Exercise 3b

Question 1.
Define * on N by m * n = 1 cm (m, n). Show that * is a binary operation which is commutative as well as associative.

Answer:
* is an operation as m*n = LCM (m, n) where m, n $\in$ N. Let m = 2 and b = 3 two natural numbers.

m*n = 2*3

= LCM (2, 3)

= 6$\in$ N

So, * is a binary operation from $N \times N \rightarrow N$.

For commutative,

n*m = 3*2

= LCM (3, 2)

= 6$\in$ N

Since m*n = n*m, hence * is commutative operation.

Again, for associative, let p = 4

m*(n*p) = 2*LCM (3, 4)

= 2*12

= LCM (2, 12)

= 12$\in$ N

(m*n) *p = LCM (2, 3) *4

= 6*4

= LCM (6, 4)

= 12$\in$ N

As m*(n*p) = (m*n) *p, hence * an associative operation.

Question 2.
Define * on Z by a * b = a – b + ab. Show that * is a binary operation on Z which is neither commutative nor associative.

Answer:
* is an operation as a*b = a-b + ab where a, b $\in$ Z. Let $a = \frac{1}{2}$ and b = 2 two integers.

$$a*b = \frac{1}{2}*2 = \frac{1}{2} - 2 + \frac{1}{2}.2 \Rightarrow \frac{1-4}{2} + 1 = \frac{-3+2}{2} \Rightarrow \frac{-1}{2} \in Z$$

So, * is a binary operation from $Z \times Z \rightarrow Z$.

For commutative,

$$b*a = 2 - \frac{1}{2} + 2.\frac{1}{2} = \frac{4-1}{2} + 1 \Rightarrow \frac{3+2}{2} = \frac{5}{2} \in Z$$

Since a*b ≠ b*a, hence * is not commutative operation.

Again for associative,

a*(b*c) = a*(b-c+ bc)

= a- (b- c+ bc) +a (b- c+ bc)

= a-b+ c- bc+ ab- ac+ abc

(a*b) *c = (a-b+ ab) *c

= a- b+ ab-c+ (a- b+ ab) c

= a-b-c+ ab+ ac- bc+ abc

As a*(b*c) ≠ (a*b) *c, hence * not an associative operation.

Question 3.
Define * on Z by a * b = a + b - ab. Show that * is a binary operation on Z which is commutative as well as associative.

Answer:
* is an operation as a*b = a+ b - ab where a, b $\in$ Z. Let $a = \frac{1}{2}$ and b = 2 two integers.

$$a*b = \frac{1}{2}*2 = \frac{1}{2}+2-\frac{1}{2}.2 \Rightarrow \frac{1+4}{2}-1 = \frac{5-2}{2} \Rightarrow \frac{3}{2} \in Z$$

So, * is a binary operation from $Z \times Z \rightarrow Z$.

For commutative,

$$b*a = 2+\frac{1}{2}-2.\frac{1}{2} = \frac{4+1}{2}-1 \Rightarrow \frac{5-2}{2} = \frac{3}{2} \in Z$$

Since a*b = b*a, hence * is a commutative binary operation.

Again for associative,

a*(b*c) = a*(b+ c- bc)

= a+ (b+ c- bc) -a (b+ c- bc)

= a+ b+ c- bc- ab- ac+ abc

(a*b) *c = (a+ b- ab) *c

= a+ b- ab+ c- (a+ b- ab) c

= a+ b+ c- ab- ac- bc+ abc

As a*(b*c) = (a*b) *c, hence * an associative binary operation.

Question 4.

Consider a binary operation on Q – {1}, defined by a * b = a + b - ab.

(i) Find the identity element in Q – {1}.

(ii) Show that each a $\in$ Q - {1} has its inverse.

Answer:
(i) For a binary operation *, e identity element exists if a*e = e*a = a. As a*b = a+ b- ab

a*e = a+ e- ae (1)

e*a = e+ a- e a (2)

using a*e = a

a+ e- ae = a

e-ae = 0

e(1-a) = 0

either e = 0 or a = 1 as operation is on Q excluding 1 so a≠1, hence e = 0.

So identity element e = 0.

(ii) for a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 0

a+ b- ab = 0

b(1-a) = -a

$$b = \frac{-a}{(1-a)} \Rightarrow \frac{a}{(a-1)}$$

$$a^{-1} = \frac{a}{(a-1)}$$

Question 5.

Let Q_0 be the set of all nonzero rational numbers. Let * be a binary operation on Q_0, defined by $a * b = \frac{ab}{4}$ for all a, b $\in Q_0$.

(i) Show that * is commutative and associative.

(ii) Find the identity element in Q_0.

(iii) Find the inverse of an element a in Q_0.

Answer:
(i) For commutative binary operation, a*b = b*a.

$$a*b = \frac{ab}{4} \text{ and } b*a = \frac{ba}{4}$$

as multiplication is commutative ab = ba so a*b = b*a. Hence * is commutative binary operation.

For associative binary operation, a*(b*c) = (a*b) *c

$$a*(b*c) = a*\frac{bc}{4} \Rightarrow \frac{a.\frac{bc}{4}}{4} = \frac{abc}{16}$$

$$(a*b)*c = \frac{ab}{4}*c \Rightarrow \frac{\frac{ab}{4}.c}{4} = \frac{abc}{16}$$

Since a*(b*c) = (a*b) *c, hence * is an associative binary operation.

(ii) For a binary operation *, e identity element exists if a*e = e*a = a. As a*b = a+ b- ab

$$a*e = \frac{ae}{4} \quad (1)$$

$$e*a = \frac{ea}{4} \quad (2)$$

using a*e = a

$$\frac{ae}{4} = a \Rightarrow \frac{ae}{4} - a = 0 \Rightarrow \frac{a}{4}(e-4) = 0$$

Either a = 0 or e = 4 as given a≠0, so e = 4.

Identity element e = 4.

(iii) For a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 4

$$\frac{ab}{4} = 4 \Rightarrow b = \frac{16}{a}$$

$$a^{-1} = \frac{16}{a}$$

Question 6.

On the set Q^+ of all positive rational numbers, define an operation * on Q^+ by $a * b = \frac{ab}{2}$ for all a, b ∈ Q^+. Show that

(i) * is a binary operation on Q^+,

(ii) * is commutative,

(iii) * is associative.

Find the identity element in Q^+ for *. What is the inverse of a ∈ Q^+?

Answer:

(i) * is an operation as $a*b = \frac{ab}{2}$ where a, b ∈ Q^+. Let $a = \frac{1}{2}$ and b = 2 two integers.

$$a*b = \frac{1}{2}*2 \Rightarrow 1 \in Q^+$$

So, * is a binary operation from $Q^+ \times Q^+ \to Q^+$.

(ii) For commutative binary operation, a*b = b*a.

$$b*a = 2.\frac{1}{2} \Rightarrow 1 \in Q^+$$

Since a*b = b*a, hence * is a commutative binary operation.

(iii) For associative binary operation, a*(b*c) = (a*b) *c.

$$a*(b*c) = a*\frac{bc}{2} \Rightarrow \frac{a.\frac{bc}{2}}{2} = \frac{abc}{4}$$

$$(a*b)*c = \frac{ab}{2}*c \Rightarrow \frac{\frac{ab}{2}.c}{2} = \frac{abc}{4}$$

As a*(b*c) = (a*b) *c, hence * is an associative binary operation.

For a binary operation *, e identity element exists if a*e = e*a = a.

$a*e = \frac{ae}{2}$ (1)

$e*a = \frac{ea}{2}$ (2)

using a*e = a

$$\frac{ae}{2} = a \Rightarrow \frac{ae}{2} - a = 0 \Rightarrow \frac{a}{2}(e-2) = 0$$

Either a = 0 or e = 2 as given a≠0, so e = 2.

For a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 2

$$\frac{ab}{2} = 2 \Rightarrow b = \frac{4}{a}$$

$$a^{-1} = \frac{4}{a}$$

Question 7.

Let Q^+ be the set of all positive rational numbers.

(i) Show that the operation * on Q^+ defined by $a * b = \frac{1}{2}(a + b)$ is a binary operation.

(ii) Show that * is commutative.

(iii) Show that * is not associative.

Answer:

(i) * is an operation as $a*b = \frac{1}{2}(a+b)$ where a, b $\in$ Q^+. Let $a = 1$ and b = 2 two integers.

$$a*b = \frac{1}{2}(1+2) \Rightarrow \frac{3}{2} \in Q^+$$

So, * is a binary operation from $Q^+ \times Q^+ \rightarrow Q^+$.

(ii) For commutative binary operation, a*b = b*a.

$$b*a = \frac{1}{2}(2+1) \Rightarrow \frac{3}{2} \in Q^+$$

Since a*b = b*a, hence * is a commutative binary operation.

(iii) For associative binary operation, a*(b*c) = (a*b) *c.

$$a*(b*c) = a*\frac{1}{2}(b+c) \Rightarrow \frac{1}{2}\left(a+\frac{b+c}{2}\right) = \frac{1}{4}(2a+b+c)$$

$$(a*b)*c = \frac{1}{2}(a+b)*c \Rightarrow \frac{1}{2}\left(\frac{a+b}{2}+c\right) = \frac{1}{4}(a+b+2c)$$

As a*(b*c) ≠(a*b) *c, hence * is not associative binary operation.

Question 8.

Let Q be the set of all rational numbers. Define an operation on Q – {-1} by a * b = a + b + ab.

Show that

(i) * is a binary operation on Q – {-1},

(ii) * is Commutative,

(iii) * is associative,

(iv) zero is the identity element in Q – {-1} for *,

(v) $a^{-1} = \left(\frac{-a}{1+a}\right)$, where a ∈ Q – {-1}.

Answer:

(i) * is an operation as a*b = a+ b+ ab where a, b ∈ Q- {-1}. Let $a = 1$ and $b = \frac{-3}{2}$ two rational numbers.

$$a*b = 1+\frac{-3}{2}+1.\frac{-3}{2} \Rightarrow \frac{2-3}{2}-\frac{3}{2} = \frac{-1-3}{2} \Rightarrow \frac{-4}{2} = -2 \in Q-\{-1\}$$

So, * is a binary operation from $Q-\{-1\} \times Q-\{-1\} \rightarrow Q$- {-1}.

(ii) For commutative binary operation, a*b = b*a.

$$b*a = \frac{-3}{2}+1+\frac{-3}{2}.1 \Rightarrow \frac{-3+2}{2}-\frac{3}{2} = \frac{-1-3}{2} \Rightarrow \frac{-4}{2} = -2 \in Q-\{-1\}$$

Since a*b = b*a, hence * is a commutative binary operation.

(iii) For associative binary operation, a*(b*c) = (a*b) *c

a+(b*c) = a*(b+ c+ bc) = a+ (b+ c+ bc) +a(b+ c+ bc)

= a+ b+ c+ bc+ ab+ ac+ abc

(a*b) *c = (a+ b+ ab) *c = a+ b+ ab+ c+ (a+ b+ ab) c

= a+ b+ c+ ab+ ac+ bc+ abc

Now as a*(b*c) = (a*b) *c, hence an associative binary operation.

(iv) For a binary operation *, e identity element exists if a*e = e*a = a. As a*b = a+ b- ab

a*e = a+ e+ ae (1)

e*a = e+ a +e a (2)

using $a*e = a$

$a+ e+ ae = a$

$e+ ae = 0$

$e(1+a) = 0$

either $e = 0$ or $a = -1$ as operation is on Q excluding -1 so $a\neq-1$, hence $e = 0$.

So identity element $e = 0$.

(v) for a binary operation * if e is identity element then it is invertible with respect to * if for an element b, $a*b = e = b*a$ where b is called inverse of * and denoted by a^{-1}.

$a*b = 0$

$a+ b+ ab = 0$

$b(1+a) = -a$

$$b = \frac{-a}{(1+a)}$$

$$a^{-1} = \frac{-a}{(a+1)}$$

Question 9.
Let $A = N \times N$. Define * on A by $(a, b) * (c, d) = (a + c, b + d)$.

Show that

(i) A is closed for *,

(ii) * is commutative,

(iii) * is associative,

(iv) identity element does not exist in A.

Answer:

(i) A is said to be closed on * if all the elements of (a, b) *(c, d) = (a+ c, b+ d) belongs to N×N for A = N×N.

Let a = 1, b = 3, c = 8, d = 2

(1, 3) * (8, 2) = (1+8, 3+2)

= (9, 5) ∈N×N

Hence A is closed for *.

(ii) For commutative,

(c, d) *(a, b) = (c+ a, d+ b)

As addition is commutative a+ c = c+ a and b+ d = d+ b, hence * is commutative binary operation.

(iii) For associative,

(a, b) *((c, d) *(e, f)) = (a, b) *(c+ e, d+ f)

= (a+ c+ e, b+ d+ f)

((a, b) *(c, d)) *(e, f) = (a+ c, b+ d) *(e, f)

= (a+ c+ e, b+ d+ f)

As (a, b) *((c, d) *(e, f)) = ((a, b) *(c, d)) *(e, f), hence * is an associative binary operation.

(iv) For identity element (e_1, e_2), (a, b) *(e_1, e_2) = (e_1, e_2) *(a, b) = (a, b) in a binary operation.

(a, b) *(e_1, e_2) = (a, b)

$(a+e_1, b+e_2)$ = (a, b)

(e_1, e_2) = (0, 0)

As (0,0) ∉N×N, hence identity element does not exist for *.

Question 10.
Let A = (1, -1, i, -i) be the set of four 4th roots of unity. Prepare the composition table for multiplication on A and show that

(i) A is closed for multiplication,

(ii) multiplication is associative on A,

(iii) multiplication is commutative on A,

(iv) 1 is the multiplicative identity,

(v) every element in A has its multiplicative inverse.

Answer:
(i) A is said to be closed on * if all the elements of a*b ∈A. composition table is

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

(as $i^2 = -1$)

As table contains all elements from set A, A is close for multiplication operation.

(ii) For associative, a× (b× c) = (a× b) ×c

1× (-i× i) = 1×1 = 1

(1× -i) ×i = -i× i = 1

$a \times (b \times c) = (a \times b) \times c$, so A is associative for multiplication.

(iii) For commutative, $a \times b = b \times a$

$1 \times -1 = -1$

$-1 \times 1 = -1$

$a \times b = b \times a$, so A is commutative for multiplication.

(iv) For multiplicative identity element e, $a \times e = e \times a = a$ where $a \in A$.

$a \times e = a$

$a(e-1) = 0$

either $a = 0$ or $e = 1$ as $a \neq 0$ hence $e = 1$.

So, multiplicative identity element $e = 1$.

(v) For multiplicative inverse of every element of A, $a*b = e$ where $a, b \in A$.

$1 \times b_1 = 1$

$b_1 = 1$

$-1 \times b_2 = 1$

$b_2 = -1$

$i \times b_3 = 1$

$$b_3 = \frac{1}{i} \Rightarrow \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} \Rightarrow \frac{i}{-1} = -i$$

$-i \times b_4 = 1$

$$b_4 = \frac{1}{-i} \Rightarrow \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} \Rightarrow \frac{i}{-(-1)} = i$$

So, multiplicative inverse of A = {1, -1, -i, i}