<u>Exercise 2a</u>

Question 1.
Define a function. What do you mean by the domain and range of a function? Give examples.

Answer:
Definition:A relation R from a set A to a set B is called a function if each element of A has a unique image in B.

It is denoted by the symbol f:A→B which reads 'f' is a function from A to B 'f' maps A to B.

Let f:A→B,then the set A is known as the domain of f & the set B is known as co - domain of f .The set of images of all the elements of A is known as the range of f.

Thus, Domain of f = {a|a ∈ A,(a,f(a)) ∈ f)

Range of f = {f(a) | a∈ A ,f(a) ∈ B }

Example: The domain of *y* = sin *x* is all values of *x* i.e. R , since there are no restrictions on the values for *x*. The range of *y is* betweeen –1 and 1. We could write this as $-1 \leq y \leq 1$.

Question 2.
Define each of the following:

(i) injective function

(ii) surjective function

(iii) bijective function

(iv) many - one function

(v) into function

Give an example of each type of functions.

Answer:
1)injective function

Definition: A function f: A → B is said to be a one - one function or injective mapping if different elements of A have different f images in B.

A function *f* is injective if and only if whenever *f(x) = f(y)*, *x = y*.

Example: $f(x) = x + 9$ from the set of real number R to R is an injective function. When $x = 3$,then :$f(x) = 12$,when $f(y) = 8$,the value of y can only be 3,so $x = y$.

(ii) surjective function

Definition: If the function $f:A \to B$ is such that each element in B (co - domain) is the 'f' image of atleast one element in A , then we say that f is a function of A 'onto' B .Thus $f: A \to B$ is surjective if, for all $b \in B$, there are some $a \in A$ such that $f(a) = b$.

Example: The function $f(x) = 2x$ from the set of natural numbers N to the set of non negative even numbers is a surjective function.

(iii) bijective function

Definition: A function f (from set A to B) is bijective if, for every y in B, there is exactly one x in A such that $f(x) = y$.Alternatively, f is bijective if it is a one - to - one correspondence between those sets, in other words, both injective and surjective.

Example: If $f(x) = x^2$,from the set of positive real numbers to positive real numbers is both injective and surjective.Thus it is a bijective function.

(iv)many - one function

Defintion : A function $f: A \to B$ is said to be a many one functions if two or more elements of A have the same f image in B.

trigonometric functions such as sinx are many - to - one since $\sin x = \sin(2\pi + x) = \sin(4\pi + x)$ and so one...

(v) into function

Definition: If $f:A \to B$ is such that there exists atleast one element in co - domain , which is not the image of any element in the domain , then $f(x)$ is into.

Let $f(x) = y = x - 1000$

$\Rightarrow x = y + 1000 = g(y)$ (say)

Here $g(y)$ is defined for each $y \in I$, but $g(y) \notin N$ for $y \leq -1000$. Hence,f is into.

Question 3.
Give an example of a function which is

(i) one - one but not onto

(ii) one - one and onto

(iii) neither one - one nor onto

(iv) onto but not one - one.

Answer:
(i) one - one but not onto

$f(x) = 6x$

For One - One

$f(x_1) = 6x_1$

$f(x_2) = 6x_2$

put $f(x_1) = f(x_2)$ we get

$6x_1 = 6x_2$

Hence, if $f(x_1) = f(x_2)$, $x_1 = x_2$

Function f is one - one

For Onto

$f(x) = 6x$

let $f(x) = y$,such that $y \in N$

$6x = y$

$\Rightarrow x = \frac{y}{6}$

If $y = 1$

x = $\frac{1}{6} = 0.166667$

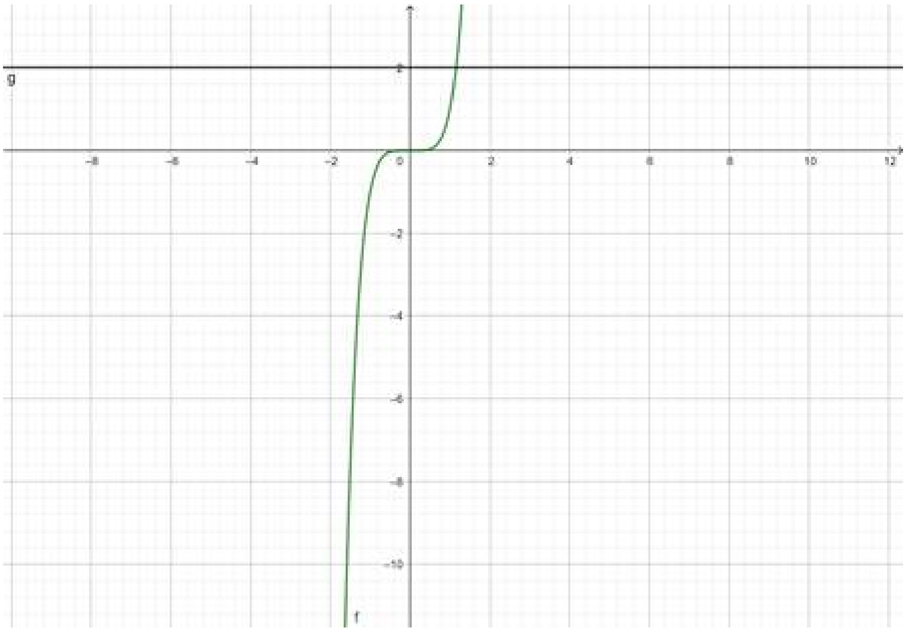
which is not possible as $x \in N$

Hence, f is not onto.

(ii) one - one and onto

$f(x) = x^5$

$\Rightarrow y = x^5$



Since the lines do not cut the curve in 2 equal valued points of y, therefore, the function f(x) is one - one.

The range of f(x) = $(-\infty,\infty)$ = R(Codomain)

$\therefore$f(x) is onto

$\therefore$f(x) is one - one and onto.

(iii) neither one - one nor onto

$f(x) = x^2$

for one one:

$f(x_1) = (x_1)^2$

$f(x_2) = (x_2)^2$

$f(x_1) = f(x_2)$

$\Rightarrow (x_1)^2 = (x_2)^2$

$\Rightarrow x_1 = x_2$ or $x_1 = - x_2$

Since x_1 does not have a unique image it is not one - one

For onto

$f(x) = y$

such that $y \in R$

$x^2 = y$

$\Rightarrow x = \pm\sqrt{y}$

If y is negative under root of a negative number is not real

Hence,f(x) is not onto.

$\therefore$f(x) is neither onto nor one - one

(iv) onto but not one - one.

Consider a function $f: Z \rightarrow N$ such that $f(x) = |x|$.

Since the Z maps to every single element in N twice, this function is onto but not one - one.

Z - integers

N - natural numbers.

Question 4.
Let f : R → R be defined by

$$f(x)=\begin{cases}2x+3, & \text{when} \quad x<-2\\ x^2-2, & \text{when} \quad -2\le x\le 3\\ 3x-1, & \text{when} \quad x>3\end{cases}$$

Find (i) f(2) (ii) f(4) (iii) f(- 1) (iv) f(- 3).

Answer:
i)f(2)

Since $f(x) = x^2 - 2$, when x = 2

$\therefore f(2) = (2)^2 - 2 = 4 - 2 = 2$

$\therefore f(2) = 2$

ii)f(4)

Since f(x) = 3x - 1 , when x = 4

$\therefore f(4) = (3\times4) - 1 = 12 - 1 = 11$

$\therefore f(4) = 11$

iii)f(- 1)

Since $f(x) = x^2 - 2$, when x = - 1

$\therefore f(-1) = (-1)^2 - 2 = 1 - 2 = -1$

$\therefore f(-1) = -1$

iv)f(- 3)

Since f(x) = 2x + 3 , when x = - 3

$\therefore f(-3) = 2\times(-3) + 3 = -6 + 3 = -3$

$\therefore f(-3) = -3$

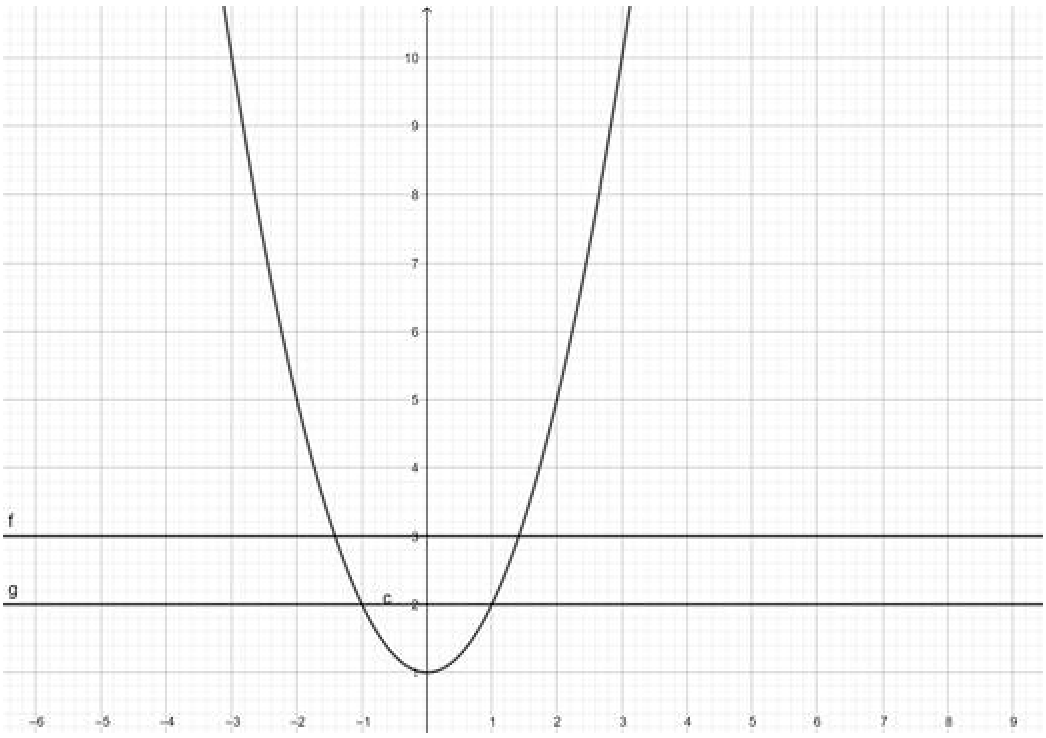
Question 5.
Show that the function f: R → R : $f(x) = 1 + x^2$ is many - one into.

Answer:
To show: f: R → R : $f(x) = 1 + x^2$ is many - one into.

Proof:

$f(x) = 1 + x^2$

$\Rightarrow y = 1 + x^2$



Since the lines cut the curve in 2 equal valued points of y therefore the function f(x) is many one.

The range of $f(x) = [1,\infty) \neq R$(Codomain)

$\therefore f(x)$ is not onto

$\Rightarrow f(x)$ is into

Hence, showed that f: R → R : $f(x) = 1 + x^2$ is many - one into.

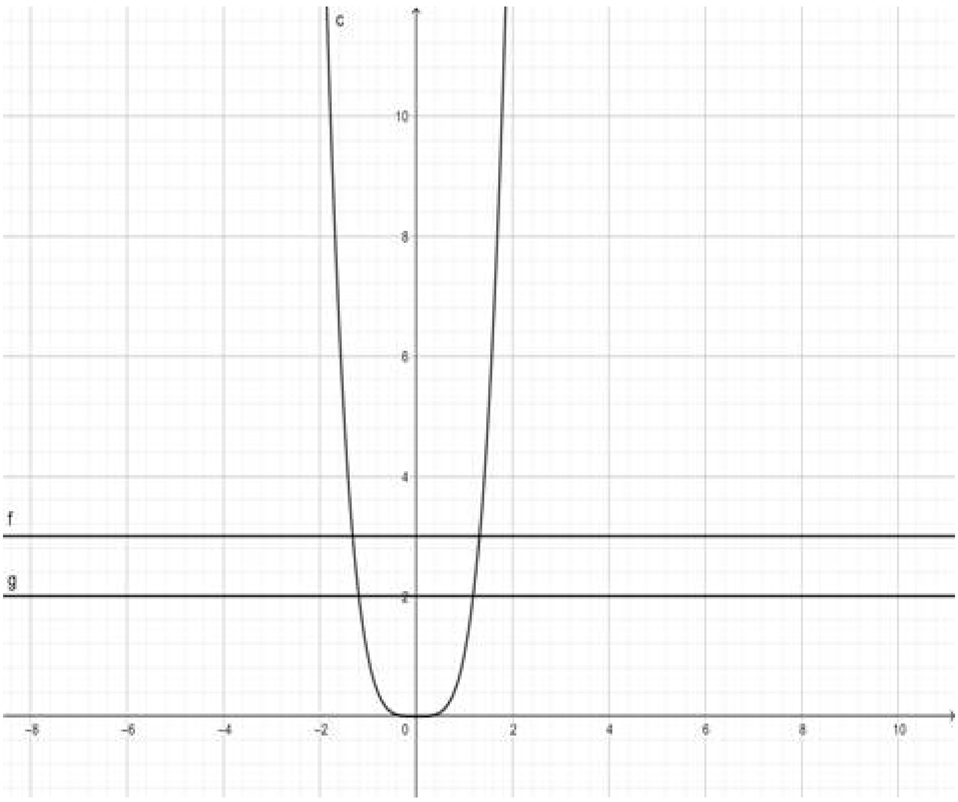
Question 6.
Show that the function f : R → R : f(x) = x^4 is many - one and into.

Answer:
To show: f: R → R : f(x) = x^4 is many - one into.

Proof:

$f(x) = x^4$

$\Rightarrow y = x^4$



Since the lines cut the curve in 2 equal valued points of y, therefore, the function f(x) is many ones.

The range of f(x) = $[0,\infty) \neq R$(Codomain)

$\therefore$f(x) is not onto

$\Rightarrow$f(x) is into

Hence, showed that f: R → R : f(x) = x^4 is many - one into.

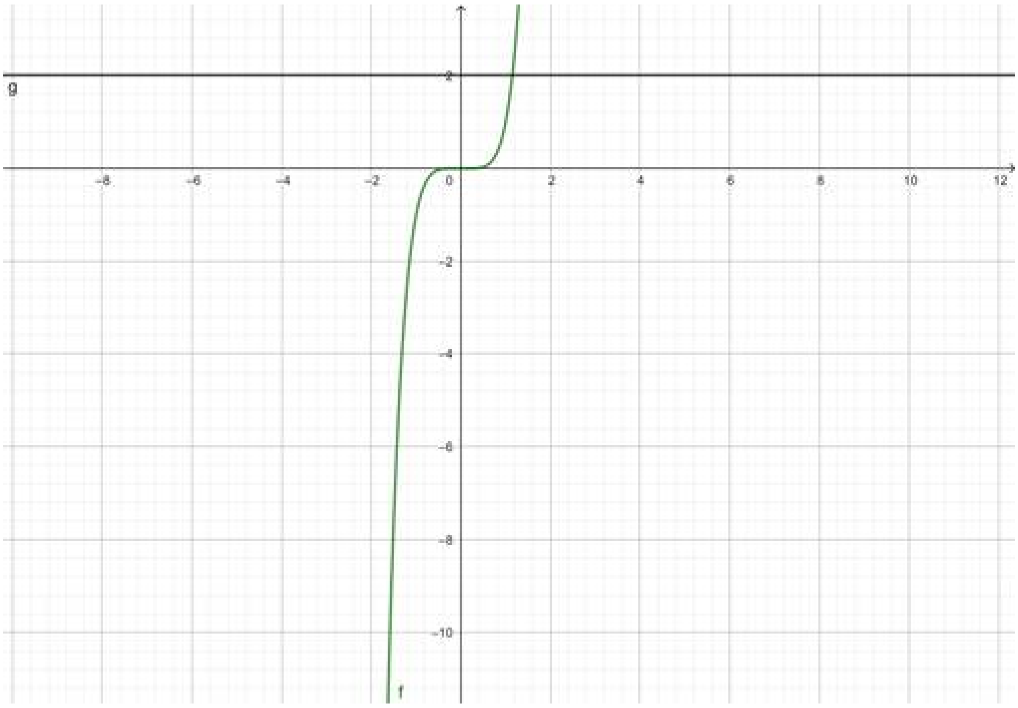
Question 7.
Show that the function f: R → R : $f(x) = x^5$ is one - one and onto.

Answer:
To show: f: R → R : : $f(x) = x^5$ is one - one and onto.

Proof:

$f(x) = x^5$

$\Rightarrow y = x^5$



Since the lines do not cut the curve in 2 equal valued points of y, therefore, the function f(x) is one - one.

The range of $f(x) = (-\infty, \infty)$ = R(Codomain)

$\therefore$f(x) is onto

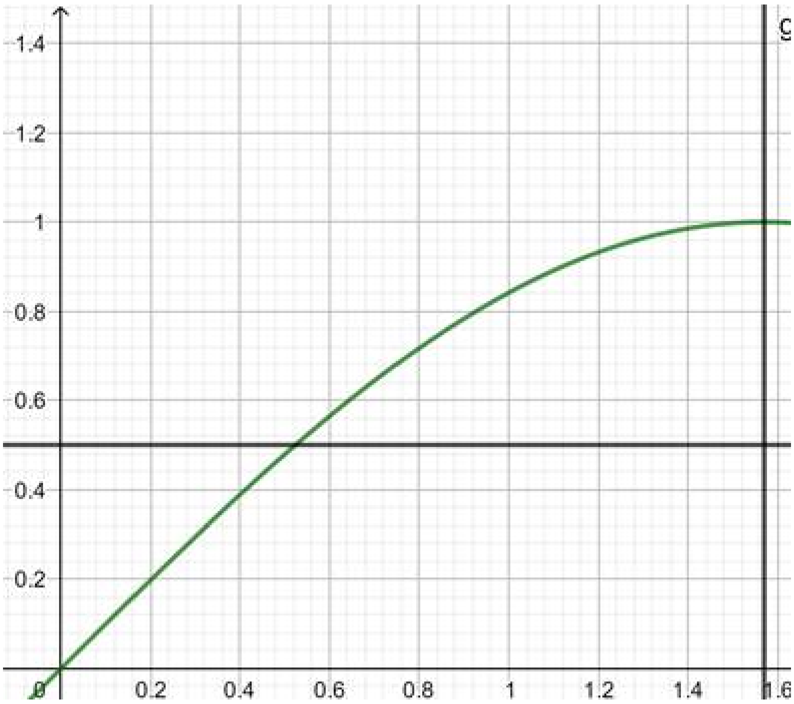
Hence, showed f: R → R : $f(x) = x^5$ is one - one and onto.

Question 8.

Let $f:\left[0,\frac{\pi}{2}\right]\to R: f(x)=\sin x$ and $g:\left[0,\frac{\pi}{2}\right]\to R: g(x)=\cos x$. Show that each one of f and g is one - one but (f + g) is not one - one.

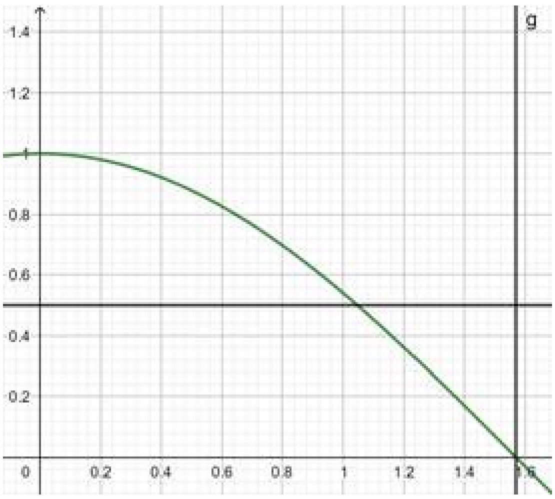
Answer:

$$f:\left[0,\frac{\pi}{2}\right]\to R: f(x)=\sin x$$



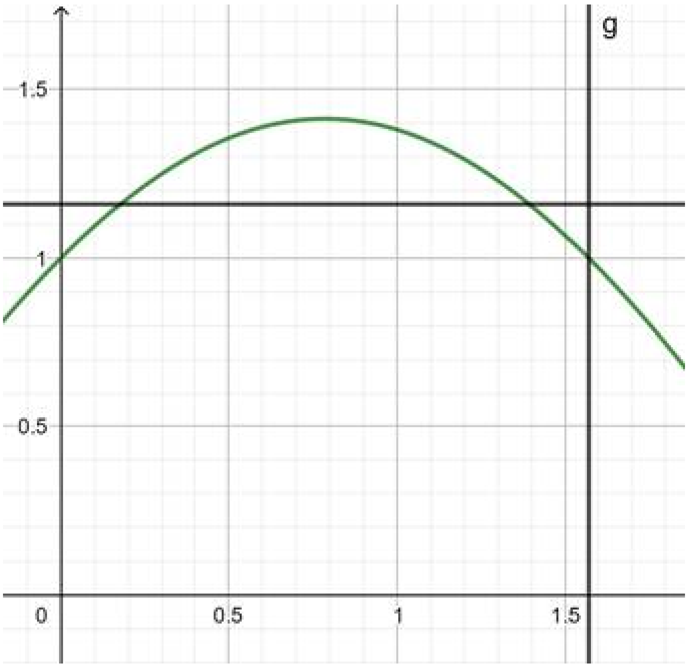
Here in this range, the lines do not cut the curve in 2 equal valued points of y, therefore, the function f(x) = sinx is one - one.

$$g:\left[0,\frac{\pi}{2}\right]\to R: g(x)=\cos x.$$



in this range, the lines do not cut the curve in 2 equal valued points of y, therefore, the function f(x) = cosx is also one - one.

(f + g):$[0,\frac{\pi}{2}]$ →R = sinx + cosx



in this range the lines cut the curve in 2 equal valued points of y, therefore, the function f(x) = cosx + sinx is not one - one.

Hence,showed that each one of f and g is one - one but (f + g) is not one - one.

Question 9.
Show that the function

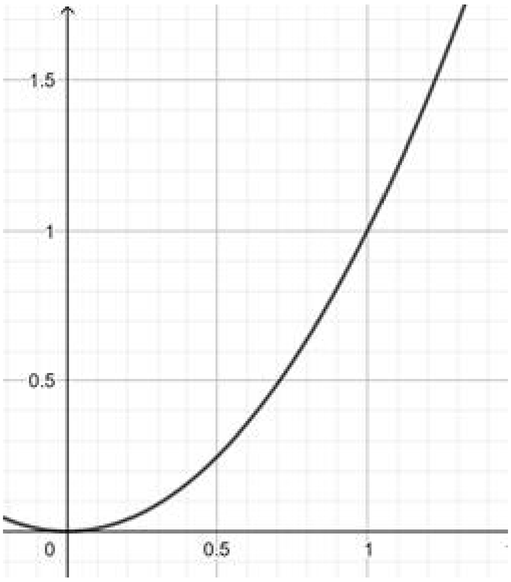
(i) f : N → N : f(x) = x^2 is one - one into.

(ii) f : Z → Z : f(x) = x^2 is many - one into

Answer:
(i) f : N → N : f(x) = x^2 is one - one into.

f(x) = x^2

⇒y = x^2



Since the function f(x) is monotonically increasing from the domain N → N

∴f(x) is one –one

Range of f(x) = $(0,\infty) \neq N$(codomain)

∴f(x) is into

∴f : N → N : f(x) = x^2 is one - one into.

(ii) f : Z → Z : f(x) = x^2 is many - one into

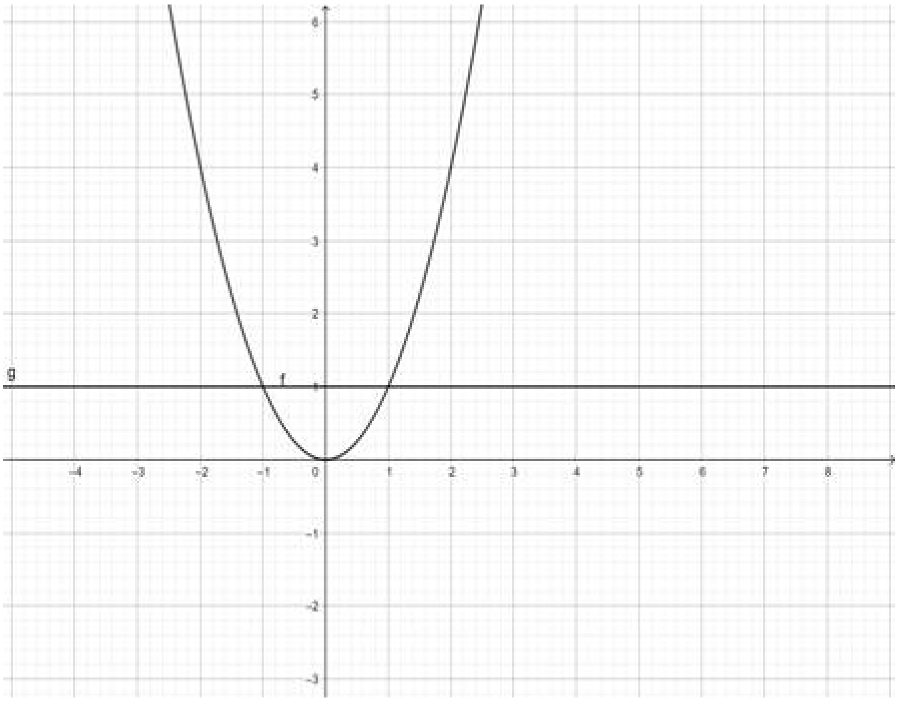
$f(x) = x^2$

$\Rightarrow y = x^2$

in this range the lines cut the curve in 2 equal valued points of y, therefore, the function f(x) = x^2 is many - one .

Range of f(x) = $(0,\infty) \neq Z$(codomain)

∴f(x) is into



$\therefore$ f : Z → Z : $f(x) = x^2$ is many - one into

Question 10.
Show that the function

(i) f : N → N : $f(x) = x^3$ is one - one into

(ii) f : Z → Z : $f(x) = x^3$ is one - one into

Answer:
(i) f : N → N : $f(x) = x^3$ is one - one into.

$f(x) = x^3$

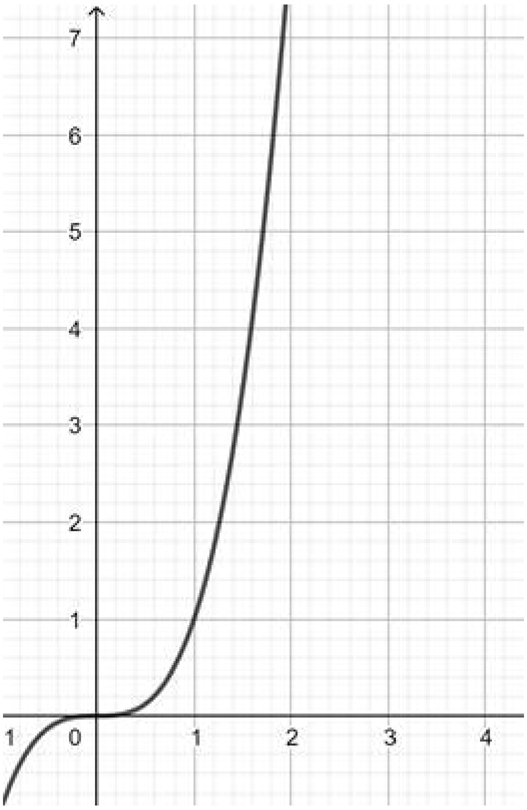
Since the function f(x) is monotonically increasing from the domain N → N

$\therefore$f(x) is one –one

Range of f(x) = $(- \infty,\infty)\neq N$(codomain)

$\therefore$f(x) is into

$\therefore$f : N → N : $f(x) = x^2$ is one - one into.



(ii) $f : Z \to Z : f(x) = x^3$ is one - one into

$f(x) = x^3$

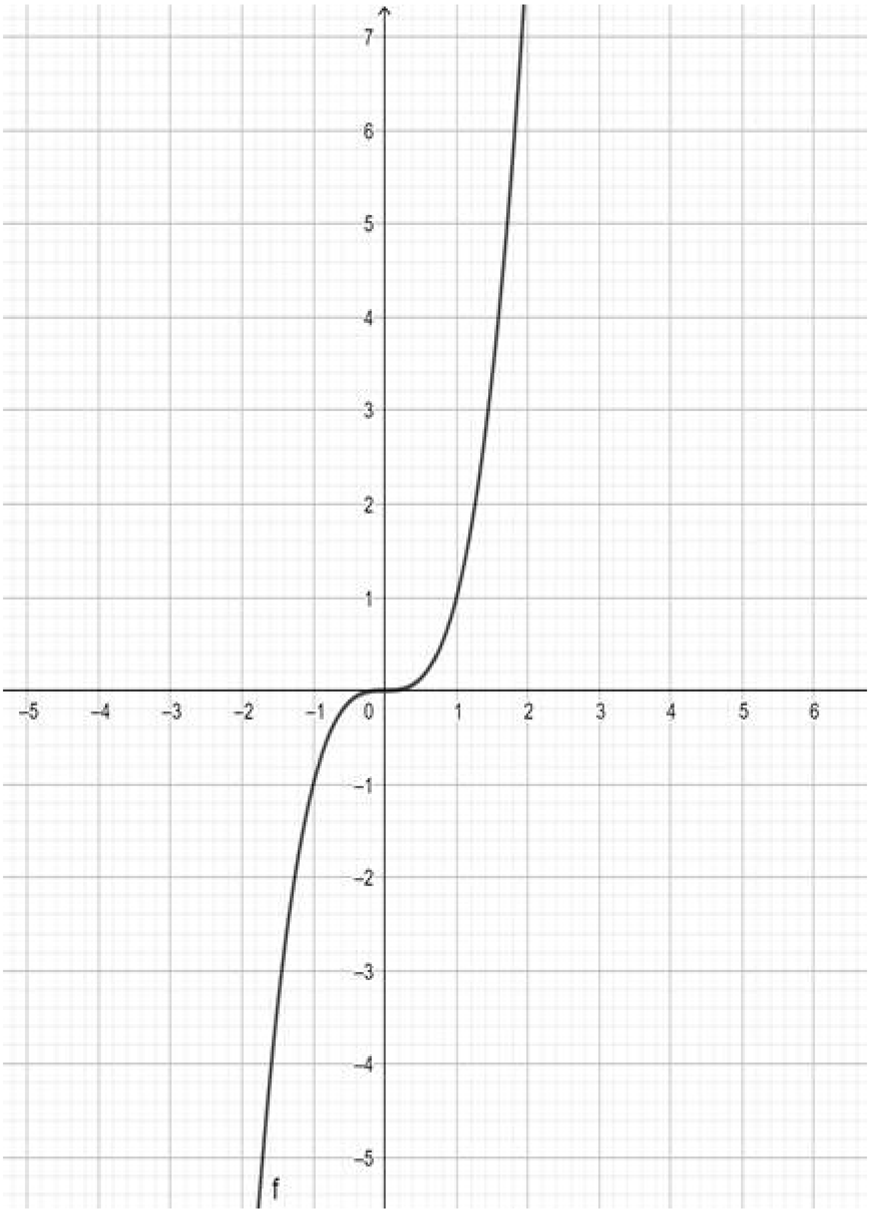
Since the function f(x) is monotonically increasing from the domain $Z \to Z$

$\therefore$f(x) is one –one

Range of $f(x) = (-\infty,\infty) \neq Z$(codomain)

$\therefore$f(x) is into

$\therefore$ $f : Z \to Z : f(x) = x^3$ is one - one into.

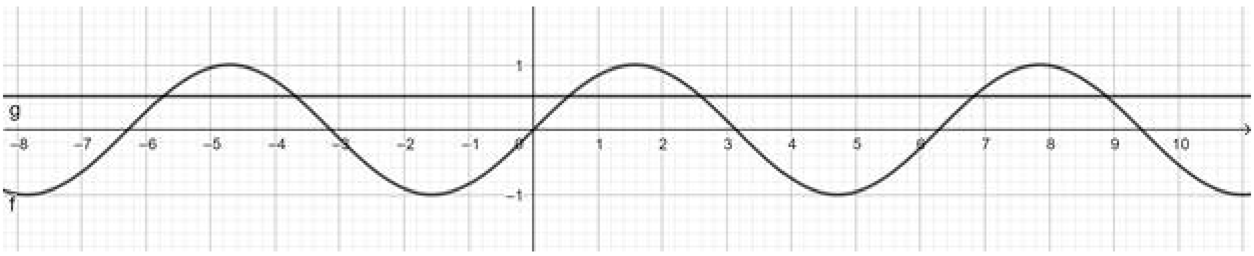


Question 11.
Show that the function f : R → R : f(x) = sin x is neither one - one nor onto.

Answer:
f(x) = sinx

y = sinx

Here in this range, the lines cut the curve in 2 equal valued points of y, therefore, the function f(x) = sinx is not one - one.



Range of $f(x) = [-1,1] \neq R$(codomain)

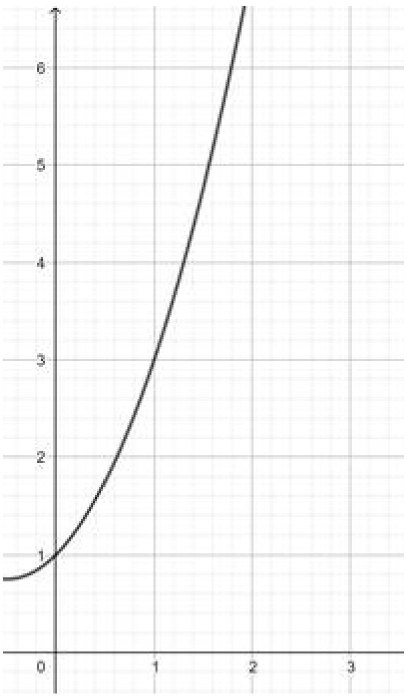
$\therefore f(x)$ is not onto.

Hence, showed that the function $f : R \rightarrow R : f(x) = \sin x$ is neither one - one nor onto.

Question 12.
Prove that the function $f : N \rightarrow N : f(n) = (n^2 + n + 1)$ is one - one but not onto.

Answer:
In the given range of N f(x) is monotonically increasing.

$\therefore f(n) = n^2 + n + 1$ is one one.



But Range of $f(n) = [0.75, \infty) \neq N$(codomain)

Hence,f(n) is not onto.

Hence, proved that the function $f : N \rightarrow N : f(n) = (n^2 + n + 1)$ is one - one but not onto.

Question 13.

Show that the function f: N → Z, defined by

$$f(n)=\begin{cases}\frac{1}{2}(n-1), \text{ when n is odd} \\ -\frac{1}{2}n, \text{ when n is even}\end{cases}$$

is both one - one and onto.

Answer:

$$f(n)=\begin{cases}\frac{1}{2}(n-1), \text{ when n is odd} \\ -\frac{1}{2}n, \text{ when n is even}\end{cases}$$

f(1) = 0

f(2) = - 1

f(3) = 1

f(4) = - 2

f(5) = 2

f(6) = - 3

Since at no different values of x we get same value of y ∴f(n) is one –one

And range of f(n) = Z = Z(codomain)

∴ the function f: N → Z, defined by

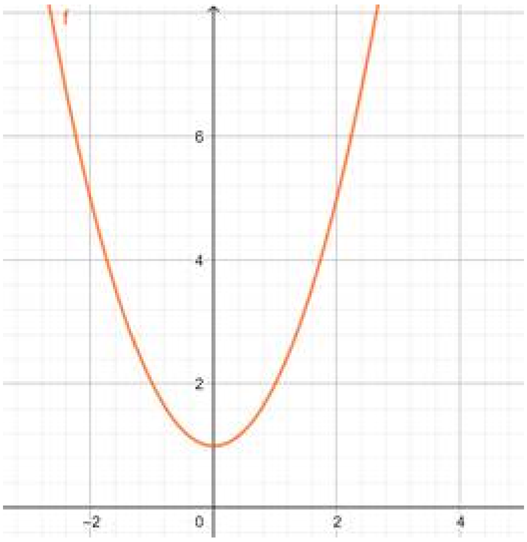
$$f(n)=\begin{cases}\frac{1}{2}(n-1), \text{ when n is odd} \\ -\frac{1}{2}n, \text{ when n is even}\end{cases}$$

is both one - one and onto.

Question 14.
Find the domain and range of the function

F : R → R : $f(x) = x^2 + 1$.

Answer:



Since the function f(x) can accept any values as per the given domain R, therefore, the domain of the function $f(x) = x^2 + 1$ is R.

The minimum value of f(x) = 1

⇒Range of f(x) = [- 1,∞]

i.e range (f) = {y ∈ R : y ≥ 1}

Ans: dom (f) = R and range (f) = {y ∈ R : y ≥ 1}

Question 15.
Which of the following relations are functions? Give reasons. In case of a function, find its domain and range.

(i) f = {(- 1, 2), (1, 8), (2, 11), (3, 14)}

(ii) g = {(1, 1), (1, - 1), (4, 2), (9, 3),
(16, 4)}

(iii) h = {(a, b), (b, c), (c, b), (d, c)}

Answer:

For a relation to be a function each element of 1^{st} set should have different image in the second set(Range)

i) (i) f = {(- 1, 2), (1, 8), (2, 11), (3, 14)}

Here, each of the first set element has different image in second set.

∴f is a function whose domain = { - 1, 1, 2, 3} and range (f) = {2, 8, 11, 14}

(ii) g = {(1, 1), (1, - 1), (4, 2), (9, 3),
(16, 4)}

Here, some of the first set element has same image in second set.

∴ g is not a function.

(iii) h = {(a, b), (b, c), (c, b), (d, c)}

Here, each of the first set element has different image in second set.

∴h is a function whose domain = {a, b, c, d} and range (h) = {b, c}

(range is the intersection set of the elements of the second set elements.)

Question 16.

Find the domain and range of the real function, defined by $f(x)=\frac{x^2}{(1+x^2)}$. Show that f is many - one.

Answer:
For domain $(1 + x^2)\neq 0$

$\Rightarrow x^2 \neq -1$

⇒dom(f) = R

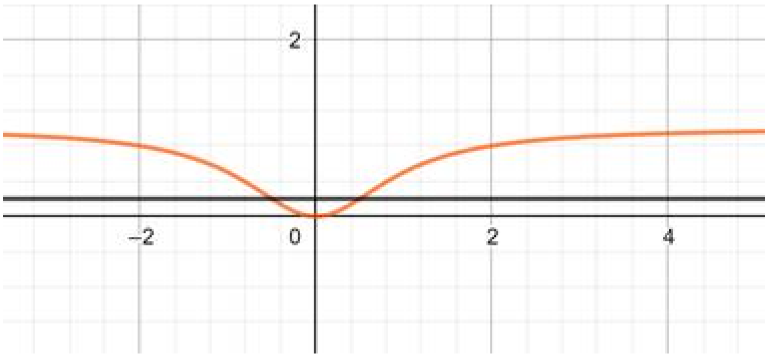
For the range of x:

$\Rightarrow y = \frac{x^2+1-1}{x^2+1} = 1-\frac{1}{x^2+1}$

$y_{min} = 0$ (when x = 0)

$y_{max} = 1$ (when $x = \infty$)

∴range of f(x) = [0,1)



For many one the lines cut the curve in 2 equal valued points of y therefore the function f(x) $= \frac{x^2}{x^2+1}$ is many - one.

Ans:

dom(f) = R

range(f) = [0,1)

function f(x) $= \frac{x^2}{x^2+1}$ is many - one.

Question 17.
Show that the function

$$f : R \rightarrow R : f(x) = \begin{cases} 1, \text{ if } x \text{ is rational} \\ -1, \text{ if } x \text{ is irrational} \end{cases}$$

is many - one into.

Find (i) $f\left(\frac{1}{2}\right)$ (ii) $f\left(\sqrt{2}\right)$ (iii) $f(\pi)$

(iv) $f\left(2+\sqrt{3}\right)$.

Answer:

(i) $f\left(\frac{1}{2}\right)$

Here, x = 1/2,which is rational

$\therefore f(1/2) = 1$

(ii) $f\left(\sqrt{2}\right)$

Here, x = $\sqrt{2}$,which is irrational

$\therefore f(\sqrt{2}) = -1$

(iii) $f(\pi)$

Here, x = ∏, which is irrational

$f(\pi) = -1$

(iv) $f\left(2+\sqrt{3}\right)$.

Here,x = 2 +$\sqrt{3}$, which is irrational

$\therefore f(2+\sqrt{3}) = -1$

Ans. (i) 1 (ii) - 1 (iii) - 1 (iv) - 1