Math 114C: Homework 1

Due on January 15, 2025 at 11:59pm

Professor Arant

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Problem 1

Part (a)

$\mathrm{Char}_R(x)$	$\mathrm{Char}_Q(x)$	$\mathrm{Char}_{R\&Q}(x)$	$\mathrm{Char}_R(x) \cdot \mathrm{Char}_Q(x)$
1	1	1	$1 \cdot 1 = \mathbf{1}$
1	0	0	$1 \cdot 0 = \mathbf{0}$
0	1	0	$0 \cdot 1 = \mathbf{0}$
0	0	0	$0 \cdot 0 = \mathbf{0}$

Part (b)

$\mathrm{Char}_R(x)$	$\mathrm{Char}_Q(x)$	$\mathrm{Char}_{R\vee Q}(x)$	$\min(1, \mathrm{Char}_R(x) + \mathrm{Char}_Q(x))$
1	1	1	$\min(1, 2) = 1$
1	0	1	$\min(1, 1) = 1$
0	1	1	$\min(1, 1) = 1$
0	0	0	$\min(1, 0) = 0$

Problem 2

Part (a)

Proof. First, we prove the forward direction. Assume $A = B$ for subsets $A, B \subseteq \mathbb{N}$. By definition of set equality, $x \in A \iff x \in B$. If $x \in A$, then $x \in B$, and thus $\mathrm{Char}_A(x) = 1$ and $\mathrm{Char}_B(x) = 1$. Similarly, if $x \notin A$, then $x \notin B$, and thus $\mathrm{Char}_A(x) = 0$ and $\mathrm{Char}_B(x) = 0$.
Now, we prove the backward direction. Assume $\mathrm{Char}_A = \mathrm{Char}_B$. We have $x \in A \iff \mathrm{Char}_A(x) = 1 \iff \mathrm{Char}_B(x) = 1 \iff x \in B$ and thus, $A = B$. □

Part (b)

Proof. First, we prove existence. Let $f : \mathbb{N} \to \{0, 1\}$ be a total function. Let's construct A as follows:

$$A = \{n \in \mathbb{N} \mid \text{if } f(n) = 1\}$$

By definition, $f = \mathrm{Char}_A$.
Now, we prove uniqueness. Assume for the sake of contradiction that there exists two distinct sets A and B with this property. Then, we have function $f = \mathrm{Char}_A = \mathrm{Char}_B$. By part (a), we have $A = B$, which is a contradiction! □

Problem 3

Part (a)

$f \circ g$ is also the unique unary partial function with empty domain. If g has an empty domain, then $f \circ g$ must also have an empty domain.

Part (b)

$$(f \circ g)(x) = \begin{cases} 1 & \text{if } x \geq 10 \text{ and } x \text{ is even} \\ \uparrow & \text{otherwise} \end{cases}$$

Applying g yields outputs greater than 20 for inputs greater than 10.

Part (c)

$$(f \circ g)(x) = \begin{cases} 1 & \text{if } x \text{ is a multiple of } 15 \\ \uparrow & \text{otherwise} \end{cases}$$

Problem 4

$$g(x) = 0$$

$$h(x, t, y) = t + x$$

Proof. We proceed with mathematical induction on y. Our base case is when $y = 0$. We get $M(x, 0) = g(0) = 0$ which is true. Now, we assume $M(x, y) = xy$, and want to show that $M(x, y+1) = x(y+1) = xy+x$. By our definition, we have:

$$M(x, y + 1) = h(x, M(x, y), y) = x + M(x, y) = x + xy = x(y + 1)$$

. □

Problem 5

$$F(x, y) = g(x)$$

$$F(x, y + 1) = h(x, F(x, y), y)$$

where

$$g(x) = x$$

$$h(x, t, y) = f(t)$$

We have $F(x, y) \downarrow \iff F(x, z) \downarrow \forall z < y$.

Problem 6

$f(1) = \mu y[g(1, y) = 0] = \uparrow$. No number bigger than 1 also divides 1.
$f(4) = \mu y[g(4, y) = 0] = 2$. 2 is the smallest non-one factor of 4.
$f(7) = \mu y[g(1, y) = 0] = \uparrow$. 7 is prime.
$f(15) = \mu y[g(1, y) = 0] = 3$. 3 is the smallest non-one factor of 15.

Problem 7

$$f(x,y) = \begin{cases} 0 & \text{if } y^2 \geq x \\ 1 & \text{if } y^2 < x \end{cases}$$

$$g(x) = \mu y[f(x,y) = 0]$$

If x is not a perfect square, g returns $\lceil \sqrt{x} \rceil$. This function is recursive by definition by minimization. Squaring a natural number (definition by primitive recursion of multiplication) and the $\geq$ and $<$ relations (defintion by primitve recursion) are recursive.

Problem 8

h is recrusive by definition by primitive recursion. We have our base case of $h(\vec{x}, 0) = g(\vec{x})$, where $g(\vec{x}) = 1$. We also have $h(\vec{x}, y+1) = \alpha(\vec{x}, h(\vec{x}, y), y)$, where $\alpha(\vec{x}, t, y) = t \cdot f(\vec{x}, y)$. The function α is recrusive because f is recursive and multiplication is recursive by defnition by primitive recursion.