Exercise 16b

Question 1.
Evaluate the following integrals

$$\int_0^1 \frac{dx}{(2x-3)}$$

Answer:
Let $I = \int_0^1 \frac{1}{2x-3} dx$

Let 2x-3=t

⇒ 2dx=dt.

Hence,

$I = \frac{1}{2}\int_0^1 \frac{1}{t} dt = \frac{1}{2}\log_e|t|$

$= \frac{1}{2}\log_e|2x-3|\Big|_0^1$

$\Rightarrow I = \frac{1}{2}\log_e 1 - \frac{1}{2}\log_e 3 = \frac{1}{2}\log_e \frac{1}{3}$

$= -\frac{1}{2}\log_e 3$

$(Since\ \ log_a \frac{1}{b} = -log_a b)$

Question 2.
Evaluate the following integrals

$$\int_0^1 \frac{2x}{(1+x^2)} dx$$

Answer:
Let $I = \int_0^1 \frac{2x}{1+x^2} dx$

Let $1+x^2=t$

$\Rightarrow 2xdx=dt.$

Also,

when x=0, t=1

and

when x=1, t=2

Hence, $I = \int_1^2 \frac{1}{t} dt = \log_e |t| \Big|_1^2$

$= \log_e 2 - \log_e 1$

$= \log_e 2$

Question 3.
Evaluate the following integrals

$$\int_1^2 \frac{3x}{(9x^2-1)} dx$$

Answer:
Let $I = \int_1^2 \frac{3x}{9x^2-1} dx$

Let $9x^2-1=t$

$\Rightarrow 18xdx=dt.$

Also,

when x=1, t=8

and

when x=2, t=35.

Hence,

$I=\frac{1}{6}\int_{8}^{35}\frac{1}{t}\,dt=\frac{1}{6}\log_e t\,\Big|_{8}^{35}=\frac{1}{6}(\log_e 35-\log_e 8)$

Question 4.
Evaluate the following integrals

$$\int_0^1 \frac{\tan^{-1} x}{\left(1+x^2\right)}\,dx$$

Answer:
Let $I=\int_0^1 \frac{tan^{-1}x}{1+x^2}\,dx$

Let $\tan^{-1}x=t$

$\Rightarrow \frac{1}{1+x^2}\,dx=dt.$

Also, when x=0, t=0

and when x=1, $t=\frac{\pi}{4}$

Hence,

$I=\int_0^{\frac{\pi}{4}} t\,dt=\frac{1}{2}t^2\,\Big|_0^{\frac{\pi}{4}}=\frac{\pi^2}{32}$

Question 5.
Evaluate the following integrals

$\int_0^1 \frac{e^x}{1+e^{2x}}\,dx$

Answer:
Let $I = \int_0^1 \frac{e^x}{1+e^{2x}} dx$

Let e^x=t

⇒ e^x dx=dt.

Also,

when x=0, t=1

and

when x=1, t=e.

Hence,

$I = \int_1^e \frac{1}{1+t^2} dt = tan^{-1} t|_1^e$

$= tan^{-1} e - \frac{\pi}{4}$

Question 6.
Evaluate the following integrals

$$\int_0^1 \frac{2x}{\left(1+x^4\right)} dx$$

Answer:
Let $I = \int_0^1 \frac{2x}{1+x^4} dx$

Let x^2=t

⇒ 2xdx=dt.

Also,

when x=0, t=0

and

when x=1, t=1.

Hence,

$I = \int_0^1 \frac{1}{1+t^2} dt$

$= tan^{-1} t \Big|_0^1$

$= \frac{\pi}{4}$

Question 7.
Evaluate the following integrals

$$\int_0^1 x e^{x^2} dx$$

Answer:
Let $I = \int_0^1 x e^{x^2}\, dx$

Let x^2=t

⇒ 2xdx=dt.

Also,

when x=0, t=0

and

when x=1, t=1.

Hence,

$I=\frac{1}{2}\int_0^1 e^t\,dt$

$$=\frac{1}{2}e^t\Big|_0^1$$

$$=\frac{1}{2}(e-1)$$

Question 8.
Evaluate the following integrals

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx$$

Answer:
Let $I=\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

Let $\frac{1}{x}=t$

$\Rightarrow \frac{-1}{x^2}dx=dt.$

Also,

when x=1, t=1

and

when x=2, $t=\frac{1}{2}.$

Hence,

$$I=-\int_1^{\frac{1}{2}} e^t\,dt$$

$$= -e^t \Big|_1^{\frac{1}{2}}$$

$$= e - \sqrt{e}$$

Question 9.
Evaluate the following integrals

$$\int_0^{\pi/6} \frac{\cos x}{(3 + 4\sin x)} dx$$

Answer:
Let $I = \int_0^{\frac{\pi}{6}} \frac{cosx}{3+4sinx} dx$

Let 3+4sinx=t

⇒ 4cosxdx=dt.

Also,

when x=0, t=3

and

when $x = \frac{\pi}{6}$, t=5.

Hence,

$$I = \frac{1}{4}\int_3^5 \frac{1}{t} dt$$

$$= \frac{1}{4}\log_e t \Big|_3^5$$

$$= \frac{1}{4}(\log_e 5 - \log_e 3)$$

Question 10.
Evaluate the following integrals

$$\int_0^{\pi/2} \frac{\sin x}{\left(1+\cos^2 x\right)} dx$$

Answer:
Let $I=\int_0^{\frac{\pi}{2}} \frac{sinx}{1+cos^2 x} dx$

Let cos x=t

⇒ -sin x dx=dt.

Also,

when x=0, t=1

and

when $x=\frac{\pi}{2}$, t=0.

Hence,

$I=-\int_1^0 \frac{1}{1+t^2} dt$

$=-tan^{-1} t \Big|_1^0$

$=\frac{\pi}{4}$

Question 11.
Evaluate the following integrals

$$\int_0^1 \frac{dx}{\left(e^x+e^{-x}\right)}$$

Answer:

Let $I = \int_0^1 \frac{1}{e^x+e^{-x}} dx = \int_0^1 \frac{e^x}{1+e^{2x}} dx$

Let $e^x=t$

$\Rightarrow e^x\,dx=dt.$

Also,

when x=0, t=1

and

when x=1, t=e.

Hence,

$$I = \int_1^e \frac{1}{1+t^2} dt$$

$$= tan^{-1}t\Big|_1^e$$

$$= tan^{-1}e - \frac{\pi}{4}$$

Question 12.

Evaluate the following integrals

$$\int_{1/e}^{e} \frac{dx}{x(\log x)^{1/3}}$$

Answer:

Let $I = \int_{\frac{1}{e}}^{e} \frac{1}{x(\log_e x)^{\frac{1}{3}}} dx$

Let $\log_e x = t$

$\Rightarrow \frac{1}{x}dx = dt.$

Also,

when $x = \frac{1}{e}$, t=-1

and

when x=e, t=1.

Hence,

$$I = \int_{-1}^{1} \frac{1}{t^{\frac{1}{3}}} dt$$

$$= \frac{3}{2} t^{\frac{2}{3}} \Big|_{-1}^{1}$$

$$= \frac{3}{2}(1 - 1)$$

=0

Question 13.
Evaluate the following integrals

$$\int_{0}^{1} \frac{\sqrt{\tan^{-1} x}}{\left(1+x^{2}\right)} dx$$

Answer:
Let $I = \int_0^1 \frac{\sqrt{tan^{-1}x}}{1+x^2} dx$

Let $\tan^{-1}x = t$

$$\Rightarrow \frac{1}{1+x^2} dx = dt.$$

Also,

when x=0, t=0

and

when x=1, $t = \frac{\pi}{4}$

Hence,

$I = \int_0^{\frac{\pi}{4}} \sqrt{t}\, dt$

$= \frac{2}{3} t^{\frac{3}{2}} \Big|_0^{\frac{\pi}{4}}$

$= \frac{\pi^{\frac{3}{2}}}{12}$

Question 14.
Evaluate the following integrals

$$\int_0^{\pi/2} \frac{\sin x}{\sqrt{1+\cos x}} dx$$

Answer:
Let $I = \int_0^{\frac{\pi}{2}} \frac{sinx}{\sqrt{1+cosx}} dx$

Let 1+cos x=t

⇒ -sin x dx=dt.

Also, when x=0, t=2

and

when $x = \frac{\pi}{2}$, t=1

Hence,

$I=-\int_2^1 \frac{1}{\sqrt{t}}dt$

$=-2\sqrt{t}\Big|_2^1$

=2(√2-1)

Question 15.
Evaluate the following integrals

$$\int_0^{\pi/2} \sqrt{\sin x} \cdot \cos^5 x \, dx$$

Answer:
Let $I=\int_0^{\frac{\pi}{2}} \sqrt{sinx}\, cos^5 x dx$

Let sinx=t

⇒ cos x dx=dt.

Also,

when x=0, t=0

and

when $x=\frac{\pi}{2}$, t=1.

Consider $\cos^5 x=\cos^4 x \times \cos x=(1-\sin^2 x)^2 \times \cos x$ (Using $\sin^2 x+\cos^2 x=1$)

Hence,

$I=\int_0^1 \sqrt{x}\,(1-x^2)^2 dx$

$$= \int_0^1 \sqrt{x}\,dx + \int_0^1 x^{\frac{9}{2}}\,dx - 2\int_0^1 x^{\frac{5}{2}}\,dx$$

$$\Rightarrow I = \frac{2}{3}t^{\frac{3}{2}}\Big|_0^1 + \frac{2}{11}t^{\frac{11}{2}}\Big|_0^1 - \frac{4}{7}t^{\frac{7}{2}}\Big|_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{64}{231}$$

Question 16.
Evaluate the following integrals

$$\int_0^{\pi/2} \frac{\sin x \cos x}{\left(1+\sin^4 x\right)}dx$$

Answer:
Let $I = \int_0^{\frac{\pi}{2}} \frac{sinxcosx}{1+sin^4x}dx$

Let $\sin^2 x=t$

$\Rightarrow$ 2sin x cos x=dt.

Also,

when x=0, t=0

and

when $x = \frac{\pi}{2}$, t=1.

Hence,

$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+t^2}dt$

$$= \frac{1}{2} \tan^{-1} t \Big|_0^1$$

$$= \frac{\pi}{8}$$

Question 17.
Evaluate the following integrals

$$\int_0^a \sqrt{a^2 - x^2}\, dx$$

Answer:
Let $I = \int_0^a \sqrt{a^2 - x^2}\, dx$

Let x=a sin t

⇒ a cos t dt=dx.

Also,

when x=0, t=0

and

when x=a, $t = \frac{\pi}{2}$.

Hence,

$$I = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \sin^2 t}\, a \cos t\, dt = a^2 \int_0^{\frac{\pi}{2}} \cos^2 t dt$$

Using $\cos^2 t = \frac{1 + \cos 2t}{2}$, we get

$$I = \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2t) dt$$

$$= \frac{a^2}{2}\left(t + \frac{\sin 2t}{2}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^2}{4}$$

Question 18.
Evaluate the following integrals

$$\int_0^{\sqrt{2}} \sqrt{2 - x^2}\, dx$$

Answer:
Let $I = \int_0^{\sqrt{2}} \sqrt{2 - x^2} dx$

Consider, $I = \int_0^a \sqrt{a^2 - x^2} dx$

Let x=a sin t

⇒ a cos t dt=dx.

Also, when x=0, t=0

and when x=a, $t = \frac{\pi}{2}$.

Hence,

$$I = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \sin^2 t}\, a\cos t\, dt = a^2 \int_0^{\frac{\pi}{2}} \cos^2 t dt$$

Using $\cos^2 t = \frac{1 + \cos 2t}{2}$, we get

$$I = \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2t) dt$$

$$= \frac{a^2}{2}\left(t + \frac{\sin 2t}{2}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^2}{4}$$

Here $a = \sqrt{2}$, hence $I = \frac{\pi}{2}$

Question 19.
Evaluate the following integrals

$$\int_0^a \frac{x^4}{\sqrt{a^2 - x^2}} dx$$

Answer:
Let $I = \int_0^a \frac{x^4}{\sqrt{a^2-x^2}} dx$

Let x=a sin t

⇒ a cos t dt=dx.

Also, when x=0, t=0

and when x=a, $t = \frac{\pi}{2}$.

Hence,

$$I = \int_0^{\frac{\pi}{2}} \frac{a^4 \sin^4 t}{\sqrt{a^2 - a^2 \sin^2 t}} a \cos t dt$$

$$= a^4 \int_0^{\frac{\pi}{2}} \sin^4 t dt$$

Using $\sin^2 t = \frac{1-\cos 2t}{2}$, we get

$$I = a^4 \int_0^{\frac{\pi}{2}} \left(\frac{1 - cos2t}{2}\right)^2 dt$$

$$= \frac{a^4}{4} \int_0^{\frac{\pi}{2}} (1 + cos^2 2t - 2cos2t) dt$$

$$\Rightarrow I = \frac{a^4}{4}\left(t\Big|_0^{\frac{\pi}{2}} - sin2t\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \left(\frac{1+cos4t}{2}\right) dt\right)$$

$$\left(Using\ cos^2 t = \frac{1 + cos2t}{2}\right)$$

Hence,

$$I = \frac{\pi a^4}{8} + \frac{a^4}{4} \times \frac{t}{2}\Big|_0^{\frac{\pi}{2}} + \frac{a^4}{32} sin4t\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{3\pi a^4}{16}$$

Question 20.
Evaluate the following integrals

$$\int_0^a \frac{x}{\sqrt{a^2 + x^2}} dx$$

Answer:
Let $I = \int_0^a \frac{x}{\sqrt{a^2+x^2}} dx$

Let $a^2+x^2=t^2$

$\Rightarrow$ x dx=t dt.

Also, when x=0, t=a

and when x=a, $t = \sqrt{2}a$.

Hence,

$I=\int_a^{\sqrt{2}a}\frac{t}{\sqrt{t^2}}dt$

$= t\Big|_{a}^{\sqrt{2}a}$

=a(√2-1)

Question 21.
Evaluate the following integrals

$$\int_0^2 x\sqrt{2-x}\,dx$$

Answer:
Let $I=\int_0^2 x\sqrt{2-x}\,dx$

Using the property that $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$, we get

$$I=\int_0^2 (2-x)\sqrt{x}dx$$

$$=\int_0^2 2\sqrt{x}dx-\int_0^2 x^{\frac{3}{2}}dx$$

$$=2\times\frac{2}{3}x^{\frac{3}{2}}\Big|_0^2-\frac{2}{5}x^{\frac{5}{2}}\Big|_0^2$$

Hence,

$$I=2\sqrt{2}\left(\frac{4}{3}-\frac{4}{5}\right)$$

$$=\frac{16}{15}\sqrt{2}$$

Question 22.

Evaluate the following integrals

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Answer:

Let $I = \int_0^1 sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Let $f(x) = sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let x=tanθ

⇒ $\theta = \tan^{-1}x$

$$\Rightarrow f(x) = sin^{-1}\left(\frac{2tan\theta}{1+tan^2\theta}\right)$$

$$= sin^{-1}\left(\frac{2tan\theta}{sec^2\theta}\right)$$

$=\sin^{-1}(2\sin\theta\cos\theta)$

$=\sin^{-1}(\sin 2\theta)$

Hence f(x)=2θ

$=2\tan^{-1}x$

Hence $I = 2\int_0^1 1 \times tan^{-1}x dx$

Using integration by parts, we get

$$I = 2xtan^{-1}x\Big|_0^1 - \int_0^1 \frac{2x}{1+x^2}dx$$

$= \frac{\pi}{2} - \int_0^1 \frac{2x}{1+x^2}dx$ -(1)

Let $I' = \int_0^1 \frac{2x}{1+x^2} dx$

Let $1+x^2=t$

$\Rightarrow$ 2x dx=dt.

Also, when x=0, t=1

and when x=1, t=2

Hence,

$I' = \int_1^2 \frac{1}{t} dt = \log_e |t| \, |_1^2$

$= \log_e 2 - \log_e 1$

$= \log_e 2$ –(2)

Substituting value of (2) in (1), we get

$I = \frac{\pi}{2} - \log_e 2$

Question 23.
Evaluate the following integrals

$$\int_0^{\pi/2} \sqrt{1+\cos x}\, dx$$

Answer:
Let $I = \int_0^{\frac{\pi}{2}} \sqrt{1 + cosx} \; dx$

Using $1 + cosx = 2cos^2 \frac{x}{2}$, we get

$I = \sqrt{2} \int_0^{\frac{\pi}{2}} \cos\left(\frac{x}{2}\right) \, dx$

$= 2\sqrt{2}\sin\left(\frac{x}{2}\right)\Big|_0^{\frac{\pi}{2}}$

=2

Question 24.
Evaluate the following integrals

$$\int_0^{\pi/2} \sqrt{1+\sin x}\, dx$$

Answer:
Let $I = \int_0^{\frac{\pi}{2}} \sqrt{1+sinx}\ \ dx$

Using $sin^2\frac{x}{2} + cos\frac{x}{2} = 1$ and $sinx = 2sin\frac{x}{2}cos\frac{x}{2}$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right)^2}\ dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right) dx$$

$$= -2\cos\left(\frac{x}{2}\right)\Big|_0^{\frac{\pi}{2}} + 2\sin\left(\frac{x}{2}\right)\Big|_0^{\frac{\pi}{2}}$$

=-(√2-2) +(√2)

=2

Question 25.
Evaluate the following integrals

25. $\displaystyle\int_0^{\pi/2} \frac{dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)}$

Answer:

Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$

Dividing by $\cos^2$x in numerator and denominator, we get

$$I = \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{a^2 + b^2 tan^2 x} dx$$

Let tan x=t

⇒ sec^2xdx=dt

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{a^2 + b^2 t^2} dt = \frac{1}{b^2} \int_0^{\frac{\pi}{2}} \frac{1}{\frac{a^2}{b^2} + t^2} dt$$

Let $t = \frac{a}{b} tan\theta = tanx$

$$I = \frac{1}{b^2} \int_0^{\frac{\pi}{2}} \frac{\frac{a}{b} sec^2 \theta}{\frac{a^2}{b^2} + \frac{a^2}{b^2} tan^2 \theta} d\theta$$

$$= \frac{1}{ab} \theta$$

$$= \frac{1}{ab} tan^{-1} \left(\frac{b}{a} tanx \right) \Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2ab}$$

Question 26.

Evaluate the following integrals

$$\int_0^{\pi/2} \frac{dx}{\left(1 + \cos^2 x\right)}$$

Answer:

Let $I=\int_0^{\frac{\pi}{2}}\frac{1}{1+cos^2x}dx$

Dividing by $\cos^2$x in numerator and denominator, we get

$$I=\int_0^{\frac{\pi}{2}}\frac{sec^2x}{sec^2x+tan^2x}dx=\int_0^{\frac{\pi}{2}}\frac{sec^2x}{1+2tan^2x}dx$$

Consider $I=\int_0^{\frac{\pi}{2}}\frac{sec^2x}{a^2+b^2tan^2x}dx$

Let tan x=t

⇒ $sec^2xdx=dt$

$$I=\int_0^{\frac{\pi}{2}}\frac{1}{a^2+b^2t^2}dt$$

$$=\frac{1}{b^2}\int_0^{\frac{\pi}{2}}\frac{1}{\frac{a^2}{b^2}+t^2}dt$$

Let $t=\frac{a}{b}tan\theta$

=tan x

$$I=\frac{1}{b^2}\int_0^{\frac{\pi}{2}}\frac{\frac{a}{b}sec^2\theta}{\frac{a^2}{b^2}+\frac{a^2}{b^2}tan^2\theta}d\theta$$

$$=\frac{1}{ab}\theta=\frac{1}{ab}tan^{-1}\left(\frac{b}{a}tanx\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2ab}$$

Here, a=1 and b=$\sqrt{2}$

Hence,

$$I=\frac{\pi}{2\sqrt{2}}$$

Question 27.
Evaluate the following integrals

$$\int_0^{\pi/2}\frac{dx}{\left(4+9\cos^2 x\right)}$$

Answer:
Let $I=\int_0^{\frac{\pi}{2}}\frac{1}{4+9cos^2x}dx$

Dividing by $\cos^2$x in numerator and denominator, we get

$$I=\int_0^{\frac{\pi}{2}}\frac{sec^2x}{4sec^2x+9tan^2x}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{sec^2x}{4+13tan^2x}dx$$

Consider $I=\int_0^{\frac{\pi}{2}}\frac{sec^2x}{a^2+b^2tan^2x}dx$

Let tan x=t

⇒ $sec^2xdx=dt$

$$I=\int_0^{\frac{\pi}{2}}\frac{1}{a^2+b^2t^2}dt$$

$$=\frac{1}{b^2}\int_0^{\frac{\pi}{2}}\frac{1}{\frac{a^2}{b^2}+t^2}dt$$

Let $t = \frac{a}{b} tan\theta$

=tan x

$$I = \frac{1}{b^2}\int_0^{\frac{\pi}{2}} \frac{\frac{a}{b} sec^2\theta}{\frac{a^2}{b^2} + \frac{a^2}{b^2} tan^2\theta} d\theta$$

$$= \frac{1}{ab}\theta$$

$$= \frac{1}{ab} tan^{-1}\left(\frac{b}{a} tanx\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2ab}$$

Here, a=2 and b=√13

Hence,

$$I = \frac{\pi}{4\sqrt{13}}$$

Question 28.
Evaluate the following integrals

$$\int_0^{\pi/2} \frac{dx}{(5 + 4\sin x)}$$

Answer:
Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{5+4sinx} dx$

Using $sinx = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$, we get

$$I=\int_0^{\frac{\pi}{2}}\frac{1}{5+4\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2\left(\frac{x}{2}\right)}{5+5\tan^2\left(\frac{x}{2}\right)+8\tan\left(\frac{x}{2}\right)}dx$$

Let $tan\left(\frac{x}{2}\right)=t$

$\Rightarrow \frac{1}{2}sec^2\left(\frac{x}{2}\right)dx=dt,$

when x=0, t=0 and when $x=\frac{\pi}{2},$ t=1.

Hence, $I=\int_0^1\frac{2}{5+5t^2+8t}dt$

$$=\frac{2}{5}\int_0^1\frac{1}{t^2+\frac{8}{5}t+\frac{16}{25}+\frac{9}{25}}dt$$

$$=\frac{2}{5}\int_0^1\frac{1}{\left(t+\frac{4}{5}\right)^2+\frac{9}{25}}dt$$

Let $t+\frac{4}{5}=u$

⇒ dt=du.

When t=0, $u=\frac{4}{5}$ and when t=1, $u=\frac{9}{5}$.

$$I=\frac{2}{5}\int_{\frac{4}{5}}^{\frac{9}{5}}\frac{1}{(u)^2+\frac{9}{25}}du$$

$$= \frac{2}{5} \times \frac{5}{3} tan^{-1}\left(\frac{5x}{3}\right)\Bigg|_{\frac{4}{5}}^{\frac{9}{5}}$$

$$= \frac{2}{3}\left(tan^{-1}3 - tan^{-1}\left(\frac{4}{3}\right)\right)$$

$$= \frac{2}{3} \times tan^{-1}\left(\frac{3-\frac{4}{3}}{5}\right)$$

$$= \frac{2}{3} tan^{-1}\left(\frac{1}{3}\right)$$

$$\left(Using\ tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)\right)$$

Question 29.
Evaluate the following integrals

$$\int_0^{\pi} \frac{dx}{(6 - \cos x)}$$

Answer:
Let $I = \int_0^{\pi} \frac{1}{6-cosx} dx$

Using $cosx = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$, we get

$$I = \int_0^{\pi} \frac{1}{6 - \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}} dx$$

$$= \int_0^{\pi} \frac{\sec^2\left(\frac{x}{2}\right)}{5 + 7\tan^2\left(\frac{x}{2}\right)} dx$$

Let $tan\left(\frac{x}{2}\right) = t$

$\Rightarrow \frac{1}{2}sec^2\left(\frac{x}{2}\right)dx = dt,$

when x=0, t=0 and when x=π, t=∞.

Hence, $I = \int_0^\infty \frac{2}{5+7t^2}dt$

$$= \frac{2}{7}\int_0^\infty \frac{1}{t^2+\frac{5}{7}}dt$$

$$= \frac{2}{7} \times \sqrt{\frac{7}{5}}tan^{-1}\left(\sqrt{\frac{7}{5}}x\right)\Big|_0^\infty$$

$$\Rightarrow I = \frac{2}{\sqrt{35}}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{\sqrt{35}}$$

Question 30.
Evaluate the following integrals

$$\int_0^\pi \frac{dx}{(5+4\cos x)}$$

Answer:
Let $I = \int_0^\pi \frac{1}{5+4cosx}dx$

Using $cosx = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$, we get

$$I=\int_0^{\pi}\frac{1}{5+4\times\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}}dx$$

$$=\int_0^{\pi}\frac{\sec^2\left(\frac{x}{2}\right)}{9+\tan^2\left(\frac{x}{2}\right)}dx$$

Let $tan\left(\frac{x}{2}\right)=t$

$\Rightarrow\frac{1}{2}sec^2\left(\frac{x}{2}\right)dx=dt,$

when x=0, t=0 and when x=π, t=∞.

Hence, $I=\int_0^{\infty}\frac{2}{9+t^2}dt$

$$=2\int_0^{\infty}\frac{1}{9+t^2}dt$$

$$=2\times\frac{1}{3}tan^{-1}\left(\frac{x}{3}\right)\Big|_0^{\infty}$$

$$\Rightarrow\ I=\frac{2}{3}\left(\frac{\pi}{2}-0\right)$$

$$=\frac{\pi}{3}$$

Question 31.
Evaluate the following integrals

$$\int_0^{\pi/2}\frac{dx}{(\cos x+2\sin x)}$$

Answer:
Let $I=\int_0^{\frac{\pi}{2}}\frac{1}{cosx+2sinx}dx$

Using $sinx = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

And

$cosx = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)},$

we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{1}{\frac{1-tan^2\left(\frac{x}{2}\right)}{1+tan^2\left(\frac{x}{2}\right)} + 2\frac{2\,tan\left(\frac{x}{2}\right)}{1+tan^2\left(\frac{x}{2}\right)}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)+4\tan\left(\frac{x}{2}\right)} dx$$

Let $tan\left(\frac{x}{2}\right) = t$

$\Rightarrow \frac{1}{2} sec^2\left(\frac{x}{2}\right) dx = dt,$

when x=0, t=0

and when $x = \frac{\pi}{2},$ t=1.

Hence,

$I = \int_0^1 \frac{2}{1-t^2+4t} dt$

$$= -2\int_0^1 \frac{1}{t^2-4t+4-5} dt$$

$$= -2\int_0^1 \frac{1}{(t-2)^2-5} dt$$

Let t-2=u

⇒ dt=du.

Also, when t=0, u=-2

and when t=1, u=-1.

$$\Rightarrow\ I = -2\int_{-2}^{-1} \frac{1}{u^2-5}\,dt$$

$$= -2 \times \tfrac{1}{2\sqrt{5}} \log_e \left|\tfrac{x-\sqrt{5}}{x+\sqrt{5}}\right| \mid_{-2}^{-1}$$

$$\left(Using\ \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\,log_e\left|\frac{x-a}{x+a}\right|\right)$$

Hence,

$$I = -\tfrac{1}{\sqrt{5}}\left(\log_e\left|\tfrac{-1-\sqrt{5}}{-1+\sqrt{5}}\right| - \log_e\left|\tfrac{-2-\sqrt{5}}{-2+\sqrt{5}}\right|\right)$$

$$= \frac{-1}{\sqrt{5}}\left(\log_e\left|\frac{\sqrt{5}+1}{\sqrt{5}-1}\right| \times \left|\frac{\sqrt{5}-2}{2+\sqrt{5}}\right|\right)$$

$$\left(Using\ \ log_e\,a - log_e\,b = log_e\frac{a}{b}\right)$$

$$\Rightarrow\ I = \frac{-1}{\sqrt{5}}\left(log_e\left|\frac{3-\sqrt{5}}{3+\sqrt{5}}\right|\right)$$

$$= \frac{-2}{\sqrt{5}}\left(\log_e\left(\frac{3-\sqrt{5}}{2}\right)\right)$$

(Using $\log_e a^b = b\log_e a$)

Question 32.

Evaluate the following integrals

$$\int_0^{\pi} \frac{dx}{(3+2\sin x+\cos x)}$$

Answer:

Let $I=\int_0^{\pi}\frac{1}{3+cosx+2sinx}dx$

Using $sinx=\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

And

$cosx=\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$,

we get

$$\Rightarrow\ I=\int_0^{\pi}\frac{1}{3+\frac{1-tan^2\left(\frac{x}{2}\right)}{1+tan^2\left(\frac{x}{2}\right)}+2\frac{2\ tan\left(\frac{x}{2}\right)}{1+tan^2\left(\frac{x}{2}\right)}}dx$$

$$=\int_0^{\pi}\frac{\sec^2\left(\frac{x}{2}\right)}{4+2\tan^2\left(\frac{x}{2}\right)+4\tan\left(\frac{x}{2}\right)}dx$$

Let $tan\left(\frac{x}{2}\right)=t$

$$\Rightarrow\ \frac{1}{2}sec^2\left(\frac{x}{2}\right)dx=dt,$$

when x=0, t=0

and when $x=\pi$, t=∞.

Hence,

$I=\int_0^{\infty}\frac{1}{(t+1)^2+1}dt$

Let t+1=u

⇒ dt=du.

Also, when t=0, u=1

and when t=∞, u=∞.

$$I=\int_1^{\infty}\frac{1}{u^2+1}dt$$

$$=tan^{-1}u\Big|_1^{\infty}$$

$$=\frac{\pi}{2}-\frac{\pi}{4}$$

$$=\frac{\pi}{4}$$

Question 33.
Evaluate the following integrals

$$\int_0^{\pi/4}\frac{\tan^3 x}{(1+\cos 2x)}dx$$

Answer:
Let $I=\int_0^{\frac{\pi}{4}}\frac{tan^3x}{1+cos2x}dx$

Using 1+cos2x=2cos^{2}x, we get

$$I=\frac{1}{2}\int_0^{\frac{\pi}{4}}tan^3xsec^2x\,dx$$

Let tan x=t

⇒ $\sec^2 x dx = dt$.

when x=0, t=0

and when $x = \frac{\pi}{4}$, t=1.

$$= \frac{1}{2}\int_0^1 t^3\, dt = \frac{t^4}{8}\bigg| \begin{matrix} 1 \\ 0 \end{matrix}$$

$$= \frac{1}{8}$$

Question 34.
Evaluate the following integrals

$$\int_0^{\pi/2} \frac{\sin x \cos x}{\left(\cos^2 x + 3\cos x + 2\right)} dx$$

Answer:
Let $I = \int_0^{\frac{\pi}{2}} \frac{sinxcosx}{cos^2x+3cosx+2} dx$

Let cos x=t

⇒ -sin x dx=dt.

Also, when x=0, t=1

and when $x = \frac{\pi}{2}$, t=0.

Hence,

$$I = -\int_1^0 \frac{t}{t^2+3t+2} dt$$

$$= -\int_1^0 \frac{2(t+1)-(t+2)}{(t+1)(t+2)} dt$$

$$= -\int_1^0 \frac{2}{(t+2)} dt + \int_1^0 \frac{1}{(t+1)} dt$$

$$\Rightarrow\ I = -2\, log_e(t+2) \Big|_1^0 + log_e(t+1) \Big|_1^0$$

$$= -2\log_e 2 + 2\log_e 3 - \log_e 2$$

Hence $I = \log_e 9 - \log_e 8$

$$(Using\ b log_e a = log_e a^b\ and\ log_e a + log_e b = log_e ab)$$

Question 35.
Evaluate the following integrals

$$\int_0^{\pi/2} \frac{\sin 2x}{\left(\sin^4 x + \cos^4 x\right)} dx$$

Answer:
Let $I = \int_0^{\frac{\pi}{2}} \frac{sin2x}{sin^4 x + cos^4 x} dx$

Using sin 2x =2 sin x cos x, we get

$$I = \int_0^{\frac{\pi}{2}} \frac{2sinxcosx}{cos^4 x(tan^4 x + 1)} dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{tanxsec^2 x}{(tan^4 x + 1)} dx$$

Let tan x=t

$\Rightarrow$ $sec^2xdx=dt$.

Also, when x=0, t=0

and when $x = \frac{\pi}{2}$, t=∞.

Hence, $2\int_0^{\infty}\frac{t}{(t^4+1)}dt$

Let x^2=t

⇒ 2xdx=dt.

Also, when x=0, t=0

and when x=∞, t=∞.

Hence, $I=\int_0^{\infty}\frac{1}{1+t^2}dt$

$=tan^{-1}t\Big|_0^{\infty}$

$=\frac{\pi}{2}$

Question 36.
Evaluate the following integrals

$$\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}dx$$

Answer:
Let $I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\sqrt{1+cosx}}{(1-cosx)^{\frac{5}{2}}}dx$

Using $1+cosx=2cos^2\left(\frac{x}{2}\right)$

And

$1-cosx=2sin^2\left(\frac{x}{2}\right)$,

we get

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\sqrt{2}\cos\left(\frac{x}{2}\right)}{4\sqrt{2}\left(\sin\left(\frac{x}{2}\right)\right)^5}dx$$

$$=\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cot\left(\frac{x}{2}\right)cosec^4\left(\frac{x}{2}\right)dx$$

Let $\cot\left(\frac{x}{2}\right)=t$

$\Rightarrow -\frac{1}{2}cosec^2\left(\frac{x}{2}\right)dx=dt.$

Also, when $x=\frac{\pi}{3}$, $t=\sqrt{3}$

and when $x=\frac{\pi}{2}$, t=1

Hence,

$I=-\frac{1}{2}\int_{\sqrt{3}}^{1}t\,(1+t^2)dt$

$$=-\frac{1}{2}\frac{t^2}{2}\Big|_{\sqrt{3}}^{1}-\frac{1}{2}\frac{t^4}{4}\Big|_{\sqrt{3}}^{1}$$

$$=\frac{1}{2}+1$$

$$=\frac{3}{2}$$

Question 37.
Evaluate the following integrals

$$\int_0^1\left(\cos^{-1}x\right)^2dx$$

Answer:

Let $I = \int_0^1 (cos^{-1}x)^2\,dx$

Let x=cost ⇒ dx=-sin t dt.

Also, when x=0, $t = \frac{\pi}{2}$

and when x=1, t=0.

Hence, $I = -\int_{\frac{\pi}{2}}^{0} t^2 sint\,dt$

Using integration by parts, we get

$$I = -\left(t^2 \times -cost \left|_{\frac{\pi}{2}}^{0} + 2\int_{\frac{\pi}{2}}^{0} tcost\,dt\right.\right)$$

$$= -\left(0 - 0 + 2t \times sint \left|_{\frac{\pi}{2}}^{0} - 2\int_{\frac{\pi}{2}}^{0} sint\,dt\right.\right)$$

$$= -\left(-\pi + 2cost \left|_{\frac{\pi}{2}}^{0}\right.\right)$$

Hence, I=π-2

Question 38.
Evaluate the following integrals

$$\int_0^1 x\left(\tan^{-1}x\right)^2 dx$$

Answer:
Let $I = \int_0^1 x(tan^{-1}x)^2\,dx$

Using integration by parts, we get

$$I = \frac{(tan^{-1}x)^2 x^2}{2}\Big|_0^1 - \int_0^1 \frac{2tan^{-1}x}{1+x^2} \times \frac{x^2}{2}dx$$

$$= \frac{\pi^2}{32} - 0 - \int_0^1 \frac{tan^{-1}x}{1+x^2} \times (1 + x^2 - 1)\, dx$$

$$= \frac{\pi^2}{32} - \int_0^1 tan^{-1}x dx + \int_0^1 \frac{tan^{-1}x}{1+x^2} dx$$

Let $\tan^{-1}x=t$

$\Rightarrow \frac{1}{1+x^2} dx = dt.$

When x=0, t=0 and when x=1, $t = \frac{\pi}{4}$.

Hence

$$I = \frac{\pi^2}{32} - tan^{-1}x \times x \Big|_0^1 + \int_0^1 \frac{x}{1+x^2} dx + \int_0^{\frac{\pi}{4}} tdt$$

$$= \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{t^2}{2} \Bigg|_0^{\frac{\pi}{4}} + \int_0^1 \frac{x}{1+x^2} dx$$

Let $1+x^2=y$

$\Rightarrow$ 2xdx=dy.

Also, when x=0, y=1

and when x=1, y=2.

$$I = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \int_1^2 \frac{1}{y} dy$$

$$= \frac{\pi}{4} \left(\frac{\pi}{4} - 1 \right) + \frac{1}{2} \log_e y \Big|_1^2$$

$$= \frac{\pi}{4} \left(\frac{\pi}{4} - 1 \right) + \frac{1}{2} \log_e 2.$$

Question 39.
Evaluate the following integrals

$$\int_0^1 \sin^{-1}\sqrt{x}\,dx$$

Answer:
Let $I = \int_0^1 \sin^{-1}\sqrt{x}\,dx$

Let $\sqrt{x}=t$

$$\Rightarrow \frac{1}{2\sqrt{x}}dx = dt$$

or

dx=2tdt.

When, x=0, t=0

and when x=1, t=1.

Hence,

$$I = 2\int_0^1 t\sin^{-1}t\,dt$$

Using integration by parts, we get

$$I = 2\left(\sin^{-1}t \times \frac{t^2}{2}\Big|_0^1 - \int_0^1 \frac{1}{\sqrt{1-t^2}} \times \frac{t^2}{2}dt\right)$$

$$= \frac{\pi}{2} - \int_0^1 \frac{t^2}{\sqrt{1-t^2}}dt$$

Let t=sin y

⇒ dt=cos y dy.

When t=0, y=0, when t=1, $y = \frac{\pi}{2}$.

$I = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} sin^2 y dy$ ..... (1)

Using, $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we get

$I = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} cos^2 y dy$ .....(2)

Adding (1) and (2), we get

$$2I = \pi - \int_0^{\frac{\pi}{2}} dy$$

$$= \pi - \frac{\pi}{2}$$

Hence,

$I = \frac{\pi}{4}$

Question 40.
Evaluate the following integrals

$$\int_0^a \sin^{-1} \sqrt{\frac{x}{a+x}} dx$$

Answer:

Let $I = \int_0^a sin^{-1} \sqrt{\frac{x}{a+x}} dx$

Let x=a $\tan^2$y

⇒ dx=2a tan y $\sec^2$y dy.

Also, when x=0, y=0

and when x=a, $y=\frac{\pi}{4}$

Hence $I=\int_0^{\frac{\pi}{4}} sin^{-1}\left(\sqrt{\frac{atan^2 y}{a+atan^2 y}}\right) 2a\tan y\, sec^2\, y\, dy = 2a\int_0^{\frac{\pi}{4}} y\tan y\ sec^2 ydy$

Using integration by parts, we get

$$I=2a\left(y\int_0^{\frac{\pi}{4}} tanysec^2 ydy-\int_0^{\frac{\pi}{4}}\left(\int tanysec^2 ydy\right)dy\right)$$

Let tan y=t

⇒ $sec^2 ydy=dt$.

Also, when y=0, t=0

and when $y=\frac{\pi}{4}$, t=1.

Also, $y=tan^{-1}t$

$$\Rightarrow\ dy=\frac{dt}{1+t^2}$$

$$I=2a\left(tan^{-1}t\int tdt\,\Big|_0^1-\int_0^1\left(\int tdt\right)\frac{dt}{1+t^2}\right)$$

$$=2a\left(\frac{tan^{-1}t\times t^2}{2}\Big|_0^1\right)-2a\int_0^1\frac{t^2}{2}\frac{dt}{1+t^2}$$

$$=\frac{a\pi}{4}-a\int_0^1\frac{t^2}{1+t^2}dt$$

Let $I'=\int_0^1\frac{t^2}{1+t^2}dt$

$$=\int_0^1\frac{1+t^2-1}{1+t^2}dt$$

$$= \int_0^1 dt - \int_0^1 \frac{1}{1+t^2} dt$$

$$= t \Big|_0^1 - tan^{-1} t \Big|_0^1$$

Hence $I' = 1 - \frac{\pi}{4}$

Substituting value of I′ in I, we get

$$I = \frac{a\pi}{4} - a\left(1 - \frac{\pi}{4}\right)$$

$$= a\left(\frac{\pi}{2} - 1\right)$$

Question 41.
Evaluate the following integrals

$$\int_0^9 \frac{dx}{\left(1+\sqrt{x}\right)}$$

Answer:
Let $I = \int_0^9 \frac{1}{1+\sqrt{x}} dx$

Let √x=u

$$\Rightarrow \frac{1}{2\sqrt{x}} dx = du$$

$= \frac{1}{2u} dx$ or dx=2udu.

Also, when x=0, u=0 and x=9, u=3.

Hence,

$$I = \int_0^3 \frac{2u}{1+u} du$$

$$= 2\left(\int_0^3 \frac{u+1-1}{1+u} du\right)$$

$$= 2\left(\int_0^3 du - \int_0^3 \frac{1}{1+u} du\right)$$

$$I = 2u \Big|_0^3 - \log_e(1+u) \Big|_0^3$$

$$= 6 - 2\log_e 4$$

$$= 6 - 4\log_e 2$$

$$(Using\ log_e a^b = b\, log_e a)$$

Question 42.
Evaluate the following integrals

$$\int_0^1 x^3 \sqrt{1+3x^4}\, dx$$

Answer:
Let $I = \int_0^1 x^3\sqrt{1+3x^4}\, dx$

Let $1+3x^4=t$

⇒ $12x^3dx=dt$.

Also, when x=0, t=1 and when x=1, t=4.

$$I = \frac{1}{12}\int_1^4 \sqrt{t}\, dt$$

$$= \frac{1}{12} \times \frac{2}{3} t^{\frac{3}{2}} \Big|_1^4$$

$$= \frac{7}{18}$$

Question 43.
Evaluate the following integrals

$$\int_0^1 \frac{\left(1-x^2\right)}{\left(1+x^2\right)^2} dx$$

Answer:
Let $I = \int_0^1 \frac{1-x^2}{(1+x^2)^2} dx$

Let $I' = \int_0^1 \frac{1}{(1+x^2)^2} dx$

Let x=tan t

⇒ dx=$\sec^2$tdt.

Also when x=0, t=0 and when x=1, $t = \frac{\pi}{4}$.

Hence, $I' = \int_0^{\frac{\pi}{4}} \frac{sec^2 t}{(1+tan^2 t)^2} dt$

$$= \int_0^{\frac{\pi}{4}} cos^2 t dt$$

Using $cos^2 t = \frac{1+cos2t}{2}$, we get

$$I' = \int_0^{\frac{\pi}{4}} \left(\frac{1+cos2t}{2}\right) dt$$

$$= \frac{t}{2}\Bigg|_0^{\frac{\pi}{4}} + \frac{sin2t}{4}\Bigg|_0^{\frac{\pi}{4}}$$

$$=\frac{\pi+2}{8}$$

Let $I''=\int_0^1 \frac{x^2}{(1+x^2)^2}dx$

$$=\int_0^1 x\times\frac{x}{(1+x^2)^2}dx$$

$$=x\int_0^1 \frac{x}{(1+x^2)^2}dx-\int_0^1\left(\int\frac{x}{(1+x^2)^2}dx\right)dx$$

Let 1+x^2=t ⇒ 2xdx=dt.

When x=0, t=1 and when x=1, t=2.

$$I''=\sqrt{t-1}\times\frac{1}{2}\int_1^2\frac{1}{t^2}dt-\int_1^2\frac{\left(\frac{1}{2}\int\frac{1}{t^2}dt\right)dt}{2\sqrt{t-1}}$$

$$=-\frac{\sqrt{t-1}}{2}\times\frac{1}{t}\Big|_1^2+\int_1^2\frac{dt}{4t\sqrt{t-1}}$$

$$=-\frac{1}{4}+\int_1^2\frac{dt}{4t\sqrt{t-1}}$$

Substituting t=1+x^2

⇒ 2xdx=dt.

When t=1, x=0 and when t=2, x=1.

$$I''=-\frac{1}{4}+\int_0^1\frac{2xdx}{4x(1+x^2)}$$

$$=-\frac{1}{4}+\frac{1}{2}tan^{-1}x\Big|_0^1$$

$$=\frac{\pi-2}{8}$$

Hence,

$$I = \frac{\pi + 2}{8} - \frac{\pi - 2}{8}$$

$$= \frac{1}{2}$$

Question 44.
Evaluate the following integrals

$$\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$$

Answer:
Let $I = \int_1^2 \frac{1}{(x+1)\sqrt{x^2-1}} dx$

Let x=sect

⇒ dx=sec t tan t dt.

Also,

when x=1, t=0 and when x=2, $t = \frac{\pi}{3}$

Hence,

$$I = \int_0^{\frac{\pi}{3}} \frac{secttant}{(sect+1)\sqrt{sec^2 t-1}} dt$$

$$= \int_0^{\frac{\pi}{3}} \frac{sect}{(sect + 1)} dt$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{(1 + cost)} dt$$

Using $1 + cost = 2cos^2\left(\frac{t}{2}\right)$, we get

$$I = \frac{1}{2}\int_0^{\frac{\pi}{3}} sec^2\left(\frac{t}{2}\right) dt$$

$$= \tan\left(\frac{t}{2}\right)\Big|_0^{\frac{\pi}{3}}$$

$$= \frac{1}{\sqrt{3}}$$

Question 45.
Evaluate the following integrals

$$\int_0^{\pi/2}\left(\sqrt{\tan x} + \sqrt{\cot x}\right)dx$$

Answer:
Let $I = \int_0^{\frac{\pi}{2}}(\sqrt{tanx} + \sqrt{cotx})dx = \int_0^{\frac{\pi}{2}}\frac{sinx+cosx}{\sqrt{sinxcosx}}dx$

Let sin x- cos x=t

⇒ (cos x + sin x)dx=dt.

When x=0, t=-1 and $x = \frac{\pi}{2}$, t=1.

Also, t^2=(sin x – cos x)^2

=sin^{2}x+cos^{2}x-2sinxcosx

=1-2sinxcosx

or

$$sincosx = \frac{1-t^2}{2}$$

Hence $I=\sqrt{2}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}dt$

Let t=sin y

⇒ dt=cos y dy.

Also, when t=-1, $y=-\frac{\pi}{2}$

and when t=1, $y=\frac{\pi}{2}$.

$$I=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{cosy}{\sqrt{1-sin^2 y}}dy$$

$$=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}dy=\pi\sqrt{2}$$

Question 46.
Evaluate the following integrals

$$\int_{2}^{3}\frac{(2-\mathrm{x})}{\sqrt{5\mathrm{x}-6-\mathrm{x}^2}}\mathrm{dx}$$

Answer:
Let $I=\int_{2}^{3}\frac{2-x}{\sqrt{5x-6-x^2}}dx$

Let,

$$2-x=a\frac{d}{dx}(5x-6-x^2)+b$$

=-2ax+5a+b

Hence -2a=-1 and 5a+b=2.

Solving these equations,

we get $a=\frac{1}{2}$ and $b=-\frac{1}{2}$.

We get,

$$I=\frac{1}{2}\int_2^3 \frac{-2x+5}{\sqrt{5x-6-x^2}}dx-\frac{1}{2}\int_2^3 \frac{1}{\sqrt{5x-6-x^2}}dx$$

Let $I'=\int_2^3 \frac{-2x+5}{\sqrt{5x-6-x^2}}dx$

Let $5x-6-x^2=t$

$\Rightarrow$ (5-2x) dx=dt.

When x=2, t=0 and when x=3, y=0.

Hence $I'=\int_0^0 \frac{1}{\sqrt{t}}dt=0$

$$\left(Since\ \int_a^a f(x)dx=0\right)$$

Let,

$$I''=\int_2^3 \frac{1}{\sqrt{5x-6-x^2}}dx$$

$$=\int_2^3 \frac{1}{\sqrt{\frac{1}{4}-\left(x-\frac{5}{2}\right)^2}}$$

$$=sin^{-1}\left(\frac{x-\frac{5}{2}}{\frac{1}{2}}\right)$$

$$=sin^{-1}(2x-5)\Big|_2^3$$

$=\pi$

Hence,

$$I = \frac{1}{2} \times 0 - \frac{1}{2} \times \pi$$

$$= -\frac{\pi}{2}$$

Question 47.
Evaluate the following integrals

$$\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^3} d\theta$$

Answer:
Let $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{cosx}{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)^3} dx$

Using $cosx = cos^2\left(\frac{x}{2}\right) - sin^2\left(\frac{x}{2}\right)$, we get

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\left(\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right)^2} dx$$

Let $\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) = t$

$\Rightarrow \frac{1}{2}\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) dx = dt.$

Also, when $x = \frac{\pi}{4}, t = \cos\left(\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{8}\right) = \alpha(Let)$

and when $x = \frac{\pi}{2}, t = \sqrt{2}$

$$I = \int_{\alpha}^{\sqrt{2}} \frac{2}{t^2} dt$$

$$= -2 \times \frac{1}{t}\Big|_{\alpha}^{\sqrt{2}}$$

$$= \frac{2}{\cos\left(\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{8}\right)} - \sqrt{2}$$

Question 48.
Evaluate the following integrals

$$\int_{0}^{(\pi/2)^{1/3}} x^2 \sin x^3 dx$$

Answer:
Let $I = \int_0^{\left(\frac{\pi}{2}\right)^{\frac{1}{3}}} x^2 \sin(x^3)\, dx$

Let $x^3=t$

$\Rightarrow 3x^2=dt.$

Also, when x=0, t=0 and when $x = \left(\frac{\pi}{2}\right)^{\frac{1}{3}}, t = \frac{\pi}{2}.$

Hence, $I = \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin(t)\, dt$

$$= \frac{-1}{3} cost \Big|_{0}^{\frac{\pi}{2}}$$

$$= -\frac{1}{3}(0-1)$$

$$= \frac{1}{3}$$

Question 49.

Evaluate the following integrals

$$\int_1^2 \frac{dx}{x(1+\log x)^2}$$

Answer:

Let $I = \int_1^2 \frac{1}{x(1+\log_e x)^2} dx$

Let $1 + \log_e x = t$

$\Rightarrow \frac{1}{x} dx = dt.$

Also, when x=1, t=1 and when x=2, $t = 1 + \log_e 2$

Hence $I = \int_1^{1+\log_e 2} \frac{1}{t^2} dt$

$$= -\frac{1}{t}\Big|_{1}^{1+\log_e 2}$$

$$= 1 - \frac{1}{1+\log_e 2}$$

$$= \frac{\log_e 2}{1+\log_e 2}$$

Question 50.

Evaluate the following integrals

$$\int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x} dx$$

Answer:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosecx cotx}{1+cosec^2 x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx}{1+sin^2 x} dx$

Let sinx=t

$\Rightarrow$ cos x dx=dt.

Also, when $x = \frac{\pi}{6}$, $t = \frac{1}{2}$ and when $x = \frac{\pi}{2}$, t=1.

$$I = \int_{\frac{1}{2}}^{1} \frac{1}{1+t^2} dt$$

$$= tan^{-1}t \Big|_{\frac{1}{2}}^{1}$$

$$= tan^{-1}1 - tan^{-1}\left(\frac{1}{2}\right)$$

$$= tan^{-1}\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)$$

$$= tan^{-1}\left(\frac{1}{3}\right)$$

(Using $tan^{-1}a - tan^{-1}b = tan^{-1}\left(\frac{a-b}{1+ab}\right)$)