Exercise 9c

Question 1.
Show that ƒ(x) = x^3 is continuous as well as differentiable at x=3.

Answer:
Given:

f(x) = x^3

If a function is differentiable at a point, it is necessarily continuous at that point.

Left hand derivative (LHD) at x = 3

$$\lim_{x\to3^-}\frac{f(x)-f(3)}{x-3} = \lim_{h\to0}\frac{f(3-h)-f(3)}{(3-h)-3}$$

$$= \lim_{h\to0}\frac{(3-h)^3-3^3}{(3-h)-3} = \lim_{h\to0}\frac{(3-h)^3-27}{-h} = \lim_{h\to0} -\frac{h\{(3-h)^2+3(3-h)+9\}}{h}$$

$$= \lim_{h\to0} -\{(3-h)^2+3(3-h)+9\} = \lim_{h\to0} -[-\{-(3-h)^2-3(3-h)-9\}]$$

$$= \lim_{h\to0} -\{-h^2+9h-27\} = \lim_{h\to0} h^2-9h+27 = 0^2-9(0)+27 = 27$$

Right hand derivative (RHD) at x = 3

$$\lim_{x\to3^+}\frac{f(x)-f(3)}{x-3} = \lim_{h\to0}\frac{f(3+h)-f(3)}{(3+h)-3}$$

$$= \lim_{h\to0}\frac{(3+h)^3-3^3}{(3+h)-3} = \lim_{h\to0}\frac{(3+h)^3-27}{h} = \lim_{h\to0}\frac{h\{(3+h)^2+3(3+h)+9\}}{h}$$

$$= \lim_{h\to0}\{(3+h)^2+3(3+h)+9\} = \lim_{h\to0}(3+h)^2+3(3+h)+9$$

$$= \lim_{h\to0}\{h^2+9h+27\} = 0^2+9(0)+27 = 27$$

LHD = RHD

Therefore, f(x) is differentiable at x = 3.

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} x^3 = 3^3 = 27$$

Also, f(3) =27

Therefore, f(x) is also continuous at x = 3.

Question 2.
Show that ƒ(x) = $(x-1)^{1/3}$ is not differentiable at x=1.

Answer:
Given function f(x) = $(x-1)^{1/3}$

LHD at x = 1

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0} \frac{f(1-h)-f(1)}{(1-h)-1} = \lim_{h\to 0} \frac{\{(1-h)-1\}^{\frac{1}{3}}(1-1)^{\frac{1}{3}}}{(1-h)-1}$$

$$= \lim_{h\to 0} \frac{(-h)^{\frac{1}{3}}(0)^{\frac{1}{3}}}{-h} = \frac{0}{0} = \text{Not defined}$$

RHD at x = 1

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{(1+h)-1} = \lim_{h\to 0} \frac{\{(1+h)-1\}^{\frac{1}{3}}(1-1)^{\frac{1}{3}}}{(1+h)-1}$$

$$= \lim_{h\to 0} \frac{(-h)^{\frac{1}{3}}(0)^{\frac{1}{3}}}{-h} = \frac{0}{0} = \text{Not defined}$$

Since, LHD and RHD doesn’t exists

Therefore, f(x) is not differentiable at x = 1.

Question 3.
Show that constant function is always differentiable

Answer:
Let a be any constant number.

Then, f(x) = a

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We know that coefficient of a linear function is

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

Since our function is constant, $y_1 = y_2$

Therefore, a = 0

Now,

$$f'(x) = \lim_{h \to 0} \frac{a-a}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

Thus, the derivative of a constant function is always 0.

Question 4.
Show that ƒ(x) = |x-5| is continuous but not differentiable at x=5

Answer:
Left hand limit at x = 5

$$\lim_{x \to 5^-} |x - 5| = \lim_{x \to 5} (5 - x) = 0$$

Right hand limit at x = 5

$$\lim_{x \to 5^+} |x - 5| = \lim_{x \to 5} (x - 5) = 0$$

Also f(5) = |5 – 5 |= 0

As,

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$$

Therefore, f(x) is continuous at x = 5

Now, lets see the differentiability of f(x)

LHD at x = 5

$$\lim_{x\to 5^-} \frac{f(x)-f(5)}{x-5} = \lim_{h\to 0} \frac{f(5-h)-f(5)}{5-h-5} = \lim_{h\to 0} \frac{|5-(5-h)|-|5-5|}{-h} = \lim_{h\to 0} -\frac{h}{h} = -1$$

RHD at x = 5

$$\lim_{x\to 5^+} \frac{f(x)-f(5)}{x-5} = \lim_{h\to 0} \frac{f(5+h)-f(5)}{5+h-5} = \lim_{h\to 0} \frac{|(5+h)-5|-|5-5|}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Since, LHD ≠ RHD

Therefore,

f(x) is not differentiable at x = 5

Question 5.

Let ƒ(x) = $\begin{cases} (2-x), & \text{when } x \geq 1; \\ x, & \text{when } 0 \leq x \leq 1. \end{cases}$

Show that ƒ(x) is continuous but not differentiable at x=1

Answer:

Left hand limit at x = 1

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x = 1$$

f(x) = x is polynomial function and a polynomial function is continuous everywhere

Right hand limit at x = 1

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} (2-x) = (2-1) = 1$$

f(x) = 2 - x is polynomial function and a polynomial function is continuous everywhere

Also, f(1) =1

As we can see that,

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$$

Therefore,

f(x) is continuous at x =1

Now,

LHD at x = 1

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x-1}{x-1} = \lim_{x\to 1} \frac{1}{1} = \lim_{x\to 1} 1 = 1$$

RHD at x = 1

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{2-x-(2-1)}{x-1} = \lim_{x\to 1} \frac{2-x-1}{x-1} = \lim_{x\to 1} \frac{-(x-1)}{x-1}$$

$$\lim_{x\to 1} -\frac{1}{1} = \lim_{x\to 1} -1 = -1$$

As, LHD ≠ RHD

Therefore,

f(x) is not differentiable at x = 1

Question 6.
Show that ƒ(x) = [x] is neither continuous nor derivable at x=2.

Answer:
Left hand limit at x = 2

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} [2-h] = \lim_{h\to 0} 1 = 1$$

Right hand limit at x = 2

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [2+h] = \lim_{h\to 0} 2 = 2$$

As left hand limit ≠ right hand limit

Therefore, f(x) is not continuous at x = 2

Lets see the differentiability of f(x):

LHD at x = 2

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{h\to 0} \frac{f(x-h)-f(2)}{(x-h)-2} = \lim_{h\to 0} \frac{f(2-h)-f(2)}{(2-h)-2}$$
$$= \lim_{h\to 0} -\frac{1-2}{h}$$

$$\lim_{h\to 0} -\frac{(-1)}{h} = \infty$$

RHD at x = 2

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{h\to 0} \frac{f(x+h)-f(2)}{(x+h)-2} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{(2+h)-2} = \lim_{h\to 0} \frac{2-2}{h}$$

$$\lim_{h\to 0} \frac{0}{h} = 0$$

As, LHD ≠ RHD

Therefore,

f(x) is not derivable at x = 2

Question 7.
Show that function

$f(x) = \begin{cases} (1-x), & \text{when } x < 1; \\ (x^2-1), & \text{when } x \geq 1. \end{cases}$ is continuous but not differentiable at x=1

Answer:

Given function f(x) = $\begin{cases}(1-x), \text{when} x < 1; \\ (x^2-1), \text{when} x \geq 1.\end{cases}$

Left hand limit at x = 1:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1}(1-x) = 1 - 1 = 0$$

Right hand limit at x = 1:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1}(x^2-1) = 1^2 - 1 = 0$$

Also, f(1) = 1^2 – 1 = 0

As,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

Therefore,

f(x) is continuous at x = 1

Now, let's see the differentiability of f(x):

LHD at x = 2:

$$\lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2} \frac{(1-x)-(1-2)}{x-2} = \lim_{x \to 2} \frac{1-x-1+2}{x-2} = \lim_{x \to 2} \frac{-(x-2)}{x-2}$$

$$= \lim_{x \to 2} -1 = -1$$

RHD at x = 2:

$$\lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2} \frac{(x^2-1)-(2^2-1)}{x-2} = \lim_{x \to 2} \frac{x^2-1-3}{x-2} = \lim_{x \to 2} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2} \frac{x^2-2^2}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2}(x+2) = 2 + 2 = 4$$

As, LHD ≠ RHD

Therefore,

f(x) is not differentiable at x = 2

Question 8.

Let ƒ(x) = $\begin{cases} (2+x), & \text{if } x \geq 0; \\ (2-x), & \text{if } x < 0. \end{cases}$ Show that ƒ(x) is not derivable at x=0.

Answer:

Given function f(x) = $\begin{cases} (2+x), \text{if} x \geq 0; \\ (2-x), \text{if} x < 0. \end{cases}$

LHD at x = 0:

$$\lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{(2-x)-(2)}{x-0} = \lim_{x \to 0} \frac{-x}{x}$$

$$= \lim_{x \to 0} -1 = -1$$

RHD at x = 0:

$$\lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{(2+x)-(2)}{x-0} = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$$

As, LHD ≠ RHD

Therefore,

f(x) is not differentiable at x = 0

Question 9.

If ƒ(x) = |x| show that ƒ′(2)=1

Answer:

Given function is f(x) = |x|

LHD at x = 2:

$$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{2-h-2} = \lim_{h\to 0}\frac{|2-h|-|2|}{-h} = \lim_{h\to 0}\frac{-h}{-h}$$

$$\lim_{h\to 0} 1 = 1$$

RHD at x = 2:

$$\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{2+h-2} = \lim_{h\to 0}\frac{|2+h|-|2|}{h} = \lim_{h\to 0}\frac{h}{h}$$

$$\lim_{h\to 0} 1 = 1$$

As, LHD = RHD

Therefore, f(x) = |x| is differentiable at x = 2

Now $f'(2) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{|2+h|-|2|}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1$

Therefore,

f′(2) = 1

Question 10.
Find the values of a and b so that the function

$f(x) = \begin{cases} (x^2+3x+a), & \text{when } x \le 1; \\ (bx+2), & \text{when } x > 1 \end{cases}$ is differentiable at each x ϵ R

Answer:
It is given that f(x) is differentiable at each x ϵ R

For x ≤ 1,

$f(x) = x^2 + 3x + a$ i.e. a polynomial

for x > 1,

f(x) = bx + 2, which is also a polynomial

Since, a polynomial function is everywhere differentiable. Therefore, f(x) is differentiable for all x > 1 and for all x < 1.

f(x) is continuous at x = 1

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$$

$$\operatorname{Lim}_{x\to1}(x^2 + 3x + a) = \lim_{x\to1}(bx + 2) = 1 + 3 + a$$

$$1^2 + 3(1) + a = b(1) + 2 = 4 + a$$

4 + a = b + 2

a – b + 2 = 0 ...(1)

As function is differentiable, therefore, LHD = RHD

LHD at x = 1:

$$\operatorname{Lim}_{x\to1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1}\frac{x^2+3x+a-(4+a)}{x-1} = \lim_{x\to1}\frac{x^2+3x-4}{x-1} = \lim_{x\to1}\frac{(x+4)(x-1)}{x-1}$$

$$= \lim_{x\to1}(x+4) = 1 + 4 = 5$$

RHD at x = 1:

$$\operatorname{Lim}_{x\to1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1}\frac{(bx+2)-(4+a)}{x-1} = \lim_{x\to1}\frac{bx-2-a}{x-1} = \lim_{x\to1}\frac{bx-b}{x-1} = \lim_{x\to1}\frac{b(x-1)}{x-1}$$

$$= \lim_{x\to1} b = b$$

As, LHD = RHD

Therefore,

5 = b

Putting b in (1), we get,

$a – b + 2 = 0$

$a - 5 + 2 =0$

$a = 3$

Hence,

$a = 3$ and $b = 5$