Exercise 9b

Question 1.

Show that function $f(x) = \begin{cases} (7x+5), & \text{when } x \geq 0; \\ (5-3x), & \text{when } x < 0 \end{cases}$ is continuous function.

Answer:
Given:

$f(x) = \begin{cases} (7x+5), \text{when} x \geq 0; \\ (5-3x), \text{when} x < 0 \end{cases}$

Let's calculate the limit of f(x) when x approaches 0 from the right

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (7x+5) = 7(0) + 5$

= 5

Therefore,

$\lim_{x \to 0^+} f(x) = 5$

Let's calculate the limit of f(x) when x approaches 0 from the left

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (5-3x) = 5 - 3(0)$

= 5

Therefore,

$\lim_{x \to 0^-} f(x) = 5$

Also, f(0) = 5

As we can see,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 5$$

Thus, we can say that f(x) is continuous function.

Question 2.

Show that function $f(x) = \begin{cases} \sin x, & \text{if } x < 0; \\ x, & \text{if } x \geq 0 \end{cases}$ is continuous.

Answer:
Given:

$$f(x) = \begin{cases} \sin x, \text{if } x < 0; \\ x, \text{if } x \geq 0 \end{cases}$$

Left hand limit at x = 0

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (\sin x) = \sin(0) = 0$$

Therefore,

$$\lim_{x \to 0^-} f(x) = 0$$

Right hand limit at x = 0

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = 0$$

Therefore,

$$\lim_{x \to 0^+} f(x) = 0$$

Also, f(0) = 0

As,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$$

Thus, we can say that f(x) is continuous function.

Question 3.

Show that function f(x) = $\begin{cases} \frac{x^n - 1}{x - 1}, & \text{when } x \neq 1; \\ n, & \text{when } x = 1 \end{cases}$ is continuous.

Answer:
Given:

$$f(x) = \begin{cases} \frac{x^n - 1}{x-1}, \text{when} x \neq 1; \\ n, \text{when} x = 1 \end{cases}$$

Left hand limit and x = 1

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \frac{(1-h)^n - 1}{(1-h) - 1}$$

$$\lim_{h \to 0} \frac{(1-h)^n - 1}{1-h-1} = \lim_{h \to 0} \frac{(1-h)^n - 1}{-h} = \lim_{h \to 0} -\frac{(1-h)^n - 1}{h}$$

$$= -\lim_{h \to 0} \frac{(1-h)^n - 1}{h} \text{(Because } \lim_{x \to a} c.f(x) = c \lim_{x \to a} f(x))$$

Applying L hospital's rule $\left(\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f\prime(x)}{g\prime(x)}\right)$

$$= -\lim_{h \to 0} \frac{-n(1-h)^{n-1}}{1} = -[-n(1-0)^{n-1}] = n$$

Right hand limit and x = 1

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \frac{(1+h)^n - 1}{(1+h) - 1}$$

$$\lim_{h \to 0} \frac{(1+h)^n - 1}{1+h-1} = \lim_{h \to 0} \frac{(1+h)^n - 1}{h}$$

Applying L hospital's rule $\left(\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f\prime(x)}{g\prime(x)}\right)$

$= \lim_{h \to 0} \frac{n(1+h)^{n-1}}{1} = [n(1+0)^{n-1}]$ = n

Also, f(x) = n at x =1

As we can see that $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(x)$

Thus, f(x) is continuous at x = 1

Question 4.
Show that sec x is a continuous function.

Answer:
Let f(x) = sec x

Therefore, f(x) = $\frac{1}{\cos x}$

f(x) is not defined when cos x = 0

And cos x = 0 when, x = $\frac{\pi}{2}$ and odd multiples of $\frac{\pi}{2}$ like $-\frac{\pi}{2}$

Let us consider the function

f(a) = cos a and let c be any real number. Then,

$\lim_{a \to c^+} f(a) = \lim_{h \to 0} f(c+h)$

$\lim_{h \to 0} \cos(c+h) = \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$

= cos c $\lim_{h \to 0} \cos h$ - sin c $\lim_{h \to 0} \sin h$

= cos c (1) – sin c (0)

Therefore,

$\lim_{a \to c^+} f(a) =$ cos c

Similarly,

$\lim\limits_{a \to c^-} f(a) =$ f(c) = cos c

Therefore,

$\lim\limits_{a \to c^-} f(a) = \lim\limits_{a \to c^+} f(a) =$ f(c) = cos c

So, f(a) is continuous at a = c

Similarly, cos x is also continuous everywhere

Therefore, sec x is continuous on the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Question 5.
Show that sec |x| is a continuous function

Answer:
Let f(x) = sec |x| and a be any real number. Then,

Left hand limit at x = a

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} \sec|x| = \lim_{h\to 0} \sec|a-h| = \sec|a|$$

Right hand limit at x = a

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} \sec|x| = \lim_{h\to 0} \sec|a+h| = \sec|a|$$

Also, f(a) = sec |a|

Therefore,

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) =$ f(a)

Thus, f(x) is continuous at x = a.

Question 6.

Show that function $f(x) = \begin{cases} (2-x), & \text{when } x \geq 1; \\ x, & \text{when } 0 \leq x \leq 1. \end{cases}$ is continuous.

Answer:
We know that sin x is continuous everywhere

Consider the point x = 0

Left hand limit:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{\sin x}{x}\right) = \lim_{h \to 0} \left(\frac{\sin(0-h)}{0-h}\right) = \lim_{h \to 0} \left(\frac{-\sin h}{-h}\right) = 1$$

Right hand limit:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right) = \lim_{h \to 0} \left(\frac{\sin(0+h)}{0+h}\right) = \lim_{h \to 0} \left(\frac{\sin h}{h}\right) = 1$$

Also we have,

f(0) = 2

As,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$$

Therefore, f(x) is discontinuous at x = 0.

Question 7.
Discuss the continuity of f(x) = [x].

Answer:
Let n be any integer

[x] = Greatest integer less than or equal to x.

Some values of [x] for specific values of x

[3] = 3

[4.4] = 4

[-1.6] = -2

Therefore,

Left hand limit at x = n

$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [x] = n - 1$

Right hand limit at x = n

$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [x] = n$

Also, f(n) = [n] = n

As $\lim_{x \to n^-} f(x) \neq \lim_{x \to n^+} f(x)$

Therefore, f(x) = [x] is discontinuous at x = n.

Question 8.

Show that $f(x) = \begin{cases} (2x-1), \text{ if } x < 2; \\ \dfrac{3x}{2}, \text{ if } x \geq 2 \end{cases}$ is continuous.

Answer:

Given function $f(x) = \begin{cases} (2x-1), \text{if } x < 2; \\ \frac{3x}{2}, \text{if } x \geq 2 \end{cases}$

Left hand limit at x = 2

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} (2x - 1) = 2(2) - 1 = 3$

Right hand limit at x = 2

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} \frac{3x}{2} = \frac{3(2)}{2} = 3$$

Also,

$$f(2) = \frac{3(2)}{2} = 3$$

As

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) = 3$$

Therefore,

The function f(x) is continuous at x = 2.

Question 9.

Show that $f(x) = \begin{cases} x, \text{ if } x \neq 0; \\ 1, \text{ if } x = 0 \end{cases}$ is continuous at each point except 0.

Answer:

Given function is $f(x) = \begin{cases} x, \text{if } x \neq 0; \\ 1, \text{if } x = 0 \end{cases}$

Left hand limit at x = 0

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h) = 0$$

Right hand limit at x = 0

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h) = 0$$

Also,

f(0) = 1

As,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \neq f(0)$$

f(x) = x for other values of x expect 0 f(x) = 1,2,3,4...

Therefore,

f(x) is not continuous everywhere expect at x = 0

Question 10.
Locate the point of discontinuity of the function

$$f(x) = \begin{cases} (x^3 - x^2 + 2x - 2), & \text{if } x \neq 1; \\ 4, & \text{if } x = 0 \end{cases}$$

Answer:
Given function $f(x) = \begin{cases} (x^3 - x^2 + 2x - 2), \text{if } x \neq 1; \\ 4, \text{if } x = 1 \end{cases}$

Left hand limit at x = 1:
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^3 - x^2 + 2x - 2)$$

$$= \lim_{h\to 0}\{(1-h)^3 - (1-h)^2 + 2(1-h) - 2\}$$

$$= \lim_{h\to 0}(1-h)^3 - \lim_{h\to 0}(1-h)^2 + 2\lim_{h\to 0}(1-h) - 2$$

= 1 – 1 + 2 – 2

= 0

Right hand limit at x = 1:
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^3 - x^2 + 2x - 2)$$

$$= \lim_{h\to 0}\{(1+h)^3 - (1+h)^2 + 2(1+h) - 2\}$$

$$= \lim_{h\to 0}(1+h)^3 - \lim_{h\to 0}(1+h)^2 + 2\lim_{h\to 0}(1+h) - 2$$

= 1 – 1 + 2 – 2

= 0

Also, f(1) = 4

As we can see that,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \neq f(1)$$

Therefore,

f(x) is not continuous at x = 1

Question 11.
Discus the continuity of the function ƒ(x)= |x|+|x-1| in the interval of [-1, 2]

Answer:
Given function f(x) = |x| + |x - 1|

A function f(x) is said to be continuous on a closed interval [a, b] if and only if,

(i) f is continuous on the open interval (a, b)

(ii) $\lim_{x \to a^+} f(x) = f(a)$

(iii) $\lim_{x \to b^-} f(x) = f(b)$

Let's check continuity on the open interval (-1, 2)

As -1 < x < 2

Left hand limit:

$$\lim_{x \to -1^-} f(x) = \lim_{h \to 0} \{|-1 - h| + |(-1 - h) - 1|\}$$

=|-1-0| + |(-1-0) – 1|

=1 + 2

= 3

Right hand limit:

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} \{|2+h| + |(2+h)-1|\}$$

=|2| + |2 – 1|

=2+1

= 3

Left hand limit = Right hand limit

Here a = -1 and b = 2

Therefore,

$$\lim_{x \to -1^+} f(x) = \lim_{h \to 0} \{|-1+h| + |(-1+h)-1|\}$$

= |-1 + 0| + |(-1 + 0) - 1|

= |- 1| + |-1 – 1|

= 1 + 2 = 3

Also f(-1) = |-1| + |-1 - 1| = 1 + 2 = 3

Now,

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} \{|2-h| + |(2-h)-1|\}$$

= |2 - 0| + |(2 - 0) - 1|

= | 2| + |2 – 1|

= 2 + 1 = 3

Also f(2) = |2| + |2 - 1| = 2 + 1 = 3

Therefore,

f(x) is continuous on the closed interval [-1, 2].