Limits and Derivatives

Short Answer type:

Evaluate:

1. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Solution:

Given $\lim_{x \to 3} \frac{x^2-9}{x-3}$

The above equation can be written as

$$= \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3}$$

On simplifying and applying limits we get

$$\Rightarrow \lim_{x \to 3}(x+3) = 6$$

$$\Rightarrow \lim_{x \to 3} \frac{x^2-9}{x-3} = 6$$

2. $\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x - 1}$

Solution:

Given $\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x - 1}$

The above equation can be written as

$$= \lim_{x \to \frac{1}{2}} \frac{(2x)^2 - 1}{2x - 1}$$

Using $a^2 - b^2$ formula and expanding we get

$$= \lim_{x \to \frac{1}{2}} \frac{(2x-1)(2x+1)}{2x-1}$$

On simplifying and applying the limits we get

$$\Rightarrow \lim_{x \to \frac{1}{2}}(2x+1) = 2$$

$$\Rightarrow \lim_{x \to \frac{1}{2}} \frac{4x^2-1}{2x-1} = 2$$

On simplifying and applying the limits we get

$$\Rightarrow \lim_{x\to\frac{1}{2}}(2x+1)=2$$

$$\Rightarrow \lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1}=2$$

6. $\lim\limits_{x\to a}\dfrac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$

Solution:

Given $\lim\limits_{x\to a}\frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$

$$\Rightarrow \lim_{x\to a}\frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$$

Now by adding and subtracting 2 to denominator for further simplification we get

$$=\lim_{x\to a}\frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{(x+2)-(a+2)}$$

Now we have $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

$$\Rightarrow \lim_{x\to a}\frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{(x+2)-(a+2)}$$

By using the above formula we get

$$=\frac{5}{2}(a+2)^{\frac{5}{2}-1}$$

Simplifying and applying the limits we get

$$=\frac{5}{2}(a+2)^{\frac{3}{2}}$$

$$\Rightarrow \lim_{x\to a}\frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}=\frac{5}{2}(a+2)^{\frac{3}{2}}$$

7. $\lim\limits_{x\to 1}\dfrac{x^4-\sqrt{x}}{\sqrt{x}-1}$

Solution:

Given $\lim\limits_{x\to1}\frac{x^4-\sqrt{x}}{\sqrt{x}-1}$

Now to rationalize the denominator by multiplying the given equation by its rationalizing factor we get

$$\Rightarrow \lim_{x\to1}\frac{x^4-\sqrt{x}}{\sqrt{x}-1}=\lim_{x\to1}\frac{x^4-\sqrt{x}}{\sqrt{x}-1}\times\left(\frac{\sqrt{x}+1}{\sqrt{x}+1}\right)$$

On simplifying the above equation we get

$$\Rightarrow \lim_{x\to1}\frac{x^4-\sqrt{x}}{\sqrt{x}-1}\times\left(\frac{\sqrt{x}+1}{\sqrt{x}+1}\right)=\lim_{x\to1}\frac{x^4\sqrt{x}+x^4-x-\sqrt{x}}{x-1}$$

Taking √x common we get

$$\Rightarrow \lim_{x\to1}\frac{x^4\sqrt{x}-x^4-x-\sqrt{x}}{x-1}=\lim_{x\to1}\frac{\sqrt{x}(x^4-1)+x(x^3-1)}{x-1}$$

Using $a^3 - b^3$ and $a^2 - b^2$ formula and expanding we get

$$\Rightarrow \lim_{x\to1}\frac{\sqrt{x}(x^4-1)+x(x^3-1)}{x-1}=\lim_{x\to1}\frac{\sqrt{x}(x-1)(x+1)(x^2+1)+x(x-1)(x^2+x+1)}{x-1}$$

On simplification we get

$$\Rightarrow \lim_{x\to1}\frac{\sqrt{x}(x-1)(x+1)(x^2+1)+x(x-1)(x^2+x+1)}{x-1}=\lim_{x\to1}\frac{(x-1)(\sqrt{x}(x+1)(x^2+1)+x(x^2+x+1))}{x-1}$$

$$\Rightarrow \lim_{x\to1}\left(\sqrt{x}\,(x+1)(x^2+1)+x\,(x^2+x+1)\right)=4+3=7$$

$$\Rightarrow \lim_{x\to1}\frac{x^4-\sqrt{x}}{\sqrt{x}-1}=7$$

8. $\lim\limits_{x\to2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$

Solution:

Given $\lim\limits_{x\to2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$

Now to rationalize the denominator by multiplying the given equation by its rationalizing factor we get

$$\Rightarrow \lim_{x\to2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}=\lim_{x\to2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}\times\left(\frac{\sqrt{3x-2}+\sqrt{x+2}}{\sqrt{3x-2}+\sqrt{x+2}}\right)$$

Taking (x – 2) (x + 2) as common we get

$$\Rightarrow \lim_{x\to2}\frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{(3x-2)-(x+2)}=\lim_{x\to2}\frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2x-4}$$

$$\Rightarrow \lim_{x\to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}} = \lim_{x\to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}} \times \left(\frac{\sqrt{3x-2}+\sqrt{x+2}}{\sqrt{3x-2}+\sqrt{x+2}}\right)$$

Taking (x – 2) (x + 2) as common we get

$$\Rightarrow \lim_{x\to 2} \frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{(3x-2)-(x+2)} = \lim_{x\to 2} \frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2x-4}$$

Taking common and simplifying we get

$$\Rightarrow \lim_{x\to 2} \frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2x-4} = \lim_{x\to 2} \frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2(x-2)}$$

Now by applying the limit we get

$$\Rightarrow \lim_{x\to 2} \frac{(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2} = 8$$

$$\Rightarrow \lim_{x\to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}} = 8$$

10. $\lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2}$

Solution:

This question can be easily solved using LH rule that is L. Hospital's rule which is given below

$$\Rightarrow \text{if } \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ then } \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Given $\lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2}$

If we apply the limit we will get in determinant form

$$\Rightarrow \lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2} = \frac{0}{0}$$

So now we have to apply L. Hospital's rule

$$\Rightarrow \lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2} = \lim_{x\to 1} \frac{\frac{d}{dx}(x^7-2x^5+1)}{\frac{d}{dx}(x^3-3x^2+2)}$$

Now by differentiating we get

$$\Rightarrow \lim_{x\to 1} \frac{\frac{d}{dx}(x^7-2x^5+1)}{\frac{d}{dx}(x^3-3x^2+2)} = \lim_{x\to 1} \frac{7x^6-10x^4}{3x^2-6x}$$

Now by applying the limit we get

$$\Rightarrow \lim_{x\to 1} \frac{7x^6-10x^4}{3x^2-6x} = \frac{7-10}{3-6} = \frac{-3}{-3} = 1$$

$$\Rightarrow \lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2} = 1$$

11. $\lim_{x\to 0} \frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}$

Solution:

Given $\lim\limits_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}$

Now to rationalize the denominator by multiplying the given equation by its rationalizing factor we get

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2} = \lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}\times\left(\frac{\sqrt{1+x^3}+\sqrt{1-x^3}}{\sqrt{1+x^3}+\sqrt{1-x^3}}\right)$$

On simplifying the above equation we get

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}\times\left(\frac{\sqrt{1+x^3}+\sqrt{1-x^3}}{\sqrt{1+x^3}+\sqrt{1-x^3}}\right) = \lim_{x\to 0}\frac{(1+x^3)-(1-x^3)}{x^2\left(\sqrt{1+x^3}+\sqrt{1-x^3}\right)}$$

The above equation can be written as

$$\Rightarrow \lim_{x\to 0}\frac{(1+x^3)-(1-x^3)}{x^2\left(\sqrt{1+x^3}+\sqrt{1-x^3}\right)} = \lim_{x\to 0}\frac{2x^3}{x^2\left(\sqrt{1+x^3}+\sqrt{1-x^3}\right)} = \lim_{x\to 0}\frac{2x}{\left(\sqrt{1+x^3}+\sqrt{1-x^3}\right)}$$

Now by applying the limit we get

$$\Rightarrow \lim_{x\to 0}\frac{2x}{\left(\sqrt{1+x^3}+\sqrt{1-x^3}\right)} = 0$$

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2} = 0$$

14. Find 'n', if $\lim\limits_{x\to 2}\frac{x^n-2^n}{x-2}=80$, $n\in\mathbf{N}$

Solution:

Given $\lim\limits_{x\to 2}\frac{x^n-2^n}{x-2} = 80$

We know that $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$

By using this formula we get

$\Rightarrow n2^{n-1}=80 = 5\times 2^4=5\times 2^{5-1}$

$\Rightarrow$ n=5

16. $\lim\limits_{x\to 0}\frac{\sin^2 2x}{\sin^2 4x}$

Solution:

Given $\lim_{x\to 0} \frac{\sin^2 2x}{\sin^2 4x}$

Multiply and divide both numerator and denominator by $4x^2/16\,x^2$ then we get

$$\Rightarrow \lim_{x\to 0} \frac{\sin^2 2x}{\sin^2 4x} = \lim_{x\to 0} \frac{(\sin^2 2x)/4x^2}{(\sin^2 4x)/16x^2} \times \frac{4x^2}{16x^2}$$

On simplifying

$$\Rightarrow \lim_{x\to 0} \left[\frac{\left(\frac{\sin 2x}{2x}\right)^2}{\left(\frac{\sin 4x}{4x}\right)^2}\right] \times \frac{4}{16}$$

Now as $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\Rightarrow \lim_{x\to 0} \left[\frac{\left(\frac{\sin 2x}{2x}\right)^2}{\left(\frac{\sin 4x}{4x}\right)^2}\right] \times \frac{4}{16} = \frac{4}{16}$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin^2 2x}{\sin^2 4x} = \frac{4}{16}$$

$= ¼$

17. $\lim_{x\to 0} \frac{1-\cos 2x}{x^2}$

Solution:

Given $\lim_{x\to 0} \frac{1-\cos 2x}{x^2}$

Now by substituting the formula $\cos 2x = 1 - 2\sin^2 x$ we get

$$\Rightarrow \lim_{x\to 0} \frac{1-\cos 2x}{x^2} = \lim_{x\to 0} \frac{1-(1-2\sin^2 x)}{x^2}$$

The above equation can be written as

$$\Rightarrow \lim_{x\to 0} \frac{1-(1-2\sin^2 x)}{x^2} = \lim_{x\to 0} \frac{2\sin^2 x}{x^2}$$

By applying the limits we get

$$\Rightarrow \lim_{x\to 0} \frac{2\sin^2 x}{x^2} = 2\lim_{x\to 0} \frac{\sin^2 x}{x^2} = 2$$

$$\Rightarrow \lim_{x\to 0} \frac{1-\cos 2x}{x^2} = 2$$

18. $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^3}$

Solution:

Given $\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}$

We know that sin 2x = 2 sin x cos x, using this formula we get

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3} = \lim_{x\to 0}\frac{2\sin x-2\sin x\cos x}{x^3}$$

Again by taking 2 sin x common we get

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x-2\sin x\cos x}{x^3} = \lim_{x\to 0}\frac{2\sin x(1-\cos x)}{x^3}$$

Now we have cos x=1-2sin^2 (x/2)

Using this identity above equation can be written as

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x(1-\cos x)}{x^3} = \lim_{x\to 0}\frac{2\sin x(1-(1-2\sin^2\left(\frac{x}{2}\right))}{x^3}$$

On simplifying we get

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x(1-(1-2\sin^2\left(\frac{x}{2}\right))}{x^3} = \lim_{x\to 0}\frac{4\sin x\ \sin^2\left(\frac{x}{2}\right)}{x^3}$$

By splitting the above equation we get

$$\Rightarrow \lim_{x\to 0}\frac{4\sin x\ \sin^2\left(\frac{x}{2}\right)}{x^3} = \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2}$$

The above equation becomes

$$\Rightarrow \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2} = \lim_{x\to 0}\frac{\sin x}{x}\left[\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right]^2$$

Now as $\lim_{x\to 0}\frac{\sin x}{x} = 1$

By applying the limit we get

$$\Rightarrow \lim_{x\to 0}\frac{\sin x}{x}\left[\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right]^2 = 1.1^2 = 1$$

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3} = 1$$

19. $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$

Solution:

Given $\lim\limits_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$

Here cos mx can be written as

$\Rightarrow \cos mx = 1 - 2\sin^2\frac{mx}{2}$

And similarly

$\Rightarrow \cos nx = 1 - 2\sin^2\frac{nx}{2}$

Using these two identities in given equation we get

$$\Rightarrow \lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} = \lim_{x\to 0}\frac{\left[1-\left(1-2\sin^2\frac{mx}{2}\right)\right]}{\left[1-\left(1-2\sin^2\frac{nx}{2}\right)\right]}$$

On simplification we get

$$\Rightarrow \lim_{x\to 0}\frac{\left[1-\left(1-2\sin^2\frac{mx}{2}\right)\right]}{\left[1-\left(1-2\sin^2\frac{nx}{2}\right)\right]} = \lim_{x\to 0}\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{nx}{2}} = \lim_{x\to 0}\frac{\sin^2\frac{mx}{2}}{\sin^2\frac{nx}{2}}$$

Again by using trigonometric identity the above equation can be written as

$$\Rightarrow \lim_{x\to 0}\frac{\sin^2\frac{mx}{2}}{\sin^2\frac{nx}{2}} = \lim_{x\to 0}\frac{\left(\frac{m}{2}\right)^2\left[\frac{\sin^2\frac{mx}{2}}{\left(\frac{m}{2}\right)^2}\right]}{\left(\frac{n}{2}\right)^2\left[\frac{\sin^2\frac{nx}{2}}{\left(\frac{n}{2}\right)^2}\right]}$$

Taking common and simplifying we get

$$\Rightarrow \lim_{x\to 0}\frac{\left(\frac{m}{2}\right)^2\left[\frac{\sin^2\frac{mx}{2}}{\left(\frac{m}{2}\right)^2}\right]}{\left(\frac{n}{2}\right)^2\left[\frac{\sin^2\frac{nx}{2}}{\left(\frac{n}{2}\right)^2}\right]} = \frac{m^2}{n^2}\frac{\lim\limits_{x\to 0}\left[\frac{\sin^2\frac{mx}{2}}{\left(\frac{m}{2}\right)^2}\right]}{\lim\limits_{x\to 0}\left[\frac{\sin^2\frac{nx}{2}}{\left(\frac{n}{2}\right)^2}\right]}$$

Now as $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

$$\Rightarrow \frac{m^2}{n^2}\frac{\lim\limits_{x\to 0}\left[\frac{\sin^2\frac{mx}{2}}{\left(\frac{m}{2}\right)^2}\right]}{\lim\limits_{x\to 0}\left[\frac{\sin^2\frac{nx}{2}}{\left(\frac{n}{2}\right)^2}\right]} = \frac{m^2}{n^2}\frac{1}{1} = \frac{m^2}{n^2}$$

By applying the limits we get

$$\Rightarrow \lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} = \frac{m^2}{n^2}$$

20. $\lim\limits_{x\to\frac{\pi}{3}}\frac{\sqrt{1-\cos 6x}}{\sqrt{2}\ \ \frac{\pi}{3}-x}$

Solution:

Given $\lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$

Now by using the formula

$\cos 6x = 1 - 2\sin^2 3x$

Then the above equation becomes,

$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-(1-2\sin^2 3x)}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$

Again using sin 3x formula the above equation can be written as

$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-(1-2\sin^2 3x)}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{2}|\sin 3x|}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$

Now we have

$\sin 3x = \sin(\pi - 3x) = \sin 3\left(\frac{\pi}{3} - x\right)$

$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{2}|\sin 3x|}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x\to\frac{\pi}{3}} \frac{\sin 3\left(\frac{\pi}{3}-x\right)}{\left(\frac{\pi}{3}-x\right)}$

On simplifying we get

$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sin 3\left(\frac{\pi}{3}-x\right)}{\left(\frac{\pi}{3}-x\right)} = \lim_{x\to\frac{\pi}{3}} \frac{3\sin 3\left(\frac{\pi}{3}-x\right)}{3\left(\frac{\pi}{3}-x\right)} = 3\lim_{x\to\frac{\pi}{3}} \frac{\sin(\pi-3x)}{(\pi-3x)}$

Now as $\lim_{x\to 0} \frac{\sin x}{x} = 1$

By substituting the limit we get

$\Rightarrow 3\lim_{x\to\frac{\pi}{3}} \frac{\sin(\pi-3x)}{(\pi-3x)} = 3.1 = 3$

$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = 3$

21. $\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$

Solution:

Given $\lim_{x\to\frac{\pi}{4}} \frac{\sin x-\cos x}{x-\frac{\pi}{4}}$

We have

$\sin x - \cos x = \sqrt{2}\left(\frac{\sin x}{\sqrt{2}} - \frac{\cos x}{\sqrt{2}}\right) = \sqrt{2}\left(\sin x \cos\left(\frac{\pi}{4}\right) - \cos x \sin\left(\frac{\pi}{4}\right)\right)$

By using this formula in given equation we get

$\Rightarrow \sqrt{2}\left(\sin x \cos\left(\frac{\pi}{4}\right) - \cos x \sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2}\sin(x - \frac{\pi}{4})$

On simplification we get

$\Rightarrow \sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$

Now substituting these values in given equation we get

$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sin x-\cos x}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}}$

$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}} = \sqrt{2}\lim_{x\to\frac{\pi}{4}} \frac{\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}}$

Now as $\lim_{x\to 0} \frac{\sin x}{x} = 1$

And by substituting the limits we get

$\Rightarrow \sqrt{2}\lim_{x\to\frac{\pi}{4}} \frac{\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}} = \sqrt{2}.1 = \sqrt{2}$

$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sin x-\cos x}{x-\frac{\pi}{4}} = \sqrt{2}$

22. $\lim_{x\to\frac{\pi}{6}} \frac{\sqrt{3}\sin x - \cos x}{x - \frac{\pi}{6}}$

Solution:

Given $\lim_{x\to\frac{\pi}{6}} \frac{\sqrt{3}\sin x-\cos x}{x-\frac{\pi}{6}}$

Consider $\sqrt{3}\sin x - \cos x = 2\left(\frac{\sqrt{3}\sin x}{2} - \frac{\cos x}{2}\right) = 2\left(\sin x \cos\left(\frac{\pi}{6}\right) - \cos x \sin\left(\frac{\pi}{6}\right)\right)$

On simplification the above equation can be written as

$\Rightarrow 2\left(\sin x\cos\left(\frac{\pi}{6}\right)-\cos x\sin\left(\frac{\pi}{6}\right)\right)=2\sin(x-\frac{\pi}{6})$

$\Rightarrow \sqrt{3}\sin x-\cos x=2\sin(x-\frac{\pi}{6})$

Now by substituting these values in given equation we get

$\Rightarrow \lim_{x\to\frac{\pi}{6}}\frac{\sqrt{3}\sin x-\cos x}{x-\frac{\pi}{6}}=\lim_{x\to\frac{\pi}{6}}\frac{2\sin(x-\frac{\pi}{6})}{x-\frac{\pi}{6}}$

Taking constant term 2 outside

$\Rightarrow \lim_{x\to\frac{\pi}{6}}\frac{2\sin(x-\frac{\pi}{6})}{x-\frac{\pi}{6}}=2\lim_{x\to\frac{\pi}{6}}\frac{\sin(x-\frac{\pi}{6})}{x-\frac{\pi}{6}}$

Now as $\lim_{x\to 0}\ \frac{\sin x}{x}=1$

Now by applying the limit we get

$\Rightarrow 2\lim_{x\to\frac{\pi}{6}}\frac{\sin(x-\frac{\pi}{6})}{x-\frac{\pi}{6}}=2.1=2$

$\Rightarrow \lim_{x\to\frac{\pi}{6}}\frac{\sqrt{3}\sin x-\cos x}{x-\frac{\pi}{6}}$ =2

23. $\lim_{x\to 0}\frac{\sin 2x+3x}{2x+\tan 3x}$

Solution:

Given $\lim_{x\to 0}\frac{\sin 2x+3x}{2x+\tan 3x}$

Multiply and divide the numerator of given equation by 2x

$\Rightarrow \lim_{x\to 0}\frac{\sin 2x+3x}{2x+\tan 3x}=\lim_{x\to 0}\frac{2x(\sin 2x)/2x+3x}{2x+3x(\tan 3x)/3x}$

Now by splitting the limits we get

$\Rightarrow \lim_{x\to 0}\frac{2x(\sin 2x)/2x+3x}{2x+3x(\tan 3x)/3x}=\frac{\lim_{x\to 0}2x.\lim_{x\to 0}\left[\frac{\sin 2x}{2x}\right]+\lim_{x\to 0}3x}{\lim_{x\to 0}2x+\lim_{x\to 0}3x.\lim_{x\to 0}\left[\frac{\tan 3x}{3x}\right]}$

Now as $\lim_{x\to 0}\left[\frac{\tan 3x}{3x}\right]$ and $\lim_{x\to 0}\left[\frac{\sin 2x}{2x}\right]$ both will be 1.

Substituting these in above equation and simplifying we get

$\Rightarrow \frac{\lim_{x\to 0}2x.\lim_{x\to 0}\left[\frac{\sin 2x}{2x}\right]+\lim_{x\to 0}3x}{\lim_{x\to 0}2x+\lim_{x\to 0}3x.\lim_{x\to 0}\left[\frac{\tan 3x}{3x}\right]}=\frac{\lim_{x\to 0}2x.1+\lim_{x\to 0}3x}{\lim_{x\to 0}2x+\lim_{x\to 0}3x.1}=\lim_{x\to 0}\frac{2x+3x}{2x+3x}=\lim_{x\to 0}1=1$

$\Rightarrow \lim_{x\to 0}\frac{\sin 2x+3x}{2x+\tan 3x}$=1

$\Rightarrow \frac{\lim_{x\to 0}2x.\lim_{x\to 0}\left[\frac{\sin 2x}{2x}\right]+\lim_{x\to 0}3x}{\lim_{x\to 0}2x+\lim_{x\to 0}3x.\lim_{x\to 0}\left[\frac{\tan 3x}{3x}\right]}=\frac{\lim_{x\to 0}2x.1+\lim_{x\to 0}3x}{\lim_{x\to 0}2x+\lim_{x\to 0}3x.1}=\lim_{x\to 0}\frac{2x+3x}{2x+3x}=\lim_{x\to 0}1=1$

$\Rightarrow \lim_{x\to 0}\frac{\sin 2x+3x}{2x+\tan 3x}$=1

24. $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

Solution:

Given $\lim\limits_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

Now we have to rationalize the denominator by multiplying the dividing by its rationalizing factor then we get

$\Rightarrow \lim\limits_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} = \lim\limits_{x \to a} \left[\frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}\right]$

On simplifying and splitting the denominator we get

$\Rightarrow \lim\limits_{x \to a} \left[\frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}\right] = \lim\limits_{x \to a} \frac{\sin x - \sin a}{x - a} \lim\limits_{x \to a} (\sqrt{x} + \sqrt{a})$

Now as $\sin x - \sin a = 2\cos\left(\frac{x+a}{2}\right) \sin\left(\frac{x-a}{2}\right)$

Substituting this in above equation we get

$\Rightarrow \lim\limits_{x \to a} \frac{\sin x - \sin a}{x - a} \lim\limits_{x \to a} (\sqrt{x} + \sqrt{a}) = \lim\limits_{x \to a} \frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{x-a} \lim\limits_{x \to a} (\sqrt{x} + \sqrt{a})$

$\Rightarrow \lim\limits_{x \to a} \frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{x-a} \lim\limits_{x \to a} (\sqrt{x} + \sqrt{a}) = 2\sqrt{a} \lim\limits_{x \to a} \frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}} \lim\limits_{x \to a} \cos\left(\frac{x+a}{2}\right)$

Now as $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

Applying the limits in above equation we get

$\Rightarrow 2\sqrt{a} \lim\limits_{x \to a} \frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}} \lim\limits_{x \to a} \cos\left(\frac{x+a}{2}\right) = 2\sqrt{a}\,.\,1\,.\cos a$

$\Rightarrow \lim\limits_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} = 2\sqrt{a} \cos a$

25. $\lim\limits_{x \to \frac{\pi}{6}} \dfrac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

Solution:

Given $\lim_{x\to\frac{\pi}{6}} \frac{\cot^2 x-3}{\operatorname{cosec} x-2}$

We know that

$\cot^2 x = \operatorname{cosec}^2 x - 1$

By using this in given equation we get

$\Rightarrow \lim_{x\to\frac{\pi}{6}} \frac{(\operatorname{cosec}^2 x-1)-3}{\operatorname{cosec} x-2} = \lim_{x\to\frac{\pi}{6}} \frac{\operatorname{cosec}^2 x-4}{\operatorname{cosec} x-2}$

Again using $a^2 - b^2$ identity the above equation can be written as

$\Rightarrow \lim_{x\to\frac{\pi}{6}} \frac{\operatorname{cosec}^2 x-4}{\operatorname{cosec} x-2} = \lim_{x\to\frac{\pi}{6}} \frac{(\operatorname{cosec} x-2)(\operatorname{cosec}x+2)}{\operatorname{cosec} x-2}$

On simplification and applying the limits we get

$\Rightarrow \lim_{x\to\frac{\pi}{6}} \frac{(\operatorname{cosec} x-2)(\operatorname{cosec}x+2)}{\operatorname{cosec} x-2} = \lim_{x\to\frac{\pi}{6}}(\operatorname{cosec}x + 2) = 2 + 2 = 4$

$\Rightarrow \lim_{x\to\frac{\pi}{6}} \frac{\cot^2 x-3}{\operatorname{cosec} x-2} = 4$

26. $\lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

Solution:

Given $\lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

Multiply and divide the given equation by $\sqrt{2}-\sqrt{1+\cos x}$

Then we get

$\Rightarrow \lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} = \lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} \times \left(\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}\right)$

Now by splitting the limits we have

$\Rightarrow \lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} \times \left(\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}\right) = \lim_{x\to 0} \frac{2-(1+\cos x)}{\sin^2 x} \lim_{x\to 0} \left(\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}\right)$

Now $\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$

Substituting this in above equation

$\Rightarrow \lim_{x\to 0} \frac{2-(1+\cos x)}{\sin^2 x} \lim_{x\to 0} \left(\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}\right) = \frac{1}{2\sqrt{2}} \lim_{x\to 0} \frac{(1-\cos x)}{(1-\cos x)(1+\cos x)}$

Now by applying the limits we get

$\Rightarrow \frac{1}{2\sqrt{2}} \lim_{x\to 0} \frac{(1-\cos x)}{(1-\cos x)(1+\cos x)} = \frac{1}{2\sqrt{2}} \lim_{x\to 0} \frac{1}{(1+\cos x)} = \frac{1}{2\sqrt{2}} . \frac{1}{2}$

$\Rightarrow \lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} = \frac{1}{4\sqrt{2}}$

$\Rightarrow \frac{1}{2\sqrt{2}} \lim_{x\to 0} \frac{(1-\cos x)}{(1-\cos x)(1+\cos x)} = \frac{1}{2\sqrt{2}} \lim_{x\to 0} \frac{1}{(1+\cos x)} = \frac{1}{2\sqrt{2}} . \frac{1}{2}$

$\Rightarrow \lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} = \frac{1}{4\sqrt{2}}$

27. $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$

Solution:

Given $\lim_{x\to0} \frac{\sin x-2\sin 3x+\sin 5x}{x}$

Now by splitting the limits in above equation we get

$\Rightarrow \lim_{x\to0} \frac{\sin x-2\sin 3x+\sin 5x}{x} = \lim_{x\to0} \frac{\sin x}{x} - \lim_{x\to0} \frac{2\sin 3x}{x} + \lim_{x\to0} \frac{\sin 5x}{x}$

Taking constant term outside the limits we get

$\Rightarrow \lim_{x\to0} \frac{\sin x}{x} - \lim_{x\to0} \frac{2\sin 3x}{x} + \lim_{x\to0} \frac{\sin 5x}{x} = \lim_{x\to0} \frac{\sin x}{x} - 2(3)\lim_{x\to0} \frac{\sin 3x}{3x} + (5)\lim_{x\to0} \frac{\sin 5x}{5x}$

Now as $\lim_{x\to0} \frac{\sin x}{x} = 1$

By substituting and applying the limits we get

$\Rightarrow \lim_{x\to0} \frac{\sin x}{x} - 2(3)\lim_{x\to0} \frac{\sin 3x}{3x} + (5)\lim_{x\to0} \frac{\sin 5x}{5x} = 1 - 6 + 5 = 0$

$\Rightarrow \lim_{x\to0} \frac{\sin x-2\sin 3x+\sin 5x}{x} = 0$

Differentiate each of the functions with respect to *x* in Exercises 29 to 42.

29. $\frac{x^4 + x^3 + x^2 + 1}{x}$

Solution:

Let $y = \frac{x^4+x^3+x^2+1}{x}$

$\Rightarrow y = \frac{x^4+x^3+x^2+1}{x}$

Dividing by x we get

$\Rightarrow y = x^3 + x^2 + x + \frac{1}{x}$

Differentiating given equation with respect to x

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(x^3 + x^2 + x + \frac{1}{x}\right)$

On differentiation we get

$\Rightarrow \frac{d}{dx}\left(x^3 + x^2 + x + \frac{1}{x}\right) = 3x^2 + 2x + 1 - \frac{1}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{3x^4+2x^3+x^2-1}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(x^3 + x^2 + x + \frac{1}{x}\right)$

On differentiation we get

$\Rightarrow \frac{d}{dx}\left(x^3 + x^2 + x + \frac{1}{x}\right) = 3x^2 + 2x + 1 - \frac{1}{x^2}$

Hence, the required answer is $3x^2 + 2x + 1 - 1/x^2$.

30. $x+\frac{1}{x}\ ^{3}$

Solution:

Let $y=\left(x+\frac{1}{x}\right)^{3}$

Now differentiating y with respect to x we get

$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x+\frac{1}{x}\right)^{3}$

Expanding the equation using $(a + b)^3$ formula then we get

$=\frac{d}{dx}\left(x^{3}+\frac{1}{x^{3}}+3x+\frac{3}{x}\right)$

Splitting the differential we get

$=\frac{d}{dx}(x^{3})+\frac{d}{dx}\left(\frac{1}{x^{3}}\right)+\frac{d}{dx}(3x)+\frac{d}{dx}\left(\frac{3}{x}\right)$

On differentiating we get

$= 3x^2 - 3x^{-4} + 3 - 3x^{-2}$

$=3x^{2}-\frac{3}{x^{4}}+3-\frac{3}{x^{2}}$

31. (3x + 5) (1 + tan x)

Solution:

Given (3x + 5) (1 + tan x)

Let y = (3x + 5) (1 + tan x)

Applying product rule of differentiation that is

$\Rightarrow \frac{d}{dx}(t.y)=y.\frac{dt}{dx}+t.\frac{dy}{dx}$

$\Rightarrow y=(3x+5)(1+\tan x)$

$\Rightarrow \frac{dy}{dx}=(1+\tan x)\frac{d}{dx}(3x+5)+(3x+5)\frac{d}{dx}(1+\tan x)$

$\Rightarrow \frac{dy}{dx}=3(1+\tan x)+(3x+5)\sec^{2}x$

$\Rightarrow \frac{dy}{dx}=3x\sec^{2}x+5\sec^{2}x+3+3\tan x$ (by using product rule)

Hence, the required answer is $3x\sec^2 x + 5\sec^2 x + 3\tan x + 3$

32. (sec x – 1) (sec x + 1)

Solution:

Given (sec x – 1) (sec x + 1)

Let $y = (\sec x - 1)(\sec x + 1)$

The above equation can be written as

$\Rightarrow y = (\sec x - 1)(\sec x + 1) = \sec^2 x - 1 = \tan^2 x$

$\Rightarrow y = \tan^2 x$

Now applying the chain rule we get

$\Rightarrow \frac{dy}{dx} = \frac{d}{d(\tan x)}(\tan^2 x).\frac{d}{dx}(\tan x)$

$\Rightarrow \frac{dy}{dx} = 2\tan x \sec^2 x$

33. $\frac{3x+4}{5x^2-7x+9}$

Solution:

Given $y = \frac{3x+4}{5x^2-7x+9}$

Applying quotient rule of differentiation that is

$\Rightarrow \frac{d}{dx}\left(\frac{t}{y}\right) = \frac{y.\frac{dt}{dx} - t.\frac{dy}{dx}}{y^2}$

$\Rightarrow y = \frac{3x+4}{5x^2-7x+9}$

Applying the rule

$$\Rightarrow \frac{dy}{dx} = \frac{(5x^2 - 7x + 9)\frac{d}{dx}(3x+4) - (3x+4)\frac{d}{dx}(5x^2 - 7x + 9)}{(5x^2 - 7x + 9)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3(5x^2 - 7x + 9) - (3x+4)(10x - 7)}{(5x^2 - 7x + 9)^2}$$

On differentiation we get

$=(15x^2 - 21x + 27 - 30x^2 + 21x - 40x + 28)/ (5x^2 - 7x + 9)^2$

$= (-15x^2 - 40x + 55)/ (5x^2 - 7x + 9)^2$

$= (55 - 40x - 15x^2) / (5x^2 - 7x + 9)^2$

Hence, the required answer is

$(55 - 40x - 15x^2) / (5x^2 - 7x + 9)^2$

34. $\frac{x^5 - \cos x}{\sin x}$

Solution:

Given $y = \frac{x^5 - \cos x}{\sin x}$

$d/dx\ (x^5 - \cos x)/\sin x = [\sin x\ .\ d/dx(x^5 - \cos x) - (x^5 - \cos x).\ d/dx\ (\sin x)]/\ \sin^2 x$

By using quotient rule,

$= [\sin x\ (5x^4 + \sin x) - (x^5 - \cos x)\ (\cos x)]/\ \sin^2 x$

$= [5x^4\ .\ \sin x + \sin^2 x - x^5 \cos x + \cos^2 x]/\ \sin^2 x$

$= [5x^4 \sin x - x^5 \cos x + (\sin^2 + \cos^2 x)]/\ \sin^2 x$

$= [5x^4 \sin x - x^5 \cos x + 1]/\ \sin^2 x$

Hence, the required answer is $[5x^4 \sin x - x^5 \cos x + 1]/\ \sin^2 x$

36. (ax^2 + cot x) (p + q cos x)

Solution:

Given $y = (ax^2 + \cot x)(p + q\cos x)$

Applying product rule of differentiation that

$\Rightarrow \frac{d}{dx}(t.y) = y.\frac{dt}{dx} + t.\frac{dy}{dx}$

$\Rightarrow y = (ax^2 + \cot x)(p + q\cos x)$

Now splitting the differentials,

$\Rightarrow \frac{dy}{dx} = (p + q\cos x)\frac{d}{dx}(ax^2 + \cot x) + (ax^2 + \cot x)\frac{d}{dx}(p + q\cos x)$

On differentiation we get

$\Rightarrow \frac{dy}{dx} = (p + q\cos x)(2ax - \operatorname{cosec}^2 x) + (ax^2 + \cot x)(-q\sin x)$

$\Rightarrow \frac{dy}{dx} = (p + q\cos x)(2ax - \operatorname{cosec}^2 x) - q\sin x(ax^2 + \cot x)$

37. $\frac{a + b\sin x}{c + d\cos x}$

Solution:

Given $y = \frac{a+b\sin x}{c+d\cos x}$

Applying division rule or quotient rule of differentiation that is

$$\Rightarrow \frac{d}{dx}\left(\frac{t}{y}\right) = \frac{y.\frac{dt}{dx} - t.\frac{dy}{dx}}{y^2}$$

$$\Rightarrow y = \frac{a + b\sin x}{c + d\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(c + d\cos x)\frac{d}{dx}(a + b\sin x) - (a + b\sin x)\frac{d}{dx}(c + d\cos x)}{(c + d\cos x)^2}$$

On differentiating we get

$$\Rightarrow \frac{dy}{dx} = \frac{(c + d\cos x)(b\cos x) - (a + b\sin x)(-d\sin x)}{(c + d\cos x)^2}$$

$= [cb \cos x + bd \cos^2 x + ad \sin x + bd \sin^2 x] / (c + d\cos x)^2$

$= [cb \cos x + ad \sin x + bd (\cos^2 x + \sin^2 x)] / (c + d\cos x)^2$

$= [cb \cos x + ad \sin x + bd] / (c + d\cos x)^2$

39. $(2x - 7)^2 (3x + 5)^3$

Solution:

Given $y = (2x - 7)^2(3x + 5)^3$

Applying product rule of differentiation that is

$$\Rightarrow \frac{d}{dx}(t.y) = y.\frac{dt}{dx} + t.\frac{dy}{dx}$$

$$\Rightarrow y = (2x - 7)^2(3x + 5)^3$$

$$\Rightarrow \frac{dy}{dx} = (3x + 5)^3\frac{d}{dx}(2x - 7)^2 + (2x - 7)^2\frac{d}{dx}(3x + 5)^3$$

On differentiating we get

$$\Rightarrow \frac{dy}{dx} = (2)(3x + 5)^3 2(2x - 7)^1 + (3)(2x - 7)^2 3(3x + 5)^2$$

$$\Rightarrow \frac{dy}{dx} = 4(3x + 5)^3 (2x - 7) + 9(2x - 7)^2 (3x + 5)^2$$

On simplification we get

$$\Rightarrow \frac{dy}{dx} = (2x - 7)(3x + 5)^2[4(3x + 5) + 9(2x - 7)]$$

$$\Rightarrow \frac{dy}{dx} = (2x - 7)(3x + 5)^2(30x - 43)$$

40. $x^2 \sin x + \cos 2x$

Solution:

Applying product rule of differentiation for given equation
That is

$$\Rightarrow \frac{d}{dx}(t.y) = y.\frac{dt}{dx} + t.\frac{dy}{dx}$$

$$\Rightarrow y = x^2 \sin x + \cos 2x$$

$$\Rightarrow \frac{dy}{dx} = \sin x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos 2x)$$

On differentiating we get

$$\Rightarrow \frac{dy}{dx} = \sin x(2x) + x^2 \cos x + (-\sin 2x)(2)$$

$$\Rightarrow \frac{dy}{dx} = 2x\sin x + x^2 \cos x - 2\sin 2x$$