Ex. Find Equation of the plane through $(2,-5,3)$ and perpendicular to the $\langle 3,6,5\rangle$ a,b,c

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ $\quad x_0, y_0, z_0$

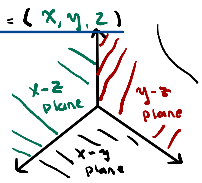
Ex. Find distance between point $a=(x_0,y_0,z_0)$ and point $b=(x_1,y_1,z_1)$

$d=\sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}$

* If point a is $(0,0,0)$ then equation is $d=\sqrt{x^2+y^2+z^2}$

Ex. Find distance of point $a=(x,y,z)$

to x-y plane: $|z|$
to x-z plane: $|y|$
to y-z plane: $|x|$



The plane intersects the
y-axis: $x=0, z=0$
z-axis: $x=0, y=0$
x-axis: $z=0, y=0$

plug in to find equation, find values of a,b,c

Ex. Determine whether all level curves of the plane $z=3x-4y$ are lines

① Find relevant curve by replacing z with k: $3x-4y=k$
② Solve for y: $y=\frac{3}{4}x-\frac{k}{4}$
③ Since this is equation of line, statement is true.

Ex. Describe set of all points that satisfy
$x^2+y^2+z^2=2x+4y+4$
① re-arrange
$x^2-2x+y^2-4y+z^2=4 \;\{(x-1)^2+(y+2)^2+z^2=9\}$
② use 3D formula
$3=\sqrt{(x-1)^2+(y+2)^2+z^2}$
③ ans: sphere of radius 3 while centered on $(1,-2,0)$
* if asking to satisfy $x^2+y^2+z^2<2x-4y+4$, is the interior

Level curves of a function $z=f(x,y)$ are given by the equations $f(x,y)=k$,

To determine Partial Derivatives

Ex. $f(x)_{xy}$ → take derivative of x, then y

$f(x)_{xy}=\frac{\partial^2 f}{\partial y\,\partial x}$

Normal vector to plane given by equation: $ax+by+cz=d$
$\langle a,b,c\rangle$

Exam Q's to review: 2015 F

Taylor Polynomial

$T_n(x)=\sum_{k=0}^{n}\frac{f^{k}(a)}{k!}(x-a)^k$

$=f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f^{(n)}(a)}{n!}(x-a)^n$

SEQUENCES & SERIES

① Integral Test
$f(x)$ positive, continuous, decreasing (first derivative)
- look at improper integral

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p>1$

→ $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p<1$

turn into improper int.

Ex. $\sum_{n=1}^{\infty}\frac{n}{n^2+1}=\int_1^t \frac{x}{x^2+1}=\lim_{t\to\infty}\int \frac{x}{x^2+1}\,dx \cdot \frac{1}{2}\int\frac{1}{u}\,du=$

1) POSITIVE
2) continuous
3) decreasing

$u=x^2+1$
$\frac{1}{2}du=x\,dx$

$\frac{1}{2}\ln(|x^2+1|)$

$\lim_{t\to\infty}\left[\frac{1}{2}\ln|x^2+1|\right]_1^t$

$\lim_{t\to\infty}\frac{1}{2}\ln|t^2+1|-\frac{1}{2}\ln|2|$

$f'(x)=\frac{(x^2+1)(1)-x\cdot 2x}{(x^2+1)^2}$

$=\frac{-x^2+1}{(x^2+1)^2}$ decreasing

$1-x^2=0$
$(1-x)(1+x)$
$x=1,-1$

f' sign

since $\ln|t|=\infty$, improper integral diverges, so then original series diverges

* Improper integrals
Limit is finite num - converges
Limit is ∞ or $-$ DNE - diverges

② P-SERIES

$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^p} \quad \text{- convert to this form} \quad p>1 \to \text{converge}, \quad p\le 1 \to \text{diverge}}$

③ Geometric Series $\quad r=\text{ratio}\left(\frac{2}{3}\right)$

$\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^i=\frac{1}{1-r}$ if $|r|<1$ converge

DNE $\quad |r|>1$ diverge

Ex. $\sum_{i=50}^{\infty}\left(\frac{2}{3}\right)^{i+17}=\left(\frac{2}{3}\right)^{67}\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^i=\left(\frac{2}{3}\right)^{67}\cdot\frac{1}{1-\frac{2}{3}}$

$50+17$

Ex. $\sum_{i=0}^{\infty}\left(-\frac{10}{9}\right)^i=$ DNE since $|r|>1$, diverges

FOR PARTIAL SUMS

$\sum_{i=0}^{n} r^i=\frac{1-r^{n+1}}{1-r}$ * plug n in, solve

⑥ Limit Comparison test

* divide simple by complex, check limit, and determine whether converge or div

Ex. $\sum\frac{1}{n^{3/2}}$ converges (p-series)

does $\sum\frac{1}{n^{3/2}-n}$ converge?

$L=\lim_{n\to\infty}\frac{\frac{1}{n^{3/2}}}{\frac{1}{n^{3/2}-n}}=\lim_{n\to\infty}\frac{n^{3/2}-n}{n^{3/2}}=\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n}}\right)=1$ one is finite,

* choose easy series to compute (consider largest pieces of top and bottom)

So by limit comparison,

$\sum\frac{1}{n^{3/2}-n}$ converges too.

Linearity Rules

$\int_1^2 f(x)\,dx=-\int_2^1 f(x)\,dx$

Integrate thru u-sub
• If you see that derivative of value is in it

$\int x^2\cdot(1-2x^3)^{\frac{1}{2}}$

$u=1-2x^3$
$\frac{1}{6}du=x^2$ etc.

(constant) < (logarithm) < (polynomial) < (exponential)
$1+\ln n+n^2+2^n$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$ memorize

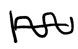
Try this 1st check nth term if

④ Divergence Test $\quad$ if $\neq 0$, then diverge

* USE RATIO

$\sum_{n=1}^{\infty}\frac{n^2+3n+1}{2n^2+4}=\frac{n^2}{n^2}\boxed{\frac{1}{2}}\neq 0$ diverges

Ex. $\sum_{n=1}^{\infty}\ln(\cos n)=\text{DNE}\neq 0$ diverges



oscillates

If $\lim=0$, try other tests

⑦ Ratio Test $\quad$ Ex. Does the series $\sum_{k=1}^{\infty}\frac{10^k}{k!}$ converge or diverge?

$r=\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$

$r<1$, then $\sum a_k$ converges
$r>1$, then $\sum a_k$ diverges
$r=1$, use other test

$\lim_{k\to\infty}\frac{10^{k+1}}{(k+1)!}\Big/\frac{10^k}{k!}$

$\lim_{k\to\infty}\left(\frac{10^{k+1}}{10^k}\right)\left(\frac{k!}{(k+1)!}\right)$

$\frac{10^{k+1}}{10^k}$ $\quad \frac{k!}{(k+1)\cdot k!}$

$10\cdot\frac{1}{k+1}=0$, so converges

* IF CONVERGE
∟ next equation smaller
DIVERGE
∟ next equation larger

⑤ Comparison Test
- works with positive terms only, → either converge or diverge to infinity

Ex. We know $0<\frac{1}{n^2+n}<\frac{1}{n^2}$

We know $\sum_{n=1}\frac{1}{n^2}$ converges (p-series)

By Comparison Test, $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$ converges as well

Ex. We know that $0<\frac{1}{n}<\frac{1}{n-0.1}$

$\sum\frac{1}{n}$ diverges (p series) ← since diverges to inf, greater diverges to inf

By comparison test $\sum_{n=1}^{\infty}\frac{1}{n-0.1}$ diverges as well

Ex. $\sum_{n=1}^{\infty}\frac{5}{2n^2+4n+3}$ converge or diverge? $\quad \frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ $p=2$

We know that $\frac{5}{2n^2}$ converges (p-series)

$0<\frac{5}{2n^2+4n+3}<\frac{5}{2n^2}$ $\quad$ by comparison test, $\frac{5}{n^2+4n+3}$ converges as well

Ex. $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ converge or diverge? $\ln x$ is greater than 1

If $n\ge 3$, then $\ln n\ge 1$

So $\frac{\ln n}{n}>\frac{1}{n}>0$ * watch out

and $\sum_{n=1}^{\infty}\frac{1}{n}$ ← (p-series) diverges

So, by direct comparison, sequence diverges

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

$\sum_{n=1}^{\infty}\frac{1}{n-\sqrt{n}}$ diverges because larger

$\sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}}$ does not work, because smaller, cannot compare

Taylor Series & radius

Ex Find Taylor series for $f(x)=e^x$ around the center $a=0$. Give radius of convergence.

$f(x)=e^x$
$f'(x)=e^x$
$f''(x)=e^x$
...
$f(0)=1$
$f''(0)=1$
...

Taylor Series
$\sum_{n=0}^{\infty}\frac{1}{n!}(x-0)^n$

$\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$

given this, make Taylor Polynomial Expression x^n x

② use ratio test for radius of convergence, since not geometric

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=0<1$

$(n+1)n!$

for any x.

So series converges for all x

radius of convergence is ∞

Taylor Series for $f(x)$ about the center a

$\boxed{\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n}$

$e^r=\sum_{k=0}^{\infty}\frac{r^k}{k!}$ $\quad$ set $r=$ num

[$\sum a_n$ diverges, then also $\sum|a_n|$ diverges
$\sum|a_n|$ then also $\sum a_n$ converges

[If $\sum a_n$ converges and if $\sum|a_n|$ also converges, $\sum a_n$ absolutely convergent

[If $\sum a_n$ converges but $\sum|a_n|$ diverges, we call $\sum a_n$ conditionally convergent

Integration by Parts formula:
$\int u\,dv=uv-\int v\,du$

* for integrals, always add $+C$

SEQUENCES & SERIES

P-SERIES

$\sum_{n=1}^{\infty} \frac{1}{n^p}$

convert

$p > 1 \rightarrow$ converge

$p \leq 1 \rightarrow$ diverge

DIVERGENCE T

if $\sum a_k$ converges: $\lim_{k\to\infty} a_k = 0$

if $\lim_{k\to\infty} a_k \neq 0$: series diverges diverge

$\lim_{n\to\infty} \frac{n^2+2n+4}{2n^2+6n+8} = \frac{1}{2} \neq 0$ diverge

$\sum_{n=1}^{\infty} \ln(\cos n) = \text{DNE} \neq 0$ diverges

RATIO T

$r = \lim_{k\to 0} \left|\frac{a_{k+1}}{a_k}\right|$

$r < 1$, $\sum a_k$ converge

$r > 1$, $\sum a_k$ diverge

$r = 1$, use other test

GEO SERIES

$\sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i = \frac{1}{1-r}$

$|r| < 1$ converge

$|r| \geq 1$ diverge

$\sum_{i=50}^{\infty} \left(\frac{2}{3}\right)^{i+17} = \left(\frac{2}{3}\right)^{67} \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i$

DIR. COMP

FIND SIMILAR

$0 < \frac{1}{n^2+n} < \frac{1}{n^2}$

$0 < \frac{1}{n^2} < \frac{1}{n^2-n}$

$1 \leq \sin k + 2 \leq 3$

USE P-SERIES

SIM LARGER $\rightarrow$ CONV

SIM SMALLER $\rightarrow$ DIV

By Direct comparison, converges as well

LIM. COMP

$\sum \frac{1}{n^{3/2}}$ converges (p-ser.)

does $\sum \frac{1}{n^{3/2}-n}$ converge?

$\lim_{n\to\infty} \left|\frac{1}{n^{3/2}} \cdot \frac{n^{3/2}-n}{1}\right| = \lim_{n\to\infty} \left|1 - \frac{1}{\sqrt{n}}\right| = 1$

DIVIDE complex by simple, check limit, determine whether converge or diverge

by lim comp, converges too

$1 < x < \ln x < x^3 < 3^x$

const < log < POLYN < EXP.

INTEGRAL T.

$f(x)$ positive, cont., decreasing

$\int_1^{\infty} \frac{1}{x^p} dx$ converges

$\Rightarrow$ if $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p < 1$

PARTIAL SUM GEO

$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$

plug n, solve

RADIUS OF CONVERGENCE

① RATIO TEST

② $|x| < 1 \rightarrow -1 < x < 1$

Ex. $\left|\frac{(x+2)^2}{5}\right| < 1$ radius of conv: $\sqrt{5}$

$|(x+2)^2| < 5$

$|x+2| < \sqrt{5}$

FINDING ENDPOINT

Ex. CONVERGES when $A \cdot |x - c| < r$

DIVERGES when $A \cdot |x - c| > r$

[ ] ← CONVERGENT

( ) ← DIVERGENT

$-2 < x < 2$ ← Plug n $\lim_{k\to\infty}$ |equation|

$|x| < 2$

Ex. $\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \int_1^t \frac{x}{x^2+1} = \lim_{t\to\infty} \int \frac{x}{x^2+1} dx = \frac{1}{2}\int \frac{1}{u} du =$

$u = x^2+1$

$\frac{1}{2} du = x\,dx$

$\frac{1}{2}\ln(|x^2+1|)$

$\lim_{t\to\infty} \left[\frac{1}{2}\ln|x^2+1|\right]_1^t$

$\lim_{t\to\infty} \frac{1}{2}\ln|t^2+1| - \frac{1}{2}\ln|1|$

since $\ln|\infty| = \infty$, improper integral diverges, so then original series diverges

TYPES OF CONV.

$\sum a_n$ diverges, $\sum |a_n|$ also diverge

$\sum |a_n|$ converges, $\sum |a_n|$ converges

If $\sum a_n$ converges, & $\sum |a_n|$ also converges, $\sum a_n$ absolutely conv.

If $\sum a_n$ converges but $\sum |a_n|$ diverges, $\sum a_n$ conditionally convergent

FIND CRIT POINTS, SADDLE, etc. given $f(x)$

1) Find $c(a,b)$

2) PLUG INTO D

3D PROBLEMS

** $g(x)$ is a $\leq$ if AREA

1) ASKING AREA? DISK? → FIND CRIT $C(a,b)$ $f_x = 0$, $f_y = 0$, SOLVE FOR X & Y

2) LAGRANGE $f_x = \lambda g_x$ $f_y = \lambda g_y$

i) SET $\lambda = \lambda$

ii) SOLVE X & Y

iii) PLUG INTO CONSTRAINT

$G(x)$? → USE LAGRANGE

3) PLUG INTO $f(x)$, TAKE MAX/MIN REFER TO SHEET FOR $D(a,b)$

TAYLOR SERIES: $\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$

$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^n(a)}{n!}(x-a)^n$

INITIAL VALUES

SOLVE $\frac{dy}{dt} = e^{y/3} \cos t$

$\int e^{-y/3} dy = \int \cos(t)\,dt$

$-3e^{-y/3} = \sin(t) + C_1$

$e^{-y/3} = -\frac{\sin t}{3} - \frac{C_1}{3}$ $C_2 = -\frac{C_1}{3}$

$e^{-y/3} = -\frac{\sin t}{3} + C_2$ * take log

$-\frac{y}{3} = \ln\left(-\frac{\sin t}{3} + C_2\right)$

$y = -3\ln\left(C - \frac{\sin t}{3}\right)$

$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!}$

$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!} = 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{\pi}\right)^n}{n!} = \sum_{k=1}^{\infty} \frac{1}{\pi^k k!}$

EVALUATE SUM * Every time evaluate, look for series

Hence $\sum_{k=1}^{\infty} \frac{1}{\pi^k k!} = e^{\frac{1}{\pi}} - 1$

If Use Simpson's rule to approximate $\int_1^2 x(1 - \ln x)\,dx$, how large so that error is not larger than $\frac{1}{400{,}000}$?

1) Find Fourth Derivative

$f'(x) = 1(1 - \ln x) - x\frac{1}{\ln}x = -\ln x$

$f''(x) = -\frac{1}{x}$ $f^4(x) = -\frac{2}{x^3}$

$f'''(x) = \frac{1}{x^2}$

2) FIND MAX ERROR OF L

$\left|-\frac{2}{x^3}\right| \leq L$ ← Plug in 1 because is largest

$|-2| = L$

3) PLUG INTO ERROR EQUATION   SOLVE N

$\frac{-2}{180} \cdot \frac{(2-1)^5}{n^4} < \frac{1}{400{,}000}$

$\frac{1}{n^4} < \frac{90}{400{,}000}$

$10{,}000 < n^4$

$10 < n^4$

FUND. THEOREM OF CALC

$\frac{d}{dx} \int_{-x}^{x} \sqrt[3]{(1+t)^2}\,dt$

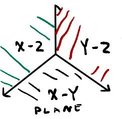
1) PLUG IN, APPLY CHAIN RULE

$\sqrt[3]{(1+x)^2} - (-1)\sqrt[3]{(1+x)^2} = 2\sqrt[3]{1+x^2}$

(x_0, y_0, z_0) $\langle a,b,c \rangle$

Y-Z PARALELL → Ex. $x - 3y + 2z = 1$

$\langle 1, -3, 2 \rangle$



Ex. Are all level curves of $f(x)$ lines?

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

L Ex. $3x - 4y = z$

1) REPLACE Z with K

2) SOLVE FOR Y

3) CHECK TO SEE KIND OF EQ.

$y = \frac{3}{4}x - \frac{k}{4}$ ←

THEOREM OF CALC 2

$\lim_{x\to 0} \frac{1}{x} \int_0^{2017x} \sqrt{2017 + t^{2017}}\,dt$

1) DEFINE AUX FUNC $f(y) = \int_0^y \sqrt{2017 + t^{2017}}\,dt$

2) Compute $\lim_{x\to 0} \frac{f(2017x) - f(0)}{x}$ $g(x) = 2017x$

3) $f(0)$ is 0, $\int_0^0 \sqrt{2017 + t^{2017}} = 0$

4) derivative of $g(x)$ is $2017 f'(2017x)$

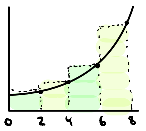
derivative of $g(0)$ is $2017 f'(0)$

$f'(0) = \sqrt{2017 + 0^{2017}} = \sqrt{2017}$ $g'(0) = 2017\sqrt{2017} = 2017^{3/2}$

RIEMANN SUM

Area = $\sum_{i=1} \Delta x\, f(x_i)$

$\Delta x = \frac{b-a}{n}$ $A_R = \Delta x[f(2) + f(4) + f(6) + f(8)]$



$[0, 8] \rightarrow [a, b]$

PS + CS = total gains

SHAPE EQ.

$(x-h)^2 + (y-k)^2 = r^2 \rightarrow$ CIRCLE

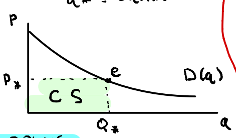
$y = a(x-h)^2 + k \rightarrow$ PARAB.

$y = ax + k \rightarrow$ LINE

CONSUMER SURPLUS

$\int_0^{q_*} d(q)\,dq - p^* q^*$

q_* = equilibrium



Ex. FIND CDF of PDF

$g(x) = e^{-x} e^{e^{-x}}$ where $-\infty < x < \infty$

$G(x) = \int_{-\infty}^{x} g(t)\,dt$

1) COMPUTE INTEGRAL

$\int_{-\infty}^{x} e^{-t} e^{e^{-t}}\,dt$ $u = -e^{-t}$, $du = e^{t}$ $= \int_{-\infty}^{x} e^u\,du$

$= \left[e^{-e^{-t}}\right]_{-\infty}^{x} = e^{-e^{x}}$

$\lim_{t\to\infty} e^{-e^{-t}} = 0$

The plane intersects the

y-axis: $x = 0, z = 0$

z-axis: $x = 0, y = 0$

x-axis: $z = 0, y = 0$

Plug in to find equation, find values of a, b, c

Point is crit when $f_x(a,b) =$ $f_y(a,b) = \text{DNE}$

$\ln x$ is greater than 1

Ex. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge or diverge?

If $n \geq 3$, then $\ln n \geq 1$,

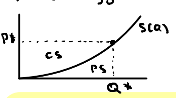
so $\frac{\ln n}{n} > \frac{1}{n} > 0$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ ← (p-series) diverges

So, by direct comparison, sequence diverges

PRODUCER SURPLUS

$P^* q^* - \int_0^{q^*} s(q)\,dq$



If you know CDF, $F(x)$

$f(x) = F'(x)$

If you know PDF, $f(x)$

$F(x) = \int_{-\infty}^{x} f(t)\,dt$

Distance from (x_1, y_1, z_1) to (x_2, y_2, z_2)

$\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (z_1-z_2)^2}$

PROBABILITY

$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$ – take integral, solve

$\int_a^b f(x)\,dx = F(b) - F(A) = Pr(a \leq V \leq b)$

$P(x > \frac{1}{2})$ cumulative distribution function

$0 \leq F(x) \leq 1$, non-decreasing

Variance

square because makes it positive

$Var(X) = \int_{-\infty}^{\infty} (X - E(X))^2 f(x)\,dx$ * Average difference squared between actual and expected

square inside   diff. between actual and expected

$Var(V) = E(V^2) - [E(V)]^2$ * Discrete Calculations

$= \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx - \left[\int_{-\infty}^{\infty} x \cdot f(x)\,dx\right]^2$

square inside   square after calculator

Ex. determine $P(a \leq V \leq b)$ →

2 ways to determine

• $\int_a^b f(x)\,dx$

• $F(b) - F(a)$

$f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

CDF maxes out to 1

$Var(x) = E(x^2) - [E(x)]^2$ * Discrete Calculation

$\sigma(x) = \sqrt{Var(x)}$ * Standard Deviation Formula

$E[x] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$ Understand that symmetrical values have an expected value of zero.

Ex. Find distance of point $a = (x, y, z)$

to x-y plane: $|z|$

to x-z plane: $|y|$

to y-z plane: $|x|$

$f_{xy} = f_{yx}$ Young's Theorem

USEFUL INT FORMULAS

$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

$\int \cos kx\,dx = \frac{1}{k}\sin kx + C$

$\int a^{kx}\,dx = \frac{1}{k}\frac{a^{kx}}{\ln a} + C$

$\int a^x\,dx = \frac{a^x}{\ln a} + C\ (a > 0, a \neq 1)$

$\int \sin kx\,dx = -\frac{1}{k}\cos kx + C$

$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$

PDF → CDF

$f(x) = F'(x)$

$\int (kx+b)^n\,dx = \frac{1}{n+1} \cdot \frac{1}{k}(kx+b)^{n+1} + C$

$\sec x\,dx = \ln|\sec x + \tan x| + C$

$\csc x = \ln|\csc x - \cot x| + C$

L'Hopital's Rule

$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

* If you know PDF

$F(x) = \int_{-\infty}^{x} f(t)\,dt$ - area of curve under distribution

Integration by Parts formula:

$\int u\,dv = uv - \int v\,du$

* for integrals, always add + C

$\int \sin^2 x\,dx = \int\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

derivatives

$\sin x = \cos x$

$\cos x = -\sin x$

$\tan x = \sec^2 x$

$\sec^2 x = \tan x \sec x$

$\csc x = -\csc x \cot x$

$\cot x = -\csc^2 x$

grade 11 trig

$\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$

$\cot x = \frac{1}{\tan x}$ $\tan x = \frac{\sin x}{\cos x}$

$\cot x = \frac{\cos x}{\sin x}$ $\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$

deriv. of exponents

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} e^u = e^u \frac{du}{dx}$

$\frac{d}{dx} a^x = a^x \ln a$

$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$

arc deriv.

$\frac{d}{dx}$ $\arcsin x = \frac{1}{\sqrt{1-x^2}}$ $\text{arc}\sec x = \frac{1}{x\sqrt{x^2-1}}$

$\arccos x = -\frac{1}{\sqrt{1-x^2}}$ $\text{arc}\csc x = -\frac{1}{x\sqrt{x^2-1}}$

$\arctan x = \frac{1}{1+x^2}$

$\text{arccot}\,x = -\frac{1}{1+x^2}$

$\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$

$\frac{d}{dx}\ln|x| = \frac{1}{x}$

$\int \frac{1}{x} = \ln|x|$

Lagrange

$f(x)$ $g(x)$

Given $f(x,y,z)=3x^2+y^2-2z^2$ and $3x+2y-8z=-50$, use Lagrange multipliers to find any max or min values

① $f(x,y,z)$ $g(x,y,z)=k$

equations you must know

variables x, y, z, λ

$f_x=\lambda\cdot g_x$

$f_y=\lambda\cdot g_y$

$f_z=\lambda\cdot g_z$

$g(x,y,z)=k$

← deriv of g with respect to x

② $f_x=\lambda g_x$

$\frac{6x}{6}=\frac{\lambda(3)}{6}$ solve for x

$\boxed{x=\frac{1}{2}\lambda}$

$f_y=\lambda g_y$

$\frac{2y}{2}=\frac{\lambda(2)}{2}$ solve for y

$\boxed{y=\lambda}$

$f_z=\lambda g_z$

$\frac{-4z}{-4}=\frac{\lambda(-8)}{-4}$

$\boxed{z=2\lambda}$

③ plug into restraint

$3x+2y-8z=-50$

$3(\frac{1}{2}\lambda)+2\lambda-8(2\lambda)=-50$

$(\frac{3}{2}\lambda-14\lambda=-50)2$

$3\lambda-28\lambda=-100$

$\lambda=\frac{-100}{-25}$

$\lambda=4$

④ Plug back in to get values for x,y,z

$x=\frac{1}{2}\lambda=2$ $y=\lambda=4$ $z=2\lambda=8$

Point $f(2,4,8)$ ← minimum

To check if min, plug other point to see if greater or lower

$f(2,4,8)=3(2)^2+4^2-2(8)^2$
$=12+16-128$
$=28-128=-100$ ← min

*other point must be in constraint

$f(4,3,7)=-41$

$f(0,11,9)=-41$

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+\sum_{k=1}^{\infty}\frac{x^k}{k!}$$

POWER SERIES

$\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$ for all x,

find convergent series whose sum

$\int_0^\pi \sin(x^2)\,dx$

STEP 1: Power SERIES FORM $\sin(x^2)$

$$\sin(x^2)=\sum_{k=0}^{\infty}\frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!}=\underbrace{\frac{(-1)^k}{(2k+1)!}}\,x^{4k+2}$$

STEP 2: Anti-differentiate

$$\int \sin(x^2)\,dx=\int\Big(\sum \underbrace{\frac{(-1)^k}{(2k+1)!}}_{\text{is constant}}x^{4k+2}\Big)dx=$$

integrated

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\cdot\frac{x^{4k+3}}{(4k+3)}+C$$

STEP 3: PLUG IN $0,\pi$

$$\int_0^\pi \sin(x^2)\,dx=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\pi^{4k+3}}{4k+3}-\cancel{\sum_{k=0}^{\infty}}\ \ x=0\ \ 0$$

$$f(x)=\begin{cases}\frac{1}{4}+\frac{1}{2}|x| & \text{if } -1\le x\le 1\\ 0 & \text{otherwise}\end{cases}$$

$f(x,y,z)=xyz$

constraint: $x^2+4y^2+16z^2=48$

①

$f_x=\lambda\cdot g_x$ $f_y=\lambda\cdot g_y$ $f_z=\lambda g_z$

$yz=\lambda\cdot g_x$ * isolate lambda

$\frac{yz}{2\lambda}=x$ ✗ $\frac{xz}{8y}=y$ ✗ $\frac{xy}{32\lambda}=z$

$\frac{yz}{2x}=\lambda$ $\frac{xz}{8x}=\lambda$ $\frac{xy}{32x}=\lambda$

set equal set equal

$\frac{y\cancel{z}}{2x}=\frac{x\cancel{z}}{8y}$ cancel z

$\frac{8y^2}{2}=\frac{2x^2}{2}$

$\boxed{4y^2=x^2}$

$\frac{xz}{8y}=\frac{xy}{32z}$

$\frac{32z^2}{2}=\frac{8y^2}{2}$

$\boxed{16z^2=4y^2}$

② Plug into constraint

$x^2+4y^2+16z^2=48$

$4y^2+4y^2+4y^2=48$

$16z^2=y(4)$

$\frac{16z^2}{16}=\frac{16}{16}$

$z=\pm1$

$f(x,y,z)=xyz$

multiplying with one another

$\frac{12y^2}{12}=\frac{48}{12}$

$y^2=4$

$4y^2=x^2$

$4(4)=x^2$

$\sqrt{16}=\sqrt{x^2}$

$x=\pm4$

③

$y=\pm2$
$x=\pm4$
$z=\pm1$

Evaluate points

$f(4,2,1)=4(2)(1)=8$

$f(-4,-2,-1)=-4(-2)(-1)=-8$

x	y	z	f
4	2	1	8
-4	-2	-1	-8
4	-2	1	-8
-4	-2	1	8

$2^3=8$ possibilities

$f=+8\to$ max

$f=-8\to$ min

many maxes & mins

Given $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$ for all x, find series converging to $\int_0^1 e^{x^2}dx$

how do you turn e^x to e^{x^2}?

$e^{x^2}=e^r=\sum_{k=0}^{\infty}\frac{r^k}{k!}$ set $r=x^2$

$e^{x^2}=\sum_{k=0}^{\infty}\frac{(x^2)^k}{k!}=\sum_{k=0}^{\infty}\frac{x^{2k}}{k!}$ convert to Power Series, easy to differentiate

integrated

$$\int e^{x^2}dx=\int\sum_{k=0}^{\infty}\frac{x^{2k}}{k!}\,dx=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)(k!)}$$

$$=\int_0^1 e^{x^2}dx=F(1)-F(0)=\sum_{k=0}^{\infty}\frac{1^{2k+1}}{(2k+1)\cdot k!}-\sum_{k=0}^{\infty}\frac{0^{2k+1}}{(2k+1)k!}$$

anti deriv

$$=\sum_{k=0}^{\infty}\frac{1}{(2k+1)\cdot k!}$$

1. Find $F(x)$ for $0<x<1$

$F(x)=\int_{-\infty}^{y}f(y)\,dy$

$=\int_{-\infty}^{-1}f(y)\,dy+\int_{-1}^{0}f(y)+\int_0^x f(y)$

$0+[\frac{y}{4}-\frac{y^2}{4}]_1^0$

$0+\frac{1}{2}+\frac{x^2}{4}+\frac{x^4}{4}$

2. Find $E(x)$

$$\int_{-\infty}^{\infty}x f(x)\,dx=\Big[\frac{x^2}{8}-\frac{x^3}{6}\Big]_{-1}^{0}+\Big[\frac{x^2}{8}+\frac{x^3}{6}\Big]_0^1$$

$=0$

TELESCOPING SERIES

$\sum_{n=1}^{N}\left(\frac{1}{n(n+1)}\right)=\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ ← terms cancel each other

TELESCOPING SERIES

Ex. Evaluate $\sum_{n=1}^{800}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{801}=\lim_{N\to\infty}\left(1-\frac{1}{n+1}\right)$ $\boxed{=1}$

cancel out

$n=1\to\frac{1}{1}-\frac{1}{2}$

$n=2\to\frac{1}{2}-\frac{1}{3}$

$n=3\to\frac{1}{3}-\frac{1}{4}$

⋮

$n=799$ $\frac{1}{799}-\frac{1}{800}$

$n=800$ $\frac{1}{800}-\frac{1}{801}$

if there is $n, n+1, n+2$ is a telescoping series

$S_n=1-\frac{1}{n-1}$

Just remove $\frac{1}{n}$ and plug ∞ only n at bottom

$\sum_{n=1}^{1000}\ln\left(\frac{n+1}{n}\right)=\lim_{n\to\infty}[\ln(N+1)]$

$=\infty$

$\sum_{n=1}^{800}\left(\frac{1}{n}-\frac{1}{n}\right)=1-\frac{1}{n+1}=$

Ex. 2 Evaluate $\sum_{n=1}^{1000}\ln\left(\frac{n+1}{n}\right)=\lim_{N\to\infty}[\ln(N+1)]=\infty$

$\arctan 0=0$

Ans:

looks like a bowl, what is min and max

$f(x,y)=4x^2+10y^2$ (objective)

$g(x,y)=x^2+y^2\le4$ ← 2^2

$x^2+y^2=r^2=4$

STEP 1: USE CRIT POINTS, then check if inside disk

$f_x=8x=0\longrightarrow x=0$
$f_y=20y=0\longrightarrow y=0$ } $(0,0)$ is point of interest

$f_x=0$, solve for it

if ever asked to find min or max of disc, check crit points

STEP 2: LaGrange Multiplier

$g(x,y)=x^2+y^2=4$

$f_x=\lambda g_x$ $f_y=\lambda g_y$

$8x=\lambda\cdot2x$ $20y=\lambda\cdot2y$

$8x-\lambda2x=0$ $20y=8y$

$2x(4-\lambda)=0$ $y=0\to(2,0),(-2,0)$

$x=+2,-2$, (plug into original)

$x=0$
$\lambda=4$ ← plug 0 into original to find y
$y=+2,-2$

$(0,+2)\ (0,-2)$

if asking for disks check for crit, or area

if asking for ring use lagrange, or along a curve on function

To check if inside a constraint, plug in points

$x^2+y^2\le4$

$3^2+0^2\le4$ ✗

$9\le4$

disc = $\le$

ring = $=$

STEP 3 Plug into original

$f(0,2)=40$
$f(0,-2)=40$
$f(2,0)=16$
$f(-2,0)=16$
$f(0,0)=0$

$(0,\pm2)\to$ MAX

$(0,0)\to$ MIN

$\int_a^b dx=b-a$

$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$

$\int_a^a f(x)\,dx=0$

$\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$

PROBABILITY

$Pr(A \text{ or } B)=Pr(A)+Pr(B)$

$Pr(a\le x\le b)=F(b)-F(a)$

$0<F(x)<1$

$F(x)$ non-decreasing

$\lim_{x\to\infty}F(x)=1$

Ex. Describe set of all points that satisfy

$x^2+y^2+z^2=2x+4y+4$

① re-arrange

$x^2-2x+y^2-4y+z^2=4=(x-1)^2+(x+2)^2+z^2=9$

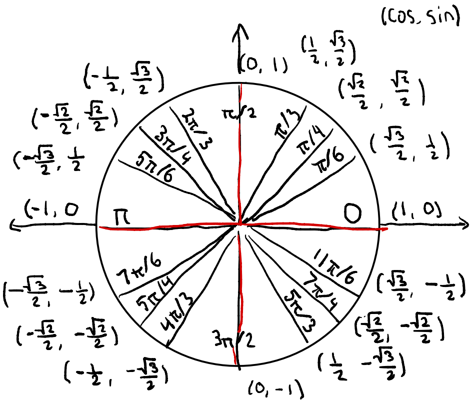
② use 3D formula

$3=\sqrt{(x-1)^2+(y+2)^2+z^2}$

③ ans: sphere of radius 3 while centered on $(1,-2,0)$

* if asking to satisfy $x^2+y^2+z^2<2x-4y+4$, is the interior

(cos, sin)



$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

$\ln(M\cdot N)=\ln M+\ln N$

$\ln\left(\frac{M}{N}\right)=\ln M-\ln N$

$\ln(M^k)=k\ln M$

$\ln 1=0$

$\ln e=1$

$\ln(e^k)=k$

$e^{\ln k}=k$

Ex. Use sigma notation to write Riemann sum for $f(x)=x+2$ on $[0,6]$ with $n=60$. DO NOT EVALUATE the right Riemann sum.

① know that $n=60$

given interval for Riemann sum is $[0,6]$.

Formula: $\sum_{i=1}^{n}f(x_i)(x_i-x_{i-1})$ ← loops thru every rectangle

width of an interval

② Determine interval size. $[0,6]\to\frac{6}{60}\to\frac{1}{10}$

← number

③ ANS: Right Riemann sum is $\sum_{i=1}^{60}\left(\frac{i}{10}+2\right)\frac{1}{10}$ $x=\frac{i}{10}$ the width of each one

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 +$$

Right Riemann Sum

$$\sum_{i=1}^{n} \Delta x \cdot f(x_i)$$

where $\Delta x = b$

and x_i

Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \sum$

Bound 4th derivative

$|f^{(4)}(x)| \le k$, $|c$

abs

$\frac{2}{x^3} \le$ —

(2)

1. Geometry in Three Dimensions

Distance between (x_1, y_1, z_1) to $(x_2, y_2, z_2) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$

Quadratic formula: $\frac{-b \pm \sqrt{b^2-4ac}}{2a} = x$

Radius of circle: $r^2 = x^2 + y^2$ w/centre (h,k): $r^2 = (x-h)^2 + (y-k)^2$

Radius of sphere: $r^2 = x^2 + y^2 + z^2$ w/centre (h,k,l): $r^2 = (x-h)^2 + (y-k)^2 + (z-l)^2$

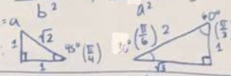
Horizontal Ellipses: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, Vertical Ellipses: $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

(h,k) = centre

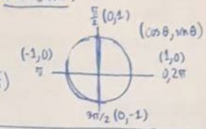
a = longer side
b = shorter side } distance from centre $(a > b)$

$f(a) = b \Leftrightarrow f^{-1}(b) = a$

Special Triangles:



Unit Circle:



2. Partial Derivatives

Ln Laws: 1. $\log_a xy = \log_a x + \log_a y$ 2. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$ 3. $\log_a b = \frac{\log_c b}{\log_c a}$ 4. $\log_a x^n = n \log_a x$ 5. $\log_a b = \frac{1}{\log_b a}$

Notation: $f_x = \frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_y = D_x f = D_1 f$

Single variable derivative: $f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ or $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Partial derivatives: $f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$ and $f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$

Trig identities:

1. $\sin^2 x + \cos^2 x = 1$ 2. $1 + \tan^2 x = \sec^2 x$ 3. $1 + \cot^2 x = \csc^2 x$ 4. $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$ 5. $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x$
6. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ 7. $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$ 8. $\sin 2x = 2 \sin x \cos x$
9. $\cos^2 x = (1 + \cos 2x)/2$ 10. $\sin^2 x = (1 - \cos 2x)/2$

Derivatives:

1. $\frac{d}{dx}(x) = 1$ 2. $\frac{d}{dx}(ax) = a$ 3. $\frac{d}{dx}(x^n) = nx^{n-1} \frac{dy}{dx}$ 4. $\frac{d}{dx}(\cos x) = -\sin(x) \frac{dy}{dx}$ 5. $\frac{d}{dx}(\sin x) = \cos(x) \frac{dy}{dx}$
6. $\frac{d}{dx}(\tan x) = \sec^2 x \frac{dy}{dx}$ 7. $\frac{d}{dx}(\cot x) = -\csc^2 x \frac{dy}{dx}$ 8. $\frac{d}{dx}(\sec x) = \sec(x)\tan(x) \frac{dy}{dx}$
9. $\frac{d}{dx}(\csc x) = -\csc x (\cot x) \frac{dy}{dx}$ 10. $\frac{d}{dx} \ln(x) = \frac{1}{x} \frac{dy}{dx}$ 11. $\frac{d}{dx} e^x = e^x \frac{dy}{dx}$ 12. $\frac{d}{dx} a^x = a^x \ln a \frac{dy}{dx}$
13. $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \frac{dy}{dx}$ 14. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \frac{dy}{dx}$ 15. $\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}} \frac{dy}{dx}$

L'hopital's Rule: $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$
Use when $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Higher Order Derivatives: $f_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ differentiate x then y

Clairaut's Theorem:
If exist and continuous, $f_{xy} = f_{yx}$

Critical points criteria:
1. $f_x(a,b)$ DNE 2. $f_y(a,b)$ DNE or 3. $f_x(a,b) = f_y(a,b) = 0$

3 Cases for Absolute Extrema:
1. (a,b) in interior, $f_x(a,b)$ or $f_y(a,b)$ DNE
2. (a,b) in interior, $f_x(a,b) = f_y(a,b) = 0$
3. (a,b) is boundary point, max or min on boundary

Lagrange Multiplers (3 equations):
1. $f_x(a,b) = \lambda g_x(a,b)$
2. $f_y(a,b) = \lambda g_y(a,b)$
3. $g(a,b) = c$

Bounded vs Unbounded Constraints:
1. Closed curve (circle, square): abs max and min.
2. Not closed, bounded: abs max and min, check endpoints
3. Not closed, unbounded: maybe abs extrema. Use limits and traces.

"more is better" = $\frac{\partial u}{\partial x} > 0$ and $\frac{\partial u}{\partial y} > 0$ Diminishing returns: $\frac{\partial^2 u}{\partial x^2} < 0$ and $\frac{\partial^2 u}{\partial y^2} < 0$

Single variable second derivative test:
1. $f''(c) > 0$, $f(c)$ is minimum
2. $f''(c) < 0$, $f(c)$ is maximum
3. $f''(c) = 0$, test fails

3. Integration

Arithmetic of Summation Notation:
1. $\sum_{i=m}^{n} ca_i = c\left(\sum_{i=m}^{n} a_i\right)$ 2. $\sum_{i=m}^{n}(a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right)$
3. $\sum_{i=m}^{n}(a_i - b_i) = \left(\sum_{i=m}^{n} a_i\right) - \left(\sum_{i=m}^{n} b_i\right)$ 4. $\sum_{i=1}^{n} c = nc$

Change Bounds:
$\sum_{i=1}^{5} 2i + 2 = \sum_{j=0}^{4} 2(j+1) + 2$

Even and Odd:
Even: $f(-x) = f(x) \Rightarrow$ symmetrical with y-axis
Odd: $f(-x) = -f(x) \Rightarrow$ symmetrical around origin
Neither: $f(-x) \neq -f(x)$ or $f(x)$

If $f(x)$ is even: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
If $f(x)$ is odd: $\int_{-a}^{a} f(x)\,dx = 0$

RH Riemann Sum:
$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x$ where $\Delta x = \frac{b-a}{n}$ (width of rectangle; upper bound; lower bound; # of rectangles)

$\lim_{n \to \infty} \sum_{i=1}^{n} f(a + i\Delta x) \cdot \Delta x = \int_a^b f(x)\,dx$

Fundamental Theorem of Calculus:
If $g(x) = \int_a^x f(t)\,dt$, $g'(x) = f(x)$ or $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

Arithmetic of Integration:
1. $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ 5. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ 6. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
3. $\int_a^b c f(x)\,dx = c \cdot \int_a^b f(x)\,dx$ 7. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
4. $\int_a^b dx = \int_a^b 1\,dx = b - a$

$y^2 = 4$

$4y^2 = x^2$

$4(4) = x^2$

$\sqrt{16} = \sqrt{x^2}$

$x = \pm 4$

(3)

$$\boxed{\begin{aligned} y &= \pm 2 \\ x &= \pm 4 \\ z &= \pm 1 \end{aligned}}$$

x	y	z	f
4	2	1	8
-4	-2	-1	-8
4	-2	1	-8
-4	-2	1	8

$2^3 = 8$ possibilities

$f = +8 \to$ max

$f = -8 \to$ min

many maxes & mins

LEvaluate points

$f(4,2,1) = 4(2)(1) = 8$

$f(-4,-2,-1) = -4(-2)(-1) = -8$

Lagrange

Given $\underbrace{f(x,y,z) = 3x^2+y^2-2z^2}_{f(x)}$ and $\underbrace{3x+2y-8z=-50}_{g(x)}$,
use Lagrange multipliers to find any max or min values

① $f(x,y,z)$ $\quad g(x,y,z)=k$

equations you must know $\quad$ variables

$f_x = \lambda \cdot g_x$ $\quad x, y, z, \lambda$

$f_y = \lambda \cdot g_y$

$f_z = \lambda \cdot g_z$

$g(x,y,z) = k$

deriv of g with respect to x

② $f_x = \lambda g_x$

$\frac{6x}{6} = \frac{\lambda(3)}{6}$ $\quad$ solve for x

$\boxed{x = \frac{1}{2}\lambda}$

$f_y = \lambda g_y$ $\quad$ $f_z = \lambda g_z$

$\frac{2y}{2} = \frac{\lambda(2)}{2}$ $\quad$ solve for y $\quad$ $\frac{-4z}{-4} = \frac{\lambda(-8)}{-4}$

$\boxed{y = \lambda}$ $\quad$ $\boxed{z = 2\lambda}$

③ plug into restraint

$3x+2y-8z=-50$

$3(\frac{1}{2}\lambda) + 2\lambda - 8(2\lambda) = -50$

$(\frac{3}{2}\lambda - 14\lambda = -50)\,2$

$3\lambda - 28\lambda = -100$

$\lambda = \frac{-100}{-25}$

$\lambda = 4$

④ Plug back in to get values for x, y, z

$x = \frac{1}{2}\lambda$ $\quad y = \lambda$ $\quad z = 2\lambda$

$= 2$ $\quad = 4$ $\quad = 8$

point $f(2,4,8)$ ← minimum

To check if min, plug other point to see if greater or lower

$f(2,4,8) = 3(2)^2 + 4^2 - 2(8)^2$

$= 12 + 16 - 128$

$= 28 - 128 = -100$ ← min

*other point must be in constraint

$f(4,3,7) = -41$

$f(0,11,9) = -41$

scenario: Imagine λ is a $\pm$ because of square root

$f(2,1,3) = 4(2) + 2(1) + 6(3) = 8 + 2 + 18 = \boxed{28}$ ← max

$f(-2,-1,-3) = -8 - 2 - 18 = \boxed{-28}$ ← min

plug into f(x) and check for max and min

$f(3,1,2)$

$3^2 + 1^2 + 2^2 = 14$ } lies within constraint

$9 + 1 + 4 = 14$

$4(3) + 2(1) + 6(2)$

$12 + 2 + 12 = 26$

$f(x,y,z) = xyz$

constraint: $x^2 + 4y^2 + 16z^2 = 48$

①

$f_x = \lambda \cdot g_x$ $\quad f_y = \lambda \cdot g_y$ $\quad f_z = \lambda g_z$

$yz = \lambda \cdot g_x$ $\quad$ * isolate lambda

$\frac{yz}{2x} = x$ $\quad \frac{xz}{8y} = y$ $\quad \frac{xy}{32\lambda} = z$

$\frac{yz}{2x} = \lambda$ $\quad \frac{xz}{8x} = \lambda$ $\quad \frac{xy}{32x} = \lambda$

set equal $\quad$ set equal

$\frac{y\cancel{z}}{2x} = \frac{x\cancel{z}}{8y}$ $\quad$ cancel z

$\frac{8y^2}{2} = \frac{2x^2}{2}$

$\boxed{4y^2 = x^2}$

$\frac{xz}{8y} = \frac{xy}{32z}$

$\frac{32z^2}{2} = \frac{8y^2}{2}$

$\boxed{16z^2 = 4y^2}$

② Plug into constraint

$x^2 + 4y^2 + 16z^2 = 48$

$4y^2 + 4y^2 + 4y^2 = 48$

$\frac{12y^2}{12} = \frac{48}{12}$

$y^2 = 4$

$4y^2 = x^2$

$4(4) = x^2$

$\sqrt{16} = \sqrt{x^2}$

$x = \pm 4$

$16z^2 = y(4)$

$\frac{16z^2}{16} = \frac{16}{16}$

$z = \pm 1$

③

$y = \pm 2$

$x = \pm 4$

$z = \pm 1$

LEvaluate points

$f(4,2,1) = 4(2)(1) = 8$

$f(-4,-2,-1) = -4(-2)(-1) = -8$

$f(x,y,z) = xyz$

multiplying with one another

x	y	z	f
4	2	1	8
-4	-2	-1	-8
4	-2	1	-8
-4	-2	1	8

$2^3 = 8$ possibilities

$f = +8 \rightarrow$ max

$f = -8 \rightarrow$ min

many maxes & mins

$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ for all x,

find convergent series whose sum

$\int_0^{\pi} \sin(x^2)\,dx$

STEP 1: Power SERIES FORM $\sin(x^2)$

$\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} = \frac{(-1)^k}{(2k+1)!} x^{4k+2}$

STEP 2: Anti-differentiate

$\int \sin(x^2)\,dx = \int \left(\sum \frac{(-1)^k}{(2k+1)!} x^{4k+2}\right) dx =$

is constant $\quad$ integrated

$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot \frac{x^{4k+3}}{(4k+3)} + C$

STEP 3: PLUG IN $0, \pi$

$\int_0^{\pi} \sin(x^2)\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot \frac{\pi^{4k+3}}{4k+3} - \cancel{\sum_{k=0}} \; x=0$

0

To Solve Lagrange multiplier Question

① Find critical points

② Solve discriminant

③ plug numbers into discriminant

Critical points of multivariable functions

Given $f(x,y) = 10 - 3x^2 - 2y^2 + 8y + 12x$,
identify any critical points, saddle points, and local extrema

* use Lagrange for crit.

① Identify points of interest

1. $C(a,b)$ $\quad f_x = 0$ $\quad f_y = 0$

2. $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

3. $D > 0$, $f_{xx} > 0$, $f(a,b) \rightarrow$ local min

$D > 0$, $f_{xx} < 0$, $f(a,b) \rightarrow$ local max

$D < 0$, $f(a,b) \rightarrow$ Neither

$P(a,b) \rightarrow$ saddle point

Telescoping Sums

* Part of sequences & series

Ex. Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{801}$ $\quad \lim_{N\to\infty} \left(1 - \frac{1}{n+1}\right)$

$\boxed{=1}$

cancel out

$n=1 \rightarrow \frac{1}{1} - \frac{1}{2}$

$n=2 \rightarrow \frac{1}{2} - \frac{1}{3}$

$n=3 \rightarrow \frac{1}{3} - \frac{1}{4}$

$\vdots$

$n=799$ $\quad \frac{1}{799} - \frac{1}{800}$

$n=800$ $\quad \frac{1}{800} - \frac{1}{801}$

$S_n = 1 - \frac{1}{n-1}$

if there is $n, n+1, n+2$ is a telescoping series

Just remove $\frac{1}{n}$ and only n at bottom

plug ∞

$\sum_{n=1}^{1000} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} [\ln(N+1)]$

$= \infty$

$\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} = 1$

Ex. 2 Evaluate $\sum_{n=1}^{1000} \ln\left(\frac{n+1}{n}\right) = \lim_{N\to\infty} [\ln(N+1)] = \infty$

divergent series

$n=1$ $\quad \ln\left(\frac{2}{1}\right) = \ln 2 - \ln 1$

$n=2$ $\quad \ln\left(\frac{3}{2}\right) = \ln 3 - \ln 2$

$n=3$ $\quad \ln\left(\frac{4}{3}\right) = \ln 4 - \ln 3$

$\vdots$

$N = n-1$ $\quad \ln N - \ln N-1$

$n = N$ $\quad \ln(N+1) - \ln N$

Ex. Find a power series representation of function $\frac{1}{1+x^2}$ when x is in interval $(-1,1)$

geometric series

① $\frac{1}{1+x^2} = \frac{1}{1-(-x)^2} = \sum_{k=0}^{\infty} (-x^2)^k$

$= \sum_{k=0}^{\infty} (-1)^k (x^2)^k$

$\boxed{= \sum_{k=0}^{\infty} (-1)^k x^{2k}}$ answer

$-1 < r < 1$

$-1 < x^2 < 1$

$-1 < x^2 < 1$

$-1 < x < 1$

For x in $(-1,1)$

Power Series

$-\ln(1-x) + c = \int \frac{1}{1-x}\,dx = \sum_{n=1}^{\infty} \frac{x^n}{n} + c$

$\ln(1-x) = \sum_{n=1}^{\infty} -\left(\frac{x^n}{n}\right)$ when $-1 < x < 1$

$\ln\left(\frac{1}{3}\right) = \ln\left(1 - \frac{2}{3}\right) = \sum_{n=1}^{\infty} \frac{-\left(\frac{2}{3}\right)^n}{n}$ $\quad$ legal to use

Power Series

$\sum_{n=0}^{\infty} c_n x^n$ $\quad \sum_{n=0}^{\infty} c_n (x-a)^n$

Possibility 1: series converges when $x = a$

Possibility 2: series converges for all x

Possibility 3: for some positive number R:

Radius of convergence

this resembles a polynomial

$|x-a| < R$ $\quad |x-a| > R$

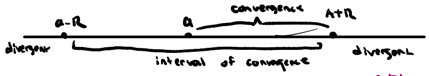
convergent $\quad$ divergent

$f(x) = \overbrace{C_0 + C_1 x + C_2 x^2}$

coefficient

* to solve power series, use ratio test

domain = set of x for which f(x) converges



Ex. $\sum_{n=0}^{\infty} C_n x^n$

$f(x) = C_0 + C_1 x + C_2 x^2 \ldots$

if all coefficients $= 1$

$f(x) = 1 + x + x^2 + \ldots$

convergent when $-1 < x < 1$

Ex. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

ratio test

$\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}$ $\quad \lim_{n\to\infty}$

$(x-3)\frac{n}{n+1} \div \frac{n}{n} = |(x-3)| \frac{1}{1+\frac{1}{n}} = 1$

absolute value is 3

$|x-3| < 1$

$-1 < x-3 < 1$

$2 < x < 4$

Ex. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^{n+1}}{n+1!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}$

Possibility 2

$(n+1)!$

$(n+1)! = n!(n+1)$

$\lim_{n\to\infty} \frac{x}{n+1} = 0$

radius of convergence $= \infty$

interval of convergence $= (\infty, -\infty)$

Ex.

Find interval of convergence where $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$

write this down

- converges when $A \cdot |x-c| < r$
- diverges when $A \cdot |x-c| > r$

$-1 < x-5 < 1$

$A = \frac{x+2^2}{n^2}$

$+5 \quad +5 \quad +5$

$4 < x < 6$

$= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

move outside

$= \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(x+2)^n} \right| = \lim_{n\to\infty} \left| \frac{(x+2)\, n^2}{(n+1)^2} \right|$

$x+2 \lim_{n\to\infty} \left| \frac{n^2}{n^2+1} \right|$

convergence

[ ] ← convergent

( ) ← divergent

$x+2\,(1)$

$| \quad |$

$| < \quad < |$

$-1 < x+2 < 1$

$\boxed{-3 < x < -1}$

endpoints

① plug into function

$\lim_{n\to\infty} \frac{(-3+2)^n}{n^2} = \frac{(-1)^n}{n^2} = 0$

alternating series that is decreasing

Converges

Decreasing

$y = \frac{1}{x}$

加油!

goodluck!

$F(x) = CDF$
$f(x) = PDF$

$P(x > \frac{1}{2})$ ← cumulative distribution function

$0 \leq F(x) \leq 1$, non-decreasing

Variance

square because makes it positive

$Var(X) = \int_{-\infty}^{\infty} (X - E(X))^2 f(x)\,dx$ * Average difference squared between actual and expected

square inside; diff. between actual and expected

$Var(V) = E(V^2) - [E(V)]^2$ * Discrete Calculations

$= \underbrace{\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx}_{\text{square inside}} - \underbrace{[\int_{-\infty}^{\infty} x \cdot f(x)\,dx]^2}_{\text{square after calculator}}$

Ex. determine $P(a \leq V \leq b) \rightarrow$ 2 ways to determine
- $\int_a^b f(x)\,dx$
- $F(b) - F(a)$

$Var(x) = E(x^2) - [E(x)]^2$ * Discrete Calculation

$\sigma(x) = \sqrt{Var(x)}$ * Standard Deviation Formula

$E[x] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$ Understand that symmetrical values have an expected value of zero.

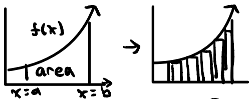
PDF → CDF

$f(x) = F'(x)$

* If you know PDF

$F(x) = \int_{-\infty}^{x} f(t)\,dt$ - area of curve under distribution

Riemann sums



n = num of rectangles

$P = \{[x_0, x_1], [x_1, x_2] \ldots, [x_{n-1}$

Ex. Use sigma notation to write Riemann sum for $f(x) = x+2$ on $[0,6]$ with $n = 60$. DO NOT EVALUATE the right Riemann sum.

① know that $n = 60$
given interval for Riemann sum is $[0,6]$.

Formula: $\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$ ← loops thru every rectangle; width of an interval

② Determine interval size. $[0,6] \rightarrow \frac{6}{60} \rightarrow \frac{1}{10}$ ← number

③ ANS: Right Riemann sum is $\sum_{i=1}^{60} (\frac{i}{10} + 2)\frac{1}{10}$

$x = \frac{i}{10}$; the width of each one

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^3} \qquad |x - a| < R \text{ is convergent}$$

ratio test

$$\lim_{n \to \infty} \left| \frac{(-1)^n x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(-1)^{n-1} x^n} \right| = \left| \frac{(-1)^n x^n x}{(n+1)^3} \cdot \frac{-n^3}{(-1)^n x^n} \right|$$

$$= \left| \frac{x}{(n+1)^3} \cdot \frac{-n}{x^n} \right|$$

$$= \frac{xn}{(n+1)^3 x^n}$$

to reach 0, or finite number, n has to be less than 1, otherwise will overpower and

Does the series $\sum_{n=10}^{\infty} \frac{3}{4n + 3\sqrt{n}}$ converge or diverge?

We know that $\sum_{n=10}^{\infty} \frac{3}{4n} = \frac{3}{4} \sum_{n=10}^{10} \frac{1}{n}$ diverges
↖ p-series diverges
↖ since diverges, equation needs to be larger, but this one smaller, so need

aside: If we want to use direct comparison, need

$\frac{3}{4n + 3\sqrt{n}} > \frac{3}{4n}$ but that's not true

$$\frac{\frac{3}{4n}}{\frac{3}{4n+3\sqrt{n}}} = \frac{\cancel{3}(4n + 3\sqrt{n})}{\cancel{3} \cdot 4n} \qquad \frac{4n + 3\sqrt{n}}{4n} = 1 + \frac{3\sqrt{n}}{4n} = \boxed{1}$$

$0 < 1 < \infty$ so by limit comparison, $\sum \frac{3}{4n+3\sqrt{n}}$ also **diverges**

Given $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all x, find series converging to x^2

how do you turn e^x to e^{x^2}?

$\int_0^1 e^{x^2}\,dx$

$\boxed{e^r = \sum_{k=0}^{\infty} \frac{r^k}{k!}}$

$e^{x^2} =$

$e^r = \sum_{k=0}^{\infty} \frac{r^k}{k!}$ Set $r = x^2$

$e^{x^2} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$

convert to power series, easy to differentiate

$\int e^{x^2}\,dx = \int \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}\,dx$

$= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)(k!)}$

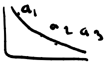
$= \int_0^1 e^{x^2}\,dx = F(1) - F(0)$
↖ anti deriv

$= \sum_{k=0}^{\infty} \frac{1^{2k+1}}{(2k+1) \cdot k!} - \sum_{k=0}^{\infty} \frac{0^{2k+1}}{(2k+1)k!}$

$= \sum_{k=0}^{\infty} \frac{1}{(2k+1) \cdot k!}$

SEQUENCE

Ex.1 $\lim_{n\to\infty} a_n = 0$

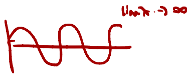


$a_n = \frac{1}{3n+1}$

$a_n = f(n)$

if $\lim_{x\to\infty} f(x) = L$,

then also $\lim_{n\to\infty} a_n = L$



$\lim_{n\to\infty}$

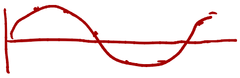
$b_n = \sin(\pi n)$

$b_1 = \sin\pi = 0$

$b_2 = \sin 2\pi = 0$

$b_3 = \sin 3\pi = 0$

$\lim_{n\to\infty}\left[\frac{1+n}{ne^n}\right] = \lim_{n\to 0}\left[\underbrace{\frac{1+n}{n}}_{0}\cdot\underbrace{e^{-n}}_{1}\right] = 1$



$\lim_{n\to 0} \sin(n)$ DNE

$a_n = (-n)^{-n}$

$= \frac{1}{-n^n} = \frac{1}{(-1\cdot n)^n} = \frac{1}{(-1)^n n^n} = \frac{(-1)^n}{n^n}$

$= \frac{(-1)^n}{n^n}$

$-1)$

$\lim_{n\to\infty} \frac{1}{n^2} = 0$

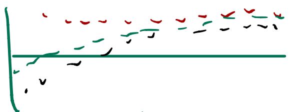
∞

$\frac{1}{n^2}\cdot 2n^2 = 2 \qquad \lim_{n\to\infty} = 2$

SQUEEZE THEOREM

Let a_n, b_n, c_n be sequences with $a_n \le b_n \le c_n$ for n sufficiently large.

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$



Ex. $\lim_{n\to\infty}\left(\frac{2n+\cos n}{n+1}\right) =$

$-1 \le \cos n \le 1$

So: $\frac{2n-1}{n+1} \le \frac{2n+\cos n}{n+1} \le \frac{2n+1}{n+1}$

$\frac{2n-1}{n+1} \le \frac{2n+\cos n}{n+1} \le 2$

$2 \le \frac{2n+\cos n}{n+1} \le 2$

limit is 2

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p>1 \to$ converge

$p \le 1 \to$ diverge

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^1}$ $p=1$

$\to$ diverge

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ $p=2$

$p>1$

$\to$ converge

Ex. $\sum_{n=1}^{\infty} n^{-\pi} = \frac{1}{n^{\pi}}$ $p=\pi$

$\to$ converges

Suppose $\lim_{n\to\infty} a_n = A$ $\lim_{n\to\infty} b_n = B$ for some real numbers A and B. Then

- $\lim_{n\to\infty} (a_n + b_n) = A + B$
- $\lim_{n\to\infty} (Ca_n) = CA$ for any constant C
- $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \ne 0$

Ex. $\lim_{n\to\infty} e^{-n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$

$\lim_{n\to\infty} \frac{1+n}{n} = \frac{1}{n} + 1 = 1$

$\lim_{n\to\infty}\left[\frac{1+n}{n} - e^n\right] \quad 1 - 0 = \boxed{1}$

If $\lim_{n\to\infty} a_n = L$ and if function $g(x)$ is continuous at L, then

$\lim_{n\to\infty} g(a_n) = g(L)$

Ex. Evaluate $\lim_{n\to\infty} \sin\left(\frac{\pi n}{2n+1}\right) = \lim_{n\to\infty} \sin\left(\frac{\pi}{2}\right) = 1$

Intermediate step: is same degree

$\lim_{n\to\infty} \frac{\pi n}{2n+1}$

* is rational
deg of top
is same
as bottom

use SQUEEZE THEOREM

* to find bound
$(-1)^n$
$\sin x$
$\cos x$

Ex. $a_n = (-n)^{-n}$, Find $\lim_{n\to\infty} a_n$

$= \frac{1}{-n^n} = \frac{1}{(-1\cdot n)^n} = \frac{1}{(-1)^n n^n} = \frac{(-1)^n}{n^n}$

$-1 \le (-1)^n \le 1$

$\frac{-1}{n^n} \le \frac{(-1)^n}{n^n} \le \frac{1}{n^n}$

$\lim_{n\to\infty} \frac{-1}{n^n} = 0 \qquad \lim_{n\to\infty} \frac{1}{n^n} = 0$

By Squeeze Theorem

$\lim_{n\to\infty} (-n)^{-n} = 0$

Compare to $\sum \frac{3n}{n^2} = \sum \frac{3}{n}$ diverges
p-series

$\sum_{n=2}^{\infty} \frac{n^2+n+1}{n^5-n} \qquad \frac{1}{n^3} \qquad p>1$

is larger $\qquad$ converges

$\frac{\frac{n^2+n+1}{n^5-n}}{\frac{1}{n^3}} = \frac{n^5 + n^4 + n^3}{n^5 - n} = 1$, since $0 < 1 < \infty$, series also converges

$\int_1^{\infty} \frac{5}{6^{\ln(x)}}\,dx = 5\int_1^{\infty} \frac{1}{6^{\ln(x)}}\,dx$

$= 5\int_1 6^{-\ln(x)}\,dx = 5\int -x\,6^u\,du = -x^1 6^u\,du$

$= -x^{1-u}\,du$

$u = -\ln x$

$du = -\frac{1}{x}$

$-x\,du = 1$

$w = 1-u$

$dw = -1$

$\int_1^{\infty} x^w\,du$

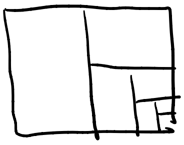
SERIES — Adding Numbers together

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \dots$$

Sequence = list of numbers
series = sum of numbers
geometric series = add em up

$$\sum_{n=1}^{\infty} (c \cdot a_n) = c \cdot \sum_{n=1}^{\infty} a_n$$

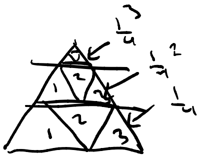
$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} (a_n + b_n) = a_1 + b_1 + a_2 + b_2 \dots$$



$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}$ geometric ratios

covered area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}$

sum of numbers, approaching one



geometric sequence
common ratio: $\frac{1}{4}$

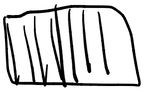
$\lim_{n\to\infty} \left(\frac{1}{4}\right)^n = 0$

$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3$

$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{3}$

"geometric series"

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{2}$



geometric series
common ratio: $\frac{1}{3}$

Terms being added

$\{a_n\}$

$a_1 = \frac{1}{5}$

$a_2 = \frac{1}{5^2}$

$a_3 = \frac{1}{5^3}$

$\lim_{n\to\infty} a_n = 0$

sequence

$\lim_{n\to\infty} S_n = \frac{1}{4}$

Partial Sums

$S_1 = a_1 = \frac{1}{5}$

$S_2 = a_1 + a_2 = \frac{1}{5} + \frac{1}{5^2}$

$S_3 = a_1 + a_2 + a_3 = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3}$

$\lim_{n\to\infty} S_n = 0.25$

"SERIES" $= \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \frac{1}{4}$

$$S_N = \sum_{N=1}^{N} a_n$$

$S_1 = a_1$	$S_1 = \sum_{n=1}^{1} a_n$
$S_2 = a_1 + a_2$	$S_2 = \sum_{n=1}^{2} a_n$
$S_3 = a_1 + a_2 + a_3$	$S_3 = \sum_{n=1}^{3} a_n$
$S_4 = a_1 + a_2 + a_3 + a_4$	$S_4 = \sum_{n=1}^{4} a_n$
Entire Series	$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{N} a_n = \lim_{N\to\infty} S_n$

$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$

$\frac{1}{1-\frac{1}{4}}$

$\frac{3}{2} - 1 = \frac{1}{2}$

$\frac{1}{2}$

$$\sum_{n=1}^{100} a_n = S_{100} = \frac{100}{100+1}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{n} a_n = \lim_{n\to\infty} \left[\frac{n}{n+1}\right] = 1$$

Calculate geometric series

r = ratio

Problem Type 1

$a_n = r^n$, $|r| < 1$

$\lim_{n\to\infty} r^n = 0$

$\sum_{n=0}^{\infty} a_n = \frac{1}{1-r}$ $\quad r = \frac{1}{4}$

Ex. $\underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n}_{} = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = \frac{4}{3} = 1 + \frac{1}{3}$

$\left(\frac{1}{4}\right)^0 + \left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots = 1 + \frac{1}{3}$

$-1 \qquad -1$

$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$

* because starts at $n=0$, need to subtract 1 $\left(\frac{1}{4}\right)^0$

$r = \frac{1}{3}$

$\sum_{n=0}^{\infty} \frac{1}{1-r}$

because $n=0$

$\sum_{n=0}^{\infty} \frac{1}{1-\frac{1}{3}} = \frac{3}{2} - 1 = \boxed{\frac{1}{2}}$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}}$$

Geometric Series Formula

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1-r} \Big\} \text{ partial sum}$$

$$\sum_{i=0}^{\infty} r^i = \lim_{n\to\infty} \left(\frac{1 - r^{n+1}}{1-r}\right) \Big\} \text{ infinity}$$

$i = 0$

$= \frac{1}{1-r}$ if $|r| < 1$

DNE if $|r| \geq 1$

use to solve formulas

Ex. $\sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^i = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$

$\sum_{i=0}^{\infty} \left(-\frac{10}{9}\right)^i$ DNE

ratio is greater then 1 so DNE

"diverges" $\lim S_n$ DNE

$\frac{1}{1-\frac{1}{3}}$

Ex. 2

$$\sum_{i=2}^{\infty} \left(\frac{2}{3}\right)^i = \left(\frac{2}{3}\right)^2 \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i = \boxed{3 \cdot \frac{4}{9}}$$

1.333

$\frac{12}{9}$

③ $\frac{1}{1+1} = \frac{1}{2} = \frac{1}{2}$

$S_1 = a_1$

$S_2 = a_1 + a_2 = \frac{2}{2+1}$

$a_2 = S_2 - S_1 = \frac{2}{3} - \frac{1}{2}$

$a_4 = S_4 - S_3$

$S_n - S_{n-1} =$
$[a_1 + a_2 + \dots + a_{n-1} + a_n]$

$\underbrace{\frac{N}{n+1}}_{S_n} - \underbrace{\frac{(n-1)}{(n-1)+1}}_{S_{n-1}} = \frac{n}{n+1} - \frac{N-1}{n}$

$= a_n$

$\frac{N}{n+1} - \frac{n-1}{N+1-1}$

$\frac{N}{N+1} - \frac{N-1}{N}$

$= a_n$

$$\sum_{n=1}^{\infty} \sqrt[5]{n^2} = \frac{(n^2)^{\frac{1}{5}}}{1} = \frac{1}{n^{-2/5}}$$

$p = -\frac{2}{5}$ will diverge

Ex. 3

$$\sum_{i=0}^{100}\left(\frac{10^i}{9^{2i}}\right) = \frac{1-r^{n+1}}{1-r} = \boxed{\frac{1-\frac{10}{81}^{101}}{1-\frac{10}{81}}}$$

$$\sum_{i=0}^{100}\frac{10^i}{9^{2i}} = 1 + \frac{10}{81} + \frac{100}{6561} \qquad \frac{1}{1-\frac{2}{3}}$$

$$\sum_{i=50}^{\infty}\left(\frac{2}{3}\right)^{i+17} = \sum_{i=67}^{\infty}\left(\frac{2}{3}\right)^{i} = \left(\frac{2}{3}\right)^{67}\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^{i}, \quad \frac{1}{1-\frac{2}{3}} \quad = \left(\frac{2}{3}\right)^{67}\cdot 3$$

* simplify

$$\sum_{i=10}^{\infty}\left(\frac{3^{i+1}}{2^{2i}}\right) = \frac{3^{i+1}}{2^{2i}} = \frac{3\cdot 3^i}{4^i} = 3\left(\frac{3}{4}\right)^i$$

$$\sum_{i=10}^{\infty} 3\left(\frac{3}{4}\right)^i = 3\left(\frac{3}{4}\right)^{10} + 3\left(\frac{3}{4}\right)^{11} + 3\left(\frac{3}{4}\right)^{12}$$

$$3\left(\frac{3}{4}\right)^{10}\cdot\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i \qquad 3\left(\frac{3}{4}\right)^{10}\cdot 4$$

$$\frac{1}{1-\frac{3}{4}}$$

Telescoping Sums

Ex. Evaluate $\sum_{n=1}^{800}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1 - \frac{1}{801}$; $\lim_{N\to\infty}\left(1-\frac{1}{n+1}\right) = 1$

cancel out

$n=1 \to \frac{1}{1} - \frac{1}{2}$

$n=2 \to \frac{1}{2} - \frac{1}{3}$

$n=3 \to \frac{1}{3} - \frac{1}{4}$

$\vdots$

$n=799 \quad \frac{1}{799} - \frac{1}{800}$

$n=800 \quad \frac{1}{800} - \frac{1}{801}$

$S_n = 1 - \frac{1}{n-1}$

if there is $n, n+1, n+2$ is a telescoping series

Ex. 2 Evaluate $\sum_{n=1}^{1000}\ln\left(\frac{n+1}{n}\right) = \lim_{N\to\infty}\left[\ln(N+1)\right] = \infty$

divergent series

$n=1 \quad \ln\left(\frac{2}{1}\right) = \ln 2 - \ln 1$

$n=2 \quad \ln\left(\frac{3}{2}\right) = \ln 3 - \ln 2$

$n=0 \quad \ln\left(\frac{4}{3}\right) = \ln 4 - \ln 3$

$\vdots$

$N = n-1 \quad \ln N - \ln N-1$

$n = N \quad \ln(N+1) - \ln N$

Arithmetic of Series

Let A, B, and C be real numbers and let the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to S and T.

$$\sum_{n=1}^{\infty}[a_n + b_n] = S+T$$

$$\sum_{n=1}^{\infty}[a_n - b_n] = S-T$$

$$\sum_{n=1}^{\infty}[Ca_n] = CS$$

for telescoping sums, look at what cancels out

Ex. Evaluate $\sum_{n=1}^{\infty}\left(\frac{2}{n} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \lim_{n\to\infty}\left[2 - \frac{1}{2} + 1\right.$

$n=1 \quad \frac{2}{1} - \frac{1}{2} - \frac{1}{3}$

$n=2 \quad \frac{2}{2} - \frac{1}{3} - \frac{1}{4}$

$n=4 \quad \frac{2}{3} - \frac{1}{4} - \frac{1}{5}$

$\vdots$

$n = N-2 \quad \frac{2}{N-2} - \frac{1}{n-1} - \frac{1}{n}$

$n = N-1 \quad \frac{2}{n-1} - \frac{1}{n} - \frac{1}{n+1}$

$n = n \quad \frac{2}{n} - \frac{1}{n+1} - \frac{1}{n+2}$

those don't cancel out

$$= \lim_{n\to\infty}\left[2.5 - \frac{2}{n+1} - \frac{1}{n+2}\right] = 2.5$$

Geometric Series
If $a_n = ar^n$, then $\sum a_n$ converges if $|r|<1$,
diverges if $|r| \geq 1$

Telescoping Series If $\sum a_n$ is telescoping, can find its partial sums

Divergence Test - If $\lim_{a\to\infty} a_n \neq 0$, then $\sum a_n$ diverges

Integral Test If $f(n) = a_n$ is continuous, positive, and decreasing, then $\sum a_n$ converges / diverges if $\int f(x)$ converges / diverges

p-Series $\sum \frac{1}{n^p}$ converges when $p>1$ and diverges when $p \leq 1$

divergent if less than 1

convergent if more than 1

p-series

$$\sum_{n=1}^{\infty} \left(n^{-2.4} + 8n^{-1.6} \right) \qquad p > 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2.4}} + 8 \sum_{n=1}^{\infty} \frac{1}{n^{1.6}} = \quad \text{converges}$$

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad p = \frac{1}{2}$$

$$a_n = \frac{1}{\sqrt{n}} \qquad p \leq 1$$

divergent

$$1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} \qquad p = \frac{2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} \qquad \text{divergent}$$

$$1 + \frac{1}{2\sqrt[3]{2}} + \frac{1}{3\sqrt[3]{3}} + \frac{1}{4\sqrt[3]{4}} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt[3]{n}} = \quad n^{\cdot \frac{4}{3}} \qquad \text{converges}$$

$$\sum_{m=1}^{\infty} \frac{3m + \sin\sqrt{m}}{m^2} = \frac{1}{m^1} \leftarrow p \text{ is } 0$$

so is divergent

$$\text{since} \quad 0 < \frac{1}{m^2} < \frac{3m + \sin\sqrt{m}}{m^2},$$

by direct comparison test,

$$\frac{3m + \sin\sqrt{m}}{m^2} \quad \text{is divergent as well}$$

$$\sum_{k=1}^{\infty} \frac{k}{3^k} \quad \text{converge or diverge?}$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{(k+1)}{3^{(k+1)}} \Big/ \frac{k}{3^k} = \lim_{k\to\infty} \frac{(k+1)}{3^{k+1}} \cdot \frac{3^k}{k} = \lim_{k\to\infty} \underbrace{\left(\frac{k+1}{k} \right)}_{1} \cdot \frac{1}{3} = \frac{1}{3}, \text{ so}$$

$$1 \cdot \frac{1}{3}$$

$$\frac{3^{\cancel{k}}}{3^{\cancel{k}+1}} = \frac{1}{3^1} = \frac{1}{3}$$

by ratio test, $\sum \frac{k}{3^k}$ converges

$$\sum_{k=1}^{\infty} \frac{k^5}{4^k} \quad \text{converge or diverge?}$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \quad \lim_{k\to\infty} \frac{(k+1)^5}{4^{(k+1)}} \Big/ \frac{k^5}{4^k} = \lim_{k\to\infty} \frac{(k+1)^5}{4^{(k+1)}} \cdot \frac{4^k}{k^5} = \frac{1}{4} \lim_{k\to\infty} = \frac{(k+1)^5}{k^5} = \frac{1}{4} \cdot \lim_{k\to\infty} \frac{k+1}{k} = 1$$

$$\frac{\cancel{4^k}}{\cancel{4^{k+1}}} \quad \frac{1}{4}$$

$$= \frac{1}{4} \quad \text{converges at } \frac{1}{4}$$

$$\sum$$

$$\frac{k^5}{4^k} \text{ converges} \qquad r = \frac{1}{4}, \text{ less than } 1, \text{ converges}$$

So by ratio test

Integral T

$$\sum_{k=1}^{\infty} \frac{k(2 + \sin k)}{k^{\sqrt{2}}} = \frac{2k}{k^{\sqrt{2}}}$$

$$1 \leq 2 + \sin k \leq 3$$

$$2\sum \frac{k}{k^{\sqrt{2}}} = \frac{1}{k^{\sqrt{2}-1}}, \qquad \text{larger}$$

all divergent

$$\frac{x^a}{x^b} = \frac{1}{x^{b-a}} \qquad p = \sqrt{2} - 1$$

$$0 < \frac{2k}{k^{\sqrt{2}}} < \frac{k(2 + \sin k)}{k^{\sqrt{2}}}$$

Since we know this, by direct comparison test

$$\frac{k(2 + \sin k)}{k^{\sqrt{2}}} \quad \text{is divergent also}$$

$$\sum$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{10^{k+1}}{(k+1)!} \Big/ \frac{10^k}{k!}$$

$$= \frac{10^{k+1}}{10^k} \cdot \frac{k!}{(k+1)!} = \frac{\cancel{k!}}{(k+1)\cdot \cancel{k!}}$$

$$10 \cdot \frac{1}{k+1} = 0 \quad 0 < 1 \text{ by ratio test } \sum \frac{k^k}{k!} \text{ converges}$$

$$\sum_{k=1}^{\infty} \frac{k^k}{k!}$$

$$\frac{k^k}{k!} = \frac{k \quad k \quad k \quad k}{k(k-1)(k-2)\dots(1)} = \left(\frac{k}{k-1}\right)\left(\frac{k}{k-2}\right)\left(\frac{k}{k-3}\right)\dots\left(\frac{k}{3}\right)\left(\frac{k}{2}\right)\frac{k}{1}$$

goes to ∞ as k goes to ∞

By divergence test,

$\sum \frac{k^k}{k!}$ divergence becaus k^k is larger then $k!$

Series diverges

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{(k+1)^{(k+1)}}{(k+1)!} \Big/ \frac{k^k}{k!} = \lim_{k\to\infty} \frac{(k+1)^{(k+1)}}{k^k} \cdot \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{(k+1)(k+1)^k}{k^k} \cdot \frac{\cancel{k!}}{(k+1)k!}$$

flip, multiply

$$= \lim_{k\to\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k$$

$= e$

$e > 1$

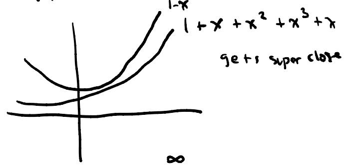
Given $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all real values of x, give a series that converges to e whose terms are all rational numbers.

$$e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$$

plug $x=1$

Power Series $\frac{1}{1-x}$



Ex. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when x in open interval of convergence $(-1,1)$

so: $\frac{2}{1-x} = 2\left(\frac{1}{1-x}\right) = 2\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 2x^n$ $R=1$ (still)

$\frac{2x^2}{1-x} = x^2\left(\frac{2}{1-x}\right) = x^2 \sum_{n=0}^{\infty} 2 \cdot x^n = \sum_{n=0}^{\infty} x^2 \cdot 2x^n = \sum_{n=0}^{\infty} 2x^{n+2}$

$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \frac{d}{dx}\left(1 + x + x^2 + \dots\right) = \underbrace{\sum_{n=0}^{\infty} n x^{n-1}}_{\text{if } n=0,\ \text{adding "0"}} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{k=0}^{\infty} (k+1)x^k$

remember

differentiate

$R=1$ $= \sum_{n=0}^{\infty} n x^{n-1}$

$= \sum_{n=0}^{\infty} (n+1)x^n$

all equivalent

Ex. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when x is in open interval of convergence $(-1,1)$

so: $\int\left[\frac{1}{1-x}\right]dx = \int \sum_{n=0}^{\infty} x^n\,dx = \left[\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\right] + C$ converges on $-1 < x < 1$

$R=1$ (still)

anti differentiate

equivalently

$\left[\sum_{n=?}^{\infty} \frac{x^n}{n}\right] + C$

$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

$(n=0)$ $(n=1)$ $(n=4)$

$n=1$ $n=2$ $n=3$

$\sum_{n=1}^{\infty} \frac{x^n}{n}$

-1

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\left|\frac{a_{n+1}}{a_n}\right|$

$x^n \cdot x$

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} = 0$

$(n+1)n!$

convergent

$x^n \cdot x$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^3} = \frac{(-1)^n x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(-1)^{n-1} x^n} = \frac{(-1)^n x^n x}{(n+1)^3} \frac{-n^3}{(-1)^n x^n}$$

$(-1)^n$

$$= \left| \frac{-x n^3}{(n+1)^3} \right|$$

$$= \lim_{n\to\infty} \frac{x n^3}{(n+1)^3}$$

$$= \frac{x \cancel{n^3}}{\cancel{n^3}}$$

radius of convergence

$= |x|$ abs value of x has to be less then 1

$-1 \le x \le 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-1)^n}{n} = \frac{(-1)^n (x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1} (x-1)^n} =$$

$$\frac{(-1)^n (x-1)^n (x-1)}{n+1} \cdot \frac{-n}{(-1)^n (x-1)^n}$$

$$\lim_{n\to\infty} \left| \frac{-n(x-1)}{n+1} \right|$$

$-1 \le \boxed{x-1} \le 1$

$0 \le x \le 2$

radius of convergence 2

$[-2, 2]$

Power Series Example

Suppose that $\frac{df}{dx} = \frac{x}{1+3x^3}$ and $f(0)=1$

a) Find power series representation for $f(x)$, centered at 0

$$\boxed{\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \quad \text{if } |r|<1} \leftarrow \text{formula}$$

① Convert to formula format

└ set $r = -3x^3$ if $|r|<1$

$$\frac{1}{1-(-3x^3)} = \sum_{n=0}^{\infty} (-3x^3)^n$$

$$\frac{1}{1+3x^3} = \sum_{n=0}^{\infty} (-3)^n x^{3n} \qquad \text{if } |-3x^3|<1$$

$|x^3| < \frac{1}{3}$

$$(x) \quad \boxed{\frac{1}{1+3x^3} = \sum_{n=0}^{\infty} (-3)^n x^{3n}} (x)$$

$|x| < \frac{1}{\sqrt[3]{3}}$

② multiply both sides by x

$$\frac{x}{1+3x^3} = \sum_{n=0}^{\infty} x \cdot (-3)^n x^{3n} = \boxed{\sum_{n=0}^{\infty} (-3)^n x^{3n+1}}$$

power series representation

③ Since we're looking for $f(x)$, determine antiderivative of $\frac{df}{dx}$

$$f(x) + C = \int \frac{df}{dx}\,dx = \int \frac{x}{1+3x^3}\,dx = \int \sum_{n=0}^{\infty} (-3)^n x^{3n+1}\,dx$$

power series representation

$\int x^{3n+1} = \frac{x^{3n+2}}{3n+2}$

$$= \sum_{n=0}^{\infty} (-3)^n \frac{x^{3n+2}}{3n+2} \quad \text{for } |x| < \frac{1}{\sqrt[3]{3}}$$

④ find C

Radius of convergence $= \frac{1}{\sqrt[3]{3}}$

Find C, plug $x=0$

$$f(0) + C = \sum_{n=0}^{\infty} (-3)^n \frac{0^{3n+2}}{3n+2}$$

$1 + C = 0$

$\boxed{C = -1}$

$$f(x) - 1 = \sum_{n=0}^{\infty} (-3)^n \frac{x^{3n+2}}{3n+2}$$

$$\boxed{f(x) = 1 + \sum_{n=0}^{\infty} (-3)^n \frac{x^{3n+2}}{3n+2}}$$

power series representation

① Mess with geometric series until it matches $\frac{df}{dx}$

② Antidifferentiate $\frac{df}{dx}$, get $+C$ and find C

③ to find radius of convergence, start with geometric series

$$\boxed{\frac{1}{1-r} = \sum r^n \quad \text{if } |r|<1}$$

mess with df dx until matches $\frac{1}{1-r}$

$$\frac{1}{1-(-3x^3)} = \sum (-3x^3)^n$$

then substitute for r

$|-3x^3| < 1$

$|3x^3| < 1$

$|x^3| = \frac{1}{3}$

$|x| < \frac{1}{\sqrt[3]{3}}$ ↙ radius of convergence of new series

odd number

$$\boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} (-1)^n = \sin x}$$

positive one negative

2016 Final Exam

BRGAK 12:54 – 12:57

Let $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k3^k}$, $g(x) = f'(x) = \frac{df}{dx}$, and R be the radius of convergence of the power series $f(x)$. Find R and $g(1)$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$f'(x) = \cos x \to 0$

$f''(x) = -\sin x \to 1$

$f'''(x) = -\cos x \to 0$

$f''''(x) = \sin x \to -1$

$$\sum_{n=0}^{\infty}$$

even powers are 0

$$0 + x + 0 + \frac{-1}{3!} x^3 + 0 + \frac{-1}{5!} x^5$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

ODD #s: $2n+1$

$(n=0)$ $n \cdot 0 + = 1$

$(n=1)$ $n \cdot 1 + 1 = 3$

$(n=2$

To get alternating signs $(-1)^n$ or $(-1)^{n+1}$