Exercise 6b

Question 1.
Evaluate :

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

Answer:

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} 67 & 19 & 21 \\ 78 & 26 & 28 \\ 81 & 24 & 26 \end{vmatrix} \text{ [}R_2' = (1/2)R_2\text{]}$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} 67 & 19 & 21 \\ -3 & 2 & 2 \\ 81 & 24 & 26 \end{vmatrix} \text{ [}R_2' = R_2 - R_3\text{]}$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} -14 & -5 & -5 \\ -3 & 2 & 2 \\ 81 & 24 & 26 \end{vmatrix} \text{ [}R_1' = R_1 - R_3\text{]}$$

$$= \begin{vmatrix} -14 & -5 & -5 \\ -3 & 2 & 2 \\ 81/2 & 12 & 13 \end{vmatrix} \text{ [}R_3' = 2R_3\text{]}$$

$= (-14)\{(2 \times 13) - (2 \times 12)\} - 5\{(2 \times 81/2) - (-3) \times 13\} - 5\{(-3) \times 12 - 2 \times 81/2\}$

[expanding by the first row]

$= -14 \times (26 - 24) - 5(81 + 39) - 5(-36 - 81)$

$= -14 \times 2 - 5 \times 120 - 5 \times (-117) = -28 - 600 + 585 =$ **-43**

Question 2.

Evaluate :

$$\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix}$$

Answer:

$$\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & -5 & -5 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix} \text{ [}R_1' = R_1 - R_2\text{]}$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} 4 & -5 & -5 \\ 50 & 62 & 54 \\ 63 & 54 & 46 \end{vmatrix} \text{ [}R_2' = 2R_2\text{]}$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} 4 & -5 & -5 \\ -13 & 8 & 8 \\ 63 & 54 & 46 \end{vmatrix} \text{ [}R_2' = R_2 - R_3\text{]}$$

$$= \begin{vmatrix} 4 & -5 & -5 \\ -13 & 8 & 8 \\ 63/2 & 27 & 23 \end{vmatrix} \text{ [}R_3' = 2R_3\text{]}$$

$= 4(8 \times 23 - 8 \times 27) - 5\{8 \times 63/2 - (-13) \times 23\} - 5\{(-13) \times 27 - 8 \times 63/2\}$

[expansion by first row]

= **132**

Question 3.
Evaluate :

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Answer:

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6 \times \begin{vmatrix} 17 & 18 & 6 \\ 1 & 6 & 4 \\ 17 & 3 & 6 \end{vmatrix} \text{ [}R_1' = R_1/6\text{]}$$

Now, for any determinant, if at least two rows are identical, then the value of the determinant becomes zero.

Here, the first and third rows are identical.

So, the value of the above determinant evaluated = **0**

Question 4.
Evaluate :

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 2^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$$

Answer:

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$

Expanding by first row, we get,

$1(9 \times 25 - 16 \times 16) + 4(16 \times 9 - 4 \times 25) + 9(4 \times 16 - 9 \times 9) = -31 + 176 - 153 = \mathbf{-8}$

Question 5.
Using properties of determinants prove that:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a).$$

Answer:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ bc-ca & ca-ab & ab \end{vmatrix}$$ [$C_1' = C_1 - C_2$ & $C_2' = C_2 - C_3$]

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ -c(a-b) & -a(b-c) & ab \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ -c & -a & ab \end{vmatrix}$$ [$C_1' = C_1/(a - b)$ & $C_2' = C_2/(b - c)$]

= (a - b)(b - c)[0 + 0 + 1{ - a - (- c)}] [expansion by first row]

= **(a - b)(b - c)(c - a)**

Question 6.
Using properties of determinants prove that:

$$\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

Answer:

$$\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-a & b^2-a^2 \\ 0 & c-b & c^2-b^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$]

$$= \begin{vmatrix} 0 & b-a & (b-a)(b+a) \\ 0 & c-b & (c-b)(c+b) \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

$$= (b-a)(c-b)\begin{vmatrix} 0 & 1 & b+a \\ 0 & 1 & c+b \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$ [$R_1' = R_1/(b - a)$ & $R_2' = R_2/(c - b)$]

= (b - a)(c - b)[0 + 0 + 1{(c + b) - (b + a)}] [expansion by first column]

= **(a - b)(b - c)(c - a)**

Question 7.
Using properties of determinants prove that:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1.$$

Answer:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -2-p & -2p-q \\ -1 & -3-p & -3p-q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$]

$$= \begin{vmatrix} 0 & 1 & p \\ -1 & -3-p & -3p-q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$$ [$R_1' = R_1 - R_2$]

$$= \left(\frac{1}{2}\right)\begin{vmatrix} 0 & 1 & p \\ -2 & -6-2p & -6p-2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$$ [$R_2' = R_2*2$]

$$= \left(\frac{1}{2}\right)\begin{vmatrix} 0 & 1 & p \\ 1 & p & 1+q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$$ [$R_2' = R_2 + R_3$]

= (1/2)[0 + 3(1 + q) - (1 + 6p + 3q) + p(6 + 3p - 3p)] [expansion by first row]

= (1/2)(3 + 3q - 1 - 6p - 3q + 6p) = **1**

Question 8.
Using properties of determinants prove that:

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z).$$

Answer:

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

$$= \begin{vmatrix} a & -a & 0 \\ 0 & a & -a \\ x & y & a+z \end{vmatrix}$$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$]

$$= a^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ x & y & a+z \end{vmatrix}$$ [$R_1' = R_1/a$ & $R_2' = R_2/a$]

$= a^2[a + z - (-y) - (-x)]$ [expansion by first row]

$=$ **$a^2(a + x + y + z)$**

Question 9.
Using properties of determinants prove that:

$$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x+2a)(x-a)^2.$$

Answer:

$$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$$

$$= \begin{vmatrix} x+2a & x+2a & x+2a \\ a & x & a \\ a & a & x \end{vmatrix}$$ [$R_1' = R_1 + R_2 + R_3$]

$$= (x + 2a)\begin{vmatrix} 1 & 1 & 1 \\ a & x & a \\ a & a & x \end{vmatrix}$$ [$R_1' = R_1/(x + 2a)$]

$$= (x + 2a)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x - a & a - x \\ a & a & x \end{vmatrix}$$ [$R_2' = R_2 - R_3$]

$$= (x + 2a)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x - a & -(x - a) \\ a & a & x \end{vmatrix}$$

$$= (x + 2a)(x - a)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ a & a & x \end{vmatrix}$$ [$R_2' = R_2/(x - a)$]

$= (x + 2a)(x - a)[x - (- a) + (- a - 0) + (- a)]$ [expansion by first row]

$= (x + 2a)(x - a)(x + a - a - a) =$ $\mathbf{(x + 2a)(x - a)^2}$

Question 10.
Using properties of determinants prove that:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(x-4)^2.$$

Answer:

$$\begin{vmatrix} x + 4 & 2x & 2x \\ 2x & x + 4 & 2x \\ 2x & 2x & x + 4 \end{vmatrix}$$

$$= \begin{vmatrix} 5x + 4 & 5x + 4 & 5x + 4 \\ 2x & x + 4 & 2x \\ 2x & 2x & x + 4 \end{vmatrix}$$ [$R_1' = R_1 + R_2 + R_3$]

$$= (5x + 4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x + 4 & 2x \\ 2x & 2x & x + 4 \end{vmatrix}$$ [$R_1' = R_1/(5x + 4)$]

$$= (5x + 4)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -x + 4 & x - 4 \\ 2x & 2x & x + 4 \end{vmatrix}$$ [$R_2' = R_2 - R_3$]

$$= (5x + 4)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -(x-4) & x-4 \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x + 4)(x-4)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 2x & 2x & x+4 \end{vmatrix} [R_2' = R_2/(x - 4)]$$

= (5x + 4)(x - 4)[- (x + 4) - 2x + 2x - 0 + 0 - (- 2x)] [expansion by first row]

= (5x + 4)(x - 4)(- x - 4 + 2x) = **(5x + 4)(x - 4)^2**

Question 11.
Using properties of determinants prove that:

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2.$$

Answer:

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 5x+\lambda & 5x+\lambda & 5x+\lambda \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} [R_1' = R_1 + R_2 + R_3]$$

$$= (5x + \lambda)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} [R_1' = R_1/(5x + \lambda)]$$

$$= (5x + \lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -x+\lambda & x-\lambda \\ 2x & 2x & x+\lambda \end{vmatrix} [R_2' = R_2 - R_3]$$

$$= (5x + \lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -(x-\lambda) & x-\lambda \\ 2x & 2x & x+\lambda \end{vmatrix}$$

$$= (5x + \lambda)(x - \lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 2x & 2x & x + \lambda \end{vmatrix} [R_2' = R_2/(x - \lambda)]$$

$= (5x + \lambda)(x - \lambda)[- (x + \lambda) - 2x + 2x - 0 + 0 - (- 2x)]$ [expansion by first row]

$= (5x + \lambda)(x - \lambda)(- x - \lambda + 2x) =$ $\mathbf{(5x + \lambda)(x - \lambda)^2}$

Question 12.
Using properties of determinants prove that:

$$\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3.$$

Answer:

$$\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 - 1 & a - 1 & 0 \\ 2a - 2 & a - 1 & 0 \\ 3 & 3 & 1 \end{vmatrix} [R_1' = R_1 - R_2 \;\&\; R_2' = R_2 - R_3]$$

$$= \begin{vmatrix} a^2 - 1 & a - 1 & 0 \\ 2(a - 1) & a - 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= (a - 1)^2\begin{vmatrix} a + 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix} [R_1' = R_1/(a - 1) \;\&\; R_2' = R_2/(a - 1)]$$

$= (a - 1)^2[a + 1 - 0 - 2]$ [expansion by first row]

$=$ $\mathbf{(a - 1)^3}$

Question 13.

Using properties of determinants prove that:

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$

Answer:

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

$$= \begin{vmatrix} 3(x+y) & 3(x+y) & 3(x+y) \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$ [$R_1' = R_1 + R_2 + R_3$]

$$= 3(x+y)\begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$ [$R_1' = R_1/3(x + y)$]

$$= 3(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & -2y & y \\ x+y & x+2y & x \end{vmatrix}$$ [$R_2' = R_2 - R_3$]

$$= 3y(x+y)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ x+y & x+2y & x \end{vmatrix}$$ [$R_2' = R_2/y$]

$$= 3y(x+y)\begin{vmatrix} 0 & 3 & 0 \\ 1 & -2 & 1 \\ x+y & x+2y & x \end{vmatrix}$$ [$R_1' = R_1 - R_2$]

$= 3y(x + y)[0 + 3(x + y) - x + 0]$ [expansion by first row]

$= 3y(x + y)(3y) =$ $\mathbf{9y^2(x + y)}$

Question 14.

Using properties of determinants prove that:

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+zx).$$

Answer:

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$ [$C_1' = C_1 + C_2 + C_3$]

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$ [$C_1' = C_1/(x + y + z)$]

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ -x+y & 3y & y-z \\ -x+z & z-y & 3z \end{vmatrix}$$ [transforming row and column]

$$= (x+y+z)\begin{vmatrix} 0 & 0 & 1 \\ -x-2y & 2y+z & y-z \\ -x+y & -y-2z & x \end{vmatrix}$$ [$C_1' = C_1 - C_2$ & $C_2' = C_2 - C_3$]

$= (x + y + z)[0 + 0 + (-x - 2y)(-y - 2z) - (-x + y)(2y + z)]$ [expansion by first row]

$= (x + y + z)(xy + 2y^2 + 2xz + 4yz + 2xy - 2y^2 + xz - yz)$

$= (x + y + z)(3xy + 3yz + 3xz)$

$= 3(x + y + z)(xy + yz + zx)$

Question 15.

Using properties of determinants prove that:

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$$

Answer:

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$ [$C_1' = C_1/x$, $C_2' = C_2/y$ & $C_3' = C_3/z$]

$$= xyz\begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$ [$C_1' = C_1 - C_2$ & $C_2' = C_2 - C_3$]

$$= xyz\begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ (x+y)(x-y) & (y+z)(y-z) & z^2 \end{vmatrix}$$

$$= xyz(x-y)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x+y & y+z & z^2 \end{vmatrix}$$ [$C_1' = C_1/(x - y)$& $C_2' = C_2/(y - z)$]

= $xyz(x - y)(y - z)(0 + 0 + y + z - x - y)$ [expansion by first row]

= $\mathbf{xyz(x - y)(y - z)(z - x)}$

Question 16.
Using properties of determinants prove that:

$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3.$$

Answer:

$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 0 & a+b+c \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} [R_1' = R_1 + R_2 + R_3]$$

$$= (a+b+c)\begin{vmatrix} 2 & 0 & 1 \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} [R_1' = R_1/(a+b+c)]$$

$= (a + b + c)[2(b - c)c - b(c - a) + (c + a)(c - a) - (a + b)(b - c)]$ [expansion by first row]

$= (a + b + c)(2bc - 2c^2 - bc + ab + c^2 - a^2 - ab - b^2 + ac + bc$

$= (a + b + c)(ab + bc + ac - a^2 - b^2 - c^2)$

$= 3abc - a^3 - b^3 - c^3$

Question 17.
Using properties of determinants prove that:

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$$

Answer:

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 2(b+c) & 2(a+c) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix} [R_1' = R_1 + R_2 + R_3]$$

$$= 2\begin{vmatrix} b+c & a+c & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} [R_1' = R_1/2]$$

$$= 2\begin{vmatrix} c & 0 & a \\ b-c & a & -a \\ c & c & a+b \end{vmatrix}$$ $[R_1' = R_1 - R_2 \ \& \ R_2' = R_2 - R_3]$

$= 2[c\{a(a + b) - (- ac)\} + 0 + a\{c(b - c) - ac\}]$ [expansion by first row]

$= 2(a^2c + abc + ac^2 + abc - ac^2 - a^2c)$

= **4abc**

Question 18.
Using properties of determinants prove that:

$$\begin{vmatrix} a & a+2b & a+2b+3c \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix} = -a^3.$$

Answer:

$$\begin{vmatrix} a & a+2b & a+2b+3c \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix}$$

$$= \left(\frac{1}{3}\right)\begin{vmatrix} 3a & 3a+6b & 3a+6b+9c \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix}$$ $[R_1' = 3R_1]$

$$= \left(\frac{1}{3}\right)\begin{vmatrix} 0 & -a & -2a-b \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix}$$ $[R_1' = R_1 - R_2]$

$$= \left(\frac{1}{6}\right)\begin{vmatrix} 0 & -a & -2a-b \\ 6a & 8a+12b & 10a+14b+18c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix}$$ $[R_2' = 2R_2]$

$$= \left(\frac{1}{6}\right)\begin{vmatrix} 0 & -a & -2a-b \\ 0 & -a & -a-b \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix}$$ $[R_2' = R_2 - R_3]$

$= (1/6)[0 + 0 + 6a\{a(a + b) - a(2a + b)]$ [expansion by first column]

$= \mathbf{-a^3}$

Question 19.
Using properties of determinants prove that:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$$

Answer:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

$$= \begin{vmatrix} a+b & a+b & -(a+b) \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} \text{[}R_1' = R_1 + R_2\text{]}$$

$$= (a+b)\begin{vmatrix} 1 & 1 & -1 \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} \text{[}R_1' = R_1/(a+b)\text{]}$$

$$= (a+b)\begin{vmatrix} 1 & 1 & -1 \\ -c-b & b+c & b+c \\ -b & -a & a+b+c \end{vmatrix} \text{[}R_2' = R_2 + R_3\text{]}$$

$$= (a+b)\begin{vmatrix} 1 & 1 & -1 \\ -(b+c) & b+c & b+c \\ -b & -a & a+b+c \end{vmatrix}$$

$$= (a+b)(b+c)\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix} \text{[}R_2' = R_1/(b+c)\text{]}$$

$$= (a+b)(b+c)\begin{vmatrix} 0 & 2 & 0 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix} \text{[}R_1' = R_1 + R_2\text{]}$$

= (a + b)(b + c){0 + 2(- b + a + b + c) + 0} [expansion by first row]

= **2(a + b)(b + c)(c + a)**

Question 20.

Using properties of determinants prove that:

$$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = \left(b^2 - ac\right)\left(ax^2 + 3bxy + cy^2\right).$$

Answer:

$$\begin{vmatrix} a & b & ax + by \\ b & c & bx + cy \\ ax + by & bx + cy & 0 \end{vmatrix}$$

$$= \left(\frac{1}{xy}\right)\begin{vmatrix} ax & bx & ax^2 + bxy \\ by & cy & bxy + cy^2 \\ ax + by & bx + cy & 0 \end{vmatrix}$$ [$R_1' = xR_1$ & $R_2' = yR_2$]

$$= \left(\frac{1}{xy}\right)\begin{vmatrix} 0 & 0 & ax^2 + 2bxy + cy^2 \\ by & cy & bxy + cy^2 \\ ax + by & bx + cy & 0 \end{vmatrix}$$ [$R_1' = R_1 + R_2 - R_3$]

= $(1/xy)[0 + 0 + (ax^2 + 2bxy + cy^2)\{by(bx + cy) - cy(ax + by)\}$[expansion by first row] .

= $(1/xy)(ax^2 + 2bxy + cy^2)(b^2xy + bcy^2 - acxy - bcy^2)$

= **$(b^2 - ac)(ax^2 + 2bxy + cy^2)$**

Question 21.
Using properties of determinants prove that:

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 4(a-b)(b-c)(c-a)$$

Answer:

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a + 1)^2 & (b + 1)^2 & (c + 1)^2 \\ (a - 1)^2 & (b - 1)^2 & (c - 1)^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & b^2 & c^2 \\ a^2 + 2a + 1 & b^2 + 2b + 1 & c^2 + 2c + 1 \\ a^2 - 2a + 1 & b^2 - 2b + 1 & c^2 - 2c + 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & b^2 & c^2 \\ 4a & 4b & 4c \\ a^2 - 2a + 1 & b^2 - 2b + 1 & c^2 - 2c + 1 \end{vmatrix}$$ [$R_2' = R_2 - R_3$]

$$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ a^2 - 2a + 1 & b^2 - 2b + 1 & c^2 - 2c + 1 \end{vmatrix}$$ [$R_2' = R_2/4$]

$$= 4\begin{vmatrix} a^2 & a & a^2 - 2a + 1 \\ b^2 & b & b^2 - 2b + 1 \\ c^2 & c & c^2 - 2c + 1 \end{vmatrix}$$ [transforming row and column]

$$= 4\begin{vmatrix} a^2 - b^2 & a - b & (a^2 - b^2) - 2(a - b) \\ b^2 - c^2 & b - c & (b^2 - c^2) - 2(b - c) \\ c^2 & c & c^2 - 2c + 1 \end{vmatrix}$$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$]

$$= 4\begin{vmatrix} (a - b)(a + b) & a - b & (a - b)(a + b - 2) \\ (b - c)(b + c) & b - c & (b - c)(b + c - 2) \\ c^2 & c & c^2 - 2c + 1 \end{vmatrix}$$

$$= 4(a - b)(b - c)\begin{vmatrix} a + b & 1 & a + b - 2 \\ b + c & 1 & b + c - 2 \\ c^2 & c & c^2 - 2c + 1 \end{vmatrix}$$ [$R_1' = R_1/(a - b)$ & $R_2' = R_2/(b - c)$]

$$= 4(a - b)(b - c)\begin{vmatrix} a - c & 0 & a - c \\ b + c & 1 & b + c - 2 \\ c^2 & c & c^2 - 2c + 1 \end{vmatrix}$$ [$R_1' = R_1 - R_2$]

$$= 4(a - b)(b - c)(a - c)\begin{vmatrix} 1 & 0 & 1 \\ b + c & 1 & b + c - 2 \\ c^2 & c & c^2 - 2c + 1 \end{vmatrix}$$ [$R_1' = R_1/(a - c)$]

$= 4(a - b)(b - c)(a - c)(c^2 - 2c + 1 - bc - c^2 + 2c + 0 + bc + c^2 - c^2)$ [expansion by first row]

$= 4(a - b)(b - c)(c - a)$

Question 22.

Using properties of determinants prove that:

$$\begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix} = -8$$

Answer:

$$\begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-4x+4 & x^2-2x+1 & x^2 \\ x^2-2x+1 & x^2 & x^2+2x+1 \\ x^2 & x^2+2x+1 & x^2+4x+4 \end{vmatrix}$$

$$= \begin{vmatrix} -2x+3 & -2x+1 & -2x-1 \\ -2x+1 & -2x-1 & -2x-3 \\ x^2 & x^2+2x+1 & x^2+4x+4 \end{vmatrix}$$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$]

$$= \begin{vmatrix} 2 & 2 & 2 \\ -2x+1 & -2x-1 & -2x-3 \\ x^2 & x^2+2x+1 & x^2+4x+4 \end{vmatrix}$$ [$R_1' = R_1 - R_2$]

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ -2x+1 & -2x-1 & -2x-3 \\ x^2 & x^2+2x+1 & x^2+4x+4 \end{vmatrix}$$ [$R_1' = R_1/2$]

$$= 2\begin{vmatrix} 1 & -2x+1 & x^2 \\ 1 & -2x-1 & x^2+2x+1 \\ 1 & -2x-3 & x^2+4x+4 \end{vmatrix}$$ [transforming row and column]

$$= 2\begin{vmatrix} 0 & 2 & -2x-1 \\ 0 & 2 & -2x-3 \\ 1 & -2x-3 & x^2+4x+4 \end{vmatrix}$$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$]

$$= 2\begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & -2x-3 \\ 1 & -2x-3 & x^2+4x+4 \end{vmatrix}$$ [$R_1' = R_1 - R_2$]

= 2{0 + 0 + 2(0 - 2)} [expansion by first row]

. = **- 8**

Question 23.
Using properties of determinants prove that:

$$\begin{vmatrix} (m+n)^2 & l^2 & mn \\ (n+l)^2 & m^2 & ln \\ (l+m)^2 & n^2 & lm \end{vmatrix} = (l^2+m^2+n^2)(l-m)$$

$(m-n)(n-l)$.

Answer:

$$\begin{vmatrix} (m+n)^2 & l^2 & mn \\ (n+l)^2 & m^2 & ln \\ (l+m)^2 & n^2 & lm \end{vmatrix}$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} m^2+2mn+n^2 & l^2 & 2mn \\ n^2+2nl+l^2 & m^2 & 2ln \\ l^2+2lm+m^2 & n^2 & 2lm \end{vmatrix}$$ [$C_3' = 2C_3$]

$$= \left(\frac{1}{2}\right)\begin{vmatrix} m^2+n^2 & l^2 & 2mn \\ n^2+l^2 & m^2 & 2ln \\ l^2+m^2 & n^2 & 2lm \end{vmatrix}$$ [$C_1' = C_1 - C_3$]

$$= \left(\frac{1}{2}\right)\begin{vmatrix} l^2+m^2+n^2 & l^2 & 2mn \\ l^2+m^2+n^2 & m^2 & 2ln \\ l^2+m^2+n^2 & n^2 & 2lm \end{vmatrix}$$ [$C_1' = C_1 + C_2$]

$$= \left(\frac{1}{2}\right)(l^2+m^2+n^2)\begin{vmatrix} 1 & l^2 & 2mn \\ 1 & m^2 & 2ln \\ 1 & n^2 & 2lm \end{vmatrix}$$ [$C_1' = C_1/(l^2+m^2+n^2)$]

$$= \left(\frac{1}{2}\right)(l^2+m^2+n^2)\begin{vmatrix} 1 & 1 & 1 \\ l^2 & m^2 & n^2 \\ 2mn & 2ln & 2lm \end{vmatrix}$$ [transforming row and column]

$$= \left(\frac{1}{2}\right)(l^2+m^2+n^2)\begin{vmatrix} 0 & 0 & 1 \\ l^2-m^2 & m^2-n^2 & n^2 \\ -2n(l-m) & -2l(m-n) & 2lm \end{vmatrix}$$ [$C_1' = C_1 - C_2$ & $C_2' = C_2 - C_3$]

$$= (l^2 + m^2 + n^2)(l - m)(m - n)\begin{vmatrix} 0 & 0 & 1 \\ l + m & m + n & n^2 \\ -n & -l & lm \end{vmatrix}$$ [$C_1' = C_1/(l - m)$ & $R_2' = C_2/(l - m)$]

$= (l^2 + m^2 + n^2)(l - m)(m - n)\{0 + 0 - l(l + m) + n(m + n)\}$ [expansion by first row]

$= (l^2 + m^2 + n^2)(l - m)(m - n)\{0 + 0 - l(l + m) + n(m + n)\}$

$= (l^2 + m^2 + n^2)(l - m)(m - n)(- l^2 - ml + mn + n^2)$

$= (l^2 + m^2 + n^2)(l - m)(m - n)\{(n^2 - l^2) + m(n - l)\}$

= $\mathbf{(l^2 + m^2 + n^2)(l - m)(m - n)(n - l)(l + m + n)}$

Question 24.
Using properties of determinants prove that:

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = \left(a^2+b^2+c^2\right)(a-b)(b-c)(c-a)(a+b+c).$$

Answer:

$$\begin{vmatrix} (b + c)^2 & a^2 & bc \\ (c + a)^2 & b^2 & ca \\ (a + b)^2 & c^2 & ab \end{vmatrix}$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} b^2 + 2bc + c^2 & a^2 & 2bc \\ c^2 + 2ac + a^2 & b^2 & 2ca \\ a^2 + 2ab + b^2 & c^2 & 2ab \end{vmatrix}$$ [$C_3' = 2C_3$]

$$= \left(\frac{1}{2}\right)\begin{vmatrix} b^2 + c^2 & a^2 & 2bc \\ c^2 + a^2 & b^2 & 2ca \\ a^2 + b^2 & c^2 & 2ab \end{vmatrix}$$ [$C_1' = C_1 - C_3$]

$$= \left(\frac{1}{2}\right)\begin{vmatrix} a^2 + b^2 + c^2 & a^2 & 2bc \\ a^2 + b^2 + c^2 & b^2 & 2ca \\ a^2 + b^2 + c^2 & c^2 & 2ab \end{vmatrix}$$ [$C_1' = C_1 + C_2$]

$$= \left(\frac{1}{2}\right)(a^2 + b^2 + c^2)\begin{vmatrix} 1 & a^2 & 2bc \\ 1 & b^2 & 2ca \\ 1 & c^2 & 2ab \end{vmatrix}$$ [$C_1' = C_1/(a^2 + b^2 + c^2)$]

$$= \left(\frac{1}{2}\right)(a^2 + b^2 + c^2)\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ 2bc & 2ca & 2ab \end{vmatrix}$$ [transforming row and column]

$$= \left(\frac{1}{2}\right)(a^2 + b^2 + c^2)\begin{vmatrix} 0 & 0 & 1 \\ a^2 - b^2 & b^2 - c^2 & c^2 \\ -2c(a-b) & -2a(b-c) & 2ab \end{vmatrix}$$ [$C_1' = C_1 - C_2$ & $C_2' = C_2 - C_3$]

$$= (a^2 + b^2 + c^2)(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ a + b & b + c & c^2 \\ -c & -a & ab \end{vmatrix}$$ [$C_1' = C_1/(a - b)$ & $C_2' = C_2/(b - c)$]

$= (a^2 + b^2 + c^2)(a - b)(b - c)\{0 + 0 - a(a + b) + c(b + c)\}$ [expansion by first row]

$= (a^2 + b^2 + c^2)(a - b)(b - c)\{0 + 0 - a(a + b) + c(b + c)\}$

$= (a^2 + b^2 + c^2)(a - b)(b - c)(- a^2 - ba + bc + c^2)$

$= (a^2 + b^2 + c^2)(a - b)(b - c)\{(c^2 - a^2) + b(c - a)\}$

$= \mathbf{(a^2 + b^2 + c^2)(a - b)(b - c)(c - a)(a + b + c)}$

Question 25.
Using properties of determinants prove that:

$$\begin{vmatrix} b^2 + c^2 & a^2 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix} = 4a^2b^2c^2.$$

Answer:

$$\begin{vmatrix} b^2 + c^2 & a^2 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 2(b^2 + c^2) & 2(c^2 + a^2) & 2(a^2 + b^2) \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$$ [$R_1' = R_1 + R_2 + R_3$]

$$= 2\begin{vmatrix} (b^2 + c^2) & (c^2 + a^2) & (a^2 + b^2) \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$$ [$R_1' = R_1/2$]

$$= 2\begin{vmatrix} c^2 & 0 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$$ [$R_1' = R_1 - R_2$]

$= 2[c^2\{(c^2 + a^2)(a^2 + b^2) - b^2c^2\} + 0 + a^2\{b^2c^2 - c^2(c^2 + a^2)\}]$ [expansion by first row]

$= 2[c^2(c^2a^2 + a^4 + b^2c^2 + a^2b^2 - b^2c^2) + a^2(b^2c^2 - c^4 - a^2c^2)]$

$= 2[a^2c^4 + a^4c^2 + a^2b^2c^2 + a^2b^2c^2 - a^2c^4 - a^4c^2]$

= $\mathbf{4a^2b^2c^2}$

Question 26.
Using properties of determinants prove that:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3.$$

Answer:
Operating $R_1 \to R_1 + bR$ �$_3$, $R_2 \to R_2 - aR_3$

$$\begin{vmatrix} 1+a^2-b^2+2b^2 & 2ab-2ab & -2b+b-a^2b-b^3 \\ 2ab-2ab & 1-a^2+b^2+2a^2 & 2a-a+a^3+ab^2 \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -b-a^2b-b^3 \\ 0 & 1+a^2+b^2 & a+a^3+ab^2 \\ 2b & -2a & 1-a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -b(1+a^2+b^2) \\ 0 & 1+a^2+b^2 & a(1+a^2+b^2) \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Taking $(1+ a^2+b^2)$ from R_1 and R_2

$$= (1 + a^2 + b^2)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1 - a^2 - b^2 \end{vmatrix}$$

Operating $R_3 \to R_3 - 2bR_1 + 2aR_2$

$$= (1 + a^2 + b^2)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 0 & 0 & 1 + a^2 + b^2 \end{vmatrix}$$

Taking $(1+a^2+b^2)$ from R_3

$$(1 + a^2 + b^2)^3 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding with respect to C_1

$= (1+a^2+b^2)^3\ 1\times[\ 1-0]$

$= (1+a^2+b^2)^3$

Hence proved

Question 27.
Using properties of determinants prove that:

$$\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = (a+b+c)\left(a^2+b^2+c^2\right).$$

Answer:
Operating $C_1 \to aC_1$

$$\frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & a+b & c \end{vmatrix}$$

Operating $C_1 \to C_1 + bC_2 + cC_3$

$$= \frac{1}{a}\begin{vmatrix} a^2 + b^2 - bc + c^2 + bc & b-c & c+b \\ a^2 + ac + b^2 + c^2 - ac & b & c-a \\ a^2 - ab + ab + b^2 + c^2 & a+b & c \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} a^2 + b^2 + c^2 & b-c & c+b \\ a^2 + b^2 + c^2 & b & c-a \\ a^2 + b^2 + c^2 & a+b & c \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$ common from C_1

$$= \frac{1}{a}(a^2 + b^2 + c^2)\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & a+b & c \end{vmatrix}$$

Operating $R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow R_2 - R_3$

$$= \frac{1}{a}(a^2 + b^2 + c^2)\begin{vmatrix} 0 & -c-a & b \\ 0 & -a & -a \\ 1 & a+b & c \end{vmatrix}$$

Operating $C_2 \rightarrow C_2 - C_3$

$$= \frac{1}{a}(a^2 + b^2 + c^2)\begin{vmatrix} 0 & -(a+b+c) & b \\ 0 & 0 & -a \\ 1 & (a+b+c) & c \end{vmatrix}$$

Taking $(a+b+c)$ common from C_2

$$= \frac{1}{a}(a^2 + b^2 + c^2)(a+b+c)\begin{vmatrix} 0 & -1 & b \\ 0 & 0 & -a \\ 1 & 1 & c \end{vmatrix}$$

Expanding with respect to C_1

$$= \frac{1}{a}(a^2 + b^2 + c^2)(a+b+c) \times 1 \times \left(0 - (-a)\right)$$

$$= \frac{1}{a}(a^2 + b^2 + c^2)(a+b+c)\ (a)$$

$= (a^2+b^2+c^2)(a+b+c)$

Question 28.
Using properties of determinants prove that:

$$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0.$$

Answer:
Expanding with R1

$=b^2c^2(a^2c+abc-abc-a^2b)-bc(a^3c^2+a^2bc^2-a^2b^2c-a^3b^2)+(b+c)(a^3bc^2-a^3b^2c)$

$=a^2b^3c^2-a^2b^3c^2-a^3bc^2-a^2b^3c^2+a^2b^3c^2+a^3b^3c+a^3b^2c^2-a^3b^3c+a^3bc^3-a^3b^2c^2$

$=0$

Question 29.
Using properties of determinants prove that:

$$\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

Answer:

$$= \begin{vmatrix} b^2+c^2+2bc & ab & ac \\ ab & a^2+c^2+2ac & bc \\ ac & bc & a^2+b^2+2ab \end{vmatrix}$$

Operating $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$

$$= \frac{1}{abc} \begin{vmatrix} a(b^2+c^2+2bc) & a^2b & a^2c \\ ab^2 & b(a^2+c^2+2ac) & b^2c \\ ac^2 & bc^2 & c(a^2+b^2+2ab) \end{vmatrix}$$

Taking a, b, c common from C_1, C_2, C_3 respectively

$$= \frac{abc}{abc}\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (a+c)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (a+c)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (a+c+b)(a+c-b) & b^2 \\ (c-a-b)(c+a+b) & (c-a-b)(c+a+b) & (a+b)^2 \end{vmatrix}$$

Taking (a+b+c) common from R_1, R_2

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

Operating $R_3 \to R_3 - R_1 - R_2$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ -2b & -2a & a^2+b^2+2ab-a^2-b^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

Operating $C_1 \to aC_1$, $C_2 \to bC_2$

$$\frac{(a+b+c)^2}{ab}\begin{vmatrix} a(b+c-a) & 0 & a^2 \\ 0 & b(a+c-b) & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}$$

Operating $C_1 \to C_1 + C_3$, $C_2 \to C_2 + C_3$

$$= \frac{(a+b+c)^2}{ab}\begin{vmatrix} a(b+c) & a^2 & a^2 \\ b^2 & b(a+c) & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

Taking a, b, 2ab from R_1, R_2, R_3

$$= \frac{(a+b+c)^2 a.b.2ab}{ab}\begin{vmatrix} b+c & a & a \\ b & a+c & b \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding with R_3

$$= 2ab(a+b+c)^2 \times 1 \times (ab+ac+bc+c^2-ab)$$

$$= 2ab(a+b+c)^2\big(c(a+b+c)\big)$$

$$= 2abc(a+b+c)^3$$

Question 30.
Using properties of determinants prove that:

$$\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix} = 0.$$

Answer:

$$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$

Taking (b-a) common from C_1, C_3

$$= (b-a)^2\begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

Operating $R_2 \rightarrow R_2 - R_1 + R_3$

$$= \begin{vmatrix} b & b-c-b+c & c \\ a & a-b-a+b & b \\ c & c-a-c+a & a \end{vmatrix}$$

$$= (b-a)^2 \begin{vmatrix} b & 0 & c \\ a & 0 & b \\ c & 0 & a \end{vmatrix}$$

[Properties of determinants say that if 1 row or column has only 0 as its elements, the value of the determinant is 0]

= 0

Hence Proved

Question 31.
Using properties of determinants prove that:

$$\begin{vmatrix} -a\left(b^2+c^2-a^2\right) & 2b^3 & 2c^3 \\ 2a^3 & -b\left(c^2+a^2-b^2\right) & 2c^3 \\ 2a^3 & ab^3 & -c\left(a^2+b^2+c^2\right) \end{vmatrix} = (abc)\left(a^2+b^2+c^2\right)^3.$$

Answer:
Taking a, b, c from C_1, C_2, C_3

$$= abc \begin{vmatrix} -b^2-c^2+a^2 & 2b^2 & 2c^2 \\ 2a^2 & b^2-c^2-a^2 & 2c^2 \\ 2a^2 & 2b^2 & -a^2-b^2+c^2) \end{vmatrix}$$

Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$$= abc \begin{vmatrix} -b^2-c^2-a^2 & 0 & a^2+b^2+c^2 \\ 0 & -(a^2+b^2+c^2) & a^2+b^2+c^2 \\ 2a^2 & 2b^2 & -a^2-b^2+c^2 \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$ common from R_1, R_2

$$= abc(a^2+b^2+c^2)^2 \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2a^2 & 2b^2 & -a^2-b^2+c^2 \end{vmatrix}$$

Operating $R_3 \to R_3 + R_1 + R_2$

$$= abc(a^2+b^2+c^2)^2 \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2a^2 & 2b^2 & a^2+b^2+c^2 \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$ common from C_3 �

$$= abc(a^2+b^2+c^2)^3 \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2a^2 & 2b^2 & 1 \end{vmatrix}$$

Expanding with C_3

$= abc(a^2+b^2+c^2)^3 \times 1 \times (1-0)$

$= abc\ (a^2+b^2+c^2)^3$

Hence proved

Question 32.
Using properties of determinants prove that:

$\begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0,$ where α, β, γ are in AP.

Answer:
Given that α, β, γ are in an AP, which means $2\beta=\alpha+\gamma$

Operating $R_3 \to R_3-2R_2+R_1$

$$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1-2x+4+x-3 & x-2-2x+6+x-4 & x-\gamma-2x+2\beta+x-\alpha \end{vmatrix}$$

$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ 0 & 0 & -\gamma+2\beta-\alpha \end{vmatrix}$ [we know that $2\beta=\alpha+\gamma$]

Operating $R_1 \to R_1-R_3$, $R_2 \to R$ �$_2-R_3$

$$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ 0 & 0 & -\gamma+\alpha+\gamma-\alpha \end{vmatrix}$$

$$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ 0 & 0 & 0 \end{vmatrix}$$

[By the properties of determinants, we know that if all the elements of a row or column is 0, then the value of the determinant is also 0]

=0

Hence proved

Question 33.
Using properties of determinants prove that:

$$\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$$

Answer:
Operating $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$= \begin{vmatrix} (a+1)(a+2)-(a+2)(a+3) & a+2-a-3 & 0 \\ (a+2)(a+3)-(a+3)(a+4) & a+3-a-4 & 0 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (a+2)(a+1-a-3) & -1 & 0 \\ (a+3)(a+2-a-4) & -1 & 0 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -2(a+2) & -1 & 0 \\ -2(a+3) & -1 & 0 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

Expanding with C_3

= (2(a+2) – 2(a+3))

= (2a+4-2a-6)

= -2

Question 34.

If x ≠ y ≠ z and $\begin{vmatrix} x & x^3 & x^4-1 \\ y & y^3 & y^4-1 \\ z & z^3 & z^4-1 \end{vmatrix} = 0,$ prove that xyz (xy + yz + zx) = (x + y + z).

Answer:
By properties of determinants, we can split the given determinant into 2 parts

$$\rightarrow 0 = \begin{vmatrix} x & x^3 & x^4 \\ y & y^3 & y^4 \\ z & z^3 & z^4 \end{vmatrix} - \begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix}$$

Taking x, y, z common from R_1, R_2, R_3 respectively

$$\rightarrow 0 = xyz \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} - \begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix}$$

Operating $R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow R_2 - R_3$

$$\rightarrow 0 = xyz \begin{vmatrix} 0 & x^2-z^2 & x^3-z^3 \\ 0 & y^2-z^2 & y^3-z^3 \\ 1 & z^2 & z^3 \end{vmatrix} - \begin{vmatrix} x-z & x^3-z^3 & 0 \\ y-z & y^3-z^3 & 0 \\ z & z^3 & 1 \end{vmatrix}$$

$$\rightarrow \begin{vmatrix} x-z & (x-z)(x^2+xz+z^2) & 0 \\ y-z & (y-z)(y^2+yx+z^2) & 0 \\ z & z^3 & 1 \end{vmatrix} = xyz \begin{vmatrix} 0 & (x-z)(x+z) & (x-z)(x^2+xz+z^2) \\ 0 & (y-z)(y+z) & (y-z)(y^2+yz+z^2) \\ 1 & z^2 & z^3 \end{vmatrix}$$

Taking (x-z) and (y-z) common from R_1, R_2

$$\rightarrow (x-z)(y-z) \begin{vmatrix} 1 & (x^2+xz+z^2) & 0 \\ 1 & (y^2+yz+z^2) & 0 \\ z & z^3 & 1 \end{vmatrix} = (x-z)(y-z) \begin{vmatrix} 0 & x+z & (x^2+xz+z^2) \\ 0 & y+z & (y^2+yz+z^2) \\ 1 & z^2 & z^3 \end{vmatrix}$$

Expanding with R_3

$\rightarrow y^2+yz+z^2-x^2-xz-z^2 = xyz(xy^2+xyz+xz^2+zy^2+yz^2+z^3-x^2y-xyz-yz^2-x^2z-xz^2-z^3)$

$\rightarrow (y-x)(y+x) +z(y-x) =xyz(xy^2+zy^2 -x^2y -x^2z)$

$\rightarrow(y-x)(x+y+z)=xyz(xy(y-x)+z(y^2-x^2))$

$\rightarrow(y-x)(x+y+z)= xyz(xy(y-x)+z(x+y)(y-x))$

$\rightarrow(y-x)(x+y+z) = xyz(xy(y-x)+(xz+yz)(y-x))$

$\rightarrow(y-x)(x+y+z)= xyz(y-x)(xy+xz+yz)$

$\rightarrow x+y+z = xyz(xy+xz+yz)$

Hence Proved

Question 35.

Prove that $\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix}$ = - (a – b) (b – c) (c – a) $(a^2 + b^2 + c^2)$.

Answer:
Operating $R_1 \rightarrow R_1-R_2$, $R_2 \rightarrow R_2-R_3$

$$= \begin{vmatrix} 0 & a^2+bc-b^2-ac & a^3-b^3 \\ 0 & b^2+ca-c^2-ab & b^3-c^3 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & (a-b)(a+b)-c(a-b) & (a-b)(a^2+ab+b^2) \\ 0 & (b-c)(b+c)-a(b-c) & (b-c)(b^2+bc+c^2) \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Taking (a-b), (b-c) common from R_1, R_2 respectively

$$= (a-b)(b-c)\begin{vmatrix} 0 & a+b-c & a^2+ab+b^2 \\ 0 & b+c-a & b^2+bc+c^2 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Operating $R_1 \to R_1 - R_2$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 2a-2c & a^2+ab-bc-c^2 \\ 0 & b+c-a & b^2+bc+c^2 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 2(a-c) & (a+c)(a-c)+b(a-c) \\ 0 & b+c-a & b^2+bc+c^2 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Taking (a-c) common from R_1

$$= (a-c)(a-b)(b-c)\begin{vmatrix} 0 & 2 & a+b+c \\ 0 & b+c-a & b^2+bc+c^2 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Expanding with C_1

$= (a-c)(a-b)(b-c)\times(2b^2+2bc+2c^2-ab-b^2-bc-ac-bc-c^2+a^2+ab+ac)$

$=-(c-a)(b-c)(a-b)(a^2+b^2+c^2)$

Hence Proved

Question 36.
Without expanding the determinant, prove that:

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Answer:
Operating $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\to \begin{vmatrix} 0 & a-b & bc-ac \\ 0 & b-c & ac-ab \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\rightarrow\begin{vmatrix}0 & a-b & -c(a-b)\\0 & b-c & -a(b-c)\\1 & c & ab\end{vmatrix}=\begin{vmatrix}0 & a-b & (a-b)(a+b)\\0 & b-c & (b-c)(b+c)\\1 & c & c^2\end{vmatrix}$$

Taking (a-b) and (b-c) from R_1, R_2

$$\rightarrow(a-b)(b-c)\begin{vmatrix}0 & 1 & -c\\0 & 1 & -a\\1 & c & ab\end{vmatrix}=(a-b)(b-c)\begin{vmatrix}0 & 1 & (a+b)\\0 & 1 & (b+c)\\1 & c & c^2\end{vmatrix}$$

Method 1:

For the two determinants to be equal, their difference must be 0.

$$=\begin{vmatrix}0 & 1 & -c\\0 & 1 & -a\\1 & c & ab\end{vmatrix}-\begin{vmatrix}0 & 1 & a+b\\0 & 1 & b+c\\1 & c & c^2\end{vmatrix}$$

$$=\begin{vmatrix}0-0 & 1-1 & -(a+b+c)\\0-0 & 1-1 & -(a+b+c)\\1-1 & c-c & ab-c^2\end{vmatrix}$$

$$=\begin{vmatrix}0 & 0 & -(a+b+c)\\0 & 0 & -(a+b+c)\\0 & 0 & ab-c^2\end{vmatrix}$$

Since 2 columns have only 0 as their elements, by properties of determinants

=0

Method 2:

Expanding both with C_1

LHS

=(a-b)(b-c)(-a+c)

RHS

=(a-b)(b-c)(b+c-a-b)

=(a-b)(b-c)(-a+c)

∴ LHS = RHS

Question 37.
Without expanding the determinant, prove that:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix}$$

Answer:
Operating $R1 \to R1 - R_3$, $R_2 \to R_2 - R3$

$$\begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & bc-ab & b+c-a-b \\ 0 & ac-ab & c+a-a-b \\ 1 & ab & a+b \end{vmatrix}$$

$$\to \begin{vmatrix} 0 & a-c & (a-c)(a+c) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & -b(a-c) & -(a-c) \\ 0 & -a(b-c) & -(b-c) \\ 1 & ab & a+b \end{vmatrix}$$

Taking (a-c) and (b-c) common from R_1, R_2

$$\to (a-c)(b-c) \begin{vmatrix} 0 & 1 & a+c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} = (a-c)(b-c) \begin{vmatrix} 0 & -b & -1 \\ 0 & -a & -1 \\ 1 & ab & a+b \end{vmatrix}$$

Method 1:

If the determinants are equal, their difference must also be equal.

(a-c) and (b-c) get cancelled.

$$= \begin{vmatrix} 0 & 1 & a+c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 0 & -b & -1 \\ 0 & -a & -1 \\ 1 & ab & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 0-0 & 1+b & a+c+1 \\ 0-0 & 1+a & b+c+1 \\ 1-1 & c-ab & c^2+a+b \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1+b & a+c+1 \\ 0 & 1+a & b+c+1 \\ 0 & c-ab & c^2+a+b \end{vmatrix}$$

Since all elements of C_1 are 0, by properties of determinants,

=0

∴ The 2 determinants are equal.

Method 2:

Expanding with C_1

→(a-c)(b-c)(b+c-a-c) = (a-c)(b-c)(b-a)

→(a-c)(b-c)(b-a)=(a-c)(b-c)(b-a)

∴ RHS and LHS are equal

Question 38.

Show that x = 2 is a root of the equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & 2+x \end{vmatrix} = 0.$

Answer:
Operating $R_1 \rightarrow R_1 - R_2$

$$0 = \begin{vmatrix} x-2 & -6+3x & -1-x+3 \\ 2 & -3x & x-3 \\ -3 & 2x & 2+x \end{vmatrix}$$

$$0 = \begin{vmatrix} x-2 & 3(x-2) & -(x-2) \\ 2 & -3x & x-3 \\ -3 & 2x & 2+x \end{vmatrix}$$

Taking (x-2) common from R_1

$$0 = (x-2)\begin{vmatrix} 1 & 1 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & 2+x \end{vmatrix}$$

Here, we can see that x-2 is a factor of the determinant.

We can say that when x-2 is put in the equation, we get 0.

x-2=0

→x=2

Question 39.
Solve the following equations:

$$\begin{vmatrix} 1 & x & x^3 \\ 1 & b & b^3 \\ 1 & c & x^3 \end{vmatrix} = 0$$

Answer:
Operating $R1 \to R1 - R_2$, $R_2 \to R2 - R_3$

$$\begin{vmatrix} 0 & x-b & x^3-b^3 \\ 0 & b-c & b^3-c^3 \\ 1 & c & c^3 \end{vmatrix} = 0$$

$$0 = \begin{vmatrix} 0 & x-c & (x-b)^3+3xb(x-b) \\ 0 & b-c & (b-c)^3+3bc(b-c) \\ 1 & c & c^3 \end{vmatrix}$$

$$0 = (x-c)(b-c)\begin{vmatrix} 0 & 1 & (x-b)^2+3xb \\ 0 & 1 & (b-c)^2+3bc \\ 1 & c & c^3 \end{vmatrix}$$

Expanding with C_1

$0=(x-c)(b-c)(b^2-2bc+c^2+3bc-x^2+2xb-b^2-3xb)$

$0= (x-c)(b-c)(bc+c^2-x^2-xb)$

$0=(x-c)(b-c)(-b(-c+x)-(c-x)(-c-x))$

$0=(x-c)^2(b-c)(-b-c-x)$

Either x-c=0 or b-c=0 or (-b-c-x)=0

∴x=c or b=c or x=-(b+c)

If b=c, x=b

∴ x=c or x=b or x=-(b+c)

Question 40.
Solve the following equations:

$$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ b & b & x+c \end{vmatrix} = 0$$

Answer:
Operating $C1 \rightarrow C1+C_2+C_3$

$$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix} = 0$$

Taking (x+a+b+c) common from C_1

$$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix} = 0$$

Operating $R_1 \rightarrow R_1-R_3$, $R_2 \rightarrow R_2-R_3$

$$(x+a+b+c)\begin{vmatrix} 0 & 0 & -x \\ 0 & x & -x \\ 1 & b & x+c \end{vmatrix} = 0$$

Expanding with C_1

$0=(x+a+b+c)(0+x^2)$

$0=x^2(x+a+b+c)$

Either $x^2=0$ or $(x+a+b+c)=0$

$\therefore$ $x=0$ or $x=-(a+b+c)$

Question 41.
Solve the following equations:

$$\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

Answer:
Operating $C1 \rightarrow C1 + C_2 + C_3$

$$0 = \begin{vmatrix} 3x-8+3+3 & 3 & 3 \\ 3+3x-8+3 & 3x-8 & 3 \\ 3+3+3x-8 & 3 & 3x-8 \end{vmatrix}$$

$$0 = \begin{vmatrix} 3x-2 & 3 & 3 \\ 3x-2 & 3x-8 & 3 \\ 3x-2 & 3 & 3x-8 \end{vmatrix}$$

Taking $(3x-2)$ common from C_1

$$0 = (3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3x-8 & 3 \\ 1 & 3 & 3x-8 \end{vmatrix}$$

Operating $R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow R_2 - R_3$

$$0 = (3x-2)\begin{vmatrix} 0 & 0 & -(3x-11) \\ 0 & 3x-11 & -3x+11 \\ 1 & 3 & 3x-8 \end{vmatrix}$$

Expanding with C_1

$0 = (3x-2)(0+(3x-11)^2)$

$0=(3x-2)(3x-11)^2$

Either 3x-2=0 or 3x-11=0

$$\therefore \mathbf{x = \frac{2}{3} \text{ or } x = \frac{11}{3}}$$

Question 42.
Solve the following equations:

$$\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$$

Answer:
Operating $C_1 \rightarrow C_1+C_2+C_3$

$$0 = \begin{vmatrix} x+9 & 3 & 5 \\ x+9 & x+2 & 5 \\ x+9 & 3 & x+4 \end{vmatrix}$$

Taking (x+9) common from C_1

$$0 = (x+9)\begin{vmatrix} 1 & 3 & 5 \\ 1 & x+2 & 5 \\ 1 & 3 & x+4 \end{vmatrix}$$

Operating $R_1 \rightarrow R_1-R_3$, $R_2 \rightarrow R_2-R_3$

$$0 = (x+9)\begin{vmatrix} 0 & 0 & 1-x \\ 0 & x-1 & 1-x \\ 1 & 3 & x+4 \end{vmatrix}$$

$0 = (x+9)(0-x+x^2+1-x)$

$0=(x+9)(x^2-2x+1)$

$0=(x+9)(x-1)^2$

∴ Either x+9=0 or x-1=0

x=-9, x=1

Question 43.
Solve the following equations:

$$\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Answer:
Operating $R1 \to R1 + R_2 + R_3$

$$0 = \begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$$

Taking (x+9) common from R_1

$$0 = (x+9)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$$

Operating $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$

$$0 = (x+9)\begin{vmatrix} 0 & 0 & 1 \\ 0 & x-2 & 2 \\ 7-x & 6-x & x \end{vmatrix}$$

Expanding with R_1

0=(x+9)(0-(x-2)(7-x))

0=(x+9)(7-x)(2-x)

Either x+9=0 or 7-x=0 or 2-x=0

∴ x=-9 or x=7 or x=2

Question 44.
Solve the following equations:

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$

Answer:
Expanding with R1

$0= x(-3x^2-6x-2x^2+6x)+6(2x+4+3x-9)-1(4x-9x)$

$0=x(-5x^2)+6(5x-5)-1(-5x)$

$0= -5x^3+30x-30+5x$

$0=-5x^3+35x-30$

$x^3-7x+6=0$

$x^3-x-6x+6=0$

$x(x^2-1)-6(x-1)=0$

$x(x-1)(x+1)-6(x-1)=0$

$(x-1)(x^2+x-6)=0$

$(x-1)(x^2+3x-2x-6)=0$

$(x-1)(x(x+3)-2(x+3)(=0$

$(x-1)(x+3)(x-2)=0$

Either $x-1=0$ or $x+3=0$ or $x-2=0$

$\therefore x=1$ or $x=-3$ or $x=2$

Question 45.

Prove that

$$\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$$

Answer:
Operating $C_1 \to aC_1$

$$= \frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & b+a & c \end{vmatrix}$$

Operating $C_1 \to C_1 + bC_2 + cC_3$

$$= \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+a & c \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$

$$= \frac{a^2+b^2+c^2}{a}\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$

Operating $C_2 \to C_2 - bC_1$, $C_3 \to C_3 - cC_3$

$$= \frac{a^2+b^2+c^2}{a}\begin{vmatrix} 1 & -c & b \\ 1 & 0 & -a \\ 1 & a & 0 \end{vmatrix}$$

Expanding with R_3

$$= \frac{a^2+b^2+c^2}{a}(ac - 0 + a^2 + ab)$$

$$= \frac{a^2+b^2+c^2}{a}a(a+b+c)$$

$= (a^2+b^2+c^2)(a+b+c)$

Hence Proved