<u>Exercise 11f</u>

Question 1.
Find two positive number whose product is 49 and the sum is minimum.

Answer:
Given,

• The two numbers are positive.

• the product of two numbers is 49.

• the sum of the two numbers is minimum.

Let us consider,

• x and y are the two numbers, such that $x > 0$ and $y > 0$

• Product of the numbers : $x \times y = 49$

• Sum of the numbers : $S = x + y$

Now as,

$x \times y = 49$

$y = \frac{49}{x}$ ------ (1)

Consider,

$S = x + y$

By substituting (1), we have

$S = x + \frac{49}{x}$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then $f'(c) = 0$.</u>

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}\left(x + \frac{49}{x}\right)$$

$$\frac{dS}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{49}{x}\right)$$

$$\frac{dS}{dx} = 1 + 49\left(\frac{-1}{x^2}\right) \text{----- (3)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 1 + 49\left(\frac{-1}{x^2}\right) = 0$$

$$= 1 - \left(\frac{49}{x^2}\right) = 0$$

$$= 1 = \left(\frac{49}{x^2}\right)$$

$= x^2 = 49$

$$= x = \pm\sqrt{49}$$

As $x > 0$, then $x = 7$

Now, for checking if the value of S is maximum or minimum at x=7, we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 7.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[1 + 49\left(\frac{-1}{x^2}\right)\right]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[1] + \frac{d}{dx}\left[49\left(\frac{-1}{x^2}\right)\right]$$

$$\frac{d^2S}{dx^2} = 0 + \left[49\left(\frac{-1 \times -2}{x^3}\right)\right]$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{d^2S}{dx^2} = 49\left(\frac{2}{x^3}\right) = \frac{98}{x^3}$$

Now when x = 7,

$$\left[\frac{d^2S}{dx^2}\right]_{x=7} = \frac{98}{7^3} = \frac{98}{343} > 0$$

As second differential is positive, hence the critical point x = 7 will be the minimum point of the function S.

Therefore, the function S = sum of the two numbers is minimum at x = 7.

From Equation (1), if x= 7

$$y = \frac{49}{7} = 7$$

Therefore, x = 7 and y = 7 are the two positive numbers whose product is 49 and the sum is minimum.

Question 2.
Find two positive numbers whose sum is 16 and the sum of whose squares is minimum.

Answer:
Given,

• The two numbers are positive.

• the sum of two numbers is 16.

- the sum of the squares of two numbers is minimum.

Let us consider,

- x and y are the two numbers, such that $x > 0$ and $y > 0$

- Sum of the numbers : $x + y = 16$

- Sum of squares of the numbers : $S = x^2 + y^2$

Now as,

$x + y = 16$

$y = (16-x)$ ------ (1)

Consider,

$S = x^2 + y^2$

By substituting (1), we have

$S = x^2 + (16-y)^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.</u>

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}[x^2 + (16 - x)^2]$$

$$\frac{dS}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}[(16 - x)^2]$$

$\frac{dS}{dx} = 2x + 2(16 - x)(-1)$ ----- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 2x + 2(16 - x)(-1) \ = 0$$

⇒ 2x – 2(16 -x) = 0

⇒ 2x – 32 + 2x = 0

= 4x = 32

$$\Rightarrow x = \frac{32}{4}$$

⇒ x = 8

As x > 0, x = 8

Now, for checking if the value of S is maximum or minimum at x=8, <u>we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 8.</u>

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[2x + 2(16 - x)(-1)\right]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[2x] - 2\frac{d}{dx}[16 - x]$$

$$\frac{d^2S}{dx^2} = 2 - \ 2[0 - 1]$$

[Since $\frac{d}{dx}$ (constant) = 0 and $\frac{d}{dx}(x^n) = \ nx^{n-1}$]

$$\frac{d^2S}{dx^2} = 2 - 0 + 2 = 4$$

Now when x = 8,

$$\left[\frac{d^2 S}{dx^2}\right]_{x=8} = 4 > 0$$

As second differential is positive, hence the critical point x = 8 will be the minimum point of the function S.

Therefore, the function S = sum of the squares of the two numbers is minimum at x = 8.

From Equation (1), if x= 8

y = 16 – 8 = 8

Therefore, x = 8 and y = 8 are the two positive numbers whose su is 16 and the sum of the squares is minimum.

Question 3.
Divide 15 into two parts such that the square of one number multiplied with the cube of the other number is maximum.

Answer:
Given,

• the number 15 is divided into two numbers.

• the product of the square of one number and cube of another number is maximum.

Let us consider,

• x and y are the two numbers

• Sum of the numbers : x + y = 15

• Product of square of the one number and cube of anther number : $P = x^3 y^2$

Now as,

x + y = 15

y = (15-x) ------ (1)

Consider,

$P = x^3y^2$

By substituting (1), we have

$P = x^3 \times (15-x)^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.

Differentiating the equation (2) with x

$$\frac{dP}{dx} = \frac{d}{dx}[x^3 \times (15-x)^2]$$

$$\frac{dP}{dx} = (15-x)^2\frac{d}{dx}(x^3) + x^3\frac{d}{dx}[(15-x)^2]$$

$$\frac{dP}{dx} = (15-x)^2(3x^2) + x^3[2(15-x)(-1)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$$\frac{dP}{dx} = (15-x)^2(3x^2) + x^3[-30+2x]$$

$= 3\times[15^2 - 2\times(15)\times(x) + x^2]\, x^2 + x^3(2x-30)$

$= x^2[3\times(225 - 30x + x^2) + x(2x - 30)]$

$= x^2[675 - 90x + 3x^2 + 2x^2 - 60x]$

$= x^2[5x^2 - 120x + 675]$

$= 5x^2[x^2 - 24x + 135]$ ----- (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dP}{dx} = 5\,x^2[x^2 - 24x + 135] = 0$$

Hence $5x^2 = 0$ (or) $x^2 - 24x + 135 = 0$

x = 0 (or) $x = \frac{-(-24)\pm\sqrt{(-24)^2-4(1)(135)}}{2\times 1}$

x = 0 (or) $x = \frac{24\pm\sqrt{576-540)}}{2}$

x = 0 (or) $x = \frac{24\pm\sqrt{36}}{2}$

x = 0 (or) $x = \frac{24\pm 6}{2}$

x = 0 (or) $x = \frac{24+6}{2}$ (or) $x = \frac{24-6}{2}$

x = 0 (or) $x = \frac{30}{2}$ (or) $x = \frac{18}{2}$

x = 0 (or) x = 15 (or) x = 9

Now considering the critical values of x = 0,9,15

Now, for checking if the value of P is maximum or minimum at x=0,9,15, <u>we will perform the second differentiation and check the value of $\frac{d^2P}{dx^2}$ at the critical value x = 0,9,15.</u>

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\,[5x^2\,(x^2 - 24x + 135)\,]$$

$$\frac{d^2P}{dx^2} = (x^2 - 24x + 135)\frac{d}{dx}\,[5x^2] + 5x^2\,\frac{d}{dx}[x^2 - 24x + 135\,]$$

$= (x^2 - 24x + 135)\,(5 \times 2x) + 5x^2\,(2x - 24 + 0)$

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$= (x^2 – 24x + 135) (10x) + 5x^2 (2x – 24)$

$= 10x^3 – 240x^2 + 1350x + 10x^3 – 120x^2$

$= 20x^3 – 360x^2 + 1350x$

$= 5x (4x^2 – 72x + 270)$

$$\frac{d^2P}{dx^2} = 5x\,(4x^2 - 72x + 270)$$

Now when x = 0,

$$\left[\frac{d^2P}{dx^2}\right]_{x=0} = 5 \times 0[\,4(0)^2 - 72\,(0) + 270]$$

= 0

So, we reject x = 0

Now when x = 15,

$$\left[\frac{d^2P}{dx^2}\right]_{x=15} = 5 \times 15\,[\,4(15)^2 - 72\,(15) + 270]$$

$= 65 [(4 \times 225) –1080+ 270]$

$= 65 [900– 1080+ 270]$

$= 65 [1170– 1080]$

$= 65\times (90) > 0$

Hence $\left[\frac{d^2P}{dx^2}\right]_{x=15} > 0$, so at x = 15, the function P is minimum

Now when x = 9,

$$\left[\frac{d^2P}{dx^2}\right]_{x=9} = 5 \times 9\,[\,4(9)^2 - 72\,(9)\; + 270]$$

= 45 [(4 × 81) – 648 + 270]

= 45 [324 – 648 + 270]

= 45 [594 – 648]

= 45 × (-54)

= -2430 < 0

As second differential is negative, hence at the critical point x = 9 will be the maximum point of the function P.

Therefore, the function P is maximum at x = 9.

From Equation (1), if x= 9

y = 15 – 9 = 6

Therefore, x = 9 and y = 6 are the two positive numbers whose sum is 15 and the product of the square of one number and cube of another number is maximum.

Question 4.
Divide 8 into two positive parts such that the sum of the square of one and the cube of the other is minimum.

Answer:
Given,

• the number 8 is divided into two numbers.

• the product of the square of one number and cube of another number is minimum.

Let us consider,

• x and y are the two numbers

• Sum of the numbers : $x + y = 8$

• Product of square of the one number and cube of anther number : $S = x^3 + y^2$

Now as,

$x + y = 8$

$y = (8-x)$ ------ (1)

Consider,

$S = x^3 + y^2$

By substituting (1), we have

$S = x^3 + (8-x)^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then $f'(c) = 0$.

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}\left[x^3 + (8-x)^2\right]$$

$$\frac{dS}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}\left[(8-x)^2\right]$$

$$\frac{dS}{dx} = (3x^2) + 2(8-x)(-1)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dS}{dx} = 3x^2 - 16 + 2x$$

$= 3x^2 + 2x - 16$ ------ (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 3x^2 + 2x - 16 = 0$$

Hence $3x^2 + 2x - 16 = 0$

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-16)}}{2 \times 3}$$

$$= \frac{-2 \pm \sqrt{4 + 192}}{6}$$

$$= \frac{-2 \pm \sqrt{196}}{6}$$

$$x = \frac{-2 \pm 14}{6}$$

$x = \frac{-2+14}{6}$ (or) $x = \frac{-2-14}{6}$

$x = \frac{12}{6}$ (or) $x = \frac{-16}{6}$

x = 2 (or) x = -2.67

Now considering the critical values of x = 2,-2.67

Now, for checking if the value of P is maximum or minimum at x=2,-2.67, we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 2,-2.67.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[3x^2 + 2x - 16]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[3x^2] + \frac{d}{dx}[2x] - \frac{d}{dx}[16]$$

= 3 (2x) + 2 (1) - 0

[Since $\frac{d}{dx}$ $(\text{constant}) = 0$ and $\frac{d}{dx}$ $(x^n) = nx^{n-1}$]

= 6x + 2

$$\frac{d^2S}{dx^2} = 6x + 2$$

Now when x = -2.67,

$$\left[\frac{d^2S}{dx^2}\right]_{x=-2.67} = 6\,(-2.67) + 2$$

= - 16.02 + 2 = -14.02

At x = -2.67 $\frac{d^2S}{dx^2} = -14.02 < 0$ hence, the function S will be maximum at this point.

Now consider x = 2,

$$\left[\frac{d^2S}{dx^2}\right]_{x=2} = 6\,(2) + 2$$

= 12 + 2 = 14

Hence $\left[\frac{d^2S}{dx^2}\right]_{x=2} = 14 > 0$, so at x = 2, the function S is minimum

As second differential is positive, hence at the critical point x = 2 will be the maximum point of the function S.

Therefore, the function S is maximum at x = 2.

From Equation (1), if x= 2

y = 8 – 2 = 6

Therefore, x = 2 and y = 6 are the two positive numbers whose sum is 8 and the sum of the square of one number and cube of another number is maximum.

Question 5.
Divide a into two parts such that the product of the pth power of one part and the qth power of the second part may be maximum.

Answer:
Given,

• the number 'a' is divided into two numbers.

• the product of the pth power of one number and qth power of another number is maximum.

Let us consider,

• x and y are the two numbers

• Sum of the numbers : $x + y = a$

• Product of square of the one number and cube of anther number : $P = x^p y^q$

Now as,

$x + y = a$

$y = (a-x)$ ------ (1)

Consider,

$P = x^p y^q$

By substituting (1), we have

$P = x^p \times (a-x)^q$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then $f'(c) = 0$.

Differentiating the equation (2) with x

$$\frac{dP}{dx} = \frac{d}{dx}[x^p \times (a-x)^q]$$

$$\frac{dP}{dx} = (a-x)^q \frac{d}{dx}(x^p) + x^p \frac{d}{dx}[(a-x)^q]$$

$$\frac{dP}{dx} = (a-x)^q\,(px^{p-1}) + x^p\,[q(a-x)^{q-1}(-1)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$$\frac{dP}{dx} = x^{p-1}(a-x)^{q-1}[(a-x)p - xq]$$

$= x^{p-1}(a-x)^{q-1}[ap-xp-xq]$

$= x^{p-1}(a-x)^{q-1}[ap – x\ (p+q)]$ ----- (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dP}{dx} = x^{p-1}(a-x)^{q-1}[ap - x\,(p+q)] = 0$$

Hence $x^{p-1} = 0$ (or) $(a-x)^{q-1}$ (or) $ap– x(p+q)= 0$

x = 0 (or) x = a (or) $x = \frac{ap}{p+q}$

Now considering the critical values of x = 0,a and $x = \frac{ap}{p+q}$

Now, using the First Derivative test,

For f, a continuous function which has a critical point c, then, function has the local maximum at c, if f′(x) changes the sign from positive to negative as x increases through c, i.e. f′(x)>0 at every point close to the left of c and f′(x)<0 at every point close to the right of c.

Now when x = 0,

$$\left[\frac{dP}{dx}\right]_{x=0} = 0$$

So, we reject x = 0

Now when x = a,

$$\left[\frac{dP}{dx}\right]_{x=a} = 0$$

Hence we reject x = a

Now when $x < \frac{ap}{p+q}$,

$$\left[\frac{dP}{dx}\right]_{x<\frac{ap}{p+q}} = \left(\frac{ap}{p+q}\right)^{p-1}\left(a - \frac{ap}{p+q}\right)^{q-1}\left[ap - \frac{ap}{p+q}(p+q)\right] > 0 \text{ ---- } (4)$$

Now when $x > \frac{ap}{p+q}$,

$$\left[\frac{dP}{dx}\right]_{x>\frac{ap}{p+q}} = \left(\frac{ap}{p+q}\right)^{p-1}\left(a - \frac{ap}{p+q}\right)^{q-1}\left[ap - \frac{ap}{p+q}(p+q)\right] < 0 \text{ ---- } (5)$$

By using first derivative test, from (4) and (5), we can conclude that, the function P has local maximum at $x = \frac{ap}{p+q}$

From Equation (1), if $x = \frac{ap}{p+q}$

$$y = a - \frac{ap}{p+q} = \frac{a(p+q) - ap}{p+q} = \frac{ap}{p+q}$$

Therefore, $x = \frac{ap}{p+q}$ and $y = \frac{aq}{p+q}$ are the two positive numbers whose sum together to give the number ‘a’ and whose product of the pth power of one number and qth power of the other number is maximum.

Question 6.

The rate of working of an engine is given by.

$$R = 15\upsilon + \frac{6000}{\upsilon}, \text{ where } 0 < \upsilon < 30$$

and υ is the speed of the engine. Show that R is the least when υ = 20.

Answer:
Given:

Rate of working of an engine R, v is the speed of the engine:

$R = 15v + \frac{6000}{v}$, where 0<v<30

For finding the maximum/ minimum of given function, we can find it by differentiating it with v and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.

Now, differentiating the function R with respect to v.

$$\frac{dR}{dv} = \frac{d}{dv}\left[15v + \frac{6000}{v}\right]$$

$$\frac{dR}{dv} = \frac{d}{dv}[15v] + \frac{d}{dv}\left[\frac{6000}{v}\right]$$

$\frac{dR}{dv} = 15 + \left[\frac{6000}{v^2}\right](-1) = 15 - \frac{6000}{v^2}$ ----- (1)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Equating equation (1) to zero to find the critical value.

$$\frac{dR}{dv} = 15 - \frac{6000}{v^2} = 0$$

$$15 = \frac{6000}{v^2}$$

$$v^2 = \frac{6000}{15} = 400$$

$v^2 = 400$

$v = \pm\sqrt{400}$

v = 20 (or) v = -20

As given in the question 0<v<30, v = 20

Now, for checking if the value of R is maximum or minimum at v=20, we will perform the second differentiation and check the value of $\frac{d^2R}{dv^2}$ at the critical value v = 20.

Differentiating Equation (1) with respect to v again:

$$\frac{d^2R}{dv^2} = \frac{d}{dx}\left[15 - \frac{6000}{v^2}\right]$$

$$= \frac{d}{dx}[15] - \frac{d}{dx}\left[\frac{6000}{v^2}\right]$$

$$= 0 - (-2)\left[\frac{6000}{v^3}\right]$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$= 2\left[\frac{6000}{v^3}\right]$$

$\frac{d^2R}{dv^2} = \left[\frac{12000}{v^3}\right]$ ----- (2)

Now find the value of $\left(\frac{d^2R}{dv^2}\right)_{v=20}$

$$\left(\frac{d^2R}{dv^2}\right)_{v=20} = \left[\frac{12000}{(20)^3}\right] = \frac{12000}{20 \times 20 \times 20} = \frac{3}{2} > 0$$

So, at critical point v = 20. The function R is at its minimum.

Hence, the function R is at its minimum at v = 20.

Question 7.

Find the dimensions of the rectangle of area 96 cm^2 whose perimeter is the least. Also, find the perimeter of the rectangle.

Answer:

Given,

- Area of the rectangle is 93 cm^2.

- The perimeter of the rectangle is also fixed.

Let us consider,

- x and y be the lengths of the base and height of the rectangle.

- Area of the rectangle = A = x × y = 96 cm^2

- Perimeter of the rectangle = P = 2 (x + y)

As,

x × y = 96

$y = \frac{96}{x}$ ------ (1)

Consider the perimeter function,

P = 2 (x + y)

Now substituting (1) in P,

$P = 2\left(x + \frac{96}{x}\right)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[2\left(x + \frac{96}{x}\right)\right]$$

$$\frac{dP}{dx} = \frac{d}{dx}(2x) + 2\frac{d}{dx}\left(\frac{96}{x}\right)$$

$$\frac{dP}{dx} = 2\,(1) + 2\left(\frac{96}{x^2}\right)(-1)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{dP}{dx} = 2 - \left(\frac{192}{x^2}\right)$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 2 - \left(\frac{192}{x^2}\right) = 0$$

$$2 = \left(\frac{192}{x^2}\right)$$

$$x^2 = \left(\frac{192}{2}\right) = 96$$

$$x = \sqrt{96}$$

$$x = \pm 4\sqrt{6}$$

As the length and breadth of a rectangle cannot be negative, hence $x = 4\sqrt{6}$

Now to check if this critical point will determine the least perimeter, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2 - \left(\frac{192}{x^2}\right)\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(2) - \frac{d}{dx}\left(\frac{192}{x^2}\right)$$

$$\frac{d^2P}{dx^2} = 0 - (-2)\left(\frac{192}{x^3}\right)$$

[Since $\frac{d}{dx}\,(\text{constant}) = 0$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

$$\frac{d^2P}{dx^2} = \left(\frac{2 \times 192}{x^3}\right) \text{------ (4)}$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=4\sqrt{6}}$

$$\frac{d^2P}{dx^2} = \left(\frac{2 \times 192}{(4\sqrt{6})^3}\right)$$

$$= \left(\frac{2 \times 192}{4\sqrt{6} \times 4\sqrt{6} \times 4\sqrt{6}}\right)$$

$$= \left(\frac{2 \times 192}{4\sqrt{6} \times 4\sqrt{6} \times 4\sqrt{6}}\right) = \frac{1}{\sqrt{6}}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=4\sqrt{6}} = \frac{1}{\sqrt{6}} > 0$, so the function P is minimum at $x = 4\sqrt{6}$.

Now substituting $x = 4\sqrt{6}$ in equation (1):

$$y = \frac{96}{4\sqrt{6}}$$

$y = \frac{96\sqrt{6}}{4 \times 6}$

[By rationalizing he numerator and denominator with $\sqrt{6}$]

$\therefore\ y = 4\sqrt{6}$

Hence, area of the rectangle with sides of a rectangle with $x = 4\sqrt{6}$ and $y = 4\sqrt{6}$ is 96cm^2 and has the least perimeter.

Now the perimeter of the rectangle is

$P = 2(4\sqrt{6} + 4\sqrt{6}) = 2(8\sqrt{6}) = 16\sqrt{6}$ cms

The least perimeter is $16\sqrt{6}$ cms.

Question 8.
Prove that the largest rectangle with a given perimeter is a square.

Answer:
Given,

• Rectangle with given perimeter.

Let us consider,

• 'p' as the fixed perimeter of the rectangle.

• 'x' and 'y' be the sides of the given rectangle.

• Area of the rectangle, A = x × y.

Now as consider the perimeter of the rectangle,

p = 2(x +y)

p = 2x + 2y

$y = \frac{p-2x}{2}$ ----- (1)

Consider the area of the rectangle,

A = x × y

Substituting (1) in the area of the rectangle,

$$A = x \times \left(\frac{p - 2x}{2}\right)$$

$$A = \frac{1}{2} \times (px - 2x^2) \text{ ----- } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{2}(px - 2x^2)\right]$$

$$\frac{dA}{dx} = \frac{1}{2}\frac{d}{dx}(px) - \frac{1}{2}\frac{d}{dx}(2x^2)$$

$$\frac{dA}{dx} = \frac{1}{2}(p) - \frac{2}{2}(2x)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dA}{dx} = \frac{p}{2} - (2x) \text{ ----- } (3)$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dA}{dx} = \frac{p}{2} - (2x) = 0$$

$$2x = \frac{p}{2}$$

$$x = \frac{p}{4}$$

Now to check if this critical point will determine the largest rectangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{p}{2} - (2x)\right]$$

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{p}{2}\right) - \frac{d}{dx}(2x)$$

$$\frac{d^2A}{dx^2} = 0 - 2 = -2$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dx^2} = -2$ ------ (4)

Now, consider the value of $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{p}{4}}$

$$\frac{d^2A}{dx^2} = -2 < 0$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{p}{4}} = -2 < 0$, so the function P is maximum at $x = \frac{p}{4}$.

Now substituting $x = \frac{p}{4}$ in equation (1):

$$y = \frac{p - 2\left(\frac{p}{4}\right)}{2}$$

$$y = \frac{p - \frac{p}{2}}{2} = \frac{p}{4}$$

$$\therefore\ y = \frac{p}{4}$$

As $= y = \frac{p}{4}$ the sides of the taken rectangle are equal, we can clearly say that a largest rectangle which has a given perimeter is a square.

Question 9.
Given the perimeter of a rectangle, show that its diagonal is minimum when it is a square.

Answer:
Given,

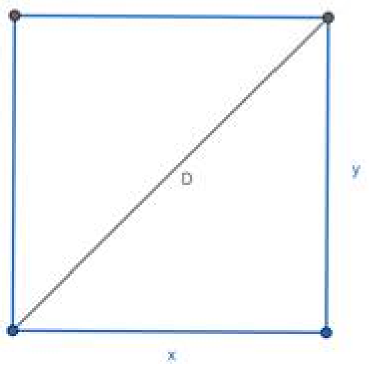
• Rectangle with given perimeter.

Let us consider,

• 'p' as the fixed perimeter of the rectangle.

• 'x' and 'y' be the sides of the given rectangle.

• Diagonal of the rectangle, $D = \sqrt{x^2 + y^2}$. (using the hypotenuse formula)



Now as consider the perimeter of the rectangle,

p = 2(x +y)

p = 2x + 2y

$y = \frac{p-2x}{2}$ ----- (1)

Consider the diagonal of the rectangle,

$D = \sqrt{x^2 + y^2}$

Substituting (1) in the diagonal of the rectangle,

$$D = \sqrt{x^2 + \left(\frac{p - 2x}{2}\right)^2}$$

[squaring both sides]

$$Z = D^2 = x^2 + \left(\frac{p-2x}{2}\right)^2 \text{ ----- } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[x^2 + \left(\frac{p - 2x}{2}\right)^2\right]$$

$$\frac{dZ}{dx} = \frac{d}{dx}(x^2) + \frac{1}{4}\frac{d}{dx}[(p - 2x)^2]$$

$$\frac{dZ}{dx} = 2x + \frac{1}{4}[2(p - 2x)(-2)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

= 2x - p + 2x

$$\frac{dZ}{dx} = 4x - p \text{ ------ } (3)$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 4x - p = 0$$

4x −p = 0

4x = p

$$x = \frac{p}{4}$$

Now to check if this critical point will determine the minimum diagonal, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[4x - p]$$

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}(4x) - \frac{d}{dx}(p)$$

= 4 + 0

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2Z}{dx^2} = 4$ ------ (4)

Now, consider the value of $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{p}{4}}$

$$\frac{d^2Z}{dx^2} = 4 > 0$$

As $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{p}{4}} = 4 > 0$, so the function Z is minimum at $x = \frac{p}{4}$.

Now substituting $x = \frac{p}{4}$ in equation (1):

$$y = \frac{p - 2\left(\frac{p}{4}\right)}{2}$$

$$y = \frac{p - \frac{p}{2}}{2} = \frac{p}{4}$$

$\therefore\ y = \frac{p}{4}$

As $x = y = \frac{p}{4}$ the sides of the taken rectangle are equal, we can clearly say that a rectangle with minimum diagonal which has a given perimeter is a square.

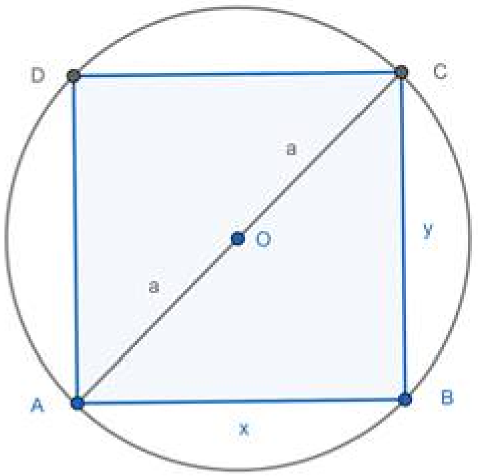
Question 10.
Show that a rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}\ a$.

Answer:
Given,

• Rectangle is of maximum perimeter.

• The rectangle is inscribed inside a circle.

• The radius of the circle is 'a'.



Let us consider,

• 'x' and 'y' be the length and breadth of the given rectangle.

• Diagonal $AC^2 = AB^2 + BC^2$ is given by $4a^2 = x^2+y^2$ (as AC = 2a)

• Perimeter of the rectangle, P = 2(x+y)

Consider the diagonal,

$4a^2 = x^2 + y^2$

$y^2 = 4a^2 - x^2$

$y = \sqrt{4a^2 - x^2}$ ---- (1)

Now, perimeter of the rectangle, P

P = 2x + 2y

Substituting (1) in the perimeter of the rectangle.

$P = 2x + 2\sqrt{4a^2 - x^2}$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[2x + 2\sqrt{4a^2 - x^2}\right]$$

$$\frac{dP}{dx} = \frac{d}{dx}(2x) + 2\frac{d}{dx}[\sqrt{4a^2 - x^2}]$$

$$\frac{dP}{dx} = 2 + 2[\frac{1}{2}(4a^2 - x^2)^{-\frac{1}{2}}(-2x)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4a^2 - x^2}}$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4a^2 - x^2}} = 0$$

$$2 = \frac{2x}{\sqrt{4a^2 - x^2}}$$

$$\sqrt{4a^2 - x^2} = x$$

[squaring on both sides]

$4a^2 - x^2 = x^2$

$2x^2 = 4a^2$

$x^2 = 2a^2$

$$x = \pm a\sqrt{2}$$

$$x = a\sqrt{2}$$

[as x cannot be negative]

<u>Now to check if this critical point will determine the maximum diagonal, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2 - \frac{2x}{\sqrt{4a^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(2) - \frac{d}{dx}\left(\frac{2x}{\sqrt{4a^2 - x^2}}\right)$$

$$\frac{d^2P}{dx^2} = 0 - \left[\frac{\sqrt{4a^2 - x^2}\,\frac{d}{dx}(2x) - (2x)\frac{d}{dx}\left(\sqrt{4a^2 - x^2}\right)}{\left(\sqrt{4a^2 - x^2}\right)^2}\right]$$

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{4a^2 - x^2}\,(2) - (2x)\frac{1}{2}(4a^2 - x^2)^{-\frac{1}{2}}(-2x)}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{4a^2 - x^2}\,(2) + (2x^2)(4a^2 - x^2)^{-\frac{1}{2}}}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{2\sqrt{4a^2 - x^2} + \frac{2x^2}{\sqrt{4a^2 - x^2}}}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{2\,(4a^2 - x^2) + 2x^2}{(4a^2 - x^2)^{\frac{3}{2}}}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{8a^2}{(4a^2 - x^2)^{\frac{3}{2}}}\right] \text{----- (4)}$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}}$

$$\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\left[\frac{8a^2}{(4a^2 - (a\sqrt{2})^2)^{\frac{3}{2}}}\right]$$

$$\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\left[\frac{8a^2}{(4a^2 - 2a^2)^{\frac{3}{2}}}\right] = -\frac{8a^2}{(2a^2)^{\frac{3}{2}}} = -\frac{8a^2}{2\sqrt{2}\,a^3} = -\frac{2\sqrt{2}}{a}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\frac{2\sqrt{2}}{a} < 0$, so the function P is maximum at $x = a\sqrt{2}$.

Now substituting $x = a\sqrt{2}$ in equation (1):

$$y = \sqrt{4a^2 - (a\sqrt{2})^2}$$

$$y = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2}$$

$\therefore\ y = a\sqrt{2}$

As $x = y = a\sqrt{2}$ the sides of the taken rectangle are equal, we can clearly say that a rectangle with maximum perimeter which is inscribed inside a circle of radius ‘a’ is a square.

Question 11.
The sum of the perimeters of a square and a circle is given. Show that the sum of their areas is least when the side of the square is equal to the diameter of the circle.

Answer:
Given,

• Sum of perimeter of square and circle.

Let us consider,

• ‘x’ be the side of the square.

• ‘r’ be the radius of the circle.

• Let ‘p’ be the sum of perimeters of square and circle.

$p = 4x + 2\pi r$

Consider the sum of the perimeters of square and circle.

$p = 4x + 2\pi r$

$4x = p - 2\pi r$

$x = \frac{p-2\pi r}{4}$ ---- (1)

Sum of the area of the circle and square is

$A = x^2 + \pi r^2$

Substituting (1) in the sum of the areas,

$$A = \left(\frac{p - 2\pi r}{4}\right)^2 + \pi r^2$$

$$A = \frac{1}{16}[p^2 + 4\pi^2 r^2 - 4\pi pr] + \pi r^2 \text{ ------ (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.</u>

Differentiating the equation (2) with respect to r:

$$\frac{dA}{dr} = \frac{d}{dr}\left[\frac{1}{16}[p^2 + 4\pi^2 r^2 - 4\pi pr] + \pi r^2\right]$$

$$\frac{dA}{dr} = \frac{1}{16}\frac{d}{dr}(p^2 + 4\pi^2 r^2 - 4\pi pr) + \frac{d}{dr}[\pi r^2]$$

$$\frac{dA}{dr} = \frac{1}{16}(0 + 8\pi^2 r - 4\pi p) + 2\pi r$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(\text{constant}) = 0$]

$$\frac{dA}{dr} = \frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r \text{ ------ (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dA}{dr} = \frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r = 0$$

$$\left(\frac{\pi^2}{2} + 2\pi\right) r - \frac{\pi p}{4} = 0$$

$$r = \frac{\frac{\pi p}{4}}{\frac{\pi^2}{2} + 2\pi} = \frac{2\pi p}{4(\pi^2 + 4\pi)} = \frac{\pi p}{2(\pi^2 + 4\pi)}$$

$$r = \frac{\pi p}{2\pi(\pi + 4)} = \frac{p}{2(\pi + 4)}$$

$$r = \frac{p}{2(\pi + 4)}$$

<u>Now to check if this critical point will determine the least of the sum of the areas of square and circle, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with r:

$$\frac{d^2A}{dr^2} = \frac{d}{dx}\left[\frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r\right]$$

$$\frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{\pi^2 r}{2}\right) - \frac{d}{dr}\left(\frac{\pi p}{4}\right) + \frac{d}{dr}(2\pi r)$$

$$\frac{d^2A}{dr^2} = \frac{\pi^2}{2} - 0 + 2\pi$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{d^2A}{dr^2} = \frac{\pi^2}{2} + 2\pi \text{ ----- } (4)$$

Now, consider the value of $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}}$

$$\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}} = \frac{\pi^2}{2} + 2\pi$$

As $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}} = \frac{\pi^2}{2} + 2\pi > 0$, so the function A is minimum at $r = \frac{p}{2(\pi+4)}$.

Now substituting $r = \frac{p}{2(\pi+4)}$ in equation (1):

$$x = \frac{p - 2\pi\left(\frac{p}{2(\pi + 4)}\right)}{4}$$

$$x = \frac{p(\pi + 4) - \pi p}{4 \times (\pi + 4)} = \frac{\pi p + 4p - \pi p}{4\pi + 16} = \frac{4p}{4(\pi + 4)}$$

$$x = \frac{p}{\pi + 4}$$

As the side of the square,

$$x = \frac{p}{\pi + 4}$$

$$x = 2\left[\frac{p}{2(\pi + 4)}\right] = 2r$$

[as $r = \frac{p}{2(\pi+4)}$]

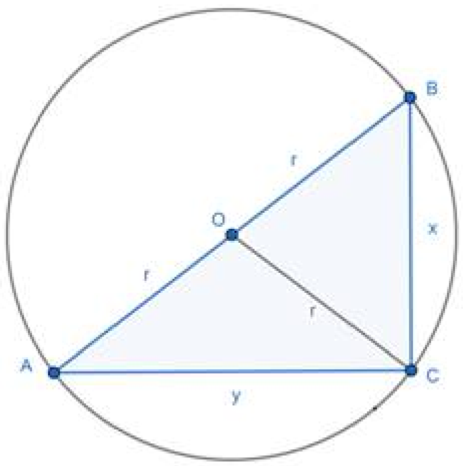
Therefore, side of the square, x = 2r = diameter of the circle.

Question 12.
Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.

Answer:
Given,

• A right angle triangle is inscribed inside the circle.

• The radius of the circle is given.



Let us consider,

• 'r' is the radius of the circle.

• ‘x’ and ‘y’ be the base and height of the right angle triangle.

• The hypotenuse of the ΔABC = $AB^2 = AC^2 + BC^2$

AB = 2r, AC = y and BC = x

Hence,

$4r^2 = x^2 + y^2$

$y^2 = 4r^2 - x^2$

$y = \sqrt{4r^2 - x^2}$ --- (1)

Now, Area of the ΔABC is

$$A = \frac{1}{2} \times \text{base} \times \text{height}$$

$$A = \frac{1}{2} \times x \times y$$

Now substituting (1) in the area of the triangle,

$$A = \frac{1}{2} x \left(\sqrt{4r^2 - x^2}\right)$$

[Squaring both sides]

$Z = A^2 = \frac{1}{4} x^2 (4r^2 - x^2)$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[\frac{1}{4} x^2 (4r^2 - x^2)\right]$$

$$\frac{dZ}{dx} = \frac{1}{4}\left[(4r^2 - x^2)\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(4r^2 - x^2)\right]$$

$$\frac{dZ}{dx} = \frac{1}{4}[(4r^2 - x^2) \times (2x) + x^2(0 - 2x)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u.v) = v\frac{du}{dx} + u\frac{dv}{dx}$]

$$\frac{dZ}{dx} = \frac{1}{4}[8r^2x - 2x^3 - 2x^3]$$

$$\frac{dZ}{dx} = \frac{1}{4}[8r^2x - 4x^3] = \frac{4x}{4}[2r^2 - x^2]$$

$\frac{dZ}{dx} = 2r^2x - x^3$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 2r^2x - x^3 = 0$$

$$2r^2x = x^3$$

$$x^2 = 2r^2$$

$$x = \pm\sqrt{2r^2}$$

$$x = r\sqrt{2}$$

[as the base of the triangle cannot be negative.]

<u>Now to check if this critical point will determine the maximum area of the triangle, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[2r^2x - x^3]$$

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}(2r^2x) - \frac{d}{dx}(x^3)$$

$\frac{d^2Z}{dx^2} = 2r^2 - 3x^2$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now, consider the value of $\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}}$

$$\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = 2r^2 - 3(r\sqrt{2})^2 = 2\,r^2 - 6r^2 = -4r^2$$

As $\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = -4r^2 < 0$, so the function A is maximum at $x = r\sqrt{2}$.

Now substituting $x = r\sqrt{2}$ in equation (1):

$$y = \sqrt{4r^2 - (r\sqrt{2}\,)^2}$$

$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2r^2} = r\sqrt{2}$$

As $x = y = r\sqrt{2}$, the base and height of the triangle are equal, which means that two sides of a right angled triangle are equal,

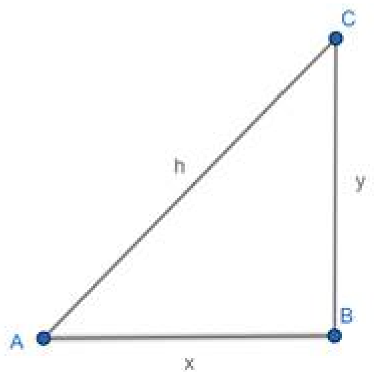
Hence the given triangle, which is inscribed in a circle, is an isosceles triangle with sides AC and BC equal.

Question 13.
Prove that the perimeter of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.

Answer:
Given,

• A right angle triangle.

• Hypotenuse of the given triangle is given.



Let us consider,

• 'h' is the hypotenuse of the given triangle.

• 'x' and 'y' be the base and height of the right angle triangle.

• The hypotenuse of the ΔABC = $AC^2 = AB^2 + BC^2$

AC = h, AB = x and BC = y

Hence,

$h^2 = x^2 + y^2$

$y^2 = h^2 - x^2$

$y = \sqrt{h^2 - x^2}$ --- (1)

Now, perimeter of the ΔABC is

P = h + x + y

Now substituting (1) in the area of the triangle,

$P = h + x + \sqrt{h^2 - x^2}$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[h + x + \sqrt{h^2 - x^2}\right]$$

$$\frac{dP}{dx} = \left[\frac{d}{dx}(h) + \frac{d}{dx}(x) + \frac{d}{dx}\left(\sqrt{h^2 - x^2}\right)\right]$$

$$\frac{dP}{dx} = 0 + 1 + \frac{1}{2}\left(\frac{-2x}{\sqrt{h^2 - x^2}}\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dP}{dx} = 1 - \frac{x}{\sqrt{h^2 - x^2}}$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 1 - \frac{x}{\sqrt{h^2 - x^2}} = 0$$

$$\frac{x}{\sqrt{h^2 - x^2}} = 1$$

$$x = \sqrt{h^2 - x^2}$$

[squaring on both sides]

$$x^2 = h^2 - x^2$$

$$x^2 = \frac{h^2}{2}$$

$$x = \pm\sqrt{\frac{h^2}{2}}$$

$$x = \frac{h}{\sqrt{2}}$$

[as the base of the triangle cannot be negative.]

Now to check if this critical point will determine the maximum perimeter of the triangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[1 - \frac{x}{\sqrt{h^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{x}{\sqrt{h^2 - x^2}}\right)$$

$$\frac{d^2P}{dx^2} = 0 - \left[\frac{\sqrt{h^2 - x^2}\,\frac{d}{dx}(x) - x\,\frac{d}{dx}\left(\sqrt{h^2 - x^2}\right)}{\left(\sqrt{h^2 - x^2}\right)^2}\right]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ if u and v are two functions of x, then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{h^2 - x^2}\,(1) - x\left(\frac{-2x}{2\sqrt{h^2 - x^2}}\right)}{h^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{(\sqrt{h^2 - x^2})^2 + x^2}{h^2 - x^2\,\sqrt{h^2 - x^2}}\right] = -\left[\frac{h^2}{(h^2 - x^2)\,\sqrt{h^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{h^2}{(h^2 - x^2\,)^{\frac{3}{2}}}\right]$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}}$

$$\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\left[\frac{h^2}{(h^2 - \left(\frac{h}{\sqrt{2}}\right)^2\,)^{\frac{3}{2}}}\right] = -\left[\frac{h^2}{(\frac{h^2}{2}\,)^{\frac{3}{2}}}\right] = -\left[\frac{h^2}{(\frac{h^2}{2}\,)^{\frac{3}{2}}}\right] = -\frac{2^{\frac{3}{2}}}{h}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\frac{2^{\frac{3}{2}}}{h} < 0$, so the function A is maximum at $x = \frac{h}{\sqrt{2}}$.

Now substituting $x = \frac{h}{\sqrt{2}}$ in equation (1):

$$y = \sqrt{h^2 - \left(\frac{h}{\sqrt{2}}\right)^2}$$

$$y = \sqrt{\frac{h^2}{2}} = \frac{h}{\sqrt{2}}$$

As $x = y = \frac{h}{\sqrt{2}}$, the base and height of the triangle are equal, which means that two sides of a right angled triangle are equal,

Hence the given triangle is an isosceles triangle with sides AB and BC equal.

Question 14.
The perimeter of a triangle is 8 cm. If one of the sides of the triangle be 3 cm, what will be the other two sides for maximum area of the triangle?

Answer:
Given,

- Perimeter of a triangle is 8 cm.

- One of the sides of the triangle is 3 cm.

- The area of the triangle is maximum.

Let us consider,

- 'x' and 'y' be the other two sides of the triangle.

Now, perimeter of the ΔABC is

8 = 3 + x + y

y = 8-3-x = 5-x

y = 5-x --- (1)

Consider the Heron's area of the triangle,

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $s = \frac{a+b+c}{2}$

As perimeter = a + b+ c = 8

$$s = \frac{8}{2} = 4$$

Now Area of the triangle is given by

$$A = \sqrt{8\,(8-3)(8-x)(8-y)}$$

Now substituting (1) in the area of the triangle,

$$A = \sqrt{4(4-3)(4-x)(4-(5-x))}$$

$$A = \sqrt{4(4-x)(x-1)}$$

$$A = \sqrt{4(\,4x-4-\,x^2+x)} = \sqrt{4(\,5x-\,x^2-4)}$$

$$A = \sqrt{4(\,5x-\,x^2-4)}$$

[squaring on both sides]

$Z = A^2 = 4(5x - x^2 - 4)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[4(5x - x^2 - 4)\right]$$

$$\frac{dZ}{dx} = 4\frac{d}{dx}(5x) - 4\frac{d}{dx}(x^2) - 4\frac{d}{dx}(4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dZ}{dx} = 4(5) - 4(2x) - 0$$

$\frac{dZ}{dx} = 20 - 8x$------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 20 - 8x = 0$$

20 – 8x = 0

8x = 20

$$x = \frac{5}{2}$$

<u>Now to check if this critical point will determine the maximum area of the triangle, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[20 - 8x]$$

$\frac{d^2Z}{dx^2} = -8$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

As $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{5}{2}} = -8 < 0$, so the function A is maximum at $x = \frac{5}{2}$.

Now substituting $x = \frac{5}{2}$ in equation (1):

y = 5 – 2.5

y = 2.5

As x = y = 2.5, two sides of the triangle are equal,

Hence the given triangle is an isosceles triangle with two sides equal to 2.5 cm and the third side equal to 3cm.

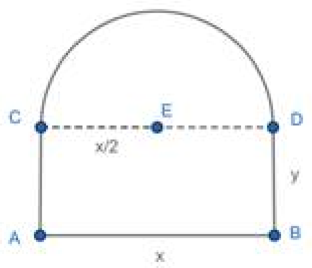
Question 15.
A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 metres. Find the dimensions of the windows to admit maximum light through it.

Answer:
Given,

• Window is in the form of a rectangle which has a semicircle mounted on it.

• Total Perimeter of the window is 10 metres.

• The total area of the window is maximum.



Let us consider,

• The breadth and height of the rectangle be ‘x’ and ‘y’.

• The radius of the semicircle will be half of the base of the rectangle.

Given Perimeter of the window is 10 meters:

$$10 = (x + 2y) + \frac{1}{2}\left[2\pi\left(\frac{x}{2}\right)\right]$$

[as the perimeter of the window will be equal to one side (x) less to the perimeter of rectangle and the perimeter of the semicircle.]

$$10 = (x + 2y) + \left(\frac{\pi x}{2}\right)$$

From here,

$$2y = 10 - x - \left(\frac{\pi x}{2}\right) = \frac{20 - 2x - \pi x}{2}$$

$$y = \frac{20-2x-\pi x}{4} \text{ ----- } (1)$$

Now consider the area of the window,

Area of the window = area of the semicircle + area of the rectangle

$$A = \frac{1}{2}\left[\pi\left(\frac{x}{2}\right)^2\right] + xy$$

Substituting (1) in the area equation:

$$A = \frac{1}{2}\left[\pi\left(\frac{x}{2}\right)^2\right] + x\left(\frac{20 - 2x - \pi x}{4}\right)$$

$$A = \frac{1}{8}\left[\pi x^2\right] + \left(\frac{20x - 2x^2 - \pi x^2}{4}\right)$$

$$A = \frac{\pi x^2 - 2\pi x^2 + 40x - 4x^2}{8}$$

$$A = \frac{1}{8}\left[x^2(\pi - 2\pi - 4) + 40x\right] \text{ ------ } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f′(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{8}\left[x^2(\pi - 2\pi - 4) + 40x\right]\right]$$

$$\frac{dA}{dx} = \frac{1}{8}\frac{d}{dx}\left(x^2(\pi - 2\pi - 4)\right) + \frac{1}{8}\frac{d}{dx}(40x)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dA}{dx} = \frac{1}{8}\left[2x(-\pi - 4)\right] + \frac{1}{8}(40)$$

$\frac{dA}{dx} = \frac{1}{4}\left[x(-\pi - 4)\right] + 5$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dA}{dx} = \frac{1}{4}\left[x(-\pi - 4)\right] + 5 = 0$$

$$\frac{1}{4}\left[x(-\pi - 4)\right] + 5 = 0$$

$$\frac{1}{4}\left[x(4 + \pi)\right] = 5$$

x (4 +π) = 20

$$x = \frac{20}{(4 + \pi)}$$

<u>Now to check if this critical point will determine the maximum area of the window, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{1}{4}\left[x(-\pi - 4)\right] + 5\right]$$

$$\frac{d^2A}{dx^2} = \frac{d}{dx}[x(-\pi - 4)] + \frac{d}{dx}(5)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dx^2} = (-\pi - 4)(1) + 0 = -(\pi + 4)$ ------ (4)

As $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{20}{(4+\pi)}} = -(\pi + 4) < 0$, so the function A is maximum at $x = \frac{20}{(4+\pi)}$.

Now substituting $x = \frac{20}{(4+\pi)}$ in equation (1):

$$y = \frac{20 - \left(\frac{20}{(4+\pi)}\right)(\pi + 2)}{4}$$

$$y = \frac{20(4+\pi) - (20)(\pi + 2)}{4(4+\pi)} = \frac{20[4 + \pi - \pi - 2]}{4(4+\pi)} = \frac{20 \times 2}{4(4+\pi)}$$

$$y = \frac{5 \times 2}{(4+\pi)} = \frac{10}{(4+\pi)}$$

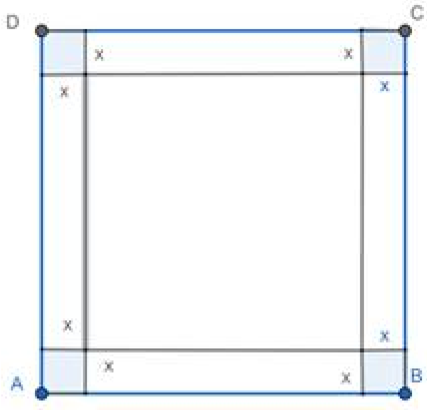
Hence the given window with maximum area has breadth, $x = \frac{20}{(4+\pi)}$ and height, $y = \frac{10}{(4+\pi)}$.

Question 16.
A square piece of tin of side 12 cm is to be made into a box without a lid by cutting a square from each corner and folding up the flaps to form the sides. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume.

Answer:
Given,

• Side of the square piece is 12 cms.

• the volume of the formed box is maximum.



Let us consider,

- 'x' be the length and breadth of the piece cut from each vertex of the piece.
- Side of the box now will be (12-2x)
- The height of the new formed box will also be 'x'.

Let the volume of the newly formed box is :

$V = (12-2x)^2 \times (x)$

$V = (144 + 4x^2 - 48x)\ x$

$V = 4x^3 - 48x^2 + 144x$ ------ (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (1) with respect to x:

$$\frac{dV}{dx} = \frac{d}{dx}\left[4x^3 - 48x^2 + 144x\right]$$

$$\frac{dV}{dx} = 12x^2 - 96x + 144 \text{ -------- } (2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dV}{dx} = 12x^2 - 96x + 144 = 0$$

$x^2 - 8x + 12 = 0$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(12)}}{2(1)} = \frac{8 \pm \sqrt{64 - 48}}{2} = \frac{8 \pm \sqrt{16}}{2}$$

$$x = = \frac{8 \pm 4}{2}$$

x = 6 or x =2

x= 2

[as x = 6 is not a possibility, because 12-2x = 12-12= 0]

<u>Now to check if this critical point will determine the maximum area of the box, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{dx^2} = \frac{d}{dx}[12x^2 - 96x + 144]$$

$$\frac{d^2V}{dx^2} = 24x - 96 \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dx^2}\right)_{x=2} = 24(2) - 96 = 48 - 96 = -48$$

As $\left(\frac{d^2V}{dx^2}\right)_{x=2} = -48 < 0$, so the function A is maximum at x = 2

Now substituting x = 2 in 12 – 2x, the side of the considered box:

Side = 12-2x = 12 - 2(2) = 12-4= 8cms

Therefore side of wanted box is 8cms and height of the box is 2cms.

Now, the volume of the box is

$V = (8)^2 \times 2 = 64 \times 2 = 128cm^3$

Hence maximum volume of the box formed by cutting the given 12cms sheet is $128cm^3$ with 8cms side and 2cms height.

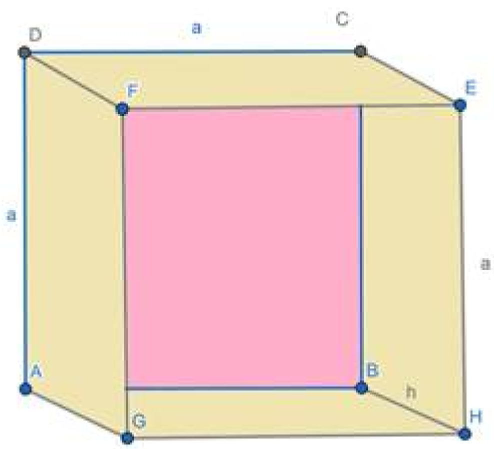
Question 17.
An open box with a square base is to be made out of a given cardboard of area c^2 (square) units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ (cubic) units.

Answer:
Given,

• The open box has a square base

• The area of the box is c^2 square units.

• The volume of the box is maximum.



Let us consider,

• The side of the square base of the box be ‘a’ units. (pink coloured in the figure)

• The breadth of the 4 sides of the box will also be ‘a’units (skin coloured part).

• The depth of the box or the length of the sides be ‘h’ units (skin coloured part).

Now, the area of the box =

(area of the base) + 4 (area of each side of the box)

So as area of the box is given c^2,

$c^2 = a^2 + 4ah$

$h = \frac{c^2 - a^2}{4a}$ ---- (1)

Let the volume of the newly formed box is :

$V = (a)^2 \times (h)$

[substituting (1) in the volume formula]

$$V = a^2 \times \left(\frac{c^2 - a^2}{4a}\right)$$

$$V = \left(\frac{ac^2 - a^3}{4}\right) \text{------ (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with a and then equating it to zero. <u>This is because if the function f(a) has a maximum/minimum at a point c then f′(c) = 0.</u>

Differentiating the equation (2) with respect to a:

$$\frac{dV}{da} = \frac{d}{da}\left[\left(\frac{ac^2 - a^3}{4}\right)\right]$$

$$\frac{dV}{da} = \frac{c^2}{4} - \frac{3a^2}{4} \text{------- (3)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{da} = \frac{c^2}{4} - \frac{3a^2}{4} = 0$$

$c^2 - 3a^2 = 0$

$$a^2 = \frac{c^2}{3}$$

$$a = \pm\sqrt{\frac{c^2}{3}}$$

$$a = \frac{c}{\sqrt{3}}$$

[as 'a' cannot be negative]

<u>Now to check if this critical point will determine the maximum Volume of the box, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{da^2} = \frac{d}{dx}\left[\frac{c^2}{4} - \frac{3a^2}{4}\right]$$

$$\frac{d^2V}{da^2} = 0 - \frac{3\times2\times a}{4} = -\frac{3a}{2} \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{da^2}\right)_{a=\frac{c}{\sqrt{3}}} = -\frac{3\left(\frac{c}{\sqrt{3}}\right)}{2} = -\frac{c\sqrt{3}}{2}$$

As $\left(\frac{d^2V}{da^2}\right)_{a=\frac{c}{\sqrt{3}}} = -48 - \frac{c\sqrt{3}}{2} < 0$, so the function V is maximum at $a = \frac{c}{\sqrt{3}}$

Now substituting a in equation (1)

$$h = \frac{c^2 - \left(\frac{c}{\sqrt{3}}\right)^2}{4\left(\frac{c}{\sqrt{3}}\right)} = \frac{\frac{2c^2}{3}}{\frac{4c}{\sqrt{3}}} = \frac{c\sqrt{3}}{6} = \frac{c}{2\sqrt{3}}$$

$$\therefore\ h = \frac{c}{2\sqrt{3}}$$

Therefore side of wanted box has a base side, $a = \frac{c}{\sqrt{3}}$ is and height of the box, $h = \frac{c}{2\sqrt{3}}$.

Now, the volume of the box is

$$V = \left(\frac{c}{\sqrt{3}}\right)^2 \times \left(\frac{c}{2\sqrt{3}}\right)$$

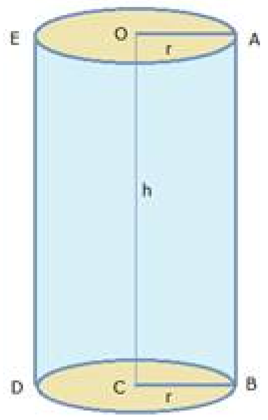
$$V = \frac{c^2}{3} \times \left(\frac{c}{2\sqrt{3}}\right) = \frac{c^3}{6\sqrt{3}}$$

$$\therefore V = \frac{c^3}{6\sqrt{3}}$$

Question 18.
A cylindrical can is to be made to hold 1 litre of oil. Find the dimensions which will minimize the cost of the metal to make the can.

Answer:
Given,

- The can is cylindrical with a circular base
- The volume of the cylinder is 1 litre = 1000 cm^2.
- The surface area of the box is minimum as we need to find the minimum dimensions.



Let us consider,

• The radius base and top of the cylinder be ‘r’ units. (skin coloured in the figure)

• The height of the cylinder be ‘h’units.

• As the Volume of cylinder is given, V = 1000cm^3

The Volume of the cylinder= $\pi r^2 h$

$1000 = \pi r^2 h$

$h = \frac{1000}{\pi r^2}$ ---- (1)

The Surface area cylinder is = area of the circular base + area of the circular top + area of the cylinder

$S = \pi r^2 + \pi r^2 + 2\pi rh$

$S = 2\pi r^2 + 2\pi rh$

[substituting (1) in the volume formula]

$$S = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$S = 2\left[\pi r^2 + \left(\frac{1000}{r}\right)\right] \text{------ (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. <u>This is because if the function f(r) has a maximum/minimum at a point c then f′(c) = 0.</u>

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\left[\pi r^2 + \left(\frac{1000}{r}\right)\right]\right]$$

$$\frac{dS}{dr} = 2\,(2\pi r) + \left(\frac{1000}{r^2}\right)(-1)$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right) \text{------- (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right) = 0$$

$$2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right) = 0$$

$$2\pi r = \frac{1000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

<u>Now to check if this critical point will determine the minimum surface area of the box, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right)\right]$$

$$\frac{d^2S}{dr^2} = 4\pi - \frac{2 \times 1000 \times (-2)}{r^3} = 4\pi + \frac{4000}{r^3} \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{500}{\pi}}} = 4\pi + \frac{4000}{\left(\sqrt[3]{\frac{500}{\pi}}\right)^3} = 4\pi + \frac{4000 \times \pi}{500} = 4\pi + 8\pi = 12\pi$$

As $\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{500}{\pi}}} = 12\pi > 0$, so the function S is minimum at $r = \sqrt[3]{\frac{500}{\pi}}$

Now substituting r in equation (1)

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi^{\frac{1}{3}}(500)^{\frac{2}{3}}}$$

$$\therefore\ h = \frac{1000}{\pi^{\frac{1}{3}}(500)^{\frac{2}{3}}}$$

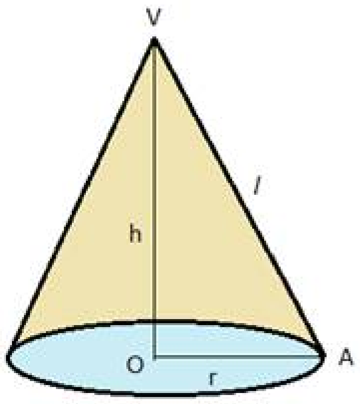
Therefore the radius of base of the cylinder, $r = \sqrt[3]{\frac{500}{\pi}}$ and height of the cylinder, $h = \frac{1000}{\pi^{\frac{1}{3}}(500)^{\frac{2}{3}}}$ where the surface area of the cylinder is minimum.

Question 19.
Show that the right circular cone of the least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

Answer:
Given,

- The volume of the cone.
- The cone is right circular cone.

• The cone has least curved surface.



Let us consider,

• The radius of the circular base be ‘r’ cms.

• The height of the cone be ‘h’ cms.

• The slope of the cone be ‘l’ cms.

Given the Volume of the cone = $\pi r^2 l$

$$V = \frac{\pi r^2 h}{3}$$

$h = \frac{3v}{\pi r^2}$ ----- (1)

The Surface area cylinder is = πrl

S = πrl

$$S = \pi r\left(\sqrt{h^2 + r^2}\right)$$

[substituting (1) in the Surface area formula]

$$S = \pi r\left[\sqrt{\left(\frac{3V}{\pi r^2}\right)^2 + r^2}\right]$$

[squaring on both sides]

$$Z = S^2 = \pi^2 r^2 (\frac{9V^2}{\pi^2 r^4} + r^2)$$

$$Z = \pi^2 \left(\frac{9V^2}{\pi^2 r^2} + r^4 \right) \text{----- (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. <u>This is because if the function Z has a maximum/minimum at a point c then Z′(c) = 0.</u>

Differentiating the equation (2) with respect to r:

$$\frac{dZ}{dr} = \frac{d}{dr}\left[\pi^2 \left(\frac{9V^2}{\pi^2 r^2} + r^4 \right)\right]$$

$$\frac{dZ}{dr} = \pi^2 \left(\frac{9V^2}{\pi^2} \right) \frac{d}{dr} \left(\frac{1}{r^2} \right) + \pi^2 \frac{d}{dr} (r^4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{dZ}{dr} = \left(\frac{-18V^2}{r^3} \right) + \pi^2 (4r^3) \text{------- (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dr} = \left(\frac{-18V^2}{r^3} \right) + \pi^2 (4r^3) = 0$$

$$\pi^2 (4r^3) = \frac{18V^2}{r^3}$$

$$2\pi^2 r^6 = 9V^2 \text{ ---- (4)}$$

<u>Now to check if this critical point will determine the minimum surface area of the cone, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with r:

$$\frac{d^2Z}{dr^2} = \frac{d}{dr}\left[\left(\frac{-18V^2}{r^3} \right) + \pi^2 (4r^3) \right]$$

$$\frac{d^2Z}{dr^2} = \frac{-18V^2\,(-3)}{r^4} + \pi^2\,(4\,\times 3\,r^2)$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

$$\frac{d^2Z}{dr^2} = \frac{54V^2}{r^4} + \pi^2\,(12\,r^2)$$

Now let us find the value of

$$\left(\frac{d^2Z}{dr^2}\right) = \frac{54V^2}{r^4} + \pi^2\,(12\,r^2) > 0$$

As $\left(\frac{d^2Z}{dr^2}\right) > 0$, so the function Z = S^2 is minimum

Now consider, the equation (4),

$$9V^2 = 2\pi^2 r^6$$

Now substitute the volume of the cone formula in the above equation.

$$9\left(\frac{\pi r^2 h}{3}\right)^2 = 2\pi^2 r^6$$

$\pi^2 r^4 h^2 = 2\,\pi^2 r^6$

$2r^2 = h^2$

$$h = r\sqrt{2}$$

Hence, the relation between h and r of the cone is proved when S is the minimum.

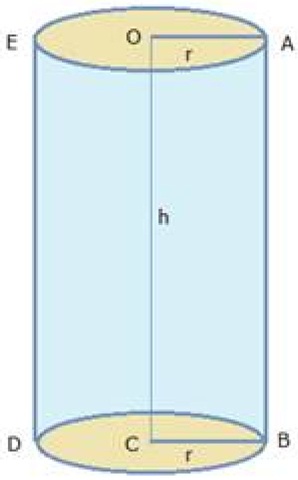
Question 20.
Find the radius of a closed right circular cylinder of volume 100 cm^3 which has the minimum total surface area.

Answer:
Given,

• The closed is cylindrical can with a circular base and top.

• The volume of the cylinder is 1 litre = 100 cm^3.

• The surface area of the box is minimum.



Let us consider,

• The radius base and top of the cylinder be 'r' units. (skin coloured in the figure)

• The height of the cylinder be 'h'units.

• As the Volume of cylinder is given, V = $100cm^3$

The Volume of the cylinder= $\pi r^2 h$

$100 = \pi r^2 h$

$h = \frac{100}{\pi r^2}$ ---- (1)

The Surface area cylinder is = area of the circular base + area of the circular top + area of the cylinder

$S = \pi r^2 + \pi r^2 + 2\pi rh$

$S = 2\pi r^2 + 2\pi rh$

[substituting (1) in the volume formula]

$$S = 2\pi r^2 + 2\pi r\left(\frac{100}{\pi r^2}\right)$$

$$S = 2\left[\pi r^2 + \left(\frac{100}{r}\right)\right]$$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. <u>This is because if the function f(r) has a maximum/minimum at a point c then f′(c) = 0.</u>

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\left[\pi r^2 + \left(\frac{100}{r}\right)\right]\right]$$

$$\frac{dS}{dr} = 2\,(2\pi r) + \left(\frac{100}{r^2}\right)(-1)$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{100}{r^2}\right)$$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{100}{r^2}\right) = 0$$

$$2\,(2\pi r) - 2\left(\frac{100}{r^2}\right) = 0$$

$$2\pi r = \frac{100}{r^2}$$ ---- (4)

<u>Now to check if this critical point will determine the minimum surface area of the box, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\,(2\pi r) - 2\left(\frac{100}{r^2}\right)\right]$$

$$\frac{d^2S}{dr^2} = 4\pi - \frac{2\times100\times(-2)}{r^3} = 4\pi + \frac{400}{r^3}$$ ----- (5)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{50}{\pi}}} = 4\pi + \frac{400}{\left(\sqrt[3]{\frac{50}{\pi}}\right)^3} = 4\pi + \frac{400\times\pi}{50} = 4\pi + 8\pi = 12\pi$$

As $\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{50}{\pi}}} = 12\pi > 0$, so the function S is minimum at $r = \sqrt[3]{\frac{50}{\pi}}$

As S is minimum from equation (4)

$$2\pi r = \frac{100}{r^2}$$

$$2\pi r = \frac{V}{r^2}$$

V = $2\pi r^3$

Now in equation (1) we have,

$$h = \frac{V}{\pi r^2} = \frac{2\pi r^3}{\pi r^2}$$

h = 2r = diameter

Therefore when the total surface area of a cone is minimum, then height of the cone is equal to twice the radius or equal to its diameter.

Question 21.
Show that the height of a closed cylinder of given volume and the least surface area is equal to its diameter.

Answer:
Let r be the radius of the base and h the height of a cylinder.

The surface area is given by,

$S = 2\pi r^2 + 2\pi rh$

$h = \frac{S-2\pi r^2}{2\pi r}$.........(1)

Let V be the volume of the cylinder.

Therefore, $V = \pi r^2 h$

$V = \pi r^2\left(\frac{S-2\pi r^3}{2\pi r}\right)$.......Using equation 1

$$V = \frac{Sr - 2\pi r^3}{2}$$

Differentiating both sides w.r.t r, we get,

$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$..........(2)

For maximum or minimum, we have,

$$\frac{dV}{dr} = 0$$

$\Rightarrow \frac{S}{2} - 3\pi r^2 = 0$

$\Rightarrow S = 6\pi r^2$

$2\pi r^2 + 2\pi rh = 6\pi r^2$

$h = 2r$

Differentiating equation 2, with respect to r to check for maxima and minima, we get,

$$\frac{d^2V}{dr^2} = -6\pi r < 0$$

Hence, V is maximum when h = 2r or h = diameter

Question 22.

Prove that the volume of the largest cone that can be inscribed in a sphere is $\frac{8}{27}$ of the volume of the sphere.

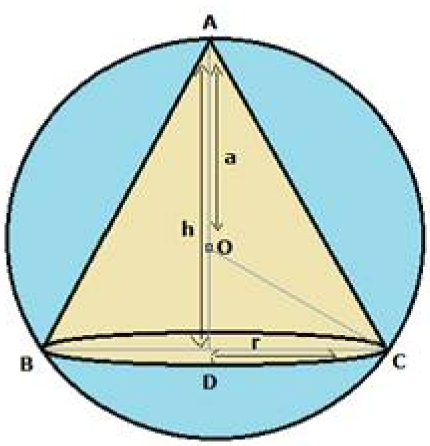
Answer:
Given,

• Volume of the sphere.

• Volume of the cone.

• Cone is inscribed in the sphere.

• Volume of cone is maximum.



Let us consider,

• The radius of the sphere be ‘a’ units.

• Volume of the inscribed cone be ‘V’.

• Height of the inscribed cone be ‘h’.

• Radius of the base of the cone is ‘r’.

Given volume of the inscribed cone is,

$$V = \frac{\pi r^2 h}{3}$$

Consider OD = (AD-OA) =(h-a)

Now let $OC^2 = OD^2 + DC^2$, here OC = a, OD = (h-a), DC = r,

So $a^2 = (h-a)^2 + r^2$

$r^2 = a^2 – (h^2 + a^2 – 2ah)$

$r^2 = h\ (2a - h)$ ----- (1)

Let us consider the volume of the cone:

$$V = \frac{1}{3}\ (\pi r^2 h)$$

Now substituting (1) in the volume formula,

$$V = \frac{1}{3}\ (\pi h(2a - h)h)$$

$V = \frac{1}{3}\ (2\pi h^2 a - \ \pi h^3)$ ---- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with h and then equating it to zero. <u>This is because if the function V(r) has a maximum/minimum at a point c then V′(c) = 0.</u>

Differentiating the equation (2) with respect to h:

$$\frac{dV}{dh} = \frac{d}{dh}\left[\frac{1}{3}\ (2\pi h^2 a - \ \pi h^3)\right]$$

$$\frac{dV}{dh} = \frac{1}{3}\ (2\pi a)(2h) - \frac{1}{3}(\pi)(3h^2)$$

[Since $\frac{d}{dx}\ (x^n) = nx^{n-1}$]

$\frac{dV}{dh} = \frac{1}{3}[\ 4\pi ah - \ 3\pi h^2]$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{dh} = \frac{1}{3}[\,4\pi ah - \,3\pi h^2] \,=\, 0$$

$4\pi ah – 3\pi h^2 = 0$

$h(4\pi a-3\pi h)= 0$

h = 0 (or) $h = \frac{4\pi a}{3\pi} = \frac{4a}{3}$

$$h = \frac{4a}{3}$$

[as h cannot be zero]

<u>Now to check if this critical point will determine the maximum volume of the inscribed cone, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with h:

$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[\frac{1}{3}[\,4\pi ah - \,3\pi h^2\,\right]$$

$\frac{d^2V}{dh^2} = \frac{1}{3}[\,4\pi a - (3\pi)(2h)] = \frac{\pi}{3}[4a - 6h]$ ----- (4)

[Since $\frac{d}{dx}\,(x^n) = \,nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4a}{3}} = \frac{\pi}{3}\left[4a - 6\left(\frac{4a}{3}\right)\right] = \frac{4a\pi}{3}\,[1-2] = \,-\frac{4a\pi}{3}$$

As $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4a}{3}} = \,-\frac{4a\pi}{3} < 0$, so the function V is maximum at $h = \frac{4a}{3}$

Substituting h in equation (1)

$$r^2 = \left(\frac{4a}{3}\right)\left(2a - \frac{4a}{3}\right)$$

$$r^2 = \left(\frac{4a}{3}\right)\left(2a - \frac{4a}{3}\right)$$

$$r^2 = \frac{8a^2}{9}$$

As V is maximum, substituting h and r in the volume formula:

$$V = \frac{1}{3}\pi\left(\frac{8a^2}{9}\right)\left(\frac{4a}{3}\right)$$

$$V = \frac{8}{27}\left(\frac{4}{3}\pi a^3\right)$$

$$V = \frac{8}{27}(\text{volume of the sphere})$$

Therefore when the volume of a inscribed cone is maximum, then it is equal to $\frac{8}{27}$ times of the volume of the sphere in which it is inscribed.

Question 23.
Which fraction exceeds its pth power by the greatest number possible?

Answer:
Given,

The pth power of a number exceeds by a fraction to be the greatest.

Let us consider,

• ‘x’ be the required fraction.

• The greatest number will be $y = x - x^p$ ------ (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function y(x)has a maximum/minimum at a point c then y′(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}(x - x^p)$$

$\frac{dy}{dx} = 1 - px^{p-1}$ ---- (2)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dy}{dx} = 1 - px^{p-1} = 0$$

1 = px^{p-1}

$$x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$$

<u>Now to check if this critical point will determine the if the number is the greatest, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (2) with x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[1 - px^{p-1}]$$

$\frac{d^2y}{dx^2} = -p(p-1)x^{p-2}$ ----- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2y}{dx^2}\right)_{x=\left(\frac{1}{p}\right)^{\frac{1}{p-1}}} = -p(p-1)\left(\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right)^{p-2}$$

As $\left(\frac{d^2y}{dx^2}\right)_{x=\left(\frac{1}{p}\right)^{\frac{1}{p-1}}} = -p(p-1)\left(\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right)^{p-2} < 0$, so the number y is greatest at $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

Hence, the y is the greatest number and exceeds by a fraction $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

Question 24.
Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, −8).

Answer:
Given,

- A point is present on a curve $y^2 = 4x$
- The point is near to the point (2,-8)

Let us consider,

- The co-ordinates of the point be P(x,y)
- As the point P is on the curve, then $y^2 = 4x$

$$x = \frac{y^2}{4}$$

- The distance between the points is given by,

$D^2 = (x-2)^2 + (y+8)^2$

$D^2 = x^2 - 4x + 4 + y^2 + 64 + 16y$

Substituting x in the distance equation

$$D^2 = \left(\frac{y^2}{4}\right)^2 - 4\left(\frac{y^2}{4}\right) + y^2 + 16y + 68$$

$Z = D^2 = \frac{y^4}{16} + 16y + 68$ ---- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with y and then equating it to zero. This is because if the function Z(x) has a maximum/minimum at a point c then Z′(c) = 0.

Differentiating the equation (2) with respect to y:

$$\frac{dZ}{dy} = \frac{d}{dy}\left(\frac{y^4}{16} + 16y + 68\right)$$

$$\frac{dZ}{dy} = \frac{4y^3}{16} + 16 = \frac{y^3}{4} + 16 \text{ ---- (2)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dZ}{dy} = \frac{y^3}{4} + 16 = 0$$

$y^3 + 64 = 0$

$(y + 4)\ (y^2 - 4y + 16) = 0$

$(y+4) = 0$ (or) $y^2 - 4y + 16 = 0$

$y = -4$

(as the roots of the $y^2 - 4y + 16$ are imaginary)

Now to check if this critical point will determine the distance is mimimum, we need to check with second differential which needs to be positive.

Consider differentiating the equation (2) with y:

$$\frac{d^2Z}{dy^2} = \frac{d}{dy}\left[\frac{y^3}{4} + 16\right]$$

$$\frac{d^2Z}{dy^2} = \frac{3y^2}{4} \text{ ----- (3)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2Z}{dy^2}\right)_{y=-4} = \frac{3}{4}(-4)^2 = 12$$

As $\left(\frac{d^2Z}{dy^2}\right)_{y=-4} = 12 > 0$, so the Distance D^2 is minimum at y = -4

Now substituting y in x, we have

$$x = \frac{(-4)^2}{4} = 4$$

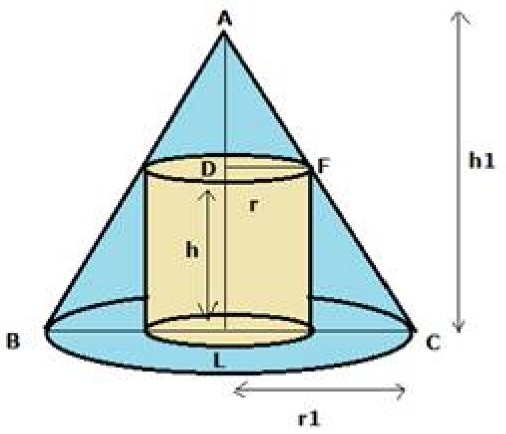
So, the point P on the curve $y^2 = 4x$ is (4,-4) which is at nearest from the (2,-8)

Question 25.
A right circular cylinder is inscribed in a cone. Show that the curved surface area of the cylinder is maximum when the diameter of the cylinder is equal to the radius of the base of the cone.

Answer:
Given,

• A right circular cylinder is inscribed inside a cone.

• The curved surface area is maximum.



Let us consider,

• 'r_1' be the radius of the cone.

• 'h_1' be the height of the cone.

• 'r' be the radius of the inscribed cylinder.

• 'h' be the height of the inscribed cylinder.

DF = r, and AD = AL – DL = $h_1 - h$

Now, here ΔADF and ΔALC are similar,

Then

$$\frac{AD}{AL} = \frac{DF}{LC} \Rightarrow \frac{h_1 - h}{h_1} = \frac{r}{r_1}$$

$$h_1 - h = \frac{rh_1}{r_1}$$

$$h = h_1 - \frac{rh_1}{r_1} = h_1\left(1 - \frac{r}{r_1}\right)$$

$$h = h_1\left(1 - \frac{r}{r_1}\right) \text{----- (1)}$$

Now let us consider the curved surface area of the cylinder,

S = 2πrh

Substituting h in the formula,

$$S = 2\pi r\left[h_1\left(1 - \frac{r}{r_1}\right)\right]$$

$$S = 2\pi r h_1 - \frac{2\pi h_1 r^2}{r_1} \text{---- (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. <u>This is because if the function S(r) has a maximum/minimum at a point c then S′(c) = 0.</u>

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\pi r h_1 - \frac{2\pi h_1 r^2}{r_1}\right]$$

$$\frac{dS}{dr} = 2\pi h_1 - \frac{2\pi h_1 (2r)}{r_1}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dS}{dr} = 2\pi h_1 - \frac{4\pi h_1 r}{r_1}$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\pi h_1 - \frac{4\pi h_1 r}{r_1} = 0$$

$$\frac{4\pi h_1 r}{r_1} = 2\pi h_1$$

$$r = \frac{2\pi h_1 r_1}{4\pi h_1}$$

$$r = \frac{r_1}{2}$$

<u>Now to check if this critical point will determine the maximum volume of the inscribed cylinder, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\pi h_1 - \frac{4\pi h_1 r}{r_1}\right]$$

$\frac{d^2S}{dr^2} = 0 - \frac{4\pi h_1}{r_1} = -\frac{4\pi h_1}{r_1}$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\frac{d^2S}{dr^2}_{r=\frac{r_1}{2}} = -\frac{4\pi h_1}{r_1}$$

As $\frac{d^2S}{dr^2}_{r=\frac{r_1}{2}} = -\frac{4\pi h_1}{r_1} < 0$, so the function S is maximum at $r = \frac{r_1}{2}$

Substituting r in equation (1)

$$h = h_1\left(1 - \frac{\frac{r_1}{2}}{r_1}\right)$$

$$h = h_1\left(1 - \frac{1}{2}\right) = \frac{h_1}{2} \text{ --- } (5)$$

As S is maximum, from (5) we can clearly say that $h_1 = 2h$ and

$r_1 = 2r$

this means the radius of the cone is twice the radius of the cylinder or equal to diameter of the cylinder.

Question 26.
Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.

Answer:
Given,

• Closed cuboid has square base.

• The volume of the cuboid is given.

• Surface area is minimum.

Let us consider,

• The side of the square base be 'x'.

• The height of the cuboid be 'h'.

• The given volume, $V = x^2h$

$h = \frac{V}{x^2}$ ----- (1)

Consider the surface area of the cuboid,

Surface Area =

2(Area of the square base) + 4(areas of the rectangular sides)

$S = 2x^2 + 4xh$

Now substitute (1) in the Surface Area formula

$$S = 2x^2 + 4x\left(\frac{V}{x^2}\right)$$

$S = 2x^2 + \left(\frac{4V}{x}\right)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function S(x) has a maximum/minimum at a point c then S′(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dS}{dx} = \frac{d}{dx}\left[2x^2 + \left(\frac{4V}{x}\right)\right]$$

$$\frac{dS}{dx} = 2(2x) + 4V\left(\frac{-1}{x^2}\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{dS}{dx} = 4x - \frac{4V}{x^2}$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dx} = 4x - \frac{4V}{x^2} = 0$$

$$4x = \frac{4V}{x^2}$$

$x^3 = V$

<u>Now to check if this critical point will determine the minimum surface area, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[4x - \frac{4V}{x^2}\right]$$

$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3}$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\frac{d^2S}{dx^2}_{x=V^{\frac{1}{3}}} = 4 + \frac{8V}{V} = 12$$

As $\frac{d^2S}{dx^2}_{x=V^{\frac{1}{3}}} = 12 > 0$, so the function S is minimum at $x = \sqrt[3]{V}$

Substituting x in equation (1)

$$h = \frac{V}{x^2} = \frac{x^3}{x^2} = x$$

h = x

As S is minimum and h = x, this means that the cuboid is a cube.

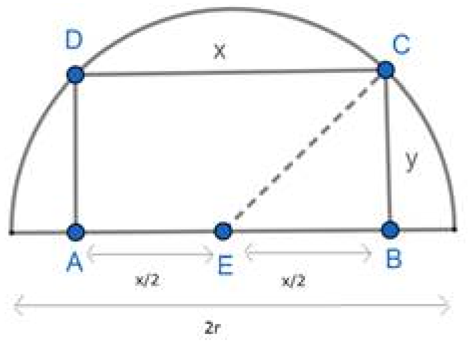
Question 27.

A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle so that its area is maximum. Find also this area.

Answer:
Given,

• Radius of the semicircle is ‘r’.

• Area of the rectangle is maximum.



Let us consider,

• The base of the rectangle be ‘x’ and the height be ‘y’.

Consider the ΔCEB,

$CE^2 = EB^2 + BC^2$

As CE = r, $EB = \frac{x}{2}$ and CB = y

$$r^2 = \left(\frac{x}{2}\right)^2 + y^2$$

$$y^2 = r^2 - \left(\frac{x}{2}\right)^2 \text{ ---- (1)}$$

Now the area of the rectangle is

$A = x \times y$

Squaring on both sides

$A^2 = x^2\, y^2$

Substituting (1) in the above Area equation

$$A^2 = x^2 \left[r^2 - \left(\frac{x}{2}\right)^2\right]$$

$$Z = A^2 = x^2r^2 - x^2\frac{x^2}{4} = x^2r^2 - \frac{x^4}{4} \text{ ----- } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function Z(x) has a maximum/minimum at a point c then Z′(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[x^2r^2 - \frac{x^4}{4}\right]$$

$$\frac{dZ}{dx} = r^2\,(2x) - \frac{4x^3}{4}$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$]

$$\frac{dZ}{dx} = 2xr^2 - x^3 \text{ ------- } (3)$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 2xr^2 - x^3 = 0$$

$x(2r^2 - x^2) = 0$

$x = 0$ (or) $x^2 = 2r^2$

$x = 0$ (or) $x = r\sqrt{2}$

$$x = r\sqrt{2}$$

[as x cannot be zero]

Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[2xr^2 - x^3]$$

$\frac{d^2Z}{dx^2} = 2r^2 - 3x^2$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\frac{d^2Z}{dx^2}_{x=r\sqrt{2}} = 2r^2 - 3(r\sqrt{2})^2 = 2r^2 - 6r^2 = -4r^2$$

As $\frac{d^2Z}{dx^2}_{x=r\sqrt{2}} = -4r^2 < 0$, so the function Z is maximum at $x = r\sqrt{2}$

Substituting x in equation (1)

$$y^2 = r^2 - \left(\frac{r\sqrt{2}}{2}\right)^2 = r^2 - \frac{r^2}{2} = \frac{r^2}{2}$$

$$y = \sqrt{\frac{r^2}{2}} = \frac{r}{\sqrt{2}} = \frac{r\sqrt{2}}{2}$$

As the area of the rectangle is maximum, and $x = r\sqrt{2}$ and $y = \frac{r\sqrt{2}}{2}$

So area of the rectangle is

$$A = r\sqrt{2} \times \frac{r\sqrt{2}}{2}$$

A = r^2

Hence the maximum area of the rectangle inscribed inside a semicircle is r^2 square units.

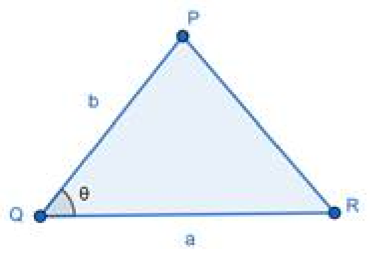
Question 28.
Two sides of a triangle have lengths a and b and the angle between them is θ. What value of θ will maximize the area of the triangle?

Answer:
Given,

• The length two sides of a triangle are 'a' and 'b'

• Angle between the sides 'a' and 'b' is θ.

• The area of the triangle is maximum.



Let us consider,

The area of the ΔPQR is given be

$A = \frac{1}{2} ab \sin\theta$ ---- (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with θ and then equating it to zero. This is because if the function A (θ) has a maximum/minimum at a point c then A′(c) = 0.

Differentiating the equation (1) with respect to θ:

$$\frac{dA}{d\theta} = \frac{d}{d\theta}\left[\frac{1}{2} ab \sin\theta\right]$$

$\frac{dA}{d\theta} = \frac{1}{2} ab \cos\theta$ ---- (2)

[Since $\frac{d}{dx}(\sin\theta) = \cos\theta$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dA}{d\theta} = \frac{1}{2}\,ab\cos\theta = 0$$

Cos θ = 0

$$\theta = \frac{\pi}{2}$$

Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.

Consider differentiating the equation (2) with θ :

$$\frac{d^2A}{d\theta^2} = \frac{d}{d\theta}\left[\frac{1}{2}\,ab\cos\theta\right]$$

$$\frac{d^2A}{d\theta^2} = -\frac{1}{2}\,ab\sin\theta$$ ----- (2)

[Since $\frac{d}{dx}(\cos\theta) = -\sin\theta$]

Now let us find the value of

$$\frac{d^2A}{d\theta^2}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\,ab\sin\left(\frac{\pi}{2}\right) = -\frac{1}{2}\,ab$$

As $\frac{d^2A}{d\theta^2}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\,ab < 0$, so the function A is maximum at $\theta = \frac{\pi}{2}$

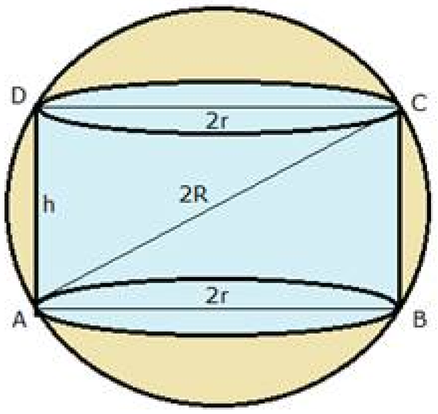
As the area of the triangle is maximum when $\theta = \frac{\pi}{2}$

Question 29.
Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is (500π) cm^3.

Answer:
Given,

• Radius of the sphere is $5\sqrt{3}$.

• Volume of cylinder is maximum.



Let us consider,

• The radius of the sphere be 'R' units.

• Volume of the inscribed cylinder be 'V'.

• Height of the inscribed cylinder be 'h'.

• Radius of the cylinder is 'r'.

Now let $AC^2 = AB^2 + BC^2$, here AC = 2R, AB =2r, BC = h,

So $4R^2 = 4r^2 + h^2$

$r^2 = \frac{1}{4}[4R^2 - h^2]$ ----- (1)

Let us consider, the volume of the cylinder:

$V = \pi r^2 h$

Now substituting (1) in the volume formula,

$$V = \pi h \left(\frac{1}{4}[4R^2 - h^2]\right)$$

$V = \frac{\pi}{4}(4R^2h - h^3)$ ---- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with h and then equating it to zero. This is because if the function V(h) has a maximum/minimum at a point c then V′(c) = 0.

Differentiating the equation (2) with respect to h:

$$\frac{dV}{dh} = \frac{d}{dh}\left[\frac{\pi}{4}(4R^2h - h^3)\right]$$

$$\frac{dV}{dh} = \frac{4R^2\pi}{4} - \frac{\pi}{4}(3h^2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dV}{dh} = R^2\pi - \frac{3h^2\pi}{4}$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{dh} = R^2\pi - \frac{3h^2\pi}{4} = 0$$

$3h^2\pi = 4R^2\pi$

$$h^2 = \frac{4}{3}R^2 = \frac{4}{3}(5\sqrt{3})^2 = \frac{4}{3}(25 \times 3) = 100$$

h = 10

[as h cannot be negative]

Now to check if this critical point will determine the maximum volume of the inscribed cone, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with h:

$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[R^2\pi - \frac{3h^2\pi}{4}\right]$$

$\frac{d^2V}{dh^2} = 0 - \frac{3\,(2h)\pi}{3} = -2h\pi$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dh^2}\right)_{h=10} = -2h\pi = -2(10)\pi = -20\pi$$

As $\left(\frac{d^2V}{dh^2}\right)_{h=10} = -20\pi < 0$, so the function V is maximum at h=10

Substituting h in equation (1)

$$r^2 = \frac{1}{4}\left[4(5\sqrt{3})^2 - (10)^2\right]$$

$$r^2 = \frac{1}{4}\left[4\,(25 \times 3) - 100\right]$$

$$r^2 = \frac{300 - 100}{4} = \frac{200}{4} = 50$$

As V is maximum, substituting h and r in the volume formula:

V = π (50) (10)

V = 500π cm^3

Therefore when the volume of a inscribed cylinder is maximum and is equal 500π cm^3

Question 30.
A square tank of capacity 250cubic meters has to be dug out. The cost of the land is Rs. 50 per square metre. The cost of digging increases with the depth and for the whole tank, it is Rs. $\left(400 \times h^2\right)$, where h metres is the depth of the tank. What should be the dimensions of the tank so that the cost is minimum?

Answer:
Given,

• Capacity of the square tank is 250 cubic metres.

• Cost of the land per square meter Rs.50.

• Cost of digging the whole tank is Rs. $(400 \times h^2)$.

• Where h is the depth of the tank.

Let us consider,

• Side of the tank is x metres.

• Cost of the digging is; $C = 50x^2 + 400h^2$ ---- (1)

• Volume of the tank is; $V = x^2h$; $250 = x^2h$

$h = \frac{250}{x^2}$ ----- (2)

Substituting (2) in (1),

$$C = 50x^2 + 400\left(\frac{250}{x^2}\right)^2$$

$C = 50x^2 + \frac{400 \times 62500}{x^4}$ ----- (3)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function C(x) has a maximum/minimum at a point c then C′(c) = 0.

Differentiating the equation (3) with respect to x:

$$\frac{dC}{dx} = \frac{d}{dx}\left[50x^2 + \frac{400 \times 62500}{x^4}\right]$$

$$\frac{dC}{dx} = 50(2x) + \frac{25000000(-4)}{x^5}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dC}{dx} = 100x - \frac{10^8}{x^5} \text{-------} (4)$$

To find the critical point, we need to equate equation (4) to zero.

$$\frac{dC}{dx} = 100x - \frac{10^8}{x^5} = 0$$

$x^6 = 10^6$

x = 10

Now to check if this critical point will determine the minimum volume of the tank, we need to check with second differential which needs to be positive.

Consider differentiating the equation (4) with x:

$$\frac{d^2C}{dx^2} = \frac{d}{dx}\left[100x - \frac{10^8}{x^5}\right]$$

$$\frac{d^2C}{dx^2} = 100 - \frac{10^8(-5)}{x^6} = 100 + \frac{10^8(5)}{x^6} \text{-----} (5)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2C}{dx^2}\right)_{x=10} = 100 + \frac{10^8(5)}{(10)^6} = 100 + 500 = 600$$

As $\left(\frac{d^2C}{dx^2}\right)_{x=10} = 600 > 0$, so the function C is minimum at x=10

Substituting x in equation (2)

$$h = \frac{250}{(10)^2} = \frac{250}{100} = \frac{5}{2}$$

h = 2.5 m

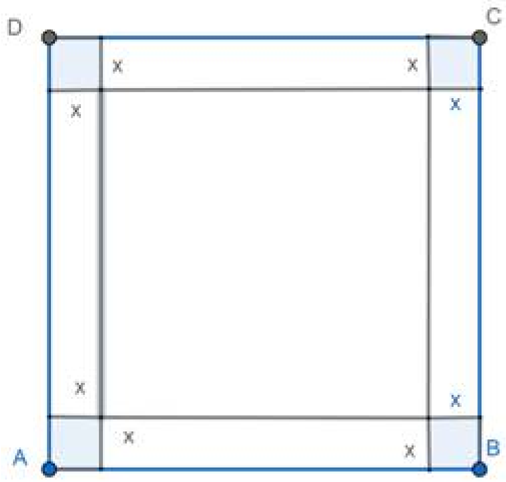
Therefore when the cost for the digging is minimum, when x = 10m and h = 2.5m

Question 31.
A square piece of tin of side 18 cm is to be made into a box without the top, by cutting a square piece from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find the maximum volume of the box.

Answer:
Given,

• Side of the square piece is 18 cms.

• the volume of the formed box is maximum.



Let us consider,

• 'x' be the length and breadth of the piece cut from each vertex of the piece.

• Side of the box now will be (18-2x)

• The height of the new formed box will also be 'x'.

Let the volume of the newly formed box is :

$V = (18-2x)^2 \times (x)$

$V = (324+ 4x^2 - 72x)\ x$

$V = 4x^3 - 72x^2 + 324x$ ------ (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function V(x) has a maximum/minimum at a point c then V′(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dV}{dx} = \frac{d}{dx}\left[4x^3 - 72x^2 + 324x\right]$$

$$\frac{dV}{dx} = 12x^2 - 144x + 324 \text{ -------- } (2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dV}{dx} = 12x^2 - 144x + 324 = 0$$

$x^2 - 12x + 27 = 0$

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(27)}}{2(1)} = \frac{12 \pm \sqrt{144 - 108}}{2} = \frac{12 \pm \sqrt{36}}{2}$$

$$x == \frac{12 \pm 6}{2}$$

x = 9 or x =3

x= 2

[as x = 9 is not a possibility, because 18-2x = 18-18= 0]

Now to check if this critical point will determine the maximum area of the box, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{dx^2} = \frac{d}{dx}\left[12x^2 - 144x + 324\right]$$

$$\frac{d^2V}{dx^2} = 24x - 144 \text{ ----- (4)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3) - 144 = 72 - 144 = -72$$

As $\left(\frac{d^2V}{dx^2}\right)_{x=3} = -72 < 0$, so the function V is maximum at x = 3cm

Now substituting x = 3 in 18 – 2x, the side of the considered box:

Side = 18-2x = 18 - 2(3) = 18-6= 12cm

Therefore side of wanted box is 12cms and height of the box is 3cms.

Now, the volume of the box is

$V = (12)^2 \times 3 = 144 \times 3 = 432cm^3$

Hence maximum volume of the box formed by cutting the given 18cms sheet is $432cm^3$ with 12cms side and 3cms height.

Question 32.
An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width.

Answer:
Given,

• The tank is square base open tank.

• The cost of the construction to be least.

Let us consider,

• Side of the tank is x metres.

• Height of the tank be 'h' metres.

• Volume of the tank is; $V = x^2h$

• Surface Area of the tank is $S = x^2 + 4xh$

• Let Rs.P is the price per square.

Volume of the tank,

$h = \frac{V}{x^2}$ ---- (1)

Cost of the construction be:

$C = (x^2 + 4xh)P$ ---- (2)

Substituting (1) in (2),

$$C = \left[x^2 + 4x\frac{V}{x^2}\right]P$$

$C = [x^2 + \frac{4V}{x}]\ P$ ----- (3)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function C(x) has a maximum/minimum at a point c then C′(c) = 0.</u>

Differentiating the equation (3) with respect to x:

$$\frac{dC}{dx} = \frac{d}{dx}\left[x^2 + \frac{4V}{x}\right]P$$

$$\frac{dC}{dx} = \left[(2x) + \frac{4V(-1)}{x^2}\right]P$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{dC}{dx} = \left[2x - \frac{4V}{x^2}\right]P$$ ------- (4)

To find the critical point, we need to equate equation (4) to zero.

$$\frac{dC}{dx} = \left[2x - \frac{4V}{x^2}\right]P = 0$$

$x^3 = 2V$

Now to check if this critical point will determine the minimum volume of the tank, we need to check with second differential which needs to be positive.

Consider differentiating the equation (4) with x:

$$\frac{d^2C}{dx^2} = P\frac{d}{dx}\left[2x - \frac{4V}{x^2}\right]$$

$$\frac{d^2C}{dx^2} = \left[2 - \frac{4V(-2)}{x^3}\right]P = \left[2 + \frac{8V}{x^3}\right]P$$ ----- (5)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2C}{dx^2}\right)_{x=(2V)^{\frac{1}{3}}} = \left[2 + \frac{8V}{2V}\right]P = [2+4]P = 6P$$

As $\left(\frac{d^2C}{dx^2}\right)_{x=(2V)^{\frac{1}{3}}} = 6P > 0$, so the function C is minimum at $x = \sqrt[3]{2V}$

Substituting x in equation (2)

$$h = \frac{V}{(2V)^{\frac{2}{3}}} = \frac{V\sqrt[3]{(2V)}}{2V} = \frac{1}{2}\sqrt[3]{2V}$$

$$h = \frac{1}{2}\sqrt[3]{2V}$$

Therefore when the cost for the digging is minimum, when $x = \sqrt[3]{2V}$ and $h = \frac{1}{2}\sqrt[3]{2V}$

Question 33.
A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum.

Answer:
Given,

- Length of the wire is 36 cm.
- The wire is cut into 2 pieces.
- One piece is made to a square.
- Another piece made into a equilateral triangle.

Let us consider,

- The perimeter of the square is x.
- The perimeter of the equilateral triangle is (36-x).
- Side of the square is $\frac{x}{4}$
- Side of the triangle is $\frac{(36-x)}{3}$

Let the Sum of the Area of the square and triangle is

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{36-x}{3}\right)^2$$

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(12 - \frac{x}{3}\right)^2 = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)$$

$$A = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right) \text{--- (1)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function A(x) has a maximum/minimum at a point c then A′(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)\right]$$

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(0 + \frac{2x}{9} - 8\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right)$ ----- (2)

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right) = 0$$

$$\frac{2x}{16} = \frac{\sqrt{3}}{4}\left(8 - \frac{2x}{9}\right)$$

$$\frac{2x}{16} = 2\sqrt{3} - \frac{\sqrt{3}x}{18}$$

$$\frac{2x}{16} + \frac{\sqrt{3}x}{18} = 2\sqrt{3}$$

$$x\left(\frac{2\,(9) + \sqrt{3}(8)}{144}\right) = 2\sqrt{3}$$

$$x\left(\frac{18 + 8\sqrt{3}}{144}\right) = 2\sqrt{3}$$

$$x = 2\sqrt{3}\left(\frac{144}{18 + 8\sqrt{3}}\right) = \frac{144\sqrt{3}}{(9 + 4\sqrt{3})}$$

<u>Now to check if this critical point will determine the minimum area, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right)\right]$$

$$\frac{d^2A}{dx^2} = \frac{1}{8} + \frac{\sqrt{3}}{4}\left(\frac{2}{9}\right) = \frac{9+4\sqrt{3}}{72} \text{-----} (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2A}{dx^2}\right)_{x=\frac{144\sqrt{3}}{(9+4\sqrt{3})}} = \frac{9+4\sqrt{3}}{72}$$

As $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{144\sqrt{3}}{(9+4\sqrt{3})}} = \frac{9+4\sqrt{3}}{72} > 0$, so the function A is minimum at

$$x = \frac{144\sqrt{3}}{(9+4\sqrt{3})}$$

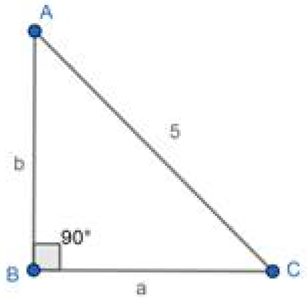
Now, the length of each piece is $x = \frac{144\sqrt{3}}{(9+4\sqrt{3})}$ cm and $36 - x = 36 - \frac{144\sqrt{3}}{(9+4\sqrt{3})} = \frac{324}{(9+4\sqrt{3})}$ cm

Question 34.
Find the largest possible area of a right-angles triangle whose hypotenuse is 5 cm.

Answer:
Given,

• The triangle is right angled triangle.

• Hypotenuse is 5cm.



Let us consider,

• The base of the triangle is ‘a’.

• The adjacent side is ‘b’.

Now $AC^2 = AB^2 + BC^2$

As AC = 5, AB = b and BC = a

$25 = a^2 + b^2$

$b^2 = 25 – a^2$ ---- (1)

Now, the area of the triangle is

$$A = \frac{1}{2}ab$$

Squaring on both sides

$$A^2 = \frac{1}{4}a^2b^2$$

Substituting (1) in the area formula

$$Z = A^2 = \frac{1}{4}a^2(25 - a^2) \text{ ----- (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with a and then equating it to zero. This is because if the function Z (x) has a maximum/minimum at a point c then Z′(c) = 0.

Differentiating the equation (2) with respect to a:

$$\frac{dZ}{da} = \frac{d}{da}\left[\frac{1}{4}\, a^2(25 - a^2)\right]$$

$$\frac{dZ}{da} = \frac{1}{4}\,[\,25\,(2a) - 4a^3]$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$]

$\frac{dZ}{da} = \frac{25a}{2} - \ a^3$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{da} = \frac{25a}{2} - \ a^3 = 0$$

$$a\left(\frac{25}{2} - \ a^2\right) = \ 0$$

a=0 (or) $a = \frac{5}{\sqrt{2}}$

$$a = \frac{5}{\sqrt{2}}$$

[as a cannot be zero]

<u>Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with a:

$$\frac{d^2Z}{da^2} = \frac{d}{da}\left[\frac{25a}{2} - \ a^3\right]$$

$\frac{d^2Z}{da^2} = \ \frac{25}{2} - 3a^2$ ----- (4)

[Since $\frac{d}{dx}\,(x^n) = \ nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2Z}{da^2}\right)_{a=\frac{5}{\sqrt{2}}} = \frac{25}{2} - 3\left(\frac{5}{\sqrt{2}}\right)^2 = \frac{25}{2} - \frac{(3)25}{2} = -25$$

As $\left(\frac{d^2Z}{da^2}\right)_{a=\frac{5}{\sqrt{2}}} = -25 < 0$, so the function A is maximum at $a = \frac{5}{\sqrt{2}}$

Substituting value of A in (1)

$$b^2 = 25 - \frac{25}{2} = \frac{25}{2}$$

$$b = \frac{5}{\sqrt{2}}$$

Now the maximum area is

$$A = \frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)\left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4}$$

$$\therefore A = \frac{25}{4}\ \text{cm}^2$$