Exercise 9a

Question 1.
Show that ƒ(x) = x^2 is continues at x=2.

Answer:
Left Hand Limit: $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2$

= 4

Right Hand Limit: $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2$

= 4

ƒ(2) = 4

Since, $\lim_{x \to 2} f(x)$ = f(2)

∴ f is continuous at x=2.

Question 2.
Show that ƒ(x) = (x^2+3x+4) is continuous at x=1.

Answer:
Left Hand Limit: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + 3x + 4$

= 7

Right Hand Limit: $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + 3x + 4$

= 7

ƒ(1) = 7

Since, $\lim_{x \to 1} f(x)$ = f(1)

∴ f is continuous at x=1.

Question 3.
Prove that

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{when } x \neq 3 \\ 5, & \text{when } x = 3 \end{cases}$$ is continuous at x=3

Answer:
LHL: $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} \frac{x^2-x-6}{x-3}$

$= \lim\limits_{x\to 3^-} \frac{(x+2)(x-3)}{x-3}$ [By middle term splitting]

$= \lim\limits_{x\to 3^-} x + 2$

= 5

RHL: $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} \frac{x^2-x-6}{x-3}$

$= \lim\limits_{x\to 3^+} \frac{(x+2)(x-3)}{x-3}$ [By middle term splitting]

$= \lim\limits_{x\to 3^+} x + 2$

= 5

f(3) = 5

Since, $\lim\limits_{x\to 3} f(x)$ = f(3)

∴ f is continuous at x=3.

Question 4.

Prove that

$$f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & \text{when } x \neq 5 \\ 10, & \text{when } x = 5 \end{cases}$$ is continuous at x=5

Answer:

LHL: $\lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^-} \frac{x^2-25}{x-5}$

$= \lim\limits_{x \to 5^-} \frac{(x+5)(x-5)}{x-5}$ [By middle term splitting]

$= \lim\limits_{x \to 5^-} x + 5$

= 10

RHL: $\lim\limits_{x \to 5^+} f(x) = \lim\limits_{x \to 5^+} \frac{x^2-25}{x-5}$

$= \lim\limits_{x \to 5^+} \frac{(x+5)(x-5)}{x-5}$ [By middle term splitting]

$= \lim\limits_{x \to 5^+} x + 5$

= 10

f(5)= 10

Since, $\lim\limits_{x \to 5} f(x)$ = f(5)

∴ f is continuous at x=5.

Question 5.

Prove that

$$f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases}$$ is discontinuous at x=0

Answer:
LHL: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin 3x}{x}$

= 3

$[\lim_{x\to a} \frac{\sin nx}{x} = n]$

RHL: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin 3x}{x}$

= 3

f(0)=1

Since, $\lim_{x\to 0} f(x) \neq f(0)$

∴ f is discontinuous at x=0.

Question 6.
Prove that

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases}$$ is discontinuous at x=0

Answer:
LHL: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos x}{x^2}$

$= \lim_{x\to 0^-} \frac{2\sin^2\frac{x}{2}}{x^2}$

$= 2 \lim_{x\to 0^-} \frac{(\sin\frac{x}{2})^2}{x^2}$

$= 2 \times \frac{1}{4}$

$= \frac{1}{2}$

RHL: $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0+} \frac{1-\cos x}{x^2}$

$= \lim\limits_{x\to 0+} \frac{2\sin^2\frac{x}{2}}{x^2}$

$=2 \lim\limits_{x\to 0+} \frac{(\sin\frac{x}{2})^2}{x^2}$

$= 2 \times \frac{1}{4}$

$= \frac{1}{2}$

f(0)= 1

Since, $\lim\limits_{x\to 0} f(x) \neq f(0)$

∴ f is discontinuous at x=0.

Question 7.
Prove that

$f(x) = \begin{cases} 2 - x, & \text{when } x < 2; \\ 2 + x, & \text{when } x \geq 2 \end{cases}$ is discontinuous at x=2

Answer:
LHL: $\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2-} 2 + x$

= 4

RHL: $\lim\limits_{x\to 2+} f(x) = \lim\limits_{x\to 2+} 2 - x$

= 0

$\lim\limits_{x\to 2-} f(x) \neq \lim\limits_{x\to 2+} f(x)$

∴ f(x) is discontinuous at x=2

Question 8.
Prove that

f(x) = $\begin{cases} 3-x, & \text{when } x \le 0; \\ x^2, & \text{when } x > 0 \end{cases}$ is discontinuous at x=0

Answer:
LHL: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3 - x$

= 3

RHL: $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2$

= 0

$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

∴ f(x) is discontinuous at x=0

Question 9.
Prove that

f(x) = $\begin{cases} 5x-4, & \text{when } 0 < x \le 1; \\ 4x^2 - 3x, & \text{when } 1 < x < 2 \end{cases}$ is continuous at x=1

Answer:
LHL: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 5x - 4$

= 1

RHL: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4x^2 - 3x$

= 1

f(x)=5x-4 [this equation is taken as equality for x=1 lies there]

f(1)= 1

Since, $\lim_{x \to 1} f(x)$ = f(1)

∴ f is continuous at x=1.

Question 10.
Prove that

f(x) = $\begin{cases} x-1, & \text{when } 1 \le x < 2; \\ 2x-3, & \text{when } 2 \le x \le 3 \end{cases}$ is continuous at x=2

Answer:
LHL: $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x - 1$

= 1

RHL: $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 2x - 3$

= 1

f(x)=2x-3 [this equation is taken as equality for x=1 lies there]

f(2)= 1

Since, $\lim_{x \to 2} f(x)$ = f(2)

∴ f is continuous at x=2.

Question 11.
Prove that

f(x) = $\begin{cases} \cos x, & \text{when } x \ge 0; \\ -\cos x, & \text{when } x < 0 \end{cases}$ is discontinuous at x=0

Answer:
LHL: $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \cos x$

= 1

RHL: $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} -\cos x$

= -1

$\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$

∴ f(x) is discontinuous at x=0

Question 12.
Prove that

$f(x) = \begin{cases} \dfrac{|x-a|}{x-a}, & \text{when } x \neq a; \\ 1, & \text{when } x = a \end{cases}$ is discontinuous at x=a

Answer:
LHL: $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^-} \frac{|x-a|}{x-a}$

$= \lim\limits_{x \to a^-} \frac{-(x-a)}{x-a}$

= -1

RHL: $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^+} \frac{|x-a|}{x-a}$

$= \lim\limits_{x \to a^+} \frac{(x-a)}{x-a}$

= 1

$\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

∴ f(x) is discontinuous at x=a

Question 13.
Prove that

f(x) = $\begin{cases} \frac{1}{2}(x - |x|), \text{ when } x \neq 0; \\ 2, \text{ when } x = 0 \end{cases}$ is discontinuous at x=0

Answer:
LHL: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{2}(x - |x|)$

$= \lim_{x\to 0^-} \frac{1}{2}(x - (-x))$

$= \lim_{x\to 0^-} 2x$

= 0

RHL: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{2}(x - |x|)$

$= \lim_{x\to 0^-} \frac{1}{2}(x - (x))$

= 0

f(0)=2

Since, $\lim_{x\to 0} f(x) \neq$ f(0)

∴ f is discontinuous at x=0.

Question 14.
Prove that

f(x) = $\begin{cases} \sin\frac{1}{x}, \text{ when } x \neq 0; \\ 0, \text{ when } x = 0; \end{cases}$ is discontinuous at x=0

Answer:
$\lim_{x\to 0} \sin\frac{1}{x} = 0$

$\sin\frac{1}{x}$ is bounded function between -1 and +1.

Also, f(0)=0

Since, $\lim_{x\to 0} f(x) = f(0)$

Hence, f is a continuous function.

Question 15.
Prove that

$$f(x) = \begin{cases} 2x, & \text{when } x < 2; \\ 2, & \text{when } x = 2; \\ x^2, & \text{when } x > 2; \end{cases}$$ is discontinuous at x=2

Answer:
LHL: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x$

=4

RHL: $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2$

=4

f(2)=2

Since, $\lim_{x\to 0} f(x) \neq f(2)$

∴ f is discontinuous at x=2.

Question 16.
Prove that

$$f(x) = \begin{cases} -x, & \text{when } x < 0; \\ 1, & \text{when } x = 0; \\ x, & \text{when } x > 0; \end{cases}$$ is discontinuous at x=0

Answer:
LHL: $\lim_{x \to 0-} f(x) = -x$

=0

RHL: $\lim_{x \to 2+} f(x) = \lim_{x \to 2+} x$

=0

f(0)=1

Since, $\lim_{x \to 0} f(x) \neq f(0)$

∴ f is discontinuous at x=0.

Question 17.
Find the value of k for which

$$f(x) = \begin{cases} \frac{\sin 2x}{5x}, & \text{when } x \neq 0; \\ \lambda, & \text{when } x = 0 \end{cases}$$ is continuous at x=0

Answer:
Since, f(x) is continuous at x=0

$\Rightarrow \lim_{x \to 0} \frac{\sin 2x}{5x} = f(0)$

$\Rightarrow \frac{1}{5} \lim_{x \to 0} \frac{\sin 2x}{x} = \lambda$

$\Rightarrow \frac{1}{5} \times 2 = \lambda$

$\Rightarrow \lambda = \frac{2}{5}$

Question 18.
Find the value of λ for which

f(x) = $\begin{cases} \frac{x^2 - 2x - 3}{x + 1}, & \text{when } x \neq -1; \\ \lambda, & \text{when } x = -1 \end{cases}$ is continuous at x=-1

Answer:
Since, f(x) is continuous at x=0

$\Rightarrow \lim_{x \to -1} \frac{x^2 - 2x - 3}{x + 1} = f(0)$

$\Rightarrow \lim_{x \to -1} \frac{(x-3)(x+1)}{x+1} = \lambda$

$\Rightarrow \lim_{x \to -1} x - 3 = \lambda$

$\Rightarrow \lambda = -4$

Question 19.
For what valve of k is the following function continuous at x=2

f(x) = $\begin{cases} 2x + 1, & \text{when } x < 2 \\ k, & \text{when } x = 2 \\ 3x - 1, & \text{when } x > 2 \end{cases}$

Answer:
Since, f(x) is continuous at x=2

$\Rightarrow \lim_{x \to 2^-} 2x + 1 = \lim_{x \to 2^+} 3x - 1 = f(2)$

$\Rightarrow \lim_{x \to 2^-} 2x + 1 = f(2)$

$\Rightarrow$k = 5

Question 20.
For what valve of k is the following function

f(x) = $\begin{cases} \frac{x^2-9}{x-3}, & \text{when } x \neq 3; \\ k, & \text{when } x = 3 \end{cases}$ is continuous at x=3

Ans. k=6

Answer:
Since, f(x) is continuous at x=3

$\Rightarrow \lim\limits_{x\to3} \frac{x^2-9}{x-3}$ =f(3)

$\Rightarrow \lim\limits_{x\to3} \frac{(x-3)(x+3)}{x-3}$ =f(3)

$\Rightarrow \lim\limits_{x\to3}(x+3)$ =f(3)

$\Rightarrow$ k = 9

Question 21.
For what valve of k is the following function

f(x) = $\begin{cases} \frac{k\cos x}{\pi-2x}, & \text{if } x \neq \frac{\pi}{2}; \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$ is continuous at x= $\frac{\pi}{2}$

Ans. k=6

Answer:
f is continuous at x = $\frac{\pi}{2}$

$\Rightarrow \lim\limits_{x\to\frac{\pi}{2}} f(x)$= f($\frac{\pi}{2}$)

$\Rightarrow \lim\limits_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = 3$

$\Rightarrow \lim\limits_{h \to 0} \frac{k\cos(\frac{\pi}{2} - h)}{\pi - 2(\frac{\pi}{2} - h)} = 3$ [Here $x = \frac{\pi}{2} - h$]

$\Rightarrow \lim\limits_{h \to 0} \frac{k\sin h}{\pi - \pi + 2h} = 3$

$\Rightarrow \lim\limits_{h \to 0} \frac{k\sin h}{2h} = 3$

$\Rightarrow \frac{k}{2} \times 1 = 3$

$\Rightarrow k = 6$

Question 22.
Show that function:

$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$ is continuous at x=0

Answer:
$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x^2 \sin\frac{1}{x}$

As $\lim\limits_{x \to 0} x^2 = 0$ and $\sin(\frac{1}{x})$ is bounded function between -1 and +1.

$\therefore \lim\limits_{x \to 0} x^2 \sin\frac{1}{x} = 0$

Also, f(0)=0

Since, $\lim\limits_{x \to 0} f(x) = f(0)$

Hence, f is a continuous function.

Question 23.

Show that: $f(x) = \begin{cases} x+1, & \text{if } x \geq 1; \\ x^2+1, & \text{if } x < 1 \end{cases}$ is continuous at x=1

Answer:

: LHL: $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 + 1$

= 2

RHL: $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 1^+} x + 1$

= 2

f(1)= 2

Since, $\lim\limits_{x \to 1} f(x)$ = f(1)

∴ f is continuous at x=1.

Question 24.

Show that: $f(x) = \begin{cases} x^3-3, & \text{if } x \leq 2; \\ x^2+1, & \text{if } x > 2 \end{cases}$ is continuous at x=2

Answer:

: LHL: $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 1^-} x^3 - 3$

= 5

RHL: $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} x^2 + 1$

= 5

f(2)= 5

Since, $\lim\limits_{x \to 2} f(x)$ = f(2)

∴ f is continuous at x=2.

Question 25.
Find the values of a and b such that the following functions continuous.

$$\begin{cases} 5, & \text{when } x \le \\ ax + b, & \text{when } 2 < x < 10 \\ 21, & \text{when } x \ge 10 \end{cases}$$

Answer:
f is continuous at x=2

$\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2+} f(x)$ = f(2)

$\lim\limits_{x\to 2-} (5) = \lim\limits_{x\to 2+} [ax + b]$ = 5

$\Rightarrow$ 2a+b=5 .... (1)

f is continuous at x=10

$\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2+} f(x)$ = f(2)

$\lim\limits_{x\to 2-} (21) = \lim\limits_{x\to 2+} [ax + b]$ = 21

$\Rightarrow$ 10a+b=21 .... (1)

(1) - (2)

-8a = -16

a = 2

Putting a in 1

b=1

Question 26.

Find the values of a and b such that the following functions ƒ, defined as

$$\begin{cases} a\sin\frac{\pi}{2}(x+1), & x \le 0; \\ \dfrac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$ is continuous at x=0

Answer:
: f is continuous at x=0

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x)$$

$$\lim_{x\to 0-}\left(a\sin\tfrac{\pi}{2}(x+1)\right) = \lim_{x\to 0+}\left[\frac{\tan x-\sin x}{x^3}\right]$$

$$\left(a\sin\tfrac{\pi}{2}(0+1)\right) = \lim_{x\to 0+}\left[\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}\right]$$

$$a = \lim_{x\to 0+}\left[\frac{\sin x\left(\frac{1}{\cos x}-1\right)}{x^3}\right]$$

$$= \lim_{x\to 0+}\left[\frac{\sin x\left(\frac{1}{\cos x}-1\right)}{x^3}\right]$$

$$= \lim_{x\to 0+}\left[\frac{\sin x(1-\cos x)}{\cos x\, x^3}\right]$$

$$= \lim_{x\to 0+}\left[\frac{\sin x \,.2\sin^2\frac{x}{2}}{\cos x\, x^3}\right]$$

$$= \lim_{x\to 0+}\left[\frac{\sin x \,.2\sin^2\frac{x}{2}}{x\,.x^2}\right] \times \frac{1}{\cos x}$$

$$= 1 \times 2 \times \frac{1}{4} \times 1$$

$$= \frac{1}{2}$$

Question 27.
Prove that the function ƒ given ƒ(x)=|x-3|, x ϵ R is continuous but not differentiable at x=3

Answer:

f(x)=|x-3|

Since every modulus function is continuous for all real x, f(x) is continuous at x=3.

f(x) = $f(x) = \begin{cases} 3 - x, x < 0 \\ x - 3, x \geq 0 \end{cases}$

To prove differentiable , we will use the following formula.

$\lim\limits_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ = $\lim\limits_{x \to a^-} \frac{f(x)-f(a)}{x-a}$ = f(a)

L.H.L $\lim\limits_{x \to a^+} \frac{f(x)-f(a)}{x-a}$

= $\lim\limits_{x \to 3^+} \frac{x-3-0}{x-3}$

= $\lim\limits_{x \to 3^+} \frac{x-3}{x-3}$

= 1

R.H.L: $\lim\limits_{x \to a^-} \frac{f(x)-f(a)}{x-a}$

= $\lim\limits_{x \to 3^+} \frac{3-x-0}{x-3}$

= $\lim\limits_{x \to 3^+} \frac{3-x}{x-3}$

= -1

Since, L.H.L $\neq$ R.H.L, f(x) is not differentiable at x=5