Exercise 11d

Question 1.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = x^2 + 2x + 3 \text{ on } [4,6]$$

Answer:
Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{(36 + 12 + 3) - (16 + 8 + 3)}{6 - 4}$$

$$= \frac{24}{2}$$

=12

⇒ f' (c)=2c+2

⇒ 2c+2=12

⇒ c=5

Question 2.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = x^2 + x - 1 \text{ on } [0,4]$$

Answer:

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [0,4].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{(16 + 4 - 1) - (0 + 0 - 1)}{4 - 0}$$

=5

⇒ f'(c)=2c+1

⇒ 2c+1=5

⇒ c=2

Question 3.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = 2x^2 - 3x + 1 \text{ on } [1, 3]$$

Answer:
Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [1,3].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\mathbf{(18 - 9 + 1) - (2 - 3 + 1)}}{\mathbf{3 - 1}}$$

=5

⇒ f'(c)=4c-3

⇒ 4c-3=5

⇒ c=2

Question 4.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = x^3 + x^2 - 6x \text{ on } [-1, 4]$$

Answer:
Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [-1,4].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{(64 + 16 - 24) - (-1 + 1 + 6)}{4 + 1}$$

$$= \frac{50}{5}$$

=10

f' (c)=3c^2+2c-6

⇒ 3 c^2+2c-6=10

⇒ 3 c^2+2c-16=0

⇒ 3 c^2-6c+8c-16=0

⇒ 3c(c-2)+8(c-2)=0

$\Rightarrow$ (3c+8)(c-2)=0

$$c = 2, \frac{-8}{3}$$

Question 5.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = (x-4)(x-6)(x-8) \text{ on } [4,6]$$

Answer:
Given:

$$\mathbf{f(x) = x^3 - 18x^2 + 104x - 192}$$

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \ f'(c) = \frac{(216 - 648 + 624 - 192) - (64 - 288 + 416 - 192)}{6 - 2}$$

=0

$\Rightarrow$ f' (c)=3c^2-36c+104

=3c^2-36c+10

=0

$$\Rightarrow \ c = \frac{36 \pm \sqrt{1296 - 1248}}{6}$$

$$\Rightarrow \ c = \frac{36 \pm \sqrt{48}}{6}$$

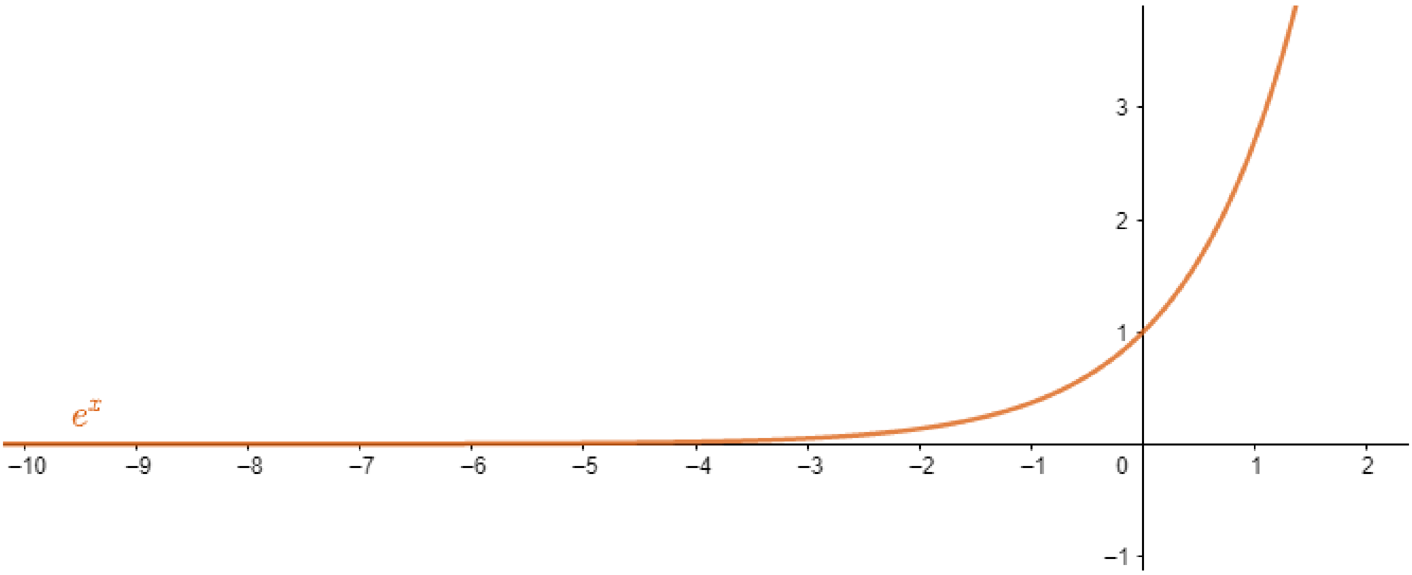
$$\Rightarrow c = 6 \pm \frac{2}{3}\sqrt{3}$$

Question 6.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = e^x \text{ on } [0,1]$$

Answer:
Given:

Since f(c) is continuous as well as differentiable in the interval [0,1].



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{e - 1}{1}$$

$\Rightarrow$ f' (c)=e^c

$\Rightarrow$ e^c =e-1

$$\Rightarrow \log_e e^c = \log_e(e - 1)$$

$\Rightarrow$ c=$\log_e$(e-1)

Question 7.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = x^{2/3} \text{ on } [0,1]$$

Answer:
Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [0,1].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{1 - 0}{1 - 0}$$

=1

$$f'(c) = \frac{2}{3} c^{\frac{1}{3}}$$

$$\Rightarrow \frac{2}{3} c^{\frac{1}{3}} = 1$$

$$\Rightarrow c^{\frac{-1}{3}} = \frac{3}{2}$$

$$\Rightarrow c^{\frac{1}{3}} = \frac{2}{3}$$

$$\Rightarrow c = \frac{8}{27}$$

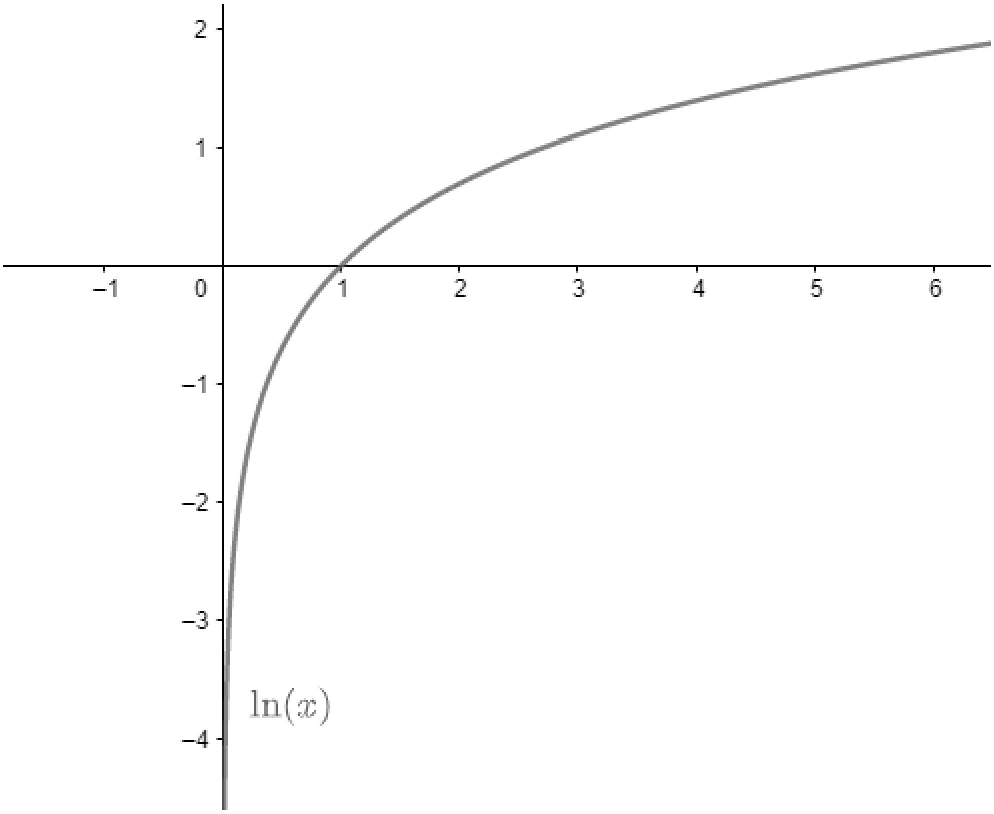
Question 8.

Verify Lagrange's mean-value theorem for the following function:

$f(x) = \log x$ on $[1, e]$

Answer:
Given:

Since log x is a continuous as well as differentiable function in the interval [1,e].



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\log e - \log 1}{e - 1}$$

$$= \frac{1}{e - 1}$$

$$f'(c) = \frac{1}{c}$$

$$\Rightarrow \frac{1}{e - 1} = \frac{1}{c}$$

c=e-1

Question 9.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = \tan^{-1} x \text{ on } [0,1]$$

Answer:
Given:

Since $\tan^{-1} x$ is a continuous as well as differentiable function in the interval [0,1].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\tan^{-1} 1 - \tan^{-1} 0}{1 - 0}$$

$$= \frac{\pi}{4}$$

$$\mathbf{f'(c)} = \frac{1}{1 + c^2}$$

$$\Rightarrow \frac{1}{1 + c^2} = \frac{\pi}{4}$$

$$\Rightarrow 1 + c^2 = \frac{4}{\pi}$$

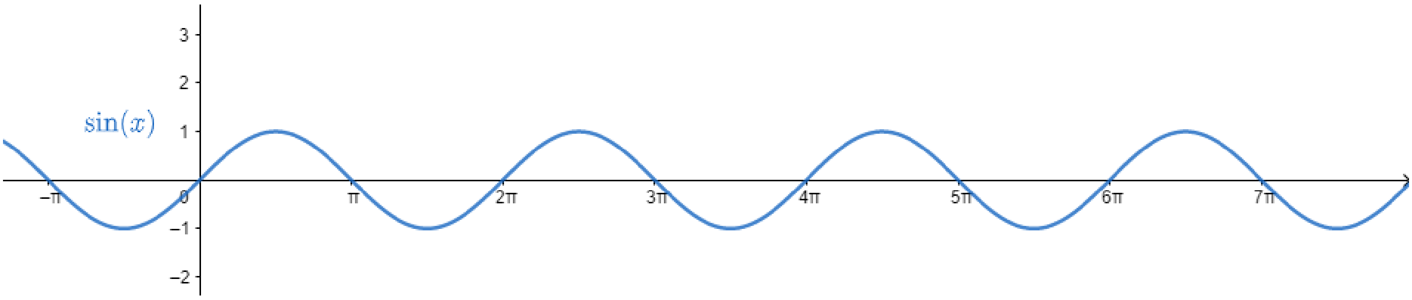
$$\Rightarrow c = \sqrt{\frac{4}{\pi} - 1}$$

Question 10.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = \sin x \text{ on } \left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$$

Answer:
Given:

Since sin x is a continuous as well as differentiable function in the interval $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sin\frac{5\pi}{2} - \sin\frac{\pi}{2}}{\frac{5\pi}{2} - \frac{\pi}{2}}$$

=0

f' (c)=cos x

cos x=0

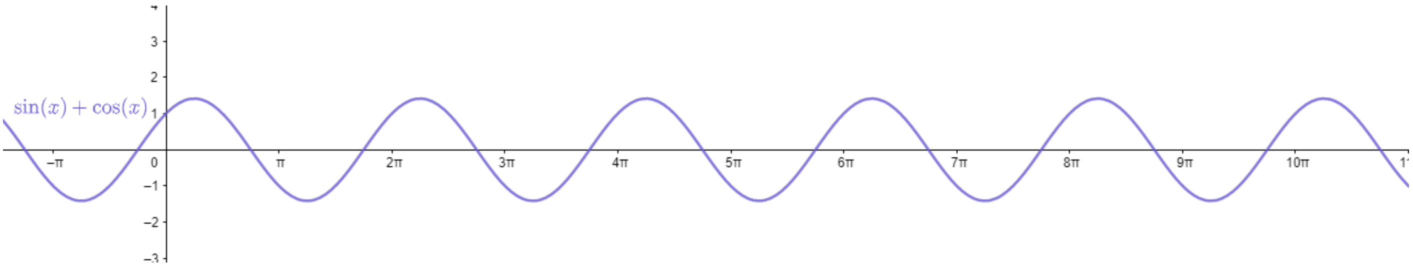
$$x = \frac{n\pi}{2}, n\epsilon\{1,2,3,4,5\}$$

Question 11.
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = (\sin x + \cos x) \text{ on } \left[0, \frac{\pi}{2}\right]$$

Answer:
Given:

Since (sin x + cos x) is a continuous as well as differentiable function in the interval $\left[0, \frac{\pi}{2}\right]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - \sin 0 - \cos 0}{\frac{\pi}{2} - 0}$$

=0

f' (c)=cos x-sin x

⇒ cos x-sin x=0

$$\Rightarrow \cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4} = 0$$

$$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \left(x + \frac{\pi}{4}\right) = \cos^{-1} 0$$

$$\Rightarrow \left(x + \frac{\pi}{4}\right) = 1$$

$$\Rightarrow \mathbf{x = 1 - \frac{\pi}{4}}$$

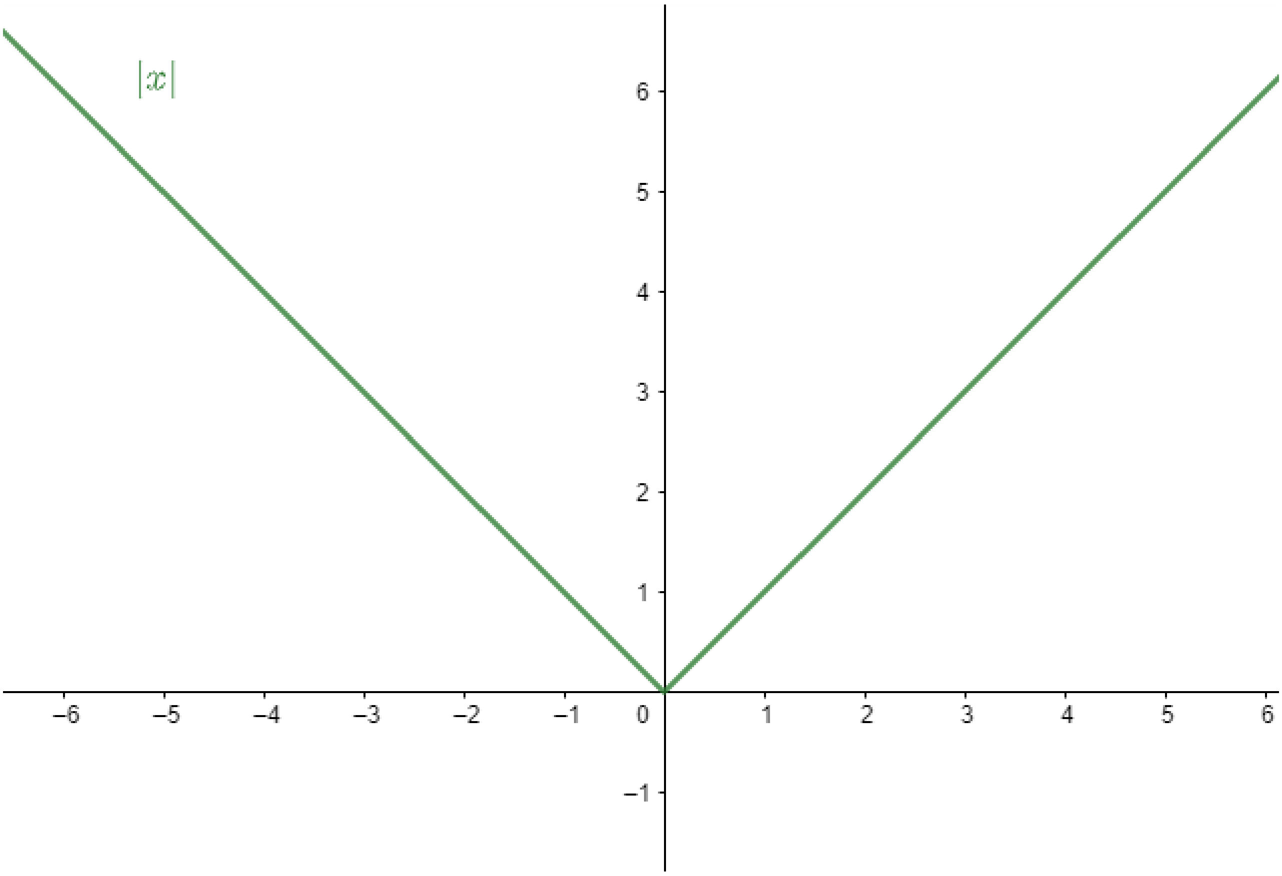
Question 12.
Show that Lagrange's mean-value theorem is not applicable to $f(x) = |x|$ on $[-1,1]$.

Answer:
Given:

Since f(x) is continuous in the interval [-1,1].

But is non differentiable at x=0 due to sharp corner.

So LMVT is not applicable to ƒ(x)=|x|

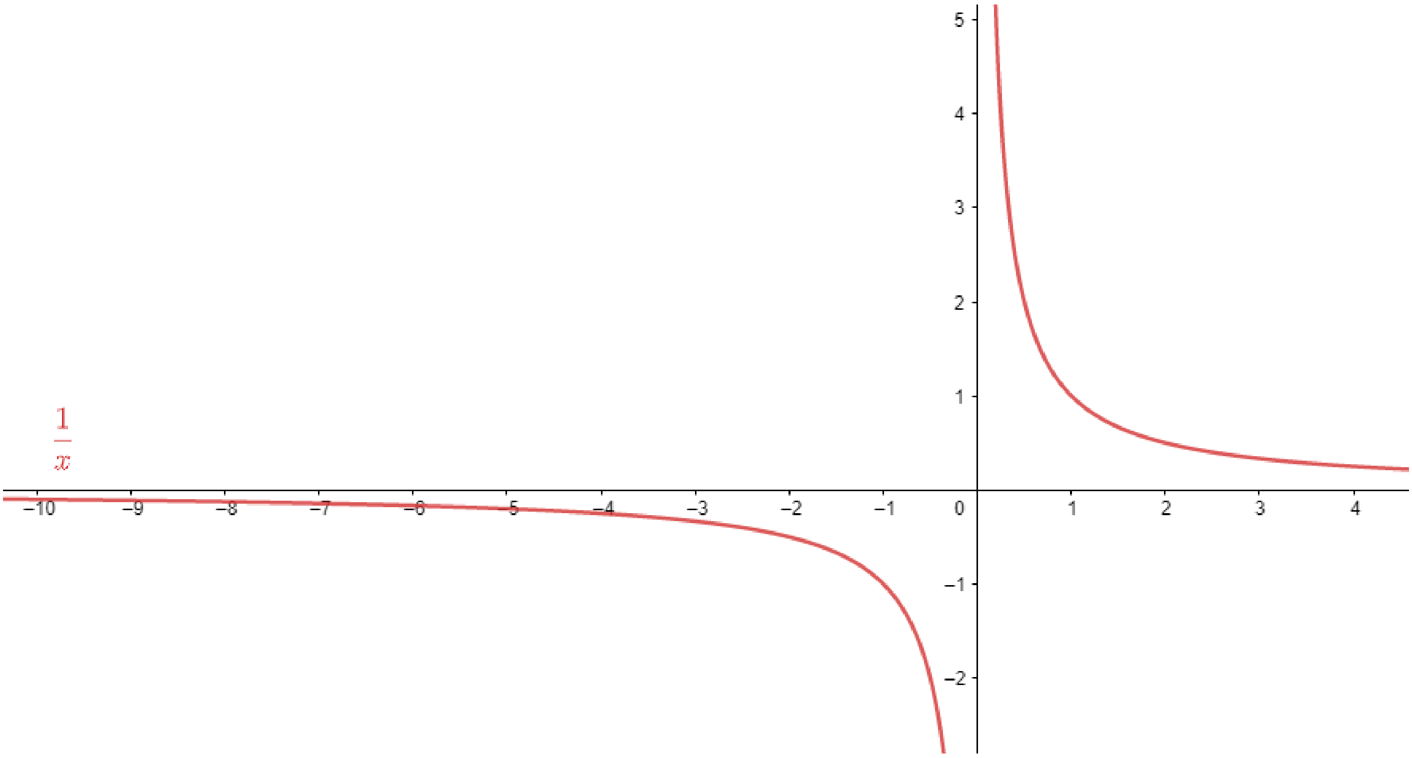


Question 13.

Show that Lagrange's mean-value theorem is not applicable to $f(x) = \frac{1}{x}$ on $[-1,1]$

Answer:
Given:

Since the graph is discontinuous at x=0 as shown in the graph.



So LMVT is not applicable to the above function.

Question 14.
Find 'c' of Lagrange's mean-value theorem for

$$f(x) = \left(x^3 - 3x^2 + 2x\right) \text{ on } \left[0, \frac{1}{2}\right]$$

Answer:
Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval $[0,\frac{1}{2}]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\frac{1}{8} - \frac{3}{4} + 1 - 0}{\frac{1}{2} - 0}$$

$$= \frac{3}{4}$$

f' (c)=3x^2 -6x+2

3 x^2-6x+2=3/4

12 x^2-24x+8=3

12 x^2-24x+5=0

$$x = \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$x = 1 \pm \sqrt{\frac{336}{576}}$$

$$x = 1 \pm \sqrt{\frac{7}{12}}$$

Question 15.
Find ‘c’ of Lagrange’s mean-value theorem for

$$f(x) = \sqrt{25 - x^2} \text{ on } [1, 5]$$

Answer:
Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [1,5].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sqrt{25 - 25} - \sqrt{25 - 1}}{5 - 1}$$

$$= \frac{-\sqrt{24}}{4}$$

$$f'(c) = \frac{1}{2\sqrt{25-c^2}}(-2c)$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{-\sqrt{24}}{4}$$

$$\Rightarrow 4c = \sqrt{24(25-c^2)}$$

$\Rightarrow 16c^2=600-24c^2$

$\Rightarrow 40c^2=600$

$\Rightarrow c^2=15$

$$\Rightarrow c = \sqrt{15}$$

Question 16.
Find 'c' of Lagrange's mean-value theorem for

$$f(x) = \sqrt{x+2} \text{ on } [4,6]$$

Answer:
Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$= \frac{\sqrt{8}-\sqrt{6}}{6-4}$$

$$= \frac{\sqrt{8}-\sqrt{6}}{2}$$

$$f'(c)=\frac{1}{2\sqrt{c+2}}$$

$$\Rightarrow \frac{1}{2\sqrt{c+2}}=\frac{\sqrt{8}-\sqrt{6}}{2}$$

$$\Rightarrow \frac{1}{\sqrt{c+2}}=\frac{\sqrt{8}-\sqrt{6}}{1}$$

$$\Rightarrow \sqrt{c+2}=\frac{1}{\sqrt{8}-\sqrt{6}}\times\frac{\sqrt{8}+\sqrt{6}}{\sqrt{8}+\sqrt{6}}$$

$$\Rightarrow \sqrt{c+2}=\frac{\sqrt{8}+\sqrt{6}}{2}$$

$$\Rightarrow c+2=\frac{1}{4}\left(8+6+2\sqrt{48}\right)$$

$$\Rightarrow c=\frac{3}{2}+2\sqrt{3}$$

$\Rightarrow$ **c=4.964**

Question 17.

Using Lagrange's mean-value theorem, find a point on the curve $y=x^2$, where the tangent is parallel to the line joining the point (1, 1) and (2, 4)

Answer:

Given:

$y=x^2$

Since y is a polynomial function.

It is continuous and differentiable in [1,2]

So, there exists a c such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{4 - 1}{2 - 1}$$

$= 3$

$\Rightarrow$ f' (c)=2c

$\Rightarrow$ 2c=3

$$c = \frac{3}{2}$$

So, the point is $\left(\frac{3}{2}, \frac{9}{4}\right)$

Question 18.
Find a point on the curve $y = x^3$, where the tangent to the curve is parallel to the chord joining the points (1, 1) and (3, 27).

Answer:
Given:

$$y = x^3$$

Since y is a polynomial function.

It is continuous and differentiable in [1,3]

So, there exists a c such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\mathbf{27 - 1}}{\mathbf{3 - 1}}$$

$= 13$

$\Rightarrow$ f' (c)=3c^2

$\Rightarrow$ 3c^2=13

$$\Rightarrow \; c = \sqrt{\frac{13}{3}}$$

$$\Rightarrow \; c = \frac{\sqrt{39}}{3}$$

So the point is $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$

Question 19.
Find the points on the curve $y = x^3 - 3x$, where the tangent to the curve is parallel to the chord joining (1, –2) and (2, 2).

Answer:
Given:

y=x^3-3x

Since y is a polynomial function.

It is continuous and differentiable in [1,2]

So, there exists a c such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\mathbf{(8-6)-(1-3)}}{\mathbf{2-1}}$$

=4

$\Rightarrow$ f' (c)=3c^2-3

$\Rightarrow 3c^2-3=4$

$\Rightarrow 3c^2=7$

$$\Rightarrow c^2=\frac{7}{3}$$

$$\Rightarrow c=\pm\sqrt{\frac{7}{3}}$$

So, the points are $\left(\sqrt{\frac{7}{3}}, \frac{-2}{3}\sqrt{\frac{7}{3}}\right), \left(\frac{-7}{3}\sqrt{\frac{7}{3}}, \frac{2}{3}\sqrt{\frac{7}{3}}\right)$

Question 20.
If $f(x)=x(1-\log x)$, where c > 0, show that $(a-b)\log c=b(1-\log b)-a(1-\log a)$, where $0 < a < c < b$.

Answer:
Given:

f(x)=x(1-log x)

Since the function is continuous as well as differentiable

So, there exists c such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow (1-\log c)-c\times\frac{1}{c}=\frac{b(1-\log b)-a(1-\log a)}{b-a}$$

$$\Rightarrow \log c=\frac{b(1-\log b)-a(1-\log a)}{b-a}$$

(b-a) log c=b(1-log b)-a(1-log a)

Hence proved.