Exercise 4c

Question 1.
Prove that:

$$\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x, x<1$$

Answer:
To Prove: $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x$

Formula Used: $\tan\left(\frac{\pi}{4}+A\right)=\frac{1+\tan A}{1-\tan A}$

Proof:

LHS $=\tan^{-1}\left(\frac{1+x}{1-x}\right)$ ... (1)

Let x = tan A ... (2)

Substituting (2) in (1),

LHS $=\tan^{-1}\left(\frac{1+\tan A}{1-\tan A}\right)$

$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+A\right)\right)$

$=\frac{\pi}{4}+A$

From (2), A = $\tan^{-1}$ x,

$\frac{\pi}{4}+A=\frac{\pi}{4}+\tan^{-1}x$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 2.
Prove that:

$$\tan^{-1}x + \cot^{-1}(x+1) = \tan^{-1}(x^2+x+1)$$

Answer:
To Prove: $\tan^{-1} x + \cot^{-1} (x + 1) = \tan^{-1} (x^2 + x + 1)$

Formula Used:

1) $\cot^{-1}x = \tan^{-1}\frac{1}{x}$

2) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof:

LHS = $\tan^{-1} x + \cot^{-1} (x + 1)$ ... (1)

$$= \tan^{-1}x + \tan^{-1}\frac{1}{(x+1)}$$

$$= \tan^{-1}\left(\frac{x + \frac{1}{(x+1)}}{1 - \left(x \times \frac{1}{(x+1)}\right)}\right)$$

$$= \tan^{-1}\frac{x(x+1)+1}{x+1-x}$$

= $\tan^{-1} (x^2 + x + 1)$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 3.
Prove that:

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x, |x| \le \frac{1}{\sqrt{2}}.$$

Answer:
To Prove: $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1} x$

Formula Used: sin 2A = 2 × sin A × cos A

Proof:

LHS $= \sin^{-1}(2x\sqrt{1-x^2})$ ... (1)

Let x = sin A ... (2)

Substituting (2) in (1),

LHS $= \sin^{-1}(2 \sin A \sqrt{1-\sin^2 A})$

= $\sin^{-1}$ (2 × sin A × cos A)

= $\sin^{-1}$ (sin 2A)

= 2A

From (2), A = $\sin^{-1}$ x,

2A = 2 $\sin^{-1}$ x

= RHS

Therefore, LHS = RHS

Hence proved.

Question 4.
Prove that:

$$\sin^{-1}\left(3x - 4x^3\right) = 3\sin^{-1} x, |x| \leq \frac{1}{2}$$

Answer:
To Prove: $\sin^{-1}(3x - 4x^3) = 3\sin^{-1} x$

Formula Used: $\sin 3A = 3\sin A - 4\sin^3 A$

Proof:

LHS = $\sin^{-1}(3x - 4x^3)$ ... (1)

Let $x = \sin A$ ... (2)

Substituting (2) in (1),

LHS = $\sin^{-1}(3\sin A - 4\sin^3 A)$

= $\sin^{-1}(\sin 3A)$

= 3A

From (2), $A = \sin^{-1} x$,

$3A = 3\sin^{-1} x$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 5.

Prove that:

$$\cos^{-1}\left(4x^3 - 3x\right) = 3\cos^{-1} x, \frac{1}{2} \le x \le 1$$

Answer:
To Prove: $\cos^{-1} (4x^3 – 3x) = 3 \cos^{-1} x$

Formula Used: $\cos 3A = 4 \cos^3 A – 3 \cos A$

Proof:

LHS = $\cos^{-1} (4x^3 – 3x)$ ... (1)

Let $x = \cos A$ ... (2)

Substituting (2) in (1),

LHS = $\cos^{-1} (4 \cos^3 A – 3 \cos A)$

= $\cos^{-1} (\cos 3A)$

= 3A

From (2), $A = \cos^{-1} x$,

$3A = 3 \cos^{-1} x$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 6.
Prove that:

$$\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = 3\tan^{-1} x, |x| < \frac{1}{\sqrt{3}}$$

Answer:

To Prove: $tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = 3\,tan^{-1}x$

Formula Used: $tan\,3A = \frac{3\,tan\,A - tan^3 A}{1-3\,tan^2 A}$

Proof:

LHS $= tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ ... (1)

Let x = tan A ... (2)

Substituting (2) in (1),

LHS $= tan^{-1}\left(\frac{3\,tan\,A - tan^3 A}{1-3\,tan^2 A}\right)$

= $\tan^{-1}$ (tan 3A)

= 3A

From (2), A = $\tan^{-1}$ x,

3A = 3 $\tan^{-1}$ x

= RHS

Therefore, LHS = RHS

Hence proved.

Question 7.

Prove that:

$$\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Answer:

To Prove: $tan^{-1}x + tan^{-1}\left(\frac{2x}{1-x^2}\right) = tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Formula Used: $tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof:

LHS $= tan^{-1}x + tan^{-1}\left(\frac{2x}{1-x^2}\right)$ ... (1)

$$= tan^{-1}\left(\frac{x+\left(\frac{2x}{1-x^2}\right)}{1-\left(x\times\left(\frac{2x}{1-x^2}\right)\right)}\right)$$

$$= tan^{-1}\left(\frac{x(1-x^2)+2x}{1-x^2-2x^2}\right)$$

$$= tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 8.
Prove that:

$$\cos^{-1}\left(1-2x^2\right) = 2\sin^{-1}x$$

Answer:
To Prove: $\cos^{-1}(1 - 2x^2) = 2\sin^{-1}x$

Formula Used: $\cos 2A = 1 - 2\sin^2 A$

Proof:

LHS = $\cos^{-1}(1 - 2x^2)$ ... (1)

Let $x = \sin A$ ... (2)

Substituting (2) in (1),

LHS = $\cos^{-1}(1 - 2\sin^2 A)$

$= \cos^{-1}(\cos 2A)$

$= 2A$

From (2), $A = \sin^{-1} x$,

$2A = 2\sin^{-1} x$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 9.
Prove that:

$$\cos^{-1}\left(2x^2 - 1\right) = 2\cos^{-1} x$$

Answer:
To Prove: $\cos^{-1}(2x^2 - 1) = 2\cos^{-1} x$

Formula Used: $\cos 2A = 2\cos^2 A - 1$

Proof:

LHS = $\cos^{-1}(2x^2 - 1)$ ... (1)

Let $x = \cos A$ ... (2)

Substituting (2) in (1),

LHS = $\cos^{-1}(2\cos^2 A - 1)$

= $\cos^{-1}(\cos 2A)$

= 2A

From (2), A = $\cos^{-1} x$,

2A = $2\cos^{-1} x$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 10.
Prove that:

$$\sec^{-1}\left(\frac{1}{2x^2-1}\right) = 2\cos^{-1} x$$

Answer:
To Prove: $\sec^{-1}\left(\frac{1}{2x^2-1}\right) = 2\cos^{-1} x$

Formula Used:

1) $\cos 2A = 2\cos^2 A - 1$

2) $\cos^{-1} A = \sec^{-1}\left(\frac{1}{A}\right)$

Proof:

LHS $= \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

= $\cos^{-1}(2x^2 - 1)$... (1)

Let x = cos A ... (2)

Substituting (2) in (1),

LHS = $\cos^{-1}(2\cos^2 A - 1)$

= $\cos^{-1}(\cos 2A)$

= 2A

From (2), A = $\cos^{-1} x$,

2A = $2\cos^{-1} x$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 11.
Prove that:

$$\cot^{-1}\left(\sqrt{1+x^2}-x\right)=\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$$

Answer:
To Prove: $\cot^{-1}\left(\sqrt{1+x^2}-x\right)=\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$

Formula Used:

1) $\tan\left(\frac{\pi}{4}+A\right)=\frac{1+\tan A}{1-\tan A}$

2) $\operatorname{cosec}^2 A = 1 + \cot^2 A$

3) $1-\cos A = 2\sin^2\left(\frac{A}{2}\right)$

4) $\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$

Proof:

LHS $= \cot^{-1}\left(\sqrt{1+x^2}-x\right)$

Let x = cot A

LHS $= \cot^{-1}\left(\sqrt{1+\cot^2 A}-\cot A\right)$

= $\cot^{-1}$(cosec A – cot A)

$$= \cot^{-1}\left(\frac{1-\cos A}{\sin A}\right)$$

$$= \cot^{-1}\left(\frac{2\sin^2\left(\frac{A}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}\right)$$

$$= \cot^{-1}\left(\tan\left(\frac{A}{2}\right)\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\tan\left(\frac{A}{2}\right)\right)$$

$$= \frac{\pi}{2} - \frac{A}{2}$$

From (2), A = $\cot^{-1}$ x,

$$\frac{\pi}{2} - \frac{A}{2} = \frac{\pi}{2} - \frac{1}{2}\cot^{-1} x$$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 12.
Prove that:

$$\tan^{-1}\left(\frac{\sqrt{x}+\sqrt{y}}{1-\sqrt{xy}}\right)=\tan^{-1}\sqrt{x}+\tan^{-1}\sqrt{y}$$

Answer:

To Prove: $\tan^{-1}(\frac{\sqrt{x}+\sqrt{y}}{1-\sqrt{xy}}) = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{y}$

We know that, $\tan A + \tan B = \frac{\tan A+\tan B}{1-\tan A \tan B}$

Also, $\tan^{-1}(\frac{A+B}{1-AB}) = \tan^{-1} A + \tan^{-1} B$

Taking A = √x and B = √y

We get,

$$\tan^{-1}(\frac{\sqrt{x}+\sqrt{y}}{1-\sqrt{xy}}) = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{y}$$

Hence, Proved.

Question 13.
Prove that:

$$\tan^{-1}\left(\frac{x+\sqrt{x}}{1-x^{3/2}}\right)=\tan^{-1}x+\tan^{-1}\sqrt{x}$$

Answer:
We know that,

$$\tan^{-1}(\frac{A+B}{1-AB}) = \tan^{-1} A + \tan^{-1} B$$

Now, taking A = x and B = √x

We get,

$$\tan^{-1} x + \tan^{-1} \sqrt{x} = \tan^{-1}\left(\frac{x + \sqrt{x}}{1 - x^{3/2}}\right)$$

As, $x.x^{1/2} = x^{3/2}$

Hence, Proved.

Question 14.
Prove that:

$$\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right) = \frac{x}{2}$$

Answer:
To Prove: $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right) = \frac{x}{2}$

Formula Used:

1) $\sin A = 2 \times \sin\frac{A}{2} \times \cos\frac{A}{2}$

2) $1 + \cos A = 2\cos^2\frac{A}{2}$

Proof:

LHS $= \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

$$= \tan^{-1}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

= RHS

Therefore LHS = RHS

Hence proved.

Question 15.
Prove that:

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{3}{4}$$

Answer:
To Prove: $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{3}{4}$

Formula Used: $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof:

LHS = $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11}$

$$= \tan^{-1}\left(\frac{\frac{1}{2} + \frac{2}{11}}{1 - \left(\frac{1}{2} \times \frac{2}{11}\right)}\right)$$

$$= \tan^{-1}\left(\frac{11+4}{22-2}\right)$$

$$= \tan^{-1}\frac{15}{20}$$

$$= \tan^{-1}\frac{3}{4}$$

= RHS

Therefore LHS = RHS

Hence proved.

Question 16.
Prove that:

$$\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$$

Answer:
To Prove: $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

Formula Used: $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof:

LHS = $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$

$$=\tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\left(\frac{2}{11}\times\frac{7}{24}\right)}\right)$$

$$=\tan^{-1}\left(\frac{48+77}{264-14}\right)$$

$$=\tan^{-1}\frac{125}{250}$$

$$=\tan^{-1}\frac{1}{2}$$

= RHS

Therefore LHS = RHS

Hence proved.

Question 17.
Prove that:

$$\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{2}$$

Answer:
To Prove: $\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{2}$

Formula Used: $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy}\right)$

Proof:

LHS = $\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$

$$= \tan^{-1} 1 + \tan^{-1} \left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - \left(\frac{1}{2} \times \frac{1}{3}\right)}\right)$$

$$= \tan^{-1} 1 \ + \ \tan^{-1} \left(\frac{5}{6 - 1}\right)$$

$$= \tan^{-1} 1 \ + \ \tan^{-1} 1$$

$$= \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

= RHS

Therefore LHS = RHS

Hence proved.

Question 18.
Prove that:

$$2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$$

Answer:
To Prove: $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

Formula Used: $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof:

LHS $=\ 2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$

$$=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{3}}{1-\left(\frac{1}{3}\times\frac{1}{3}\right)}\right)+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left(\frac{6}{9-1}\right)+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\left(\frac{3}{4}\times\frac{1}{7}\right)}\right)$$

$$=\tan^{-1}\left(\frac{21+4}{28-3}\right)$$

$$=\tan^{-1}\frac{25}{25}$$

$= \tan^{-1} 1$

$$= \frac{\pi}{4}$$

= RHS

Therefore LHS = RHS

Hence proved.

Question 19.
Prove that:

$$\tan^{-1} 2 - \tan^{-1} 1 = \tan^{-1} \frac{1}{3}$$

Answer:
To Prove: $\tan^{-1} 2 - \tan^{-1} 1 = \tan^{-1} \frac{1}{3}$

Formula Used: $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy}\right)$ where $xy > -1$

Proof:

LHS = $\tan^{-1} 2 - \tan^{-1} 1$

$$= \tan^{-1} \left(\frac{2-1}{1+2}\right)$$

$$= \tan^{-1} \left(\frac{1}{3}\right)$$

= RHS

Therefore LHS = RHS

Hence proved.

Question 20.
Prove that:

$$\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$$

Answer:
To Prove: $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\ \pi$

Formula Used: $\tan^{-1}x+\tan^{-1}y=\pi+\ \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ where $xy>1$

Proof:

LHS $=\ \tan^{-1}1+\tan^{-1}2+\tan^{-1}3$

$=\ \frac{\pi}{4}+\ \pi+\tan^{-1}\left(\frac{2+3}{1-(2\times3)}\right)$ {since 2 × 3 = 6 > 1}

$=\dfrac{5\pi}{4}+\tan^{-1}\left(\dfrac{5}{-5}\right)$

$=\dfrac{5\pi}{4}+\tan^{-1}(-1)$

$=\dfrac{5\pi}{4}-\dfrac{\pi}{4}$

= π

= RHS

Therefore LHS = RHS

Hence proved.

Question 21.
Prove that:

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$$

Answer:

To Prove: $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Formula Used: $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ where $xy<1$

Proof:

LHS $=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}$

$$=\tan^{-1}\frac{1}{2}+\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\left(\frac{1}{5}\times\frac{1}{8}\right)}\right)$$

$$=\tan^{-1}\frac{1}{2}+\tan^{-1}\left(\frac{8+5}{40-1}\right)$$

$$=\tan^{-1}\frac{1}{2}+\tan^{-1}\left(\frac{13}{39}\right)$$

$$=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$$

$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\times\frac{1}{3}\right)}\right)$$

$$=\tan^{-1}\left(\frac{3+2}{6-1}\right)$$

$= \tan^{-1} 1$

$$=\frac{\pi}{4}$$

= RHS

Therefore LHS = RHS

Hence proved.

Question 22.
Prove that:

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\tan^{1}\frac{4}{3}$$

Answer:
To Prove: $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\tan^{-1}\frac{4}{3} \Rightarrow 2\left(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}\right)=\tan^{-1}\frac{4}{3}$

Formula Used: $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ where $xy<1$

Proof:

LHS $= 2(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9})$

$$=2\left(\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}\times\frac{2}{9}\right)}\right)\right)$$

$$=2\ \tan^{-1}\left(\frac{9+8}{36-2}\right)$$

$$=2\tan^{-1}\frac{17}{34}$$

$$=2\tan^{-1}\frac{1}{2}$$

$$=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{2}$$

$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{2}}{1-\left(\frac{1}{2}\times\frac{1}{2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{1}{\frac{4-1}{4}}\right)$$

$$= \tan^{-1}\frac{4}{3}$$

= RHS

Therefore LHS = RHS

Hence proved.

Question 23.
Prove that:

$$\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$$

Answer:
To Prove: $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Formula Used: $\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1-x^2} \times \sqrt{1-y^2})$

Proof:

LHS $= \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}$

$$= \cos^{-1}\left(\frac{4}{5} \times \frac{12}{13} - \sqrt{1-\left(\frac{4}{5}\right)^2} \times \sqrt{1-\left(\frac{12}{13}\right)^2}\right)$$

$$= \cos^{-1}\left(\frac{48}{65} - \sqrt{1-\frac{16}{25}} \times \sqrt{1-\frac{144}{169}}\right)$$

$$= \cos^{-1}\left(\frac{48}{65} - \left(\sqrt{\frac{25-16}{25}} \times \sqrt{\frac{169-144}{169}}\right)\right)$$

$$= \cos^{-1}\left(\frac{48}{65} - \left(\sqrt{\frac{9}{25}} \times \sqrt{\frac{25}{169}}\right)\right)$$

$$= \cos^{-1}\left(\frac{48}{65} - \frac{3}{13}\right)$$

$$= \cos^{-1}\left(\frac{48-15}{65}\right)$$

$$= \cos^{-1}\frac{33}{65}$$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 24.
Prove that:

$$\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}} = \frac{\pi}{2}$$

Answer:
To Prove: $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}} = \frac{\pi}{2}$

Formula Used: $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x \times \sqrt{1-y^2} + y \times \sqrt{1-x^2})$

Proof:

$\text{LHS} = \sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}}$

$$= \sin^{-1}\left(\frac{1}{\sqrt{5}} \times \sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2} + \frac{2}{\sqrt{5}} \times \sqrt{1-\left(\frac{1}{\sqrt{5}}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{5}} \times \sqrt{1-\frac{4}{5}} + \frac{2}{\sqrt{5}} \times \sqrt{1-\frac{1}{5}}\right)$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} \times \frac{2}{\sqrt{5}}\right)$$

$$= \sin^{-1}\left(\frac{1}{5} + \frac{4}{5}\right)$$

$$= \sin^{-1}\frac{5}{5}$$

$= \sin^{-1} 1$

$$= \frac{\pi}{2}$$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 25.
Prove that:

$$\cos^{-1}\frac{3}{5} + \sin^{-1}\frac{12}{13} = \sin^{-1}\frac{56}{65}$$

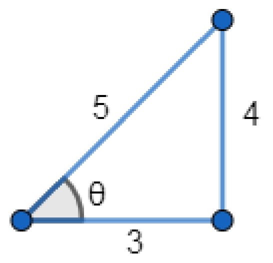
Answer:
To Prove: $\cos^{-1}\frac{3}{5} + \sin^{-1}\frac{12}{13} = \sin^{-1}\frac{56}{65}$

Formula Used: $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x \times \sqrt{1-y^2} + y \times \sqrt{1-x^2})$

Proof:

LHS $= \cos^{-1}\frac{3}{5} + \sin^{-1}\frac{12}{13}$ ... (1)

Let $\cos\theta = \frac{3}{5}$



Therefore $\theta = \cos^{-1}\frac{3}{5}$ ... (2)

From the figure, $\sin\theta = \frac{4}{5}$

$\Rightarrow \theta = \sin^{-1}\frac{4}{5}$ ... (3)

From (2) and (3),

$$\cos^{-1}\frac{3}{5} = \sin^{-1}\frac{4}{5}$$

Substituting in (1), we get

LHS $= \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{12}{13}$

$$= \sin^{-1}\left(\frac{4}{5} \times \sqrt{1 - \left(\frac{12}{13}\right)^2} + \frac{12}{13} \times \sqrt{1 - \left(\frac{4}{5}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{4}{5} \times \sqrt{1 - \frac{144}{169}} + \frac{12}{13} \times \sqrt{1 - \frac{16}{25}}\right)$$

$$= \sin^{-1}\left(\frac{4}{5} \times \sqrt{\frac{25}{169}} + \frac{12}{13} \times \sqrt{\frac{9}{25}}\right)$$

$$= \sin^{-1}\left(\frac{4}{5} \times \frac{5}{13} + \frac{12}{13} \times \frac{3}{5}\right)$$

$$= \sin^{-1}\left(\frac{20}{65} + \frac{36}{65}\right)$$

$$= \sin^{-1}\frac{56}{65}$$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 26.
Prove that:

$$\cos^{-1}\frac{4}{5} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{27}{11}$$

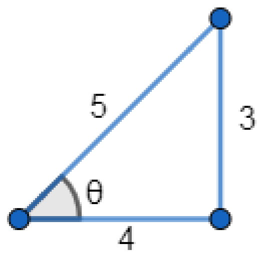
Answer:
To Prove: $\cos^{-1}\frac{4}{5} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{27}{11}$

Formula Used: $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x \times \sqrt{1-y^2} + y \times \sqrt{1-x^2})$

Proof:

LHS $= \cos^{-1}\frac{4}{5} + \sin^{-1}\frac{3}{5}$ ... (1)

Let $\cos\theta = \frac{4}{5}$



Therefore $\theta = \cos^{-1}\frac{4}{5}$ ... (2)

From the figure, $\sin\theta = \frac{3}{5}$

$\Rightarrow \theta = \sin^{-1}\frac{3}{5}$ ... (3)

From (2) and (3),

$$\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{3}{5}$$

Substituting in (1), we get

LHS $= \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{3}{5}$

$$= \sin^{-1}\left(2 \times \frac{3}{5} \times \sqrt{1 - \left(\frac{3}{5}\right)^2}\right)$$

$$= \sin^{-1}\left(2 \times \frac{3}{5} \times \sqrt{1 - \frac{9}{25}}\right)$$

$$= \sin^{-1}\left(2 \times \frac{3}{5} \times \sqrt{\frac{16}{25}}\right)$$

$$= \sin^{-1}\left(2 \times \frac{3}{5} \times \frac{4}{5}\right)$$

$$= \sin^{-1}\frac{24}{25}$$

Question 27.
Prove that:

$$\tan^{-1}\frac{1}{3}+\sec^{-1}\frac{\sqrt{5}}{2}=\frac{\pi}{4}$$

Answer:
To Prove: $\tan^{-1}\frac{1}{3}+\sec^{-1}\frac{\sqrt{5}}{2}=\frac{\pi}{4}$

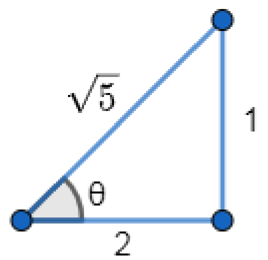
Formula Used: $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ where $xy<1$

Proof:

LHS $=\tan^{-1}\frac{1}{3}+\sec^{-1}\frac{\sqrt{5}}{2}$ ... (1)

Let $\sec\theta=\frac{\sqrt{5}}{2}$

Therefore $\theta=\sec^{-1}\frac{\sqrt{5}}{2}$ ... (2)



From the figure, $\tan\theta=\frac{1}{2}$

$\Rightarrow \theta=\tan^{-1}\frac{1}{2}$ ... (3)

From (2) and (3),

$$\sec^{-1}\frac{\sqrt{5}}{2}=\tan^{-1}\frac{1}{2}$$

Substituting in (1), we get

LHS $= \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$

$$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\times\frac{1}{2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{2+3}{6-1}\right)$$

$$= \tan^{-1}\frac{5}{5}$$

$= \tan^{-1} 1$

$$= \frac{\pi}{4}$$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 28.
Prove that:

$$\sin^{-1}\frac{1}{\sqrt{17}} + \cos^{-1}\frac{9}{\sqrt{85}} = \tan^{-1}\frac{1}{2}$$

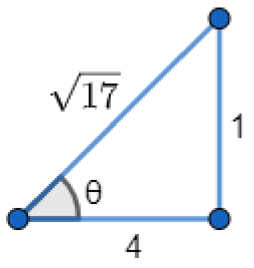
Answer:
To Prove: $\sin^{-1}\frac{1}{\sqrt{17}} + \cos^{-1}\frac{9}{\sqrt{85}} = \tan^{-1}\frac{1}{2}$

Formula Used: $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ where $xy < 1$

Proof:

LHS $= \sin^{-1}\frac{1}{\sqrt{17}} + \cos^{-1}\frac{9}{\sqrt{85}}$ ... (1)

Let $\sin\theta = \frac{1}{\sqrt{17}}$



Therefore $\theta = \sin^{-1}\frac{1}{\sqrt{17}}$ ... (2)

From the figure, $\tan\theta = \frac{1}{4}$

$\Rightarrow \theta = \tan^{-1}\frac{1}{4}$ ... (3)

From (2) and (3),

$\sin^{-1}\frac{1}{\sqrt{17}} = \tan^{-1}\frac{1}{4}$ ... (3)

Now, let $\cos\theta = \frac{9}{\sqrt{85}}$

Therefore $\theta = \cos^{-1}\frac{9}{\sqrt{85}}$ ... (4)

From the figure, $\tan\theta = \frac{2}{9}$

$\Rightarrow \theta = \tan^{-1}\frac{2}{9}$ ... (5)

From (4) and (5),

$\cos^{-1}\frac{9}{\sqrt{85}} = \tan^{-1}\frac{2}{9}$ ... (6)

Substituting (3) and (6) in (1), we get

LHS $= \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}$

$= \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}\times\frac{2}{9}\right)}\right)$

$= \tan^{-1}\left(\frac{9+8}{36-2}\right)$

$= \tan^{-1}\frac{17}{34}$

$= \tan^{-1}\frac{1}{2}$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 29.
Prove that:

$$2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$$

Answer:
To Prove: $2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

Formula Used:

1) $2\sin^{-1}x = \sin^{-1}(2x \times \sqrt{1-x^2})$

2) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ where $xy < 1$

Proof:

LHS $= 2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31}$ ... (1)

$$2\sin^{-1}\frac{3}{5} = \sin^{-1}\left(2 \times \frac{3}{5} \times \sqrt{1-\left(\frac{3}{5}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{6}{5} \times \frac{4}{5}\right)$$

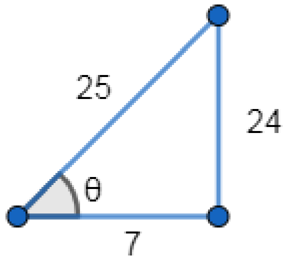
$= \sin^{-1}\frac{24}{25}$ ... (2)

Substituting (2) in (1), we get

LHS $= \sin^{-1}\frac{24}{25} - \tan^{-1}\frac{17}{31}$ ... (3)

Let $\sin\theta = \frac{24}{25}$

Therefore $\theta = \sin^{-1}\frac{24}{25}$ ... (4)



From the figure, $\tan\theta = \frac{24}{7}$

$\Rightarrow \theta = \tan^{-1}\frac{24}{7}$ ... (5)

From (4) and (5),

$\sin^{-1}\frac{24}{25} = \tan^{-1}\frac{24}{7}$ ... (6)

Substituting (6) in (3), we get

LHS $= \tan^{-1}\frac{24}{7} - \tan^{-1}\frac{17}{31}$

$= \tan^{-1}\left(\frac{\frac{24}{7} - \frac{17}{31}}{1 + \left(\frac{24}{7} \times \frac{17}{31}\right)}\right)$

$= \tan^{-1}\left(\frac{744 - 119}{217 + 408}\right)$

$= \tan^{-1}\frac{625}{625}$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$

= RHS

Therefore, LHS = RHS

Hence proved.

Question 30.
Solve for x:

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$$

Answer:
To find: value of x

Formula Used: $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ where $xy < 1$

Given: $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$

LHS $= \tan^{-1}\left(\frac{x+1+x-1}{1-\{(x+1)\times(x-1)\}}\right)$

$$= \tan^{-1}\frac{2x}{1-(x^2-x+x-1)}$$

$$= \tan^{-1}\frac{2x}{2-x^2}$$

Therefore, $\tan^{-1}\frac{2x}{2-x^2} = \tan^{-1}\frac{8}{31}$

Taking tangent on both sides, we get

$$\frac{2x}{2-x^2} = \frac{8}{31}$$

$\Rightarrow 62x = 16 - 8x^2$

$\Rightarrow 8x^2 + 62x - 16 = 0$

$\Rightarrow 4x^2 + 31x - 8 = 0$

$\Rightarrow 4x^2 + 32x - x - 8 = 0$

$\Rightarrow 4x \times (x + 8) - 1 \times (x + 8) = 0$

$\Rightarrow (4x - 1) \times (x + 8) = 0$

$$\Rightarrow x = \frac{1}{4} \text{ or } x = -8$$

Therefore, $x = \frac{1}{4}$ or $x = -8$ are the required values of x.

Question 31.
Solve for x:

$$\cos\left(\sin^{-1}x\right) = \frac{1}{9}$$

Answer:

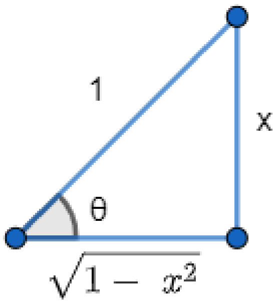
To find: value of x

Given: $\cos(\sin^{-1} x) = \frac{1}{9}$

LHS = $\cos(\sin^{-1} x)$ ... (1)

Let $\sin \theta = x$

Therefore $\theta = \sin^{-1} x$ ... (2)



From the figure, $\cos\theta = \sqrt{1-x^2}$

$\Rightarrow \theta = \cos^{-1}\sqrt{1-x^2}$ ... (3)

From (2) and (3),

$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$ ... (4)

Substituting (4) in (1), we get

LHS $= \cos(\cos^{-1}\sqrt{1-x^2})$

$= \sqrt{1-x^2}$

Therefore, $\sqrt{1-x^2} = \frac{1}{9}$

Squaring and simplifying,

$\Rightarrow 81 - 81x^2 = 1$

$\Rightarrow 81x^2 = 80$

$$\Rightarrow x^2 = \frac{80}{81}$$

$$\Rightarrow x = \pm\frac{4\sqrt{5}}{9}$$

Therefore, $x = \pm\frac{4\sqrt{5}}{9}$ are the required values of x.

Question 32.
Solve for x:

$$\cos\left(2\sin^{-1}x\right) = \frac{1}{9}$$

Answer:
To find: value of x

Formula Used: $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

Given: $\cos(2\sin^{-1}x) = \frac{1}{9}$

LHS = $\cos(2\sin^{-1} x)$

Let $\theta = \sin^{-1} x$

So, $x = \sin\theta$ ... (1)

LHS = $\cos(2\theta)$

= $1 - 2\sin^2\theta$

Substituting in the given equation,

$$1 - 2\sin^2\theta = \frac{1}{9}$$

$$2\sin^2\theta = \frac{8}{9}$$

$$\sin^2\theta = \frac{4}{9}$$

Substituting in (1),

$$x^2 = \frac{4}{9}$$

$$x = \pm\frac{2}{3}$$

Therefore, $x = \pm\frac{2}{3}$ are the required values of x.

Question 33.
Solve for x:

$$\sin^{-1}\frac{8}{x} + \sin^{-1}\frac{15}{x} = \frac{\pi}{2}$$

Answer:
To find: value of x

Given: $\sin^{-1}\frac{8}{x} + \sin^{-1}\frac{15}{x} = \frac{\pi}{2}$

We know $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Let $\sin^{-1}\frac{8}{x} = P$

$$\Rightarrow \sin P = \frac{8}{x}$$

Therefore, $\cos P = \frac{\sqrt{x^2-64}}{x}$

$$P = \cos^{-1}\frac{\sqrt{x^2-64}}{x}$$

$$\cos^{-1}\frac{\sqrt{x^2-64}}{x}+\sin^{-1}\frac{15}{x}=\frac{\pi}{2}$$

Therefore, $\frac{\sqrt{x^2-64}}{x}=\frac{15}{x}$

$\Rightarrow\ \sqrt{x^2-64}=15$

Squaring both sides,

$\Rightarrow x^2 - 64 = 225$

$\Rightarrow x^2 = 289$

$\Rightarrow x = \pm 17$

Therefore, $x = \pm 17$ are the required values of x.

Question 34.
Solve for x :

$$\cos\left(\sin^{-1}x\right)=\frac{1}{2}$$

Answer:
To find: value of x

Given: $\cos(\sin^{-1}x)=\frac{1}{2}$

LHS $=\cos(\sin^{-1}x)$

$=\cos(\cos^{-1}(\sqrt{1-x^2}))$

$=\ \sqrt{1-x^2}$

Therefore, $\sqrt{1-x^2}=\frac{1}{2}$

Squaring both sides,

$$1 - x^2 = \frac{1}{4}$$

$$x^2 = 1 - \frac{1}{4}$$

$$x^2 = \frac{3}{4}$$

$$x = \pm\frac{\sqrt{3}}{2}$$

Therefore, $x = \pm\frac{\sqrt{3}}{2}$ are the required values of x.

Question 35.
Solve for x :

$$\tan^{-1} x = \sin^{-1}\frac{1}{\sqrt{2}}$$

Answer:
To find: value of x

Given: $\tan^{-1} x = \sin^{-1}\frac{1}{\sqrt{2}}$

We know that $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Therefore, $\frac{\pi}{4} = \sin^{-1}\frac{1}{\sqrt{2}}$

Substituting in the given equation,

$$\tan^{-1} x = \frac{\pi}{4}$$

$$x = \tan\frac{\pi}{4}$$

$\Rightarrow$ x = 1

Therefore, x = 1 is the required value of x.

Question 36.
Solve for x :

$$\sin^{-1} x - \cos^{-1} x = \frac{\pi}{6}$$

Answer:
Given: $\sin^{-1} x - \cos^{-1} x = \frac{\pi}{6}$

We know that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

So, $\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$

Substituting in the given equation,

$$\frac{\pi}{2} - \cos^{-1} x - \cos^{-1} x = \frac{\pi}{6}$$

Rearranging,

$$2 \cos^{-1} x = \frac{\pi}{2} - \frac{\pi}{6}$$

$$2 \cos^{-1} x = \frac{\pi}{3}$$

$$\cos^{-1} x = \frac{\pi}{6}$$

$$x = \frac{\sqrt{3}}{2}$$

Therefore, $x = \frac{\sqrt{3}}{2}$ is the required value of x.