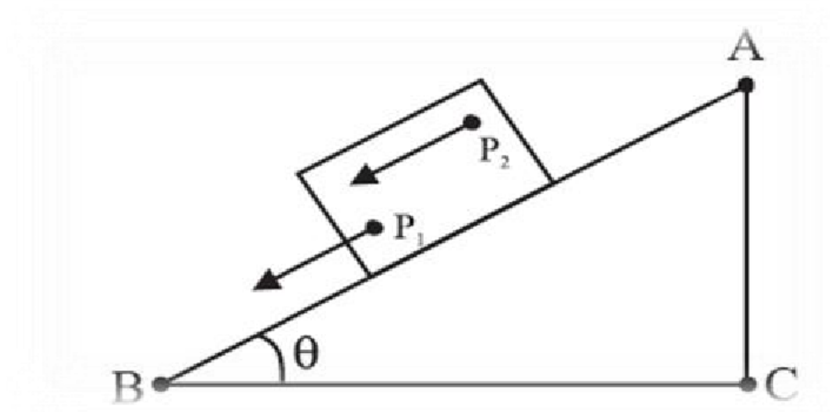
Chapter - 7 Systems Of Particles And Rotational Motion

RIGID BODY

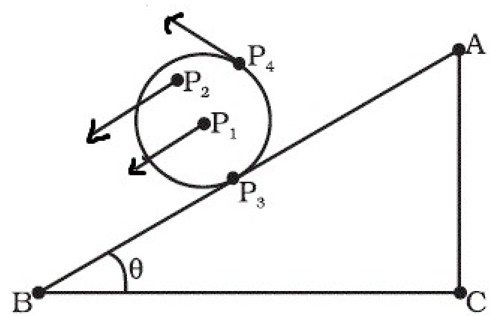
- A rigid body is a body with a perfectly definite and unchanging shape.
- The distances between different pairs of such a body do not change

MOTIONS OF A RIGID BODY PURE TRANSLATION



• In pure translational motion **at any instant of time every particle of the body** has the **same velocity**.

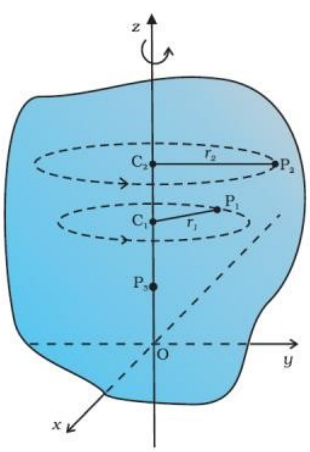
TRANSLATION AND ROTATION

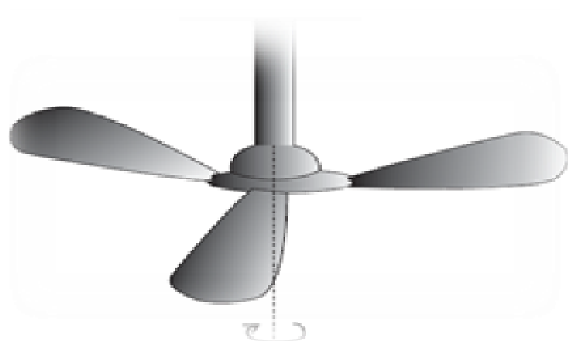


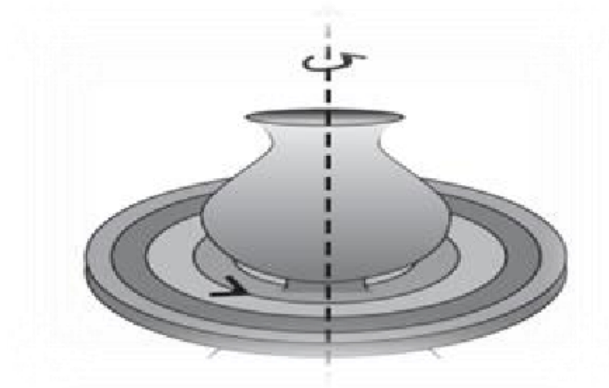
- Points P1, P2, P3 and P4 have different velocities at any instant of time.
- The line along which the body is fixed is termed as its axis of rotation**.**

ROTATION ABOUT A FIXED AXIS

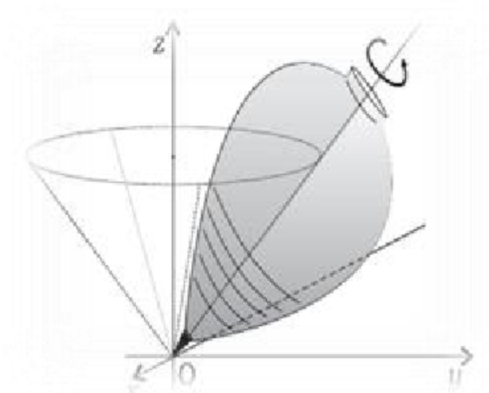
- In rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.

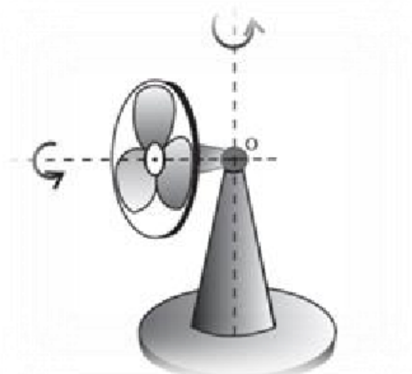






ROTATION ABOUT AN AXIS NOT FIXED





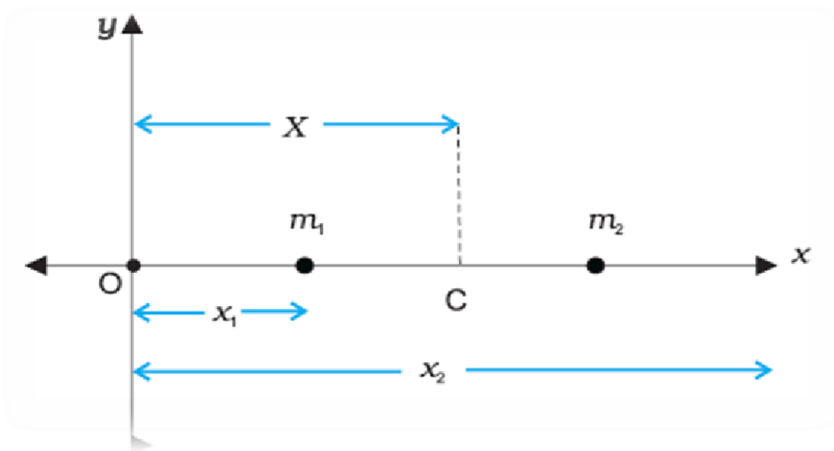
- The axis of a spinning top moves around the vertical through its point of contact with the ground, sweeping out a cone.
- The movement of the axis of rotation is termed precession.

•	**The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation.**

•	**The motion of a rigid body which is pivoted or fixed in some way is rotation.**

CENTRE OF MASS

- **Centre of mass is the point at which the entire mass of the body can be assumed to be concentrated.**



- For two particle system as shown in fig. the centre of mass is given by

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

- If the two particles have the same mass , *then*

$$X = \frac{mx_1 + mx_2}{m + m} = \frac{x_1 + x_2}{2}$$

- If we have *n particle*

$$X = \frac{m_1 x_1 + m_2 x_2 + \ldots\ldots + m_n x_n}{m_1 + m_2 + \ldots\ldots + m_n} = \frac{\sum m_i x_i}{\sum m_i}$$

- The centre of mass C of the system of the three particles is defined and located by the coordinates (*X, Y) given by*

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

- For the particles of equal mass *m*

$$X = \frac{mx_1 + mx_2 + mx_3}{m + m + m} = \frac{x_1 + x_2 + x_3}{3}$$

$$Y = \frac{my_1 + my_2 + my_3}{m + m + m} = \frac{y_1 + y_2 + y_3}{3}$$

- Thus, for three particles of equal mass, the centre of mass coincides with the **centroid of the triangle** formed by the particles.

- In terms of position vectors, the centre of mass is given by

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{M}$$

- Where $\vec{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$

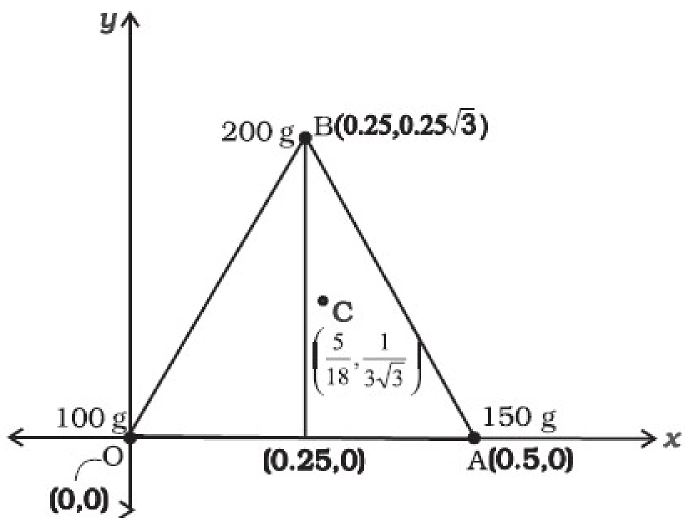
$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

- The centre of mass of **homogeneous bodies of regular shapes like rings, discs, spheres**, **rods** lie at their **geometric centers**.

PROBLEM

• Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

Solution



• We have

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{[100(0) + 150(0.5) + 200(0.25)] \text{ g m}}{(100 + 150 + 200) \text{ g}}$$

$$= \frac{75 + 50}{450} \text{m} = \frac{125}{450} \text{m} = \frac{5}{18} \text{m}$$

$$Y = \frac{[100(0) + 150(0) + 200(0.25\sqrt{3})] \text{ g m}}{450 \text{ g}}$$

$$= \frac{50\sqrt{3}}{450} \text{m} = \frac{\sqrt{3}}{9} \text{m} = \frac{1}{3\sqrt{3}} \text{m}$$

MOTION OF CENTRE OF MASS

- We have , $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n$
- Differentiating with respect to time

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + ... + m_n\frac{d\vec{r}_n}{dt}$$

- Thus

$$M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n$$

- Where V – velocity of centre of mass
- Again differentiating with respect to time

$$M\frac{d\vec{V}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + ... + m_n\frac{d\vec{v}_n}{dt}$$

$$M\vec{A} = m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n$$

- A – acceleration of centre of mass
- From Newton's second law,

$$M\vec{A} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n$$

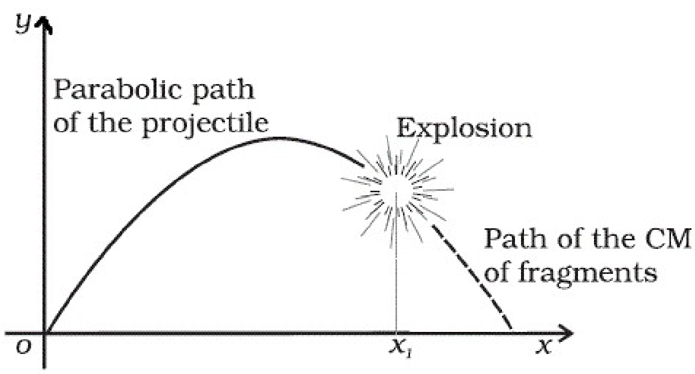
- Thus the **total mass of a system of particles times the acceleration of its centre of mass is the vector sum of all the forces acting on the system of particles.**
- Only the external forces contribute to the equation

$$M\vec{A} = \vec{F}_{ext}$$

- **The centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.**

Illustration- projectile explodes into fragments

- Consider a projectile, following the usual parabolic trajectory, explodes into fragments midway in air.
- The forces leading to the explosion are internal forces. They contribute nothing to the motion of the centre of mass.
- The total external force, namely, the force of gravity acting on the body, is the same before and after the explosion.
- The centre of mass under the influence of the external force continues, therefore, along the same parabolic trajectory as it would have followed if there were no explosion.



LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

- The total momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass

$$\vec{P} = M\vec{V}$$

- Differentiating above equation with respect to time,

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{V}}{dt} = M\vec{A}$$

$$\frac{d\vec{P}}{dt} = F_{ext}$$

- This is the statement of Newton's second law extended to a system of particles.
- When the total external force acting on a system of particles is zero, the total linear momentum of the system is constant.

$$\frac{d\vec{P}}{dt} = 0$$

$$\vec{P} = \text{constant}$$

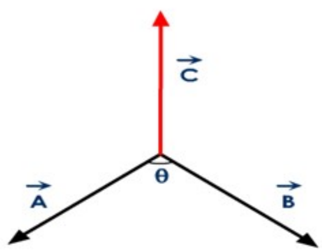
• **When the total external force on the system is zero the velocity of the centre of mass remains constant.**

VECTOR PRODUCT OF TWO VECTORS

- The vector product or cross product of two vectors is given by

$$\vec{A} \times \vec{B} = AB\sin\theta\hat{n}$$

- Where $A = |\vec{A}|, B = |\vec{B}|,\ \hat{n}$ **- unit vector perpendicular to A and B.**



- The vector product of two vectors is another vector perpendicular to the given vectors.

Properties of vector product

- Vector product is not commutative

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

$$\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$$

- Vector product is distributive

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

- Also $\vec{A} \times \vec{A} = \vec{0}$**, null vector**
- Vector products of orthogonal unit vectors are

$$\hat{i} \times \hat{i} = 0, \quad \hat{j} \times \hat{j} = 0, \quad \hat{k} \times \hat{k} = 0$$

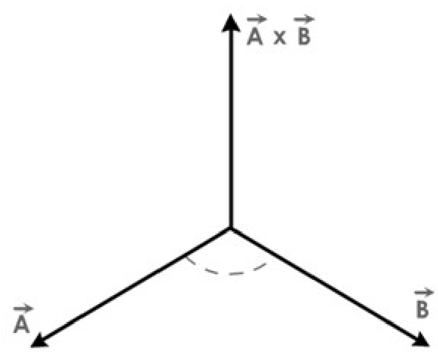
$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} \times = -\hat{j}$$

RULES TO FIND DIRECTION OF VECTOR PRODUCT

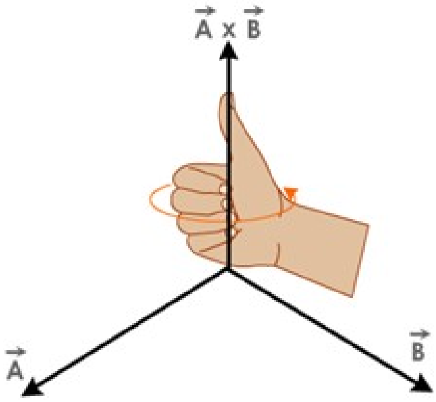
i) Right hand screw rule

- If a right-handed screw is rotated from vector A to vector B, through a small angle, the direction of the advancing screw gives the direction of the cross product of vectors A and B.



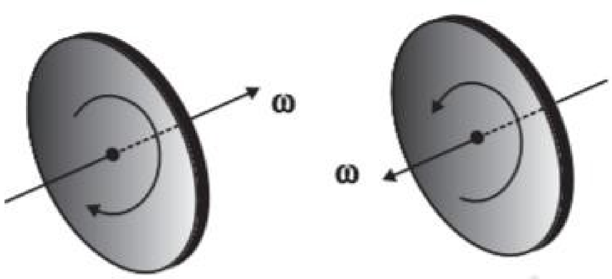
ii) Right hand thumb rule

- If the fingers of the right hand are curled in such a way that they point along the direction of rotation from vector A to vector B through a small angle, then the thumb points in the direction of the cross product of vector A and B.



ANGULAR VELOCITY AND ITS RELATION WITH LINEAR VELOCITY

- The average angular velocity of the particle over the interval Δt *is* $\Delta\theta / \Delta t$.
- The instantaneous angular velocity $\omega = d\theta/dt$.



- The general relation connecting angular velocity and linear velocity is given by

$$\vec{v} = \vec{\omega} \times \vec{r}$$

- The angular velocity is a vector quantity.

Angular acceleration

• Angular acceleration α is the time rate of change of angular velocity.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

- If the axis of rotation is fixed, the direction of ω and hence, that of α is fixed.

Moment of force (Torque)

- The **rotational analogue of force** is moment of force or torque.
- Torque is given by

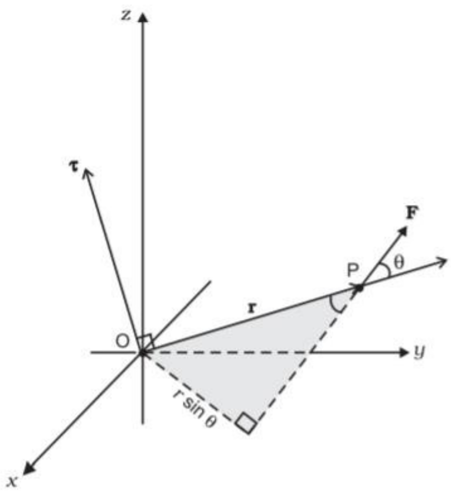
$$\vec{\tau} = \vec{r} \times \vec{F}$$

- The moment of force (or torque) is a vector quantity.
- The symbol τ stands for the Greek letter tau.
- The magnitude of τ is

$$\tau = rF\sin\theta$$

- Moment of force has dimensions same as those of work or energy **[ML2T-2]**
- Moment of a force is a vector, while work is a scalar.

- The SI unit of moment of force is **Newton Meter** (Nm).



Angular momentum of a particle

- Angular momentum is the **rotational analogue of linear momentum**.
- The angular momentum is given by

$$\vec{l} = \vec{r} \times \vec{p}$$

- The magnitude of the angular momentum vector is

$$l = rp\sin\theta$$

Relation Between Angular Momentum and Torque

- **We have**

$$\vec{l} = \vec{r} \times \vec{p}$$

- Differentiating with respect to time,

$$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

- But ,

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = (\vec{r} \times \frac{d\vec{p}}{dt}) + (\frac{d\vec{r}}{dt} \times \vec{p})$$

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = (\vec{r} \times \vec{F}) + (\vec{v} \times m\vec{v})$$

- Here F= (dp/dt) and p= mv
- Since (**v x v) = 0**

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = (\vec{r} \times \vec{F}) + (\vec{v} \times m\vec{v})$$

- **Thus**

$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F}$$

- Therefore

$$\frac{d\vec{l}}{dt} = \vec{\tau}$$

- Thus, the time rate of change of the angular momentum of a particle is equal to the torque acting on it.

Torque and angular momentum for a system of particles

• For a system of *n particles, the total angular momentum is*

$$\vec{L} = l_1 + l_2 + \ldots\ldots + l_n$$

$$\vec{L} = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i$$

- Thus

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$$

Conservation of Angular Momentum

- If the external torque acting on a system is zero, then

$$\frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = \text{constant}$$

- Thus, **if total external torque on a system is zero the angular**

momentum is conserved.

EQUILIBRIUM OF A RIGID BODY

• A rigid body is said to be in mechanical equilibrium, if both its linear momentum and angular momentum are not changing with time.

Translational equilibrium

- **The vector sum of the forces, on the rigid body is zero – linear momentum is conserved.**

$$\vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = 0$$

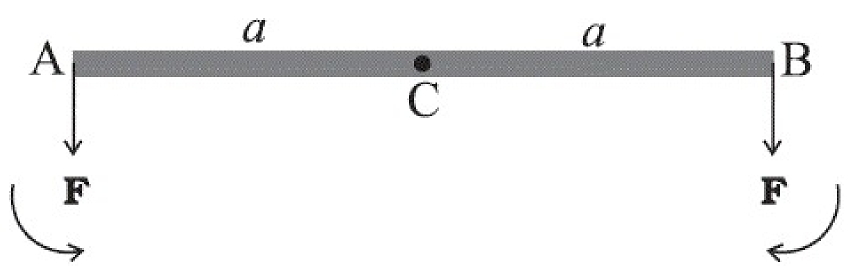
Rotational equilibrium

- **The vector sum of the torques on the rigid body is zero – angular momentum is conserved.**

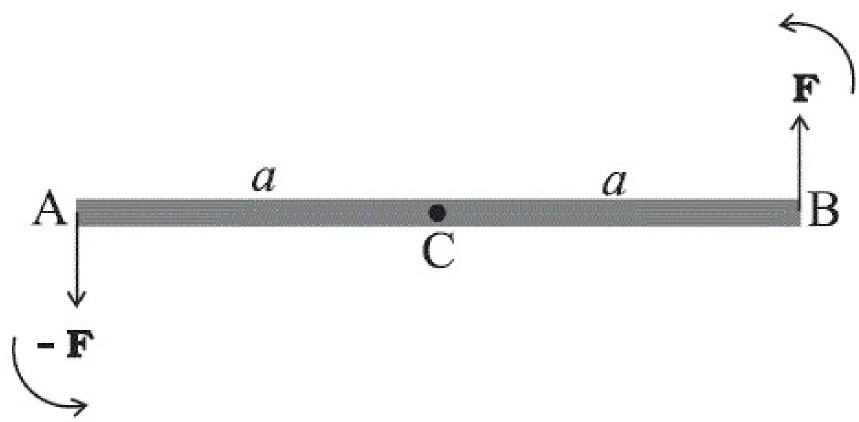
$$\vec{\tau}_1 + \vec{\tau}_2 + ... + \vec{\tau}_n = 0$$

Partial equilibrium

- When two parallel forces both equal in magnitude are applied perpendicular to alight rod , **the system will be in rotational equilibrium, and not in translational equilibrium**

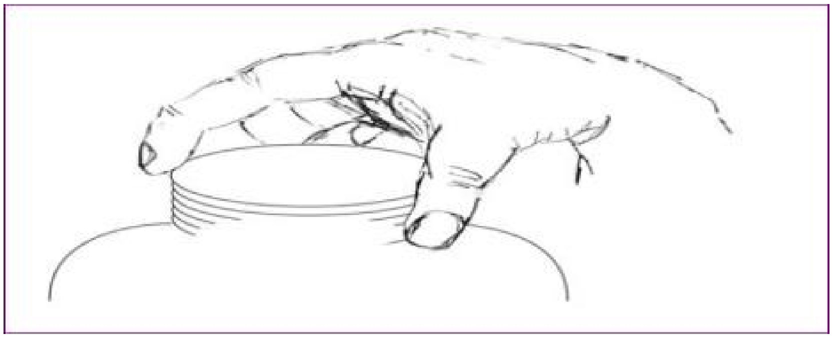


- When two forces are applied perpendicular in two opposite directions, the body is in translational equilibrium; but not in rotational equilibrium.

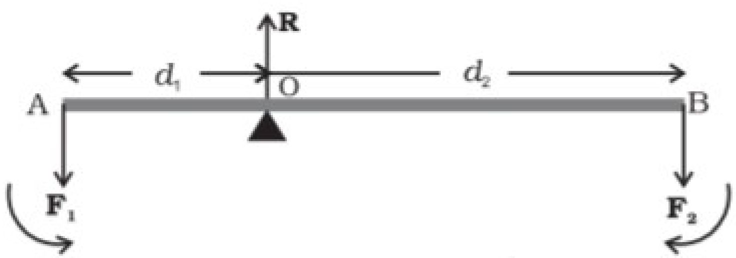


Couple

- A pair of equal and opposite forces with different lines of action is known as a **couple*.***
- **A couple produces rotation without translation**.
- When we open the lid of a bottle by turning it, our fingers are applying a ***couple*** to the lid.



Principle of moments



- For a lever at equilibrium the moment on the left = moment on the right **load arm × load = effort arm × effort**

$$d_1F_1 = d_2F_2$$

- This is the ***principle of moments for a lever.***
- A **lever** is a light rod pivoted at a point along its length. This point is called the **fulcrum.**
- A seesaw on the children's playground is a typical example of a lever.
- **Anticlockwise moments – positive Clockwise moments – negative**
- In the case of the lever force *F1 is usually* some weight to be lifted. It is called the ***load*** and its distance from the fulcrum *d1 is called* the

load arm.

Mechanical Advantage (M.A.)

• The ratio **F1/F2** is called **the Mechanical**

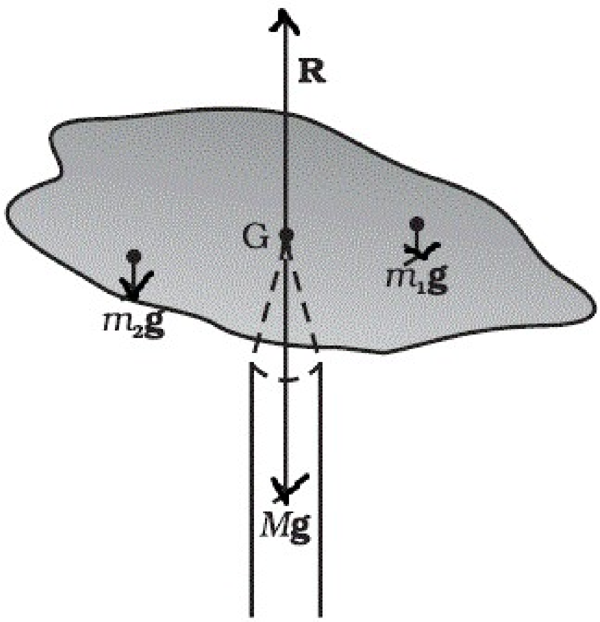
Advantage (M.A.)

$$M.A. = \frac{F_1}{F_2} = \frac{d_2}{d_1}$$

- If the effort arm *d2 is larger than the load* arm, the mechanical advantage is greater than one.
- Mechanical advantage greater than one means that a small effort can be used to lift a large load.

Centre of gravity

- The CG of a body is the point where the **total gravitational torque on the body is zero.**
- If acceleration due to gravity is same at all parts of a body, its centre of gravity coincides with centre of mass.
- If g varies centre of gravity and centre of mass are different.



MOMENT OF INERTIA

- Moment of inertia is the rotational analogue of mass of a body.
- The moment of inertia given by

$$I = \sum_{i=1}^{n} m_i r_i^2$$

- It is independent of the magnitude of the angular velocity.
- It is regarded as a measure of **rotational inertia of the body**
- Unit is kgm2.

The moment of inertia of a rigid body depends on :

- the mass of the body,
- its shape and size distribution of mass about the axis of rotation,

- The position and orientation of the axis of rotation.

Rotational kinetic energy

- The kinetic energy in terms of moment of inertia is
- We have kinetic energy of a particle

- The kinetic energy in terms of moment of inertia is
- We have kinetic energy of a particle

$$k_i = \frac{1}{2} m_i v_i^{\ 2}$$

- The velocity is given by

$$v_i = r_i \omega$$

- Thus for a system of particles

$$K = \frac{1}{2} \sum_{i=1}^{n} m_i r_i^{\ 2} \omega^2$$

- Therefore

$$K = \frac{1}{2} I \omega^2$$

- where ω – angular velocity, I – moment of inertia
- or

$$K = \frac{L^2}{2I}$$

• where L – angular momentum

Moment of Inertia of a thin Ring

- Consider a thin ring of radius *R and mass M,* rotating in its own plane around its centre with angular velocity ω.
- Each mass element of the ring is at a distance R from the axis, and moves with a speed *Rω.*
- The kinetic energy is therefore,

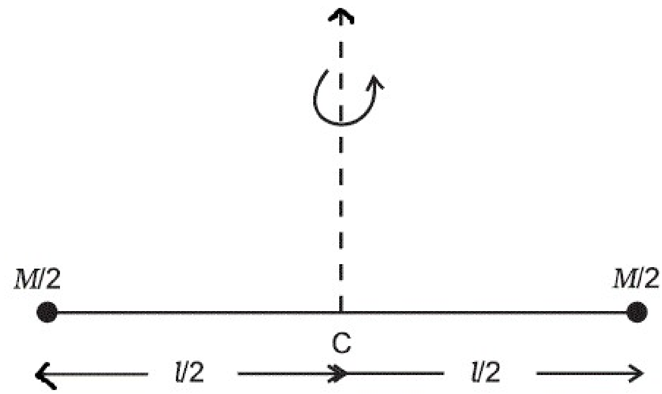
$$K = \frac{1}{2} M v^2 = \frac{1}{2} M R^2 \omega^2$$

- Therefore, comparing the equation with

• We get *I* = *MR*2

Moment of Inertia of a rigid Rod

- Consider a rigid massless rod of length with a pair of small masses, rotating about an axis through the centre of mass perpendicular to the rod.



- Each mass *M/2 is at a distance l/2 from the axis.*
- The moment of inertia of the masses is therefore given by

$$I = \frac{M}{2}\left(\frac{l}{2}\right)^2 + \frac{M}{2}\left(\frac{l}{2}\right)^2$$

- Thus

$$I = \frac{Ml^2}{4}$$

Radius of Gyration

- In general moment of inertia can be written as

I =Mk2

- Here the length *k is a* geometric property of the body and axis of rotation. It is called the **radius of gyration.**

- **The radius of gyration**

k =√I/M

- **The radius of gyration of a body about an axis** may be defined as the distance from the axis of a mass point whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.

Moment of inertia of different bodies

No	Body	Axis	I
1	Thin circular ring radius R	Perpendicular to plane ,at centre	MR^2
2	Thin circular ring radius R	Diameter	$MR^2/2$
3	Thin rod ,length L	Perpendicular to rod ,at mid point	*ML*2 /12

4	Circular disc radius R	Perpendicular disc at centre	
5	Circular disc radius R	diameter	
6	Hollow cylinder radius R	Axis of cylinder	MR^2
7	Solid cylinder radius R	Axis of cylinder	*MR*2/2
8	Solid sphere radius R	Diameter	$\frac{2}{5}$ *MR*2

Practical uses of moment of inertia

- The machines, such as steam engine and the automobile engine, etc., that produce rotational motion have a disc with a large moment of inertia, called a **flywheel.**
- Because of its large moment of inertia, the flywheel resists the sudden increase or decrease of the speed of the vehicle.
- It allows a gradual change in the speed and prevents jerky motions, thereby ensuring a smooth ride for the passengers on the vehicle.

Theorem of Perpendicular Axes

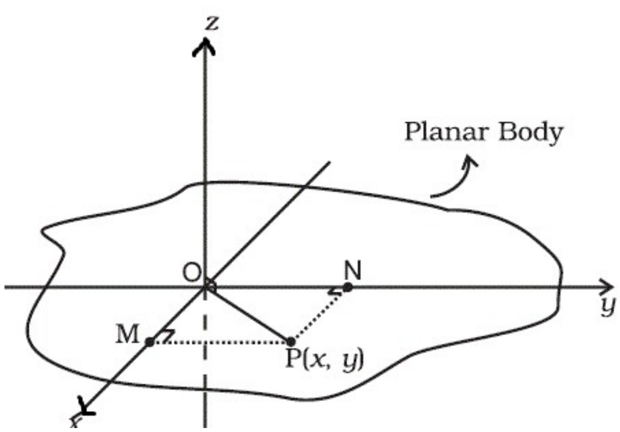
- **It states that the moment of inertia of a planar body (lamina)**

about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

- **Thus**

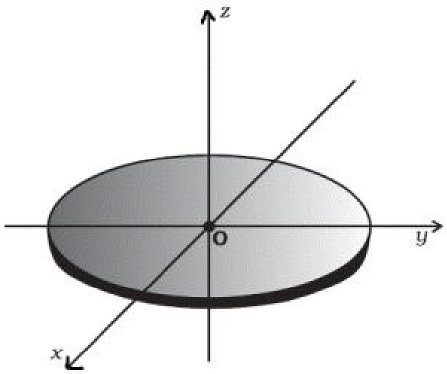
$$I_z = I_x + I_y$$

- This theorem is applicable to bodies which are planar.



PROBLEM

• What is the moment of inertia of a disc about one of its diameters?



Solution

- The moment of inertia of the disc about an axis perpendicular to it and through its centre is

$$I_z = \frac{MR^2}{2}$$

- Where M –mass, R – radius
- By symmetry of the disc, the moment of inertia about any diameter is same.

$$I_x = I_y$$

- Using perpendicular axis theorem

$$I_z = I_x + I_y = 2I_x$$

$$2I_x = \frac{MR^2}{2}$$

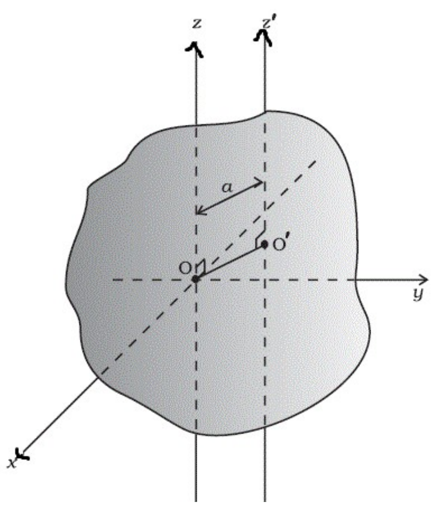
$$I_x = \frac{MR^2}{4}$$

Theorem of parallel axes

- **The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.**

Iz‘ = *Iz* + *Ma*2

- Where a –distance between two parallel axes.
- This theorem is applicable to a body of any shape.



PROBLEM – 1

• What is the moment of inertia of a rod of mass *M,* length *l* about an axis perpendicular to it through one end?

Solution

- The moment of inertia about an axis perpendicular and through the midpoint of the rod is

$$I_z = \frac{Ml^2}{12}$$

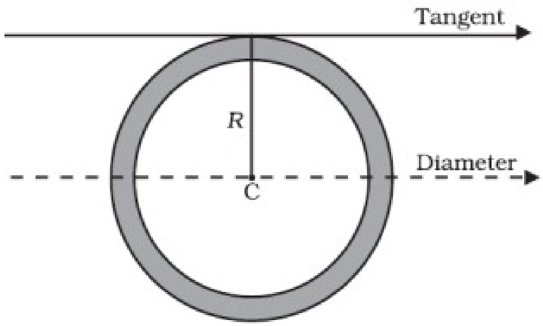
- Thus, using parallel axis theorem, the moment of inertia about an axis perpendicular through one end is

$$I_{z'} = I_z + Ma^2 = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$$

POBLEM – 2

• What is the moment of inertia of a ring about a tangent to the circle of the ring?

Solution



$$I_{\text{tangent}} = I_{dia} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

KINEMATICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

- The kinematical equations of linear motion with uniform (i.e. constant) acceleration are:

$$v = v_0 + at$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^{\ 2} + 2a(x - x_0)$$

- The kinematic equations for ***rotational motion*** with uniform angular acceleration are:

$$\omega = \omega_0 + \alpha t$$

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0{}^2 + 2\alpha(\theta - \theta_0)$$

PROBLEM

• The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds.

1. What is its angular acceleration, assuming the acceleration to be uniform?
2. How many revolutions does the engine make during this time?

Solution

i)

We have

$$\omega = \omega_0 + \alpha t$$

ω_0 = initial angular speed in rad/s

$= 2\pi \times$ angular speed in rev/s

$$= \frac{2\pi \times \text{angular speed in rev/min}}{60 \text{ s/min}}$$

$$= \frac{2\pi \times 1200}{60} \text{rad/s}$$

$= 40\pi$ rad/s

- Thus

Angular acceleration

$$\alpha = \frac{\omega - \omega_0}{t} = 4\ \pi\ \text{rad/s}^2$$

ii)The angular displacement in time *t* is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= (40\pi \times 16 + \frac{1}{2} \times 4\pi \times 16^2) \text{ rad}$$

$$= (640\pi + 512\pi) \text{ rad}$$

$$= 1152\pi \text{ rad}$$

$$\text{Number of revolutions} = \frac{1152\pi}{2\pi} = 576$$

Work done by a torque

• The work done by the total (external) torque τ which acts on the body rotating about a fixed

axis is given by

$$dW = \tau d\theta$$

- The instantaneous power is given by

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$

$$P = \tau\omega$$

Relation connecting torque and moment of inertia

- In a perfectly rigid body there is no internal motion. The work done by external torques is goes on to increase the kinetic energy of the body.
- Thus the rate of work done is equal to the rate of change of kinetic energy.
- Thus

$$\frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = \frac{dW}{dt} = \tau\omega$$

$$\frac{1}{2}I(2\omega)\frac{d\omega}{dt} = I\alpha\omega = \tau\omega$$

$$I\alpha\omega = \tau\omega$$

- Since

$$\frac{d\omega}{dt} = \alpha$$

• The torque is given by

$$\tau = I\alpha$$

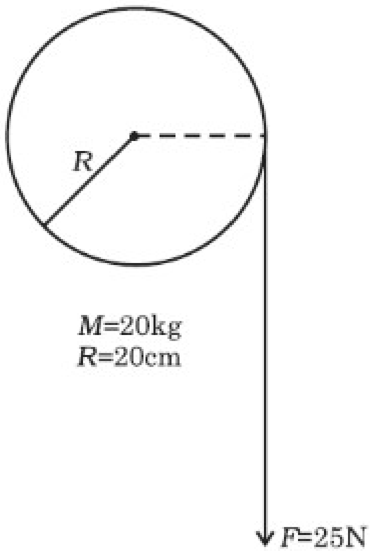
- Where I –moment of inertia, α- angular acceleration

PROBLEM

- A cord of negligible mass is wound round the rim of a fly wheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in figure. The flywheel is mounted on a horizontal axle with frictionless bearings.

1. Compute the angular acceleration of the wheel.
2. Find the work done by the pull, when 2mof the cord is unwound.
3. Find also the kinetic energy of the wheel at this point. Assume that

the wheel starts from rest.



Solution

1. We have the torque

$$\tau = FR = 25 \times 0.20 = 5.0Nm$$

- But $\tau = I\alpha$
- Where

$$I = \frac{MR^2}{2} = \frac{20 \times 0.20^2}{2} = 0.4kgm^2$$

- Therefore

$$0.4 \times \alpha = FR = 5$$

$$\alpha = \frac{5}{0.4} = 12.5s^{-2}$$

1. Work done by the pull unwinding 2m of the cord is

$$W = Force \times displacement$$
$$= 25 \times 2 = 50J$$

1. The kinetic energy gained is

$$K = \frac{1}{2} I\omega^2$$

- The angular displacement θ = length of unwound string / radius of wheel = 2m/0.2 m = 10 rad

- Thus

$$\omega^2 = \omega_0^{\ 2} + 2\alpha\theta$$

$$\omega^2 = 0 + 2 \times 12.5 \times 10 = 250(rad/s)^2$$

- Therefore

$$K = \frac{1}{2} \times 0.4 \times 250 = 50J$$

Relation connecting angular momentum and moment of inertia

- The angular momentum of a particle is given by

$$\vec{l}_i = \vec{r}_i \times \vec{p}_i = \vec{r}_i \times m_i v_i$$

- In rotation about a fixed axis velocity and radius will be perpendicular, the angular momentum is

$$\vec{l}_i = m_i r_i v_i \hat{k}$$

- **Where k – unit vector perpendicular to r and v.**
- But

$$v_i = r_i\omega$$

- Thus

$$\vec{l}_i = m_i r_i^2 \omega \hat{k}$$

- There for the total angular momentum of a system of particle is given by

$$\vec{L} = \sum_{i=1}^{n} m_i r_i^2 \omega \hat{k}$$

$$\vec{L} = I\omega\hat{k}$$

- Where

$$I = \sum_{i=1}^{n} m_i r_i^2$$

- The magnitude of angular momentum is given by

$$L = I\omega$$

Principle of Conservation of Angular momentum

- The angular momentum is given by

L = I ω

- If the external torque is zero,

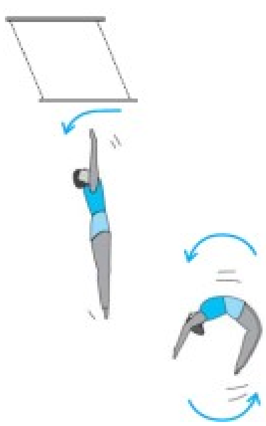
$L = I\omega =$ constant

Examples of principle of conservation of angular momentum

- When we stretch hand angular speed is reduced (moment of inertia is increased) and when hand is closed angular speed is increased

(moment inertia is decreased)

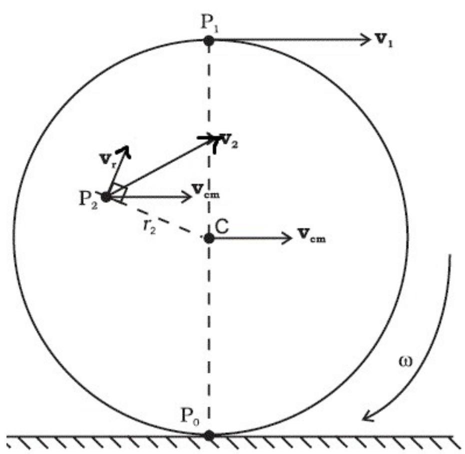
- A circus acrobat and a diver take advantage of this principle.



- Also, skaters and classical, Indian or western, dancers etc , use this principle.

ROLLING MOTION

- Rolling motion is the combination of translation and rotation.
- All wheels used in transportation have rolling motion.
- When a disc rolls without slipping , at any instant of time the bottom of the disc which is in contact with the surface is at rest on the surface



- Thus, for the disc , the condition for rolling without slipping is

$$v_{cm} = R\omega$$

- Where vcm – velocity of centre of mass
- Thus, the velocity of point P1 at the top of the disc (**v1)** has a magnitude

$$v_1 = v_{cm} + R\omega = 2v_{cm}$$

• V1 is directed parallel to the level surface.

Kinetic Energy of Rolling Motion

- The kinetic energy of a rolling body = kinetic energy of rotation + kinetic energy of translation .
- Thus
- $$K = \frac{1}{2} I\omega^2 + \frac{1}{2} m{v_{cm}}^2$$
- Substituting *I* = *m k*2 and *vcm* =*R*ω
- $$K = \frac{1}{2}\frac{mk^2R^2\omega^2}{R^2} + \frac{1}{2} m{v_{cm}}^2 = \frac{1}{2}\frac{mk^2{v_{cm}}^2}{R^2} + \frac{1}{2} m{v_{cm}}^2$$

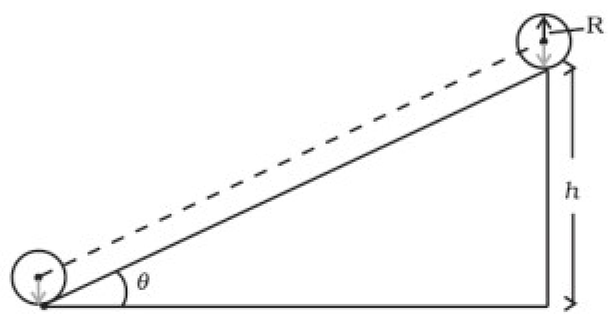
• Thus

$$K = \frac{1}{2} m{v_{cm}}^2\left(1 + \frac{k^2}{R^2}\right)$$

PROBLEM

Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. Which of the bodies reaches the ground with maximum velocity?

Solution



- Applying conservation of energy

Kinetic energy gained = potential energy

- We have
  $$K = \frac{1}{2} m {v_{cm}}^2 \left(1 + \frac{k^2}{R^2}\right)$$
-
- v – velocity of centre of mass
- Potential energy = mgh
- Therefore

$$\frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)=mgh$$

$$v^2=\frac{2gh}{\left(1+\frac{k^2}{R^2}\right)}$$

- **Thus**

$$v=\sqrt{\frac{2gh}{\left(1+\frac{k^2}{R^2}\right)}}$$

Velocity of ring

- For a ring k2 = R2

- Therefore

$$v_{ring}=\sqrt{\frac{2gh}{\left(1+\frac{R^2}{R^2}\right)}}=\sqrt{gh}$$

Velocity of a solid cylinder

- For a solid cylinder , $k^2 = R^2/2$

$$v_{cylinder} = \sqrt{\frac{2gh}{\left(1+\frac{R^2/2}{R^2}\right)}} = \sqrt{\frac{4gh}{3}}$$

Velocity of a sphere

For a solid sphere $k^2 = 2R^2/5$

$$v_{sphere} = \sqrt{\frac{2gh}{\left(1+\frac{2R^2/5}{R^2}\right)}} = \sqrt{\frac{10gh}{7}}$$

- Thus $v_{sphere} > v_{cylinder} > v_{ring}$
- The sphere has the greatest velocity.