H > x\}$
.000	1.000
.036	.994
.109	.982
.146	.956
.182	.944
.327	.920
.400	.885
.436	.872
.546	.847
.582	.802
.727	.792
.836	.771
.909	.752
.982	.716
1.127	.669
1.200	.646

$n_1 = 1, n_2 = 5, n_3 = 5$

x	$P_0\{H > x\}$
1.309	.630
1.346	.605
1.600	.584
1.636	.571
1.709	.509
1.746	.493
1.782	.468
1.927	.462
2.000	.438
2.146	.422
2.182	.411
2.327	.379
2.436	.374
2.509	.361
2.582	.314
2.727	.286
2.909	.242
2.946	.227
3.236	.188
3.346	.168
3.382	.161
3.527	.141
3.600	.132
3.636	.116
3.927	.113
4.036	.105
4.109	.086
4.182	.082
4.400	.076
4.546	.074
4.800	.056
4.909	.053
5.127	.046
5.236	.039
5.636	.033
5.709	.030
5.782	.027
6.000	.022
6.146	.019
6.509	.018
6.546	.015
6.582	.014
6.727	.012
6.836	.011

$n_1 = 1, n_2 = 5, n_3 = 5$

x	$P_0\{H > x\}$
7.309	.009
7.527	.008
7.746	.005
8.182	.002

$n_1 = 2, n_2 = 2, n_3 = 2$

x	$P_0\{H > x\}$
.000	1.000
.286	.933
.857	.800
1.143	.667
2.000	.533
2.571	.400
3.429	.333
3.714	.200
4.571	.067

$n_1 = 2, n_2 = 2, n_3 = 3$

x	$P_0\{H > x\}$
.000	1.000
.179	.971
.214	.895
.500	.857
.607	.800
.714	.743
.857	.686
1.179	.657
1.357	.619
1.464	.562
1.607	.524
1.929	.467
2.000	.438
2.214	.419
2.429	.381
2.464	.362
2.750	.324
2.857	.286
3.179	.267
3.429	.248

$n_1 = 2, n_2 = 2, n_3 = 3$

x	$P_0\{H \geq x\}$
3.607	.238
3.750	.219
3.929	.181
4.464	.105
4.500	.067
4.714	.048
5.357	.029

$n_1 = 2, n_2 = 2, n_3 = 4$

x	$P_0\{H \geq x\}$
.000	1.000
.125	.971
.167	.914
.333	.890
.458	.862
.500	.814
.667	.757
.792	.733
1.000	.695
1.125	.657
1.333	.581
1.500	.552
1.792	.514
1.833	.486
2.000	.448
2.125	.410
2.458	.362
2.667	.333
2.792	.314
2.833	.295
3.000	.276
3.125	.248
3.167	.229
3.458	.210
3.667	.190
4.000	.181
4.125	.152
4.167	.105
4.458	.100
4.500	.090
5.125	.052

$n_1 = 2, n_2 = 2, n_3 = 4$

x	$P_0\{H \geq x\}$
5.333	.033
5.500	.024
6.000	.014

$n_1 = 2, n_2 = 2, n_3 = 5$

x	$P_0\{H \geq x\}$
.000	1.000
.093	.984
.133	.937
.240	.913
.360	.881
.373	.844
.533	.807
.573	.791
.773	.759
.840	.722
.893	.685
.960	.653
1.093	.638
1.200	.606
1.373	.590
1.440	.563
1.493	.542
1.533	.516
1.693	.495
1.800	.474
2.133	.452
2.160	.444
2.173	.402
2.293	.381
2.333	.365
2.373	.344
2.693	.317
2.760	.296
2.973	.275
3.093	.265
3.133	.254
3.240	.238
3.333	.206
3.360	.196
3.573	.185

$n_1 = 2, n_2 = 2, n_3 = 5$

x	$P_0\{H \geq x\}$
3.773	.175
3.840	.164
3.973	.159
4.093	.148
4.200	.138
4.293	.122
4.373	.090
4.573	.085
4.800	.063
4.893	.061
5.040	.056
5.160	.034
5.693	.029
6.000	.019
6.133	.013
6.533	.008

$n_1 = 2, n_2 = 3, n_3 = 3$

x	$P_0\{H \geq x\}$
.028	1.000
.111	.968
.222	.946
.250	.896
.472	.864
.556	.807
.694	.757
1.000	.686
1.111	.671
1.139	.600
1.361	.564
1.444	.539
1.806	.511
1.889	.446
2.000	.425
2.028	.396
2.250	.368
2.472	.357
2.694	.329
2.778	.307
2.889	.286
3.139	.243

$n_1 = 2, n_2 = 3, n_3 = 3$

x	$P_0\{H \geq x\}$
3.222	.221
3.361	.207
3.778	.200
3.806	.179
4.028	.164
4.111	.129
4.250	.121
4.556	.100
4.694	.093
5.000	.075
5.139	.061
5.361	.032
5.556	.025
6.250	.011

$n_1 = 2, n_2 = 3, n_3 = 4$

x	$P_0\{H \geq x\}$
.000	1.000
.078	.987
.100	.965
.111	.944
.244	.922
.278	.902
.311	.881
.344	.862
.400	.844
.444	.829
.544	.811
.600	.794
.611	.770
.700	.756
.778	.722
.811	.703
.900	.689
.978	.673
1.000	.660
1.078	.627
1.111	.614
1.178	.602
1.244	.586
1.344	.571

$n_1 = 2, n_2 = 3, n_3 = 4$

x	$P_0\{H \geq x\}$
1.378	.559
1.411	.548
1.500	.537
1.600	.511
1.611	.502
1.678	.478
1.711	.468
1.778	.457
1.844	.448
1.944	.437
2.144	.417
2.178	.406
2.200	.398
2.211	.376
2.244	.368
2.378	.357
2.400	.346
2.411	.338
2.444	.329
2.500	.321
2.778	.294
2.800	.284
2.911	.271
2.944	.262
3.011	.256
3.100	.251
3.111	.238
3.244	.232
3.278	.225
3.300	.216
3.311	.203
3.444	.197
3.478	.190
3.544	.184
3.600	.175
3.811	.168
3.844	.163
3.911	.159
3.978	.156
4.000	.149
4.078	.140
4.200	.137
4.278	.124
4.311	.108

$n_1 = 2, n_2 = 3, n_3 = 4$

x	$P_0\{H \geq x\}$
4.378	.105
4.444	.102
4.511	.098
4.544	.086
4.611	.083
4.711	.079
4.811	.076
4.878	.073
4.900	.071
4.978	.059
5.078	.057
5.144	.054
5.378	.052
5.400	.051
5.444	.046
5.500	.040
5.611	.032
5.800	.030
6.000	.024
6.111	.021
6.144	.014
6.300	.011
6.444	.008
7.000	.005

$n_1 = 2, n_2 = 3, n_3 = 5$

x	$P_0\{H \geq x\}$
.014	1.000
.069	.981
.113	.966
.131	.951
.142	.932
.273	.917
.276	.901
.306	.886
.331	.869
.364	.855
.451	.823
.549	.807
.567	.794
.622	.781
.636	.769

$n_1 = 2, n_2 = 3, n_3 = 5$

x	$P_0\{H \geq x\}$
.713	.743
.724	.714
.767	.703
.887	.692
.942	.680
1.014	.659
1.058	.648
1.091	.638
1.149	.616
1.178	.593
1.276	.579
1.324	.569
1.378	.537
1.451	.529
1.586	.519
1.596	.510
1.614	.502
1.713	.483
1.727	.474
1.760	.459
1.814	.451
1.858	.444
1.876	.429
2.022	.420
2.033	.403
2.076	.396
2.106	.389
2.196	.382
2.251	.375
2.294	.368
2.367	.362
2.454	.356
2.458	.350
2.469	.336
2.487	.330
2.546	.321
2.633	.294
2.749	.287
2.818	.279
2.894	.269
2.924	.263
2.949	.257
2.978	.252
3.022	.248

$n_1 = 2, n_2 = 3, n_3 = 5$

x	$P_0\{H \geq x\}$
3.069	.243
3.167	.237
3.186	.233
3.273	.222
3.331	.211
3.342	.206
3.386	.201
3.414	.193
3.506	.189
3.546	.183
3.604	.175
3.676	.171
3.767	.167
3.778	.159
3.822	.156
3.909	.152
3.942	.146
3.996	.139
4.058	.137
4.069	.132
4.204	.129
4.214	.125
4.233	.122
4.258	.120
4.331	.117
4.378	.113
4.494	.101
4.651	.091
4.694	.089
4.724	.087
4.727	.085
4.814	.071
4.869	.067
4.913	.063
4.942	.062
5.076	.060
5.087	.053
5.106	.052
5.251	.049
5.349	.046
5.513	.044
5.524	.043
5.542	.041
5.727	.037

$n_1 = 2, n_2 = 3, n_3 = 5$

x	$P_0\{H \geq x\}$
5.742	.034
5.786	.033
5.804	.033
5.949	.026
6.004	.025
6.033	.024
6.091	.021
6.124	.020
6.294	.017
6.386	.016
6.414	.015
6.818	.012
6.822	.010
6.909	.009
6.949	.006
7.182	.004
7.636	.002

$n_1 = 2, n_2 = 4, n_3 = 4$

x	$P_0\{H \geq x\}$
.000	1.000
.054	.988
.082	.970
.191	.940
.218	.910
.273	.893
.327	.879
.409	.848
.491	.820
.627	.779
.736	.757
.764	.712
.873	.685
.954	.671
1.091	.651
1.146	.638
1.173	.596
1.282	.577
1.309	.559
1.364	.537
1.582	.526
1.636	.510
1.718	.488
1.827	.441
1.964	.426
2.046	.400
2.236	.386
2.264	.375
2.373	.363
2.454	.338
2.509	.317
2.673	.301
2.809	.281
2.918	.272
2.946	.263
3.054	.239
3.136	.228
3.327	.220
3.354	.210
3.464	.192
3.491	.185
3.682	.180
3.764	.166
3.818	.152
4.009	.142
4.364	.125
4.418	.120
4.446	.103
4.554	.098
4.582	.094
4.691	.080
4.773	.075
4.854	.071
4.991	.065
5.127	.057
5.236	.052
5.454	.046
5.509	.044
5.536	.042
5.646	.039
5.727	.034
5.946	.028
6.082	.025
6.327	.024
6.409	.022
6.546	.020
6.600	.017
6.627	.016
6.873	.011
7.036	.006
7.282	.004
7.854	.002

$n_1 = 2, n_2 = 4, n_3 = 5$

x	$P_0\{H \geq x\}$
.000	1.000
.041	.992
.064	.979
.068	.965
.141	.952
.154	.939
.164	.926
.223	.913
.254	.902
.273	.891
.300	.880
.323	.866
.368	.855
.404	.832
.504	.823
.518	.812
.541	.801
.564	.791
.573	.781
.614	.759
.618	.749
.654	.740
.723	.730
.791	.720
.841	.710
.864	.701
.891	.691
.904	.683
.914	.674
.950	.657
.954	.649
1.018	.640
1.023	.632
1.050	.623
1.091	.614
1.200	.607
1.204	.599
1.268	.592
1.291	.576
1.314	.569
1.318	.562
1.391	.554
1.414	.537
1.450	.529
1.473	.521
1.518	.507
1.591	.499
1.618	.491
1.641	.485
1.664	.479
1.704	.472
1.750	.465
1.754	.459
1.814	.452
1.823	.432
1.973	.427
2.004	.420
2.018	.403
2.073	.398
2.114	.392
2.118	.387
2.141	.381
2.164	.375
2.223	.371
2.254	.366
2.291	.361
2.318	.351
2.323	.346
2.391	.335
2.454	.329
2.473	.324
2.504	.320
2.550	.315
2.618	.311
2.700	.306
2.723	.301
2.754	.296

$n_1 = 2, n_2 = 4, n_3 = 5$

x	$P_0\{H \geq x\}$
2.768	.285
2.773	.273
2.868	.267
2.891	.262
2.904	.258
2.914	.249
2.973	.246
3.023	.237
3.050	.234
3.064	.231
3.118	.226
3.164	.221
3.268	.217
3.314	.214
3.341	.208
3.364	.200
3.414	.197
3.454	.193
3.523	.190
3.564	.187
3.568	.184
3.573	.181
3.618	.178
3.641	.175
3.654	.170
3.700	.164
3.704	.160
3.791	.157
3.800	.151
3.818	.148
3.823	.145
3.864	.143
4.041	.139
4.064	.135
4.073	.133
4.091	.130
4.141	.128
4.154	.126
4.200	.123
4.223	.121
4.250	.119
4.323	.116
4.364	.114
4.368	.112
4.404	.110
4.500	.104
4.518	.101
4.541	.098
4.614	.090
4.664	.088
4.768	.079
4.791	.078
4.800	.076
4.818	.074
4.841	.072
4.868	.071
4.950	.063
5.073	.061
5.154	.059
5.164	.053
5.254	.052
5.268	.051
5.273	.049
5.300	.048
5.314	.046
5.414	.045
5.518	.043
5.523	.042
5.564	.038
5.641	.037
5.664	.036
5.754	.035
5.823	.034
5.891	.032
5.954	.030
5.973	.029
6.004	.026
6.041	.025
6.068	.025
6.118	.024
6.141	.023
6.223	.022
6.368	.021
6.391	.021
6.473	.020
6.504	.020
6.541	.017
6.550	.017
6.564	.016
6.654	.016
6.723	.015
6.904	.014
6.914	.013
7.000	.013
7.018	.012
7.064	.012
7.118	.010
7.204	.009
7.254	.009
7.291	.008
7.450	.007
7.500	.007
7.568	.006
7.573	.005
7.773	.004
7.814	.003
8.018	.002
8.114	.001
8.591	.001

$n_1 = 2, n_2 = 5, n_3 = 5$

x	$P_0\{H \geq x\}$
.008	1.000
.046	.988
.069	.978
.077	.966
.169	.947
.192	.928
.254	.896
.323	.877
.377	.859
.415	.830
.446	.822
.538	.807
.562	.775
.623	.759
.692	.749
.746	.735
.808	.719
.815	.688
.908	.674
.931	.661
1.115	.638
1.154	.611
1.185	.593
1.277	.569
1.300	.558
1.362	.552
1.431	.539
1.485	.528
1.523	.516
1.554	.506
1.646	.496
1.669	.486
1.731	.463
1.854	.445
1.915	.434
1.923	.424
2.015	.407
2.038	.398
2.223	.379
2.262	.374
2.285	.363
2.292	.353
2.385	.345
2.408	.330
2.469	.323
2.538	.315
2.592	.300
2.662	.292
2.754	.286
2.777	.279
2.908	.276
2.962	.270
3.023	.243
3.031	.234
3.123	.228
3.146	.218
3.331	.210
3.369	.203
3.392	.198
3.492	.190
3.515	.186
3.577	.181

$n_1 = 2, n_2 = 5, n_3 = 5$

x	$P_0\{H \geq x\}$
3.646	.169
3.738	.165
3.769	.163
3.862	.150
3.885	.146
4.015	.136
4.069	.132
4.131	.130
4.138	.127
4.231	.124
4.254	.114
4.438	.106
4.477	.103
4.508	.100
4.623	.097
4.685	.092
4.754	.084
4.808	.081
4.846	.073
4.877	.068
4.992	.066
5.054	.060
5.177	.057
5.238	.054
5.246	.051
5.338	.047
5.546	.045
5.585	.041
5.608	.040
5.615	.039
5.708	.037
5.731	.036
5.792	.032
5.915	.030
5.985	.028
6.077	.027
6.231	.026
6.346	.025
6.354	.021
6.446	.020
6.469	.019
6.654	.017
6.692	.016
6.815	.015
6.838	.014

$n_1 = 2, n_2 = 5, n_3 = 5$

x	$P_0\{H \geq x\}$
6.969	.013
7.023	.013
7.185	.012
7.208	.011
7.269	.010
7.338	.010
7.392	.009
7.462	.008
7.577	.007
7.762	.007
7.923	.006
8.008	.006
8.077	.006
8.131	.005
8.169	.003
8.292	.003
8.377	.002
8.562	.002
8.685	.001
8.938	.001
9.423	.000

$n_1 = 3, n_2 = 3, n_3 = 3$

x	$P_0\{H \geq x\}$
.000	1.000
.089	.993
.267	.929
.356	.879
.622	.829
.800	.721
1.067	.664
1.156	.629
1.422	.543
1.689	.511
1.867	.439
2.222	.382
2.400	.361
2.489	.339
2.756	.296
3.200	.254
3.289	.232

$n_1 = 3, n_2 = 3, n_3 = 3$

x	$P_0\{H \geq x\}$
3.467	.196
3.822	.168
4.267	.139
4.356	.132
4.622	.100
5.067	.086
5.422	.071
5.600	.050
5.689	.029
5.956	.025
6.489	.011
7.200	.004

$n_1 = 3, n_2 = 3, n_3 = 4$

x	$P_0\{H \geq x\}$
.018	1.000
.046	.984
.118	.970
.164	.941
.200	.925
.336	.895
.346	.869
.409	.842
.454	.830
.482	.817
.636	.791
.700	.764
.746	.717
.891	.690
1.064	.656
1.073	.633
1.136	.611
1.182	.602
1.209	.582
1.427	.541
1.473	.523
1.573	.513
1.618	.497
1.654	.481
1.791	.447
1.800	.433

$n_1 = 3, n_2 = 3, n_3 = 4$

x	$P_0\{H \geq x\}$
1.864	.415
2.091	.402
2.200	.389
2.227	.368
2.300	.351
2.382	.326
2.518	.314
2.527	.303
2.664	.291
2.882	.281
2.927	.273
2.954	.253
3.027	.244
3.073	.234
3.109	.220
3.254	.212
3.364	.203
3.391	.196
3.609	.188
3.682	.180
3.754	.178
3.800	.165
3.836	.150
3.973	.143
4.046	.132
4.091	.126
4.273	.123
4.336	.117
4.382	.111
4.564	.106
4.700	.101
4.709	.092
4.818	.085
4.846	.081
5.000	.074
5.064	.070
5.109	.068
5.254	.064
5.436	.062
5.500	.056
5.573	.053
5.727	.050
5.791	.046
5.936	.036

$n_1 = 3, n_2 = 3, n_3 = 4$

x	$P_0\{H \geq x\}$
5.982	.034
6.018	.027
6.154	.025
6.300	.023
6.564	.017
6.664	.014
6.709	.013
6.746	.010
7.000	.006
7.318	.004
7.436	.002
8.018	.001

$n_1 = 3, n_2 = 3, n_3 = 5$

x	$P_0\{H \geq x\}$
.000	1.000
.048	.994
.061	.970
.133	.958
.170	.948
.194	.926
.242	.902
.315	.890
.376	.868
.412	.847
.436	.826
.533	.804
.546	.794
.594	.783
.679	.765
.776	.725
.848	.686
.970	.668
1.042	.641
1.079	.624
1.103	.609
1.200	.594
1.212	.587
1.261	.571
1.442	.539
1.503	.526

$n_1 = 3, n_2 = 3, n_3 = 5$

x	$P_0\{H \geq x\}$
1.515	.512
1.527	.505
1.576	.491
1.648	.478
1.746	.450
1.770	.437
1.867	.425
2.012	.414
2.048	.403
2.061	.393
2.133	.382
2.170	.367
2.182	.358
2.194	.352
2.315	.342
2.376	.334
2.594	.315
2.667	.306
2.679	.298
2.715	.291
2.836	.267
2.861	.258
2.970	.242
3.079	.239
3.103	.232
3.333	.218
3.382	.215
3.394	.209
3.442	.196
3.467	.184
3.503	.179
3.576	.173
3.648	.167
3.709	.162
3.879	.156
3.927	.149
4.012	.144
4.048	.139
4.170	.135
4.194	.126
4.242	.122
4.303	.117
4.315	.113
4.412	.109

$n_1 = 3, n_2 = 3, n_3 = 5$

x	$P_0\{H \geq x\}$
4.533	.097
4.679	.094
4.776	.090
4.800	.087
4.848	.085
4.861	.082
4.909	.079
5.042	.077
5.079	.069
5.103	.067
5.212	.065
5.261	.062
5.346	.058
5.442	.055
5.503	.053
5.515	.051
5.648	.049
5.770	.047
5.867	.042
6.012	.040
6.061	.033
6.109	.032
6.194	.027
6.303	.026
6.315	.021
6.376	.020
6.533	.019
6.594	.019
6.715	.014
6.776	.013
6.861	.012
6.982	.011
7.079	.009
7.333	.008
7.467	.008
7.503	.006
7.515	.005
7.636	.004
7.879	.003
8.048	.002
8.242	.001
8.727	.001

$n_1 = 3, n_2 = 4, n_3 = 4$

x	$P_0\{H \geq x\}$
.000	1.000
.046	.993
.053	.981
.144	.959
.167	.937
.182	.925
.212	.913
.326	.890
.348	.870
.386	.850
.409	.829
.477	.819
.576	.799
.598	.779
.659	.761
.667	.742
.712	.731
.727	.713
.848	.704
.894	.685
.932	.668
.962	.651
1.053	.635
1.076	.620
1.136	.604
1.144	.597
1.296	.582
1.303	.568
1.326	.553
1.394	.539
1.417	.524
1.500	.510
1.546	.503
1.598	.490
1.636	.477
1.682	.470
1.750	.457
1.803	.444
1.909	.421
1.962	.409
2.053	.388
2.144	.378
2.227	.368
2.296	.354

$n_1=3, n_2=4, n_3=4$	
x	$P_0\{H \geq x\}$
2.303	.344
2.326	.334
2.394	.325
2.417	.315
2.598	.306
2.636	.290
2.667	.281
2.712	.276
2.848	.269
2.894	.261
2.909	.254
2.932	.250
2.962	.243
3.076	.230
3.136	.218
3.326	.212
3.386	.207
3.394	.201
3.417	.195
3.477	.190
3.576	.184
3.598	.178
3.659	.173
3.682	.162
3.727	.160
3.803	.154
3.848	.150
3.932	.145
3.962	.140
4.144	.135
4.167	.131
4.212	.129
4.296	.125
4.303	.121
4.326	.116
4.348	.113
4.409	.106
4.477	.102
4.546	.099
4.576	.097
4.598	.093
4.712	.090
4.750	.087
4.894	.084

$n_1=3, n_2=4, n_3=4$	
x	$P_0\{H \geq x\}$
5.053	.078
5.144	.073
5.182	.068
5.212	.066
5.296	.063
5.303	.061
5.326	.058
5.386	.054
5.500	.052
5.576	.051
5.598	.049
5.667	.047
5.803	.045
5.932	.043
5.962	.041
6.000	.040
6.046	.039
6.053	.035
6.144	.032
6.167	.031
6.182	.030
6.348	.027
6.386	.026
6.394	.025
6.409	.023
6.417	.022
6.546	.021
6.659	.020
6.712	.019
6.727	.018
6.962	.017
7.000	.016
7.053	.014
7.076	.011
7.136	.011
7.144	.010
7.212	.009
7.477	.006
7.598	.004
7.636	.004
7.682	.003
7.848	.003
8.227	.002
8.326	.001

$n_1=3, n_2=4, n_3=4$	
x	$P_0\{H \geq x\}$
8.909	.001

$n_1=3, n_2=4, n_3=5$	
x	$P_0\{H \geq x\}$
.010	1.000
.030	.990
.060	.981
.081	.972
.092	.963
.118	.953
.138	.944
.173	.935
.180	.926
.214	.917
.241	.908
.256	.900
.265	.891
.276	.882
.323	.874
.337	.865
.368	.857
.426	.841
.430	.833
.462	.825
.491	.817
.503	.809
.542	.784
.549	.777
.626	.769
.645	.754
.692	.746
.727	.738
.737	.716
.799	.709
.830	.696
.831	.689
.856	.667
.953	.660
1.004	.654
1.041	.641
1.045	.628

$n_1=3, n_2=4, n_3=5$	
x	$P_0\{H \geq x\}$
1.062	.621
1.103	.615
1.106	.609
1.118	.602
1.137	.590
1.164	.584
1.188	.578
1.241	.572
1.246	.566
1.260	.553
1.349	.548
1.414	.542
1.445	.537
1.465	.522
1.472	.516
1.487	.506
1.506	.495
1.558	.490
1.568	.479
1.599	.475
1.615	.465
1.718	.460
1.733	.455
1.753	.450
1.780	.446
1.814	.441
1.856	.437
1.906	.427
1.927	.423
1.938	.418
1.964	.400
1.968	.396
1.985	.391
2.019	.387
2.030	.383
2.060	.379
2.103	.375
2.112	.366
2.169	.358
2.272	.354
2.308	.350
2.337	.346
2.349	.343
2.368	.335

$n_1=3, n_2=4, n_3=5$	
x	$P_0\{H \geq x\}$
2.388	.332
2.395	.321
2.472	.318
2.481	.311
2.491	.307
2.522	.301
2.573	.294
2.580	.291
2.641	.288
2.645	.284
2.676	.281
2.677	.278
2.737	.271
2.830	.266
2.887	.263
2.908	.260
2.949	.251
2.953	.248
2.964	.240
3.010	.238
3.035	.235
3.087	.232
3.092	.222
3.106	.219
3.137	.216
3.195	.214
3.256	.209
3.260	.206
3.312	.204
3.318	.199
3.353	.197
3.414	.194
3.445	.192
3.462	.190
3.496	.188
3.503	.183
3.506	.181
3.568	.179
3.580	.177
3.599	.173
3.626	.169
3.703	.165
3.722	.163

$n_1=3, n_2=4, n_3=5$	
x	$P_0\{H \geq x\}$
3.753	.161
3.773	.159
3.785	.156
3.810	.152
3.831	.150
3.865	.148
3.876	.146
3.958	.144
4.015	.140
4.030	.137
4.060	.134
4.122	.132
4.154	.131
4.180	.125
4.195	.124
4.235	.121
4.241	.119
4.276	.117
4.318	.115
4.327	.112
4.368	.110
4.419	.109
4.426	.107
4.487	.106
4.522	.105
4.523	.103
4.549	.099
4.564	.097
4.645	.095
4.676	.093
4.754	.091
4.789	.089
4.810	.088
4.830	.083
4.856	.082
4.881	.081
4.891	.078
4.939	.075
4.953	.074
4.983	.073
5.041	.072
5.045	.071
5.106	.070

$n_1=3, n_2=4, n_3=5$	
x	$P_0\{H \geq x\}$
5.137	.068
5.158	.067
5.180	.065
5.291	.063
5.308	.062
5.342	.061
5.349	.061
5.353	.059
5.414	.058
5.426	.057
5.549	.054
5.568	.052
5.619	.051
5.631	.050
5.656	.049
5.660	.048
5.677	.047
5.718	.046
5.722	.045
5.753	.044
5.780	.043
5.804	.041
5.814	.040
5.862	.040
5.876	.039
5.964	.038
6.026	.038
6.030	.037
6.060	.037
6.087	.035
6.164	.035
6.173	.034
6.231	.033
6.265	.032
6.272	.030
6.337	.030
6.368	.029
6.369	.029
6.395	.026
6.410	.025
6.491	.025
6.522	.024
6.542	.023

$n_1=3, n_2=4, n_3=5$	
x	$P_0\{H \geq x\}$
6.580	.021
6.635	.020
6.676	.020
6.703	.019
6.780	.019
6.785	.018
6.799	.016
6.830	.016
6.891	.015
7.004	.015
7.010	.015
7.096	.014
7.106	.014
7.188	.013
7.195	.012
7.256	.012
7.260	.012
7.272	.012
7.291	.011
7.318	.011
7.395	.011
7.445	.010
7.465	.010
7.477	.009
7.523	.007
7.568	.007
7.641	.007
7.708	.006
7.753	.006
7.810	.006
7.876	.006
7.887	.006
7.906	.005
7.927	.005
8.030	.005
8.060	.004
8.077	.004
8.118	.004
8.122	.004
8.215	.003
8.256	.003
8.430	.002
8.446	.002

$n_1 = 3, n_2 = 4, n_3 = 5$

x	$P_0\{H \geq x\}$
8.481	.002
8.503	.001
8.573	.001
8.626	.001
8.795	.001
9.035	.001
9.118	.001
9.199	.000
9.692	.000

$n_1 = 3, n_2 = 5, n_3 = 5$

x	$P_0\{H \geq x\}$
.000	1.000
.026	.996
.035	.989
.088	.974
.106	.959
.114	.951
.141	.944
.193	.930
.220	.916
.237	.902
.264	.895
.316	.880
.352	.866
.422	.840
.457	.819
.484	.813
.536	.800
.563	.788
.580	.763
.659	.751
.695	.745
.721	.733
.774	.721
.791	.698
.826	.686
.879	.675
.950	.653
1.029	.648

$n_1 = 3, n_2 = 5, n_3 = 5$

x	$P_0\{H \geq x\}$
1.037	.643
1.055	.632
1.064	.611
1.116	.602
1.134	.592
1.143	.583
1.248	.573
1.266	.563
1.292	.554
1.371	.550
1.407	.541
1.450	.514
1.459	.506
1.512	.497
1.565	.480
1.688	.472
1.723	.460
1.741	.453
1.750	.445
1.802	.438
1.829	.431
1.855	.420
1.934	.413
1.978	.393
2.066	.386
2.136	.380
2.145	.377
2.163	.370
2.198	.364
2.250	.351
2.321	.339
2.374	.327
2.409	.321
2.479	.315
2.488	.310
2.514	.305
2.593	.299
2.620	.294
2.637	.289
2.716	.276
2.752	.271
2.778	.267
2.848	.262

$n_1 = 3, n_2 = 5, n_3 = 5$

x	$P_0\{H \geq x\}$
2.857	.257
2.884	.255
2.936	.246
2.963	.241
3.094	.237
3.112	.224
3.121	.220
3.165	.216
3.191	.208
3.279	.206
3.306	.202
3.429	.195
3.464	.191
3.516	.187
3.622	.173
3.648	.167
3.666	.164
3.745	.161
3.780	.158
3.798	.152
3.807	.147
3.912	.144
3.965	.142
3.991	.139
4.114	.136
4.141	.135
4.150	.132
4.202	.127
4.220	.125
4.255	.117
4.308	.112
4.352	.110
4.378	.107
4.457	.105
4.466	.104
4.536	.102
4.545	.100
4.571	.098
4.694	.094
4.774	.092
4.826	.089
4.835	.088
4.888	.082

$n_1 = 3, n_2 = 5, n_3 =$

x	$P_0\{H \geq x\}$
4.914	.079
4.941	.077
4.993	.075
5.020	.072
5.064	.070
5.152	.067
5.169	.065
5.222	.065
5.284	.063
5.363	.062
5.407	.059
5.486	.057
5.494	.056
5.521	.055
5.574	.053
5.600	.051
5.626	.051
5.706	.046
5.802	.045
5.837	.042
5.934	.040
5.943	.039
6.022	.038
6.048	.037
6.198	.035
6.207	.034
6.250	.034
6.259	.033
6.286	.031
6.312	.030
6.365	.030
6.391	.028
6.435	.027
6.488	.025
6.550	.024
6.593	.024
6.655	.022
6.734	.022
6.752	.021
6.866	.019
6.892	.018
6.945	.018
6.963	.017

$n_1 = 3, n_2 = 5, n_3 = 5$

x	$P_0\{H \geq x\}$
6.998	.015
7.050	.015
7.121	.014
7.209	.014
7.226	.012
7.288	.012
7.306	.012
7.314	.011
7.437	.011
7.543	.010
7.578	.010
7.622	.009
7.736	.009
7.763	.008
7.780	.008
7.859	.007
7.894	.007
7.912	.007
8.026	.006
8.079	.006
8.106	.006
8.237	.005
8.264	.005
8.316	.005
8.334	.005
8.545	.004
8.571	.004
8.580	.004
8.650	.003
8.659	.003
8.791	.002
8.809	.002
8.950	.002
9.002	.002
9.055	.001
9.284	.001
9.336	.001
9.398	.001
9.521	.000
9.635	.000
9.916	.000
10.057	.000
10.550	.000

$n_1 = 4, n_2 = 4, n_3 = 4$

x	$P_0\{H \geq x\}$
.000	1.000
.038	.994
.115	.968
.154	.941
.269	.913
.346	.864
.462	.840
.500	.815
.615	.770
.731	.746
.808	.706
.962	.667
1.038	.648
1.077	.630
1.192	.592
1.385	.557
1.423	.540
1.500	.510
1.654	.480
1.846	.452
1.885	.436
2.000	.397
2.192	.370
2.346	.348
2.423	.327
2.462	.307
2.577	.296
2.808	.277
2.885	.260
2.923	.252
3.038	.234
3.115	.219
3.231	.212
3.500	.197
3.577	.173
3.731	.162
3.846	.151
3.962	.145
4.154	.136
4.192	.131
4.269	.122
4.308	.114
4.500	.104

$n_1 = 4, n_2 = 4, n_3 = 4$

x	$P_0\{H \geq x\}$
4.654	.097
4.769	.094
4.885	.086
4.962	.080
5.115	.074
5.346	.063
5.538	.057
6.654	.055
5.692	.049
5.808	.044
6.000	.040
6.038	.037
6.269	.033
6.500	.030
6.577	.026
6.615	.024
6.731	.021
6.962	.019
7.038	.018
7.269	.016
7.385	.015
7.423	.013
7.538	.011
7.654	.008
7.731	.007
8.000	.005
8.115	.003
8.346	.002
8.654	.001
8.769	.001
9.269	.001
9.846	.000

$n_1 = 4, n_2 = 4, n_3 = 5$

x	$P_0\{H \geq x\}$
.000	1.000
.030	.996
.033	.981
.086	.974
.096	.967

$n_1 = 4, n_2 = 4, n_3 = 5$

x	$P_0\{H \geq x\}$
.119	.952
.132	.937
.201	.930
.218	.916
.228	.903
.267	.889
.297	.875
.343	.869
.376	.862
.382	.849
.399	.836
.425	.823
.475	.811
.528	.798
.544	.792
.597	.780
.610	.768
.613	.757
.640	.745
.689	.734
.742	.723
.771	.711
.804	.706
.824	.695
.860	.690
.870	.679
.903	.668
.910	.658
.940	.647
1.019	.637
1.058	.627
1.068	.617
1.124	.607
1.167	.598
1.187	.589
1.190	.584
1.203	.574
1.256	.565
1.272	.556
1.299	.548
1.371	.539
1.404	.534
1.414	.526

$n_1=4, n_2=4, n_3=5$		$n_1=4, n_2=4, n_3=5$		$n_1=4, n_2=4, n_3=5$		$n_1=4, n_2=4, n_3$	
x	$P_0\{H \geq x\}$	x	$P_0\{H \geq x\}$	x	$P_0\{H \geq x\}$	x	$P_0\{H \geq x$
1.454	.518	3.013	.228	4.701	.094	6.214	.034
1.503	.509	3.086	.224	4.711	.092	6.228	.033
1.530	.501	3.119	.221	4.728	.091	6.267	.032
1.533	.493	3.129	.217	4.747	.089	6.310	.031
1.586	.485	3.168	.214	4.760	.088	6.343	.030
1.596	.477	3.218	.210	4.813	.086	6.382	.029
1.615	.469	3.260	.206	4.830	.084	6.399	.028
1.668	.465	3.297	.202	4.833	.082	6.462	.027
1.701	.458	3.330	.200	4.896	.081	6.544	.027
1.718	.450	3.382	.197	4.975	.077	6.547	.026
1.744	.443	3.432	.190	5.014	.076	6.597	.026
1.810	.436	3.442	.187	5.024	.074	6.672	.024
1.876	.429	3.481	.183	5.028	.073	6.676	.024
1.899	.422	3.511	.180	5.090	.071	6.804	.023
1.929	.414	3.590	.176	5.172	.069	6.860	.022
1.942	.408	3.613	.170	5.196	.068	6.870	.022
1.958	.401	3.630	.167	5.225	.066	6.887	.021
2.047	.388	3.640	.164	5.344	.065	6.890	.021
2.110	.375	3.656	.160	5.360	.063	6.943	.020
2.140	.371	3.696	.157	5.370	.062	6.953	.020
2.143	.365	3.758	.154	5.387	.061	6.976	.019
2.176	.362	3.828	.151	5.410	.060	7.058	.018
2.196	.356	3.910	.146	5.440	.059	7.075	.017
2.275	.344	3.986	.143	5.476	.057	7.101	.017
2.387	.338	3.989	.141	5.486	.056	7.124	.016
2.390	.332	4.025	.139	5.489	.056	7.190	.016
2.403	.327	4.042	.134	5.519	.054	7.203	.015
2.440	.316	4.068	.132	5.568	.052	7.233	.015
2.443	.310	4.075	.130	5.571	.051	7.240	.014
2.453	.305	4.118	.127	5.618	.050	7.256	.014
2.558	.299	4.170	.125	5.657	.049	7.418	.014
2.575	.293	4.200	.122	5.687	.048	7.467	.013
2.601	.288	4.233	.121	5.756	.047	7.470	.013
2.667	.283	4.253	.119	5.782	.046	7.497	.013
2.670	.279	4.272	.117	5.815	.045	7.503	.012
2.733	.271	4.289	.114	5.819	.043	7.586	.012
2.756	.267	4.332	.112	5.914	.042	7.596	.012
2.799	.262	4.381	.108	6.003	.042	7.714	.011
2.881	.257	4.447	.106	6.013	.041	7.744	.011
2.904	.253	4.497	.104	6.030	.040	7.760	.009
2.918	.249	4.553	.102	6.096	.039	7.767	.009
2.967	.245	4.619	.100	6.119	.038	7.797	.009
2.987	.240	4.668	.098	6.132	.037	7.810	.009
2.997	.236	4.685	.096	6.201	.036	7.833	.008

$n_1=4, n_2=4, n_3=5$	
x	$P_0\{H \geq x\}$
7.942	.007
7.981	.007
8.047	.006
8.113	.006
8.130	.006
8.140	.005
8.156	.005
8.189	.005
8.403	.004
8.440	.004
8.456	.004
8.525	.003
8.558	.003
8.571	.003
8.575	.003
8.604	.003
8.703	.003
8.733	.002
8.782	.002
8.868	.002
8.997	.001
9.053	.001
9.099	.001
9.129	.001
9.168	.001
9.396	.001
9.528	.001
9.590	.001
9.613	.000
9.758	.000
10.118	.000
10.187	.000
10.681	.000

$n_1=4, n_2=5, n_3=5$	
x	$P_0\{H \geq x\}$
.006	1.000
.020	.994
.043	.988
.051	.976
.086	.970

$n_1=4, n_2=5, n_3=5$		$n_1=4, n_2=5, n_3=5$		$n_1=4, n_2=5, n_3=5$	
x	$P_0\{H \geq x\}$	x	$P_0\{H \geq x\}$	x	$P_0\{H \geq x\}$
.111	.958	1.366	.525	2.783	.272
.131	.946	1.411	.518	2.786	.268
.143	.935	1.423	.512	2.831	.257
.180	.929	1.483	.505	2.840	.254
.203	.923	1.551	.498	2.886	.250
.223	.912	1.560	.492	2.931	.246
.226	.901	1.606	.485	2.946	.239
.271	.890	1.620	.479	2.966	.236
.280	.879	1.643	.470	2.991	.232
.326	.874	1.651	.458	3.023	.229
.360	.863	1.686	.455	3.083	.224
.371	.852	1.711	.449	3.103	.221
.386	.841	1.731	.443	3.160	.218
.463	.821	1.743	.437	3.240	.215
.500	.805	1.803	.431	3.243	.211
.523	.800	1.826	.425	3.266	.209
.543	.790	1.871	.420	3.286	.203
.546	.781	1.963	.414	3.311	.200
.591	.771	1.971	.409	3.343	.197
.600	.752	1.986	.398	3.380	.188
.691	.742	2.006	.393	3.403	.187
.706	.738	2.031	.382	3.471	.184
.726	.729	2.051	.377	3.540	.176
.751	.720	2.063	.372	3.571	.174
.771	.711	2.100	.369	3.586	.170
.783	.693	2.143	.364	3.651	.167
.843	.684	2.191	.354	3.743	.162
.863	.675	2.246	.349	3.746	.160
.866	.667	2.280	.344	3.791	.155
.966	.658	2.306	.339	3.800	.153
.980	.654	2.351	.335	3.846	.151
1.000	.650	2.371	.330	3.883	.148
1.003	.642	2.383	.326	3.891	.144
1.011	.626	2.420	.322	3.906	.142
1.046	.617	2.443	.319	3.926	.140
1.071	.610	2.463	.307	3.951	.137
1.140	.594	2.466	.302	3.971	.135
1.183	.587	2.511	.298	4.043	.133
1.186	.579	2.520	.294	4.063	.131
1.286	.572	2.566	.292	4.166	.127
1.300	.553	2.600	.288	4.200	.124
1.323	.547	2.626	.284	4.203	.122
1.331	.540	2.691	.280	4.246	.120
1.346	.532	2.740	.276	4.271	.118

$n_1 = 4, n_2 = 5, n_3 = 5$	
x	$P_0\{H \geq x\}$
4.291	.115
4.303	.113
4.363	.111
4.383	.110
4.386	.108
4.486	.106
4.500	.105
4.520	.101
4.523	.099
4.531	.098
4.591	.096
4.611	.095
4.660	.093
4.706	.092
4.806	.089
4.843	.088
4.851	.086
4.866	.084
4.886	.083
4.911	.079
4.943	.078
4.980	.076
5.023	.075
5.071	.074
5.126	.073
5.163	.070
5.171	.069
5.186	.068
5.206	.067
5.231	.066
5.263	.064
5.323	.063
5.400	.061
5.446	.059
5.460	.058
5.483	.057
5.491	.056
5.526	.056
5.571	.055
5.583	.052
5.620	.051
5.643	050
5.666	.049

$n_1 = 4, n_2 = 5, n_3 = 5$	
x	$P_0\{H \geq x\}$
5.711	.048
5.780	.048
5.803	.047
5.811	.046
5.871	.045
5.903	.043
5.963	.042
5.983	.042
5.986	.041
6.031	.040
6.086	.040
6.100	.038
6.123	.037
6.146	.037
6.166	.035
6.211	.035
6.223	.034
6.283	.034
6.303	.033
6.351	.032
6.406	.031
6.440	.030
6.451	.029
6.486	.029
6.531	.028
6.543	.028
6.603	.027
6.623	.026
6.626	.026
6.671	.025
6.760	.025
6.763	.024
6.771	.024
6.786	.023
6.806	.022
6.831	.022
6.900	.021
6.943	.020
7.000	.019
7.046	.019
7.080	.018
7.106	.018
7.171	.018

$n_1 = 4, n_2 = 5, n_3 = 5$	
x	$P_0\{H \geq x\}$
7.183	.017
7.220	.017
7.243	.017
7.266	.016
7.311	.015
7.320	.015
7.426	.015
7.446	.014
7.471	.014
7.491	.014
7.503	.013
7.563	.013
7.586	.012
7.631	.012
7.640	.011
7.686	.011
7.720	.011
7.766	.010
7.791	.010
7.823	.010
7.860	.010
7.903	.009
7.906	.009
8.006	.009
8.043	.009
8.051	.008
8.066	.008
8.086	.008
8.131	.008
8.143	.008
8.223	.007
8.226	.007
8.271	.007
8.280	.006
8.340	.006
8.363	.006
8.371	.005
8.386	.005
8.431	.005
8.463	.005
8.523	.005
8.543	.005
8.546	.004

$n_1 = 4, n_2 = 5, n_3 = 5$	
x	$P_0\{H \geq x\}$
8.683	.004
8.691	.004
8.726	.004
8.751	.004
8.771	.004
8.969	.003
8.980	.003
9.000	.003
9.011	.003
9.026	.003
9.071	.002
9.103	.002
9.163	.002
9.231	.002
9.286	.002
9.323	.001
9.411	.001
9.503	.001
9.506	.001
9.606	.001
9.643	.001
9.651	.001
9.686	.001
9.926	.001
9.986	.000
10.051	.000
10.063	.000
10.100	.000
10.260	.000
10.511	.000
10.520	.000
10.566	.000
10.646	.000
11.023	.000
11.083	.000
11.571	.000

$n_1 = 5, n_2 = 5, n_3 = 5$	
x	$P_0\{H \geq x\}$
.000	1.000
.020	.998

$n_1 = 5, n_2 = 5, n_3 = 5$	
x	$P_0\{H \geq x\}$
.060	.983
.080	.968
.140	.954
.180	.925
.240	.911
.260	.898
.320	.871
.380	.858
.420	.832
.500	.807
.540	.794
.560	.783
.620	.759
.720	.736
.740	.725
.780	.703
.860	.681
.960	.660
.980	.650
1.040	.620
1.140	.601
1.220	.582
1.260	.564
1.280	.547
1.340	.538
1.460	.521
1.500	.505
1.520	.497
1.580	.481
1.620	.466
1.680	.459
1.820	.444
1.860	.416
1.940	.403
2.000	.390
2.060	.383
2.160	.371
2.180	.365
2.220	.353
2.240	.342

$n_1 = 5, n_2 = 5, n_3 = 5$	
x	$P_0\{H \geq x\}$
2.340	.330
2.420	.319
2.480	.314
2.540	.304
2.580	.294
2.660	.284
2.780	.265
2.880	.256
2.940	.252
2.960	.239
3.020	.231
3.120	.223
3.140	.216
3.260	.208
3.380	.201
3.420	.190
3.440	.184
3.500	.177
3.620	.171
3.660	.165
3.780	.159
3.840	.153
3.860	.150
3.920	.145
3.980	.137
4.020	.132
4.160	.127
4.220	.123
4.340	.118
4.380	.110
4.460	.105
4.500	.102
4.560	.100
4.580	.096
4.740	.092
4.820	.089
4.860	.085
4.880	.084
4.940	.081
5.040	.075

$n_1 = 5, n_2 = 5, n_3 = 5$	
x	$P_0\{H \geq x\}$
5.120	.072
5.180	.070
5.360	.065
5.420	.063
5.460	.060
5.540	.055
5.580	.053
5.660	.051
5.780	.049
5.820	.048
5.840	.046
6.000	.044
6.020	.043
6.080	.040
6.140	.038
6.180	.036
6.260	.035
6.320	.033
6.480	.032
6.500	.031
6.540	.030
6.620	.028
6.660	.027
6.720	.026
6.740	.025
6.860	.024
6.980	.021
7.020	.020
7.220	.019
7.260	.018
7.280	.018
7.340	.016
7.440	.015
7.460	.015
7.580	.014
7.620	.013
7.740	.012
7.760	.012
7.940	.011
7.980	.011

$n_1 = 5, n_2 = 5, n_3 = 5$	
x	$P_0\{H \geq x\}$
8.000	.009
8.060	.009
8.180	.008
8.240	.008
8.340	.007
8.420	.007
8.540	.006
8.640	.006
8.660	.006
8.720	.005
8.780	.005
8.820	.005
8.880	.004
8.960	.004
9.060	.004
9.140	.003
9.260	.003
9.360	.003
9.380	.003
9.420	.002
9.500	.002
9.620	.002
9.680	.001
9.740	.001
9.780	.001
9.920	.001
9.980	.001
10.140	.001
10.220	.001
10.260	.000
10.500	.000
10.580	.000
10.640	.000
10.820	.000
11.060	.000
11.180	.000
11.520	.000
11.580	.000
12.020	.000
12.500	.000

APPENDIX 13

CRITICAL VALUES FOR ALL TREATMENTS MULTIPLE COMPARISONS RASED ON KRUSKAL -WALLIS RANK SUMS

Source: Hollander and Wolfe, 1973, **328-29**

$k = 3, n = 2(1)6$; $k = 6, 7, 8, n = 2, 3$;
$k = 4, 5, n = 2, 3, 4$; $k = 9(1)15, n = 2$

For a given k and n, the entries in the table correspond to $P_0\{|R_u - R_v| < y(\alpha, k, n), u = 1, \ldots, k-1, v = u+1, \ldots, k\} \approx 1 - \alpha$.

	n = 2		3		4		5		6	
k	$y(\alpha, k, 2)$	α	$y(\alpha, k, 3)$	α	$y(\alpha, k, 4)$	α	$y(\alpha, k, 5)$	α	$y(\alpha, k, 6)$	α
3	8	.067	15*	.064	24*	.045	33*	.048	43*	.049
			16	.029	25	.031	35	.031	51*	.011
			17*	.011	27*	.011	39*	.009		
4	12	.029	22	.043	34	.049				
			23	.023	36	.026				
			24	.012	38	.012				
5	15	.048	28	.060	44	.056				
	16	.016	30	.023	46	.033				
			32	.007	50	.010				
6	19	.030	35	.055						
	20	.010	37	.024						
			39	.009						
7	22	.056	42	.054						
	23	.021	44	.026						
	24	.007	46	.012						
8	26	.041	49	.055						
	28	.005	51	.029						
			54	.010						
9	29	.063								
	30	.031								
	31	.012								
10	33	.050								
	34	.025								
	35	.009								
11	37	.040								
	38	.020								
	39	.008								

	$n=2$	
k	$y(\alpha, k, 2)$	α
12	40	.062
	41	.033
	43	.006
13	44	.052
	45	.028
	46	.014
14	48	.044
	49	.024
	50	.012
15	52	.038
	54	.010

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