Math 114C: Homework 3

Due on February 8, 2026 at 11:59pm

Professor Arant

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Problem 1

Claim. For every $m, n \geq 1$, there is a recursive total $(m+1)$-ary function $s_n^m(e, \vec{x})$ such that

$$\varphi_e^{(m+n)}(\vec{x}, \vec{y}) \cong \varphi_{s_n^m(e,\vec{x})}^{(n)}(\vec{y})$$

for all $\vec{x} \in \mathbb{N}^m$ and $\vec{y} \in \mathbb{N}^n$.

Proof. We proceed by induction on m.
Base case ($m = 1$). This is exactly the s-m-n theorem proved in class (with parameter n): there exists a recursive total function $s_n^1(e, x)$ such that

$$\varphi_e^{(1+n)}(x, \vec{y}) \cong \varphi_{s_n^1(e,x)}^{(n)}(\vec{y}).$$

Inductive step. Suppose the result holds for m; we prove it for $m+1$. Let $\vec{x} = (x_1, \dots, x_{m+1}) \in \mathbb{N}^{m+1}$ and $\vec{y} \in \mathbb{N}^n$. Write

$$\varphi_e^{((m+1)+n)}(x_1, x_2, \dots, x_{m+1}, \vec{y}) = \varphi_e^{(1+(m+n))}(x_1,\ x_2, \dots, x_{m+1}, \vec{y}).$$

By the base case (applied with n replaced by $m+n$), there is a recursive total function s_{m+n}^1 such that

$$\varphi_e^{(1+(m+n))}(x_1,\ x_2, \dots, x_{m+1}, \vec{y}) \cong \varphi_{s_{m+n}^1(e,x_1)}^{(m+n)}(x_2, \dots, x_{m+1}, \vec{y}).$$

By the inductive hypothesis (applied to the index $s_{m+n}^1(e, x_1)$), there is a recursive total function s_n^m such that

$$\varphi_{s_{m+n}^1(e,x_1)}^{(m+n)}(x_2, \dots, x_{m+1}, \vec{y}) \cong \varphi_{s_n^m(s_{m+n}^1(e,x_1),\, x_2,\dots,x_{m+1})}^{(n)}(\vec{y}).$$

Define

$$s_n^{m+1}(e, x_1, \dots, x_{m+1}) \;=\; s_n^m\big(s_{m+n}^1(e, x_1),\ x_2, \dots, x_{m+1}\big).$$

This is a composition of recursive total functions, so it is recusive and total. Combining the chain of Kleene equalities gives

$$\varphi_e^{((m+1)+n)}(\vec{x}, \vec{y}) \cong \varphi_{s_n^{m+1}(e,\vec{x})}^{(n)}(\vec{y}),$$

completing the induction. □

Problem 2

Claim (Rogers' Fixed Point Theorem). If $f : \mathbb{N} \to \mathbb{N}$ is a total recursive function, then for any $n \geq 1$ there exists $e^* \in \mathbb{N}$ such that $\varphi_{e^*}^{(n)} = \varphi_{f(e^*)}^{(n)}$.

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be total recursive and $n \geq 1$. Define a partial function $g :\subseteq \mathbb{N}^{n+1} \to \mathbb{N}$ by

$$g(e, \vec{y}) \;=\; \varphi_{f(e)}^{(n)}(\vec{y}).$$

Since f is total recursive and the universal partial recursive function $(e, \vec{y}) \mapsto \varphi_e^{(n)}(\vec{y})$ is partial recursive, the composition g is partial recursive (as a fucntion of $n+1$ variables).
By Kleene's second recursion theorem, there exists $e^* \in \mathbb{N}$ such that

$$\varphi_{e^*}^{(n)}(\vec{y}) \;\cong\; g(e^*, \vec{y}) \;=\; \varphi_{f(e^*)}^{(n)}(\vec{y}) \qquad \text{for all } \vec{y} \in \mathbb{N}^n.$$

Therefore $\varphi_{e^*}^{(n)} = \varphi_{f(e^*)}^{(n)}$. □

Problem 3

Claim (Kleene's Second Recursion Theorem). For every $n \geq 1$ and every partial recursive function $\Phi :\subseteq \mathbb{N}^{n+1} \to \mathbb{N}$, there exists $e^* \in \mathbb{N}$ such that

$$\varphi_{e^*}^{(n)}(\vec{y}) \;\cong\; \Phi(e^*, \vec{y}) \qquad \text{for all } \vec{y} \in \mathbb{N}^n.$$

Proof. Let $\Phi :\subseteq \mathbb{N}^{n+1} \to \mathbb{N}$ be partial recursive. Since Φ is partial recursive, it has an index: there exists $a \in \mathbb{N}$ with $\varphi_a^{(n+1)} = \Phi$, i.e. $\varphi_a^{(n+1)}(e, \vec{y}) \cong \Phi(e, \vec{y})$ for all $e \in \mathbb{N}$, $\vec{y} \in \mathbb{N}^n$.
By the s-m-n theorem (with $m = 1$), there is a recursive total function $s : \mathbb{N}^2 \to \mathbb{N}$ such that

$$\varphi_{s(a,e)}^{(n)}(\vec{y}) \;\cong\; \varphi_a^{(n+1)}(e, \vec{y}) \;=\; \Phi(e, \vec{y}) \qquad \text{for all } e,\ \vec{y} \in \mathbb{N}^n.$$

Define $h : \mathbb{N} \to \mathbb{N}$ by $h(e) = s(a, e)$. Since s is recursive total and a is a fixed constant, h is a total recursive function.
By Rogers' fixed point theorem, there exists $e^* \in \mathbb{N}$ such that

$$\varphi_{e^*}^{(n)} \;=\; \varphi_{h(e^*)}^{(n)} \;=\; \varphi_{s(a,e^*)}^{(n)}.$$

Therefore, for all $\vec{y} \in \mathbb{N}^n$,

$$\varphi_{e^*}^{(n)}(\vec{y}) \;\cong\; \varphi_{s(a,e^*)}^{(n)}(\vec{y}) \;\cong\; \Phi(e^*, \vec{y}),$$

which is exactly the conclusion of Kleene's second recurison theorem. □

Problem 4

Let $W_e = \mathrm{dom}(\varphi_e^{(1)})$ for every $e \in \mathbb{N}$.

(a) We show there is a recursive total function $u : \mathbb{N} \to \mathbb{N}$ with $W_{u(x)} = \{y \in \mathbb{N} : x \mid y\}$.

Proof. Define $g : \mathbb{N}^2 \to \mathbb{N}$ by

$$g(x,y) = \begin{cases} 0 & \text{if } x \mid y, \\ \uparrow & \text{if } x \nmid y, \end{cases}$$

where $\uparrow$ means "undefined." Concretely, $g(x,y) = 0 \cdot \mu z\,[\mathrm{Div}(x,y) = 0]$, where $\mathrm{Div}(x,y)$ is defined to equal 0 if $x \mid y$ (i.e. if $(\exists k \leq y)\,[k \cdot x = y]$) and to go into an infinite search otherwise. More precisely, the relation "x divides y" is primitive recursive (decidable by bounded search: check whether any k with $0 \leq k \leq y$ satisfies $k \cdot x = y$), so we can define

$$g(x,y) = \begin{cases} 0 & \text{if } (\exists k \leq y)\,[k \cdot x = y], \\ \uparrow & \text{otherwise,} \end{cases}$$

which is partial recursive. (If $x \mid y$, the bounded search succeeds and we output 0; otherwise we invoke $\mu z\,[z \neq z]$, which diverges.)
Since g is partial recursive, there is an index a with $\varphi_a^{(2)}(x,y) \cong g(x,y)$. By the s-m-n theorem, there is a recursive total function $u : \mathbb{N} \to \mathbb{N}$ with

$$\varphi_{u(x)}^{(1)}(y) \;\cong\; \varphi_a^{(2)}(x,y) \;=\; g(x,y).$$

Then

$$W_{u(x)} = \mathrm{dom}(\varphi_{u(x)}^{(1)}) = \{y \in \mathbb{N} : g(x,y)\downarrow\} = \{y \in \mathbb{N} : x \mid y\}.$$ □

(b) We show there exists $e^* \in \mathbb{N}$ with $W_{e^*} = \{y \in \mathbb{N} : e^* \mid y\}$.

Proof. By part (a), the function $u : \mathbb{N} \to \mathbb{N}$ is total recursive and satisfies $W_{u(x)} = \{y : x \mid y\}$ for all x. By Rogers' fixed point theorem, there exists $e^* \in \mathbb{N}$ such that

$$\varphi_{e^*}^{(1)} = \varphi_{u(e^*)}^{(1)}.$$

Therefore

$$W_{e^*} = \mathrm{dom}(\varphi_{e^*}^{(1)}) = \mathrm{dom}(\varphi_{u(e^*)}^{(1)}) = W_{u(e^*)} = \{y \in \mathbb{N} : e^* \mid y\}.$$ □

Problem 5

For any $e \in \mathbb{N}$, let $R_e = \{y \in \mathbb{N} : (\exists x)\ \varphi_e^{(1)}(x)\downarrow= y\}$ denote the range of $\varphi_e^{(1)}$.

(a) We show there is a computable total function $S : \mathbb{N} \to \mathbb{N}$ with $R_{S(x)} = \{y \in \mathbb{N} : y \text{ is a power of } x\}$.

Proof. Define $f : \mathbb{N}^2 \to \mathbb{N}$ by $f(x, k) = x^k$. This is a total function, and it is primitive recursive (exponentiation is primitive recursive).
Since f is (total) recursive, there exists an index a with $\varphi_a^{(2)}(x, k) = f(x, k) = x^k$ for all x, k. By the s-m-n theorem, there is a recursive total function $S : \mathbb{N} \to \mathbb{N}$ with

$$\varphi_{S(x)}^{(1)}(k) \ \cong\ \varphi_a^{(2)}(x, k) \ =\ x^k \qquad \text{for all } k \in \mathbb{N}.$$

In particular $\varphi_{S(x)}^{(1)}$ is total and its range is

$$R_{S(x)} = \{x^k : k \in \mathbb{N}\} = \{y \in \mathbb{N} : y \text{ is a power of } x\}.$$

□

(b) We show there exists $x^* \in \mathbb{N}$ with $R_{x^*} = \{y \in \mathbb{N} : y \text{ is a power of } x^*\}$.

Proof. By part (a), $S : \mathbb{N} \to \mathbb{N}$ is total recursive with $R_{S(x)} = \{y : y \text{ is a power of } x\}$ for all x. By Rogers' fixed point theorem, there exists $x^* \in \mathbb{N}$ such that

$$\varphi_{x^*}^{(1)} = \varphi_{S(x^*)}^{(1)}.$$

Therefore

$$R_{x^*} = \text{range}(\varphi_{x^*}^{(1)}) = \text{range}(\varphi_{S(x^*)}^{(1)}) = R_{S(x^*)} = \{y \in \mathbb{N} : y \text{ is a power of } x^*\}.$$

□