<u>Exercise 14c</u>

Question 1.
Evaluate the following integrals:

$$\int \sqrt{4-x^2}\,dx$$

Answer:
To Find : $\int \sqrt{4-x^2}\,dx$

Now, $\int \sqrt{4-x^2}\,dx$ can be written as $\int \sqrt{2^2-x^2}\,dx$

Formula Used: $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int \sqrt{2^2-x^2}\,dx$ is of the form $\int \sqrt{a^2-x^2}\,dx$,

Hence, $\int \sqrt{2^2-x^2}\,dx = \frac{1}{2}x\sqrt{2^2-x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} + C$

$= \frac{1}{2}x\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} + C$

$= \frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$

Therefore, $\int \sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$

Question 2.
Evaluate the following integrals:

$$\int \sqrt{4-9x^2}\,dx$$

Answer:
To Find : $\int \sqrt{4-9x^2}\,dx$

Now, $\int \sqrt{4-9x^2}\,dx$ can be written as $\int \sqrt{2^2-(3x)^2}\,dx$

Formula Used: $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$ + C

Since $\int \sqrt{2^2-(3x)^2}\,dx$ is of the form $\int \sqrt{a^2-x^2}\,dx$,

Hence, $\int \sqrt{2^2-(3x)^2}\,dx = \frac{1}{2}(3x)\sqrt{2^2-(3x)^2} + \frac{2^2}{2}\sin^{-1}\frac{3x}{2}$ + C

$= \frac{x}{2}\sqrt{4-9x^2} + \frac{4}{6}\sin^{-1}\frac{3x}{2}$ + C

$= \frac{x}{2}\sqrt{4-9x^2} + \frac{2}{3}\sin^{-1}\frac{3x}{2}$ + C

Therefore, $\int \sqrt{4-9x^2}\,dx = \frac{x}{2}\sqrt{4-9x^2} + \frac{2}{3}\sin^{-1}\frac{3x}{2}$ + C

Question 3.
Evaluate the following integrals:

$$\int \sqrt{x^2-2}dx$$

Answer:
To Find : $\int \sqrt{x^2-2}\,dx$

Now, $\int \sqrt{x^2-2}\,dx$ can be written as $\int \sqrt{x^2-(\sqrt{2})^2}\,dx$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}$ log |x + $\sqrt{x^2-a^2}$|+ C

Since $\int \sqrt{x^2-(\sqrt{2})^2}\,dx$ is of the form $\int \sqrt{x^2-a^2}\,dx$,

Hence, $\int \sqrt{x^2-(\sqrt{2})^2}\,dx = \frac{x}{2}\sqrt{x^2-(\sqrt{2})^2} - \frac{(\sqrt{2})^2}{2}$ log |x + $\sqrt{x^2-(\sqrt{2})^2}$|+ C

$= \frac{x}{2}\sqrt{x^2-2} - \frac{2}{2}$ log |x + $\sqrt{x^2-2}$|+ C

$= \frac{x}{2}\sqrt{x^2-2}$ − log |x + $\sqrt{x^2-2}$|+ C

Therefore, $\int \sqrt{x^2-2}\,dx = \frac{x}{2}\sqrt{x^2-2}$ − log |x + $\sqrt{x^2-2}$|+ C

Question 4.
Evaluate the following integrals:

$$\int \sqrt{2x^2-3}dx$$

Answer:
To Find : $\int \sqrt{2x^2-3}dx$

Now, $\int \sqrt{2x^2-3}\,dx$ can be written as $\int \sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2}\,dx$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}$ log |x + $\sqrt{x^2-a^2}$|+ C

Since $\int \sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2}\,dx$ is of the form $\int \sqrt{x^2-a^2}\,dx$,

Hence, $\int \sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2}\,dx = \frac{\sqrt{2}x}{2}\sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2} - \frac{(\sqrt{3})^2}{2}$ log |$\sqrt{2}$x + $\sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2}$ |+C

$= \frac{\sqrt{2}x}{2}\sqrt{2x^2-3} - \frac{3}{2}$ log |$\sqrt{2}$x + $\sqrt{2x^2-3}$|+ C

$= \frac{x}{2}\sqrt{2x^2-3} - \frac{3}{2\sqrt{2}}$ log |$\sqrt{2}$x + $\sqrt{2x^2-3}$|+ C

Therefore, $\int \sqrt{2x^2-3}\,dx = \frac{x}{2}\sqrt{2x^2-3} - \frac{3}{2\sqrt{2}}$ log |$\sqrt{2}$x + $\sqrt{2x^2-3}$|+ C

Question 5.

Evaluate the following integrals:

$$\int \sqrt{x^2+5}dx$$

Answer:
To Find : $\int \sqrt{x^2+5}\,dx$

Now, $\int \sqrt{x^2+5}\,dx$ can be written as $\int \sqrt{x^2+(\sqrt{5})^2}\,dx$

Formula Used: $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$

Since $\int \sqrt{x^2+(\sqrt{5})^2}\,dx$ is of the form $\int \sqrt{x^2+a^2}\,dx$,

Hence, $\int \sqrt{x^2+(\sqrt{5})^2}\,dx = \frac{x}{2}\sqrt{x^2+(\sqrt{5})^2} + \frac{(\sqrt{5})^2}{2}\log|x+\sqrt{x^2+(\sqrt{5})^2}| + C$

$= \frac{x}{2}\sqrt{x^2+5} + \frac{5}{2}\log|x+\sqrt{x^2+5}| + C$

Therefore, $\int \sqrt{x^2+5}\,dx = \frac{x}{2}\sqrt{x^2+5} + \frac{5}{2}\log|x+\sqrt{x^2+5}| + C$

Question 6.
Evaluate the following integrals:

$$\int \sqrt{4x^2+9}dx$$

Answer:
To Find : $\int \sqrt{4x^2+9}\,dx$

Now, $\int \sqrt{4x^2+9}\,dx$ can be written as $\int \sqrt{(2x)^2+3^2}\,dx$

Formula Used: $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$

Since $\int \sqrt{(2x)^2 + 3^2}\, dx$ is of the form $\int \sqrt{x^2 + a^2}\, dx$,

Hence, $\int \sqrt{(2x)^2 + 3^2}\, dx = \frac{2x}{2}\sqrt{(2x)^2 + 3^2} + \frac{3^2}{2} \log |2x + \sqrt{(2x)^2 + 3^2}| + C$

$= \frac{2x}{2}\sqrt{4x^2 + 9} + \frac{9}{2} \log |2x + \sqrt{4x^2 + 9}\,| + C$

$= \frac{x}{2}\sqrt{4x^2 + 9} + \frac{9}{4} \log |2x + \sqrt{4x^2 + 9}| + C$

Therefore, $\int \sqrt{4x^2 + 9}\, dx = \frac{x}{2}\sqrt{4x^2 + 9} + \frac{9}{4} \log |2x + \sqrt{4x^2 + 9}| + C$

Question 7.
Evaluate the following integrals:

$$\int \sqrt{3x^2 + 4}dx$$

Answer:
To Find : $\int \sqrt{3x^2 + 4}\, dx$

Now, $\int \sqrt{3x^2 + 4}\, dx$ can be written as $\int \sqrt{(\sqrt{3}x)^2 + 2^2}\, dx$

Formula Used: $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$

Since $\int \sqrt{(\sqrt{3}x)^2 + 2^2}\, dx$ is of the form $\int \sqrt{x^2 + a^2}\, dx$,

Hence, $\int \sqrt{(\sqrt{3}x)^2 + 2^2}\, dx = \frac{\sqrt{3}x}{2}\sqrt{(\sqrt{3}x)^2 + 2^2} + \frac{2^2}{2} \log |\sqrt{3}x + \sqrt{(\sqrt{3}x)^2 + 2^2}| + C$

$= \frac{\sqrt{3}x}{2}\sqrt{3x^2 + 4} + \frac{4}{2} \log |\sqrt{3}x + \sqrt{3x^2 + 4}\,| + C$

$= \frac{x}{2}\sqrt{3x^2 + 4} + \frac{2}{\sqrt{3}} \log |\sqrt{3}x + \sqrt{3x^2 + 4}| + C$

Therefore, $\int \sqrt{3x^2+4}\, dx = \frac{x}{2}\sqrt{3x^2+4} + \frac{2}{\sqrt{3}} \log|\sqrt{3}x + \sqrt{3x^2+4}| + C$

Question 8.
Evaluate the following integrals:

$$\int \cos x\sqrt{9-\sin^2 x}dx$$

Answer:
To Find : $\int cosx\sqrt{9-\sin^2 x}\, dx$

Now, let sin x = t

$\Rightarrow$cosx dx = dt

Therefore, $\int cosx\sqrt{9-\sin^2 x}\, dx$ can be written as $\int \sqrt{3^2-t^2}\, dt$

Formula Used: $\int \sqrt{a^2-x^2}\, dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since , $\int \sqrt{3^2-t^2}\, dt$ is in the form of $\int \sqrt{a^2-x^2}\, dx$ with t as a variable instead of x .

$\Rightarrow \int \sqrt{3^2-t^2}\, dt = \frac{1}{2}t\sqrt{3^2-t^2} + \frac{3^2}{2}\sin^{-1}\frac{t}{3} + C$

$= \frac{t}{2}\sqrt{9-t^2} + \frac{9}{2}\sin^{-1}\frac{t}{3} + C$

Now since sin x = t and cosx dx = dt

$\Rightarrow \int cosx\sqrt{9-\sin^2 x}\, dx = \frac{\sin x}{2}\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}(\frac{\sin x}{3}) + C$

Question 9.
Evaluate the following integrals:

$$\int \sqrt{x^2-4x+2}dx$$

Answer:
To Find : $\int \sqrt{x^2 - 4x + 2}\, dx$

Now, $\int \sqrt{x^2 - 4x + 2}\, dx$ can be written as $\int \sqrt{x^2 - 4x + 2^2 - 2^2 + 2}\, dx$

i.e., $\int \sqrt{(x-2)^2 - 2}\, dx$

Here , let x – 2 = y ⇒dx = dy

Therefore, $\int \sqrt{(x-2)^2 - 2}\, dx$ can be written as $\int \sqrt{y^2 - (\sqrt{2})^2}\, dy$

Formula Used: $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}$ log |x + $\sqrt{x^2 - a^2}$|+ C

Since $\int \sqrt{y^2 - (\sqrt{2})^2}\, dy$ is of the form $\int \sqrt{x^2 - a^2}\, dx$ with change in variable.

⇒ $\int \sqrt{y^2 - (\sqrt{2})^2}\, dy = \frac{y}{2}\sqrt{y^2 - (\sqrt{2})^2} - \frac{(\sqrt{2})^2}{2}$ log |y + $\sqrt{y^2 - (\sqrt{2})^2}$|+ C

= $\frac{y}{2}\sqrt{y^2 - 2} - \frac{4}{2}$ log |y + $\sqrt{y^2 - 2}$|+ C

= $\frac{y}{2}\sqrt{y^2 - 2} - 2$ log |y + $\sqrt{y^2 - 2}$|+ C

Since , x – 2 = y and dx = dy

⇒ $\int \sqrt{(x-2)^2 - 2}\, dx = \frac{(x-2)}{2}\sqrt{(x-2)^2 - 2} - 2$ log |(x-2) + $\sqrt{(x-2)^2 - 2}$|+C Therefore,

$\int \sqrt{x^2 - 4x + 2}\, dx = \frac{(x-2)}{2}\sqrt{x^2 - 4x + 2} - 2$ log |(x - 2) + $\sqrt{x^2 - 4x + 2}$|+ C

Question 10.
Evaluate the following integrals:

$$\int \sqrt{x^2 + 6x - 4}dx$$

Answer:

To Find : $\int \sqrt{x^2 + 6x - 4}\, dx$

Now, $\int \sqrt{x^2 + 6x - 4}\, dx$ can be written as $\int \sqrt{x^2 + 6x + 3^2 - 3^2 - 4}\, dx$

i.e, $\int \sqrt{(x+3)^2 - 13}\, dx$

Here , let x + 3 = y ⇒dx = dy

Therefore, $\int \sqrt{(x+3)^2 - 13}\, dx$ can be written as $\int \sqrt{y^2 - (\sqrt{13})^2}\, dy$

Formula Used: $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$

Since $\int \sqrt{y^2 - (\sqrt{13})^2}\, dy$ is of the form $\int \sqrt{x^2 - a^2}\, dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2 - (\sqrt{13})^2}\, dy = \frac{y}{2}\sqrt{y^2 - (\sqrt{13})^2} - \frac{(\sqrt{13})^2}{2} \log |y + \sqrt{y^2 - (\sqrt{13})^2}| + C$

$= \frac{y}{2}\sqrt{y^2 - 13} - \frac{13}{2} \log |y + \sqrt{y^2 - 13}| + C$

Since , x + 3 = y and dx = dy

$\Rightarrow \int \sqrt{(x+3)^2 - 13}\, dx = \frac{(x+3)}{2}\sqrt{(x+3)^2 - 13} - \frac{13}{2} \log |(x+3) + \sqrt{(x+3)^2 - 13}| + C$

Therefore,

$\int \sqrt{x^2 + 6x - 4}\, dx = \frac{(x+3)}{2}\sqrt{x^2 + 6x - 4} - \frac{13}{2} \log |(x+3) + \sqrt{x^2 + 6x - 4}| + C$

Question 11.

Evaluate the following integrals:

$$\int \sqrt{2x - x^2} dx$$

Answer:

To Find : $\int \sqrt{2x - x^2}$ dx

Now, $\int \sqrt{2x - x^2}\, dx$ can be written as $\int \sqrt{2x - x^2 - 1^2 + 1^2}\, dx$

i.e, $\int \sqrt{1 - (x-1)^2}\, dx$

Let x – 1 = y ⇒ dx = dy

Therefore , $\int \sqrt{1 - (x-1)^2}\, dx$ becomes $\int \sqrt{1^2 - y^2}\, dy$

Formula Used: $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a}$ + C

Since $\int \sqrt{1^2 - y^2}\, dy$ is of the form $\int \sqrt{a^2 - x^2}\, dx$ with change in variable,

Hence $\int \sqrt{1^2 - y^2}\, dy = \frac{1}{2} y\sqrt{1^2 - y^2} + \frac{1^2}{2} \sin^{-1}\frac{y}{1}$ + C

$= \frac{y}{2}\sqrt{1 - y^2} + \frac{1}{2} \sin^{-1}\frac{y}{1}$ + C

Here we have x – 1 = y and dx = dy

⇒ $\int \sqrt{1 - (x-1)^2}\, dx = \frac{(x-1)}{2}\sqrt{1 - (x-1)^2} + \frac{1}{2} \sin^{-1}\frac{(x-1)}{1}$ + C

Therefore , $\int \sqrt{2x - x^2}\, dx = \frac{(x-1)}{2}\sqrt{2x - x^2} + \frac{1}{2} \sin^{-1}(x-1)$ + C

Question 12.

Evaluate the following integrals:

$$\int \sqrt{1 - 4x - x^2} dx$$

Answer:

To Find : $\int \sqrt{1-4x-x^2}$ dx

Now, $\int \sqrt{1-4x-x^2}\,dx$ can be written as $\int \sqrt{1-4x-x^2-2^2+2^2}\,dx$

i.e, $\int \sqrt{5-(x+2)^2}\,dx$

Let x + 2 = y ⇒ dx = dy

Therefore , $\int \sqrt{5-(x+2)^2}\,dx$ becomes $\int \sqrt{(\sqrt{5})^2-y^2}\,dy$

Formula Used: $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int \sqrt{(\sqrt{5})^2-y^2}\,dy$ is of the form $\int \sqrt{a^2-x^2}\,dx$ with change in variable,

Hence $\int \sqrt{(\sqrt{5})^2-y^2}\,dy = \frac{1}{2}y\sqrt{(\sqrt{5})^2-y^2} + \frac{(\sqrt{5})^2}{2}\sin^{-1}\frac{y}{\sqrt{5}} + C$

$= \frac{y}{2}\sqrt{5-y^2} + \frac{5}{2}\sin^{-1}\frac{y}{\sqrt{5}} + C$

Here we have x + 2 = y and dx = dy

$\Rightarrow \int \sqrt{5-(x+2)^2}\,dx = \frac{(x+2)}{2}\sqrt{5-(x+2)^2} + \frac{5}{2}\sin^{-1}(\frac{x+2}{\sqrt{5}}) + C$

Therefore , $\int \sqrt{1-4x-x^2}\,dx = \frac{(x+2)}{2}\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}(\frac{x+2}{\sqrt{5}}) + C$

Question 13.

Evaluate the following integrals:

$\int \sqrt{2ax-x^2}\,dx$

Answer:
To Find : $\int \sqrt{2ax - x^2}$ dx

Now, $\int \sqrt{2ax - x^2}\, dx$ can be written as $\int \sqrt{2ax - x^2 - a^2 + a^2}\, dx$

i.e, $\int \sqrt{a^2 - (x-a)^2}\, dx$

Let x - a = y ⇒ dx = dy

Therefore , $\int \sqrt{a^2 - (x-a)^2}\, dx$ becomes $\int \sqrt{a^2 - y^2}\, dy$

Formula Used: $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$

Since $\int \sqrt{a^2 - y^2}\, dy$ is of the form $\int \sqrt{a^2 - x^2}\, dx$ with change in variable,

Hence $\int \sqrt{a^2 - y^2}\, dy = \frac{1}{2} y\sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1}\frac{y}{a} + C$

$= \frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1}\frac{y}{a} + C$

Here we have x - a = y and dx = dy

⇒ $\int \sqrt{a^2 - (x-a)^2}\, dx = \frac{(x-a)}{2}\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1}(\frac{x-a}{a}) + C$

Therefore , $\int \sqrt{2ax - x^2}\, dx = \frac{(x-a)}{2}\sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}(\frac{x-a}{a}) + C$

Question 14.
Evaluate the following integrals:

$$\int \sqrt{2x^2 + 3x + 4dx}$$

Answer:
To Find : $\int \sqrt{2x^2 + 3x + 4dx}$

Now , consider $\int \sqrt{2x^2+3x+4}dx$ = $\int \sqrt{2[x^2+\frac{3}{2}x+2]}dx$

= $\sqrt{2}\int \sqrt{x^2+\frac{3}{2}x+2}dx$

= $\sqrt{2}\int \sqrt{x^2+\frac{3}{2}x+(\frac{3}{4})^2-\left(\frac{3}{4}\right)^2+2}dx$

= $\sqrt{2}\int \sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}dx$

Let x + $\frac{3}{4}$ = y ⇒dx = dy

Hence $\sqrt{2}\int \sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}dx$ becomes $\sqrt{2}\int \sqrt{y^2+(\frac{\sqrt{23}}{4})^2}dy$

Formula Used: $\int \sqrt{x^2+a^2}\,dx$ = $\frac{x}{2}\sqrt{x^2+a^2}$ + $\frac{a^2}{2}$ log |x + $\sqrt{x^2+a^2}$|+ C

Now consider $\int \sqrt{y^2+(\frac{\sqrt{23}}{4})^2}dy$ which is in the form of $\int \sqrt{x^2+a^2}\,dx$ with change in variable.

⇒ $\int \sqrt{y^2+(\frac{\sqrt{23}}{4})^2}dy$ = $\frac{y}{2}\sqrt{y^2+(\frac{\sqrt{23}}{4})^2}$ + $\frac{(\frac{\sqrt{23}}{4})^2}{2}$ log |y + $\sqrt{y^2+(\frac{\sqrt{23}}{4})^2}$|+ C

= $\frac{y}{2}\sqrt{y^2+\frac{23}{16}}$ + $\frac{23}{32}$ log |y + $\sqrt{y^2+\frac{23}{16}}$|+ C

Since x + $\frac{3}{4}$ = y and dx = dy

⇒ $\int \sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}dx$ = $\frac{1}{8}(4x+3)\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}$ + $\frac{23}{32}$ log | $x+\frac{3}{4}$ + $\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}$|+ C

Now , $\sqrt{2}\int \sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}dx$ = $\frac{\sqrt{2}}{8}(4x+3)\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}$ + $\frac{23\sqrt{2}}{32}$ log | $x+\frac{3}{4}$ + $\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}$ |+ C

Therefore,

$\int \sqrt{2x^2 + 3x + 4}dx = \frac{1}{8}(4x + 3)\sqrt{2x^2 + 3x + 4} + \frac{23}{32}\log |(x + \frac{3}{4}) + \sqrt{2x^2 + 3x + 4}| + C$

Question 15.
Evaluate the following integrals:

$$\int \sqrt{x^2 + x}dx$$

Answer:
To Find : $\int \sqrt{x^2 + x}\,dx$

Now, $\int \sqrt{x^2 + x}\,dx$ can be written as $\int \sqrt{x^2 + x + (\frac{1}{2})^2 - (\frac{1}{2})^2}\,dx$

i.e, $\int \sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}}\,dx$

Here , let x + $\frac{1}{2}$ = y ⇒dx = dy

Therefore, $\int \sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}}\,dx$ can be written as $\int \sqrt{y^2 - (\frac{1}{2})^2}\,dy$

Formula Used: $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log |x + \sqrt{x^2 - a^2}| + C$

Since $\int \sqrt{y^2 - (\frac{1}{2})^2}\,dy$ is of the form $\int \sqrt{x^2 - a^2}\,dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2 - (\frac{1}{2})^2}\,dy = \frac{y}{2}\sqrt{y^2 - (\frac{1}{2})^2} - \frac{(\frac{1}{2})^2}{2}\log |y + \sqrt{y^2 - (\frac{1}{2})^2}| + C$

$= \frac{y}{2}\sqrt{y^2 - \frac{1}{4}} - \frac{1}{8}\log |y + \sqrt{y^2 - \frac{1}{4}}| + C$

Since , x + $\frac{1}{2}$ = y and dx = dy

$\Rightarrow \int \sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}\,dx = \frac{1}{4}(2x+1)\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}} - \frac{1}{8}\log|(x+\frac{1}{2})+\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}|+C$

Therefore,

$\int \sqrt{x^2+x}\,dx = \frac{1}{4}(2x+1)\sqrt{x^2+x} - \frac{1}{8}\log|x+\frac{1}{2}+\sqrt{x^2+x}|+C$

Question 16.
Evaluate the following integrals:

$$\int \sqrt{x^2+x+1}dx$$

Answer:
To Find : $\int \sqrt{x^2+x+1}\,dx$

Now, $\int \sqrt{x^2+x+1}\,dx$ can be written as $\int \sqrt{x^2+x+(\frac{1}{2})^2-\left(\frac{1}{2}\right)^2+1}\,dx$

i.e, $\int \sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$

Here , let $x+\frac{1}{2}=y \Rightarrow dx = dy$

Therefore, $\int \sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$ can be written as $\int \sqrt{y^2+(\frac{\sqrt{3}}{2})^2}\,dy$

Formula Used: $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+C$

Since $\int \sqrt{y^2+(\frac{\sqrt{3}}{2})^2}\,dy$ is of the form $\int \sqrt{x^2+a^2}\,dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2+(\frac{\sqrt{3}}{2})^2}\,dy = \frac{y}{2}\sqrt{y^2+(\frac{\sqrt{3}}{2})^2} + \frac{(\frac{\sqrt{3}}{2})^2}{2}\log|y+\sqrt{y^2+(\frac{\sqrt{3}}{2})^2}|+C$

$= \frac{y}{2}\sqrt{y^2 + \frac{3}{4}} + \frac{3}{8}\log|y + \sqrt{y^2 + \frac{3}{4}}| + C$

Since , $x + \frac{1}{2} = y$ and $dx = dy$

$\Rightarrow \int \sqrt{(x+\frac{1}{2})^2 + \frac{3}{4}}\,dx = \frac{1}{4}(2x+1)\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{1}{8}\log|(x+\frac{1}{2}) + \sqrt{(x+\frac{1}{2})^2 + \frac{3}{4}}| + C$

Therefore,

$\int \sqrt{x^2+x+1}\,dx = \frac{1}{4}(2x+1)\sqrt{x^2+x+1} + \frac{3}{8}\log|x + \frac{1}{2} + \sqrt{x^2+x+1}| + C$

Question 17.
Evaluate the following integrals:

$$\int (2x-5)\sqrt{x^2-4x+3}dx$$

Answer:
To Find : $\int (2x-5)\sqrt{x^2-4x+3}\,dx$

Now, let $2x - 5$ be written as (2x – 4) -1 and split

Therefore ,

$\int (2x-5)\sqrt{x^2-4x+3}\,dx = \int\{(2x-4)\sqrt{x^2-4x+3} - 1\sqrt{x^2-4x+3}\}dx$

$= \int (2x-4)\sqrt{x^2-4x+3}\,dx - \int \sqrt{x^2-4x+3}dx$

Now solving, $\int (2x-4)\sqrt{x^2-4x+3}\,dx$

Let $x^2 - 4x + 3 = u \Rightarrow dx = \frac{du}{(2x-4)}$

Thus, $\int (2x-4)\sqrt{x^2-4x+3}\,dx$ becomes $\int \sqrt{u}\,du$

Now , $\int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}u^{\frac{3}{2}}$

$=\frac{2}{3}(x^2-4x+3)^{\frac{3}{2}}$

Now solving, $\int \sqrt{x^2-4x+3}dx$

$\int \sqrt{x^2-4x+3}dx = \int \sqrt{x^2-4x+2^2-2^2+3}dx$

$=\int \sqrt{(x-2)^2-1}dx$

Let x – 2 = y ⇒ dx = dy

Then $\int \sqrt{(x-2)^2-1}dx$ becomes $\int \sqrt{y^2-1^2}dy$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x + \sqrt{x^2-a^2}| + C$

Since $\int \sqrt{y^2-1^2}dy$ is in the form of $\int \sqrt{x^2-a^2}\,dx$ with change in variable.

Hence $\int \sqrt{y^2-1^2}dy = \frac{y}{2}\sqrt{y^2-1^2} - \frac{1^2}{2}\log|y + \sqrt{y^2-1^2}| + C$

$= \frac{y}{2}\sqrt{y^2-1} - \frac{1}{2}\log|y + \sqrt{y^2-1}| + C$

Now, since x – 2 = y and dx = dy

$\int \sqrt{(x-2)^2-1}dx = \frac{(x-2)}{2}\sqrt{(x-2)^2-1} - \frac{1}{2}\log|(x-2) + \sqrt{(x-2)^2-1}| + C$

Hence $\int \sqrt{x^2-4x+3}dx = \frac{(x-2)}{2}\sqrt{x^2-4x+3} - \frac{1}{2}\log|(x-2) + \sqrt{x^2-4x+3}| + C$

Therefore , $\int (2x-4)\sqrt{x^2-4x+3}\,dx - \int \sqrt{x^2-4x+3}dx = \frac{2}{3}(x^2-4x+3)^{\frac{3}{2}}$

$-\frac{(x-2)}{2}\sqrt{x^2-4x+3} + \frac{1}{2}\log|(x-2) + \sqrt{x^2-4x+3}| + C$

$i.e, \int (2x - 5)\sqrt{x^2 - 4x + 3}\, dx = \frac{2}{3}(x^2 - 4x + 3)^{\frac{3}{2}}$

$- \frac{(x-2)}{2}\sqrt{x^2 - 4x + 3} + \frac{1}{2}$ log |x - 2 + $\sqrt{x^2 - 4x + 3}$|+ C

Question 18.
Evaluate the following integrals:

$$\int (x+2)\sqrt{x^2 + x + 1}dx$$

Answer:
To Find : $\int (x + 2)\sqrt{x^2 + x + 1}\, dx$

Now, let x + 2 be written as $\frac{1}{2}$(2x + 1) + $\frac{3}{2}$ and split

Therefore ,

$\int (x + 2)\sqrt{x^2 + x + 1}\, dx = \int \{\frac{(2x + 1)\sqrt{x^2 + x + 1}}{2} + \frac{3}{2}\sqrt{x^2 + x + 1}\}dx$

$= \frac{1}{2}\int (2x + 1)\sqrt{x^2 + x + 1}dx + \frac{3}{2}\int \sqrt{x^2 + x + 1}dx$

Now solving, $\frac{1}{2}\int (2x + 1)\sqrt{x^2 + x + 1}dx$

Let $x^2 + x + 1$ = u ⇒ dx = $\frac{du}{(2x+1)}$

Thus, $\frac{1}{2} \int (2x + 1)\sqrt{x^2 + x + 1}dx$ becomes $\frac{1}{2}\int \sqrt{u}\, du$

Now , $\frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}\int u^{\frac{1}{2}}\, du = \frac{1}{2}(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}) = \frac{1}{3}u^{\frac{3}{2}}$

$=\frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}}$

Now solving , $\int \sqrt{x^2 + x + 1} dx$

Now, $\int \sqrt{x^2 + x + 1}\, dx$ can be written as $\int \sqrt{x^2 + x + (\frac{1}{2})^2 - \left(\frac{1}{2}\right)^2 + 1}\, dx$

i.e, $\int \sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}\, dx$

Here , let x + $\frac{1}{2}$ = y ⇒dx = dy

Therefore, $\int \sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}\, dx$ can be written as $\int \sqrt{y^2 + (\frac{\sqrt{3}}{2})^2}\, dy$

Formula Used: $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}$ log |x + $\sqrt{x^2 + a^2}$|+ C

Since $\int \sqrt{y^2 + (\frac{\sqrt{3}}{2})^2}\, dy$ is of the form $\int \sqrt{x^2 + a^2}\, dx$ with change in variable.

⇒ $\int \sqrt{y^2 + (\frac{\sqrt{3}}{2})^2}\, dy = \frac{y}{2}\sqrt{y^2 + (\frac{\sqrt{3}}{2})^2} + \frac{(\frac{\sqrt{3}}{2})^2}{2}$ log |y + $\sqrt{y^2 + (\frac{\sqrt{3}}{2})^2}$|+ C

= $\frac{y}{2}\sqrt{y^2 + \frac{3}{4}} + \frac{3}{8}$ log |y + $\sqrt{y^2 + \frac{3}{4}}$|+ C

Since , x + $\frac{1}{2}$ = y and dx = dy

⇒ $\int \sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}\, dx = \frac{1}{4}(2x + 1)\sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{1}{8}$ log |(x + $\frac{1}{2}$) + $\sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}$|+C

Therefore,

$\int \sqrt{x^2 + x + 1}\, dx = \frac{1}{4}(2x + 1)\sqrt{x^2 + x + 1} + \frac{3}{8}$ log |x + $\frac{1}{2}$ + $\sqrt{x^2 + x + 1}$|+C

Hence ,

$\frac{1}{2}\int(2x + 1)\sqrt{x^2+x+1}dx + \frac{3}{2}\int\sqrt{x^2+x+1}dx = \frac{1}{3}(x^2+x+1)^{\frac{3}{2}} +$
$\frac{3}{8}(2x+1)\sqrt{x^2+x+1} + \frac{9}{16}$ log $|(x + \frac{1}{2}) + \sqrt{x^2+x+1}|+C$

Therefore , $\int(x+2)\sqrt{x^2+x+1}\,dx = \frac{1}{3}(x^2+x+1)^{\frac{3}{2}} + \frac{3}{8}(2x+1)\sqrt{x^2+x+1} + \frac{9}{16}$ log $|(x + \frac{1}{2}) + \sqrt{x^2+x+1}|+C$

Question 19.
Evaluate the following integrals:

$$\int(x-5)\sqrt{x^2+x}dx$$

Answer:
To Find : $\int(x-5)\sqrt{x^2+x}\,dx$

Now, let x - 5 be written as $\frac{1}{2}(2x + 1) - \frac{11}{2}$ and split

Therefore ,

$$\int(x-5)\sqrt{x^2+x}\,dx = \int\{\frac{(2x+1)\sqrt{x^2+x}}{2} - \frac{11}{2}\sqrt{x^2+x}\}dx$$

$$= \frac{1}{2}\int(2x + 1)\sqrt{x^2+x}dx - \frac{11}{2}\int\sqrt{x^2+x}dx$$

Now solving, $\frac{1}{2}\int(2x + 1)\sqrt{x^2+x}dx$

Let $x^2+x = u \Rightarrow dx = \frac{du}{(2x+1)}$

Thus, $\frac{1}{2}\int(2x + 1)\sqrt{x^2+x}dx$ becomes $\frac{1}{2}\int\sqrt{u}\,du$

Now , $\frac{1}{2}\int\sqrt{u}\,du = \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{2}\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) = \frac{1}{3}u^{\frac{3}{2}}$

$= \frac{1}{3}(x^2+x)^{\frac{3}{2}}$

Now solving, $\int \sqrt{x^2+x}\,dx$

Now, $\int \sqrt{x^2+x}\,dx$ can be written as $\int \sqrt{x^2+x+(\frac{1}{2})^2-(\frac{1}{2})^2}\,dx$

i.e, $\int \sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}\,dx$

Here , let x + $\frac{1}{2}$ = y ⇒dx = dy

Therefore, $\int \sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}\,dx$ can be written as $\int \sqrt{y^2-(\frac{1}{2})^2}\,dy$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}$ log |x + $\sqrt{x^2-a^2}$|+ C

Since $\int \sqrt{y^2-(\frac{1}{2})^2}\,dy$ is of the form $\int \sqrt{x^2-a^2}\,dx$ with change in variable.

⇒ $\int \sqrt{y^2-(\frac{1}{2})^2}\,dy = \frac{y}{2}\sqrt{y^2-(\frac{1}{2})^2} - \frac{(\frac{1}{2})^2}{2}$ log |y + $\sqrt{y^2-(\frac{1}{2})^2}$|+ C

= $\frac{y}{2}\sqrt{y^2-\frac{1}{4}} - \frac{1}{8}$ log |y + $\sqrt{y^2-\frac{1}{4}}$|+ C

Since , x + $\frac{1}{2}$ = y and dx = dy

⇒ $\int \sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}\,dx = \frac{1}{4}(2x+1)\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}} - \frac{1}{8}$ log |(x + $\frac{1}{2}$) + $\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}$|+C

Therefore,

$\int \sqrt{x^2+x}\,dx = \frac{1}{4}(2x+1)\sqrt{x^2+x} - \frac{1}{8}$ log |x + $\frac{1}{2}$ + $\sqrt{x^2+x}$|+C

Now ,

$\frac{1}{2}\int(2x + 1)\sqrt{x^2+x}dx - \frac{11}{2}\int\sqrt{x^2+x}dx = \frac{1}{3}(x^2+x)^{\frac{3}{2}}$ - $\frac{11}{8}(2x+1)\sqrt{x^2+x} + \frac{11}{16}$log |x + $\frac{1}{2}$ + $\sqrt{x^2+x}$|+C

Therefore ,

$\int(x-5)\sqrt{x^2+x}\,dx = \frac{1}{3}(x^2+x)^{\frac{3}{2}}$ - $\frac{11}{8}(2x+1)\sqrt{x^2+x} + \frac{11}{16}$log |x + $\frac{1}{2}$ + $\sqrt{x^2+x}$|+C

Question 20.
Evaluate the following integrals:

$$\int(4x+1)\sqrt{x^2-x-2}dx$$

Answer:
To Find : $\int(4x+1)\sqrt{x^2-x-2}\,dx$

Now, let 4x + 1 be written as 2(2x - 1) + 3 and split

Therefore ,

$\int(4x+1)\sqrt{x^2-x-2}\,dx = \int\{2(2x-1)\sqrt{x^2-x-2} + 3\sqrt{x^2-x-2}\}\,dx$

= $2\int(2x - 1)\sqrt{x^2-x-2}dx + 3\int\sqrt{x^2-x-2}dx$

Now solving, $2\int(2x - 1)\sqrt{x^2-x-2}dx$

Let x^2-x-2 = u ⇒ dx = $\frac{du}{(2x-1)}$

Thus, $2\int(2x - 1)\sqrt{x^2-x-2}dx$ becomes $2\int\sqrt{u}\,du$

Now , $2\int\sqrt{u}\,du = 2\int u^{\frac{1}{2}}\,du = 2\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) = \frac{4}{3}u^{\frac{3}{2}}$

$=\frac{4}{3}(x^2-x-2)^{\frac{3}{2}}$

Now solving, $\int \sqrt{x^2 - x - 2}dx$

Now, $\int \sqrt{x^2 - x - 2}\, dx$ can be written as $\int \sqrt{x^2 - x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 2}\, dx$

i.e, $\int \sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}}\, dx$

Here , let x - $\frac{1}{2}$ = y ⇒dx = dy

Therefore, $\int \sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}}\, dx$ can be written as $\int \sqrt{y^2 - (\frac{3}{2})^2}\, dy$

Formula Used: $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

Since $\int \sqrt{y^2 - (\frac{3}{2})^2}\, dy$ is of the form $\int \sqrt{x^2 - a^2}\, dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2 - (\frac{3}{2})^2}\, dy = \frac{y}{2}\sqrt{y^2 - (\frac{3}{2})^2} - \frac{(\frac{3}{2})^2}{2}\log\left|y + \sqrt{y^2 - (\frac{3}{2})^2}\right| + C$

$= \frac{y}{2}\sqrt{y^2 - \frac{9}{4}} - \frac{9}{8}\log\left|y + \sqrt{y^2 - \frac{9}{4}}\right| + C$

Since , x - $\frac{1}{2}$ = y and dx = dy

$\Rightarrow \int \sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}}\, dx = \frac{1}{4}(2x - 1)\sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}} - \frac{9}{8}\log\left|(x - \frac{1}{2}) + \sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}}\right| + C$

Therefore,

$\int \sqrt{x^2 - x - 2}\, dx = \frac{1}{4}(2x - 1)\sqrt{x^2 - x - 2} - \frac{9}{8}\log\left|x - \frac{1}{2} + \sqrt{x^2 - x - 2}\right| + C$

Hence ,

$2\int(2x-1)\sqrt{x^2-x-2}dx + 3\int\sqrt{x^2-x-2}dx = \frac{4}{3}(x^2-x-2)^{\frac{3}{2}} + \frac{3}{4}(2x-1)\sqrt{x^2-x-2} - \frac{27}{8}\log|x - \frac{1}{2} + \sqrt{x^2-x-2}| + C$

Therefore ,

$\int(4x+1)\sqrt{x^2-x-2}\,dx = \frac{4}{3}(x^2-x-2)^{\frac{3}{2}} + \frac{3}{4}(2x-1)\sqrt{x^2-x-2} - \frac{27}{8}\log|x - \frac{1}{2} + \sqrt{x^2-x-2}| + C$

Question 21.
Evaluate the following integrals:

$$\int(x+1)\sqrt{2x^2+3}dx$$

Answer:
To Find : $\int(x+1)\sqrt{2x^2+3}\,dx$

Now, $\int(x+1)\sqrt{2x^2+3}\,dx$ can be written as

$\int(x+1)\sqrt{2x^2+3}\,dx = \int\{x\sqrt{2x^2+3} + \sqrt{2x^2+3}\}\,dx$

$= \int x\sqrt{2x^2+3}dx + \int\sqrt{2x^2+3}dx$

Now solving, $\int x\sqrt{2x^2+3}dx$

Let $2x^2+3 = u \Rightarrow dx = \frac{1du}{4x}$

Thus, $\int x\sqrt{2x^2+3}dx$ becomes $\frac{1}{4}\int\sqrt{u}\,du$

Now , $\frac{1}{4}\int\sqrt{u}\,du = \frac{1}{4}\int u^{\frac{1}{2}}\,du = \frac{1}{4}\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) = \frac{1}{6}u^{\frac{3}{2}}$

$= \frac{1}{6}(2x^2+3)^{\frac{3}{2}}$

Now solving, $\int \sqrt{2x^2+3}dx$

Now, $\int \sqrt{2x^2+3}\,dx$ can be written as $\int \sqrt{(\sqrt{2x})^2+(\sqrt{3})^2}\,dx$

Formula Used: $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$

Since $\int \sqrt{2x^2+3}\,dx$ is of the form $\int \sqrt{x^2+a^2}\,dx$.

$\Rightarrow \int \sqrt{2x^2+3}\,dx = \frac{\sqrt{2}x}{2}\sqrt{(\sqrt{2x})^2+(\sqrt{3})^2} + \frac{(\sqrt{3})^2}{2}\log|\sqrt{2}x + \sqrt{(\sqrt{2x})^2+(\sqrt{3})^2}| + C$

$= \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\log|\sqrt{2}x + \sqrt{2x^2+3}| + C$

Therefore,

$\int x\sqrt{2x^2+3}dx + \int \sqrt{2x^2+3}dx = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\log|\sqrt{2}x + \sqrt{2x^2+3}| + C$

Hence ,

$\int (x+1)\sqrt{2x^2+3}\,dx = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\log|\sqrt{2}x + \sqrt{2x^2+3}| + C$

Question 22.
Evaluate the following integrals:

$$\int x\sqrt{1+x-x^2}\,dx$$

Answer:
To Find : $\int x\sqrt{1+x-x^2}\,dx$

Now, let x be written as $\frac{1}{2} - \frac{1}{2}(1 - 2x)$ and split

Therefore ,

$\int x\sqrt{1+x-x^2}\,dx = \int\{\frac{\sqrt{-x^2+x+1}}{2} - \frac{(1-2x)\sqrt{-x^2+x+1}}{2}\}\,dx$

$= \frac{1}{2}\int(2x-1)\sqrt{-x^2+x+1}\,dx + \frac{1}{2}\int\sqrt{-x^2+x+1}\,dx$

Now solving, $\frac{1}{2}\int(2x-1)\sqrt{-x^2+x+1}\,dx$

Let $-x^2+x+1 = u \Rightarrow dx = \frac{du}{(1-2x)}$

Thus, $\frac{1}{2}\int(2x-1)\sqrt{-x^2+x+1}\,dx$ becomes $-\frac{1}{2}\int\sqrt{u}\,du$

Now, $-\frac{1}{2}\int\sqrt{u}\,du = -\frac{1}{2}\int u^{\frac{1}{2}}\,du = -\frac{1}{2}\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) = -\frac{1}{3}u^{\frac{3}{2}}$

$= -\frac{1}{3}(-x^2+x+1)^{\frac{3}{2}}$

Now solving, $\int\sqrt{-x^2+x+1}\,dx$

$\int\sqrt{-x^2+x+1}\,dx$ can be written as $\int\sqrt{-x^2+x-\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+1}\,dx$

i.e, $\int\sqrt{\frac{5}{4}-(x-\frac{1}{2})^2}\,dx = \frac{1}{2}\int\sqrt{5-(2x-1)^2}\,dx$

let $2x-1 = y \Rightarrow dx = \frac{1dy}{2}$

Therefore, $\frac{1}{4}\int\sqrt{5-(2x-1)^2}\,dx$ becomes $\frac{1}{4}\int\sqrt{(\sqrt{5})^2-y^2}\,dy$

Formula Used: $\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int\sqrt{(\sqrt{5})^2-y^2}\,dy$ is of the form $\int\sqrt{a^2-x^2}\,dx$ with change in variable .

Hence, $\int \sqrt{(\sqrt{5})^2 - y^2}\, dy = \frac{1}{2} y \sqrt{(\sqrt{5})^2 - y^2} + \frac{(\sqrt{5})^2}{2} \sin^{-1}\frac{y}{\sqrt{5}}$ + C

$= \frac{1}{2} y \sqrt{5 - y^2} + \frac{5}{2} \sin^{-1}\frac{y}{\sqrt{5}}$ + C

Since , 2x – 1 = y and dx = $\frac{1dy}{2}$

Therefore,

$\frac{1}{4}\int \sqrt{5 - (2x-1)^2}\, dx = \frac{1}{8}(2x-1)\sqrt{5-(2x-1)^2} + \frac{5}{8} \sin^{-1}\frac{(2x-1)}{\sqrt{5}}$ + C

i.e, $\int \sqrt{-x^2 + x + 1}\, dx = \frac{1}{8}(2x-1)\sqrt{-x^2+x+1} + \frac{5}{8} \sin^{-1}\frac{(2x-1)}{\sqrt{5}}$ + C

hence , $\int x\sqrt{1+x-x^2}\, dx = \frac{1}{2} \int (2x-1)\sqrt{-x^2+x+1}\, dx + \frac{1}{2}\int \sqrt{-x^2+x+1}\, dx$ = = $-\frac{1}{3}(-x^2+x+1)^{\frac{3}{2}} + \frac{1}{16}(2x-1)\sqrt{-x^2+x+1} + \frac{5}{16} \sin^{-1}(\frac{2x-1}{\sqrt{5}})$ + C

Question 23.
Evaluate the following integrals:

Answer:

To Find : $\int (2x-5)\sqrt{2+3x-x^2}\, dx \int (2x-5)\sqrt{2+3x-x^2} dx$

Now, let 2x - 5 be written as (2x – 3) -2 and split

Therefore ,

$\int (2x-5)\sqrt{2+3x-x^2}\, dx = \int \{(2x-3)\sqrt{-x^2+3x+2} - 2\sqrt{-x^2+3x+2}\}\, dx$

= $\int (2x-3)\sqrt{-x^2+3x+2}\, dx$ - 2 $\int \sqrt{-x^2+3x+2}\, dx$

Now solving, $\int (2x-3)\sqrt{-x^2+3x+2}\, dx$

Let $-x^2 + 3x + 2$= u ⇒ dx = $\frac{du}{(3-2x)}$

Thus, $\int (2x - 3)\sqrt{-x^2 + 3x + 2}\,dx$ becomes $-\int \sqrt{u}\,du$

Now , $-\int \sqrt{u}\,du = -\int u^{\frac{1}{2}}\,du = -\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) = -\frac{2}{3}u^{\frac{3}{2}}$

$= -\frac{2}{3}(-x^2 + 3x + 2)^{\frac{3}{2}}$

Now solving, $\int \sqrt{-x^2 + 3x + 2}\,dx$

$\int \sqrt{-x^2 + 3x + 2}\,dx$ can be written as $\int \sqrt{-x^2 + 3x - \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 2}\,dx$

i.e, $\int \sqrt{\frac{17}{4} - (x - \frac{3}{2})^2}\,dx$

let x $-\frac{3}{2}$ = y ⇒ dx = dy

Therefore , $\int \sqrt{\frac{17}{4} - (x - \frac{3}{2})^2}\,dx$ becomes $\int \sqrt{(\frac{\sqrt{17}}{2})^2 - y^2}\,dy$

Formula Used: $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int \sqrt{(\frac{\sqrt{17}}{2})^2 - y^2}\,dy$ is of the form $\int \sqrt{a^2 - x^2}\,dx$ with change in variable .

Hence, $\int \sqrt{(\frac{\sqrt{17}}{2})^2 - y^2}\,dy = \frac{1}{2}y\sqrt{(\frac{\sqrt{17}}{2})^2 - y^2} + \frac{(\frac{\sqrt{17}}{2})^2}{2}\sin^{-1}\frac{y}{\frac{\sqrt{17}}{2}} + C$

$= \frac{1}{2}y\sqrt{\frac{17}{4} - y^2} + \frac{17}{8}\sin^{-1}\frac{y}{\frac{\sqrt{17}}{2}} + C$

Since , x $-\frac{3}{2}$ = y and dx = dy

Therefore,

$\int \sqrt{\frac{17}{4} - (x - \frac{3}{2})^2}\, dx = \frac{1}{4}(2x - 3)\sqrt{\frac{17}{4} - (x - \frac{3}{2})^2} + \frac{17}{8} \sin^{-1}(\frac{2x-3}{\sqrt{17}})$ + C

i.e, $\int \sqrt{-x^2 + 3x + 2}\, dx = \frac{1}{4}(2x - 3)\sqrt{-x^2 + 3x + 2} + \frac{17}{8} \sin^{-1}(\frac{2x-3}{\sqrt{17}})$ + C

hence ,

$\int (2x - 5)\sqrt{2 + 3x - x^2}\, dx = \int (2x - 3)\sqrt{-x^2 + 3x + 2}\, dx$ - 2 $\int \sqrt{-x^2 + 3x + 2}\, dx$ =
$-\frac{2}{3}(-x^2 + 3x + 2)^{\frac{3}{2}}$ - $\frac{1}{2}(2x - 3)\sqrt{-x^2 + 3x + 2}$ - $\frac{17}{4} \sin^{-1}(\frac{2x-3}{\sqrt{17}})$ + C

Question 24.
Evaluate the following integrals:

$$\int (6x + 5)\sqrt{6 + x - 2x^2}\, dx$$

Answer:
To Find : $\int (6x + 5)\sqrt{6 + x - 2x^2}\, dx$

Now, let 6x + 5 be written as $\frac{13}{2}$ - $\frac{3}{2}$(1 - 4x)and split

Therefore ,

$\int (6x + 5)\sqrt{6 + x - 2x^2}\, dx = \int \{\frac{13\sqrt{-2x^2+x+6}}{2} - \frac{3(1-4x)\sqrt{-2x^2+x+6}}{2}\}\, dx$

= $\frac{3}{2}\int (4x - 1)\sqrt{-2x^2 + x + 6}\, dx + \frac{13}{2} \int \sqrt{-2x^2 + x + 6}\, dx$

Now solving, $\int (4x - 1)\sqrt{-2x^2 + x + 6} dx$

Let $-2x^2 + x + 6$ = u ⇒ dx = $\frac{du}{(1-4x)}$

Thus, $\int (4x - 1)\sqrt{-2x^2 + x + 6}\, dx$ becomes $- \int \sqrt{u}\, du$

Now , $-\int \sqrt{u}\, du = -\int u^{\frac{1}{2}}\, du = -\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) = -\frac{2}{3}u^{\frac{3}{2}}$

$= -\frac{2}{3}(-2x^2 + x + 6)^{\frac{3}{2}}$

Now solving, $\int \sqrt{-2x^2 + x + 6}\, dx$

$\int \sqrt{-2x^2 + x + 6}\, dx$ can be written as $\int \sqrt{-(\sqrt{2}x)^2 + x - \left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 + 6}\, dx$

i.e, $\int \sqrt{\frac{49}{8} - (\sqrt{2}x - \frac{1}{2\sqrt{2}})^2}\, dx$

let $\sqrt{2}x - \frac{1}{2\sqrt{2}} = y \Rightarrow dx = \frac{dy}{\sqrt{2}}$

Therefore , $\int \sqrt{\frac{49}{8} - (\sqrt{2}x - \frac{1}{2\sqrt{2}})^2}\, dx$ becomes $\int \sqrt{(\frac{7}{2\sqrt{2}})^2 - y^2}\, dy$

Formula Used: $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int \sqrt{(\frac{7}{2\sqrt{2}})^2 - y^2}\, dy$ is of the form $\int \sqrt{a^2 - x^2}\, dx$ with change in variable .

Hence, $\int \sqrt{(\frac{7}{2\sqrt{2}})^2 - y^2}\, dy = \frac{1}{2}y\sqrt{(\frac{7}{2\sqrt{2}})^2 - y^2} + \frac{(\frac{7}{2\sqrt{2}})^2}{2}\sin^{-1}\frac{y}{\frac{7}{2\sqrt{2}}} + C$

$= \frac{1}{2}y\sqrt{\frac{49}{8} - y^2} + \frac{7}{16}\sin^{-1}\frac{y}{\frac{\sqrt{17}}{2}} + C$

Since , $\sqrt{2}x - \frac{1}{2\sqrt{2}} = y$ and $dx = \frac{dy}{\sqrt{2}}$

Therefore,

$\int \sqrt{\frac{49}{8} - (\sqrt{2}x - \frac{1}{2\sqrt{2}})^2}\, dx = \frac{1}{4\sqrt{2}}(4x-1)\sqrt{\frac{49}{8} - (\sqrt{2}x - \frac{1}{2\sqrt{2}})^2} + \frac{49}{16}\sin^{-1}(\frac{4x-1}{7}) + C$

i.e, $\int \sqrt{-2x^2+x+6}\, dx = \frac{1}{4\sqrt{2}}(4x-1)\sqrt{-2x^2+x+6} + \frac{49}{16}\sin^{-1}(\frac{4x-1}{7}) + C$

hence ,

$\int(6x+5)\sqrt{6+x-2x^2}\, dx = \frac{3}{2}\int(4x-1)\sqrt{-2x^2+x+6}\, dx + \frac{13}{2}\int\sqrt{-2x^2+x+6}\, dx =$
$-(-2x^2+x+6)^{\frac{3}{2}} + \frac{13}{16}(4x-1)\sqrt{-2x^2+x+6} + \frac{637}{32\sqrt{2}}\sin^{-1}(\frac{4x-1}{7}) + C$

Question 25.
Evaluate the following integrals:

$$\int(x+1)\sqrt{1-x-x^2}dx$$

Answer:
To Find : $\int(x+1)\sqrt{1-x-x^2}\, dx$

Now, let x + 1 be written as $\frac{1}{2}$ - $\frac{1}{2}$(-2x - 1) and split

Therefore ,

$\int(x+1)\sqrt{1-x-x^2}\, dx = \int\{\frac{\sqrt{-x^2-x+1}}{2} - \frac{(-2x-1)\sqrt{-x^2-x+1}}{2}\}\, dx$

$= \frac{1}{2}\int(2x-1)\sqrt{-x^2-x+1}\, dx + \frac{1}{2}\int\sqrt{-x^2-x+1}\, dx$

Now solving, $\int(2x-1)\sqrt{-x^2-x+1}dx$

Let $-x^2-x+1$ = u ⇒ dx = $\frac{du}{-2x-1}$

Thus, $\int(2x-1)\sqrt{-x^2-x+1}\, dx$ becomes $-\int\sqrt{u}\, du$

Now , $-\int\sqrt{u}\, du = -\int u^{\frac{1}{2}}\, du = -(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}) = -\frac{2}{3}u^{\frac{3}{2}}$

$=-\frac{2}{3}(-x^2-x+1)^{\frac{3}{2}}$

Now solving, $\int\sqrt{-x^2-x+1}\,dx$

$\int\sqrt{-x^2-x+1}\,dx$ can be written as $\int\sqrt{-x^2-x-\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+1}\,dx$

i.e, $\int\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}\,dx$

let $x+\frac{1}{2}$= y ⇒ dx = dy

Therefore , $\int\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}\,dx$ becomes $\int\sqrt{(\frac{\sqrt{5}}{2})^2-y^2}\,dy$

Formula Used: $\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$ + C

Since $\int\sqrt{(\frac{\sqrt{5}}{2})^2-y^2}\,dy$ is of the form $\int\sqrt{a^2-x^2}\,dx$ with change in variable .

Hence, $\int\sqrt{(\frac{\sqrt{5}}{2})^2-y^2}\,dy=\frac{1}{2}y\sqrt{(\frac{\sqrt{5}}{2})^2-y^2}+\frac{(\frac{\sqrt{5}}{2})^2}{2}\sin^{-1}\frac{y}{\frac{\sqrt{5}}{2}}$ + C

$=\frac{1}{2}y\sqrt{\frac{5}{4}-y^2}+\frac{5}{8}\sin^{-1}\frac{y}{\frac{\sqrt{5}}{2}}$ + C

Since , $x+\frac{1}{2}$= y and dx = dy

Therefore,

$\int\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}\,dx=\frac{1}{4}(2x+1)\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}+\frac{5}{8}\sin^{-1}(\frac{2x+1}{\sqrt{5}})$ + C

i.e, $\int\sqrt{-x^2-x+1}\,dx=\frac{1}{4}(2x+1)\sqrt{-x^2-x+1}+\frac{5}{8}\sin^{-1}(\frac{2x+1}{\sqrt{5}})$ + C

hence ,

$\int (x+1)\sqrt{1-x-x^2}\,dx = \frac{1}{2}\int (2x-1)\sqrt{-x^2-x+1}\,dx + \frac{1}{2}\int \sqrt{-x^2-x+1}\,dx =$
$-\frac{1}{3}(-x^2-x+1)^{\frac{3}{2}} + \frac{1}{8}(2x+1)\sqrt{-x^2-x+1} + \frac{5}{16}\sin^{-1}(\frac{2x+1}{\sqrt{5}}) + C$

Question 26.
Evaluate the following integrals:

$$\int (x-3)\sqrt{x^2+3x-18}dx$$

Answer:
To Find : $\int (x-3)\sqrt{x^2+3x-18}\,dx$

Now, let x - 3 be written as $\frac{1}{2}(2x+3) - \frac{9}{2}$ and split

Therefore ,

$\int (x-3)\sqrt{x^2+3x-18}\,dx = \int \{\frac{(2x+3)\sqrt{x^2+3x-18}}{2} - \frac{9\sqrt{x^2+3x-18}}{2}\}dx$

$= \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\,dx - \frac{9}{2}\int \sqrt{x^2+3x-18}\,dx$

Now solving, $\int (2x+3)\sqrt{x^2+3x-18}dx$

Let $x^2+3x-18 = u \Rightarrow dx = \frac{du}{2x+3}$

Thus, $\int (2x+3)\sqrt{x^2+3x-18}\,dx$ becomes $\int \sqrt{u}\,du$

Now , $\int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du = (\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}) = \frac{2}{3}u^{\frac{3}{2}}$

$= \frac{2}{3}(x^2+3x-18)^{\frac{3}{2}}$

Now solving, $\int \sqrt{x^2+3x-18}\,dx$

$\int \sqrt{x^2+3x-18}\,dx$ can be written as $\int \sqrt{x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-18\,dx}$

i.e, $\int \sqrt{(x+\frac{3}{2})^2-\frac{81}{4}}\,dx$

let $x+\frac{3}{2}$= y ⇒ dx = dy

Therefore, $\int \sqrt{(x+\frac{3}{2})^2-\frac{81}{4}}\,dx$ can be written as $\int \sqrt{y^2-(\frac{9}{2})^2}\,dy$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}$ log |x + $\sqrt{x^2-a^2}$|+ C

Since $\int \sqrt{y^2-(\frac{9}{2})^2}\,dy$ is of the form $\int \sqrt{x^2-a^2}\,dx$ with change in variable.

⇒ $\int \sqrt{y^2-(\frac{9}{2})^2}\,dy = \frac{y}{2}\sqrt{y^2-(\frac{9}{2})^2} - \frac{(\frac{9}{2})^2}{2}$ log |y + $\sqrt{y^2-(\frac{9}{2})^2}$|+ C

$= \frac{y}{2}\sqrt{y^2-\frac{81}{4}} - \frac{81}{8}$ log |y + $\sqrt{y^2-\frac{81}{4}}$|+ C

Since , x + $\frac{3}{2}$ = y and dx = dy

⇒ $\int \sqrt{(x+\frac{3}{2})^2-\frac{81}{4}}\,dx = \frac{1}{4}(2x+3)\sqrt{(x+\frac{3}{2})^2-\frac{81}{4}} - \frac{81}{8}$ log |(x + $\frac{3}{2}$) + $\sqrt{(x+\frac{3}{2})^2-\frac{81}{4}}$|+C

Therefore,

$\int \sqrt{x^2+3x-18}\,dx = \frac{1}{4}(2x+3)\sqrt{x^2+3x-18} - \frac{81}{8}$ log |x + $\frac{3}{2}$ + $\sqrt{x^2+3x-18}$|+C

Hence ,

$\int (x-3)\sqrt{x^2+3x-18}\,dx = \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\,dx$ - $\frac{9}{2}$ $\int \sqrt{x^2+3x-18}\,dx$ =
$\frac{1}{3}(x^2+3x-18)^{\frac{3}{2}} - \frac{9}{8}(2x+3)\sqrt{x^2+3x-18} + \frac{726}{16}$ log |x + $\frac{3}{2}$ + $\sqrt{x^2+3x-18}$|+C