

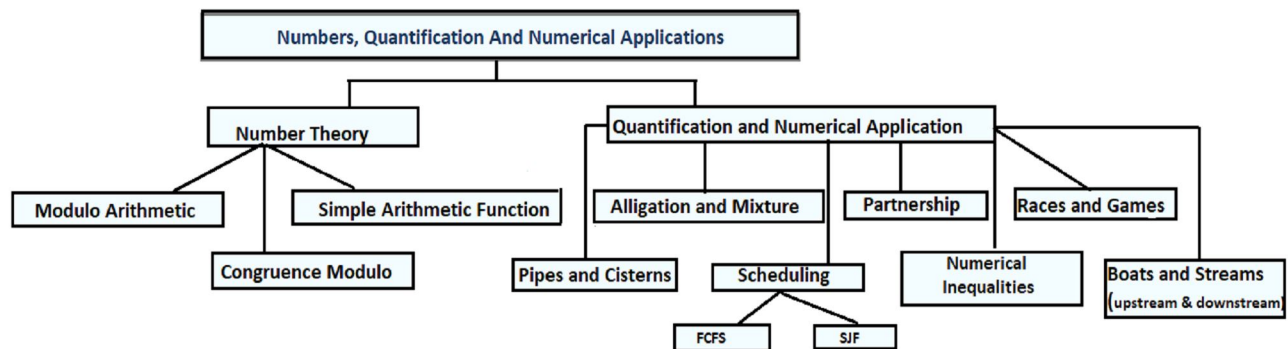
Numbers, Quantification and Numerical Applications

1.0 LEARNING OUTCOMES

After completion of this unit the students will be able to

- ❖ Apply arithmetic operations using modular arithmetic rules
- ❖ Apply the definition of congruence modulo in various problems
- ❖ Enlist Simple Arithmetic functions
- ❖ Apply the rule of alligation, to produce a mixture at a given price
- ❖ Determine the mean price of a mixture
- ❖ Express the problem of upstream and downstream in the form of an equation
- ❖ Differentiate between active partner and sleeping partner
- ❖ Determine the gain or loss to be divided among the partners in the ratio of their investment with due consideration of time
- ❖ Determine the time taken by two or more pipes to fill or empty the tank
- ❖ Compare the performance of two players w.r.t. time, distance
- ❖ Differentiate between First Come First Serve (FCFS) & Shortest Job First(SJF)
- ❖ Describe the basic concepts of numerical inequalities
- ❖ Demonstrate understanding and ability to write numerical inequality

CONCEPT MAP:



INTRODUCTION:

Recall how we classify a number as either odd or even, but not both. Consider a set of ten consecutive positive integers, and classify them as even or odd integers. This is classification of the integers into two “sets” as the numbers selected are either even or odd.

Above statement can also be understood as dividing the numbers into two classes on the basis of their divisibility by 2, i.e. if the number leaves the remainder **zero** then it is classified as an even number and if it is leaving the remainder as 1 then it is classified as an odd number.

Thus, we are only dealing with remainders as the basis to classify the numbers as even or odd. Let us explore the concept of classification in the following examples.

Illustration1: If today is a Wednesday then what will be the day after 68 days?

A week is cyclic in nature and repeat itself after every 7 days.

When we divide 68 by 7, the remainder is 5

$\Rightarrow 68 \text{ days} = 9 \text{ weeks and } 5 \text{ days}$

Thus after 5 days from Wednesday it will be Monday.

Thus, the day after 68 days from today will be same as 5 days after Wednesday.

Hence, 68th day from today (Wednesday) will be Monday.

1.1 MODULAR ARITHMETIC:

A system of arithmetic for integers, which deals with remainders only, is known as **Modular Arithmetic**.

We have learnt Euclid’s division algorithm in previous classes. It states that:

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

Given two positive integers X and Y such that $X > Y$ and $Y \neq 0$ then

$$\begin{aligned} X &= Y \times Q + R \\ \text{Where, } X &= \text{Dividend} \\ Y &= \text{Divisor} \\ Q &= \text{Quotient} \\ R &= \text{Remainder, } 0 \leq R < Y \end{aligned}$$

MODULO OPERATOR:

Given two integers X and Y ; $X > Y$ and $Y \neq 0$

$$X = Y \times Q + R$$

where $0 \leq R < Y$

Then there exist two unique integers Q and R such that

The same expression can be written as:

$$X \bmod Y = R \quad \dots(i)$$



Where *mod* (modulo) gives the remainder after X is divided by Y

For example: $29 \bmod 3 = 2$ (when 29 is divided by 3 it leaves remainder 2)

Consider a 12-hour display wall clock. Here, 13 O'clock in the afternoon is displayed as 1 O'clock as $13 = 12 \times 1 + 1$

Or we can say, $13 \bmod 12 = 1$

Basically, in such cases, it does not matter how many times hour hand goes around, but where it ends up!

Modulo operator is a mathematical operator which gives the value of remainder when a positive integer X is divided by another positive integer Y . It is written as "modulo" or "mod" and is represented as equation (i)

Note that X , Y and Q can either be positive, negative or even 0 (if we exclude Y) whereas R can only be non-negative.

Let's understand the condition(s) with the help of examples:

Cases		
If $X > 0$, $Y > 0$	$68 \bmod 12 = 8$	$R = 8 > 0$
If $X < 0$, $Y > 0$	$-68 \bmod 12 = 4$	$R = 4 > 0$
If $X > 0$, $Y < 0$	$68 \bmod (-12) = 8$	$R = 8 > 0$
If $X < 0$, $Y < 0$:	$-68 \bmod (-12) = 4$	$R = 4 > 0$

- When $X = Y$, then
 $X \bmod Y = 0$
For example:
 $3 \bmod 3 = 0$
- When $X < Y$, then
 $X \bmod Y = X$
For example:
 $7 \bmod 13 = 7$

Example 1

Find

(i) $21 \bmod 8$	(ii) $-31 \bmod 7$
(iii) $18 \bmod 18$	(iv) $7 \bmod 12$

- Solution:**
- (i) $21 = 8 \times 2 + 5$; remainder = 5
 $21 \bmod 8 = 5$
 - (ii) $-31 = 7 \times (-5) + 4$
 $-31 \bmod 7 = 4$
 - (iii) $18 \bmod 18 = 0$
 - (iv) $7 \bmod 12 = 7$

PROPERTIES OF MODULO OPERATOR:

Property 1: If the value of X is increased by a multiple of Y (say, kY , where k is any integer) then
 $X \bmod Y = (X + kY) \bmod Y$; where k is any integer

Example 2

If $X = 7$ and $Y = 3$, then verify that $X \bmod Y = (X + kY) \bmod Y$ for $k = 6$

Solution : For given $X = 7$ and $Y = 3$

$$X \bmod Y = 7 \bmod 3 = 1$$

$$\text{Also, } (X + kY) \bmod Y = (7 + 6 \times 3) \bmod 3 = 25 \bmod 3 = 1$$

$$\text{As } 25 = 8 \times 3 + 1$$

$$\text{Therefore, } 7 \bmod 3 = 25 \bmod 3 = 1$$

Hence verified.

Illustration 2: Use information of table 1 to complete Table 2:

X	Y	$X \bmod Y$
18	5	3

Table 1

k	kY	$X + kY$	$(X + kY) \bmod Y$	Observation
4	20			Compare the value of $X \bmod Y$ with $(X + kY) \bmod Y$
7	35	53	3	
8		40		
-2	-10	8	3	
-4				
12	60			

Table 2

Property 2: Addition property of modulo arithmetic is if A , B and C are positive integers then,
 $(A + B) \bmod C = (A \bmod C + B \bmod C) \bmod C$

Example 3

Verify that $(14 + 8) \bmod 5 = (14 \bmod 5 + 8 \bmod 5) \bmod 5$

Solution: **L.H.S.** $(14 + 8) \bmod 5 = 22 \bmod 5 = 2$

R.H.S. $(14 \bmod 5 + 8 \bmod 5) \bmod 5 = (4 + 3) \bmod 5 = 7 \bmod 5 = 2$

L.H.S. = R.H.S. Hence verified

Illustration 3: Complete the following table, where A , B and C are positive integers:

A	B	C	$(A+B) \bmod C=Y$	$A \bmod C$	$B \bmod C$	$Z = A \bmod C + B \bmod C$	$Z \bmod C$
17	25	4	$42 \bmod 4 = 2$	1	1	2	2
84	37	9					
39	34	17					
171	245	3					

Table 3

Property 3: Subtraction property of modulo arithmetic is if A , B and C are positive integers then,
 $(A - B) \bmod C = (A \bmod C - B \bmod C) \bmod C$

Illustration 4: Complete the following table, If A , B and C are positive integers:

A	B	C	$(A - B) \bmod C=Y$	$A \bmod C$	$B \bmod C$	$Z = A \bmod C - B \bmod C$	$Z \bmod C$
37	25	4	$12 \bmod 4 = 0$	1	1	0	0
84	37	9	$47 \bmod 9 = 2$	3	1	2	2
39	34	17					
245	171	3					

Table 4

Property 4: Multiplication property of modulo arithmetic is if A , B and C are positive integers then,
 $(A \times B) \bmod C = (A \bmod C \times B \bmod C) \bmod C$

Illustration 5 : Complete the following table, If A , B and C are positive integers:

A	B	C	$(A \times B) \bmod C=Y$	$A \bmod C$	$B \bmod C$	$Z = A \bmod C \times B \bmod C$	$Z \bmod C$
17	25	4	$425 \bmod 4 = 1$	1	1	1	1
84	37	9					
39	34	17					
171	245	3					

Table 5

Example 4

Verify $(14 \times 8) \bmod 5 = (14 \bmod 5 \times 8 \bmod 5) \bmod 5$

Solution: **L.H.S.** $(14 \times 8) \bmod 5 = 112 \bmod 5 = 2$

R.H.S. $(14 \bmod 5 \times 8 \bmod 5) \bmod 5 = (4 \times 3) \bmod 5 = 12 \bmod 5 = 2$

L.H.S. = R.H.S. Hence verified

Example 5

Find the remainder when $(127 \times 137 \times 23 \times 50 \times 235 \times 15)$ is divided by 7.

Solution: As we know that $((A \times B) \bmod C = (A \bmod C \times B \bmod C) \bmod C) \dots (i)$

Therefore, for calculating $(127 \times 137 \times 23 \times 50 \times 235 \times 15) \bmod 7$, let us find

$$127 \bmod 7 = 1$$

$$137 \bmod 7 = 4$$

$$23 \bmod 7 = 2$$

$$50 \bmod 7 = 1$$

$$235 \bmod 7 = 4$$

$$15 \bmod 7 = 1$$

using equation (i) repeatedly we have:

$$(127 \times 137 \times 23 \times 50 \times 235 \times 15) \bmod 7 = (1 \times 4 \times 2 \times 1 \times 4 \times 1) \bmod 7$$

$$= 32 \bmod 7 = 4$$

Hence remainder is 4

ADDITION MODULO m :

Given two positive integers a and b , the addition modulo m denoted as $+_m$ (where m is any positive integer) is defined for set of integers as:

$$a +_m b = \begin{cases} (a + b) \bmod m, & \text{if } (a + b) \geq m \\ a + b, & \text{if } (a + b) < m \end{cases}$$

Note: $a +_m b$ is read as addition modulo m to $(a+b)$

Example 6

(i) Find addition modulo 8 if a and b are 3 and 11 respectively.

(ii) Find addition modulo 25 where a and b are 15 and 6 respectively.

(iii) Apply addition modulo to positive integers 17 and 13 for modulo 30.

Solution: (i) Given $a = 3$, $b = 11$, $m = 8$

$$\text{Hence } a +_m b = 3 +_8 11 = 6 \quad \text{as } a + b = 14 > 8$$

(ii) Here $a = 15$, $b = 6$, $m = 25$

$$\text{Therefore } a +_m b = 15 +_{25} 6 = 21 \quad (\text{as } a + b = 21 < 25)$$

Solution: Let $a = 17$, $b = 13$ and $m = 30$

$$a +_m b = 17 +_{30} 13 = 0 \quad (\text{as } a + b = 17 + 13 = 30; 30 \bmod 30 = 0)$$

SUBTRACTION MODULO m :

Given two positive integers a and b , the subtraction modulo m denoted as ' $-_m$ ' (where m is any positive integer) is defined for set of integers as:

$$a -_m b = \begin{cases} (a - b) \bmod m, & \text{if } (a - b) \geq m \\ a - b, & \text{if } (a - b) < m \end{cases}$$

Note: $a -_m b$ is read as subtraction modulo m to $(a - b)$

Example 7

- (i) Find subtraction modulo 8 if a and b are 11 and 3 respectively.
- (ii) Apply subtraction modulo to positive integers 15 and 6 for modulo 25
- (iii) Find $a -_m b$, if $a = 17$, $b = 13$ and $m = 3$
- (iv) Find $a -_m b$, if $a = 11$, $b = 43$ and $m = 7$

Solution:

- (i) Given $a = 11$, $b = 3$, $m = 8$
 $a -_m b = 11 -_8 3 = 0$ (as $a - b = 8 = m$)
- (ii) Here $a = 15$, $b = 6$, $m = 25$
then, $a -_m b = 15 -_{25} 6 = 9$ (as $a - b = 9 < 25$)
- (iii) As, $a = 17$, $b = 13$ and $m = 3$
then, $a -_m b = 17 -_3 13 = 1$ (as $a - b = 17 - 13 = 4 > 3$)
- (iv) As, $a = 11$, $b = 43$ and $m = 7$
then, $a -_m b = 11 -_7 43 = (11 - 43) \bmod 7$
 $-32 \bmod 7 = 3$ (as $-32 = 7 \times (-5) + 3$)

MULTIPLICATION MODULO m :

Given two positive integers a and b , the multiplication modulo m denoted as ' \odot_m ' (where m is any positive integer) is defined for set of integers as:

$$a \odot_m b = \begin{cases} (ab) \bmod m, & \text{if } (ab) \geq m \\ ab, & \text{if } ab < m \end{cases}$$

Note: $a \odot_m b$ is read as multiplication modulo m to $(a \times b)$

Example 8

- (i) If $a = 3$, $b = 11$, $m = 8$, then find $a \odot_m b$
- (ii) Apply multiplication modulo on positive integers 5 and 3 for modulo 25

Solution:

- (i) Given $a = 3$, $b = 11$, $m = 8$
 $a \odot_m b = 3 \odot_8 11 = 1$
- (ii) Given $a = 5$, $b = 3$, $m = 25$
then $a \odot_m b = 5 \odot_{25} 3 = 15$ (as $ab = 15 < 25$)

APPLICATION OF MODULO ARITHMETIC

There are many application of modulo, ranging from everyday life to Math and Science

A practical application of modulo arithmetic is to calculate checksums within serial number identifiers. For example, International Standard Book Number (ISBN) uses modulo 11 (for 10 digit ISBN) or modulo 10 (for 13 digit ISBN) arithmetic for error detection. Likewise, International Bank Account Numbers (IBANs), for example, make use of modulo 97 arithmetic to spot user input errors in bank account numbers.

In computer science, modular arithmetic is often applied in bitwise operations and other operations. The modulo operation, as implemented in many programming languages and calculators, is an application of modular arithmetic.

In music, arithmetic modulo 12 is used in the consideration of the system of twelve-tone equal temperament.

On a calendar, arithmetic modulo 7 is used in algorithms that determine the day of the week for a given date.

Modular arithmetic also has wide application in disciplines such as law (e.g., apportionment), economics (e.g., game theory) and other areas of the social sciences. (source: Wikipedia)

On a wrist watch, when reckoning hours, we count up to 12 and start again from one. Thus, five hours after 10 o' clock is 3 o' clock.

Let's check how it can be done by using modulo arithmetic.

Five hours after 10 o' clock means $5 + 10 = 15$

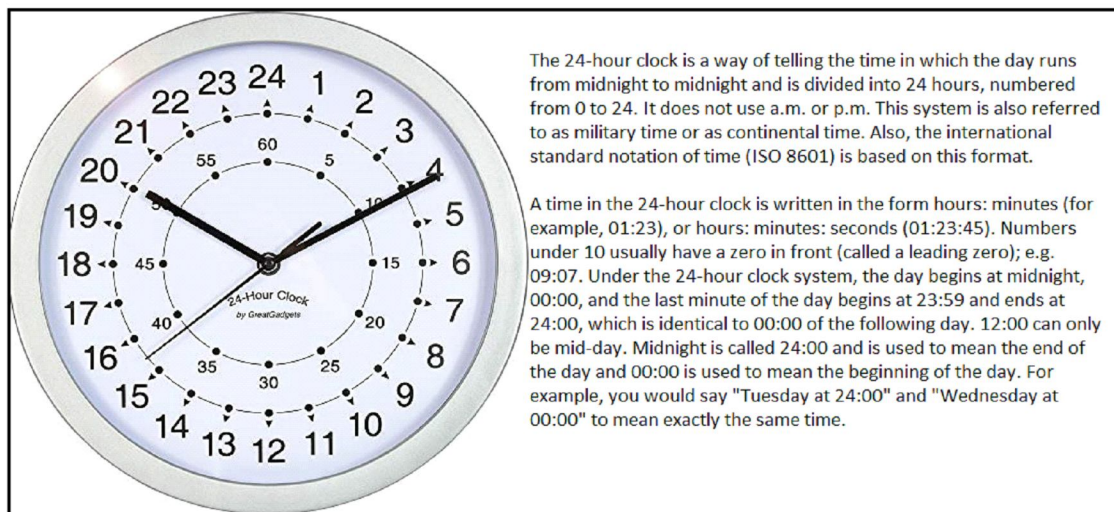
$$15 \bmod 12 = 3$$

So, it will be 3 o' clock.

But the question arises here that why are we using modulo 12 here?

The answer lies in understanding of rotation and revolution of earth, and how this affects time elapse to repeat after a fixed interval of time, meaning that time forms a particular type of periodicity. Modular Arithmetic, also known as Clock Arithmetic acts as a powerful tool as it works on the principle of cyclic repetitions of phenomenon related to numbers

The terms "**a.m.**" and "**p.m.**" are abbreviations of the Latin ante meridiem (before midday) and post meridiem (after midday).



When speaking in military time, 07:00 is stated as "zero seven hundred". Also, in the military, these time stamps are often written without the colon, so 07:52 would rather be written 0752.

Example 9

What time will it be after 1250 hours, If the present time is 9:00 pm?

Solution: Number of hours in a day = 24 hours

$$1250 \pmod{24} = 2 \quad (\text{as } 1250 \text{ hours} = 52 \text{ days} + 2 \text{ hours})$$

Therefore, it will be 11:00 pm after 1250 hours.

Illustration6 : Take two positive integers 262 and 137 and divide them by 5.

$$262 = 5 \times 52 + 2 \quad \dots(i)$$

$$137 = 5 \times 27 + 2 \quad \dots(ii)$$

Subtracting (ii) equation from (i), we get

$$262 - 137 = (5 \times 52 + 2) - (5 \times 27 + 2)$$

$$125 = 5 (52 - 27)$$

125 is a multiple of 5

If we generalise the above result as: a and b are two positive integers which leaves the same remainder r when divided by positive integer m .

$$a = ms + r \quad (a \pmod{m} = r) \quad \dots(iii)$$

$$b = mt + r \quad (b \pmod{m} = r) \quad \dots(iv)$$

Subtracting (iv) equation from (iii), we get

$$a - b = m(s - t)$$

Here, m divides $(a-b)$ which can be written mathematically as $m \mid (a - b)$

Thus, $a \bmod m = b \bmod m$

which can be represented as $a \equiv b \pmod{m}$ where $a - b$ is a multiple of m

Also \equiv symbol is used for expressing congruence as $a \equiv b \pmod{m}$, which is read as “ a congruence b modulo m ” or “ a is congruent to b w. r.t. modulo m ”.

1.2 CONGRUENCE MODULO:

Two positive integers a and b are said to be congruence modulo m ($m \neq 0$) if a and b satisfy the following conditions:

i) $(a-b)$ is divisible by m

ii) $a \bmod m = b \bmod m$

Notation used for congruence modulo is:

$$a \equiv b \pmod{m}$$

Example 10

Verify $a \equiv b \pmod{m}$, if $a = 41$, $b = 21$ and $m = 5$

Solution: $a - b = 41 - 21 = 20$, which is divisible by 5

$$a \bmod m = 41 \bmod 5 = 1$$

$$b \bmod m = 21 \bmod 5 = 1$$

therefore, $41 \equiv 21 \pmod{5}$

Example 11

If $a = 31$, $b = 21$, $m = 5$, then check $a \equiv b \pmod{m}$.

Solution: $a - b = 31 - 21 = 10$ which is divisible by 5

and $a \bmod m = b \bmod m$

$31 \bmod 5 = 21 \bmod 5 = 1$

$$31 \equiv 21 \pmod{5}$$

Hence, $a \equiv b \pmod{m}$ is true

Property 5: If $a \equiv b \pmod{m}$ where a , b and m are positive integers then $a^k \equiv b^k \pmod{m}$ for any positive integer k .

Illustration7 : Complete the following table:

a	b	m	$a \equiv b \pmod{m}$	k	$a^k \bmod m$	$b^k \bmod m$	Observation for relation $a^k \equiv b^k \bmod m$
5	2	3	$5 \equiv 2 \pmod{3} = 2$	3	$5^3 \bmod 3 = 2$	$2^3 \bmod 3 = 2$	Holds true
5	2	3	$5 \equiv 2 \pmod{3} = 2$	5			
18	32	7		2			
18	32	7		4			
17	32	5		3			

Table 6

Example12

Find $5^6 \pmod{4}$.

Solution: $5 \equiv 1 \pmod{4} = 1$

$$5^6 \equiv 1^6 \pmod{4} \quad (\text{Using property 5})$$

Hence, $5^6 \pmod{4} = 1$

ACTIVITY:

- Write first 50 positive integers.
- Find mod 6 for all the integers listed above.
- Our result will be as follows:

$$1 \bmod 6 = 1$$

$$2 \bmod 6 = 2$$

$$3 \bmod 6 = 3$$

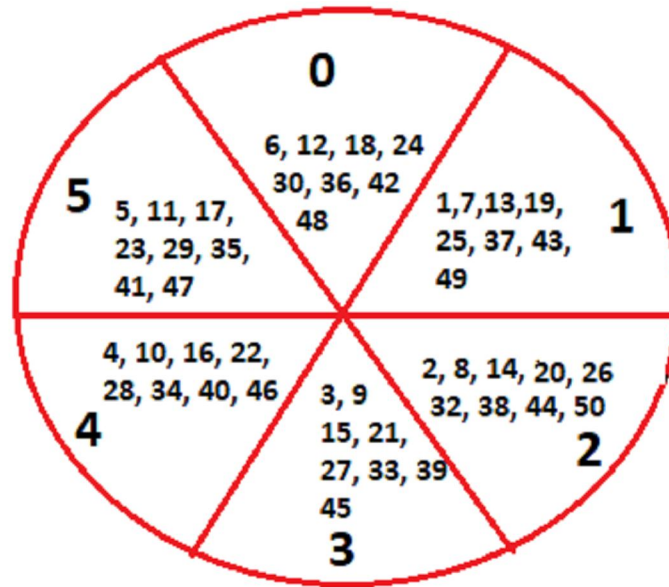
$$4 \bmod 6 = 4$$

$$5 \bmod 6 = 5$$

$$6 \bmod 6 = 0$$

Similarly we can find modulo 6 for the remaining positive integers.

- Segregate the above results on the basis of their remainders for mod 6.



Here, the set of 50 integers is divided into 6 classes for modulo 6 and the elements i.e. numbers lying in each subset (each sector) are congruent to each other.

The numbers belonging to the same subset leaving the same remainder, forms the equivalence class.

Theorem : Let $m \in \mathbb{Z}^+$ and consider the relation R_m defined by $a R_m b$ if and only if $a \equiv b \pmod{m}$. Then R_m is an equivalence relation

Proof

- (i) Since $m \mid (a - a) = 0$, we have $a \equiv a \pmod{m}$
therefore, R_m is reflexive: for all $a \in \mathbb{Z}$
- (ii) If $m \mid (a - b)$, then $m \mid (-1)(a - b) = (b - a)$. Thus $a \equiv b \pmod{m}$ implies $b \equiv a \pmod{m}$
therefore, R_m is symmetric: for all $a, b \in \mathbb{Z}$
- (iii) Suppose $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

Then by the previous theorem we can write:

$$b = a + k_m \text{ for some } k$$

$$\text{and } c = b + k'_m \text{ for some } k'.$$

$$\text{But then } c = b + k'_m = a + km + k'_m = a + (k + k')m$$

$$\text{thus } a \equiv c \pmod{m}$$

therefore, R_m is transitive: for all $a \in \mathbb{Z}$

If $a, m \in \mathbb{Z}$ then the equivalence class of $a \pmod{m}$ (denoted by $[a]_m$) is the set of all integers that are congruent to $a \pmod{m}$.

EQUIVALENCE CLASS:

For $x \in \mathbb{Z}$ we define the equivalence class of X w.r.t. (\pmod{m}) by the following:

$$[X] = \{a \in \mathbb{Z} \mid a \equiv X \pmod{m}\}$$

For example, if $m=3$, $X = 0$,

Then, $[0] = \{a \in \mathbb{Z} \mid a \equiv 0 \pmod{3}\} = \{0, \pm 3, \pm 6, \dots\}$

Similarly for $X = 1$

$[1] = \{a \in \mathbb{Z} \mid a \equiv 1 \pmod{3}\} = \{\dots, -5, -2, 1, 4, 7, \dots\}$

And for $X = 2$,

$[2] = \{a \in \mathbb{Z} \mid a \equiv 2 \pmod{3}\} = \{\dots, -4, -1, 2, 5, 8, 11, \dots\}$

we can find all the integers in these sets and none of these sets overlap.

Fact: There are exactly m equivalence classes for mod m .
 $[0], [1], [2], [3], \dots, [m-1]$

Example 13

Find all the positive integers less than 30 forming the equivalence class of 5 for modulo 7.

Solution: The smallest positive integer divisible by 7 is 7.

Therefore, $5 \bmod 7 = 5$

$7 + 5 = 12$

So, $12 \bmod 7 = 5$

$19 \bmod 7 = 5$

$26 \bmod 7 = 5$

Hence, positive integers less than 30 forming the equivalence class 5 for modulo 7 are 5, 12, 19, and 26.

Therefore $[5] = \{5, 12, 19, 26\}$

EXERCISE-1

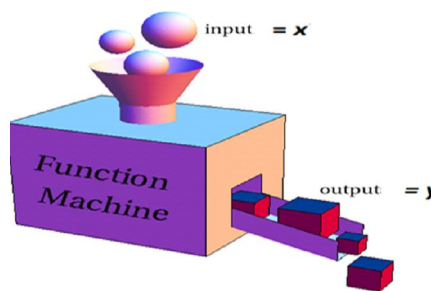
1. Find the sum of 132 and 121 mod 23.
2. Find $7^6 \pmod{3}$.
3. Find the value of x , given that $x \equiv 23 \pmod{7}$; if $21 \leq x < 31$
4. Find the remainder when $226 \times 369 \times 122 \times 461 \times 1025$ is divided by 8.
5. Evaluate $(16 \times 29) \pmod{7}$
6. If $x \pmod{9} = 2$, find all the possible values of x ; where $0 < x < 47$.
7. Find the positive integers less than 50 forming the equivalence class 4 for modulo 6
8. What time will it be after 200 hours, if the present time is 5:00 am?
9. There are 81 boxes with 21 articles in each. When we rearrange all of the articles so that each box has 5 articles, how many articles will be left out without a box.
10. Find $3^{128} \bmod 7$.

1.3 SIMPLE ARITHMETIC FUNCTIONS:

From previous classes, recall that a function is an expression, rule or law that defines a relationship between two variables, i.e. how one variable is dependent on the other variable.

For example, $y = x^2$ is a function where x is an element, known as *pre-image* from the given domain of function and y is an element, known as *image* of the range.

We can also understand function as a machine which follows the instructed operation and converts the input(pre-image) to required output (image).



A Simple Arithmetic function is a function whose domain is positive integers and the range is real or complex numbers, also known as **Number theoretic function**. Such functions describe arithmetic properties of numbers and are widely useful in number theory.

For example, a function f defined as, $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ where $f(x) = x^2$ is an arithmetic function.

TYPES OF SIMPLE ARITHMETIC FUNCTION:

EULER'S TOTIENT FUNCTION OR EULER'S PHI FUNCTION:

Euler's totient function counts the number of coprime integers to n from domain of integers from 1 to n .

Euler's totient function is also known as Euler's phi function and denoted by $\phi(n)$.

Euler's totient function $\phi(n)$ for an input of n is the count of numbers in $\{1, 2, 3, \dots, n\}$ that are coprime (relatively prime) to n , i.e., the numbers whose gcd (greatest common divisor) with n is 1.

$\phi(n)$ = the number of strictly positive integers $\leq n$ which are coprime to n

Example 14

Find $\phi(n)$ for $n = 3$.

Solution: If $n = 3$,

as, $\gcd(1, 3) = 1$ and $\gcd(2, 3) = 1$ (\gcd = greatest common divisor)

Therefore, numbers coprime to 3 are 1 and 2

Hence $\phi(3) = 2$

JUGGLE YOUR BRAIN

Complete the following table:

n	$\phi(n)$
1	1
2	1
3	2
4	
5	
6	
7	6
8	

$\phi(n)$ is a function which counts the number of positive integers $\leq n$ which are coprime to n ; $1 \leq \phi(n) < n$

9	
10	
12	4
15	
19	

Additive function: A Simple Arithmetic function f is said to be additive if

$$f(xy) = f(x) + f(y), \quad \gcd(x, y) = 1 \text{ i.e. } x \text{ and } y \text{ are coprime}$$

For example: $\log(xy) = \log x + \log y$

Multiplicative function: A Simple Arithmetic function f is said to be multiplicative if

$$f(xy) = f(x) \times f(y), \quad \gcd(x, y) = 1 \text{ i.e. } x \text{ and } y \text{ are coprime}$$

For example : $\phi(6) = \phi(2) \times \phi(3) = 1 \times 2 = 2$

PROPERTIES OF EULER'S TOTIENT FUNCTION:

1. If p is a prime number then $\phi(p) = p - 1$.
2. If x and y are coprime numbers then $\phi(xy) = \phi(x) \cdot \phi(y)$.
Hence $\phi(p)$ is known as a multiplicative function where $p = xy$
3. If a number n can be expressed as the product of prime positive integers, i.e. $n = x^a y^b z^c$; where x, y, z are the primes and a, b, c are the powers of primes,

$$\text{then } \phi(n) = n \left(1 - \frac{1}{x}\right) \left(1 - \frac{1}{y}\right) \left(1 - \frac{1}{z}\right)$$

Example15

Find $\phi(12)$

$$12 = 2^2 \times 3$$

$$\text{Solution: } \phi(12) = 12 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right)$$

$$= 12 \times \frac{1}{2} \times \frac{2}{3} = 4$$

NUMBER OF DIVISOR FUNCTION $\tau(n)$

As evident from the name itself, this function, denoted by $\tau(n)$ counts the number of divisors of n from the domain of 1 to n .

The greek symbol τ is read as 'tau'

For example,

$$\tau(1) = 1 \quad (1 \text{ has only } 1 \text{ positive divisor i.e. } 1 \text{ itself})$$

$$\tau(3) = 2 \quad (3 \text{ has two positive divisors i.e. } 1 \text{ and } 3)$$

$$\tau(4) = 3 \quad (4 \text{ has three positive divisors i.e. } 1, 2 \text{ and } 4)$$

We can define $\tau(n)$ as:

Given a positive integer n , the number of divisor function $\tau(n)$ on \mathbb{Z}^+ is defined as

$$\tau(n) = \text{the number of positive divisors of } n$$

Example 16

Find $\tau(12)$

Solution: The positive divisors of 12 are = 1, 2, 3, 4, 6, 12

$$\tau(12) = 6$$

DIVISOR SUM FUNCTION $\sigma(n)$

The function $\sigma(n)$ is the sum of the positive divisors of n has, including 1 and n itself.

The greek symbol σ is read as 'sigma'

For example,

$$\sigma(1) = 1 \quad (1 \text{ has only 1 positive divisor})$$

$$\sigma(3) = 4 \quad (3 \text{ has two positive divisors 1 and 3, and there sum} = 1+3)$$

$$\sigma(4) = 7 \quad (4 \text{ has three positive divisors 1, 2 and 4, and there sum} = 1+2+4)$$

We can define $\sigma(n)$ as:

Given a positive integer n , the divisor sum function $\sigma(n)$ on \mathbb{Z}^+ is defined as

$$\sigma(n) = \text{the sum of positive divisors of } n$$

Example 17

Find $\sigma(12)$

Solution: Positive divisors of 12 are = 1, 2, 3, 4, 6, 12

$$\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$$

Illustration 8 : $\tau(1) = 1$ (positive divisors of 1 is 1)

$$\sigma(1) = 1 \quad (\text{sum of positive divisors of 1 is 1})$$

So, we get : $\sigma(n) = \sigma(n)$ for $n = 1$

Illustration 9: Complete the following table:

n (prime number)	$\tau(n)$	$\sigma(n)$
2	2	3
5		6
17	2	
29		

After analysing the above table, we deduce:

If n is a prime number, then

$$\tau(n) = 2 \quad (\text{prime numbers have two positive divisors 1 and number itself})$$

and

$$\sigma(n) = n + 1 \quad (\text{sum of positive divisors of } n \text{ is } n + 1)$$

THE MOBIUS FUNCTION $\mu(n)$

Given a positive integer n , the mobius function is defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \end{cases}$$

Here the Greek symbol μ is pronounced as 'mu'.

Example 18

Find $\mu(12)$ and $\mu(35)$

Solution: As $12 = 2^2 \times 3$
thus $\mu(12) = 0$
and $35 = 5 \times 7$
therefore $\mu(35) = (-1)^2 = 1$

EXERCISE-2

1. Find $\phi(35)$.
2. Find $\mu(75) + \mu(85)$.
3. Calculate $\phi(90) + \tau(42) + \sigma(72) + \mu(70)$
4. Verify that the relation $\phi(p) + \tau(p) = \sigma(p)$ holds true for $p = 24$.
5. If p is a prime then $\phi(p) + \tau(p) = \sigma(p)$, Using the information complete the following table:

P	$\phi(p)$	$\tau(p)$	$\sigma(p)$	$\phi(p) + \tau(p) = \sigma(p)$
2				
3				
5				
7				
11				

6. If $n = 2p$ and p is an odd prime number less than 14, then:
 - i) Evaluate $\phi(n)$, $\tau(n)$, $\sigma(n)$ and represent the results in tabular form.
 - ii) Verify the relation $n + \phi(n) + \tau(n) = \sigma(n)$.
7. Show that $\tau(n)$ and $\sigma(n)$ are multiplicative function for $n = 24$.

1.4 ALLIGATION AND MIXTURE

MIXTURE:

Mixing two or more ingredients together in certain ratio is known as a mixture.

ALLIGATION:

It is the rule that enables us to find the ratio in which two or more ingredients are mixed at the given price to produce a mixture of desired price.

When different quantities of different costs are mixed together to produce a mixture, then cost of mixture is known as mean price.

MEAN PRICE:

Mean price is the cost price of a unit quantity of the mixture.

RULE OF ALLIGATION:

When different quantities of different ingredients are mixed together to produce a mixture of mean price, the ratio of their quantities is inversely proportional to the differences in their cost from the mean price.

$$\frac{\text{Quantity of cheaper ingredient}}{\text{Quantity of dearer ingredient}} = \frac{\text{CP of dearer} - \text{Mean Price}}{\text{Mean Price} - \text{CP of cheaper}}$$

Dearer ingredient is the ingredient whose cost is more (greater)

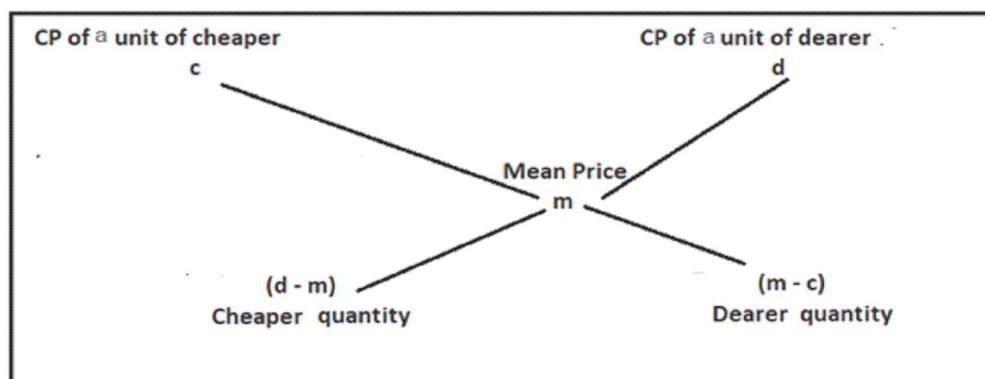
Cheaper ingredient is the ingredient whose cost is less

Above formula can also be understood through the following diagram :

Let Mean Price of mixture be ' m '

Cost Price of cheaper ingredient be ' c '

Cost Price of dearer ingredient be ' d '



Ratio of cheaper and dearer = $(d - m) : (m - c)$

Suppose a container contains x unit of a liquid from which y unit are taken out and replaced by water. If we repeat the same operation n times then the quantity of pure liquid is given by

$$x \left(1 - \frac{y}{x}\right)^n \text{ unit}$$

Note: The cost price of water is considered to be zero.

Example 19

A retailer has 250 kg of rice, a part of which he sells at 10% profit. The remaining quantity of rice is of low quality and he sold it at 5% loss. Overall he made a profit of 7%. Find the quantity of rice sold at 5% loss.

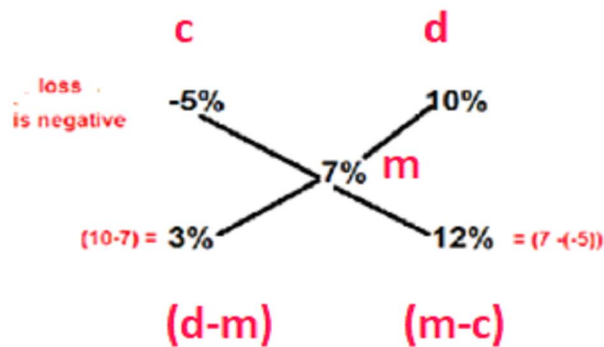
Solution : Total quantity of rice = 250 kg

$$\text{Loss} = 5\% = -5$$

$$m = 7\%$$

$$c = -5\%$$

$$d = 10\%$$



Therefore, ratio of selling = 1 : 4

$$\text{Quantity of rice sold at 5\% loss} = \frac{1}{5} \times 250 = 50 \text{ kg}$$

Example 20

A container contains 40 litre milk. From this container 4 litre milk was taken out and replaced with water. This process was repeated further two more times. How much milk is there in the container now?

Solution : Total milk in container = 40 l

Milk taken out = 4 l

No. of times process repeated = 3

$$\text{Milk contained by the container now} = x \left(1 - \frac{y}{x} \right)^n \text{ unit}$$

where x is total quantity, y is quantity removed, n is no. of times operation repeated.

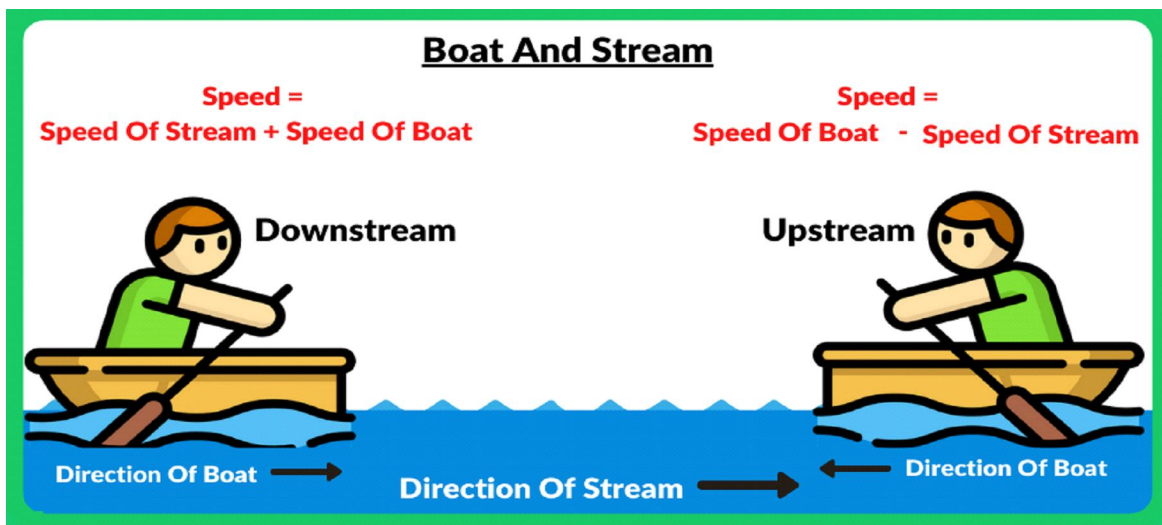
$$\begin{aligned} &= 40 \left(1 - \frac{4}{40} \right)^3 \\ &= 40 \left(\frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} \right) \\ &= 40 \times \frac{729}{1000} \\ &= 29.16 \text{ l} \end{aligned}$$

EXERCISE-3

1. In what ratio must rice at ₹ 69 per kg be mixed with rice ₹ 100 per kg so that the mixture be worth ₹ 80 per kg?
2. The average salary per head of the entire staff of a small factory including the supervisor and labours is ₹ 5750. The average salary per head of the supervisor is ₹ 20,000 and that of the labours is ₹ 5000. Find the number of labours in the factory if there are 4 supervisors.
3. A container contains 70 l of orange squash. The squash being too concentrated 7 l of squash was taken out from this container and replaced by water. This process was repeated thrice to reduce the concentration of squash. How much quantity of orange squash is left in the container?
4. Cost of two types of pulses is ₹ 55 per kg and ₹ 90 per kg. If both the pulses are mixed together in the ratio 2:3, what should be the price of the mixed variety of pulses per kg?
5. A shopkeeper has 1 quintal of wheat, part of which she sells at 18% gain and the rest at 28% gain. In total she gains 24 %. Find the quantity of wheat sold at 18% and 28%.
6. 600 gm of jaggery syrup has 40% jaggery in it. How much jaggery should be added to make it 50% in the syrup?
7. In what ratio, water must be added to dilute honey costing ₹ 240 per litre so that the resulted syrup would be worth ₹ 200 per l?
8. A container has 50 l of juice in it. 5 l of juice is taken out and is replaced by 5 l of water. This process is repeated 4 more times. What is the amount of juice in the container after final replacement?

1.5.1 BOATS AND STREAMS: (Upstream and downstream):

If the speed of a boat in still water is x km/hr and the speed of stream be y km/hr then the resultant speed of the boat along the stream (downstream) is $(x+y)$ km/hr and the resultant speed of the boat in the opposite direction to the stream (upstream) is $(x-y)$ km/hr.



Furthermore, If the speed of the boat downstream is a km/hr and upstream is b km/hr then,

Speed of the boat = $\frac{1}{2} (a+b)$ km/hr

Speed of the stream = $\frac{1}{2} (a-b)$ km/hr

Example 21

A man rows 15 km upstream and 25 km downstream in 5 hours each time. What is the speed of the current?

Solution : Let the speed of the boat in still water be x km/hr and the speed of the current be y km/hr then $(x+y) \times 5 = 25$ and $(x-y) \times 5 = 15$ (Distance = Speed \times Time)

$$\Rightarrow x + y = 5$$

and $x - y = 3$

On solving we get, $x = 4$ and $y = 1$

Thus, speed of the current is 1 km/hr.

Example 22

A person can row a boat 5 km an hour in still water. It takes him thrice as long to row upstream as to row downstream. Find the rate at which the stream is flowing.

Solution: Let the rate at which the stream is flowing be x km/hr and let the distance covered by the boat be y km.

According to the question

$$\frac{3y}{5+x} = \frac{y}{5-x}$$

$$\Rightarrow 3(5-x) = 5+x$$

$$\Rightarrow 15 - 3x = 5+x$$

$$\Rightarrow 4x = 10$$

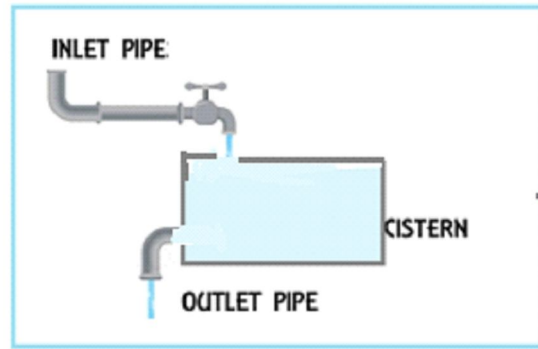
$$\therefore x = 2.5$$

The stream is flowing at the rate of 2.5 km/hr.

EXERCISE-4

1. Find the speed of the boat, if a boat moves downstream at the rate of 16 km/hr and upstream at the rate of 10 km/hr.
2. The speed of a boat in still water is 14 km per hour. While going downstream it moves at the rate of 24 km per hour. Find the speed of the boat against the stream.
3. A boat goes 8 km upstream and then returns. Total time taken is 4 hours 16 minutes. If the speed of current is 1 km/hr, find the actual speed of the boat.
4. A man can row 7 km per hour in still water. If the stream is flowing at the rate of 5 km per hour, it takes him 7 hours to row to a place and return, how far is the place?
5. A boat covers 32 km upstream and 36 km downstream in 7 hours. Also it covers 40 km upstream and 48 km downstream in 9 hours. Find the speed of the boat in still water and that of the stream.
6. A man can row $7\frac{1}{2}$ km/h in still water. If in a river running at 1.5 km an hour, it takes him 50 minutes to row to a place and back, how far off is the place?
7. The speed of a motor boat and that of the current of water is 36:5. The boat goes along with the current in 5 hours 10 minutes. How much time will it take to come back?

1.5.2 PIPES AND CISTERNS



A pipe connected to a tank or cistern which fills it is known as inlet pipe and the pipe connected to the tank which drains or empties it is known as outlet pipe.

When a tank is connected to many pipes (inlets and outlets), then the difference between the sum of the work done by inlets and the sum of the work done by outlets gives the filled part of the tank.

Let a pipe fill a tank in x number of hours, then it can fill $(1/x)^{\text{th}}$ portion of the tank in one hour.

If a pipe can empty a tank in y number of hours, then it can empty out $(1/y)^{\text{th}}$ portion of the tank in one hour.

The portion of tank they can fill together in one hour = $(1/x - 1/y)^{\text{th}}$

Example 23

Two pipes can fill a cistern in 8 and 12 hours respectively. The pipes are opened simultaneously and it takes 12 minutes more to fill the cistern due to leakage. If the cistern is full, what will be the time taken by the leakage to empty it?

Solution: Portion of cistern filled (work done) by both the pipes in 1 hour = $\frac{1}{8} + \frac{1}{12} = \frac{5}{24}$

Time taken by both the pipes to fill the tank = $\frac{24}{5}$ hr = 4 hrs 48 minutes

Time taken to fill the tank due to leakage = 4 hrs 48 minutes + 12 minutes = 5 hrs

Net work done by the pipes and the leakage in 1 hr = $\frac{1}{5}$

Work done by leakage in 1 hr = $\frac{5}{24} - \frac{1}{5} = \frac{1}{120}$

Time taken by leakage to empty the tank = 120 hrs.

Example 24

A cistern can be filled by two pipes A and B in 12 minutes and 15 minutes respectively. Another tap C can empty the full tank in 20 minutes. If the tap C is opened 5 minutes after the pipes A and B are opened, find when the cistern becomes full?

Solution: Cistern filled by A and B in 5 minutes = $5 \left(\frac{1}{12} + \frac{1}{15} \right) = \frac{3}{4}$

Unfilled part of the tank = $\frac{1}{4}$

$$\text{Portion of cistern filled by } A, B \text{ and } C \text{ in 1 minute} = \frac{1}{12} + \frac{1}{15} - \frac{1}{20} = \frac{1}{10}$$

$$\text{So, } \frac{1}{4} \text{ of the tank will be filled in } = \frac{1}{4} \times 10 = 2 \text{ mins } 30 \text{ secs}$$

$$\text{Total time taken to fill the tank} = 5 + 2 \text{ mins } 30 \text{ secs} = 7 \text{ mins } 30 \text{ secs}$$

EXERCISE-5

1. Pipe A can fill a tank in 30 hours and pipe B in 45 hours. If both the pipes are opened in an empty tank, how much time will it take to fill the tank?
2. A pipe can fill a cistern in 6 hours. Due to a leakage in the tank the cistern is just full in 9 hours. How much time the leakage will take to empty the tank?
3. A cistern can be filled by pipes A and B in 4 hours and 6 hours respectively. When full, the cistern can be emptied by pipe C in 8 hours. If all the pipes were turned on at the same time, in how much time will the cistern be filled?
4. A cistern can be filled in 8 hours but due to a leakage in its bottom, it takes 2 hours more to fill the tank. If the cistern is full, how much time will the leakage take to empty it?
5. A cistern can be filled by an inlet pipe in 20 hours and can be emptied by an outlet pipe in 25 hours. Both the pipes are opened. After 10 hours, the outlet pipe is closed, find the total time taken to fill the tank.
6. Two pipes A and B can fill a tank in 24 minutes and 32 minutes respectively. If both the pipes are opened simultaneously, after how much time B should be closed so that the tank is full in 18 minutes?
7. A tank is fitted with 3 taps A, B and C. All the three taps, if opened together, can drain the full tank in $1\frac{1}{2}$ minutes. Taps B and C together take 2 minutes to drain the tank while A and C together take $2\frac{4}{13}$ minutes to drain it. How long will taps A and B together take to drain the tank?

1.5.3 RACES AND GAMES



A race is a competition in which the competitors have to cover a certain distance in the minimum possible time. A race includes activities like running, riding, driving, sailing, rowing etc. The path followed by the competitor during the race is called a racecourse. The initial point from where a race starts is called the starting point, and the final point or the finishing point is the point where a race finishes.

Races and Games is sub-topic of Time, Speed and Distance; therefore the formulas of time, speed and distance are applied to the questions of Races and Games.

Terms related to races and games

'A gives B a start of x metres': This means that, while A starts the race from the starting point, B starts x metres ahead of A. To cover a race of y metres in this case, A will have to cover y metres while B will have to cover only $(y - x)$ metres.

'A beats B by x metres': This implies that while A reached the finishing point, B is behind A by x metres in the same time. To cover a race of y metres in this case, A has covered y metres while B has covered only $(y - x)$ metres.

'A gives B a start of t minutes': It implies that A will start t minutes after B starts from the starting point but both A and B will reach the finishing point at the same time.

'A beats B by x metres and t minutes': This statement implies that A and B start from the starting point at the same instant, but while A reaches the finishing point, B is behind by x metres, and, B takes t minutes more compared to A to complete the race. So, B covers remaining x metres in extra t minutes. This gives the speed of B as x/t .

Dead Heat: When all the participants of the race finish the race at the same instant of time, such a situation is known as dead heat situation.

Example 25

In a 1000 metres race. A, B, and C get the gold, silver, and bronze medals, respectively. If A beats B by 100 metres and B beats C by 100 metres, then by how many metres does A beat C?

Solution: A beats B by 100 metres, means A travels 1000 metres in the same time in which B travels 900 metres.

B beats C by 100 metres, means B travels 1000 metres in the same time in which C travels 900 metres.

$$A : B = 10 : 9$$

$$B : C = 10 : 9$$

$$\Rightarrow \text{The ratio of } A : B : C = 100 : 90 : 81$$

A travels 100 metres and in the same time C travels 81 metres.

Therefore when A travels 1000 metres, in the same time C travels 810 metres.

Thus A beats C by 190 metres.

Example 26

Priya can run 1 km in 3 min 10 sec and Preeti can cover the same distance in 3 min 20 sec. By what distance can Priya beat Preeti?

Solution: Priya beats Preeti by $(200-190)$ sec = 10 sec

$$\text{Distance covered by Preeti in 10 sec} = \frac{1000}{200} \times 10 = 50 \text{ m}$$

Therefore, Priya beats Preeti by 50 meters.

Example 27

Two runners A and B complete a 100 metres race in 36 seconds and 48 seconds respectively, by how many metres will A defeat B?

Solution: The distance covered by B, after A has finished the race i.e. distance covered by B in $(48 - 36 = 12\text{s})$.

$$\text{Distance} = (100/48) \times 12 = 25 \text{ meters}$$

A beats B by 25 metres.

EXERCISE-6

1. In a 1000 metres race, A defeats B by 300 metres and B defeats C by 200metres. In the same race by how many metres will A defeat C ?
2. In a 1000 metres race, A can give a start of 100 metres to B and a start of 280 metres to C . In the same race, how much start can B give to C ?
3. In a 500 metres race, A defeats B by 60 metres (or) 12 seconds. What is the time taken by A to complete the race?
4. In a 900 metres race, A gives B a start of 150 metres and defeats him by 50 seconds. If the speed of A is 4.5m/sec then find the speed of B .
5. A runs 3 times as fast as B . If A gives B a start of 40metres, how far must the goal on the race course be so that A and B reach the goal at the same time?
6. A team played 40 games in a season and lost in 16 of them. What percent of games played did the team win?
7. In a 200 metresrace , Prateek beats Samarth by 35 metres or 7 seconds. How much time did Prateek take to cover the race?

1.5.4 PARTNERSHIP

What is business partnership?

- A specific kind of legal relationship
- Business partnerships are formed by the agreement between two or more individuals to carry on a business as co - owners
- Types of partnership and levels of involvement vary

When two or more persons run a business jointly, they are called partners and the deal is known as partnership.

If all the partners invest their capitals for the same period of time, then it is called a simple partnership and if they invest their capitals for different time period, it is called a compound partnership.

Profits earned or losses incurred in the business are divided among the partners in the ratio of their investments with due consideration of time for which they invested in the business.

Ratio of investment \times Time = Ratio of profit

(A's investment \times A's time) : (B's investment \times B's time) = A's profit : B's Profit

Example 28

P, Q and R jointly invested a total of ₹ 64000 in a business. P invested ₹ 6000 more than Q and Q invested ₹ 5000 more than R . Find the ratio of their capitals?

Solution: Let the investment of R is ₹ x , then the investment of Q and P is ₹ $(x+5000)$ and ₹ $(x+11000)$ respectively.

According to the question, $x+x+5000+x+11000 = 64000$

$$3x = 48000$$

$$x = 16000$$

The ratio of the capitals of P, Q and R is 27000:21000:16000 or 27:21:16

Example 29

Divide a net profit of ₹ 21000 among three partners A,B,C in a business whose capital investments were ₹ 36000, ₹ 42000 and ₹ 48000 respectively.

Solution: Ratio of the capitals is 36000:42000:48000 or 6:7:8

The profit is divided in the ratio of capitals, therefore

$$A's \text{ share } ₹ \frac{21000 \times 6}{21} = ₹ 6000$$

$$B's \text{ share } ₹ \frac{21000 \times 7}{21} = ₹ 7000$$

$$C's \text{ share } ₹ \frac{21000 \times 8}{21} = ₹ 8000$$

Example 30

Three friends A,B and C enter into a partnership to run a café business. A puts in ₹ 5000 per month for the whole year, B contributes ₹ 3000 per month at first and increases his contribution to ₹ 4500 at the end of 4 months, while C puts in at first ₹ 4000 per month and withdraws ₹ 1000 at the end of nine months. How should they divide a profit of ₹ 10200 at the end of the year?

Solution: A's equivalent capital for 1 month = ₹ (5000×12)= ₹ 60000

B's equivalent capital for 1 month = ₹ (3000×4 + 4500×8)= ₹ 48000

C's equivalent capital for 1 month = ₹ (4000×9+3000×3)= ₹ 45000

Ratio of the capitals is 60000:48000:45000 or 20:16:15

$$A's \text{ share } ₹ \frac{10200 \times 20}{51} = ₹ 4000$$

$$B's \text{ share } ₹ \frac{10200 \times 16}{51} = ₹ 3200$$

$$C's \text{ share } ₹ \frac{10200 \times 15}{51} = ₹ 3000$$

EXERCISE-7

1. A,B and C enter into a partnership. B contributes 1/3 of the capital, while A contributes as much as B and C together contribute. Find the ratio of their capitals.
2. Rahul got ₹ 5000 as his share out of the total profit of ₹ 9000. Ramesh had invested ₹ 3000 for 6 months, while Rahul invested for the whole year. Find the amount invested by Rahul.
3. Priya started a tiffin service business by investing ₹ 40000. After 4 months her friend Rekha joined her business and invested ₹ 50000 in it. What will be the share of Rekha in the profit if they earn ₹ 220000 as profit in the entire year?
4. What is the investment ratio of A,B and C if their profit ratio is 6:7:8 respectively and their investment time period ratio is 2:3:4 respectively?
5. A and B start a trading business with a capital of ₹ 30000 and ₹ 60000 respectively. From the profit of entire year which is ₹ 18000, A gets ₹ 8500 as his share of profit and salary for managing the business. What is the amount that A receives as his salary?

6. Prakash, Sachin and Anil started a business jointly investing ₹ 11 lakh, ₹ 16.5 lakh and ₹ 8.25 lakh respectively. The profit earned by them in the business at the end of 3 years was ₹ 19.5 lakh. What will be 50% of Anil's share in the profit?
7. Nikhil, Priyesh and Ritik took a house on rent for ₹ 13824. They remained together for 4 months and then Ritik left the house. After 5 more months, Priyesh also left the house. How much rent should each pay?
8. Arun started a business investing ₹ 38000 . After 5 months Bakul joined him with a capital of ₹ 55000. At the end of the year the total profit was ₹ 22000. What is the approximate difference between the shares of Arun and Bakul?

1.5.5 SCHEDULING:

Scheduling is an art of planning the activities to achieve goals and prioritise time. Activities in the production unit are pre-planned and scheduled to minimise the production time and cost.

Scheduling problems arise in almost all areas of human activity. The Viking mission to Mars required coordination of the activities of more than 20,000 people. Even preparation of a multicourse dinner can present a scheduling problem.

A **processor** is the person or machine that works on the tasks in the schedule.

The **process** is basically an instance of the computer program that is being executed.

Over the past few years several mathematical models of scheduling processes have been devised. The basic scheduling model consists of a system of n identical processors $P_1, P_2, P_3, \dots, P_n$ and a set of tasks A, B, C, \dots to be performed by the processors. A scheduling algorithm is the set of rules by which tasks are assigned to the individual processors.

A **priority list** is a list of tasks in order in which they should be completed.

Priority scheduling algorithm can be of two types:

1. **Pre-emptive Priority Scheduling:** If the new process arrived at the ready queue has a higher priority than the currently running process, the CPU is pre-empted, which means the processing of the current process is stopped and the incoming new process with higher priority gets the CPU for its execution.
2. **Non-Pre-emptive Priority Scheduling:** In case of non-pre-emptive priority scheduling algorithm if a new process arrives with a higher priority than the current running process, the incoming process is put at the head of the ready queue, which means after the execution of the current process it will be processed.

Finishing time is the amount of time it will take to complete all the tasks in the schedule.

An **optimal schedule** is a schedule with the shortest possible finishing time.

Idle time is time in which a processor has no tasks to work on.

CPU - Central Processing Unit

Burst time is the amount of time required by a process for executing on CPU.

Completion time is the time at which process completes its execution.

Turnaround time (TAT) = completion time (CT) - arrival time (AT)

Waiting time is the time difference between turnaround time and burst time

Waiting time (WT) = turnaround time (TAT) - burst time (BT)

For every scheduling algorithm, **Average waiting time** is a crucial parameter to judge its performance. Average waiting time (AWT) is the average of the waiting times of the processes in

the queue, waiting for the scheduler to pick them for execution. Lower the Average Waiting Time, better the scheduling algorithm.

Response time is the time at which a process gets the CPU first time. In case of non-pre-emptive, response time is same as waiting time.

GANTT CHART : It is a type of bar chart that illustrates a project schedule, named after its inventor Henry Gantt.

Note : In this chapter we are only going to study about two non-pre-emptive scheduling algorithms- First come first serve (FCFS) and Shortest job first (SJF)

First Come First Serve (FCFS):

It is a scheduling algorithm that automatically executes queued request and processes it in the order of their arrival. The process that comes first will be executed first and the next process starts only after the previous one gets fully executed. It is the easiest and simplest CPU scheduling algorithm, also known as FIFO (First in first out). FCFS is a non-pre-emptive scheduling algorithm. FCFS can cause long waiting times (Average Waiting Time is high), especially when the first job takes too much CPU time. A real-life example of the FCFS method is buying a movie ticket at the ticket counter. Arrival time is the criteria in FCFS. Short processes that are at the back of the queue have to wait for the long process at the front to finish.

Example 31

The processes P1, P2, P3, P4 given in the table below, arrives for execution in the same order, with Arrival Time 0, and given Burst Time. Find the average waiting time using the FCFS scheduling algorithm.

Process	Burst Time
P1	25
P2	4
P3	7
P4	3

Solution: The GANTT chart for given process is

P1	P2	P3	P4
0	25	29	36 39

The average waiting time will be = $(0+25+29+36)/4 = 22.5$ units

For the above given processes, first **P1** will be executed hence, waiting time for **P1** will be 0. **P1** requires 25 units for completion, hence waiting time for **P2** will be 25 units. Similarly, the waiting time for process **P3** will be execution time of **P1** + execution time for **P2**, which will be $(25+4)=29$ units. For process **P4** it will be the sum of execution times of **P1**, **P2** and **P3** i.e. $(0+25+4+7) = 36$ units

The **GANTT chart** above perfectly represents the waiting time for each process.

In FCFS Average Waiting Time is not optimal. Resources utilization in parallel is not possible, which leads to **Convoy Effect**.

What is Convoy Effect?

Convoy Effect is a situation where many processes, which need to use a resource for short time are blocked by one process holding that resource for a long time.

Example 32

Consider the available processes given below in the ready queue for execution and with given burst time.

Process No	Arrival Time	Burst time
P1	0	2
P2	1	2
P3	5	3
P4	6	4

- What is the time at which all the processes get executed?
- Find the average waiting time and average turnaround time using the non - pre-emptive FCFS scheduling algorithm.

Solution: Mode: Non-Pre-emptive Criteria: Arrival time

Process No	Arrival Time(AT)	Burst time(BT)	Completion time (CT)	Turnaround time (TAT) = CT-AT	Waiting time (WT)=TAT-BT
P1	0	2	2	2	0
P2	1	2	4	3	1
P3	5	3	8	3	0
P4	6	4	12	6	2

GANTT CHART

P1	P2	IDLE TIME	P3	P4	
0	2	4	5	8	12

Arrival time of P1 is 0 and it gets executed for 2 units of time. While P1 is being executed P2 has already arrived in ready queue after 1 unit of time. Completion time for P2 is 4 units. Since P3 arrives after 5 units of time, the processor remains idle for 1 unit of time. When time is 5 units P3 arrives and get executed for 3 units of time. After 6 units of time P4 is in ready queue. Thus, its completion time is 12 units.

a) All the processes got executed at 12 units of time.

b) Average waiting time = $\frac{0+1+0+2}{4} = 0.75$ units

Average turnaround time = $\frac{2+3+3+6}{4} = 3.5$ units

Shortest Job First (SJF):

It is a scheduling algorithm that automatically first executes the process which have the shortest burst time. If two processes have the same burst time then FCFS is used to break the tie. It is a non-pre-emptive scheduling algorithm. In SJF scheduling, average waiting time is minimum.

Problem with Non-Pre-emptive SJF

If the **arrival time** for processes is different, which means all the processes are not available in the ready queue at time 0, and some jobs arrive after some time, in such situation, sometimes the process with short burst time have to wait for the current process's execution to finish, because in Non- Pre-emptive SJF, on arrival of a process with short duration, the existing job/process's execution is not halted/stopped to execute the short job first.

This leads to the problem of **Starvation**, where a shorter process has to wait for a long time until the current longer process gets executed. This happens if shorter jobs keep coming.

Example 33

Consider the below processes available in the ready queue for execution, with **arrival time** as 0 for all and given burst time. Find the average waiting time using the SJF scheduling algorithm.

Process	Burst Time
P1	25
P2	4
P3	7
P4	3

Solution: In SJF, the shortest process is executed first, the GANTT chart for given process is:

P4	P2	P3	P1
0	3	7	14
			39

Average waiting time = $(0+3+7+14)/4 = 6$ units

As you can see in the **GANTT chart** above, the process **P4** will be executed first as it has the shortest burst time, then **P2**, followed by **P3** and finish at **P1**.

We scheduled the same set of processes using the First come first serve algorithm in the previous question, and got average waiting time to be 22.5 units, whereas with SJF, the average waiting time comes out 6 units.

Example 34

Consider the below processes available in the ready queue for execution and with given burst time.

Process No	Arrival Time	Burst time
P1	1	3
P2	2	4
P3	1	2
P4	4	4

- What is the time at which all the processes get executed?
- Find the average waiting time and average turnaround time using the non- pre-emptive SJF scheduling algorithm.

Solution: Mode: Non-Pre-emptive Criteria: Burst time

Process No	Arrival Time(AT)	Burst time(BT)	Completion time (CT)	Turnaround time (TAT) = CT-AT	Waiting time (WT)=TAT-BT
P1	1	3	6	5	2
P2	2	4	10	8	4
P3	1	2	3	2	0
P4	4	4	14	10	6

GANTT CHART

IDLE TIME	P3	P1	P2	P4	
0	1	3	6	10	14

Arrival time of both P1 and P3 is 1 unit of time. Thus, the processor remains idle for 1 unit of time. Since both P1 and P3 arrive at the same time, priority will be given to the one with lesser burst time therefore P3 gets executed first for 2 units of time. P3 is over after 3 units of time and now P1 gets executed for 3 units of time. P2 has already arrived in ready queue after 2 units of time and the completion time for P2 is 10 units. When time is 4 units P4 arrives and gets executed for 4 units of time. Thus, its completion time is 14 units.

a) All the processes got executed at 14 units of time.

b) Average waiting time = $\frac{2+4+0+6}{4} = 3$ units

Average turnaround time = $\frac{5+8+2+10}{4} = 6.25$ units

EXERCISE-8

- Which of the following is false about SJF?
S1: It causes minimum average waiting time
S2: It can cause starvation
(A) Only S1
(B) Only S2
(C) Both S1 and S2
(D) Neither S1 nor S2
- What is the relationship between arrival time, burst time, completion time and waiting time?
- The processes P1, P2, P3, P4 given in the table below, arrives for execution in the same order, with **Arrival Time 0**, and given **Burst Time**, find the average waiting time using the FCFS scheduling algorithm.

Process	Burst Time
P1	20
P2	5
P3	8
P4	44

4. Consider the available processes given below in the ready queue for execution, with arrival time as 0 for all and given burst time. Find the average waiting time using the SJF scheduling algorithm.

Process	Burst Time
P1	20
P2	5
P3	9
P4	4

5. Consider the available processes given below in the ready queue for execution and with given burst time.

Process No	Arrival Time	Burst time
P1	1	3
P2	2	4
P3	3	2
P4	4	4

- a) What is the time at which all the processes get executed?
b) Find the average turnaround time using the non- pre-emptive FCFS scheduling algorithm.
6. Consider the available processes given below in the ready queue for execution and with given burst time.

Process No	Arrival Time	Burst time
P1	0	2
P2	1	3
P3	5	3
P4	6	4

- a) What is the time at which all the processes get executed?
b) Find the average waiting time and average turnaround time using the non- pre-emptive SJF scheduling algorithm.

1.6 NUMERICAL INEQUALITIES

Inequalities are statements where two quantities are unequal but a relationship exists between them. An inequality compares any two values and shows that one value is less than, greater than, or equal to the value on the other side of the equation.

Any two real numbers associated by the symbol ' $>$ ', ' $<$ ', ' \leq ' or ' \geq ' form a numerical inequality.

For example, $9 < 11$, $18 > 17$ are examples of numerical inequalities.

Let's Check:

Is it possible to construct the triangle whose sides measure 4cm, 11cm and 15 cm respectively? Justify your answer and name the property used.

Relationship between AM and GM:

If a, b are positive numbers and A and G are their arithmetic mean and geometric mean respectively, then

$$\begin{aligned}
 A &= \frac{a+b}{2}, \text{ and} \\
 G &= \sqrt{ab} \\
 A - G &= \frac{a+b}{2} - \sqrt{ab} \\
 &= \frac{a+b-2\sqrt{ab}}{2} \\
 &= \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0 \\
 A - G &\geq 0 \\
 A &\geq G
 \end{aligned}$$

RULES FOR OPERATIONS ON INEQUALITIES

To understand the rules of operations on inequalities let's complete the following table:

INEQUALITY→ OPERATIONS	5 < 7 Using (+3)	Change in direction or sign of inequality	5 < 7 Using (-2)	Change in direction or sign of inequality
ADDITION	5+3 < 7+3 => 8 < 10	No change		
SUBTRACTION	5-3 < 7-3 => 2 < 4	No change		
MULTIPLICATION	5 x 3 < 7 x 3 => 15 < 21	No change	5 x -2 = -10 7 x -2 = -14 -10 > -14	Sign changes
DIVISION	5 ÷ 3 < 7 ÷ 3 => $\frac{5}{3} < \frac{7}{3}$	No change		

By using above table we can enlist the properties as under:

PROPERTIES:

For three real numbers a , b and c :

- If $a > b$, and $b > c$, then $a > c$
- If $a > b$, and $c > 0$, then $a + c > b + c$
- If $a > b$, and $c > 0$, then $a - c > b - c$
- If $a > b$, and $c > 0$, then $a \times c > b \times c$
- If $a > b$, and $c > 0$, then $a \div c > b \div c$
- If $a > b$, and $c < 0$, then $ac < bc$
- If $a > b$, and $c < 0$, then $a \div c < b \div c$

Example 35

If $B > A$, then which expression will have the highest value, given that A and B are positive integers:

- a) $A - B$
- b) $A \times B$
- c) $A + B$
- d) can't say

Solution: As, $B > A$

$$\Rightarrow A < B$$

$$\Rightarrow A - B < 0$$

$$A + B > 0 \text{ and } AB > 0$$

$$\text{If } A = 1, B = 4 \text{ then, } AB < A+B$$

$$\text{If } A = 2, B = 4 \text{ then, } AB > A+B$$

Thus, we can't say which one of $A+B$ and AB has higher value.

Hence, option d is the correct answer

Example 36

If $0 < x < 1$, which of the following is greatest?

- a) x
- b) x^2
- c) $\frac{1}{x}$
- d) $\frac{1}{x^2}$

Solution: $0 < x < 1$

between 0 and 1, $x^2 < x < 1$

$$\frac{1}{x^2} > \frac{1}{x} > 1, \text{ also } 1 > x > 0 \text{ (given)}$$

$$\text{therefore, } \frac{1}{x^2} > \frac{1}{x} > 1 > x > 0$$

$$\text{thus, } \frac{1}{x^2} \text{ is greatest.}$$

Hence, option d is the correct answer

EXERCISE-9

1. Mr. X and Mr. Y have net worth of ₹ 1.93 crores INR and -0.22 crores INR. Represent the above information in the form of inequality.
2. Two players: Player A and Player B are playing a game by rolling a dice. They decided that the player who will get the higher total will be the winner. In total they rolled the dice three times and the observations were recorded as follows:

PLAYER A	2	5	1
PLAYER B	1	4	6

Answer the following questions on the basis of the information given above:

- a) Who is winner of the game.
- b) Represent the above information as numerical inequality.

3. Solve: $4x - 2 < 8$, when $x \in \mathbb{Z}$
4. Show that the numbers 16 and 4, satisfy the numerical inequality $AM \geq GM$.
5. Solve the following inequality:
 - (i) $(-2 \leq -6) < 10$
 - (ii) $2a < a - 4 \leq 3a + 8$
 - (iii) $\frac{(y-1)}{3} + 4 < \frac{(y-5)}{5} - 2$
6. Prove that the following inequality holds true:

$$\sqrt{5} + \sqrt{3} > \sqrt{6} + \sqrt{2}$$
7. Satyarth and Swarit are brothers, Satyarth owns a house which is worth ₹ 3 crores and Swarit owns a farmhouse which is worth ₹ 2.75 crores. But Satyarth has a debt of ₹ 55 lakhs, if they both sell their properties then which of the following statement(s) holds true to represent the above data mathematically:
 - a) Satyarth's net worth is more than Swarit's net worth.
 - b) Swarit's net worth is more than Satyarth's net worth.
 - c) $2.55 < 2.75$
8. Insert the appropriate sign of inequality:

$$\sqrt{3}(\sqrt{50} - \sqrt{32}) \text{ _____ } 3\sqrt{54} + 2\sqrt{24}$$
9. If a and b are positive integers and $\frac{a-b}{6.25} = \frac{8}{2.5}$
 - (i) $b > a$
 - (ii) $b < a$
 - (iii) $b = a$
 - (iv) $b \geq a$
10. If $p > q$ and $r < 0$, then which of the following is true?
 - (i) $pr < qr$
 - (ii) $p - r < q - r$
 - (iii) $p + r < q + r$
 - (iv) None of these

ANSWERS:

Exercise – 1

1. 0 (zero)
2. 1
3. 30
4. 4
5. 2
6. 11, 20, 29, 38
7. 4, 10, 16, 22, 28, 34, 40, 46
8. 1:00 pm
9. 1
10. 2

Exercise – 2

- 24
- 1
- 226
-
-

P				$\varphi(p) + \tau(p) =$
2	1	2	3	3 YES
3	2	2	4	4 YES
5	4	2	6	6 YES
7	6	2	8	8 YES
11	10	2	12	12 YES

6.	P	n				$n + \varphi(n) + \tau(n) =$
	3	6	2	4	12	Yes
	5	10	4	4	18	Yes
	7	14	6	4	24	Yes
	11	22	10	4	36	Yes
	13	26	12	4	42	Yes

EXERCISE – 3

- 20:11
- 76
- 51.03 litres
- ₹ 76
- 400 kg , 600 kg
- 120 g
- 1 : 5
- 29.5 litres

EXERCISE – 4

- 13 km/hr
- 4 km/h
- 4 km/hr
- 12 km
- 10km/hr, 2 km/hr
- 3 km
- $\frac{41}{6}$ hours or $6\frac{5}{6}$ hours

EXERCISE – 5

- 18 hrs
- 18 hrs
- 3 hrs 25 minutes
- 40 hours

5. 28 hours
6. 8 minutes
7. $\frac{5}{2}$ minutes

EXERCISE – 6

1. 440 meters
2. 200meters
3. 88 seconds
4. 3 m/sec
5. 60meters
6. 60%
7. 33 seconds

EXERCISE – 7

1. 3:2:1
2. ₹ 1875
3. ₹ 1,00,000
4. 9:7:6
5. ₹ 2500
6. ₹ 2.25 lakh
7. Nikhil, Priyesh and Ritik should pay ₹ 7872, ₹ 4416 and ₹ 1536 respectively.
8. ₹ 1857

EXERCISE – 8

1. option (D)
S1 is true SJF will always give minimum average waiting time.
S2 is true SJF can cause starvation.
2. **Waiting time = Completion time- Arrival time - Burst time**
WT = CT - AT - BT
3. 19.5 units
4. **7.75 units**
5. 14 units, 3.25 units, 6.5 units
6. 12 units, 0.75 units, 3.75 units

EXERCISE – 9

1. Income of Mr. X > Income of Mr. Y ; $1.93 > -0.22$
2. a) Player B b) $8 < 11$
3. {..., -2, -1, 0, 1, 2}
4. $10 > 8$ satisfied
5. i) $z > -8$
ii) $-6 \leq x < -4$
iii) $(-\infty, -50)$
7. b, c are true
8. $\sqrt{3}(\sqrt{50} - \sqrt{32}) \underline{\hspace{1cm}} < \underline{\hspace{1cm}} 3\sqrt{54} + 2\sqrt{24}$
9. ii) $b < a$
10. i) $pr < qr$

□□□