Exercise 16c

Question 1.

Prove that

$$\int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)} dx = \frac{\pi}{4}$$

Answer:

$$y = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{2\cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos x + \cos x - \sin x + \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\left((x)_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\sin x + \cos x} dx\right)$$

Let, sin x + cos x = t

⇒ (cos x – sin x) dx = dt

At x = 0, t = 1

At x = π/2, t = 1

$$y = \frac{1}{2}\left(\frac{\pi}{2} + \int_1^1 \frac{1}{t} dt\right)$$

$$y = \frac{1}{2}(\frac{\pi}{2} + (\ln t)_1^1$$

$$y = \frac{\pi}{4}$$

Question 2.
Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)} dx = \frac{\pi}{4}$$

Answer:
$y = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\left(\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}\right)} dx$$

$y = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\left(\sqrt{\cos x}+\sqrt{\sin x}\right)} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\left(\sqrt{\cos x} + \sqrt{\sin x}\right)} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)} dx$$

$$= \int_0^{\pi/2} 1\, dx$$

$$= (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 3.
Prove that

$$\int_0^{\pi/2} \frac{\sin^3 x}{\left(\sin^3 x + \cos^3 x\right)} dx = \frac{\pi}{4}$$

Answer:
$y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{sin^3\left(\frac{\pi}{2} - x\right)}{sin^3\left(\frac{\pi}{2} - x\right) + cos^3\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x} dx + \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x} dx$$

$$= \int_0^{\pi/2} \frac{sin^3 x + cos^3 x}{sin^3 x + cos^3 x} dx$$

$$= \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 4.
Prove that

$$\int_0^{\pi/2} \frac{\cos^3 x \, dx}{\left(\sin^3 x + \cos^3 x\right)} = \frac{\pi}{4}$$

Answer:
$y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$$

$$y = \int_0^{\pi/2} \frac{cos^3\left(\frac{\pi}{2} - x\right)}{sin^3\left(\frac{\pi}{2} - x\right) + cos^3\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x} dx + \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x} dx$$

$$2y = \int_0^{\pi/2} \frac{cos^3 x + sin^3 x}{sin^3 x + cos^3 x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 5.
Prove that

$$\int_0^{\pi/2} \frac{\sin^7 x}{\left(\sin^7 x + \cos^7 x\right)} dx = \frac{\pi}{4}$$

Answer:

$y = \int_0^{\pi/2} \frac{sin^7 x}{sin^7 x + cos^7 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{sin^7\left(\frac{\pi}{2} - x\right)}{sin^7\left(\frac{\pi}{2} - x\right) + cos^7\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{cos^7 x}{sin^7 x + cos^7 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{sin^7 x}{sin^7 x + cos^7 x} dx + \int_0^{\pi/2} \frac{cos^7 x}{sin^7 x + cos^7 x} dx$$

$$2y = \int_0^{\pi/2} \frac{sin^7 x + cos^7 x}{sin^7 x + cos^7 x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 6.
Prove that

$$\int_0^{\pi/2} \frac{\cos^4 x}{\left(\sin^4 x + \cos^4 x\right)} dx = \frac{\pi}{4}$$

Answer:

$y = \int_0^{\pi/2} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\cos^4\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} dx + \int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$$

$$2y = \int_0^{\pi/2} \frac{\cos^4 x + \sin^4 x}{\sin^4 x + \cos^4 x} dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 7.
Prove that

$$\int_0^{\pi/2} \frac{\cos^4 x}{\left(\sin^4 x + \cos^4 x\right)} dx = \frac{\pi}{4}$$

Answer:
$y = \int_0^{\pi/2} \frac{cos^4 x}{sin^4 x + cos^4 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{cos^4\left(\frac{\pi}{2} - x\right)}{sin^4\left(\frac{\pi}{2} - x\right) + cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{sin^4 x}{sin^4 x + cos^4 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cos^4 x}{sin^4 x + cos^4 x} dx + \int_0^{\pi/2} \frac{sin^4 x}{sin^4 x + cos^4 x} dx$$

$$2y = \int_0^{\pi/2} \frac{cos^4 x + sin^4 x}{sin^4 x + cos^4 x} dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 8.
Prove that

$$\int_0^{\pi/2} \frac{\cos^{1/4} x}{\left(\sin^{1/4} x + \cos^{1/4} x\right)} dx = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\pi/2} \frac{cos^{\frac{1}{4}} x}{sin^{\frac{1}{4}} x + cos^{\frac{1}{4}} x} dx \text{ ...(1)}$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{cos^{\frac{1}{4}}\left(\frac{\pi}{2} - x\right)}{sin^{\frac{1}{4}}\left(\frac{\pi}{2} - x\right) + cos^{\frac{1}{4}}\left(\frac{\pi}{2} - x\right)} dx$$

$$y = \int_0^{\pi/2} \frac{sin^{\frac{1}{4}} x}{sin^{\frac{1}{4}} x + cos^{\frac{1}{4}} x} dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cos^{\frac{1}{4}} x}{sin^{\frac{1}{4}} x + cos^{\frac{1}{4}} x} dx + \int_0^{\pi/2} \frac{sin^{\frac{1}{4}} x}{sin^{\frac{1}{4}} x + cos^{\frac{1}{4}} x} dx$$

$$2y = \int_0^{\pi/2} \frac{cos^{\frac{1}{4}} x + sin^{\frac{1}{4}} x}{sin^{\frac{1}{4}} x + cos^{\frac{1}{4}} x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 9.
Prove that

$$\int_0^{\pi/2} \frac{\sin^{3/2} x}{\left(\sin^{3/2} x + \cos^{3/2} x\right)} dx = \frac{\pi}{4}$$

Answer:

$y = \int_0^{\pi/2} \frac{sin^{\frac{3}{2}} x}{sin^{\frac{3}{2}} x + cos^{\frac{3}{2}} x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}{sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{cos^{\frac{3}{2}} x}{sin^{\frac{3}{2}} x + cos^{\frac{3}{2}} x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{sin^{\frac{3}{2}} x}{sin^{\frac{3}{2}} x + cos^{\frac{3}{2}} x} dx + \int_0^{\pi/2} \frac{cos^{\frac{3}{2}} x}{sin^{\frac{3}{2}} x + cos^{\frac{3}{2}} x} dx$$

$$2y = \int_0^{\pi/2} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 10.
Prove that

$$\int_0^{\pi/2} \frac{\sin^n x}{\left(\sin^n x + \cos^n x\right)} dx = \frac{\pi}{4}$$

Answer:
$y = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{\sin^n \left(\frac{\pi}{2} - x\right)}{\sin^n \left(\frac{\pi}{2} - x\right) + \cos^n \left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx + \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$$

$$2y = \int_0^{\pi/2} \frac{sin^n x + cos^n x}{sin^n x + cos^n x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 11.
Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\left(\sqrt{\tan x} + \sqrt{\cot x}\right)} dx = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{\sin x}{\cos x}}}{\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}} dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx \text{ ...(1)}$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$$

$$y = \int_0^{\pi/2} \frac{sin\left(\frac{\pi}{2} - x\right)}{sin\left(\frac{\pi}{2} - x\right) + cos\left(\frac{\pi}{2} - x\right)} dx$$

$$y = \int_0^{\pi/2} \frac{cosx}{sinx + cosx} dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{sinx}{sinx + cosx} dx + \int_0^{\pi/2} \frac{cosx}{sinx + cosx} dx$$

$$2y = \int_0^{\pi/2} \frac{sinx + cosx}{sinx + cosx} dx$$

$$2y = \int_0^{\pi/2} 1 \, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 12.
Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\left(\sqrt{\tan x} + \sqrt{\cot x}\right)} dx = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{\cos x}{\sin x}}}{\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}} dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx \text{ ...(1)}$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{cos\left(\frac{\pi}{2} - x\right)}{sin\left(\frac{\pi}{2} - x\right) + cos\left(\frac{\pi}{2} - x\right)} dx$$

$$y = \int_0^{\pi/2} \frac{sinx}{sinx + cosx} dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cosx}{sinx + cosx} dx + \int_0^{\pi/2} \frac{sinx}{sinx + cosx} dx$$

$$2y = \int_0^{\pi/2} \frac{sinx + cosx}{sinx + cosx} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 13.
Prove that

$$\int_0^{\pi/2} \frac{dx}{(1+\tan x)} = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\sin x}{\cos x}} dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x} dx \text{ ...(1)}$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{cos\left(\frac{\pi}{2} - x\right)}{sin\left(\frac{\pi}{2} - x\right) + cos\left(\frac{\pi}{2} - x\right)} dx$$

$$y = \int_0^{\pi/2} \frac{sinx}{sinx+cosx} dx \ ...(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cosx}{sinx + cosx} dx + \int_0^{\pi/2} \frac{sinx}{sinx + cosx} dx$$

$$2y = \int_0^{\pi/2} \frac{sinx + cosx}{sinx + cosx} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 14.
Prove that

$$\int_0^{\pi/2} \frac{dx}{(1+\cot x)} = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\cos x}{\sin x}} dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x} dx \ ...(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$y = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$$

$$2y = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 15.
Prove that

$$\int_0^{\pi/2} \frac{dx}{\left(1 + \tan^3 x\right)} = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{\sin^3 x}{\cos^3 x}} dx$$

$$y = \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x + \cos^3 x} dx \text{ ...(1)}$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{cos^3\left(\frac{\pi}{2}-x\right)}{sin^3\left(\frac{\pi}{2}-x\right)+cos^3\left(\frac{\pi}{2}-x\right)}dx$$

$$y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x+cos^3 x}dx \;...(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x}dx + \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x}dx$$

$$2y = \int_0^{\pi/2} \frac{cos^3 x + sin^3 x}{sin^3 x + cos^3 x}dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 16.
Prove that

$$\int_0^{\pi/2} \frac{dx}{\left(1+\cot^3 x\right)} = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{cos^3 x}{sin^3 x}}dx$$

$$y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x+cos^3 x}dx \;...(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{sin^3\left(\frac{\pi}{2}-x\right)}{sin^3\left(\frac{\pi}{2}-x\right)+cos^3\left(\frac{\pi}{2}-x\right)}dx$$

$$y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x+cos^3 x}dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x}dx + \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x}dx$$

$$2y = \int_0^{\pi/2} \frac{sin^3 x + cos^3 x}{sin^3 x + cos^3 x}dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 17.
Prove that

$$\int_0^{\pi/2} \frac{dx}{\left(1+\sqrt{\tan x}\right)} = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt{\frac{\sin x}{\cos x}}} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx \text{ ...(1)}$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\left(\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}\right)} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\cos x}+\sqrt{\sin x})} dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\cos x}+\sqrt{\sin x})} dx$$

$$2y = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 18.

Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\left(1+\sqrt{\cot x}\right)} dx = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{\cos x}{\sin x}}}{1+\sqrt{\frac{\cos x}{\sin x}}} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)} dx \text{ ...(1)}$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\left(\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}\right)} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\left(\sqrt{\cos x}+\sqrt{\sin x}\right)} dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)} dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\left(\sqrt{\cos x}+\sqrt{\sin x}\right)} dx$$

$$2y = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 19.
Prove that

$$\int_{0}^{\pi/2} \frac{\sqrt{\tan x}}{\left(1+\sqrt{\tan x}\right)} dx = \frac{\pi}{4}$$

Answer:

$$y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{\sin x}{\cos x}}}{1+\sqrt{\frac{\sin x}{\cos x}}} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx \text{ ...(1)}$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\left(\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}\right)} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\cos x}+\sqrt{\sin x})} dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\cos x} + \sqrt{\sin x})} dx$$

$$2y = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 20.
Prove that

$$\int_0^{\pi/2} \frac{(\sin x - \cos x)}{(1 + \sin x \cos x)} dx = 0$$

Answer:
$y = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx$$

$$2y = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x + \cos x - \sin x}{1 + \cos x \sin x} dx$$

$$2y = \int_0^{\frac{\pi}{2}} 0 \, dx$$

y = 0

Question 21.
Prove that

$$\int_0^1 x(1-x)^5 \, dx = \frac{1}{42}$$

Answer:
$y = \int_0^1 x(1-x)^5 \, dx$

Use King theorem of definite integral

$$\int_a^b f(x) \, dx = \int_a^b f(a+b-x) \, dx$$

$$y = \int_0^1 (1-x)x^5 \, dx$$

$$y = \int_0^1 x^5 - x^6 \, dx$$

$$y = \left(\frac{x^6}{6} - \frac{x^7}{7}\right)_0^1$$

$$y = \frac{1}{6} - \frac{1}{7}$$

$$= \frac{1}{42}$$

Question 22.
Prove that

$$\int_0^2 x\sqrt{2-x}\,dx = \frac{16\sqrt{2}}{15}$$

Answer:
$y = \int_0^2 x\sqrt{2-x}\,dx$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^2 (2-x)\sqrt{x}\,dx$$

$$y = \int_0^2 2x^{\frac{1}{2}} - x^{\frac{3}{2}}\,dx$$

$$y = \left(2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_0^2$$

$$y = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} = \frac{16\sqrt{2}}{15}$$

Question 23.
Prove that

$$\int_0^{\pi} x\cos^2 x\,dx = \frac{\pi^2}{4}$$

Answer:

$y = \int_0^{\pi} x\, cos^2 x\, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi} (\pi - x) cos^2(\pi - x)\, dx$$

$y = \int_0^{\pi} \pi\, cos^2 x - x\, cos^2 x\, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} x\, cos^2 x\, dx + \int_0^{\pi} \pi\, cos^2 x - x\, cos^2 x\, dx$$

$$2y = \int_0^{\pi} \pi\, cos^2 x\, dx$$

$$y = \frac{\pi}{2} \int_0^{\pi} \frac{1+\cos 2x}{2} dx$$

$$y = \frac{\pi}{2} \left(\frac{x}{2} + \frac{\sin 2x}{4} \right)_0^{\pi}$$

$$y = \frac{\pi}{2} \left(\frac{\pi}{2} + \frac{\sin 2\pi}{4} \right) = \frac{\pi^2}{4}$$

Question 24.

Prove that

$$\int_0^{\pi} \frac{x \tan x}{(\sec x \operatorname{cosec} x)} dx = \frac{\pi^2}{4}$$

Answer:

$y=\int_0^{\pi} \frac{x \tan x}{\sec x \; cosec\, x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

$$y=\int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)\ \ cosec\ (\pi-x)} dx$$

$$y=\int_0^{\pi} \frac{-(\pi-x)\tan x}{-\sec x\ \ cosec\ x} dx$$

$y=\int_0^{\pi} \frac{\pi \tan x - x \tan x}{\sec x \; cosec\, x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y=\int_0^{\pi} \frac{x\tan x}{\sec x\ \ cosec\ x} dx+\int_0^{\pi} \frac{\pi\tan x - x\tan x}{\sec x\ \ cosec\ x} dx$$

$$2y=\int_0^{\pi} \frac{\pi\tan x}{\sec x\ \ cosec\ x} dx$$

$$y=\frac{\pi}{2}\int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}\times\frac{1}{\sin x}} dx$$

$$y=\frac{\pi}{2}\int_0^{\pi} \frac{1-\cos 2x}{2} dx$$

$$y=\frac{\pi}{2}\left(\frac{x}{2}-\frac{\sin 2x}{4}\right)_0^{\pi}$$

$$y=\frac{\pi}{2}\left(\frac{\pi}{2}-\frac{\sin 2\pi}{4}\right)=\frac{\pi^2}{4}$$

Question 25.
Prove that

$$\int_0^{\pi/2}\frac{\cos^2 x}{(\sin x+\cos x)}dx=\frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

Answer:
$y=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\sin x+\cos x}dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

$$y=\int_0^{\frac{\pi}{2}}\frac{\cos^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

$y=\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\sin x+\cos x}dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\sin x+\cos x}dx+\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\sin x+\cos x}dx$$

$$2y=\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+\cos x}dx$$

$$2y=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{1}{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x}dx$$

$$2y = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\sin\left(x+\frac{\pi}{4}\right)}dx$$

$$y = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} cosec\ \left(x+\frac{\pi}{4}\right)dx$$

$$y = \frac{1}{2\sqrt{2}}\left(\ln\left(\operatorname{cosec}\left(x+\frac{\pi}{4}\right)-\cot\left(x+\frac{\pi}{4}\right)\right)\right)_0^{\frac{\pi}{2}}$$

$$y = \frac{1}{2\sqrt{2}}\left(\ln\left(\operatorname{cosec}\frac{3\pi}{4}-\cot\frac{3\pi}{4}\right)-\ln\left(\operatorname{cosec}\frac{\pi}{4}-\cot\frac{\pi}{4}\right)\right)$$

$$y = \frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$y = \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)^2 = \frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)$$

Question 26.
Prove that

$$\int_0^{\pi}\frac{x\tan x}{(\sec x+\cos x)}dx = \frac{\pi^2}{4}$$

Answer:

$$y = \int_0^{\pi}\frac{x\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\cos x}dx$$

$$y = \int_0^{\pi}\frac{x\sin x}{1+cos^2 x}dx$$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + cos^2(\pi - x)} dx$$

$y = \int_0^{\pi} \frac{\pi \sin x - x \sin x}{1 + cos^2 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} \frac{x \sin x}{1 + cos^2 x} dx + \int_0^{\pi} \frac{\pi \sin x - x \sin x}{1 + cos^2 x} dx$$

$$2y = \int_0^{\pi} \frac{\pi \sin x}{1 + cos^2 x} dx$$

Let, cos x = t

⇒ -sin x dx = dt

At x = 0, t = 1

At x = π, t = -1

$$y = -\frac{\pi}{2} \int_1^{-1} \frac{1}{1 + t^2} dt$$

$$y = -\frac{\pi}{2} (\tan^{-1} t)_1^{-1}$$

$$y = -\frac{\pi}{2} (\tan^{-1}(-1) - \tan^{-1} 1)$$

$$y = \frac{\pi^2}{4}$$

Question 27.

Prove that

$$\int_0^{\pi} \frac{x \sin x}{(1+\sin x)} dx = \pi\left(\frac{\pi}{2}-1\right)$$

Answer:

$y = \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)} dx$$

$y = \int_0^{\pi} \frac{\pi \sin x}{1+\sin x} - \frac{x \sin x}{1+\sin x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx + \int_0^{\pi} \frac{\pi \sin x}{1+\sin x} - \frac{x \sin x}{1+\sin x} dx$$

$$2y = \int_0^{\pi} \frac{\pi(\sin x + 1 - 1)}{1+\sin x} dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} 1 - \frac{1}{1+\sin x} dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} 1 - \frac{1-\sin x}{cos^2 x} dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} 1 - sec^2 x + \frac{\sin x}{cos^2 x} dx$$

Let, cos x = t

⇒ -sin x dx = dt

At x = 0, t = 1

At x = π, t = -1

$$y = \frac{\pi}{2}\left((x - \tan x)_0^{\pi} - \int_1^{-1} \frac{1}{t^2}\,dt\right)$$

$$y = \frac{\pi}{2}\left(\pi - \tan\pi - \left(\frac{-1}{t}\right)_1^{-1}\right)$$

$$y = \frac{\pi}{2}(\pi - 2) = \pi\left(\frac{\pi}{2} - 1\right)$$

Question 28.
Prove that

$$\int_0^{\pi} \frac{x}{\left(1+\sin^2 x\right)}\,dx = \frac{\pi^2}{2\sqrt{2}}$$

Answer:
$y = \int_0^{\pi} \frac{x}{1+sin^2x}\,dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} \frac{(\pi - x)}{1 + sin^2(\pi - x)}dx$$

$y = \int_0^{\pi} \frac{\pi}{1+sin^2x} - \frac{x}{1+sin^2x}\,dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} \frac{x}{1+sin^2x}dx + \int_0^{\pi} \frac{\pi}{1+sin^2x} - \frac{x}{1+sin^2x}dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} \frac{1}{1+sin^2x}dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} \frac{\frac{1}{cos^2x}}{\frac{1+sin^2x}{cos^2x}}dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} \frac{sec^2x}{sec^2x+tan^2x}dx$$

We break it in two parts

$$y = \frac{\pi}{2}\int_0^{\pi} \frac{sec^2x}{sec^2x+tan^2x}dx$$

Let, tan x = t

⇒ $sec^2x\ dx = dt$

At x = 0, t = 0

At x = π, t = 0

$$y = \frac{\pi}{2}\int_0^{0} \frac{1}{1+2t^2}dt$$

We know that when upper and lower limit is same in definite

integral then value of integration is 0.

So, y = 0

Question 29.
Prove that

$$\int_0^{\pi/2}(2\log\cos x-\log\sin 2x)dx=-\frac{\pi}{4}(\log 2)$$

Answer:

$y = \int_0^{\frac{\pi}{2}}\log\frac{cos^2 x}{\sin 2x}dx$

$$y=\int_0^{\frac{\pi}{2}}\log\frac{cos^2 x}{2\sin x\cos x}dx$$

$y = \int_0^{\frac{\pi}{2}}\log\left(\frac{1}{2}\cot x\right)dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y=\int_0^{\frac{\pi}{2}}\log\left(\frac{1}{2}\cot\left(\frac{\pi}{2}-x\right)\right)dx$$

$y = \int_0^{\frac{\pi}{2}}\log\left(\frac{1}{2}\tan x\right)dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y=\int_0^{\frac{\pi}{2}}\log\left(\frac{1}{2}\cot x\right)dx+\int_0^{\frac{\pi}{2}}\log\left(\frac{1}{2}\tan x\right)dx$$

$y = \frac{1}{2}\int_0^{\frac{\pi}{2}}\log\left(\frac{1}{4}\cot x\tan x\right)dx$ [Use cot x tan x = 1]

$$y=\frac{1}{2}\int_{0}^{\frac{\pi}{2}}\log\left(\frac{1}{4}\right)dx$$

$$y=\frac{1}{2}\log\left(\frac{1}{4}\right)(x)_{0}^{\frac{\pi}{2}}$$

$$y=-\frac{\pi}{4}\log 4$$

Question 30.
Prove that

$$\int_{0}^{\infty}\frac{x}{(1+x)(1+x^{2})}dx=\frac{\pi}{4}$$

Answer:
$y=\int_{0}^{\infty}\frac{x}{(1+x)(1+x^2)}dx$

Let, x = tan t

⇒ dx = $\sec^2$t dt

At x = 0, t = 0

At x = ∞, t = π/2

$$y=\int_{0}^{\frac{\pi}{2}}\frac{\tan t}{(1+\tan t)(1+tan^2 t)}sec^2 t\,dt$$

$$y=\int_{0}^{\frac{\pi}{2}}\frac{\tan t}{(1+\tan t)}\,dt$$

$y=\int_{0}^{\frac{\pi}{2}}\frac{\sin t}{(\cos t+\sin t)}\,dt$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-t\right)}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)}dt$$

$$y = \int_0^{\pi/2} \frac{\cos t}{\sin t+\cos t}dt \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sin t}{\sin t+\cos t}dx + \int_0^{\pi/2} \frac{\cos t}{\sin t+\cos t}dx$$

$$2y = \int_0^{\pi/2} \frac{\sin t+\cos t}{\sin t+\cos t}dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

Question 31.
Prove that

$$\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$$

Answer:
Let, x = a sin t

⇒ dx = a cos t dt

At x = 0, t = 0

At x = a, t = π/2

$$y = \int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\sin t + \sqrt{a^2 - a^2 sin^2 t}} dt$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t} dt$$

$$y = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos t + \cos t - \sin t + \sin t}{\sin t + \cos t} dt$$

$$y = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \frac{\cos t - \sin t}{\sin t + \cos t} dt$$

$$y = \frac{1}{2}\left((t)_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos t - \sin t}{\sin t + \cos t} dt \right)$$

Again, sin t + cos t = z

⇒ (cos t – sin t) dt = dz

At t = 0, z = 1

At t = π/2, z = 1

$$y = \frac{1}{2}\left(\frac{\pi}{2} + \int_1^1 \frac{1}{z} dz \right)$$

$$y = \frac{1}{2}(\frac{\pi}{2} + (\ln z)_1^1$$

$$y = \frac{\pi}{4}$$

Question 32.

$$\int_0^a \frac{\sqrt{x}}{\left(\sqrt{x} + \sqrt{a - x}\right)} dx = \frac{\pi}{4}$$

Answer:

$y = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$$

$y = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx + \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} dx$$

$$2y = \int_0^a \frac{\sqrt{x} + \sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} dx$$

$$y = \frac{1}{2} \int_0^a dx$$

$$y = \frac{1}{2} (x)_0^a$$

$$y = \frac{a}{2}$$

Question 33.

Prove that

$$\int_0^{\pi} \sin^2 x \cos^3 x \, dx = 0$$

Answer:
$y = \int_0^{\pi} sin^2 x \, cos^3 x \, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} sin^2(\pi - x)\, cos^3(\pi - x)\, dx$$

$y = -\int_0^{\pi} sin^2 x \, cos^3 x \, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} sin^2 x \, cos^3 x \, dx + \left(-\int_0^{\pi} sin^2 x \, cos^3 x \, dx\right)$$

y = 0

Question 34.
Prove that

$\int_0^{\pi} \sin^{2m} x \cos^{2m+1} x \, dx = 0,$ where m is a positive integer

Answer:
$y = \int_0^{\pi} sin^{2m} x \, cos^{2m+1} x \, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} sin^{2m}(\pi - x)\ cos^{2m+1}(\pi - x)\,dx$$

$y = -\int_0^{\pi} sin^{2m}x\ cos^{2m+1}x\,dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} sin^{2m}x\ cos^{2m+1}x\,dx + \left(-\int_0^{\pi} sin^{2m}x\ cos^{2m+1}x\,dx\right)$$

y = 0

Question 35.
Prove that

$$\int_0^{\pi/2} (\sin x - \cos x)\log(\sin x + \cos x)\,dx = 0$$

Answer:
Let, sin x + cos x = t

⇒ cos x – sin x dx = dt

At x = 0, t = 1

At x = π/2, t = 1

$$y = \int_1^1 -\log t\,dt$$

We know that when upper and lower limit in definite integral is

equal then value of integration is zero.

So, y = 0

Question 36.
Prove that

$$\int_0^{\pi/2} \log(\sin 2x)\,dx = -\frac{\pi}{2}(\log 2)$$

Answer:
$y = \int_0^{\frac{\pi}{2}} \log(2\sin x \cos x)\,dx$

$$y = \int_0^{\frac{\pi}{2}} \log 2 + \log\sin x + \log\cos x\,dx$$

Let, $I = \int_0^{\frac{\pi}{2}} \log\sin x\,dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right)dx$$

$I = \int_0^{\frac{\pi}{2}} \log\cos x\,dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2I = \int_0^{\frac{\pi}{2}} \log\sin x\,dx + \int_0^{\frac{\pi}{2}} \log\cos x\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log\frac{2\sin x\cos x}{2}dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log\sin 2x - \log 2\, dx$$

Let, 2x = t

⇒ 2 dx = dt

At x = 0, t = 0

At x = π/2, t = π

$$2I = \frac{1}{2}\int_0^{\pi} \log\sin t\, dt - \frac{\pi}{2}\log 2$$

$$2I = \frac{2}{2}\int_0^{\frac{\pi}{2}} \log\sin x\, dx - \frac{\pi}{2}\log 2$$

$$2I = I - \frac{\pi}{2}\log 2$$

$$I = \int_0^{\frac{\pi}{2}} \log\sin x\, dx \;=\; -\frac{\pi}{2}\log 2$$

Similarly, $\int_0^{\frac{\pi}{2}} \log\cos x\, dx = -\frac{\pi}{2}\log 2$

$$y = \int_0^{\frac{\pi}{2}} \log 2\, dx + \int_0^{\frac{\pi}{2}} \log\sin x\, dx + \int_0^{\frac{\pi}{2}} \log\cos x\, dx$$

$$y = \frac{\pi}{2}\log 2 - \frac{\pi}{2}\log 2 - \frac{\pi}{2}\log 2$$

$y = -\frac{\pi}{2}\log 2$

Question 37.
Prove that

$$\int_0^{\pi} x\log(\sin x)dx = -\frac{\pi^2}{2}(\log 2)$$

Answer:
$y = \int_0^{\pi} x \log\sin x \, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} (\pi - x)\log\sin(\pi - x)\,dx$$

$y = \int_0^{\pi} \pi\log\sin x - x\log\sin x\,dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} x\log\sin x\,dx + \int_0^{\pi} \pi\log\sin x - x\log\sin x\,dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} \log\sin x\,dx$$

$y = \frac{2\pi}{2}\int_0^{\frac{\pi}{2}} \log\sin x\,dx$ ...(3)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \pi \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right) dx$$

$y = \ \pi \int_0^{\frac{\pi}{2}} \log\cos x\, dx$ ...(4)

Adding eq.(3) and eq.(4)

$$2y = \ \pi\left(\int_0^{\frac{\pi}{2}} \log\sin x\, dx + \int_0^{\frac{\pi}{2}} \log\cos x\, dx\right)$$

$$2y = \ \pi\left(\int_0^{\frac{\pi}{2}} \log\frac{2\sin x\cos x}{2} dx\right)$$

$$2y = \ \pi\left(\int_0^{\frac{\pi}{2}} \log\sin 2x - \log 2\, dx\right)$$

Let, 2x = t

⇒ 2 dx = dt

At x = 0, t = 0

At x = π/2, t = π

$$2y = \frac{\pi}{2}\int_0^{\pi} \log\sin t\, dt - \frac{\pi^2}{2}\log 2$$

$$2y = \frac{2\pi}{2}\int_0^{\frac{\pi}{2}} \log\sin x\, dx - \frac{\pi^2}{2}\log 2$$

$$2y = y - \frac{\pi^2}{2}\log 2$$

$$y = -\frac{\pi^2}{2}\log 2$$

Question 38.
Prove that

$$\int_0^{\pi} \log(1+\cos x)\,dx = -\pi(\log 2)$$

Answer:
$y = \int_0^{\pi} \log(1+\cos x)\, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} \log(1+\cos(\pi - x))\,dx$$

$y = \int_0^{\pi} \log(1-\cos x)\, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} \log(1+\cos x)\,dx + \int_0^{\pi} \log(1-\cos x)\,dx$$

$$2y = \int_0^{\pi} \log sin^2 x\,dx$$

$y = 2\int_0^{\frac{\pi}{2}} \log \sin x \, dx$ ...(3)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

$y = 2\int_0^{\frac{\pi}{2}} \log \cos x \, dx$ ...(4)

Adding eq.(3) and eq.(4)

$$2y = 2\left(\int_0^{\frac{\pi}{2}} \log \sin x \, dx + \int_0^{\frac{\pi}{2}} \log \cos x \, dx\right)$$

$$2y = 2\left(\int_0^{\frac{\pi}{2}} \log \frac{2\sin x \cos x}{2} \, dx\right)$$

$$2y = 2\left(\int_0^{\frac{\pi}{2}} \log \sin 2x - \log 2 \, dx\right)$$

Let, 2x = t

⇒ 2 dx = dt

At x = 0, t = 0

At x = π/2, t = π

$$2y = \frac{2}{2}\int_0^{\pi} \log \sin t \, dt - \frac{2\pi}{2}\log 2$$

$$2y = \frac{4}{2}\int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{2\pi}{2}\log 2$$

$$2y = y - \pi \log 2$$

$$y = -\pi \log 2$$

Question 39.
Prove that

$$\int_0^{\pi/2} \log(\tan x + \cot x) dx = \pi(\log 2)$$

Answer:
$y = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) dx$

$$y = \int_0^{\frac{\pi}{2}} \log \frac{1}{\sin x \cos x} dx$$

$$y = -\left(\int_0^{\frac{\pi}{2}} \log \sin x \, dx + \int_0^{\frac{\pi}{2}} \log \cos x \, dx\right)$$

Let, $I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x) \, dx = \int_a^b f(a + b - x) \, dx$$

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right) dx$$

$$I = \int_0^{\frac{\pi}{2}} \log\cos x \, dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2I = \int_0^{\frac{\pi}{2}} \log\sin x \, dx + \int_0^{\frac{\pi}{2}} \log\cos x \, dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log\frac{2\sin x \cos x}{2} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log\sin 2x - \log 2 \, dx$$

Let, 2x = t

⇒ 2 dx = dt

At x = 0, t = 0

At x = π/2, t = π

$$2I = \frac{1}{2}\int_0^{\pi} \log\sin t \, dt - \frac{\pi}{2}\log 2$$

$$2I = \frac{2}{2}\int_0^{\frac{\pi}{2}} \log\sin x \, dx - \frac{\pi}{2}\log 2$$

$$2I = I - \frac{\pi}{2}\log 2$$

$$I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx = -\frac{\pi}{2} \log 2$$

Similarly, $\int_0^{\frac{\pi}{2}} \log \cos x \, dx = -\frac{\pi}{2} \log 2$

$$y = -\left(\int_0^{\frac{\pi}{2}} \log \sin x \, dx + \int_0^{\frac{\pi}{2}} \log \cos x \, dx \right)$$

$$y = \frac{\pi}{2} \log 2 + \frac{\pi}{2} \log 2$$

$$y = \pi \log 2$$

Question 40.
Prove that

$$\int_{\pi/8}^{3\pi/8} \frac{\cos x}{(\cos x + \sin x)} dx = \frac{\pi}{8}$$

Answer:
$y = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\cos x}{\cos x + \sin x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x) \, dx = \int_a^b f(a + b - x) \, dx$$

$$y = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{cos\left(\frac{3\pi}{8} + \frac{\pi}{8} - x\right)}{sin\left(\frac{3\pi}{8} + \frac{\pi}{8} - x\right) + cos\left(\frac{3\pi}{8} + \frac{\pi}{8} - x\right)} dx$$

$y = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{sinx}{sinx + cosx} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{cosx}{sinx + cosx} dx + \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{sinx}{sinx + cosx} dx$$

$$2y = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{sinx + cosx}{sinx + cosx} dx$$

$$2y = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} 1\, dx$$

$$2y = (x)_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$2y = \frac{3\pi}{8} - \frac{\pi}{8}$$

$$y = \frac{\pi}{8}$$

Question 41.
Prove that

$$\int_{\pi/6}^{\pi/3} \frac{1}{\left(1+\sqrt{\tan x}\right)} dx = \frac{\pi}{12}$$

Answer:

$$y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\left(\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}\right)} dx$$

$$y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{(\sqrt{\cos x}+\sqrt{\sin x})} dx \text{ ...(2)}$$

Adding eq.(1) and eq.(2)

$$2y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{(\sqrt{\cos x}+\sqrt{\sin x})} dx$$

$$2y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx$$

$$2y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\, dx$$

$$2y = (x)_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$y = \frac{\pi}{12}$$

Question 42.
Prove that

$$\int_{\pi/4}^{3\pi/4} \frac{dx}{(1+\cos x)} = 2$$

Answer:

$$y = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2\cos^2\frac{x}{2}} dx$$

$$y = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} sec^2\frac{x}{2}dx$$

$$y = \frac{1}{2}\left(\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right)_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$y = \tan\frac{3\pi}{8} - \tan\frac{\pi}{8}$$

$$y = \left(\sqrt{2}+1\right) - \left(\sqrt{2}-1\right) = 2$$

Question 43.
Prove that

$$\int_{\pi/4}^{3\pi/4} \frac{x}{(1+\sin x)}dx = \pi\left(\sqrt{2}-1\right)$$

Answer:

$y = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{x}{1+\sin x}dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\left(\frac{3\pi}{4}+\frac{\pi}{4}-x\right)}{1+\sin\left(\frac{3\pi}{4}+\frac{\pi}{4}-x\right)}dx$$

$y = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\pi - x}{1+\sin x}dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{x}{1+\sin x} dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\pi - x}{1+\sin x} dx$$

$$y = \frac{\pi}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1+\sin x} dx$$

$$y = \frac{\pi}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$

$$y = \frac{\pi}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1-\sin x}{cos^2 x} dx$$

$$y = \frac{\pi}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} sec^2 x - \frac{\sin x}{cos^2 x} dx$$

Let, cos x = t

⇒ -sin x dx = dt

At x = π/4, t = $\frac{1}{\sqrt{2}}$

At x = 3π/4, t =$\frac{-1}{\sqrt{2}}$

$$y = \frac{\pi}{2} \left((\tan x)_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \int_{\frac{1}{\sqrt{2}}}^{\frac{-1}{\sqrt{2}}} \frac{1}{t^2} dt \right)$$

$$y = \frac{\pi}{2}(\tan\frac{3\pi}{4} - \tan\frac{\pi}{4} + \left(\frac{-1}{t}\right)_{\frac{1}{\sqrt{2}}}^{\frac{-1}{\sqrt{2}}}$$

$$y = \frac{\pi}{2}\left(-1 - 1 + \sqrt{2} + \sqrt{2}\right) = \pi\left(\sqrt{2} - 1\right)$$

Question 44.
Prove that

$$\int_{\alpha/4}^{3\alpha/4} \frac{\sqrt{x}}{\left(\sqrt{a-x} + \sqrt{x}\right)} dx = \frac{a}{4}$$

Answer:

$$y = \int_{\frac{a}{4}}^{\frac{3a}{4}} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx \text{ ...(1)}$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_{\frac{a}{4}}^{\frac{3a}{4}} \frac{\sqrt{\frac{3a}{4} + \frac{a}{4} - x}}{\sqrt{\frac{3a}{4} + \frac{a}{4} - x} + \sqrt{x}} dx$$

$$y = \int_{\frac{a}{4}}^{\frac{3a}{4}} \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$$

Adding eq.(1) and eq.(2)

$$2y = \int_{\frac{a}{4}}^{\frac{3a}{4}} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx + \int_{\frac{a}{4}}^{\frac{3a}{4}} \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$$

$$2y = \int_{\frac{a}{4}}^{\frac{3a}{4}} \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$$

$$y = \frac{1}{2}\int_{\frac{a}{4}}^{\frac{3a}{4}} 1\,dx$$

$$y = \frac{1}{2}(x)_{\frac{a}{4}}^{\frac{3a}{4}}$$

$$y = \frac{a}{4}$$

Question 45.
Prove that

$$\int_1^4 \frac{\sqrt{x}}{\left(\sqrt{5-x} + \sqrt{x}\right)} dx = \frac{3}{2}$$

Answer:

$$y = \int_1^4 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_1^4 \frac{\sqrt{4+1-x}}{\sqrt{4+1-x}+\sqrt{x}} dx$$

$$y = \int_1^4 \frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}} dx$$

Adding eq.(1) and eq.(2)

$$2y = \int_1^4 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}dx + \int_1^4 \frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$$

$$2y = \int_1^4 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$$

$$y = \frac{1}{2}\int_1^4 1\,dx$$

$$y = \frac{1}{2}(x)_1^4$$

$$y = \frac{3}{2}$$

Question 46.
Prove that

$$\int_0^{\pi/2} x \cot x\, dx = \frac{\pi}{4}(\log 2)$$

Answer:
Use integration by parts

$$\int I \times II\, dx = I\int II\, dx - \int \frac{d}{dx}I\left(\int II\, dx\right)dx$$

$$y = x\int \cot x\, dx - \int \frac{d}{dx}x\left(\int \cot x\, dx\right)dx$$

$$y = (x\log\sin x)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log\sin x\, dx$$

Let, $I = \int_0^{\frac{\pi}{2}} \log\sin x\, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right) dx$$

$$I = \int_0^{\frac{\pi}{2}} \log\cos x\,dx \cdots(2)$$

Adding eq.(1) and eq.(2)

$$2I = \int_0^{\frac{\pi}{2}} \log\sin x\,dx + \int_0^{\frac{\pi}{2}} \log\cos x\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log\frac{2\sin x\cos x}{2}\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log\sin 2x - \log 2\,dx$$

Let, 2x = t

⇒ 2 dx = dt

At x = 0, t = 0

At x = π/2, t = π

$$2I = \frac{1}{2}\int_0^{\pi} \log\sin t\,dt - \frac{\pi}{2}\log 2$$

$$2I = \frac{2}{2}\int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2}\log 2$$

$$2I = I - \frac{\pi}{2}\log 2$$

$$I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx \ = -\frac{\pi}{2}\log 2$$

$$y = (x \log \sin x)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log \sin x \, dx$$

$$y = \frac{\pi}{2}\log \sin\frac{\pi}{2} - \left(-\frac{\pi}{2}\log 2\right)$$

$$y = \frac{\pi}{2}\log 2$$

Question 47.
Prove that

$$\int_0^1 \left(\frac{\sin^{-1} x}{x}\right) dx = \frac{\pi}{2}(\log 2)$$

Answer:
Let, x = sin t

⇒ dx = cos t dt

At x = 0, t = 0

At x = 1, t = π/2

$$y = \int_0^{\frac{\pi}{2}} \frac{\sin^{-1} \sin t}{\sin t} \cos t \, dt$$

$$y = \int_0^{\frac{\pi}{2}} \frac{t\cos t}{\sin t} dt$$

$$y = \int_0^{\frac{\pi}{2}} t\cot t\, dt$$

Use integration by parts

$$\int I \times II\, dt = I\int II\, dt - \int \frac{d}{dt} I\left(\int II\, dt\right) dt$$

$$y = t\int \cot t\, dt - \int \frac{d}{dt} t\left(\int \cot t\, dt\right) dt$$

$$y = (t\log\sin t)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log\sin t\, dt$$

Let, $I = \int_0^{\frac{\pi}{2}} \log\sin t\, dt$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(t)\, dt = \int_a^b f(a+b-t)\, dt$$

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - t\right) dt$$

$I = \int_0^{\frac{\pi}{2}} \log\cos t\, dt$ ...(2)

Adding eq.(1) and eq.(2)

$$2I = \int_0^{\frac{\pi}{2}} \log \sin t \, dt + \int_0^{\frac{\pi}{2}} \log \cos t \, dt$$

$$2I = \int_0^{\frac{\pi}{2}} \log \frac{2 \sin t \cos t}{2} \, dt$$

$$2I = \int_0^{\frac{\pi}{2}} \log \sin 2t - \log 2 \, dt$$

Let, 2t = z

⇒ 2 dt = dz

At t = 0, z = 0

At t = π/2, z = π

$$2I = \frac{1}{2} \int_0^{\pi} \log \sin z \, dz - \frac{\pi}{2} \log 2$$

$$2I = \frac{2}{2} \int_0^{\frac{\pi}{2}} \log \sin z \, dz - \frac{\pi}{2} \log 2$$

$$2I = I - \frac{\pi}{2} \log 2$$

$$I = \int_0^{\frac{\pi}{2}} \log \sin z \, dz \; = -\frac{\pi}{2} \log 2$$

$$y = (t \log \sin t)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log t \, dt$$

$$y = \frac{\pi}{2}\log\sin\frac{\pi}{2} - \left(-\frac{\pi}{2}\log 2\right)$$

$$y = \frac{\pi}{2}\log 2$$

Question 48.
Prove that

$$\int_0^1 \frac{\log x}{\sqrt{1-x^2}}dx = -\frac{\pi}{2}(\log 2)$$

Answer:
Use integration by parts

$$\int I \times II\, dx = I\int II\, dx - \int \frac{d}{dx}I\left(\int II\, dx\right)dx$$

$$y = \log x\int \frac{1}{\sqrt{1-x^2}}dx - \int \frac{d}{dx}\log x\left(\int \frac{1}{\sqrt{1-x^2}}dx\right)dx$$

$$y = (\log x \sin^{-1} x)_0^1 - \int_0^1 \frac{\sin^{-1} x}{x}dx$$

$$y = -\int_0^1 \frac{\sin^{-1} x}{x}dx$$

Let, x = sin t

⇒ dx = cos t dt

At x = 0, t = 0

At x = 1, t = π/2

$$y = -\int_0^{\frac{\pi}{2}} \frac{\sin^{-1}\sin t}{\sin t}\cos t\, dt$$

$$y = -\int_0^{\frac{\pi}{2}} \frac{t\cos t}{\sin t} dt$$

$$y = -\int_0^{\frac{\pi}{2}} t\cot t\, dt$$

Use integration by parts

$$\int I \times II\, dt = I \int II\, dt - \int \frac{d}{dt} I \left(\int II\, dt\right) dt$$

$$y = -\left(t \int \cot t\, dt - \int \frac{d}{dt} t \left(\int \cot t\, dt\right) dt\right)$$

$$y = -\left((t \log\sin t)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log\sin t\, dt \right)$$

Let, $I = \int_0^{\frac{\pi}{2}} \log\sin t\, dt$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(t)\, dt = \int_a^b f(a+b-t)\, dt$$

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - t\right) dt$$

$I = \int_0^{\frac{\pi}{2}} \log\cos t\, dt$ ...(2)

Adding eq.(1) and eq.(2)

$$2I = \int_0^{\frac{\pi}{2}} \log \sin t \, dt + \int_0^{\frac{\pi}{2}} \log \cos t \, dt$$

$$2I = \int_0^{\frac{\pi}{2}} \log \frac{2 \sin t \cos t}{2} dt$$

$$2I = \int_0^{\frac{\pi}{2}} \log \sin 2t - \log 2 \, dt$$

Let, 2t = z

⇒ 2 dt = dz

At t = 0, z = 0

At t = π/2, z = π

$$2I = \frac{1}{2} \int_0^{\pi} \log \sin z \, dz - \frac{\pi}{2} \log 2$$

$$2I = \frac{2}{2} \int_0^{\frac{\pi}{2}} \log \sin z \, dz - \frac{\pi}{2} \log 2$$

$$2I = I - \frac{\pi}{2} \log 2$$

$$I = \int_0^{\frac{\pi}{2}} \log \sin z \, dz \ = -\frac{\pi}{2} \log 2$$

$$y = -\left((t \log \sin t)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log t \, dt \right)$$

$$y = \frac{-\pi}{2}\log\sin\frac{\pi}{2} + \left(-\frac{\pi}{2}\log 2\right)$$

$$y = \frac{-\pi}{2}\log 2$$

Question 49.
Prove that

$$\int_0^1 \frac{\log(1+x)}{(1+x^2)}dx = \frac{\pi}{8}(\log\ 2)$$

Answer:
Let x = tan t

⇒ dx = $\sec^2$t dt

At x = 0, t = 0

At x = 1, t = π/4

$$y = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan t)}{1+tan^2 t} sec^2 t\, dt$$

$y = \int_0^{\frac{\pi}{4}} \log(1+\tan t)\, dt$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(t)\, dt = \int_a^b f(a+b-t)\, dt$$

$$y = \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-t\right)\right) dt$$

$$y = \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - \tan t}{1 + \tan t}\right) dt$$

$y = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan t}\right) dt$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\frac{\pi}{4}} \log(1 + \tan t)\, dt + \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1 + \tan t}\right) dt$$

$$2y = \int_0^{\frac{\pi}{4}} \log(1 + \tan t)\left(\frac{2}{1 + \tan t}\right) dt$$

$$2y = \int_0^{\frac{\pi}{4}} \log 2\, dt$$

$$y = \frac{\pi}{8} \log 2$$

Question 50.
Prove that

$$\int_{-a}^{a} x^3 \sqrt{a^2 - x^2}\, dx = 0$$

Answer:
$y = \int_{-a}^{a} x^3 \sqrt{a^2 - x^2}\, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(t)\, dt = \int_a^b f(a + b - t)\, dt$$

$$y = \int_{-a}^{a} (a - a - x)^3 \sqrt{a^2 - (a - a - x)^2}\, dx$$

$y = \int_{-a}^{a} -x^3\sqrt{a^2 - x^2}\, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_{-a}^{a} x^3\sqrt{a^2 - x^2}\, dx + \left(-\int_{-a}^{a} x^3\sqrt{a^2 - x^2}\, dx\right)$$

y = 0

Question 51.
Prove that

$$\int_{-\pi}^{\pi} \left(\sin^{75} x + x^{125}\right) dx = 0$$

Answer:
$y = \int_{-\pi}^{\pi} sin^{75} x + x^{125}\, dx$ ...(1)

Use King theorem of definite integral

$$\int_{a}^{b} f(t)\, dt = \int_{a}^{b} f(a + b - t)\, dt$$

$$y = \int_{-\pi}^{\pi} sin^{75}(\pi - \pi - x) + (\pi - \pi - x)^{125}\, dx$$

$y = \int_{-\pi}^{\pi} -sin^{75} x - x^{125}\, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_{-\pi}^{\pi} sin^{75}x + x^{125}\, dx + \left(- \int_{-\pi}^{\pi} sin^{75}x + x^{125}\, dx \right)$$

y = 0

Question 52.
Prove that

$$\int_{-\pi}^{\pi} x^{12} \sin^{9} x \; dx = 0$$

Answer:
$y = \int_{-\pi}^{\pi} x^{12} sin^9 x\, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(t)\, dt = \int_a^b f(a+b-t)\, dt$$

$$y = \int_{-\pi}^{\pi} (\pi - \pi - x)^{12} sin^9(\pi - \pi - x)\, dx$$

$y = \int_{-\pi}^{\pi} -x^{12} sin^9 x\, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_{-\pi}^{\pi} x^{12} sin^9 x\, dx + \left(- \int_{-\pi}^{\pi} x^{12} sin^9 x\, dx \right)$$

y = 0

Question 53.

Prove that

$$\int_{-1}^{1} e^{|x|} dx = 2(e-1)$$

Answer:
We know that

|x| = -x in [-1, 0)

|x| = x in [0, 1]

$$y = \int_{-1}^{0} e^{|x|} dx + \int_{0}^{1} e^{|x|} dx$$

$$y = \int_{-1}^{0} e^{-x} dx + \int_{0}^{1} e^{x} dx$$

$$y = (-e^{-x})_{-1}^{0} + (e^{x})_{0}^{1}$$

y= -(1-e)+(e-1)

y = 2(e – 1)

Question 54.

$$\int_{-2}^{2} |x+1| dx = 6$$

Answer:
We know that

|x+1| = -(x+1) in [-2, -1)

|x+1| = (x+1) in [-1, 2]

$$y = \int_{-2}^{-1} |x+1| \, dx + \int_{-1}^{2} |x+1| \, dx$$

$$= -\int_{-2}^{-1}(x+1)\,dx + \int_{-1}^{2}(x+1)\,dx$$

$$= -\left(\frac{x^2}{2}+x\right)_{-2}^{-1} + \left(\frac{x^2}{2}+x\right)_{-1}^{2}$$

$$= -\left(\frac{1}{2}-1-2+2\right) + \left(2+2-\frac{1}{2}+1\right)$$

=5

Question 55.
Prove that

$$\int_{0}^{8}|x-5|dx = 17$$

Answer:
We know that

|x – 5| = -(x – 5) in [0, 5)

|x – 5| = (x – 5) in [5, 8]

$$y = \int_{0}^{5}|x-5|\,dx + \int_{5}^{8}|x-5|\,dx$$

$$y = -\int_{0}^{5}(x-5)\,dx + \int_{5}^{8}(x-5)\,dx$$

$$y = -\left(\frac{x^2}{2}-5x\right)_{0}^{5} + \left(\frac{x^2}{2}-5x\right)_{5}^{8}$$

$$y = -\left(\frac{25}{2}-25\right) + \left(32-40-\frac{25}{2}+25\right)$$

=17

Question 56.
Prove that

$$\int_0^{2\pi} |\cos x| dx = 4$$

Answer:
We know that

|cos x| = cos x in [0, π/2)

|cos x| = -cos x in [π/2, 3π/2)

|cos x| = cos x in [3π/2, 2π]

$$y = \int_0^{\frac{\pi}{2}} |\cos x|\, dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\cos x|\, dx + \int_{\frac{3\pi}{2}}^{2\pi} |\cos x|\, dx$$

$$y = \int_0^{\frac{\pi}{2}} \cos x\, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\, dx$$

$$y = (\sin x)_0^{\frac{\pi}{2}} - (\sin x)_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + (\sin x)_{\frac{3\pi}{2}}^{2\pi}$$

y=(1-0)—1-1+(0+1)

=4

Question 57.
Prove that

$$\int_{-\pi/4}^{\pi/4} |\sin x| dx = \left(2 - \sqrt{2}\right)$$

Answer:
We know that

|sin x| = -sin x in [-π/4, 0)

|sin x| = sin x in [0, π/4]

$$y = \int_{\frac{-\pi}{4}}^{0} |\sin x|\, dx + \int_{0}^{\frac{\pi}{4}} |\sin x|\, dx$$

$$y = -\int_{\frac{-\pi}{4}}^{0} \sin x\, dx + \int_{0}^{\frac{\pi}{4}} \sin x\, dx$$

$$y = -(-\cos x)_{\frac{-\pi}{4}}^{0} + (-\cos x)_{0}^{\frac{\pi}{4}}$$

$$y = \left(1 - \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= 2 - \frac{1}{\sqrt{2}}$$

Question 58.
Prove that

Let $f(x) = \begin{cases} 2x+1, \text{ when } 1 \le x \le 2 \\ x^2+1, \text{ when } 2 \le x \le 3 \end{cases}$

Show that $\int_1^3 f(x)dx = \frac{34}{3}$.

Answer:
$y = \int_1^3 f(x)\, dx$

$$y = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$y = \int_1^2 2x + 1\,dx + \int_2^3 x^2 + 1\,dx$$

$$y = (x^2 + x)_1^2 + \left(\frac{x^3}{3} + x\right)_2^3$$

$$y = (4 + 2 - 1 - 1) + \left(9 + 3 - \frac{8}{3} - 2\right)$$

$$= \frac{34}{3}$$

Question 59.
Prove that

Let $f(x) = \begin{cases} 3x^2 + 4, \text{ when } 0 \le x \le 2 \\ 9x - 2, \quad \text{when } 2 \le x \le 4 \end{cases}$

Show that $\int_0^4 f(x)dx = 66$

Answer:
$y = \int_0^4 f(x)\,dx$

$$y = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx$$

$$y = \int_0^2 3x^2 + 4\,dx + \int_2^4 9x - 2\,dx$$

$$y = (x^3 + 4x)_0^2 + \left(\frac{9x^2}{2} - 2x\right)_2^4$$

y=(8+8)+(72-8-18+4)

=66

Question 60.
Prove that

$$\int_{0}^{4}\{|x|+|x-2|+|x-4|dx\}=20$$

Answer:
$y = \int_0^4 |x| + |x-2| + |x-4|\, dx$

$$y = \int_0^2 |x| + |x-2| + |x-4|\, dx + \int_2^4 |x| + |x-2| + |x-4|\, dx$$

$$y = \int_0^2 x - (x-2) - (x-4)\, dx + \int_2^4 x + (x-2) - (x-4)\, dx$$

$$y = \left(-\frac{x^2}{2} + 6x\right)_0^2 + \left(\frac{x^2}{2} + 2x\right)_2^4$$

y=(-2+12)+(8+8-2-4)

=20