Exercise 18c

Question 1.
Form the differential equation of the family of straight lines y=mx+c, where m and c are arbitrary constants.

Answer:
The equation of a straight line is represented as,

Y = mx + c

Differentiating the above equation with respect to x,

$$\frac{dy}{dx} = m$$

Differentiating the above equation with respect to x,

$$\frac{d^2y}{dx^2} = 0$$

This is the differential equation of the family of straight lines y=mx+c, where m and c are arbitrary constants

Question 2.
Form the differential equation of the family of concentric circles $x^2+y^2=a^2$, where a>0 and a is a parameter.

Answer:
Now, in the general equation of of the family of concentric circles $x^2+y^2=a^2$, where a>0, 'a' represents the radius of the circle and is an arbitrary constant.

The given equation represents a family of concentric circles centered at the origin.

$x^2+y^2=a^2$

Differentiating the above equation with respect to x on both sides, we have,

$2x + 2y\frac{dy}{dx} = 0$ (As a>0, derivative of a with respect to x is 0.)

$$x + y\frac{dy}{dx} = 0$$

Question 3.
Form the differential equation of the family of curves, y=a sin (bx+c), Where a and c are parameters.

Answer:
Equation of the family of curves, y=a sin (bx+c), Where a and c are parameters.

Differentiating the above equation with respect to x on both sides, we have,

$y = a\sin(bx + c)$ (1)

$$\frac{dy}{dx} = ab\cos(bx + c)$$

$\frac{d^2y}{dx^2} = -ab^2\sin(bx + c)$ (Substituting equation 1 in this equation)

$$\frac{d^2y}{dx^2} = -b^2y$$

$$\frac{d^2y}{dx^2} + b^2y = 0$$

This is the required differential equation.

Question 4.
Form the differential equation of the family of curves x=A cos nt+ Bsin nt, where A and B are arbitrary constants.

Answer:
Equation of the family of curves, x=A cos nt+ Bsin nt, where A and B are arbitrary constants.

Differentiating the above equation with respect to t on both sides, we have,

$x = A\cos(nt) + B\sin(nt)$ (1)

$$\frac{dx}{dt} = -An\sin(nt) + Bn\cos(nt)$$

$$\frac{d^2x}{dt^2} = -An^2\cos(nt) - Bn^2\sin(nt)$$

$\frac{d^2x}{dt^2} = -n^2\left(A\cos(nt) + B\sin(nt)\right)$ (Substituting equation 1 in this equation)

$$\frac{d^2x}{dt^2} = -n^2x$$

$$\frac{d^2x}{dt^2} + n^2x = 0$$

This is the required differential equation.

Question 5.
Form the differential equation of the family of curves $y=ae^{bx}$, where a and b are arbitrary constants.

Answer:
Equation of the family of curves, $y=ae^{bx}$, where a and b are arbitrary constants.

Differentiating the above equation with respect to x on both sides, we have,

$y = ae^{bx}$ (1)

$\frac{dy}{dx} = abe^{bx}$ (2)

$$\frac{d^2y}{dx^2} = ab^2e^{bx}$$

$y\frac{d^2y}{dx^2} = ab^2e^{bx}(ae^{bx})$ (Multiplying both sides of the equation by y)

$y\frac{d^2y}{dx^2} = \left(abe^{bx}\right)^2$ (Substituting equation 2 in this equation)

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

This is the required differential equation.

Question 6.
Form the differential equation of the family of curves $y^2=m(a^2-x^2)$, where a and m are parameters.

Answer:
Equation of the family of curves, $y^2=m(a^2-x^2)$, where a and m are parameters.

Differentiating the above equation with respect to x on both sides, we have,

$$2y\frac{dy}{dx} = m(-2x)$$

$$y\frac{dy}{dx} = -mx$$

$m = -\frac{y}{x}\frac{dy}{dx}$ (1)

Differentiating the above equation with respect to x on both sides,

$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -m$ (2)

From equations (1) and (2),

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

This is the required differential equation.

Question 7.
Form the differential equation of the family of curves given by $(x-a)^2+2y^2=a^2$, where a is an arbitrary constant.

Answer:
Equation of the family of curves, $(x-a)^2+2y^2=a^2$, where a is an arbitrary constant.

$$x^2 - 2ax + a^2 + 2y^2 = a^2$$

$x^2 - 2ax + 2y^2 = 0$ (1)

Differentiating the above equation with respect to x on both sides, we have,

$$2x - 2a + 4y\frac{dy}{dx} = 0$$

$$x - a + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{a - x}{2y}$$

$$\frac{dy}{dx} = \frac{a - x}{2y}\left(\frac{2x}{2x}\right)$$

$\frac{dy}{dx} = \frac{2ax - 2x^2}{4xy}$ (Substituting 2ax from equation 1)

$$\frac{dy}{dx} = \frac{x^2 + 2y^2 - 2x^2}{4xy}$$

$$\frac{dy}{dx} = \frac{2y^2 - x^2}{4xy}$$

This is the required differential equation.

Question 8.
Form the differential equation of the family of curves given by $x^2+y^2-2ay=a^2$, where a is an arbitrary constant.

Answer:
Equation of the family of curves, $x^2+y^2-2ay=a^2$, where a is an arbitrary constant.

$$x^2 - 2ax + a^2 + 2y^2 = a^2$$

$x^2 - 2ax + 2y^2 = 0$ (1)

Differentiating the above equation with respect to x on both sides, we have,

$$2x - 2a + 4y\frac{dy}{dx} = 0$$

$$x - a + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}=\frac{a-x}{2y}$$

$$\frac{dy}{dx}=\frac{a-x}{2y}\left(\frac{2x}{2x}\right)$$

$\frac{dy}{dx}=\frac{2ax-2x^2}{4xy}$ (Substituting 2ax from equation 1)

$$\frac{dy}{dx}=\frac{x^2+\ 2y^2\ -\ 2x^2}{4xy}$$

$$\frac{dy}{dx}=\frac{2y^2\ -\ x^2}{4xy}$$

This is the required differential equation.

Question 9.
Form the differential equation of the family of all circles touching the y-axis at the origin.

Answer:
Equation of the family of all circles touching the y-axis at the origin can be represented by

$(x-a)^2+y^2=a^2$, where a is an arbitrary constants.

$\mathbf{(x-a)^2+y^2\ =\ a^2}$ (1)

Differentiating the above equation with respect to x on both sides, we have,

$$2(x-a)+2y\frac{dy}{dx}=0$$

$$x-a+y\frac{dy}{dx}=0$$

$$a=x+y\frac{dy}{dx}$$

Substituting the value of a in equation (1)

$$\left(y\frac{dy}{dx}\right)^2+y^2=\left(x+y\frac{dy}{dx}\right)^2$$

$$\left(y\frac{dy}{dx}\right)^2+y^2=x^2+xy\frac{dy}{dx}+\left(y\frac{dy}{dx}\right)^2$$

Rearranging the above equation

$$y^2-x^2-xy\frac{dy}{dx}=0$$

This is the required differential equation.

Question 10.
From the differential equation of the family of circles having centers on y-axis and radius 2 units.

Answer:
Equation of the family of circles having centers on y-axis and radius 2 units can be represented by

$(x)^2+(y-a)^2=4$, where a is an arbitrary constant.

$$(y-a)^2+x^2 = 4 \text{ (1)}$$

Differentiating the above equation with respect to x on both sides, we have,

$$\mathbf{2}(x)+2(y-a)\frac{dy}{dx}=0$$

$$x-a\frac{dy}{dx}+y\frac{dy}{dx}=0$$

$$a=\frac{x+y\frac{dy}{dx}}{\frac{dy}{dx}}$$

Substituting the value of a in equation (1)

$$x^2+\left(y-\frac{x+y\frac{dy}{dx}}{\frac{dy}{dx}}\right)^2 = 4$$

$$x^2 + \left(\frac{y\frac{dy}{dx} - x - y\frac{dy}{dx}}{\frac{dy}{dx}} \right)^2 = 4$$

$$x^2 + \left(\frac{x}{\frac{dy}{dx}} \right)^2 = 4$$

Rearranging the above equation

$$x^2 \left(1 + \frac{1}{\left(\frac{dy}{dx}\right)^2} \right) = 4$$

This is the required differential equation.

Question 11.
Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes

Answer:
Equation of the family of circles in the second quadrant and touching the coordinate axes

can be represented by

$(x – (-a))^2+(y – a)^2= a^2$, where a is an arbitrary constants.

$\mathbf{(x + a)^2 + (y - a)^2 = a^2}$ (1)

Differentiating the above equation with respect to x on both sides, we have,

$$2(x + a) + 2(y - a)\frac{dy}{dx} = 0$$

$$x + a - a\frac{dy}{dx} + y\frac{dy}{dx} = 0$$

$$a = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx} - 1}$$

Substituting the value of a in equation (1)

$$\left(x + \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx} - 1}\right)^2 + \left(y - \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx} - 1}\right)^2 = \left(\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx} - 1}\right)^2$$

$$\left(\frac{x\frac{dy}{dx} - x + x + y\frac{dy}{dx}}{\frac{dy}{dx} - 1}\right)^2 + \left(\frac{y\frac{dy}{dx} - y - x - y\frac{dy}{dx}}{\frac{dy}{dx} - 1}\right)^2 = \left(\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx} - 1}\right)^2$$

$$\left(x\frac{dy}{dx} - x + x + y\frac{dy}{dx}\right)^2 + \left(y\frac{dy}{dx} - y - x - y\frac{dy}{dx}\right)^2 = \left(x + y\frac{dy}{dx}\right)^2$$

$$\left(\frac{dy}{dx}\right)^2 (x + y)^2 + (-y - x)^2 = \left(x + y\frac{dy}{dx}\right)^2$$

$$\left(\frac{dy}{dx}\right)^2 (x + y)^2 + (y + x)^2 = \left(x + y\frac{dy}{dx}\right)^2$$

Rearranging the above equation

$$(x + y)^2\{\left(\frac{dy}{dx}\right)^2 + 1\} = \left(x + y\frac{dy}{dx}\right)^2$$

This is the required differential equation.

Question 12.
Form the differential equation of the family of circles having centers on the x-axis and radius unity.

Answer:
Equation of the family of circles having centers on the x-axis and radius unity can be represented by

$(x - a)^2+(y)^2= 1$, where a is an arbitrary constants.

$(\mathbf{x}-\mathbf{a})^2+\mathbf{y}^2 = \mathbf{1}$ (1)

Differentiating the above equation with respect to x on both sides, we have,

$$2(x-a)+2(y)\frac{dy}{dx}=0$$

$$x-a+y\frac{dy}{dx}=0$$

$$a=x+y\frac{dy}{dx}$$

Substituting the value of a in equation (1)

$$\left(x-x-y\frac{dy}{dx}\right)^2+y^2 = 1$$

$$\left(y\frac{dy}{dx}\right)^2+y^2 = 1$$

This is the required differential equation.

Question 13.
Form the differential equation of the family of circles passing through the fixed point (a,0) and (-a,0), where a is the parameter.

Answer:
Now, it is not necessary that the centre of the circle will lie on origin in this case. Hence let us assume the coordinates of the centre of the circle be (0, h). Here, h is an arbitrary constant.

Also, the radius as calculated by the Pythagoras theorem will be $a^2 + h^2$.

Hence, the equation of the family of circles passing through the fixed point (a,0) and (-a,0), where a is the parameter can be represented by

$(x)^2+(y-h)^2= a^2 + h^2$, where a is an arbitrary constants.

$x^2+(y-h)^2= a^2+h^2$ (1)

Differentiating the above equation with respect to x on both sides, we have,

$$2(x) + 2(y - h)\frac{dy}{dx} = 0$$

$$x - h\frac{dy}{dx} + y\frac{dy}{dx} = 0$$

$$h = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}$$

Substituting the value of a in equation (1)

$$x^2 + \left(y - \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}\right)^2 = a^2 + \left(\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}\right)^2$$

$$x^2 + \left(\frac{y\frac{dy}{dx} - x - y\frac{dy}{dx}}{\frac{dy}{dx}}\right)^2 = a^2 + \left(\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}\right)^2$$

$$x^2\left(\frac{dy}{dx}\right)^2 + (x)^2 = a^2\left(\frac{dy}{dx}\right)^2 + \left(x + y\frac{dy}{dx}\right)^2$$

$$x^2\left(\frac{dy}{dx}\right)^2 + (x)^2 = a^2\left(\frac{dy}{dx}\right)^2 + x^2 + 2xy\frac{dy}{dx} + \left(y\frac{dy}{dx}\right)^2$$

$$x^2\left(\frac{dy}{dx}\right)^2 = a^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + \left(y\frac{dy}{dx}\right)^2$$

$$(x^2 - a^2 - y^2)\left(\frac{dy}{dx}\right) = 2xy$$

This is the required differential equation.

Question 14.
Form the differential equation of the family of parabolas having a vertex at the origin and axis along positive y-axis.

Answer:

Equation of the family of parabolas having a vertex at the origin and axis along positive y-axis can be represented by

$(x)^2$ = 4ay, where a is an arbitrary constants.

$x^2 = 4ay$ (1)

Differentiating the above equation with respect to x on both sides, we have,

$$2(x) = 4(a)\frac{dy}{dx}$$

$$x = 2a\frac{dy}{dx}$$

$$a = \frac{x}{2\frac{dy}{dx}}$$

Substituting the value of a in equation (1)

$$x^2 = 4\frac{x}{2\frac{dy}{dx}}y$$

$$x\frac{dy}{dx} = 2y$$

This is the required differential equation.

Question 15.
Form the differential equation of the family of an ellipse having foci on the y-axis and centers at the origin.

Answer:
Equation of the family of an ellipse having foci on the y-axis and centers at the origin can be represented by

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (1)

Differentiating the above equation with respect to x on both sides, we have,

$$\frac{2x}{b^2}+\frac{2y}{a^2}\frac{dy}{dx}=0$$

$$\frac{x}{b^2}+\frac{y}{a^2}\frac{dy}{dx}=0$$

$$\frac{y}{a^2}\frac{dy}{dx}=-\frac{x}{b^2}$$

$$\frac{y}{x}\frac{dy}{dx}=-\frac{a^2}{b^2}$$

Again differentiating the above equation with respect to x on both sides, we have,

$$\frac{y}{x}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{\frac{dy}{dx}x-y\frac{dx}{dx}}{x^2}\right)=0$$

$$xy\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{dy}{dx}x-y\frac{dx}{dx}\right)=0$$

Rearranging the above equation

$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

This is the required differential equation.

Question 16.
Form the differential equation of the family of hyperbolas having foci on the x-axis and centers at the origin.

Answer:
Equation of the family of an ellipse having foci on the y-axis and centers at the origin can be represented by

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (1)

Differentiating the above equation with respect to x on both sides, we have,

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{x}{a^2} - \frac{y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{y}{b^2}\frac{dy}{dx} = \frac{x}{a^2}$$

$$\frac{y}{x}\frac{dy}{dx} = \frac{b^2}{a^2}$$

Again differentiating the above equation with respect to x on both sides, we have,

$$\frac{y}{x}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{\frac{dy}{dx}x - y\frac{dx}{dx}}{x^2}\right) = 0$$

$$xy\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{dy}{dx}x - y\frac{dx}{dx}\right) = 0$$

Rearranging the above equation

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

This is the required differential equation.