Exercise 1b

Question 1.
Define a relation on a set. What do you mean by the domain and range of a relation? Give an example.

Answer:
Relation: Let A and B be two sets. Then a relation R from set A to set B is a subset of A x B. Thus, R is a relation to A to B $\Leftrightarrow$ R $\subseteq$ A x B.

If R is a relation from a non-void set B and if (a,b) $\in$ R, then we write a R b which is read as 'a is related to b by the relation R'. if (a,b) $\notin$ R, then we write a R b, and we say that a is not related to b by the relation R.

Domain: Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R.

Thus, domain of R ={a : (a,b) $\in$ R}. The domain of R $\subseteq$ A.

Range: let R be a relation from a set A to a set B. then the set of all second component or coordinates of the ordered pairs belonging to R is called the range of R.

Example 1: R = {(-1, 1), (1, 1), (-2, 4), (2, 4)}.

dom (R) = {-1, 1, -2, 2} and range (R) = {1, 4}

Example 2: R ={(a, b): a, b $\in$ N and a + 3b = 12}

dom (R) = {3, 6, 9} and range (R) = {3, 2, 1}

Question 2.
Let A be the set of all triangles in a plane. Show that the relation

R = $\{(\Delta_1, \Delta_2) : \Delta_1 \sim \Delta_2\}$ is an equivalence relation on A.

Answer:
Let R = $\{(\Delta_1, \Delta_2) : \Delta_1 \sim \Delta_2\}$ be a relation defined on A.

Now,

R is Reflexive if $(\Delta, \Delta) \in R \forall \Delta \in A$

We observe that for each $\Delta \in A$ we have,

$\Delta \sim \Delta$ since, every triangle is similar to itself.

$\Rightarrow (\Delta, \Delta) \in R \ \forall \ \Delta \in A$

$\Rightarrow$ R is reflexive.

R is Symmetric if $(\Delta_1, \Delta_2) \in R \Rightarrow (\Delta_2, \Delta_1) \in R \forall \Delta_1, \Delta_2 \in A$

Let $(\Delta_1, \Delta_2) \in R \ \forall \ \Delta_1, \Delta_2 \in A$

$\Rightarrow \Delta_1 \sim \Delta_2$

$\Rightarrow \Delta_2 \sim \Delta_1$

$\Rightarrow (\Delta_2, \Delta_1) \in R$

$\Rightarrow$ R is symmetric

R is Transitive if $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow (\Delta_1, \Delta_3) \in R \forall \Delta_1, \Delta_2, \Delta_3 \in A$

Let $(\Delta_1, \Delta_2) \in R$ and $((\Delta_2, \Delta_3) \in R \ \forall \ \Delta_1, \Delta_2, \Delta_3 \in A$

$\Rightarrow \Delta_1 \sim \Delta_2$ and $\Delta_2 \sim \Delta_3$

$\Rightarrow \Delta_1 \sim \Delta_3$

$\Rightarrow (\Delta_1, \Delta_3) \in R$

$\Rightarrow$ R is transitive.

Since R is reflexive, symmetric and transitive, it is an equivalence relation on A.

Question 3.

Let R = {(a, b) : a, b $\in$ Z and (a + b) is even}.

Show that R is an equivalence relation on Z.

Answer:
In order to show R is an equivalence relation, we need to show R is Reflexive, Symmetric and Transitive.

Given that, $\forall$ a, b $\in$ Z, R = {(a, b) : (a + b) is even }.

Now,

<u>R is Reflexive if (a,a)$\in$R$\forall$a$\in$Z</u>

For any a $\in$ A, we have

a+a = 2a, which is even.

$\Rightarrow$ (a,a) $\in$ R

Thus, R is reflexive.

<u>R is Symmetric if (a,b)$\in$R$\Rightarrow$(b,a)$\in$R$\forall$a,b$\in$Z</u>

(a,b) $\in$ R

$\Rightarrow$ a+b is even.

$\Rightarrow$ b+a is even.

$\Rightarrow$ (b,a) $\in$ R

Thus, R is symmetric .

<u>R is Transitive if (a,b)$\in$R and (b,c)$\in$R$\Rightarrow$(a,c)$\in$R$\forall$a,b,c$\in$Z</u>

Let (a,b) $\in$ R and (b,c) $\in$ R $\forall$ a, b,c $\in$ Z

$\Rightarrow$ a+b = 2P and b+c = 2Q

Adding both, we get

$a+c+2b = 2(P+Q)$

$\Rightarrow a+c = 2(P+Q)-2b$

$\Rightarrow$ a+c is an even number

$\Rightarrow (a, c) \in R$

Thus, R is transitive on Z.

Since R is reflexive, symmetric and transitive it is an equivalence relation on Z.

Question 4.
Let R = {(a, b) : a, b $\in$ Z and (a - b) is divisible by 5}.

Show that R is an equivalence relation on Z.

Answer:
In order to show R is an equivalence relation, we need to show R is Reflexive, Symmetric and Transitive.

Given that, $\forall$ a, b $\in$Z, aRb if and only if a – b is divisible by 5.

Now,

<u>R is Reflexive if $(a,a)\in R \forall a\in Z$</u>

aRa $\Rightarrow$ (a-a) is divisible by 5.

$a-a = 0 = 0 \times 5$ [since 0 is multiple of 5 it is divisible by 5]

$\Rightarrow$ a-a is divisible by 5

$\Rightarrow (a,a) \in R$

Thus, R is reflexive on Z.

R is Symmetric if $(a,b) \in R \Rightarrow (b,a) \in R \forall a,b \in Z$

$(a,b) \in R \Rightarrow (a-b)$ is divisible by 5

$\Rightarrow (a-b) = 5z$ for some $z \in Z$

$\Rightarrow -(b-a) = 5z$

$\Rightarrow b-a = 5(-z)$ [$\because z \in Z \Rightarrow -z \in Z$]

$\Rightarrow (b-a)$ is divisible by 5

$\Rightarrow (b,a) \in R$

Thus, R is symmetric on Z.

R is Transitive if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R \forall a,b,c \in Z$

$(a,b) \in R \Rightarrow (a-b)$ is divisible by 5

$\Rightarrow a-b = 5z_1$ for some $z_1 \in Z$

$(b,c) \in R \Rightarrow (b-c)$ is divisible by 5

$\Rightarrow b-c = 5z_2$ for some $z_2 \in Z$

Now,

$a-b = 5z_1$ and $b-c = 5z_2$

$\Rightarrow (a-b) + (b-c) = 5z_1 + 5z_2$

$\Rightarrow a-c = 5(z_1 + z_2) = 5z_3$ where $z_1 + z_2 = z_3$

$\Rightarrow a-c = 5z_3$ [$\because z_1, z_2 \in Z \Rightarrow z_3 \in Z$]

$\Rightarrow (a-c)$ is divisible by 5.

⇒ (a, c) ∈ R

Thus, R is transitive on Z.

Since R is reflexive, symmetric and transitive it is an equivalence relation on Z.

Question 5.
Show that the relation R defined on the set A = (1, 2, 3, 4, 5), given by

R = {(a, b) : |a – b| is even} is an equivalence relation.

Answer:
In order to show R is an equivalence relation we need to show R is Reflexive, Symmetric and Transitive.

Given that, ∀ a, b ∈A, R = {(a, b) : |a – b| is even}.

Now,

<u>R is Reflexive if (a,a)∈R∀a∈A</u>

For any a ∈ A, we have

|a-a| = 0, which is even.

⇒ (a,a) ∈ R

Thus, R is reflexive.

<u>R is Symmetric if (a,b)∈R⇒(b,a)∈R∀a,b∈A</u>

(a,b) ∈ R

⇒ |a-b| is even.

⇒ |b-a| is even.

⇒ (b,a) ∈ R

Thus, R is symmetric .

R is Transitive if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R \forall a,b,c \in A$

Let $(a,b) \in R$ and $(b,c) \in R \ \forall\ a, b,c \in A$

$\Rightarrow |a - b|$ is even and $|b - c|$ is even

$\Rightarrow$ (a and b both are even or both odd) and (b and c both are even or both odd)

Now two cases arise:

Case 1 : when b is even

Let $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow |a - b|$ is even and $|b - c|$ is even

$\Rightarrow$ a is even and c is even [$\because$ b is even]

$\Rightarrow |a - c|$ is even [$\because$ difference of any two even natural numbers is even]

$\Rightarrow (a, c) \in R$

Case 2 : when b is odd

Let $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow |a - b|$ is even and $|b - c|$ is even

$\Rightarrow$ a is odd and c is odd [$\because$ b is odd]

$\Rightarrow |a - c|$ is even [$\because$ difference of any two odd

natural numbers is even]

$\Rightarrow (a, c) \in R$

Thus, R is transitive on Z.

Since R is reflexive, symmetric and transitive it is an equivalence relation on Z.

Question 6.
Show that the relation R on N × N, defined by

(a, b) R (c, d) ⇔ a + d = b + c

is an equivalent relation.

Answer:
In order to show R is an equivalence relation we need to show R is Reflexive, Symmetric and Transitive.

Given that, R be the relation in N ×N defined by (a, b) R (c, d) if a + d = b + c for (a, b), (c, d) in N ×N.

<u>R is Reflexive if (a, b) R (a, b) for (a, b) in N ×N</u>

Let (a,b) R (a,b)

⇒ a+b = b+a

which is true since addition is commutative on N.

⇒ R is reflexive.

<u>R is Symmetric if (a,b) R (c,d)⇒(c,d) R (a,b) for (a, b), (c, d) in N ×N</u>

Let (a,b) R (c,d)

⇒ a+d = b+c

⇒ b+c = a+d

⇒ c+b = d+a [since addition is commutative on N]

⇒ (c,d) R (a,b)

⇒ R is symmetric.

R is Transitive if (a,b) R (c,d) and (c,d) R (e,f)⇒(a,b) R (e,f) for (a, b), (c, d), (e,f) in N ×N

Let (a,b) R (c,d) and (c,d) R (e,f)

⇒ a+d = b+c and c+f = d+e

⇒ (a+d) − (d+e) = (b+c) − (c+f)

⇒ a-e= b-f

⇒ a+f = b+e

⇒ (a,b) R (e,f)

⇒ R is transitive.

Hence, R is an equivalence relation.

Question 7.
Let S be the set of all real numbers and let

R = {(a, b) : a, b ∈ S and a = ± b}.

Show that R is an equivalence relation on S.

Answer:
In order to show R is an equivalence relation we need to show R is Reflexive, Symmetric and Transitive.

Given that, ∀ a, b ∈ S, R = {(a, b) : a = ± b }

Now,

R is Reflexive if (a,a)∈R∀a∈S

For any a ∈ S, we have

$a = \pm a$

$\Rightarrow (a,a) \in R$

Thus, R is reflexive.

R is Symmetric if $(a,b) \in R \Rightarrow (b,a) \in R \forall a,b \in S$

$(a,b) \in R$

$\Rightarrow a = \pm b$

$\Rightarrow b = \pm a$

$\Rightarrow (b,a) \in R$

Thus, R is symmetric .

R is Transitive if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R \forall a,b,c \in S$

Let $(a,b) \in R$ and $(b,c) \in R \forall a, b,c \in S$

$\Rightarrow a = \pm b$ and $b = \pm c$

$\Rightarrow a = \pm c$

$\Rightarrow (a, c) \in R$

Thus, R is transitive.

Hence, R is an equivalence relation.

Question 8.
Let S be the set of all points in a plane and let R be a relation in S defined by $R = \{(A, B) : d(A, B) < 2 \text{ units}\}$, where d(A, B) is the distance between the points A and B.

Show that R is reflexive and symmetric but not transitive.

Answer:
Given that, $\forall$ A, B $\in$ S, R = {(A, B) : d(A, B) < 2 units}.

Now,

R is Reflexive if $(A,A) \in R \forall A \in S$

For any A $\in$ S, we have

d(A,A) = 0, which is less than 2 units

$\Rightarrow (A,A) \in R$

Thus, R is reflexive.

R is Symmetric if $(A, B) \in R \Rightarrow (B,A) \in R \forall A,B \in S$

$(A, B) \in R$

$\Rightarrow d(A, B) < 2$ units

$\Rightarrow d(B, A) < 2$ units

$\Rightarrow (B,A) \in R$

Thus, R is symmetric .

R is Transitive if $(A, B) \in R$ and $(B,C) \in R \Rightarrow (A,C) \in R \forall A,B,C \in S$

Consider points A(0,0),B(1.5,0) and C(3.2,0).

d(A,B)=1.5 units < 2 units and d(B,C)=1.7 units < 2 units

$d(A,C) = 3.2 \nless 2$

$\Rightarrow (A, B) \in R$ and $(B,C) \in R \Rightarrow (A,C) \notin R$

Thus, R is not transitive.

Thus, R is reflexive, symmetric but not transitive.

Question 9.
Let S be the set of all real numbers. Show that the relation R = {(a, b) : $a^2 + b^2 = 1$} is symmetric but neither reflexive nor transitive.

Answer:
Given that, ∀ a, b ∈ S, R = {(a, b) : $a^2 + b^2 = 1$ }

Now,

<u>R is Reflexive if (a,a)∈R∀a∈S</u>

For any a ∈ S, we have

$a^2+a^2 = 2a^2 \neq 1$

⇒ (a,a) ∉ R

Thus, R is not reflexive.

<u>R is Symmetric if (a,b)∈R⇒(b,a)∈R∀a,b∈S</u>

(a,b) ∈ R

⇒ $a^2 + b^2 = 1$

⇒ $b^2 + a^2 = 1$

⇒ (b,a) ∈ R

Thus, R is symmetric .

<u>R is Transitive if (a,b)∈R and (b,c)∈R⇒(a,c)∈R∀a,b,c∈S</u>

Let (a,b) ∈ R and (b,c) ∈ R ∀ a, b,c ∈ S

⇒ $a^2 + b^2 = 1$ and $b^2 + c^2 = 1$

Adding both, we get

$a^2 + c^2 + 2b^2 = 2$

$\Rightarrow a^2 + c^2 = 2 - 2b^2 \neq 1$

$\Rightarrow (a, c) \notin R$

Thus, R is not transitive.

Thus, R is symmetric but neither reflexive nor transitive.

Question 10.
Let $R = \{(a, b) : a = b^2\}$ for all $a, b \in N$.

Show that R satisfies none of reflexivity, symmetry and transitivity.

Answer:
We have, $R = \{(a, b) : a = b^2\}$ relation defined on N.

Now,

We observe that, any element $a \in N$ cannot be equal to its square except 1.

$\Rightarrow (a,a) \notin R \ \forall a \in N$

For e.g. $(2,2) \notin R \because 2 \neq 2^2$

$\Rightarrow$ R is not reflexive.

Let $(a,b) \in R \ \forall a, b \in N$

$\Rightarrow a = b^2$

But b cannot be equal to square of a if a is equal to square of b.

$\Rightarrow (b,a) \notin R$

For e.g., we observe that $(4,2) \in R$ i.e $4 = 2^2$ but $2 \neq 4^2 \Rightarrow (2,4) \notin R$

$\Rightarrow$ R is not symmetric

Let $(a,b) \in R$ and $(b,c) \in R \ \forall \ a, b,c \in N$

$\Rightarrow a = b^2$ and $b = c^2$

$\Rightarrow a \neq c^2$

$\Rightarrow (a,c) \notin R$

For e.g., we observe that

$(16,4) \in R \Rightarrow 16 = 4^2$ and $(4,2) \in R \Rightarrow 4 = 2^2$

But $16 \neq 2^2$

$\Rightarrow (16,2) \notin R$

$\Rightarrow$ R is not transitive.

Thus, R is neither reflexive nor symmetric nor transitive.

Question 11.
Show that the relation $R = \{(a, b) : a > b\}$ on N is transitive but neither reflexive nor symmetric.

Answer:
We have, $R = \{(a, b) : a > b\}$ relation defined on N.

Now,

We observe that, any element $a \in N$ cannot be greater than itself.

$\Rightarrow (a,a) \notin R \ \forall \ a \in N$

$\Rightarrow$ R is not reflexive.

Let $(a,b) \in R \ \forall \ a, b \in N$

$\Rightarrow$ a is greater than b

But b cannot be greater than a if a is greater than b.

$\Rightarrow (b,a) \notin R$

For e.g., we observe that $(5,2) \in R$ i.e $5 > 2$ but $2 \ngtr 5 \Rightarrow (2,5) \notin R$

$\Rightarrow$ R is not symmetric

Let $(a,b) \in R$ and $(b,c) \in R \ \forall \ a, b,c \in N$

$\Rightarrow a > b$ and $b > c$

$\Rightarrow a > c$

$\Rightarrow (a,c) \in R$

For e.g., we observe that

$(5,4) \in R \Rightarrow 5 > 4$ and $(4,3) \in R \Rightarrow 4 > 3$

And we know that $5 > 3 \therefore (5,3) \in R$

$\Rightarrow$ R is transitive.

Thus, R is transitive but not reflexive not symmetric.

Question 12.
Let A = {1, 2, 3} and R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)}.

Show that R is reflexive but neither symmetric nor transitive.

Answer:
Given that, A = {1, 2, 3} and R = {1, 1), (2, 2), (3, 3), (1, 2), (2, 3)}.

Now,

R is reflexive $\because$ $(1,1),(2,2),(3,3) \in R$

R is not symmetric $\because$ $(1,2),(2,3) \in R$ but $(2,1),(3,2) \notin R$

R is not transitive $\because$ $(1,2) \in R$ and $(2,3) \in R \Rightarrow (1,3) \notin R$

Thus, R is reflexive but neither symmetric nor transitive.

Question 13.
Let A = (1, 2, 3, 4) and R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (3, 2)}. Show that R is reflexive and transitive but not symmetric.

Answer:
Given that, A = {1, 2, 3} and R = {1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (3, 2)}.

Now,

R is reflexive $\because$ $(1,1),(2,2),(3,3),(4,4) \in R$

R is not symmetric $\because$ $(1,2),(1,3),(3,2) \in R$ but $(2,1),(3,1),(2,3) \notin R$

R is transitive $\because$ $(1,3) \in R$ and $(3,2) \in R \Rightarrow (1,2) \in R$

Thus, R is reflexive and transitive but not symmetric.