Exercise 3a

Question 1.
Let * be a binary operation on the set I of all integers, defined by a * b = 3a + 4b - 2. Find the value of 4 * 5.

Answer:
To find: 4*5

a*b = 3a + 4b - 2

Here a = 4 and b = 5

⇒ 4*5 = 3 × 4 + 4 × 5 - 2 = 12 + 20 - 2 = 30

⇒ 4*5 = 30

Question 2.
The binary operation * on R is defined by a * b = 2a + b. Find (2 * 3) * 4.

Answer:
To find: (2*3)*4

Given: a*b = 2a + b

⇒ 2*3 = 2 × 2 + 3 = 7

Now 7*4 = 2 × 7 + 4 = 14 + 4 = 18

⇒ (2*3)*4 = 18

Question 3.

Let * be a binary operation on the set of all nonzero real numbers, defined by $a * b = \frac{ab}{5}$. Find the value of x given that
2 * (x * 5) = 10.

Answer:
To find: value of x

Given: $a*b = \frac{ab}{5}$

$\Rightarrow x*5 = \frac{5x}{5} = x$

Now $(2*x) = \frac{2x}{5}$

$\Rightarrow \frac{2x}{5} = 10$
$\Rightarrow x = 25$

Question 4.
Let $*: R \times R \rightarrow R$ be a binary operation given by $a * b = a + 4b^2$. Then, compute $(-5) * (2 * 0)$.

Answer:
To find: $(-5)*(2*0)$

Given: $a*b = a + 4b^2$

$\Rightarrow (2*0) = 2 + 4 \times 0^2 = 2$

Now $(-5)*2 = -5 + 4 \times 2^2 = -5 + 16 = 11$

Question 5.
Let be a binary operation on the set Q of all rational numbers given as $a * b = (2a - b)^2$ for all $a, b \in Q$. Find $3 * 5$ and $5 * 3$. Is $3 * 5 = 5 * 3$?

Answer:
To find: $3*5$ and $5*3$

Given: $a*b = (2a - b)^2$

$\Rightarrow 3*5 = (6 - 5)^2 = 1$

Now $5*3 = (10 - 3)^2 = 49$

$\Rightarrow$ $3*5$ is not equal to $5*3$

Question 6.
Let * be a binary operation on N given by a *b = 1 cm of a and b. Find the value of 20 * 16.

Is * (i) commutative, (ii) associative?

Answer:
To find: LCM of 20 and 16

Prime factorizing 20 and 16 we get.

$20 = 2^2 \times 5$

$16 = 2^4$

⇒ LCM of 20 and 16 = $2^4 \times 5 = 80$

(i) To find LCM highest power of each prime factor has been taken from both the numbers and multiplied.

So it is irrelevant in which order the number are taken as their prime factors will remain the same.

So LCM(a,b) = LCM(b,a)

So * is commutative

(ii) Let us assume that * is associative

⇒ LCM[LCM(a,b),c] = LCM[a,LCM(b,c)]

Let the prime factors of a be p_1, p_2

Let the prime factors of b be p_2, p_3

Let the prime factors of c be p_3, p_4

Let the higher factor of p_i be q_i for i = 1,2,3,4

LCM (a,b) = $p_1^{q1} \times p_2^{q2} \times p_3^{q3}$

$LCM[LCM(a,b),c] = p_1^{q1} \times p_2^{q2} \times p_3^{q3} \times p_4^{q4}$

$LCM\ (b,c) = p_2^{q2} \times p_3^{q3} \times p_4^{q4}$

$LCM[a,LCM(b,c)] = p_1^{q1} \times p_2^{q2} \times p_3^{q3} \times p_4^{q4}$

⇒ * is associative

Question 7.
If * be the binary operation on the set Z of all integers defined by a * b ($a + 3b^2$), find 2 * 4

Answer:
To find: 2*4

Given: $a*b = a + 3b^2$

⇒ $2*4 = (2 + 3 \times 4^2) = 2 + 48 = 50$

Question 8.
Show that * on Z^+ defined by a *b = |a –b| is not a binary operation.

Answer:
To prove: * is not a binary operation

Given: a and b are defined on positive integer set

And a*b = |a - b|

⇒ a*b = (a - b), when a>b

= b - a when b>a

= 0 when a = b

But 0 is neither positive nor negative.

So 0 does not belong to Z^+ .

So a*b = |a - b| does not belong to Z^+ for all values of a and b

So * is not a binary operation.

Hence proved

Question 9.
Let * be a binary operation on N, defined by a * b = a^b for all a. b ∈ N.

Show that * is neither commutative nor associative.

Answer:
To prove: * is neither commutative nor associative

Let us assume that * is commutative

⇒ $a^b = b^a$ for all a,b ∈ N

This is valid only for a = b

For example take a = 1, b = 2

$1^2 = 1$ and $2^1 = 2$

So * is not commutative

Let us assume that * is associative

⇒ $(a^b)^c = a^{b^c}$ for all a,b,c ∈ N

⇒ $a^{bc} = a^{b^c}$ for all a,b,c ∈ N

This is valid in the following cases:

(i) a = 1

(ii) b = 0

(iii) bc = b^c

For example, let a = 2,b = 1,c = 3

$a^{bc} = 2^{(1 \times 3)} = 2^3 = 8$

$a^{b^c} = 2^{1^3} = 2$

So * is not associative

Question 10.
Let a * b = 1 cm (a, b) for all values of
a, b $\in$ N.

(i) Find (12 * 16).

(ii) Show that * is commutative on N.

(iii) Find the identity element in N.

(iv) Find all invertible elements in N.

Answer:
To find: (i)

LCM of 12 and 16

Prime factorizing 12 and 16 we get.

$20 = 2^2 \times 3$

$16 = 2^4$

$\Rightarrow$ LCM of 20 and 16 = $2^4 \times 3 = 48$

(ii) To find LCM highest power of each prime factor has been taken from both the numbers and multiplied.

So it is irrelevant in which order the number are taken as their prime factors will remain the same.

So LCM(a,b) = LCM(b,a)

So * is commutative.

(iii)let $x \in N$ and $x*1 = lcm(x,1) = x = lcm(1,x)$

1 is the identity element.

(iv)let there exist y in n such that $x*y = e = y*x$

Here $e = 1$,

$Lcm(x,y) = 1$

This happens only when $x = y = 1$.

1 is the invertible element of n with respect to *.

Question 11.
Let Q be the set of all positive rational numbers.

(i) Show that the operation * on Q^+ defined by $a * b = \frac{1}{2}(a + b)$ is a binary operation.

(ii) Show that * is commutative.

(iii) Show that * is not associative.

Answer:
(i)Let $a = 1$, $b = 2 \in Q^+$

$a*b = \frac{1}{2}(1 + 2) = 1.5 \in Q^+$

* is closed and is thus a binary operation on Q^+

(ii) $a*b = \frac{1}{2}(1 + 2) = 1.5$

And $b*a = \frac{1}{2}(2 + 1) = 1.5$

Hence * is commutative.

(iii)let $c = 3$.

(a*b)*c = 1.5*c = $\frac{1}{2}(1.5 + 3) = 2.75$

a*(b*c) = a*$\frac{1}{2}(2 + 3)$ = 1*2.5 = $\frac{1}{2}(1 + 2.5)$ = 1.75

hence * is not associative.

Question 12.
Show that the set A = { - 1, 0, 1) is not closed for addition.

Answer:
For a set to be closed for addition,

For any 2 elements of the set,say a and b, a + b must also be a member of the given set, where a and b may be same or distinct

In the given problem let a = 1 and b = 1

a + b = 2 which is not in the given in set

So the set is not closed for addition.

Hence proved.

Question 13.

Show that * on R –{ - 1}, defined by $(a * b) = \frac{a}{(b+1)}$ is neither commutative nor associative.

Answer:
let a = 1,b = 0∈R - { - 1}

a*b = $\frac{1}{0 + 1}$ = 1

And b*a = $\frac{0}{1 + 1}$ = 0

Hence * is not commutative.

Let c = 3.

(a*b)*c = 1*c = $\frac{1}{3+1} = \frac{1}{4}$

a*(b*c) = a*$\frac{0}{3+1}$ = 1*0 = $\frac{1}{0+1}$ = 1

Hence * is not associative.

Question 14.
For all a, b ∈ R, we define a * b = |a – b|.

Show that * is commutative but not associative.

Answer:
a*b = a - b if a>b

= - (a - b) if b>a

b*a = a - b if a>b

= - (a - b) if b>a

So a*b = b*a

So * is commutative

To show that * is associative we need to show

(a*b)*c = a*(b*c)

Or ||a - b| - c| = |a - |b - c||

Let us consider c>a>b

Eg a = 1,b = - 1,c = 5

LHS:

|a - b| = |1 + 1| = 2

$||a - b| - c| = |2 - 5| = 3$

RHS

$|b - c| = | - 1 - 5| = 6$

$|a - |b - c|| = |1 - 6| = | - 5| = 5$

As LHS is not equal to RHS * is not associative

Question 15.
For all a, b $\in$ N, we define $a * b = a^3 + b^3$.

Show that * is commutative but not associative.

Answer:
let a = 1,b = 2$\in$N

$a*b = 1^3 + 2^3 = 9$

And $b*a = 2^3 + 1^3 = 9$

Hence * is commutative.

Let c = 3

$(a*b)*c = 9*c = 9^3 + 3^3$

$a*(b*c) = a*(2^3 + 3^3) = 1*35 = 1^3 + 35^3$

$(a*b)*c \neq a*(b*c)$

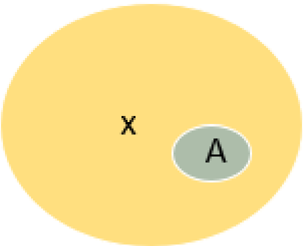
Hence * is not associative.

Question 16.
Let X be a nonempty set and * be a binary operation on P(X), the power set of X, defined by $A * B = A \cap B$ for all A, B $\in$ P(X).

(i) Find the identity element in P(X).

(ii) Show that X is the only invertible element in P(X).

Answer:
e is the identity of * if e*a = a



From the above Venn diagram,

$A*X = A \cap X = A$

$X*A = X \cap A = A$

⇒ X is the identity element for binary operation *

Let B be the invertible element

⇒ $A*B = X$

⇒ $A \cap B = X$

This is only possible if A = B = X

Thus X is the only invertible element in P(X)

Hence proved.

Question 17.
A binary operation * on the set (0, 1, 2, 3, 4, 5) is defined as

$$a * b = \begin{cases} a+b; & \text{if } a+b<6 \\ a+b-6; & \text{if } a+b \geq 6 \end{cases}$$

Show that 0 is the identity for this operation and each element a has an inverse (6 - a)

Answer:
To find: identity and inverse element

For a binary operation if a*e = a, then e s called the right identity

If e*a = a then e is called the left identity

For the given binary operation,

e*b = b

⇒ e + b = b

⇒ e = 0 which is less than 6.

b*e = b

⇒ b + e = b

⇒ e = 0 which is less than 6

For the 2nd condition,

e*b = b

⇒ e + b - 6 = b

⇒ e = 6

But e = 6 does not belong to the given set (0,1,2,3,4,5)

So the identity element is 0

An element c is said to be the inverse of a, if a*c = e where e is the identity element (in our case it is 0) a*c = e

⇒ a + c = e

⇒ a + c = 0

⇒ c = - a

a belongs to (0,1,2,3,4,5)

- a belongs to (0, - 1, - 2, - 3, - 4, - 5)

So c belongs to (0, - 1, - 2, - 3, - 4, - 5)

So c = - a is not the inverse for all elements a

Putting in the 2^{nd} condition

a*c = e

⇒ a + c - 6 = 0

⇒ c = 6 - a

$0 \le a < 6$

⇒ $-6 \le -a < 0$
⇒ $0 \le 6 - a < 6$
$0 \le c < 5$

So c belongs to the given set

Hence the inverse of the element a is (6 - a)

Hence proved