Exercise 10j

Question 1.
Find the second derivate of :

(i) x^{11} (ii) 5^x

(iii) tan x (iv) $\cos^{-1}x$

Answer:
(i) x^{11}

Differentiating with respect to x

$f'(x) = 11x^{11-1}$

$f'(x)=11x^{10}$

Differentiating with respect to x

$f''(x) = 110x^{10-1}$

$f''(x)= 110x^9$

(ii) 5^x

Differentiating with respect to x

$f'(x)= 5^x \log_e 5$ [Formula: $a^x = a^x \log_e a$]

Differentiating with respect to x

$f''(x)= \log_e 5 \,.\, 5^x \log_e 5$

$= 5^x(\log_e 5)^2$

(iii) tan x

Differentiating with respect to x

$f'(x) = \sec^2 x$

Differentiating with respect to x

$f''(x) = 2\sec x \,.\, \sec x \tan x$

$= 2\sec^2 x \tan x$

(iv) $\cos^{-1}x$

Differentiating with respect to x

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

Differentiating with respect to x

$$f''(x) = \frac{-1}{2} \times \frac{-1}{(1-x^2)^{\frac{3}{2}}} \times -2x$$

$$= \frac{-x}{(1-x^2)^{\frac{3}{2}}}$$

Question 2.
Find the second derivative of:

(i) x sin x

(ii) $e^{2x} \cos 3x$

(iii) $x^3 \log x$

Answer:
Differentiating with respect to x

$f'(x) = \sin x + x\cos x$

Differentiating with respect to x

$f''(x) = \cos x + \cos x - x\sin x$

$= -\sin x + 2\cos x$

(ii) $e^{2x} \cos 3x$

Differentiating with respect to x

$f'(x) = 2e^{2x}\cos 3x + e^{2x}(-\sin 3x).3$

$= 2e^{2x}\cos 3x - 3e^{2x}\sin 3x$

Differentiating with respect to x

$f''(x) = 2.2e^{2x}\cos 3x + 2e^{2x}(-\sin 3x).3 - 3.2e^{2x}\sin 3x - 3e^{2x}\cos 3x.3$

$= 4e^{2x}\cos 3x - 6e^{2x}\sin 3x - 6e^{2x}\sin 3x - 9e^{2x}\cos 3x$

$=-12e^{2x}\sin 3x - 5e^{2x}\cos 3x$

(iii) $x^3 \log x$

Differentiating with respect to x

$f'(x) = 3x^2 \log x + \frac{x^3}{x}$

$f'(x) = 3x^2\log x + x^2$

Differentiating with respect to x

$$f''(x) = 6x \log x + \frac{3x^2}{x} + 2x$$

$= 6x \log x + 3x + 2x$

$= 6x \log x + 5x$

Question 3.

If $y = x + \tan x$, show that $\cos^2 . \frac{d^2y}{dx^2} - 2y + 2x = 0$.

Answer:
y = x + tan x, ⇒ tan x = y-x.... (i)

Differentiating with respect to x

$$\frac{dy}{dx} = 1 + \sec^2 x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = 2\sec x\ .\sec x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\sec^2 x\ \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2\tan x}{\cos^2 x}$$

$\Rightarrow \cos^2 x\ \frac{d^2y}{dx^2} = 2\tan x$ [putting value of tan x from (i)]

$$\Rightarrow \cos^2 x\ \frac{d^2y}{dx^2} = 2y - 2x$$

$$\Rightarrow \cos^2 x\ \frac{d^2y}{dx^2} - 2y + 2x = 0$$

Question 4.

If $\mathscr{Y}$ = 2 sin x + 3 cos x, s how that $\mathscr{Y}$ + $\frac{d^2y}{dx^2}$=0.

Answer:
Differentiating with respect to x

$$\frac{dy}{dx} = 2\cos x - 3\sin x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = -2\sin x - 3\cos x$$

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

Hence Proved

Question 5.

If y = 3 cos (log x) + 4 sin (log x), prove that $x^2 y_2 + x y_1 + y = 0$.

Answer:

Differentiating with respect to x

$$y_1 = -3\sin(\log x)\frac{1}{x} + 4\cos(\log x)\frac{1}{x}$$

$\Rightarrow y_1 = \frac{-3\sin(\log x) + 4\cos(\log x)}{x}$ [we can also write this as xy_1 = -3sin(log x) +4cos(log x)

Differentiating with respect to x

$$y_2 = \frac{x\left(-3\cos(\log x)\frac{1}{x} - 4\sin(\log x)\frac{1}{x}\right) - (-3\sin(\log x) + 4\cos(\log x))}{x^2}$$

$$\Rightarrow x^2 y_2 = \frac{-x}{x}(3\cos(\log x) - 4\sin(\log x)) - (y_1 x)$$

$$\Rightarrow x^2 y_2 = -y - xy_1$$

$\Rightarrow x^2 y_2 + xy_1 + y = 0$

Hence Proved

Question 6.

If y = e⁻ cos x , show that $\frac{d^2y}{dx^2}$ = $2e^{-x}\sin x$.

Answer:
Differentiating with respect to x

$$\frac{dy}{dx}\frac{dy}{dx} = -e^{-x}\cos xx + e^{-x}(-\sin xx)$$

$$\Rightarrow \frac{dy}{dx} = -ee^{-xx}\cos x - e^{-x}\sin x$$

$$\Rightarrow \frac{dy}{dx} = -e^{-x}(\cos x + \sin x)$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = e^{-x}(\cos x + \sin x) - e^{-x}(-\sin x + \cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^{-x}(\cos x + \sin x - (-\sin x) - \cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^{-x}(\sin x + \sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2e^{-x}\sin x$$

Hence proved

Question 7.

If $\mathscr{Y}$ = sec x – tan x. show that (cos x) $\frac{d^2y}{dx^2}$ = $\mathscr{Y}^2$.

Answer:
Differentiating with respect to x

$$\frac{dy}{dx} = \sec x\tan x - \sec^2 x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec x\ \tan x \times \tan x + \sec x \times \sec^2 x - 2\sec x \times \sec x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec x \tan^2 x + \sec^3 x - 2\sec^2 x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec x\,(\tan^2 x + \sec^2 x\ - 2\sec x \tan x\,)$$

$$\Rightarrow \frac{1}{\sec x}\frac{d^2y}{dx^2} = (\sec x - \tan x)^2$$

$$\Rightarrow \cos x \frac{d^2y}{dx^2} = y^2$$

Hence Proved

Question 8.

If y = (cosec x + cot x), prove that (sin x) $\frac{d^2y}{dx^2}$ - y^2 = 0.

Answer:

$$\frac{dy}{dx} = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \operatorname{cosec} x \cot^2 x + \operatorname{cosec}^3 x + 2\operatorname{cosec} x \times \operatorname{cosec} x \cot x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \operatorname{cosec} x\,(\cot^2 x + \operatorname{cosec}^2 x + 2\operatorname{cosec} x \cot x)$$

$$\Rightarrow \frac{1}{\operatorname{cosec} x}\frac{d^2y}{dx^2} = (\cot x + \operatorname{cosec} x)^2$$

$$\Rightarrow \sin x \frac{d^2y}{dx^2} = y^2$$

$$\Rightarrow \sin x \frac{d^2y}{dx^2} - y^2 = 0$$

Hence proved

Question 9.

If $y = \tan^{-1}$, show that $(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$.

Answer:
Differentiating with respect to x

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$\Rightarrow (1 + x^2)\frac{dy}{dx} = 1$$

Differentiating with respect to x

$$(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$$

Hence Proved

Question 10.

If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + (\tan)\frac{dy}{dx} + y\cos^2 x = 0$.

Answer:
Differentiating with respect to x

$$\frac{dy}{dx} = \cos(\sin x)\ \cos x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = -\sin(\sin x)\cos x\cos x - \sin x\cos(\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y\cos^2 x - \sin x\,\frac{\frac{dy}{dx}}{\cos x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y\cos^2 x - \tan x\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} + y\cos^2 x + \tan x\frac{dy}{dx} = 0$$

Hence Proved

Question 11.
If $\mathscr{Y}$ = a cos (log x) + b sin (log x), prove that $x^2\mathscr{Y}_2 + x\mathscr{Y}_1 + \mathscr{Y} = 0$.

Answer:
Differentiating with respect to x

$y_1 = -a\sin(\log x)\frac{1}{x}$ [can also be written as $-xy_1 =$ a sin (log x)]

Differentiating with respect to x

$$y_2 = \frac{-x\,a\cos(\log x)\frac{1}{x} + a\sin(\log x)}{x^2}$$

$$\Rightarrow x^2y_2 = -y - xy_1$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Hence Proved

Question 12.
Find the second derivative of e^{3x} sin 4x .

Answer:
Differentiating with respect to x

$$\frac{dy}{dx} = 3e^{3x}\sin 4x + 4e^{3x}\cos 4x$$

Differentiating with respect to x

$$\Rightarrow \frac{d^2y}{dx^2} = 9e^{3x}\sin 4x + 12e^{3x}\cos 4x + 12e^{3x}\cos 4x - 16e^{3x}\sin 4x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 24e^{3x}\cos 4x - 7e^{3x}\sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^{3x}(24\cos x - 7\sin x)$$

Question 13.
Find the second derivative of sin 3 x cos 5x.

Answer:

$$y = \frac{1}{2}[\sin(5x + 3x) + \sin(5x - 3x)]$$

$$y = \frac{1}{2}\sin 8x + \frac{1}{2}\sin 2x$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{8}{2}\cos 8x + \frac{2}{2}\cos 2x$$

$$\Rightarrow \frac{dy}{dx} = 4\cos 8x + \cos 2x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = -32\sin 8x - 2\sin 2x$$

Hence Proved

Question 14.

If $y = e^{\tan}$, prove that $(\cos^2 x)\dfrac{d^2y}{dx^2}$ - $(1+\sin 2x).\dfrac{dy}{dx} = 0$.

Answer:
Differentiating with respect to x

$$\frac{dy}{dx} = \sec^2 x\, e^{\tan x}$$

$$\Rightarrow \frac{1}{\sec^2 x}\frac{dy}{dx} = e^{\tan x}$$

$$\Rightarrow \cos^2 x\,\frac{dy}{dx} = e^{\tan x}$$

Differentiating with respect to x

$$(\cos^2 x)\frac{d^2y}{dx^2} - (2\cos x \sin x)\frac{dy}{dx} = \sec^2 x\, e^{\tan x}$$

$$\Rightarrow (\cos^2 x)\frac{d^2y}{dx^2} - \sin 2x\frac{dy}{dx} = \frac{dy}{dx}$$

$$\Rightarrow (\cos^2 x)\frac{d^2y}{dx^2} - \sin 2x\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\Rightarrow (\cos^2 x)\frac{d^2y}{dx^2} - (\sin 2x + 1)\frac{dy}{dx} = 0$$

Hence Proved

Question 15.
If $y = \dfrac{\log x}{x}$, show that $\dfrac{d^2y}{dx^2} = \dfrac{(2\log x - 3)}{x^3}$.

Answer:
Differentiating with respect to x

$$\frac{dy}{dx} = \frac{\frac{1}{x}\times x - \log x}{x^2}$$

$\Rightarrow \frac{dy}{dx} = \frac{1-\log x}{x^2}$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{\frac{-1}{x} \times x^2 - 2x(1-\log x)}{x^4}$$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x-2x(1-\log x)}{x^4}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1-2+2\log x}{x^3}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{(2\log x - 3)}{x^3}$

Hence proved

Question 16.

If $y = e^{ax}\cos bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2) = 0$.

Answer:
Differentiating with respect to x

$$\frac{dy}{dx} = ae^{ax}\cos bx - be^{ax}\sin bx$$

$$be^{ax}\sin bx = ae^{ax}\cos bx - \frac{dy}{dx}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = a^2e^{ax}\cos bx - abe^{ax}\sin bx - abe^{ax}\sin bx - b^2e^{ax}\cos bx$$

$\Rightarrow \frac{d^2y}{dx^2} = a^2e^{ax}\cos bx - 2abe^{ax}\sin bx - b^2e^{ax}\cos bx$

$$\Rightarrow \frac{d^2y}{dx^2} = a^2e^{ax}\cos bx - 2a\left(ae^{ax}\cos bx - \frac{dy}{dx}\right) - b^2e^{ax}\cos bx$$

$$\Rightarrow \frac{d^2y}{dx^2} = a^2e^{ax}\cos bx - 2a^2e^{ax}\cos bx + 2a\frac{dy}{dx} - b^2e^{ax}\cos bx$$

$$\Rightarrow \frac{d^2y}{dx^2} = -a^2e^{ax}\cos bx - b^2e^{ax}\cos bx + 2a\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(a^2 + b^2)(e^{ax}\cos bx) + 2a\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(a^2 + b^2)y + 2a\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$$

Hence Proved

Question 17.

If $\mathscr{Y} = e^{a\cos^{-1}x}$, $-1 \le x \le 1$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2\mathscr{Y} = 0$.

Answer:
Taking log on both sides

log y = $a\cos^{-1}x$ log e

log y = $a\cos^{-1}x$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{-a}{\sqrt{1 - x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ae^{a\cos^{-1}x}}{\sqrt{1-x^2}}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{\frac{a^2e^{a\cos^{-1}x}}{\sqrt{1-x^2}} \times \sqrt{1-x^2} - ae^{\cos^{-1}x} \times \frac{2x}{2\sqrt{1-x^2}}}{(1-x^2)}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = a^2e^{\cos^{-1}x} - \frac{axe^{\cos^{-1}x}}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = a^2y + x\frac{dy}{dx}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - a^2y - x\frac{dy}{dx} = 0$$

Hence Proved

Question 18.

If $x = at^2$ and $y = 2at$, find $\frac{d^2y}{dx^2}$ at $t = 2$.

Answer:
Differentiating with t

$$\frac{dx}{dt} = 2at \quad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{-1}{t^2}\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4} \times \frac{1}{2at}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4} \times \frac{1}{4a}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{16a}$$

Question 19.

If x = a(θ – sin θ) and y = a(1 – cos θ), find $\frac{d^2y}{dx^2}$ at θ = π.

Answer:
Differentiating with respect to θ

$$\frac{dx}{d\theta} = a(1 - \cos\theta)\ \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{\cos\theta(1 - \cos\theta) - \sin^2\theta}{(1 - \cos\theta)^2} \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^2} \times \frac{1}{a(1-\cos\theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos\theta - 1}{(1-\cos\theta)^2} \times \frac{1}{a(1-\cos\theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-(1-\cos\theta)}{(1-\cos\theta)^2} \times \frac{1}{a(1-\cos\theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{a(1-\cos\theta)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{a(1-(-1))^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4a}$$

Question 20.

If y = sin (log x), prove that $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

Answer:
Differentiating with respect to

$$\frac{dy}{dx} = \cos(\log x)\ \frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = \cos(\log x)$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{-\sin(\log x)\frac{1}{x}x - \cos(\log x)}{x^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = -\sin(\log x) - \cos(\log x)$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = -y - x\frac{dy}{dx}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + y + x\frac{dy}{dx} = 0$$

Hence Proved

Question 21.

If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, show that $(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$.

Answer:
$\sqrt{1-x^2}\, y = \sin^{-1} x$

Differentiating with respect to x

$$\sqrt{1-x^2}\frac{dy}{dx} - \frac{2xy}{2\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{dy}{dx} - xy = 1$$

Differentiating with respect to x

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$$

Hence Proved

Question 22.

If $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

Answer:
$y = e^x \sin x$

Differentiating with respect to x

$$\frac{dy}{dx} = e^x \sin x + e^x \cos x$$

$$\left[e^x \cos x = \frac{dy}{dx} - e^x \sin x\right]$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2e^x \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 2e^x \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 2y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Question 23.

If x = a $\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$ and y = a sin θ, show that the value of $\frac{d^2y}{dx^2}$ at θ = $\frac{\pi}{4}$ is $\frac{2\sqrt{2}}{a}$.

Answer:

$$\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{\sec^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}\right) \frac{dy}{d\theta} = a\cos\theta$$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right)$$

$$= a\left(\frac{-\sin^2\theta + 1}{\sin\theta}\right)$$

$$= \frac{a\cos^2\theta}{\sin\theta}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\frac{dy}{dx} = a\cos\theta \times \frac{\sin\theta}{a\cos^2\theta}$$

$$\frac{dy}{dx} = \tan\theta$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec^2\theta \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\sqrt{2}\right)^2 \times \frac{\sin\theta}{a\cos^2\theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \frac{\frac{1}{\sqrt{2}}}{a\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2\sqrt{2}}{a}$$

Question 24.

If x = cos t + log tan $\frac{t}{2}$, $\mathscr{Y}$ = sin t then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at t = $\frac{\pi}{4}$.

Answer:

$$\frac{dx}{dt} = -\sin t + \frac{\sec^2\frac{t}{2}}{2\tan\frac{t}{2}} \frac{dy}{dt} = \cos t$$

$$= -\sin t + \frac{1}{\sin t}$$

$$= \frac{-\sin^2 t + 1}{\sin t}$$

$$= \frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = \cos t$$

Differentiating with respect to t

$\Rightarrow \frac{d^2y}{dt^2} = -\sin t$ [Putting t = π /4]

$\Rightarrow \frac{d^2y}{dt^2} = -\frac{1}{\sqrt{2}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dx}{dt}$

$\Rightarrow \frac{dy}{dx} = \cos t \times \frac{\sin t}{\cos^2 t}$

$\Rightarrow \frac{dy}{dx} = \tan t$

Differentiating with respect to x

$\frac{d^2y}{dx^2} = \sec^2 t \frac{dt}{dx}$ [Putting t = π /4]

$\Rightarrow \frac{d^2y}{dx^2} = (\sqrt{2}) \times \frac{\sin t}{\cos^2 t}$

$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = 2\sqrt{2}$

Question 25.

If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.

Answer:
$y = x^x$

Taking log on both sides

log y = x log x

Differentiating with respect to x

$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$ ...(i)

$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$

Differentiating with respect to x

$\frac{d^2y}{dx^2} = \frac{y}{x} + (1 + \log x)\frac{dy}{dx}$ [putting value of (1 + log x) from (i)]

$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2$

$\Rightarrow \frac{d^2y}{dx^2} - \frac{y}{x} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 0$

Hence Proved

Question 26.

If $\mathscr{Y} = (\cot^{-1})^2$, then show that $(x^2 + 1)^2 \frac{d^2y}{dx^2} + 2x\,(x^2 + 1)\,\frac{dy}{dx} = 2$.

Answer:
$y = (\cot^{-1} x)^2$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{-2\cot^{-1} x}{1 + x^2}$$

$\Rightarrow -2\cot^{-1} x = (1 + x^2)\frac{dy}{dx}$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{2 + 4x\ \cot^{-1} x}{(1 + x^2)^2}$$

$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} - 4x\cot^{-1}x = 2$

$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} - 2x\left(-(1+x^2)\frac{dy}{dx}\right) = 2$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$

Hence Proved

Question 27.

If $y = \left\{x + \sqrt{x^2+1}\right\}^m$, then show that $(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$.

Answer:

Differentiating with respect to x

$$\frac{dy}{dx} = m\left\{x + \sqrt{x^2+1}\right\}^{m-1}\left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = m\{x + \sqrt{x^2+1}\}^{m-1}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = m\frac{\{x+\sqrt{x^2+1}\}^m}{\sqrt{x^2+1}}$$

$$\Rightarrow \frac{dy}{dx} = m\frac{y}{\sqrt{x^2+1}}$$

$$[\frac{dy}{dx}\sqrt{x^2+1} = my]$$

Differentiating with respect to x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{m\frac{dy}{dx}\sqrt{1+x^2} - \frac{2xmy}{2\sqrt{x^2+1}}}{(1+x^2)}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} = m^2y - x\frac{dy}{dx}$$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$

Hence Proved

Question 28.

If y = log [x + $\left[x+\sqrt{x^2+a^2}\right]$, then prove that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Answer:

$$\frac{dy}{dx} = \frac{1+\frac{2x}{2\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sqrt{x^2+a^2}+2x}{2\sqrt{x^2+a^2}} \times \frac{1}{x+\sqrt{x^2+a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2+a^2}}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{-2x}{2(x^2+a^2)\sqrt{x^2+a^2}}$$

$$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2} = \frac{-x}{\sqrt{x^2+a^2}}$$

$$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2} = -x\frac{dy}{dx}$$

$$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Hence Proved

Question 29.

If x = a(cos θ + θ sin θ) and y = a(sin θ - θ cos θ), show that $\frac{d^2y}{dx^2} = \frac{1}{a}\left(\frac{\sec^3\theta}{\theta}\right)$

Answer:
Differentiating with respect to θ

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)\ \frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a\theta\cos\theta \Rightarrow \frac{dy}{d\theta} = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \frac{dy}{dx} = \tan\theta$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec^2\theta \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2\theta \times \frac{1}{a\theta\cos\theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2\theta \times \frac{\sec\theta}{a\theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta}$$

Hence Proved

Question 30.

If x = a cos θ + b sin θ and y = a sin θ – b cos θ , show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

Answer:

$$\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta \quad \frac{dy}{d\theta} = a\cos\theta + b\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\cos\theta + b\sin\theta}{-a\sin\theta + b\cos\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

Differentiating with respect to x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2}$$

$$\Rightarrow y^2\frac{d^2y}{dx^2} = y - x\frac{dy}{dx}$$

$$\Rightarrow y^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Hence Proved