Exercise 27e

Question 1.
Find the length and the equations of the line of shortest distance between the lines given by:

$$\frac{x-3}{3}=\frac{y-8}{-1}=z-3 \text{ and } \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}.$$

Answer:
Given: Cartesian equations of lines

$$L1: \frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$$

$$L2: \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

Formulae:

1. Condition for perpendicularity :

If line L1 has direction ratios (a_1, a_2, a_3) and that of line L2 are (b_1, b_2, b_3) then lines L1 and L2 will be perpendicular to each other if

$$(a_1 \times b_1)+(a_2 \times b_2)+(a_3 \times b_3)=0$$

2. Distance formula :

Distance between two points A≡(a_1, a_2, a_3) and B≡(b_1, b_2, b_3) is given by,

$$d=\sqrt{(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2}$$

3. Equation of line :

Equation of line passing through points A≡(x_1, y_1, z_1) and B≡(x_2, y_2, z_2) is given by,

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$$

Answer:

Given equations of lines

$$L1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

$$L2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

Direction ratios of L1 and L2 are (3, -1, 1) and (-3, 2, 4) respectively.

Let, general point on line L1 is $P \equiv (x_1, y_1, z_1)$

$x_1 = 3s+3$, $y_1 = -s+8$, $z_1 = s+3$

and let, general point on line L2 is $Q \equiv (x_2, y_2, z_2)$

$x_2 = -3t - 3$, $y_2 = 2t - 7$, $z_2 = 4t + 6$

$$\therefore \overline{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$= (-3t - 3 - 3s - 3)\hat{i} + (2t - 7 + s - 8)\hat{j} + (4t + 6 - s - 3)\hat{k}$$

$$\therefore \overline{PQ} = (-3t - 3s - 6)\hat{i} + (2t + s - 15)\hat{j} + (4t - s + 3)\hat{k}$$

Direction ratios of $\overline{PQ}$ are ((-3t – 3s - 6), (2t + s - 15), (4t – s + 3))

PQ will be the shortest distance if it perpendicular to both the given lines

Therefore, by the condition of perpendicularity,

3(-3t – 3s - 6) – 1(2t + s - 15) + 1(4t – s + 3) = 0 and

-3(-3t – 3s - 6) + 2(2t + s - 15) + 4(4t – s + 3) = 0

⇒ -9t – 9s – 18 – 2t – s + 15 + 4t – s + 3 = 0 and

9t + 9s + 18 + 4t + 2s – 30 + 16t – 4s + 12 = 0

⇒ -7t – 11s = 0 and

29t + 7s = 0

Solving above two equations, we get,

t = 0 and s = 0

therefore,

P ≡ (3, 8, 3) and Q ≡ (-3, -7, 6)

Now, distance between points P and Q is

$$d = \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2}$$

$$= \sqrt{(6)^2 + (15)^2 + (-3)^2}$$

$$= \sqrt{36 + 225 + 9}$$

$$= \sqrt{270}$$

$$= 3\sqrt{30}$$

Therefore, the shortest distance between two given lines is

$$d = 3\sqrt{30}\, units$$

Now, equation of line passing through points P and Q is,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\therefore \frac{x-3}{3+3} = \frac{y-8}{8+7} = \frac{z-3}{3-6}$$

$$\therefore \frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$$

$$\therefore \frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

Therefore, equation of line of shortest distance between two given lines is

$$\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

Question 2.
Find the length and the equations of the line of shortest distance between the lines given by:

$$\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1} \text{ and } \frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}.$$

Answer:
Given: Cartesian equations of lines

$$L1 : \frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}$$

$$L2 : \frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}$$

Formulae:

1. Condition for perpendicularity :

If line L1 has direction ratios (a_1, a_2, a_3) and that of line L2 are (b_1, b_2, b_3) then lines L1 and L2 will be perpendicular to each other if

$$(a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3) = 0$$

2. Distance formula :

Distance between two points A≡(a_1, a_2, a_3) and B≡(b_1, b_2, b_3) is given by,

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

3. Equation of line :

Equation of line passing through points A≡(x_1, y_1, z_1) and B≡(x_2, y_2, z_2) is given by,

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$$

Answer:

Given equations of lines

$$L1: \frac{x-3}{-1}=\frac{y-4}{2}=\frac{z+2}{1}$$

$$L2: \frac{x-1}{1}=\frac{y+7}{3}=\frac{z+2}{2}$$

Direction ratios of L1 and L2 are (-1, 2, 1) and (1, 3, 2) respectively.

Let, general point on line L1 is P≡(x_1, y_1, z_1)

x_1 = -s+3 , y_1 = 2s+4 , z_1 = s-2

and let, general point on line L2 is Q≡(x_2, y_2, z_2)

x_2 = t+1 , y_2 = 3t – 7 , z_2 = 2t - 2

$$\therefore \overline{PQ}=(x_2-x_1)\hat{\imath}+(y_2-y_1)\hat{\jmath}+(z_2-z_1)\hat{k}$$

$$=(t+1+s-3)\hat{\imath}+(3t-7-2s-4)\hat{\jmath}+(2t-2-s+2)\hat{k}$$

$$\therefore \overline{PQ}=(t+s-2)\hat{\imath}+(3t-2s-11)\hat{\jmath}+(2t-s)\hat{k}$$

Direction ratios of $\overline{PQ}$ are ((t + s – 2), (3t – 2s – 11), (2t – s))

PQ will be the shortest distance if it perpendicular to both the given lines

Therefore, by the condition of perpendicularity,

-1(t + s – 2) + 2(3t – 2s – 11) + 1(2t – s) = 0 and

1(t + s – 2) + 3(3t – 2s – 11) + 2(2t – s) = 0

⇒ - t – s + 2 + 6t – 4s – 22 + 2t – s = 0 and

t + s – 2 + 9t – 6s – 33 + 4t – 2s = 0

⇒ 7t – 6s = 20 and

14t - 7s = 35

Solving above two equations, we get,

t = 2 and s = -1

therefore,

P ≡ (4, 2, -3) and Q ≡ (3, -1, 2)

Now, distance between points P and Q is

$$d = \sqrt{(4-3)^2 + (2+1)^2 + (-3-2)^2}$$

$$= \sqrt{(1)^2 + (3)^2 + (-5)^2}$$

$$= \sqrt{1 + 9 + 25}$$

$$= \sqrt{35}$$

Therefore, the shortest distance between two given lines is

$$d = \sqrt{35}\, units$$

Now, equation of line passing through points P and Q is,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\therefore \frac{x - 4}{4 - 3} = \frac{y - 2}{2 + 1} = \frac{z + 3}{-3 - 2}$$

$$\therefore \frac{x-4}{1}=\frac{y-2}{3}=\frac{z+3}{-5}$$

$$\therefore \frac{x-4}{-1}=\frac{y-2}{-3}=\frac{z+3}{5}$$

Therefore, equation of line of shortest distance between two given lines is

$$\frac{x-4}{-1}=\frac{y-2}{-3}=\frac{z+3}{5}$$

Question 3.
Find the length and the equations of the line of shortest distance between the lines given by:

$$\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3} \text{ and } \frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}.$$

Answer:
Given: Cartesian equations of lines

$$L1: \frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}$$

$$L2: \frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}$$

Formulae:

1. Condition for perpendicularity :

If line L1 has direction ratios (a_1, a_2, a_3) and that of line L2 are (b_1, b_2, b_3) then lines L1 and L2 will be perpendicular to each other if

$$(a_1 \times b_1)+(a_2 \times b_2)+(a_3 \times b_3)=0$$

2. Distance formula :

Distance between two points A≡(a_1, a_2, a_3) and B≡(b_1, b_2, b_3) is given by,

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

3. Equation of line :

Equation of line passing through points A≡(x_1, y_1, z_1) and B≡(x_2, y_2, z_2) is given by,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

Answer:

Given equations of lines

$$L1 : \frac{x + 1}{2} = \frac{y - 1}{1} = \frac{z - 9}{-3}$$

$$L2 : \frac{x - 3}{2} = \frac{y + 15}{-7} = \frac{z - 9}{5}$$

Direction ratios of L1 and L2 are (2, 1, -3) and (2, -7, 5) respectively.

Let, general point on line L1 is P≡(x_1, y_1, z_1)

x_1 = 2s-1 , y_1 = s+1 , z_1 = -3s+9

and let, general point on line L2 is Q≡(x_2, y_2, z_2)

x_2 = 2t+3 , y_2 = -7t – 15 , z_2 = 5t + 9

$$\therefore \overline{PQ} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

$$= (5t + 9 - 2s + 1)\hat{\imath} + (-7t - 15 - s - 1)\hat{\jmath} + (5t + 9 + 3s - 9)\hat{k}$$

$$\therefore \overline{PQ} = (5t - 2s + 10)\hat{\imath} + (-7t - s - 16)\hat{\jmath} + (5t + 3s)\hat{k}$$

Direction ratios of $\overline{PQ}$ are ((5t - 2s + 10), (-7t – s – 16), (5t + 3s))

PQ will be the shortest distance if it perpendicular to both the given lines

Therefore, by the condition of perpendicularity,

2(5t - 2s + 10) + 1(-7t – s – 16) - 3(5t + 3s) = 0 and

2(5t - 2s + 10) – 7(-7t – s – 16) + 5(5t + 3s) = 0

⇒ 10t – 4s + 20 - 7t – s - 16 - 15t – 9s = 0 and

10t - 4s + 20 + 49t + 7s + 112 + 25t + 15s = 0

⇒ -12t – 14s = -4 and

84t + 18s = -132

Solving above two equations, we get,

t = -2 and s = 2

therefore,

P ≡ (3, 3, 3) and Q ≡ (-1, -1, -1)

Now, distance between points P and Q is

$$\mathrm{d} = \sqrt{(3+1)^2 + (3+1)^2 + (3+1)^2}$$

$$= \sqrt{(4)^2 + (4)^2 + (4)^2}$$

$$= \sqrt{16 + 16 + 16}$$

$$= \sqrt{48}$$

$$= 4\sqrt{3}$$

Therefore, the shortest distance between two given lines is

$$d = 4\sqrt{3}\, units$$

Now, equation of line passing through points P and Q is,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\therefore \frac{x-3}{3+1} = \frac{y-3}{3+1} = \frac{z-3}{3+1}$$

$$\therefore \frac{x-3}{4} = \frac{y-3}{4} = \frac{z-3}{4}$$

$$\therefore x - 3 = y - 3 = z - 3$$

$$\Rightarrow x = y = z$$

Therefore, equation of line of shortest distance between two given lines is

x = y = z

Question 4.
Find the length and the equations of the line of shortest distance between the lines given by:

$$\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1} \text{ and } \frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4}.$$

Answer:
<u>Given</u>: Cartesian equations of lines

$$L1 : \frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1}$$

$$L2 : \frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4}$$

<u>Formulae</u>:

1. Condition for perpendicularity :

If line L1 has direction ratios (a_1, a_2, a_3) and that of line L2 are (b_1, b_2, b_3) then lines L1 and L2 will be perpendicular to each other if

$$(a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3) = 0$$

2. Distance formula :

Distance between two points A≡(a_1, a_2, a_3) and B≡(b_1, b_2, b_3) is given by,

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

3. Equation of line :

Equation of line passing through points A≡(x_1, y_1, z_1) and B≡(x_2, y_2, z_2) is given by,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

Answer:

Given equations of lines

$$L1 : \frac{x - 6}{3} = \frac{y - 7}{-1} = \frac{z - 4}{1}$$

$$L2 : \frac{x}{-3} = \frac{y + 9}{2} = \frac{z - 2}{4}$$

Direction ratios of L1 and L2 are (3, -1, 1) and (-3, 2, 4) respectively.

Let, general point on line L1 is P≡(x_1, y_1, z_1)

x_1 = 3s+6 , y_1 = -s+7 , z_1 = s+4

and let, general point on line L2 is Q≡(x_2, y_2, z_2)

x_2 = -3t , y_2 = 2t – 9 , z_2 = 4t + 2

$$\therefore \overline{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$= (-3t - 3s - 6)\hat{i} + (2t - 9 + s - 7)\hat{j} + (4t + 2 - s - 4)\hat{k}$$

$\therefore \overline{PQ} = (-3t - 3s - 6)\hat{i} + (2t + s - 16)\hat{j} + (4t - s - 2)\hat{k}$

Direction ratios of $\overline{PQ}$ are ((-3t - 3s - 6), (2t + s – 16), (4t – s – 2))

PQ will be the shortest distance if it perpendicular to both the given lines

Therefore, by the condition of perpendicularity,

3(-3t - 3s - 6) - 1(2t + s – 16) + 1(4t – s – 2) = 0 and

-3(-3t - 3s - 6) + 2(2t + s – 16) + 4(4t – s – 2) = 0

⇒ -9t - 9s - 18 - 2t – s + 16 + 4t – s – 2 = 0 and

9t + 9s + 18 + 4t + 2s – 32 + 16t – 4s – 8 = 0

⇒ -7t – 11s = 4 and

29t + 7s = -22

Solving above two equations, we get,

t = 1 and s = -1

therefore,

P ≡ (3, 8, 3) and Q ≡ (-3, -7, 6)

Now, distance between points P and Q is

$$d = \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2}$$

$$= \sqrt{(6)^2 + (15)^2 + (-3)^2}$$

$$= \sqrt{36 + 225 + 9}$$

$$= \sqrt{270}$$

$= 3\sqrt{30}$

Therefore, the shortest distance between two given lines is

$d = 3\sqrt{30}\ units$

Now, equation of line passing through points P and Q is,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\therefore \frac{x - 3}{3 + 3} = \frac{y - 8}{8 + 7} = \frac{z - 3}{3 - 6}$$

$$\therefore \frac{x - 3}{6} = \frac{y - 8}{15} = \frac{z - 3}{-3}$$

$$\therefore \frac{x - 3}{2} = \frac{y - 8}{5} = \frac{z - 3}{-1}$$

Therefore, equation of line of shortest distance between two given lines is

$$\frac{x - 3}{2} = \frac{y - 8}{5} = \frac{z - 3}{-1}$$

Question 5.

Show that the lines $\frac{x}{1} = \frac{y - 2}{2} = \frac{z + 3}{3}$ and $\frac{x - 2}{2} = \frac{y - 6}{3} = \frac{z - 3}{4}$ intersect and find their point of intersection.

Answer:
Given: Cartesian equations of lines

$$L1 : \frac{x}{1} = \frac{y - 2}{2} = \frac{z + 3}{3}$$

$$L2 : \frac{x - 2}{2} = \frac{y - 6}{3} = \frac{z - 3}{4}$$

To Find: distance d

Formulae:

1. Equation of line :

Equation of line passing through point A (a_1, a_2, a_3) and having direction ratios (b_1, b_2, b_3) is

$\bar{r} = \bar{a} + \lambda\bar{b}$

Where, $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

And $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

2. Cross Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3. Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

4. Shortest distance between two lines :

The shortest distance between the skew lines $\bar{r} = \overline{a_1} + \lambda\overline{b_1}$ and

$\bar{r} = \overline{a_2} + \lambda\overline{b_2}$ is given by,

$$d = \left|\frac{\left(\overline{b_1} \times \overline{b_2}\right).\left(\overline{a_2} - \overline{a_1}\right)}{\left|\overline{b_1} \times \overline{b_2}\right|}\right|$$

<u>**Answer**</u>**:**

Given Cartesian equations of lines

$$L1 : \frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$$

Line L1 is passing through point (0, 2, -3) and has direction ratios (1, 2, 3)

Therefore, vector equation of line L1 is

$$\bar{r} = \left(0\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) + \lambda\left(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right)$$

And

$$L2 : \frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$$

Line L2 is passing through point (2, 6, 3) and has direction ratios (2, 3, 4)

Therefore, vector equation of line L2 is

$$\bar{r} = \left(2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}\right) + \mu\left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right)$$

Now, to calculate distance between the lines,

$$\bar{r} = \left(0\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) + \lambda\left(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right)$$

$$\bar{r} = \left(2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}\right) + \mu\left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right)$$

Here,

$\overline{a_1} = 0\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

$\overline{b_1} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\overline{a_2} = 2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$

$\overline{b_2} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$

Therefore,

$$\overline{b_1} \times \overline{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}$$

$= \hat{\imath}(8 - 9) - \hat{\jmath}(4 - 6) + \hat{k}(3 - 4)$

$\therefore \overline{b_1} \times \overline{b_2} = -\hat{\imath} + 2\hat{\jmath} - \hat{k}$

$\therefore \left|\overline{b_1} \times \overline{b_2}\right| = \sqrt{(-1)^2 + 2^2 + (-1)^2}$

$= \sqrt{1 + 4 + 1}$

$= \sqrt{6}$

$\overline{a_2} - \overline{a_1} = (2 - 0)\hat{\imath} + (6 - 2)\hat{\jmath} + (3 + 3)\hat{k}$

$\therefore \overline{a_2} - \overline{a_1} = 2\hat{\imath} + 4\hat{\jmath} + 6\hat{k}$

Now,

$\left(\overline{b_1} \times \overline{b_2}\right).\left(\overline{a_2} - \overline{a_1}\right) = \left(-\hat{\imath} + 2\hat{\jmath} - \hat{k}\right).\left(2\hat{\imath} + 4\hat{\jmath} + 6\hat{k}\right)$

$= ((-1) \times 2) + (2 \times 4) + ((-1) \times 6)$

= - 2 + 8 – 6

= 0

Therefore, the shortest distance between the given lines is

$$d = \left|\frac{(\overline{b_1} \times \overline{b_2}).(\overline{a_2} - \overline{a_1})}{|\overline{b_1} \times \overline{b_2}|}\right|$$

$$\therefore d = \left|\frac{0}{\sqrt{14}}\right|$$

$$\therefore d = 0 \text{ units}$$

As d = 0

Hence, given lines intersect each other.

Now, general point on L1 is

$x_1 = \lambda$, $y_1 = 2\lambda+2$, $z_1 = 3\lambda-3$

let, $P(x_1, y_1, z_1)$ be point of intersection of two given lines.

Therefore, point P satisfies equation of line L2.

$$\therefore \frac{\lambda - 2}{2} = \frac{2\lambda + 2 - 6}{3} = \frac{3\lambda - 3 - 3}{4}$$

$$\therefore \frac{\lambda - 2}{2} = \frac{2\lambda - 4}{3}$$

$\Rightarrow 3\lambda - 6 = 4\lambda - 8$

$\Rightarrow \lambda = 2$

Therefore, $x_1 = 2$, $y_1 = 2(2)+2$, $z_1 = 3(2)-3$

$\Rightarrow x_1 = 2$, $y_1 = 6$, $z_1 = 3$

Hence point of intersection of given lines is (2, 6, 3).

Question 6.

Show that the lines $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$ and $\frac{x-2}{2}=\frac{y-1}{3}=\frac{z+1}{-2}$ do not intersect each other.

Answer:
Given: Cartesian equations of lines

$$L1:\ \frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$$

$$L2:\ \frac{x-2}{2}=\frac{y-1}{3}=\frac{z+1}{-2}$$

To Find: distance d

Formulae:

1. Equation of line :

Equation of line passing through point A (a_1, a_2, a_3) and having direction ratios (b_1, b_2, b_3) is

$$\bar{r}=\bar{a}+\lambda\bar{b}$$

Where, $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

And $\bar{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

2. Cross Product :

If $\bar{a}\ \&\ \bar{b}$ are two vectors

$$\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$$

$$\bar{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$$

then,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3. Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

4. Shortest distance between two lines :

The shortest distance between the skew lines $\bar{r} = \bar{a_1} + \lambda\overline{b_1}$ and

$\bar{r} = \overline{a_2} + \lambda\overline{b_2}$ is given by,

$$d = \left|\frac{(\overline{b_1} \times \overline{b_2}).(\overline{a_2} - \overline{a_1})}{|\overline{b_1} \times \overline{b_2}|}\right|$$

<u>Answer</u>:

Given Cartesian equations of lines

$$L1 : \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$$

Line L1 is passing through point (1, -1, 1) and has direction ratios (3, 2, 5)

Therefore, vector equation of line L1 is

$$\bar{r} = (\hat{i} - \hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 5\hat{k})$$

And

$$L2 : \frac{x-2}{2} = \frac{y-1}{3} = \frac{z+1}{-2}$$

Line L2 is passing through point (2, 1, -1) and has direction ratios (2, 3, -2)

Therefore, vector equation of line L2 is

$$\bar{r} = \left(2\hat{\imath} + \hat{\jmath} - \hat{k}\right) + \mu\left(2\hat{\imath} + 3\hat{\jmath} - 2\hat{k}\right)$$

Now, to calculate distance between the lines,

$$\bar{r} = \left(\hat{\imath} - \hat{\jmath} + \hat{k}\right) + \lambda\left(3\hat{\imath} + 2\hat{\jmath} + 5\hat{k}\right)$$

$$\bar{r} = \left(2\hat{\imath} + \hat{\jmath} - \hat{k}\right) + \mu\left(2\hat{\imath} + 3\hat{\jmath} - 2\hat{k}\right)$$

Here,

$$\overline{a_1} = \hat{\imath} - \hat{\jmath} + \hat{k}$$

$$\overline{b_1} = 3\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$$

$$\overline{a_2} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$$

$$\overline{b_2} = 2\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$$

Therefore,

$$\overline{b_1} \times \overline{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 2 & 5 \\ 2 & 3 & -2 \end{vmatrix}$$

$$= \hat{\imath}(-4 - 15) - \hat{\jmath}(-6 - 10) + \hat{k}(9 - 4)$$

$$\therefore \overline{b_1} \times \overline{b_2} = -19\hat{\imath} + 16\hat{\jmath} + 5\hat{k}$$

$$\therefore \left|\overline{b_1} \times \overline{b_2}\right| = \sqrt{(-19)^2 + 16^2 + 5^2}$$

$= \sqrt{361 + 256 + 25}$

$= \sqrt{642}$

$\overline{a_2} - \overline{a_1} = (2 - 1)\hat{i} + (1 + 1)\hat{j} + (-1 - 1)\hat{k}$

$\therefore \overline{a_2} - \overline{a_1} = \hat{i} + 2\hat{j} - 2\hat{k}$

Now,

$\left(\overline{b_1} \times \overline{b_2}\right).(\overline{a_2} - \overline{a_1}) = \left(-19\hat{i} + 16\hat{j} + 5\hat{k}\right).\left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$

$= ((-19) \times 1) + (16 \times 2) + (5 \times (-2))$

= - 19 + 32 – 10

= 3

Therefore, the shortest distance between the given lines is

$$d = \left|\frac{\left(\overline{b_1} \times \overline{b_2}\right).(\overline{a_2} - \overline{a_1})}{\left|\overline{b_1} \times \overline{b_2}\right|}\right|$$

$$\therefore d = \left|\frac{3}{\sqrt{642}}\right|$$

$$\therefore d = \frac{3}{\sqrt{642}} \text{ units}$$

As d ≠ 0

Hence, given lines do not intersect each other.