Exercise 28b

Question 1.
Find the vector and Cartesian equations of a plane which is at a distance of 5 units from the origin and which has $\hat{k}$ as the unit vector normal to it.

Answer:
Given :

d = 5

$\hat{n} = \hat{k}$

To Find : Equation of a plane

Formulae :

1) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

2) Equation of plane :

Equation of plane which is at a distance of 5 units from the origin and having $\hat{n}$ as a unit vector normal to it is

$\bar{r}.\hat{n} = d$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given d = 5 and $\hat{n} = \hat{k}$,

Equation of plane is

$\bar{r}.\hat{n} = d$

$\therefore \bar{r}.\hat{k} = 5$

This is a vector equation of the plane

Now,

$\bar{r}.\hat{k} = \left(x\hat{\imath} + y\hat{\jmath} + \ z\hat{k}\right).\hat{k}$

$= \ (x \times 0) + (y \times 0) + (z \times 1)$

= z

$\therefore \bar{r}.\hat{k} = z$

Therefore, the equation of the plane is

This is - the Cartesian z = 5 equation of the plane.

Question 2.
Find the vector and Cartesian equations of a plane which is at a distance of 7 units from the origin and whose normal vector from the origin is $\left(3\hat{i} + 5\hat{j} - 6\hat{k}\right)$.

Answer:
Given :

d = 7

$\bar{n} = 3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}$

To Find : Equation of plane

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

then,

$$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

3) Equation of plane :

Equation of plane which is at a distance of 5 units from the origin and having $\hat{n}$ as a unit vector normal to it is

$$\bar{r}.\hat{n} = d$$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given normal vector

$$\bar{n} = 3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}$$

Unit vector normal to the plane is

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$\therefore \hat{n} = \frac{3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}}{\sqrt{3^2 + 5^2 + (-6)^2}}$$

$$\therefore \hat{n} = \frac{3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}}{\sqrt{9 + 25 + 36}}$$

$$\therefore \hat{n} = \frac{3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}}{\sqrt{70}}$$

Equation of the plane is

$$\bar{r}.\hat{n} = d$$

$$\therefore \bar{r}.\left(\frac{3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}}{\sqrt{70}}\right) = 7$$

$$\therefore \bar{r}.\left(3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}\right) = 7\sqrt{70}$$

This is a vector equation of the plane.

Now,

$$\bar{r}.\left(3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}\right) = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}\right)$$

= (x × 3) + (y × 5) + (z × (-6))

= 3x + 5y – 6z

Therefore equation of the plane is

$$3x + 5y - 6z = 7\sqrt{70}$$

This is the Cartesian equation of the plane.

Question 3.

Find the vector and Cartesian equations of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and whose normal vector from the origin is $(2\hat{i} - 3\hat{j} + 4\hat{k})$.

Answer:

Given :

$$d = \frac{6}{\sqrt{29}}$$

$$\bar{n} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$$

To Find : Equation of a plane

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then the unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

3) Equation of plane :

Equation of plane which is at a distance of 5 units from the origin and having $\hat{n}$ as a unit vector normal to it is

$\bar{r}.\hat{n} = d$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given normal vector

$\bar{n} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$

Unit vector normal to the plane is

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}}{\sqrt{2^2 + (-3)^2 + 4^2}}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}}{\sqrt{4 + 9 + 16}}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}}{\sqrt{29}}$$

Equation of the plane is

$\bar{r}.\hat{n} = d$

$$\therefore \bar{r}.\left(\frac{2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}}{\sqrt{29}}\right) = \frac{6}{\sqrt{29}}$$

$$\therefore \bar{r}.\left(2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}\right) = 6$$

This is a vector equation of the plane.

Now,

$$\bar{r}.\left(2\hat{\imath}-3\hat{\jmath}+4\hat{k}\right)=\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right).\left(2\hat{\imath}-3\hat{\jmath}+4\hat{k}\right)$$

= (x × 2) + (y × (-3)) + (z × 4)

= 2x - 3y + 4z

Therefore equation of the plane is

2x - 3y + 4z = 6

This is the Cartesian equation of the plane.

Question 4.
Find the vector and Cartesian equations of a plane which is at a distance of 6 units from the origin and which has a normal with direction ratios 2, -1, -2.

Answer:
Given :

d = 6

direction ratios of $\bar{n}$ are (2, -1, -2)

$$\therefore \bar{n}=2\hat{\imath}-\hat{\jmath}-2\hat{k}$$

To Find : Equation of plane

Formulae :

1) Unit Vector :

Let $\bar{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$ be any vector

Then the unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

then,

$$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

3) Equation of plane :

Equation of plane which is at a distance of 5 units from the origin and having $\hat{n}$ as a unit vector normal to it is

$$\bar{r}.\hat{n} = d$$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given normal vector

$$\bar{n} = 2\hat{\imath} - \hat{\jmath} - 2\hat{k}$$

Unit vector normal to the plane is

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{\sqrt{4+1+4}}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{\sqrt{9}}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}$$

Equation of the plane is

$$\bar{r}.\hat{n} = d$$

$$\therefore \bar{r}.\left(\frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}\right) = 6$$

$$\therefore \bar{r}.\left(2\hat{\imath} - \hat{\jmath} - 2\hat{k}\right) = 18$$

This is vector equation of the plane.

Now,

$$\bar{r}.\left(2\hat{\imath} - \hat{\jmath} - 2\hat{k}\right) = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(2\hat{\imath} - \hat{\jmath} - 2\hat{k}\right)$$

= (x × 2) + (y × (-1)) + (z × (-2))

= 2x - y – 2z

Therefore equation of the plane is

$$2x - y - 2z = 18$$

This is Cartesian equation of the plane.

Question 5.
Find the vector, and Cartesian equations of a plane which passes through the point (1, 4, 6) and the normal vector to the plane is $\left(\hat{i} - 2\hat{j} + \hat{k}\right)$.

Answer:
Given :

A = (1, 4, 6)

$\bar{n} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

To Find : Equation of plane.

Formulae :

1) Position Vector :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

3) Equation of plane :

Equation of plane passing through point A and having $\bar{n}$ as a unit vector normal to it is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Position vector of point A = (1, 4, 6) is

$\bar{a} = \hat{\imath} + 4\hat{\jmath} + 6\hat{k}$

Now,

$\bar{a}.\bar{n} = (\hat{\imath} + 4\hat{\jmath} + 6\hat{k}).(\hat{\imath} - 2\hat{\jmath} + \hat{k})$

= (1×1) + (4×(-2)) + (6×1)

= 1 – 8 + 6

= - 1

Equation of plane is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

$\therefore \bar{r}.(\hat{\imath} - 2\hat{\jmath} + \hat{k}) = -1$

This is vector equation of the plane.

As $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Therefore

$\bar{r}.(\hat{\imath} - 2\hat{\jmath} + \hat{k}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(\hat{\imath} - 2\hat{\jmath} + \hat{k})$

= (x × 1) + (y × (-2)) + (z × 1)

= x - 2y + z

Therefore equation of the plane is

$\mathrm{x} - 2\mathrm{y} + \mathrm{z} = -1$

This is Cartesian equation of the plane.

Question 6.

Find the length of the perpendicular from the origin to the plane $\bar{r}\cdot\left(3\hat{i}-12\hat{j}-4\hat{k}\right)+39=0$. Also write the unit normal vector from the origin to the plane.

Answer:
Given :

Equation of plane : $\bar{r}.\left(3\hat{\imath}-12\hat{\jmath}-4\hat{k}\right)+39=0$

To Find :

i) Length of perpendicular = d

ii) Unit normal vector = $\hat{n}$

Formulae :

1) Unit Vector :

Let $\bar{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$$\hat{a}=\frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}|=\sqrt{{a_1}^2+{a_2}^2+{a_3}^2}$

2) Length of perpendicular :

The length of the perpendicular from the origin to the plane

$\bar{r}.\bar{n}=p$ is given by,

$$d=\frac{p}{|\bar{n}|}$$

Given the equation of the plane is

$$\bar{r}.\left(3\hat{\imath}-12\hat{\jmath}-4\hat{k}\right)+39=0$$

$\therefore \bar{r}.\left(3\hat{\imath} - 12\hat{\jmath} - 4\hat{k}\right) = -39$

$\therefore \bar{r}.\left(-3\hat{\imath} + 12\hat{\jmath} + 4\hat{k}\right) = 39$

Comparing the above equation with

$\bar{r}.\bar{n} = p$

We get,

$\bar{n} = -3\hat{\imath} + 12\hat{\jmath} + 4\hat{k}$ & p = 39

Therefore,

$|\bar{n}| = \sqrt{(-3)^2 + 12^2 + 4^2}$

$= \sqrt{9 + 144 + 16}$

$= \sqrt{169}$

= 13

The length of the perpendicular from the origin to the given plane is

$d = \frac{p}{|\bar{n}|}$

$\therefore d = \frac{39}{13}$

$\therefore d = 3$

Vector normal to the plane is

$\bar{n} = -3\hat{\imath} + 12\hat{\jmath} + 4\hat{k}$

Therefore, the unit vector normal to the plane is

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$\therefore \hat{n} = \frac{-3\hat{\imath} + 12\hat{\jmath} + 4\hat{k}}{13}$$

$$\therefore \hat{n} = \frac{-3\hat{\imath}}{13} + \frac{12\hat{\jmath}}{13} + \frac{4\hat{k}}{13}$$

Question 7.
Find the Cartesian equation of the plane whose vector equation is $\bar{r} \cdot \left(3\hat{i} + 5\hat{j} - 9\hat{k}\right) = 8$.

Answer:
Given :

Vector equation of the plane is

$$\bar{r}.\left(3\hat{\imath} + 5\hat{\jmath} - 9\hat{k}\right) = 8$$

To Find : Cartesian equation of the given plane.

Formulae :

1) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

then,

$$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

Given the equation of the plane is

$\bar{r}.(3\hat{\imath} + 5\hat{\jmath} - 9\hat{k}) = 8$

Here,

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\therefore \bar{r}.(3\hat{\imath} + 5\hat{\jmath} - 9\hat{k}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(3\hat{\imath} + 5\hat{\jmath} - 9\hat{k})$

= (x × 3) + (y × 5) + (z × (-9))

= 3x + 5y – 9z

Therefore equation of the plane is

3x + 5y – 9z = 8

This is the Cartesian equation of the given plane.

Question 8.
Find the vector equation of a plane whose Cartesian equation is 5x - 7y + 2z + 4 = 0.

Answer:
Given :

Cartesian equation of the plane is

5x – 7y + 2z + 4 = 0

To Find : Vector equation of the given plane.

Formulae :

1) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

Given the equation of the plane is

5x – 7y + 2z + 4 = 0

⇒ 5x – 7y + 2z = - 4

The term (5x – 7y + 2z) can be written as

$$(5x - 7y + 2z) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(5\hat{\imath} - 7\hat{\jmath} + 2\hat{k})$$

But $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\therefore (5x - 7y + 2z) = \bar{r}.(5\hat{\imath} - 7\hat{\jmath} + 2\hat{k})$$

Therefore the equation of the plane is

$$\bar{r}.(5\hat{\imath} - 7\hat{\jmath} + 2\hat{k}) = -4$$

or

$$\bar{r}.(-5\hat{\imath} + 7\hat{\jmath} - 2\hat{k}) = 4$$

This is Vector equation of the given plane.

Question 9.
Find a unit vector normal to the plane
x – 2y + 2z = 6.

Answer:
Given :

Equation of plane : x – 2y + 2z = 6

To Find : unit normal vector $= \hat{n}$

Formula :

Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then the unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

From the given equation of a plane

x – 2y + 2z = 6

direction ratios of vector normal to the plane are (1, -2, 2).

Therefore, the equation of normal vector is

$$\bar{n} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$$

Therefore unit normal vector is given by

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$\therefore \hat{n} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{\sqrt{1^2 + (-2)^2 + 2^2}}$$

$$\therefore \hat{n} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{\sqrt{1 + 4 + 4}}$$

$$\therefore \hat{n} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{\sqrt{9}}$$

$$\therefore \hat{n} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{3}$$

$$\therefore \hat{n} = \frac{\hat{\imath}}{3} - \frac{2\hat{\jmath}}{3} + \frac{2\hat{k}}{3}$$

Question 10.
Find the direction cosines of the normal to the plane 3x – 6y + 2z = 7.

Answer:
Given :

Equation of plane : 3x – 6y + 2z = 7

To Find : Direction cosines of the normal, i.e. $l, m \;\&\; n$

Formula :

1) Direction cosines :

If a, b & c are direction ratios of the vector, then its direction cosines are given by

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

For the given equation of a plane

3x – 6y + 2z = 7

Direction ratios of normal vector are (3, -6, 2)

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{3^2 + (-6)^2 + 2^2}$$

$= \sqrt{9 + 36 + 4}$

$= \sqrt{49}$

$= \pm 7$

Therefore, direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{3}{7}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \mp \frac{6}{7}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{2}{7}$$

$$(l, m, n) = \pm \left(\frac{3}{7}, \frac{-6}{7}, \frac{2}{7}\right)$$

Question 11.
For each of the following planes, find the direction cosines of the normal to the plane and the distance of the plane from the origin:

(i) 2x + 3y - z = 5

(ii) z = 3

(iii) 3y + 5 = 0

Answer:
(i) 2x + 3y – z = 5

Given :

Equation of plane : 2x + 3y – z = 5

To Find :

Direction cosines of the normal i.e. $l, m \;\&\; n$

Distance of the plane from the origin = d

Formulae :

1) Direction cosines :

If a, b & c are direction ratios of the vector then its direction cosines are given by

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

2) The distance of the plane from the origin :

Distance of the plane from the origin is given by,

$$d = \frac{p}{|\bar{n}|}$$

For the given equation of plane

2x + 3y – z = 5

Direction ratios of normal vector are (2, 3, -1)

Therefore, equation of normal vector is

$$\bar{n} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$$

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{2^2 + 3^2 + (-1)^2}$$

$$= \sqrt{4 + 9 + 1}$$

$$= \sqrt{14}$$

Therefore, direction cosines are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{14}}$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{3}{\sqrt{14}}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{-1}{\sqrt{14}}$$

$$(l, m, n) = \left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right)$$

Now, the distance of the plane from the origin is

$$d = \frac{p}{|\bar{n}|}$$

$$\therefore d = \frac{5}{\sqrt{14}}$$

(ii) Given :

Equation of plane : z = 3

To Find :

Direction cosines of the normal, i.e. $l, m \;\&\; n$

The distance of the plane from the origin = d

Formulae :

3) Direction cosines :

If a, b & c are direction ratios of the vector, then its direction cosines are given by

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

4) The distance of the plane from the origin :

Distance of the plane from the origin is given by,

$$d = \frac{p}{|\bar{n}|}$$

For the given equation of a plane

z = 3

Direction ratios of normal vector are (0, 0, 1)

Therefore, equation of normal vector is

$$\bar{n} = \hat{k}$$

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{0^2 + 0^2 + 1^2}$$

$$= \sqrt{1}$$

$$= 1$$

Therefore, direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{0}{1} = 0$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{0}{1} = 0$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{1} = 1$$

$$(l, m, n) = (0,0,1)$$

Now, the distance of the plane from the origin is

$$d = \frac{p}{|\bar{n}|}$$

$$\therefore d = \frac{3}{1}$$

$$\therefore d = 3$$

(iii) Given :

Equation of plane : 3y + 5 = 0

To Find :

Direction cosines of the normal, i.e. $l, m \,\&\, n$

The distance of the plane from the origin = d

Formulae :

1) Direction cosines :

If a, b & c are direction ratios of the vector, then its direction cosines are given by

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

2) Distance of the plane from the origin :

Distance of the plane from the origin is given by,

$$d = \frac{p}{|\bar{n}|}$$

For the given equation of a plane

3y + 5 = 0

⇒-3y = 5

Direction ratios of normal vector are (0, -3, 0)

Therefore, equation of normal vector is

$$\bar{n} = -3\hat{j}$$

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{0^2 + (-3)^2 + 0^2}$$

$$= \sqrt{9}$$

$$= 3$$

Therefore, direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{0}{3} = 0$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{-3}{3} = -1$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{0}{3} = 0$$

$$(l, m, n) = (0, -1, 0)$$

Now, distance of the plane from the origin is

$$d = \frac{p}{|\bar{n}|}$$

$$\therefore d = \frac{5}{3}$$

Question 12.
Find the vector and Cartesian equations of the plane passing through the point (2, -1, 1) and perpendicular to the line having direction ratios 4, 2, -3.

Answer:
Given :

A = (2, -1, 1)

Direction ratios of perpendicular vector = (4, 2, -3)

To Find : Equation of a plane

Formulae :

1) Position vectors :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

then,

$$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

3) Equation of plane :

If a plane is passing through point A, then the equation of a plane is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

Where, $\bar{a} = position\ vector\ of\ A$

$\bar{n} = vector\ perpendicular\ to\ the\ plane$

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For point A = (2, -1, 1), position vector is

$\bar{a} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

Vector perpendicular to the plane with direction ratios (4, 2, -3) is

$\bar{n} = 4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

Now, $\bar{a}.\bar{n} = (2 \times 4) + ((-1) \times 2) + (1 \times (-3))$

= 8 – 2 – 3

= 3

Equation of the plane passing through point A and perpendicular to vector $\bar{n}$ is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

$\therefore \bar{r}.\left(4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) = 3$

As $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\therefore \bar{r}.\left(4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right)$

= 4x + 2y – 3z

Therefore, the equation of the plane is

4x + 2y – 3z = 3

Or

4x + 2y – 3z - 3 = 0

Question 13.
Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

(i) 2x + 3y + 4z -12 = 0

(ii) 5y + 8 = 0

Answer:
(i) 2x + 3y + 4z - 12 = 0

Given :

Equation of plane : 2x + 3y + 4z + 12 = 0

To Find :

coordinates of the foot of the perpendicular

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

From the given equation of the plane

2x + 3y + 4z – 12 = 0

⇒ 2x + 3y + 4z = 12

Direction ratios of the vector normal to the plane are (2, 3, 4)

Let, P = (x, y, z) be the foot of perpendicular perpendicular drawn from origin to the plane.

Therefore perpendicular drawn is $\overline{OP}$.

$\therefore \overline{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Let direction ratios of $\overline{OP}$ are (x, y, z)

As normal vector and $\overline{OP}$ are parallel

$\therefore \frac{x}{2} = \frac{y}{3} = \frac{z}{4} = k(say)$

⇒x = 2k, y = 3k, z = 4k

As point P lies on the plane, we can write

2(2k) + 3(3k) + 4(4k) = 12

⇒ 4k + 9k + 16k = 12

⇒ 29k = 12

$\therefore k = \frac{12}{29}$

$\therefore x = 2k = \frac{24}{29},$

$y = 3k = \frac{36}{29}$

$z = 4k = \frac{48}{29}$

Therefore co-ordinates of the foot of perpendicular are

P(x, y, z) = $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$

P = $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$

(ii) Given :

Equation of plane : 5y + 8 = 0

To Find :

coordinates of the foot of the perpendicular

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

From the given equation of the plane

5y + 8 = 0

⇒ 5y = - 8

Direction ratios of the vector normal to the plane are (0, 5, 0)

Let, P = (x, y, z) be the foot of perpendicular perpendicular drawn from origin to the plane.

Therefore perpendicular drawn is $\overline{OP}$.

$\therefore \overline{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Let direction ratios of $\overline{OP}$ are (x, y, z)

As normal vector and $\overline{OP}$ are parallel

$$\therefore \frac{0}{x} = \frac{5}{y} = \frac{0}{z} = \frac{1}{k}\,(say)$$

⇒x = 0, y = 5k, z = 0

As point P lies on the plane, we can write

5(5k) = -8

⇒ 25k = -8

$$\therefore k = \frac{-8}{25}$$

$$\therefore x = 0,$$

$$y = 5k = 5 \times \frac{-8}{25} = \frac{-8}{5}$$

$$z = 0$$

Therefore co-ordinates of the foot of perpendicular are

P(x, y, z) = $\left(0, \frac{-8}{5}, 0\right)$

P = $\left(0, \frac{-8}{5}, 0\right)$

Question 14.
Find the length and the foot of perpendicular drawn from the point (2, 3, 7) to the plane 3x – y – z = 7.

Answer:
Given :

Equation of plane : 3x – y – z = 7

A = (2, 3, 7)

To Find :

i) Length of perpendicular = d

ii) coordinates of the foot of the perpendicular

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Length of perpendicular :

The length of the perpendicular from point A with position vector $\bar{a}$ to the plane is given by,

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Given equation of the plane is

3x – y – z = 7 ....eq(1)

Therefore direction ratios of normal vector of the plane are

(3, -1, -1)

Therefore normal vector of the plane is

$$\bar{n} = 3\hat{\imath} - \hat{\jmath} - \hat{k}$$

$\therefore |\bar{n}| = \sqrt{3^2 + (-1)^2 + (-1)^2}$

$= \sqrt{9 + 1 + 1}$

$= \sqrt{11}$

From eq(1), p = 7

Given point A = (2, 3, 7)

Position vector of A is

$\bar{a} = 2\hat{\imath} + 3\hat{\jmath} + 7\hat{k}$

Now,

$\bar{a}.\bar{n} = \left(2\hat{\imath} + 3\hat{\jmath} + 7\hat{k}\right).\left(3\hat{\imath} - \hat{\jmath} - \hat{k}\right)$

= (2×3) + (3×(-1)) + (7×(-1))

= 6 – 3 – 7

= -4

Length of the perpendicular from point A to the plane is

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

$$\therefore d = \frac{|-4 - 7|}{\sqrt{11}}$$

$$\therefore d = \frac{11}{\sqrt{11}}$$

$$\therefore d = \sqrt{11}$$

Let P be the foot of perpendicular drawn from point A to the given plane, Let P = (x, y, z)

$$\overline{AP} = (x-2)\hat{\imath} + (y-3)\hat{\jmath} + (z-7)\hat{k}$$

As normal vector and $\overline{AP}$ are parallel

$$\therefore \frac{x-2}{3} = \frac{y-3}{-1} = \frac{z-7}{-1} = k(say)$$

⇒x = 3k+2, y = - k+3, z = -k+7

As point P lies on the plane, we can write

3(3k+2) - (- k+3) - (-k+7) = 7

⇒ 9k + 6 + k – 3 + k – 7 = 7

⇒ 11k = 11

$\therefore k = 1$

$\therefore x = 3k + 2 = 5,$

$y = -k + 3 = 2$

$z = -k + 7 = 6$

Therefore co-ordinates of the foot of perpendicular are

P(x, y, z) = $(5, 2, 6)$

P = (5, 2, 6)

Question 15.
Find the length and the foot of the perpendicular drawn from the point (1, 1, 2) to the plane $\bar{r}\cdot\left(2\hat{i}-2\hat{j}+4\hat{k}\right)+5=0.$

Answer:
Given :

Equation of plane : $\bar{r}.\left(2\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)+5=0$

A = (1, 1, 2)

To Find :

i) Length of perpendicular = d

ii) coordinates of the foot of the perpendicular

Formulae :

1) Unit Vector :

Let $\bar{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$$\hat{a}=\frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}|=\sqrt{{a_1}^2+{a_2}^2+{a_3}^2}$

2) Length of perpendicular :

The length of the perpendicular from point A with position vector $\bar{a}$ to the plane is given by,

$$d=\frac{|\bar{a}.\bar{n}-p|}{|\bar{n}|}$$

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$$

Given equation of the plane is

$\bar{r}.\left(2\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)+5=0$ ....eq(1)

$\therefore \bar{r}.\left(2\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)=-5$

As $\bar{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

Therefore equation of plane is

2x – 2y + 4z = -5 .... eq(2)

From eq(1) normal vector of the plane is

$\bar{n}=2\hat{\imath}-2\hat{\jmath}+4\hat{k}$

$\therefore |\bar{n}|=\sqrt{2^2+(-2)^2+4^2}$

$=\sqrt{4+4+16}$

$=\sqrt{24}$

From eq(1), p = -5

Given point A = (1, 1, 2)

Position vector of A is

$\bar{a}=\hat{\imath}+\hat{\jmath}+2\hat{k}$

Now,

$\bar{a}.\bar{n}=\left(\hat{\imath}+\hat{\jmath}+2\hat{k}\right).\left(2\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)$

= (1×2) + (1×(-2)) + (2×4)

= 2 – 2 + 8

= 8

Length of the perpendicular from point A to the plane is

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

$$\therefore d = \frac{|8+5|}{\sqrt{24}}$$

$$\therefore d = \frac{13}{\sqrt{24}}$$

$$\therefore d = \frac{13\sqrt{6}}{\sqrt{24}.\sqrt{6}}$$

$$\therefore d = \frac{13\sqrt{6}}{\sqrt{144}}$$

$$\therefore d = \frac{13\sqrt{6}}{12}$$

Let P be the foot of perpendicular drawn from point A to the given plane,

Let P = (x, y, z)

$$\overline{AP} = (x-1)\hat{\imath} + (y-1)\hat{\jmath} + (z-2)\hat{k}$$

As normal vector and $\overline{AP}$ are parallel

$$\therefore \frac{x-1}{2} = \frac{y-1}{-2} = \frac{z-2}{4} = k(say)$$

⇒x = 2k+1, y = -2k+1, z = 4k+2

As point P lies on the plane, we can write

2(2k+1) – 2(-2k+1) + 4(4k+2) = -5

⇒ 4k + 2 + 4k – 2 + 16k + 8 = -5

⇒ 24k = -13

$\therefore k = \frac{-13}{24}$

$\therefore x = 2\left(\frac{-13}{24}\right) + 1 = \frac{-1}{12},$

$y = -2\left(\frac{-13}{24}\right) + 1 = \frac{25}{12}$

$z = 4\left(\frac{-13}{24}\right) + 2 = \frac{-1}{6}$

Therefore co-ordinates of the foot of perpendicular are

P(x, y, z) = $\left(\frac{-1}{12}, \frac{25}{12}, \frac{-1}{6}\right)$

P ≡ $\left(\frac{-1}{12}, \frac{25}{12}, \frac{-1}{6}\right)$

Question 16.
From the point P(1, 2, 4), a perpendicular is drawn on the plane 2x + y - 2z + 3 = 0. Find the equation, the length and the coordinates of the foot of the perpendicular.

Answer:
Given :

Equation of plane : 2x + y – 2z + 3 = 0

P = (1, 2, 4)

To Find :

i) Equation of perpendicular

ii) Length of perpendicular = d

iii) coordinates of the foot of the perpendicular

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Length of perpendicular :

The length of the perpendicular from point A with position vector $\bar{a}$ to the plane is given by,

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Given equation of the plane is

2x + y – 2z + 3 = 0

⇒2x + y – 2z = -3 ....eq(1)

From eq(1) direction ratios of normal vector of the plane are

(2, 1, -2)

Therefore, equation of normal vector is

$$\bar{n} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$$

$$\therefore |\bar{n}| = \sqrt{2^2 + 1^2 + (-2)^2}$$

$$= \sqrt{4 + 1 + 4}$$

$$= \sqrt{9}$$

= 3

From eq(1), p = -3

Given point P = (1, 2, 4)

Position vector of A is

$$\bar{p} = \hat{\imath} + 2\hat{\jmath} + 4\hat{k}$$

Here, $\bar{a} = \bar{p}$

Now,

$$\therefore \bar{a}.\bar{n} = (\hat{\imath} + 2\hat{\jmath} + 4\hat{k}).(2\hat{\imath} + \hat{\jmath} - 2\hat{k})$$

= (1×2) + (2×1) + (4×(-2))

= 2 + 2 - 8

= -4

Length of the perpendicular from point A to the plane is

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

$$\therefore d = \frac{|-4 + 3|}{3}$$

$$\therefore d = \frac{1}{3}$$

Let Q be the foot of perpendicular drawn from point P to the given plane,

Let Q = (x, y, z)

$$\overline{PQ} = (x-1)\hat{\imath} + (y-2)\hat{\jmath} + (z-4)\hat{k}$$

As normal vector and $\overline{PQ}$ are parallel, we can write,

$$\therefore \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-4}{-2}$$

This is the equation of perpendicular.

$$\therefore \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-4}{-2} = k(say)$$

⇒x = 2k+1, y = k+2, z = -2k+4

As point Q lies on the plane, we can write

2(2k+1) + (k+2) - 2(-2k+4) = -3

⇒ 4k + 2 + k + 2 + 4k - 8 = -3

⇒ 9k = 1

$$\therefore k = \frac{1}{9}$$

$$\therefore x = 2\left(\frac{1}{9}\right) + 1 = \frac{11}{9},$$

$$y = \frac{1}{9} + 2 = \frac{19}{9}$$

$$z = -2\left(\frac{1}{9}\right) + 4 = \frac{34}{9}$$

Therefore co-ordinates of the foot of perpendicular are

Q(x, y, z) = $\left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$

Q ≡ $\left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$

Question 17.
Find the coordinates of the foot of the perpendicular and the perpendicular distance from the point P(3, 2, 1) to the plane 2x – y + z + 1 = 0.

Find also the image of the point P in the plane.

Answer:
Given :

Equation of plane : 2x - y + z + 1 = 0

P = (3, 2, 1)

To Find :

i) Length of perpendicular = d

ii) coordinates of the foot of the perpendicular

iii) Image of point P in the plane.

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Length of perpendicular :

The length of the perpendicular from point A with position vector $\bar{a}$ to the plane is given by,

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Given equation of the plane is

2x - y + z + 1 = 0

⇒2x - y + z = -1 ....eq(1)

From eq(1) direction ratios of normal vector of the plane are

(2, -1, 1)

Therefore, equation of normal vector is

$$\bar{n} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$$

$$\therefore |\bar{n}| = \sqrt{2^2 + (-1)^2 + 1^2}$$

$$= \sqrt{4 + 1 + 1}$$

$$= \sqrt{6}$$

From eq(1), p = -1

Given point P = (3, 2, 1)

Position vector of A is

$\bar{p} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$

Here, $\bar{a} = \bar{p}$

Now,

$\therefore \bar{a}.\bar{n} = \left(3\hat{\imath} + 2\hat{\jmath} + \hat{k}\right).\left(2\hat{\imath} - \hat{\jmath} + \hat{k}\right)$

= (3×2) + (2×(-1)) + (1×1)

= 6 - 2 + 1

= 5

Length of the perpendicular from point A to the plane is

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

$$\therefore d = \frac{|5 + 1|}{\sqrt{6}}$$

$$\therefore d = \frac{6}{\sqrt{6}}$$

$\therefore d = \sqrt{6}$

Let Q be the foot of perpendicular drawn from point P to the given plane,

Let Q = (x, y, z)

$\overline{PQ} = (x - 3)\hat{\imath} + (y - 2)\hat{\jmath} + (z - 1)\hat{k}$

As normal vector and $\overline{PA}$ are parallel, we can write,

$\therefore \frac{x-3}{2}=\frac{y-2}{-1}=\frac{z-1}{1}=k(say)$

⇒x = 2k+3, y = -k+2, z = k+1

As point A lies on the plane, we can write

2(2k+3) - (-k+2) + (k+1) = -1

⇒ 4k + 6 + k – 2 + k + 1 = -1

⇒ 6k = -6

$\therefore k=-1$

$\therefore x=2(-1)+3=1,$

$y=-(-1)+2=3$

$z=(-1)+1=0$

Therefore, co-ordinates of the foot of perpendicular are

Q(x, y, z) = $(1,3,0)$

Q ≡ $(1,3,0)$

Let, R(a, b, c) be image of point P in the given plane.

Therefore, the power of points P and R in the given plane will be equal and opposite.

2a – b + c + 1 = - (2(3) – 2 + 1 + 1)

⇒2a – b + c + 1 = - 6

⇒2a – b + c = - 7 ....eq(2)

Now, $\overline{PR}=(a-3)\hat{\imath}+(b-2)\hat{\jmath}+(c-1)\hat{k}$

As $\overline{PR}$ & $\bar{n}$ are parallel

$$\therefore \frac{a-3}{2}=\frac{b-2}{-1}=\frac{c-1}{1}=k(say)$$

⇒a = 2k+3, b = -k+2, c = k+1

substituting a, b, c in eq(2)

2(2k+3) - (-k+2) + (k+1) = -7

⇒ 4k + 6 + k – 2 + k + 1 = -7

⇒ 6k = -12

$\therefore k=-2$

$\therefore a=2(-2)+3=-1,$

$b=-(-2)+2=4$

$c=(-2)+1=-1$

Therefore, co-ordinates of the image of P are

R(a, b, c) = $(-1,4,-1)$

R ≡ $(-1,4,-1)$

Question 18.
Find the coordinates of the image of the point P(1, 3, 4) in the plane 2x - y + z + 3 = 0.

Answer:
Given :

Equation of plane : 2x - y + z + 3 = 0

P = (1, 3, 4)

To Find : Image of point P in the plane.

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Given equation of the plane is

2x - y + z + 3 = 0

⇒2x - y + z = -3 ....eq(1)

From eq(1) direction ratios of normal vector of the plane are

(2, -1, 1)

Therefore, equation of normal vector is

$$\bar{n} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$$

Given point is P = (1, 3, 4)

Let, R(a, b, c) be image of point P in the given plane.

Therefore, the power of points P and R in the given plane will be equal and opposite.

⇒2a – b + c + 3 = - (2(1) – 3 + 4 + 3)

⇒2a – b + c + 3 = - 6

⇒2a – b + c = - 9 ....eq(2)

Now, $\overline{PR} = (a-1)\hat{\imath} + (b-3)\hat{\jmath} + (c-4)\hat{k}$

As $\overline{PR}$ & $\bar{n}$ are parallel

$\therefore \frac{a-1}{2} = \frac{b-3}{-1} = \frac{c-4}{1} = k(say)$

⇒a = 2k+1, b = -k+3, c = k+4

substituting a, b, c in eq(2)

2(2k+1) - (-k+3) + (k+4) = -9

⇒ 4k + 2 + k – 3 + k + 4 = -9

⇒ 6k = -12

$\therefore k = -2$

$\therefore a = 2(-2) + 1 = -3,$

$b = -(-2) + 3 = 5$

$c = (-2) + 4 = 2$

Therefore, co-ordinates of the image of P are

R(a, b, c) = $(-3, 5, 2)$

Question 19.
Find the point where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane 2x + 4y – z = 1.

Answer:
Given :

Equation of plane : 2x + 4y - z = 1

Equation of line :

$$\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$$

To Find : Point of intersection of line and plane.

Let P(a, b, c) be point of intersection of plane and line.

As point P lies on the line, we can write,

$$\frac{a-1}{2}=\frac{b-2}{-3}=\frac{c+3}{4}=k(say)$$

⇒a = 2k+1, b = -3k+2, c = 4k-3 ....(1)

Also point P lies on the plane

2a + 4b – c = 1

⇒2(2k+1) + 4(-3k+2) – (4k-3) = 1 .....from (1)

⇒4k + 2 – 12k + 8 – 4k + 3 = 1

⇒-12k = -12

⇒k = 1

$\therefore a = 2(1) + 1 = 3,$

$b = -3(1) + 2 = -1$

$c = 4(1) - 3 = 1$

Therefore, co-ordinates of point of intersection of given line and plane are

P ≡ $(3,-1,1)$

Question 20.
Find the coordinates of the point where the line through (3, -4, -5) and (2, -3, 1) crosses the plane 2x + y + z = 7.

Answer:

Given :

Equation of plane : 2x + y + z = 7

Points :

A = (3, -4, -5)

B = (2, -3, 1)

To Find : Point of intersection of line and plane.

Formula :

Equation of line passing through A = (x_1, y_1, z_1) &

B = (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$$

Equation of line passing through A = (3, -4, -5) & B = (2, -3, 1) is

$$\frac{x-3}{3-2}=\frac{y+4}{-4+3}=\frac{z+5}{-5-1}$$

$$\therefore \frac{x-3}{1}=\frac{y+4}{-1}=\frac{z+5}{-6}$$

Let P(a, b, c) be point of intersection of plane and line.

As point P lies on the line, we can write,

$$\frac{a-3}{1}=\frac{b+4}{-1}=\frac{c+5}{-6}=k(say)$$

⇒a = k+3, b = -k - 4, c = -6k-5 ....(1)

Also point P lies on the plane

2a + b + c = 7

⇒2(k+3) + (-k-4) + (-6k-5) = 7 .....from (1)

⇒2k + 6 – k - 4 – 6k - 5 = 7

⇒-5k = 10

⇒k = -2

$\therefore a = (-2) + 3 = 1,$

$b = -(-2) - 4 = -2$

$c = -6(-2) - 5 = 7$

Therefore, co-ordinates of point of intersection of given line and plane are

P ≡ $(1, -2, 7)$

Question 21.
Find the distance of the point (2, 3, 4) from the plane 3x + 2y + 2z +5 =0, measured parallel to the line $\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$.

Answer:
Given :

Equation of plane : 3x + 2y + 2z + 5 = 0

Equation of line :

$$\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$$

Point : P = (2, 3, 4)

To Find : Distance of point P from the given plane parallel to the given line.

Formula :

1) Equation of line :

Equation of line passing through A = (x_1, y_1, z_1) & having direction ratios (a, b, c) is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

2) Distance formula :

The distance between two points A = (a_1, a_2, a_3) & B = (b_1, b_2, b_3) is

$$d=\sqrt{(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2}$$

For the given line,

$$\frac{x+3}{3}=\frac{y-2}{6}=\frac{z}{2}$$

Direction ratios are (a, b, c) = (3, 6, 2)

Let Q be the point on the plane such that $\overline{PQ}$ is parallel to the given line.

Therefore direction ratios of given line and line PQ will be same.

Therefore equation of line PQ with point P = (2, 3, 4) and with direction ratios (3, 6, 2) is

$$\frac{x-2}{3}=\frac{y-3}{6}=\frac{z-4}{2}$$

Let co-ordinates of Q be (u, v, w)

As point Q lies on the line PQ, we can write,

$$\frac{u-2}{3}=\frac{v-3}{6}=\frac{w-4}{2}=k(say)$$

⇒u = 3k+2, v = 6k+3, w = 2k+4 ....(1)

Also point Q lies on the plane

3u + 2v + 2w = -5

⇒3(3k+2) + 2(6k+3) + 2(2k+4) = -5 .....from (1)

⇒9k + 6 + 12k + 6 + 4k + 8 = -5

⇒25k = -25

⇒k = -1

$\therefore u = 3(-1) + 2 = -1,$

$v = 6(-1) + 3 = -3$

$w = 2(-1) + 4 = 2$

Therefore, co-ordinates of point Q are

Q = $(-1, -3, 2)$

Now distance between points P and Q by distance formula is

$d = \sqrt{(2+1)^2 + (3+3)^2 + (4-2)^2}$

$= \sqrt{(3)^2 + (6)^2 + (2)^2}$

$= \sqrt{9 + 36 + 4}$

$= \sqrt{49}$

= 7

Therefore distance of point P from the given plane measured parallel to the given line is

d = 7 units

Question 22.
Find the distance of the point (0, -3, 2) from the plane x + 2y -z = 1, measured parallel to the line $\frac{x+1}{3} = \frac{y+1}{2} = \frac{z}{3}$.

Answer:
Given :

Equation of plane : x + 2y - z = 1

Equation of line :

$$\frac{x+1}{3} = \frac{y+1}{2} = \frac{z}{3}$$

Point : P = (0, -3, 2)

To Find : Distance of point P from the given plane parallel to the given line.

Formula :

1) Equation of line :

Equation of line passing through A = (x_1, y_1, z_1) & having direction ratios (a, b, c) is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

2) Distance formula :

The distance between two points A = (a_1, a_2, a_3) & B = (b_1, b_2, b_3) is

$$d = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}$$

For the given line,

$$\frac{x+1}{3} = \frac{y+1}{2} = \frac{z}{3}$$

Direction ratios are (a, b, c) = (3, 2, 3)

Let Q be the point on the plane such that $\overline{PQ}$ is parallel to the given line.

Therefore direction ratios of given line and line PQ will be same.

Therefore equation of line PQ with point P = (0, -3, 2) and with direction ratios (3, 2, 3) is

$$\frac{x-0}{3}=\frac{y+3}{2}=\frac{z-2}{3}$$

Let co-ordinates of Q be (u, v, w)

As point Q lies on the line PQ, we can write,

$$\frac{u}{3}=\frac{v+3}{2}=\frac{w-2}{3}=k(say)$$

⇒u = 3k, v = 2k-3, w = 3k+2 ....(1)

Also point Q lies on the plane

u + 2v - w = 1

⇒(3k) + 2(2k-3) - (3k+2) = 1 .....from (1)

⇒3k + 4k - 6 - 3k - 2 = 1

⇒4k = 9

$\Rightarrow k=\frac{9}{4}$

$\therefore u=3\left(\frac{9}{4}\right)=\frac{27}{4},$

$$v=2\left(\frac{9}{4}\right)-3=\frac{6}{4}$$

$$w=3\left(\frac{9}{4}\right)+2=\frac{35}{4}$$

Therefore, co-ordinates of point Q are

$Q \equiv \left(\frac{27}{4}, \frac{6}{4}, \frac{35}{4}\right)$

Now distance between points P and Q by distance formula is

$$d = \sqrt{\left(0 - \frac{27}{4}\right)^2 + \left(-3 - \frac{6}{4}\right)^2 + \left(2 - \frac{35}{4}\right)^2}$$

$$= \sqrt{\left(\frac{-27}{4}\right)^2 + \left(\frac{-18}{4}\right)^2 + \left(\frac{-27}{4}\right)^2}$$

$$= \sqrt{45.5625 + 20.25 + 45.5625}$$

$$= \sqrt{111.375}$$

= 10.55

Therefore distance of point P from the given plane measured parallel to the given line is

d = 10.55 units

Question 23.
Find the equation of the line passing through the point P(4, 6, 2) and the point of intersection of the line $\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{7}$ and the plane x + y –Z = 8.

Answer:
Given :

Equation of plane : x + y - z = 8

Equation of line :

$$\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{7}$$

Point : P = (4, 6, 2)

To Find : Equation of line.

Formula :

Equation of line passing through A = (x_1, y_1, z_1) &

B = (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$$

let Q (a, b, c) be point of intersection of plane and line.

As point Q lies on the line, we can write,

$$\frac{a-1}{3}=\frac{b}{2}=\frac{c+1}{7}=k(say)$$

⇒a =3k+1, b= 2k, c= 7k-1

Also point Q lies on the plane,

a + b – c = 8

⇒(3k+1) + (2k) – (7k-1) = 8

⇒3k + 1 + 2k – 7k + 1 = 8

⇒-2k = 6

⇒k = -3

$\therefore a = 3(-3) + 1 = -8,$

$b = -2(-3) = -6$

$c = 7(-3) - 1 = -22$

Therefore, co-ordinates of point of intersection of given line and plane are Q = (-8, -6, -22)

Now, equation of line passing through P(4,6,2) and

Q(-8, -6, -22) is

$$\frac{x-4}{4+8}=\frac{y-6}{6+6}=\frac{z-2}{2+22}$$

$$\therefore \frac{x-4}{12}=\frac{y-6}{12}=\frac{z-2}{24}$$

$$\therefore \frac{x-4}{1}=\frac{y-6}{1}=\frac{z-2}{2}$$

This is the equation of required line

Question 24.
Show that the distance of the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane x - y + z = 5 from the point (-1, -5 -10) is 13 units.

Answer:
Given :

Equation of plane : x - y + z = 5

Equation of line :

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$$

Point : P = (-1, -5, -10)

To Prove : Distance of point P from the given plane parallel to the given line is 13 units.

Formula :

1) Equation of line :

Equation of line passing through A = (x_1, y_1, z_1) & having direction ratios (a, b, c) is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

2) Distance formula :

The distance between two points A = (a_1, a_2, a_3) & B = (b_1, b_2, b_3) is

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

For the given line,

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$$

Direction ratios are (a, b, c) = (3, 4, 12)

Let Q be the point on the plane such that $\overline{PQ}$ is parallel to the given line.

Therefore direction ratios of given line and line PQ will be same.

Therefore equation of line PQ with point P = (-1, -5, -10) and with direction ratios (3, 4, 12) is

$$\frac{x+1}{3} = \frac{y+5}{4} = \frac{z+10}{12}$$

Let co-ordinates of Q be (u, v, w)

As point Q lies on the line PQ, we can write,

$$\frac{u+1}{3} = \frac{v+5}{4} = \frac{w+10}{12} = k(say)$$

⇒u = 3k-1, v = 4k-5, w = 12k-10 ....(1)

Also point Q lies on the plane

u - v + w = 5

⇒(3k-1) - (4k-5) + (12k-10) = 5 .....from (1)

⇒3k - 1 - 4k + 5 + 12k - 10 = 5

$\Rightarrow$11k = 11

$\Rightarrow k = 1$

$\therefore u = 3(1) - 1 = 2,$

$v = 4(1) - 5 = -1$

$w = 12(1) - 10 = 2$

Therefore, co-ordinates of point Q are

Q ≡ $(2, -1, 2)$

Now distance between points P and Q by distance formula is

$d = \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$

$= \sqrt{(-3)^2 + (-4)^2 + (-12)^2}$

$= \sqrt{9 + 16 + 144}$

$= \sqrt{169}$

= 13

Therefore distance of point P from the given plane measured parallel to the given line is

d = 13 units

Hence proved.

Question 25.
Find the distance of the point A(-1, -5, -10) from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

HINT: Convert the equations of the line and the plane to Cartesian form.

Answer:
Given :

Equation of plane : $\bar{r}.(\hat{\imath} - \hat{\jmath} + \hat{k}) = 5$

Equation of line :

$$\bar{r} = (2\hat{\imath} - \hat{\jmath} + 2\hat{k}) + \lambda(3\hat{\imath} + 4\hat{\jmath} + 2\hat{k})$$

Point : P = (-1, -5, -10)

To Find : Distance of point P from the given plane parallel to the given line.

Formula :

1) Equation of line :

Equation of line passing through A = (x_1, y_1, z_1) & having direction ratios (a, b, c) is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

2) Distance formula :

The distance between two points A = (a_1, a_2, a_3) & B = (b_1, b_2, b_3) is

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

for the given plane,

$$\bar{r}.(\hat{\imath} - \hat{\jmath} + \hat{k}) = 5$$

Here, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(\hat{\imath} - \hat{\jmath} + \hat{k}) = 5$$

$\Rightarrow x - y + z = 5$ ....eq(1)

For the given line,

$$\bar{r} = \left(2\hat{\imath} - \hat{\jmath} + 2\hat{k}\right) + \lambda\left(3\hat{\imath} + 4\hat{\jmath} + 2\hat{k}\right)$$

Here, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\therefore \left(3\hat{\imath} + 4\hat{\jmath} + 2\hat{k}\right)\lambda = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) - \left(2\hat{\imath} - \hat{\jmath} + 2\hat{k}\right)$$

$$\therefore 3\lambda\hat{\imath} + 4\lambda\hat{\jmath} + 2\lambda\hat{k} = (x-2)\hat{\imath} + (y+1)\hat{\jmath} + (z-2)\hat{k}$$

Comparing coefficients of $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$

$\Rightarrow 3\lambda = (x-2), 4\lambda = (y+1)$ & $2\lambda = (z-2)$

$\Rightarrow \lambda = \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ ....eq(2)

Direction ratios for above line are (a, b, c) = (3, 4, 2)

Let Q be the point on the plane such that $\overline{PQ}$ is parallel to the given line.

Therefore direction ratios of given line and line PQ will be same.

Therefore equation of line PQ with point P = (-1, -5, -10) and with direction ratios (3, 4, 2) is

$$\frac{x+1}{3} = \frac{y+5}{4} = \frac{z+10}{2}$$

Let co-ordinates of Q be (u, v, w)

As point Q lies on the line PQ, we can write,

$$\frac{u+1}{3} = \frac{v+5}{4} = \frac{w+10}{2} = k(say)$$

$\Rightarrow$u = 3k-1, v = 4k-5, w = 2k-10 ....(3)

Also point Q lies on the given plane

Therefore from eq(1), we can write,

u - v + w = 5

⇒(3k-1) - (4k-5) + (2k-10) = 5 .....from (3)

⇒3k - 1 - 4k + 5 + 2k - 10 = 5

⇒k = 11

$\Rightarrow k = 11$

$\therefore u = 3(11) - 1 = 32,$

$v = 4(11) - 5 = 39$

$w = 2(11) - 10 = 12$

Therefore, co-ordinates of point Q are

Q ≡ $(32, 39, 12)$

Now the distance between points P and Q by distance formula is

$d = \sqrt{(-1-32)^2 + (-5-39)^2 + (-10-12)^2}$

$= \sqrt{(-33)^2 + (-44)^2 + (-22)^2}$

$= \sqrt{1089 + 1936 + 484}$

$= \sqrt{3509}$

= 59.24

Therefore distance of point P from the given plane measured parallel to the given line is

d = 59.24 units

Question 26.
Prove that the normals to the planes 4x + 11y + 2z + 3 = 0 and 3x - 2y + 5z = 8 are perpendicular to each other.

Answer:
Given :

Equations of plane are :

4x + 11y + 2z + 3 = 0

3x – 2y + 5z = 8

To Prove : $\overline{n_1}$ & $\overline{n_2}$ are perpendicular.

Formula :

1) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

then,

$$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

Note :

Direction ratios of the plane given by

ax + by + cz = d

are (a, b, c).

For plane

4x + 11y + 2z + 3 = 0

direction ratios of normal vector are (4, 11, 2)

therefore, equation of normal vector is

$$\overline{n_1} = 4\hat{\imath} + 11\hat{\jmath} + 2\hat{k}$$

And for plane

3x – 2y + 5z = 8

direction ratios of the normal vector are (3, -2, 5)

therefore, the equation of normal vector is

$$\overline{n_2} = 3\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$$

Now,

$$\overline{n_1}.\overline{n_2} = \left(4\hat{\imath} + 11\hat{\jmath} + 2\hat{k}\right).\left(3\hat{\imath} - 2\hat{\jmath} + 5\hat{k}\right)$$

$$= (4 \times 3) + (11 \times (-2)) + (2 \times 5)$$

= 12 – 22 + 10

= 0

$$\therefore \overline{n_1}.\overline{n_2} = 0$$

Therefore, normals to the given planes are perpendicular.

Question 27.
Show that the line $\bar{r} = \left(2\hat{i} - 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + 4\hat{k}\right)$ is parallel to the plane $\bar{r} \cdot \left(\hat{i} + 5\hat{j} + \hat{k}\right) = 7$.

Answer:
Given :

Equation of plane : : $\bar{r}.\left(\hat{\imath}+5\hat{\jmath}+\hat{k}\right)=7$

Equation of a line :

$$\bar{r}=\left(2\hat{\imath}-2\hat{\jmath}+3\hat{k}\right)+\lambda\left(\hat{\imath}-\hat{\jmath}+4\hat{k}\right)$$

To Prove : Given line is parallel to the given plane.

Comparing given plane i.e.

$$\bar{r}.\left(\hat{\imath}+5\hat{\jmath}+\hat{k}\right)=7$$

with $\bar{r}.\bar{n}=\bar{a}.\bar{n}$, we get,

$$\bar{n}=\hat{\imath}+5\hat{\jmath}+\hat{k}$$

This is the vector perpendicular to the given plane.

Now, comparing the given the equation of line i.e.

$$\bar{r}=\left(2\hat{\imath}-2\hat{\jmath}+3\hat{k}\right)+\lambda\left(\hat{\imath}-\hat{\jmath}+4\hat{k}\right)$$

with $\bar{r}=\bar{a}+\lambda\bar{b}$, we get,

$$\bar{b}=\hat{\imath}-\hat{\jmath}+4\hat{k}$$

Now,

$$\bar{n}.\bar{b}=\left(\hat{\imath}+5\hat{\jmath}+\hat{k}\right).\left(\hat{\imath}-\hat{\jmath}+4\hat{k}\right)$$

$$=(1\times 1)+(5\times(-1))+(1\times 4)$$

= 1 – 5 + 4

= 0

$$\therefore \bar{n}.\bar{b}=0$$

Therefore, a vector normal to the plane is perpendicular to the vector parallel to the line.

Hence, the given line is parallel to the given plane.

Question 28.
Find the equation of a plane which is at a distance of $3\sqrt{3}$ units from the origin and the normal to which is equally inclined to the coordinate axes.

Answer:
Given :

$$d = 3\sqrt{3}$$

$$\alpha = \beta = \gamma$$

To Find : Equation of plane

Formulae :

1) Distance of plane from the origin :

If $\bar{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ is the vector normal to the plane, then distance of the plane from the origin is

$$d = \frac{p}{|\bar{n}|}$$

Where, $|\bar{n}| = \sqrt{a^2 + b^2 + c^2}$

2) $l^2 + m^2 + n^2 = 1$

Where $l = \cos\alpha\,, m = \cos\beta\,, n = \cos\gamma$

3) Equation of plane :

If $\bar{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ is the vector normal to the plane, then equation of the plane is

$$\bar{r}.\bar{n} = p$$

As $\alpha = \beta = \gamma$

$\therefore \cos\alpha = \cos\beta = \cos\gamma$

⇒ l = m = n

$l^2 + m^2 + n^2 = 1$

$\therefore 3l^2 = 1$

$\therefore l = \frac{1}{\sqrt{3}}$

Therefore equation of normal vector of the plane having direction cosines l, m, n is

$\bar{n} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$

$\therefore \bar{n} = \frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k}$

$\therefore |\bar{n}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2}$

$= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}$

= $\sqrt{1}$

= 1

Now,

distance of the plane from the origin is

$d = \frac{p}{|\bar{n}|}$

$\therefore 3\sqrt{3} = \frac{p}{1}$

$\therefore p = 3\sqrt{3}$

Therefore equation of required plane is

$\bar{r}.\bar{n} = p$

$\therefore (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).\left(\frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k}\right) = 3\sqrt{3}$

$\therefore \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 3\sqrt{3}$

$\therefore x + y + z = 3\sqrt{3}.\sqrt{3}$

$\therefore x + y + z = 9$

This is the required equation of the plane.

Question 29.
A vector $\bar{n}$ of magnitude 8 units is inclined to the x-axis at 45°, y-axis at 60° and an acute angle with the z-axis, if a plane passes through a point (√2, -1, 1) and is normal to find its equation in vector form.

Answer:
Given :

$|\bar{n}| = 8$

$\alpha = 45°$

$\beta = 60°$

P = (√2, -1, 1)

To Find : Equation of plane

Formulae :

1)$l^2 + m^2 + n^2 = 1$

Where $l = \cos\alpha\,, m = \cos\beta\,, n = \cos\gamma$

2) Equation of plane :

If $\bar{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ is the vector normal to the plane, then equation of the plane is

$\bar{r}.\bar{n} = p$

As $\alpha = 45° \;\&\; \beta = 60°$

$\therefore l = \cos\alpha = \cos 45° = \frac{1}{\sqrt{2}}$ and

$$m = \cos\beta = \cos 60° = \frac{1}{2}$$

But, $l^2 + m^2 + n^2 = 1$

$$\therefore \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + n^2 = 1$$

$$\therefore \frac{1}{2} + \frac{1}{4} + n^2 = 1$$

$$\therefore n^2 = 1 - \frac{3}{4}$$

$$\therefore n^2 = \frac{1}{4}$$

$$\therefore n = \frac{1}{2}$$

Therefore direction cosines of the normal vector of the plane are (l, m, n)

Hence direction ratios are (kl, km, kn)

Therefore the equation of normal vector is

$\bar{n} = kl\hat{\imath} + km\hat{\jmath} + kn\hat{k}$

$\therefore |\bar{n}| = \sqrt{(kl)^2 + (km)^2 + (kn)^2}$

$\therefore |\bar{n}| = \sqrt{\left(\frac{k}{\sqrt{2}}\right)^2 + \left(\frac{k}{2}\right)^2 + \left(\frac{k}{2}\right)^2}$

$\therefore 8 = \sqrt{\frac{k^2}{2} + \frac{k^2}{4} + \frac{k^2}{4}}$

$\therefore 8 = \sqrt{k^2}$

$\therefore k = 8$

$\bar{n} = \left(\frac{8}{\sqrt{2}}\right)\hat{\imath} + \left(\frac{8}{2}\right)\hat{\jmath} + \left(\frac{8}{2}\right)\hat{k}$

$\therefore \bar{n} = 4\sqrt{2}\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$

Now, equation of the plane is

$\bar{r}.\bar{n} = p$

$\therefore \bar{r}.\left(4\sqrt{2}\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\right) = p$ ....eq(1)

But $\bar{r} = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$\therefore \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(4\sqrt{2}\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\right) = p$

$\Rightarrow 4\sqrt{2}x + 4y + 4z = p$

As point P ($\sqrt{2}$, -1, 1) lies on the plane by substituting it in above equation,

$4\sqrt{2}(\sqrt{2}) + 4(-1) + 4(1) = p$

$\Rightarrow 8 - 4 + 4 = p$

$\Rightarrow P = 8$

From eq(1)

$\therefore \bar{r}.\left(4\sqrt{2}\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\right) = 8$

Dividing throughout by 4

$\therefore \bar{r}.\left(\sqrt{2}\hat{\imath} + \hat{\jmath} + \hat{k}\right) = 2$

This is the equation of required plane.

Question 30.
Find the vector equation of a line passing through the point $\left(2\hat{i} - 3\hat{j} - 5\hat{k}\right)$ and perpendicular to the plane $\vec{r} \cdot \left(6\hat{i} - 3\hat{j} + 5\hat{k}\right) + 2 = 0.$

Also, find the point of intersection of this line and the plane.

Answer:
Given :

$\bar{a} = 2\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$

Equation of plane : $\bar{r}.\left(6\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right) = -2$

To Find :

Equation of line

Point of intersection

Formula :

Equation of line passing through point A with position vector $\bar{a}$ and parallel to vector $\bar{b}$ is

$\bar{r} = \bar{a} + \lambda\bar{b}$

Where, $\bar{r} = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

From the given equation of the plane

$\bar{r}.\left(6\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right) = -2$ ....eq(1)

The normal vector of the plane is

$\bar{n} = 6\hat{\imath} - 3\hat{\jmath} + 5\hat{k}$

As the given line is perpendicular to the plane therefore $\bar{n}$ will be parallel to the line.

$\therefore \bar{n} = \bar{b}$

Now, the equation of the line passing through $\bar{a} = \left(2\hat{\imath} - 3\hat{\jmath} - 5\hat{k}\right)$ and parallel to $\bar{b} = \left(6\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right)$ is

$\bar{r} = \bar{a} + \lambda\bar{b}$

$\therefore \bar{r} = \left(2\hat{\imath} - 3\hat{\jmath} - 5\hat{k}\right) + \lambda\left(6\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right)$

.........eq(2)

This is the required equation line.

Substituting $\bar{r} = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$ in eq(1)

$\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(6\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right) = -2$

⇒6x – 3y + 5z = -2 ....eq(3)

Also substituting $\bar{r} = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$ in eq(2)

$\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = \left(2\hat{\imath} - 3\hat{\jmath} - 5\hat{k}\right) + \lambda\left(6\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right)$

$\therefore (6\hat{\imath} - 3\hat{\jmath} + 5\hat{k})\lambda = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) - (2\hat{\imath} - 3\hat{\jmath} - 5\hat{k})$

$\therefore 6\lambda\hat{\imath} - 3\lambda\hat{\jmath} + 5\lambda\hat{k} = (x-2)\hat{\imath} + (y+3)\hat{\jmath} + (z+5)\hat{k}$

Comparing coefficients of $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$

$\Rightarrow 6\lambda = (x-2), -3\lambda = (y+3)$ & $5\lambda = (z+5)$

$\lambda = \frac{x-2}{6} = \frac{y+3}{-3} = \frac{z+5}{5}$ ....eq(4)

Let Q(a, b, c) be the point of intersection of given line and plane

As point Q lies on the given line.

Therefore from eq(4)

$$\frac{a-2}{6} = \frac{b+3}{-3} = \frac{c+5}{5} = k(say)$$

⇒a = 6k+2, b = -3k-3, c = 5k-5

Also point Q lies on the plane.

Therefore from eq(3)

6a – 3b + 5c = -2

⇒6(6k+2) – 3(-3k-3) + 5(5k-5) = -2

⇒36k + 12 + 9k + 9 + 25k – 25 = -2

⇒70k = 2

$\Rightarrow k = \frac{1}{35}$

$$\therefore a = 6\left(\frac{1}{35}\right) + 2 = \frac{76}{35}$$

$$b = -3\left(\frac{1}{35}\right) - 3 = \frac{-108}{35}$$

$$c = 5\left(\frac{1}{35}\right) - 5 = \frac{-170}{35} = \frac{-34}{7}$$

Therefore co-ordinates of the point of intersection of line and plane are

$$Q \equiv \left(\frac{76}{35}, \frac{-108}{35}, \frac{-34}{7}\right)$$